Closed Categories *

By

SAMUEL EILENBERG ** and G. MAX KELLY

Introduction

In the usual theory of categories, with any two objects A, B of a category $\mathscr{A}$ there is associated a *set* $\mathscr{A}(A\,B)$ of morphisms of A into B. Frequently the set $\mathscr{A}(A\,B)$ is endowed with an additional structure such as a privileged element or an abelian group structure. It has become clear that as the ramifications of the theory of categories increase, the structures that $\mathscr{A}(A\,B)$ will carry will be richer and more complex. The need for a general theory has been widely felt for some time, and beginnings have been made in various directions and often under restrictive hypotheses; e.g. by MACLANE [*15*], KELLY [*10*], BÉNABOU [*3*], LINTON [*12*].

In order to gain sufficient generality one should assume that $\mathscr{A}(A\,B)$ is an object of some category $\mathscr{V}_0$, that this category $\mathscr{V}_0$ is equipped with a functor $V: \mathscr{V}_0 \to \mathscr{S}$ into the category $\mathscr{S}$ of sets, and that $V\mathscr{A}(A\,B)$ is the set of morphisms $A \to B$ in $\mathscr{A}$. One then can write $\mathscr{A}_0(A\,B)$ for $V\mathscr{A}(A\,B)$, and distinguish the "enriched category" $\mathscr{A}$ from the ordinary category $\mathscr{A}_0$ that underlies it. Upon inspection it turned out that the categories $\mathscr{V}_0$ which occur in this connexion are endowed with a structure considerably richer than that of a category. We propose calling these "closed categories", and we may best describe them by citing two examples.

Let $\mathscr{B}$ be the category of real or complex Banach spaces. In order to ensure that an isomorphism is an isometry we take the morphisms $f: A \to B$ to be the linear transformations with norm $\|f\| \leqq 1$; these then form the set $\mathscr{B}(A\,B)$. In addition however we may consider *all* the bounded linear transformations $A \to B$; these form in a natural fashion a Banach space $(A\,B)$. This yields an "internal Hom-functor" $\mathscr{B}^* \times \mathscr{B} \to \mathscr{B}$. The set $\mathscr{B}(A\,B)$ is obtained from the Banach space $(A\,B)$ by applying the functor $\mathscr{B} \to \mathscr{S}$ which to each Banach space assigns its unit ball considered as a set. In addition we have a special "unit"

* Received March 7, 1966.

** Supported by Office of Naval Research.

Banach space I, namely $\boldsymbol{R}$ or $\boldsymbol{C}$ as appropriate, and a natural isomorphism $i: A \cong (IA)$. There is also a composition law that will be discussed later.

As a second example consider a topological space X and let $\mathscr{S}h\, X$ be the category of sheaves of sets over X. For any two such sheaves A and B, we then have the set $\mathscr{S}h\, X(A\,B)$. Given any open subset U of X, we may also consider the set $\mathscr{S}h\, U(A\,|\,U,\ B\,|\,U)$ where $A\,|\,U$ is the restriction of the sheaf A to U. These sets form a pre-sheaf on X and define a sheaf that we shall denote by $(A\,B)$. This again yields an "internal Hom-functor". The set $\mathscr{S}h\, X(A\,B)$ is obtained from $(A\,B)$ by applying the functor $\Gamma: \mathscr{S}h\, X \to \mathscr{S}$ which to each sheaf assigns its set of sections. Again there is a privileged unit sheaf I and a natural isomorphism $i: A \cong (IA)$.

The basic elements of the structure of a closed category now become clear. First there is an ordinary category $\mathscr{V}_0$, represented by $\mathscr{B}$ or $\mathscr{S}h X$ in the examples above. Next there is a functor $V: \mathscr{V}_0 \to \mathscr{S}$. Then an internal Hom-functor $\mathscr{V}_0^* \times \mathscr{V}_0 \to \mathscr{V}_0$, denoted by (AB), and such that $V(AB)$ is the set $\mathscr{V}_0(AB)$ of morphisms $A \to B$. Further there is a unit object I and a natural isomorphism $i: A \cong (IA)$. What is still lacking is a composition law that generalizes the ordinary composition law in $\mathscr{V}_0$. The notion of composition is usually linked with a notion of "product". However the need of a product for defining composition is only superficial. Indeed if we consider an ordinary category $\mathscr{A}$ and for a fixed $A \in \mathscr{A}$ we wish to consider the left represented functor $L^A = \mathscr{A}(A\,-): \mathscr{A} \to \mathscr{S}$, then we must indicate the effect of L^A on morphisms; i.e. we must give a morphism

$$L^A_{BC}: \mathscr{A}(BC) \to \mathscr{S}(\mathscr{A}(AB), \mathscr{A}(A\,C))\,;$$

and this morphism is nothing but the composition law

$$(L^A_{BC} f)g = fg\,.$$

Generalizing this we define the composition law in a closed category to be a morphism

$$L^A_{BC}: (BC) \to ((AB), (AC))\,.$$

This is the last needed primitive term for a closed category, and we denote the whole set of data $(\mathscr{V}_0, V, (AB), I, i, L)$, i.e. the closed category, by $\mathscr{V}$. There are five axioms, but as they involve a term j derived from the other terms, we preferred to include j as a primitive term and add a sixth axiom to fix its value (§ I.2).

In Chapter I we give a precise definition of closed category, and define the corresponding notions of closed functor and closed natural transformation. Then we consider for a closed category $\mathscr{V}$ the notion of a $\mathscr{V}$-category $\mathscr{A}$, i.e. a "category" whose Hom-functor has values $\mathscr{A}(A\,B)$

in $\mathscr{V}_0$. With such a $\mathscr{V}$-category $\mathscr{A}$ is associated an ordinary underlying category $\mathscr{A}_0$ with $\mathscr{A}_0(A\,B) = V\mathscr{A}(A\,B)$ as indicated above. There is a corresponding notion of $\mathscr{V}$-functor, and also of $\mathscr{V}$-natural transformation. The notations $\mathscr{V}$, $\mathscr{V}_0$ suggest that $\mathscr{V}$ itself is a $\mathscr{V}$-category with underlying category $\mathscr{V}_0$, which is indeed the case. Each object A of a $\mathscr{V}$-category $\mathscr{A}$ determines a "left represented" $\mathscr{V}$-functor $L^A : \mathscr{A} \to \mathscr{V}$, and this leads to the key representation theorem (Theorem I.8.6.) which is the generalization of the YONEDA theorem for ordinary categories. We here thank JOHN GRAY for impressing upon us the importance of such a theorem; actually the one he wanted was a still higher form which must await a later paper on functor categories. We note throughout the chapter various gross simplifications that ensue when the basic functor $V : \mathscr{V}_0 \to \mathscr{S}$ is faithful.

In Chapter II we consider closed categories which possess a tensor product defined by the adjointness relation

$$(A \otimes B, C) \cong (A\,(B C)) .$$

These considerations lead us to the notion of a monoidal category, which is a *catégorie avec multiplication* in the terminology of BÉNABOU ([*1*], [*2*], [*3*]). A key result here is Theorem II.5.8 which allows us to reconstruct the closed structure on $\mathscr{V}$ from the monoidal structure. A similar result is to be found in [*3*]; cf. also [*10*].

The theory as developed thus far does not allow for a consideration of dual $\mathscr{V}$-categories and therefore all $\mathscr{V}$-functors must remain covariant. In order to introduce contravariance one needs a notion of *symmetry.* In the presence of a tensor product a symmetry takes the form of a natural isomorphism

$$A \otimes B \cong B \otimes A$$

satisfying suitable conditions. This is the subject of Chapter III, where we also show that a symmetry allows us to introduce $\mathscr{V}$-functors of many variables and the appropriate generalized natural transformations. In a separate paper we shall study closed categories with symmetry but without assuming the tensor product. The symmetry then takes the form of an isomorphism

$$(A\,(B C)) \cong (B (A C)) .$$

Chapter IV is devoted to examples. These show the frequency with which closed categories appear in various parts of mathematics. The examples were also chosen to illustrate the various points treated in Chapters I—III. Certain classes of examples will form the subject matter of subsequent papers and such examples have been either completely omitted or treated very sketchily. This in particular applies to the construction of "functor categories" which form an indispensable continuation of the theory presented in this paper.

Chapter I

Closed Categories

1. Notation and Preliminaries

In our notation we use brackets no more than is necessary (logically or psychologically); in particular fx denotes the value of the function f at the argument x. Then $(Kf)x$ denotes the value of Kf at x, $f(xy)$ denotes the value of f at xy, and $f(x,y)$ denotes the value of f at (x,y). We often use dots in place of brackets, as in $Kg.\ Kf.\ x$ for $(Kg)\ (Kf)x$. We are similarly sparing of commas: for a bifunctor T we write $T(AB)$, not $T(A, B)$; but we write $T(f, g)$ since $T(fg)$ would be confusing.

We use $\mathscr{A}^*$ for the dual of a category $\mathscr{A}$; then a functor $T: \mathscr{A} \to \mathscr{B}$ has a dual $T^*: \mathscr{A}^* \to \mathscr{B}^*$, and a natural transformation $\alpha: T \to S: \mathscr{A} \to \mathscr{B}$ has a dual $\alpha^*: S^* \to T^*: \mathscr{A}^* \to \mathscr{B}^*$. We reserve the symbol $\mathscr{S}$ for the category of sets, and we denote by $\mathrm{Hom}\,\mathscr{A}$ the Hom-functor $\mathscr{A}^* \times \mathscr{A} \to \mathscr{S}$; however we abbreviate the values $\mathrm{Hom}\,\mathscr{A}(AB)$ and $\mathrm{Hom}\,\mathscr{A}(f, g)$ of $\mathrm{Hom}\,\mathscr{A}$ to $\mathscr{A}(AB)$ and $\mathscr{A}(f, g)$. Note that we do not require of a category $\mathscr{A}$ that the various $\mathscr{A}(AB)$ be disjoint.

If $\alpha = (\alpha_A)_{A \in \mathscr{A}}$ is a family of morphisms, where say $\alpha_A: TA \to SA$, we often abbreviate to $\alpha: TA \to SA$. Where there are several variables as in $L^A_{BC}: (BC) \to ((AB)\ (AC))$, we may abbreviate L^A_{BC} totally to L, or partially to, say, L^A if we wish to emphasize A. We also use L^A at times to denote the partial family got by fixing A and letting B and C vary.

The reader should note that the criterion for a family of morphisms $\alpha_A: TA \to SA$ to be a natural transformation $\alpha: T \to S$, where T, S: $\mathscr{A} \to \mathscr{B}$, is the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{A}(AB) \arrow[r, "T_{AB}"] \arrow[d, "S_{AB}"'] & \mathscr{B}(TA, TB) \arrow[d, "\mathscr{B}(1, \alpha_B)"] \\
\mathscr{B}(SA, SB) \arrow[r, "\mathscr{B}(\alpha_A, 1)"'] & \mathscr{B}(TA, SB)\ ;
\end{tikzcd} \tag{1.1}$$

the more usual criterion, got by evaluating (1.1) at $f \in \mathscr{A}(AB)$, is the commutativity of the diagram

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha_A"'] & TB \arrow[d, "\alpha_B"] \\ SA \arrow[r, "Sf"'] & SB \end{tikzcd}$$

As our discourse concerns generalizations of categories, functors, and natural transformations, it will be convenient to use the abstract language of *hypercategories* (the 2-categories of EHRESMANN [6]). A hypercategory $\mathfrak{A}$ consists of

(i) a class of *objects* $\mathscr{A}, \mathscr{B}, \ldots$;

(ii) for each pair of objects $\mathscr{A}, \mathscr{B}$ a set of *morphisms*

$$T, S, \ldots : \mathscr{A} \to \mathscr{B};$$

(iii) for each $\mathscr{A}, \mathscr{B}$ and for each pair $T, S : \mathscr{A} \to \mathscr{B}$ a set of *hypermorphisms* $\lambda, \mu, \ldots : T \to S : \mathscr{A} \to \mathscr{B}$;

together with four kinds of composition law:

(i) if $T : \mathscr{A} \to \mathscr{B}$ and $S : \mathscr{B} \to \mathscr{C}$ then $ST : \mathscr{A} \to \mathscr{C}$;

(ii) if $T : \mathscr{A} \to \mathscr{B}$ and $\lambda : S \to R : \mathscr{B} \to \mathscr{C}$ then $\lambda T : ST \to RT : \mathscr{A} \to \mathscr{C}$;

(iii) if $\lambda : S \to R : \mathscr{A} \to \mathscr{B}$ and $T : \mathscr{B} \to \mathscr{C}$ then $T\lambda : TS \to TR : \mathscr{A} \to \mathscr{C}$;

(iv) if $\lambda : T \to S : \mathscr{A} \to \mathscr{B}$ and $\mu : S \to R : \mathscr{A} \to \mathscr{B}$ then $\mu\lambda : T \to R : \mathscr{A} \to \mathscr{B}$;

and two kinds of identity:

(i) $1_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$;

(ii) $1_T : T \to T : \mathscr{A} \to \mathscr{B}$.

These data are to satisfy the following five axioms:

HC1. The objects and the morphisms form a category $\mathfrak{A}_0$.

HC2. For each $\mathscr{A}, \mathscr{B}$ the morphisms $\mathscr{A} \to \mathscr{B}$ and the hypermorphisms between them form a category $\mathfrak{A}(\mathscr{A}\mathscr{B})$.

HC3. If $\lambda : T \to T' : \mathscr{A} \to \mathscr{B}$, $\mathscr{A}'' \xrightarrow[R']{} \mathscr{A}' \xrightarrow[R]{} \mathscr{A}$, and $\mathscr{B} \xrightarrow[S]{} \mathscr{B}' \xrightarrow[S']{} \mathscr{B}''$, we have

(a) $1_{\mathscr{B}}\lambda = \lambda$, (b) $\lambda 1_{\mathscr{A}} = \lambda$,

(c) $(S'S)\lambda = S'(S\lambda)$, (d) $\lambda(RR') = (\lambda R)R'$,

(e) $(S\lambda)R = S(\lambda R)$.

HC4. If $R : \mathscr{A}' \to \mathscr{A}$ and $S : \mathscr{B} \to \mathscr{B}'$ the assignments $T \mapsto STR$, $\lambda \mapsto S\lambda R$ constitute a functor $\mathfrak{A}(R, S) : \mathfrak{A}(\mathscr{A}\mathscr{B}) \to \mathfrak{A}(\mathscr{A}'\mathscr{B}')$.

HC5. If $\lambda : T \to S : \mathscr{A} \to \mathscr{B}$ and $\mu : P \to Q : \mathscr{B} \to \mathscr{C}$, the following diagram commutes:

$$\begin{tikzcd} PT \arrow[r, "P\lambda"] \arrow[d, "\mu T"'] & PS \arrow[d, "\mu S"] \\ QT \arrow[r, "Q\lambda"'] & QS \end{tikzcd}$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are hypercategories, a *hyperfunctor* $\Phi : \mathfrak{A} \to \mathfrak{B}$ consists of functions assigning

(i) to each object $\mathscr{A}$ of $\mathfrak{A}$ an object $\Phi\mathscr{A}$ of $\mathfrak{B}$;

(ii) to each morphism $T : \mathscr{A} \to \mathscr{B}$ in $\mathfrak{A}$ a morphism $\Phi T : \Phi\mathscr{A} \to \Phi\mathscr{B}$ in $\mathfrak{B}$;

(iii) to each hypermorphism $\lambda : T \to S : \mathscr{A} \to \mathscr{B}$ in $\mathfrak{A}$ a hypermorphism $\Phi\lambda : \Phi T \to \Phi S : \Phi\mathscr{A} \to \Phi\mathscr{B}$ in $\mathfrak{B}$.

These are to satisfy the axioms:

HF1. $\Phi(ST) = \Phi S . \Phi T$ and $\Phi 1 = 1$.
HF2. $\Phi(\lambda T) = \Phi\lambda . \Phi T$.
HF3. $\Phi(T\lambda) = \Phi T . \Phi\lambda$.
HF4. $\Phi(\mu\lambda) = \Phi\mu . \Phi\lambda$ and $\Phi 1 = 1$.

If $\Phi, \Psi : \mathfrak{A} \to \mathfrak{B}$ are hyperfunctors, a *hypernatural transformation* $\eta : \Phi \to \Psi$ is a function assigning to each object $\mathscr{A}$ of $\mathfrak{A}$ a morphism $\eta_{\mathscr{A}} : \Phi\mathscr{A} \to \Psi\mathscr{A}$, and satisfying the axioms:

HN1. If $T : \mathscr{A} \to \mathscr{B}$ in $\mathfrak{A}$, the following diagram commutes:

$$\begin{tikzcd} \Phi\mathscr{A} \arrow[r, "\Phi T"] \arrow[d, "\eta_{\mathscr{A}}"'] & \Phi\mathscr{B} \arrow[d, "\eta_{\mathscr{B}}"] \\ \Psi\mathscr{A} \arrow[r, "\Psi T"'] & \Psi\mathscr{B} \end{tikzcd}$$

HN2. If $\lambda : T \to S : \mathscr{A} \to \mathscr{B}$ in $\mathfrak{A}$, then

$$\eta_{\mathscr{B}} . \Phi\lambda : \eta_{\mathscr{B}} . \Phi T \to \eta_{\mathscr{B}} . \Phi S : \Phi\mathscr{A} \to \Psi\mathscr{B}$$

coincides with

$$\Psi\lambda . \eta_{\mathscr{A}} : \Psi T . \eta_{\mathscr{A}} \to \Psi S . \eta_{\mathscr{A}} : \Phi\mathscr{A} \to \Psi\mathscr{B}.$$

It is clear that small categories, functors, and natural transformations form a hypercategory; so do small hypercategories, hyperfunctors, and hypernatural transformations, if we use the obvious definitions of composition. Since however we shall use hypercategories purely as a convenient language at the formal level, we shall not hesitate to speak of the "hypercategory" $\mathscr{C}at$ of *all* categories and the "hypercategory" $\mathscr{H}yp$ of *all* hypercategories, sometimes as here using quotation marks to emphasize this purely formal use. In fact when we speak in this way we suppose $\mathscr{C}at$ to contain not merely all categories but also all "categories". Note that from any hypercategory $\mathfrak{A}$ we get a category $\mathfrak{A}_0$ by discarding the hypermorphisms; indeed $\mathfrak{A} \mapsto \mathfrak{A}_0$ is clearly the object-function of a forgetful hyperfunctor $\mathscr{H}yp \to \mathscr{C}at$.

We recall some special properties of the hypercategory $\mathscr{C}at$ that provide at once a guideline for our generalizations of categories and a tool for our investigations. The underlying category $\mathscr{C}at_0$ of $\mathscr{C}at$ has an initial object, the empty category; and a terminal object, the category $\mathscr{I}$ with a single object and a single morphism. The objects and the morphisms of a category $\mathscr{A}$ may be identified with the functors $\mathscr{I} \to \mathscr{A}$ and the natural transformations between these. The category $\mathscr{S}$ of sets plays a special role in $\mathscr{C}at$; to each object A in the category $\mathscr{A}$ there is the left represented functor $L^A = \operatorname{Hom}_{\mathscr{A}}(A\ —) : \mathscr{A} \to \mathscr{S}$, and for any functor $T : \mathscr{A} \to \mathscr{S}$ we obtain a bijection between the natural transformations $\alpha : L^A \to T$ and the elements of TA by sending α to $\alpha_A 1_A \in TA$. This representation theorem (YONEDA [*17*]) occurs most frequently in our applications in the following form, in which we formally state it:

Theorem 1.1. *Let $T : \mathscr{A} \to \mathscr{B}$ be a functor and let $K \in \mathscr{A}$, $M \in \mathscr{B}$. Denote by $\{p\}$ the class of natural transformations*

$$p = p_A : \quad \mathscr{A}(KA) \to \mathscr{B}(M, TA),$$

and define a map $\Gamma : \{p\} \to \mathscr{B}(M, TK)$ by

$$\Gamma p = p_K 1_K. \tag{1.2}$$

Then Γ is a bijection, with inverse $\Omega : \mathscr{B}(M, TK) \to \{p\}$, where $\Omega\theta$ is the composite

$$\Omega\theta : \mathscr{A}(KA) \xrightarrow[T_{KA}]{} \mathscr{B}(TK, TA) \xrightarrow[\mathscr{B}(\theta, 1)]{} \mathscr{B}(M, TA). \tag{1.3}$$

Proof. $\Omega\theta$ is indeed natural, since T_{KA} is natural in A and $\mathscr{B}(\theta, 1_B)$ in B. That $\Gamma\Omega = 1$ is obvious, and that $\Omega\Gamma = 1$ follows by a simple naturality argument.

We record the form of (1.3) obtained by evaluating at $f \in \mathscr{A}(KA)$; setting $p = \Omega\theta$ we have the commutative diagram

$$\begin{tikzcd} M \arrow[rr, "pf"] \arrow[dr, "\theta"'] & & TA \\ & TK \arrow[ur, "Tf"'] & \end{tikzcd} \tag{1.4}$$

Note that if we take $\mathscr{B} = \mathscr{S}$ and M to be a single point we regain the usual form of the theorem.

The hypercategory $\mathscr{C}at$ is further enriched by its product hyperfunctor $\mathscr{A}, \mathscr{B} \mapsto \mathscr{A} \times \mathscr{B}$ and its duality hyperfunctor $\mathscr{A} \mapsto \mathscr{A}^*$, which allow us to define functors of many variables and both variances. There is a corresponding extension of the concept of natural transformation, which the authors have described in [7], and with which we shall assume familiarity. We record here the appropriate extension to the representation theorem:

Proposition 1.2. *Let $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ be categories and let $T : \mathscr{D} \times \mathscr{A} \to \mathscr{B}$, $K : \mathscr{C} \times \mathscr{D}^* \to \mathscr{A}$, $M : \mathscr{C} \to \mathscr{B}$ be functors. Let*

$$p = p_{CDA} : \mathscr{A}(K(C\,D), A) \to \mathscr{B}(M\,C, T(D\,A))$$

be a family of morphisms, natural in A for each fixed C, D; and let

$$\theta = \theta_{CD} : M\,C \to T(D, K(C\,D))$$

be Γp_{CD}. Then p is natural in C (resp. D) if and only if θ is.

Proof. If θ is natural so is p by (1.3). If p is natural so is the composite

$$* \xrightarrow{j} \mathscr{A}(K(C\,D), K(C\,D)) \xrightarrow{p} \mathscr{B}(M\,C, T(D, K(C\,D)))$$

where $*$ is a single point and $j\,* = 1$; for $j : * \to \mathscr{A}(AA)$ is clearly natural in A. It is easy to see that this implies the naturality of $p j *$, which is θ.

2. Closed Categories

We begin by axiomatizing those structures, called *closed categories*, in which the Hom-functors of our generalized categories will take their values.

A closed category $\mathscr{V} = (\mathscr{V}_0, V, \hom \mathscr{V}, I, i, j, L)$ consists of the following seven data:

(i) a category $\mathscr{V}_0$;

(ii) a functor $V : \mathscr{V}_0 \to \mathscr{S}$;

(iii) a functor $\hom \mathscr{V} : \mathscr{V}_0^* \times \mathscr{V}_0 \to \mathscr{V}_0$
(we write (AB) for $\hom \mathscr{V}(AB)$ and (f, g) for $\hom \mathscr{V}(f, g)$);

(iv) an object I of $\mathscr{V}_0$;

(v) a natural *isomorphism* $i = i_A : A \to (I\,A)$ in $\mathscr{V}_0$;

(vi) a natural transformation $j = j_A : I \to (A\,A)$ in $\mathscr{V}_0$;

(vii) a natural transformation $L = L^A_{BC} : (BC) \to ((AB)\,(A\,C))$ in $\mathscr{V}_0$.

These data are to satisfy the following six axioms:

CC0. The following diagram of functors commutes:

$$\begin{tikzcd} \mathscr{V}_0{}^* \times \mathscr{V}_0 \arrow[r, "\hom \mathscr{V}"] \arrow[dr, "\mathrm{Hom}\,\mathscr{V}_0"'] & \mathscr{V}_0 \arrow[d, "V"] \\ & \mathscr{S} \end{tikzcd}$$

CC1. The following diagram commutes:

$$\begin{tikzcd}
(B\,B) \arrow[rr, "L^A"] & & ((A\,B)\,(A\,B)) \\
& I \arrow[ul, "j"] \arrow[ur, "j"'] &
\end{tikzcd}$$

CC2. The following diagram commutes:

$$\begin{tikzcd}
(A\,C) \arrow[r, "L^A"] \arrow[dr, "i"'] & ((A\,A)\,(A\,C)) \arrow[d, "{(j, 1)}"] \\
& (I\,(A\,C))
\end{tikzcd}$$

CC3. The following diagram commutes:

$$\begin{tikzcd}
(C\,D) \arrow[rr, "L^B"] \arrow[d, "L^A"'] & & ((B\,C)\,(B\,D)) \arrow[dd, "{(1, L^A)}"] \\
((A\,C)\,(A\,D)) \arrow[d, "L^{(A\,B)}"'] & & \\
(((A\,B)\,(A\,C)), ((A\,B)\,(A\,D))) \arrow[rr, "{(L^A, 1)}"'] & & ((B\,C), ((A\,B)\,(A\,D)))
\end{tikzcd}$$

CC4. The following diagram commutes:

$$\begin{tikzcd}
(B\,C) \arrow[r, "L^I"] \arrow[dr, "{(1, i)}"'] & ((I\,B)\,(I\,C)) \arrow[d, "{(i, 1)}"] \\
& (B\,(I\,C))
\end{tikzcd}$$

CC5. The map

$$V\,i_{(AA)} : V(A\,A) \to V(I\,(A\,A)),$$

which by CC0 may also be written

$$V\,i_{(AA)} : \mathscr{V}_0(AA) \to \mathscr{V}_0(I, (AA)),$$

sends $1_A \in \mathscr{V}_0(AA)$ to $j_A \in \mathscr{V}_0(I, (AA))$.

We consider some properties of closed categories that follow directly from the above axioms. Note that by CC0 we have

$$V(A\,B) = \mathscr{V}_0(A\,B), \tag{2.1}$$

$$V(f,g) = \mathscr{V}_0(f,g). \tag{2.2}$$

Define a natural isomorphism

$$\iota = \iota_A : VA \to V(I\,A)$$

by

$$\iota_A = V i_A. \tag{2.3}$$

Proposition 2.1. *ι provides a representation of the functor $V : \mathscr{V}_0 \to \mathscr{S}$.* Axiom CC5 may be written as

$$j_A = \iota\, 1_A; \tag{2.4}$$

of course we could drop j as a primitive term, and drop axiom CC5, using (2.4) as a definition of j; note that j so defined is automatically natural. Any statement about composition with j may be turned into a statement about the image of 1 by means of:

Lemma 2.2. *For any $f : (AA) \to X$ in $\mathscr{V}_0$, the composite*

$$I \xrightarrow{j} (AA) \xrightarrow{f} X$$

is the image of $1 \in V(AA)$ under the composite map

$$V(AA) \xrightarrow{Vf} VX \xrightarrow{\iota} V(IX).$$

Proof. Evaluate at $1 \in V(AA)$ the diagram

$$\begin{tikzcd} V(A\,A) \arrow[r, "Vf"] \arrow[d, "\iota"'] & VX \arrow[d, "\iota"] \\ V(I(A\,A)) \arrow[r, "V(1,f)"'] & V(IX), \end{tikzcd}$$

which commutes by the naturality of ι.

Proposition 2.3. *In the presence of* CC0 *and* CC5, *the axiom* CC1 *is equivalent to any of the following:*

(a) $(V L^A_{BB})\, 1_B = 1_{(AB)}$; (2.5)

(b) $(V L^A_{BC})\, f = (1, f) \in V((A\,B), (A\,C))$ for $f \in V(BC)$; (2.6)

(c) $V L^A_{BC} = (A\ -) : V(BC) \to V((A\,B), (A\,C))$. (2.7)

Proof. Lemma 2.2 shows the equivalence of CC1 with (a), while (b) is merely the evaluated form of (c). The equivalence of (c) with (a) follows by applying the representation theorem, Theorem 1.1, to the natural transformations VL^A_{BC} and $(A\,-)$ in (2.7), since $(A,\, 1_B) = 1$.

Proposition 2.4. *For any $f \in V(A\,B)$ we have a commutative diagram*

$$\begin{tikzcd} I \arrow[r, "j_A"] \arrow[d, "j_B"'] \arrow[dr, "\iota f"] & (A\,A) \arrow[d, "(1,f)"] \\ (B\,B) \arrow[r, "(f,1)"'] & (A\,B), \end{tikzcd} \tag{2.8}$$

the diagonal being the image of f under

$$\iota_{(AB)} : V(A\,B) \to V(I(A\,B))\,.$$

Proof. By Lemma 2.2 we have

$$(1,f)\,j = \iota.\,V(1,f).\,1\,,$$

but $V(1,f)\,1 = \mathscr{V}_0(1,f)\,1 = f$; similarly $(f,1)\,j = \iota f$.

Proposition 2.5. $i_{(IA)} = (1, i_A) : (I\,A) \to (I\,(I\,A))$.

Proof. From the naturality of i we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i"'] & (I\,A) \arrow[d, "(1,i)"] \\ (I\,A) \arrow[r, "i"'] & (I(I\,A)), \end{tikzcd}$$

whence the result since i is an isomorphism.

Proposition 2.6. *For any $f \in V(I\,A)$, the composite*

$$I \xrightarrow[f]{} A \xrightarrow[i]{} (I\,A)$$

is the image of f under ι: $V(I\,A) \to V(I,\,(I\,A))$.

Proof. Apply V to Proposition 2.5 and evaluate at f.

Proposition 2.7. $j_I = i_I : I \to (I\,I)$.

Proof. Take $A = I$ and $f = 1$ in Proposition 2.6.

Proposition 2.8. *For $f \in V(I\,I)$ we have* $(1,f) = (f,1) : (I\,I) \to (I\,I)$.

Proof. In (2.8) put $A = B = I$; the result follows because $j_I = i_I$ is an isomorphism.

Proposition 2.9. *The monoid $\mathscr{V}_0(I\,I)$ of endomorphisms of I is commutative.*

Proof. Applying V to Proposition 2.8 gives

$$V(1, f) = V(f, 1) : V(I\,I) \to V(I\,I);$$

evaluating at $g \in V(I\,I)$ now gives $fg = gf$.

Proposition 2.10. *If V is faithful, the axioms* CC2, CC3, CC4 *are consequences of* CC0, CC1, CC5.

Proof. First note that we have made no use of CC2, CC3, CC4 in the deductions above. If V is faithful, a diagram commutes if and only if V of it does so. Applying V to the diagram of CC2 and evaluating at $f \in V(A\,C)$, using (2.6), we get $\iota f = (1, f)\, j$, which is true by Proposition 2.4. Applying V to the diagram of CC3 and using (2.6), we obtain the diagram asserting the naturality in C of L^A_{BC}, which obtains by hypothesis. Similarly V of CC4 is the assertion of the naturality of i.

Proposition 2.11. *Let there be given a category $\mathscr{V}_0$, a faithful functor $V : \mathscr{V}_0 \to \mathscr{S}$, a representation $\iota : VA \cong \mathscr{V}_0(I\,A)$ of V, and, for each A, B in $\mathscr{V}_0$, an object $(A\,B)$ of $\mathscr{V}_0$ with*

$$V(A\,B) = \mathscr{V}_0(A\,B).$$

Then there is a closed category $\mathscr{V} = (\mathscr{V}_0, V, \mathrm{hom}\,\mathscr{V}, I, i, j, L)$ with

$$\mathrm{hom}\,\mathscr{V}(A\,B) = (A\,B)$$

and

$$V i = \iota$$

if and only if

(i) *for each $f : A' \to A$ and $g : B \to B'$, the morphism*

$$\mathscr{V}_0(f, g) : \mathscr{V}_0(A\,B) \to \mathscr{V}_0(A'\,B')$$

is $V(f, g)$ for some morphism $(f, g) : (A\,B) \to (A'\,B')$;

(ii) *for each A, $\iota : VA \to \mathscr{V}_0(I\,A)$ is $V i$ for some isomorphism*

$$i : A \to (I\,A);$$

(iii) *for each A, B, C the map $h \to (1, h) : V(BC) \to V((A\,B)\,(A\,C))$ is $V L^A_{BC}$ for some $L^A_{BC} : (BC) \to ((A\,B)\,(A\,C))$;*

and if these conditions are satisfed $\mathscr{V}$ is unique.

Proof. The conditions are clearly necessary, in view of (2.2) and (2.6). Suppose they are satisfied; then (f, g), i, and L are unique by the faithfulness of V.

For $A'' \xrightarrow[f']{} A' \xrightarrow[f]{} A$ and $B \xrightarrow[g]{} B' \xrightarrow[g']{} B''$ we have

$$\begin{aligned} V(f'f, gg') &= \mathscr{V}_0(f'f, gg') \\ &= \mathscr{V}_0(f,g)\, \mathscr{V}_0(f',g') \\ &= V(f,g)\, V(f',g') \\ &= V((f,g)\,(f',g')), \end{aligned}$$

whence $(f'f, gg') = (f,g)\,(f',g')$ by the faithfulness of V. Similarly $(1,1) = 1$, and thus $(A\,B)$ and (f,g) are the values of a functor hom $\mathscr{V}$ satisfying CC0.

Again since V is faithful the naturality of i follows from that of $Vi = \iota$, and that of L^A follows from that of $VL^A = (A\ -)$. We define j by (2.4), so that CC5 is satisfied; then CC1 is satisfied by Proposition 2.3, and the remaining axioms follow by Proposition 2.10.

Proposition 2.12. *We obtain a closed category, which we denote by* $\mathscr{S}$, *if we set* $\mathscr{V}_0 = \mathscr{S}$, $V = 1$ *and* hom $\mathscr{V}$ = Hom $\mathscr{S}$; *take for* I *a set* $*$, *chosen once for all, consisting of a single point* $*$; *and define* i, j, L *by:*

$$(i\,a) * = a, \qquad a \in A; \tag{2.9}$$

$$j * = 1; \tag{2.10}$$

$$(Lf)\,g = fg, \qquad f \in (BC), \quad g \in (A\,B). \tag{2.11}$$

Proof. It is clear that i, j, L are natural. Verification of CC0, CC1 (in the form (2.6)), and CC5 is immediate, and the other axioms follow by Proposition 2.10.

Remark 2.13. For a closed category $\mathscr{V}$ we shall call $\mathscr{V}_0$ the *underlying category*, $V : \mathscr{V}_0 \to \mathscr{S}$ the *basic functor*, and hom $\mathscr{V}$ the *internal* Hom-*functor*.

3. Closed Functors

Let $\mathscr{V} = (\mathscr{V}_0, V, \hom\mathscr{V}, I, i, j, L)$ and $\mathscr{V}' = (\mathscr{V}'_0, V', \hom\mathscr{V}', I', i', j', L')$ be closed categories; we write $(X\,Y)$ for hom $\mathscr{V}'(X\,Y)$ as well as $(A\,B)$ for hom $\mathscr{V}(A\,B)$. A *closed functor* $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ consists of

(i) a functor $\phi : \mathscr{V}_0 \to \mathscr{V}'_0$;

(ii) a natural transformation $\hat{\phi} = \hat{\phi}_{AB} : \phi(A\,B) \to (\phi A, \phi B)$;

(iii) a morphism $\phi^0 : I' \to \phi I$.

These data are to satisfy the following three axioms:

CF 1. The following diagram commutes:

$$\begin{tikzcd} \phi I \arrow[r, "\phi j"] & \phi(A A) \arrow[d, "\hat{\phi}"] \\ I' \arrow[u, "\phi^0"] \arrow[r, "j'"'] & (\phi A, \phi A) \end{tikzcd}$$

CF 2. The following diagram commutes:

$$\begin{tikzcd} \phi(I A) \arrow[r, "\hat{\phi}"] & (\phi I, \phi A) \arrow[d, "(\phi^0, 1)"] \\ \phi A \arrow[u, "\phi i"] \arrow[r, "i'"'] & (I', \phi A) \end{tikzcd}$$

CF 3. The following diagram commutes:

$$\begin{tikzcd} \phi(B C) \arrow[r, "\phi L"] \arrow[d, "\hat{\phi}"'] & \phi((A B), (A C)) \arrow[r, "\hat{\phi}"] & (\phi(A B), \phi(A C)) \arrow[d, "(1, \hat{\phi})"] \\ (\phi B, \phi C) \arrow[r, "L'"'] & ((\phi A, \phi B), (\phi A, \phi C)) \arrow[r, "(\hat{\phi}, 1)"'] & (\phi(A B), (\phi A, \phi C)) \end{tikzcd}$$

Theorem 3.1. *Closed categories and closed functors form a "category" $\mathscr{C}\ell_0$ if we define the composite of*

$$\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}' \quad \textit{and} \quad \Psi = (\psi, \hat{\psi}, \psi^0) : \mathscr{V}' \to \mathscr{V}''$$

to be $X = (\chi, \hat{\chi}, \chi^0) : \mathscr{V} \to \mathscr{V}''$ where

(i) *χ is the composite* $\mathscr{V}_0 \xrightarrow[\phi]{} \mathscr{V}'_0 \xrightarrow[\psi]{} \mathscr{V}''_0$; (3.1)

(ii) *$\hat{\chi}$ is the composite* $\psi\phi(A B) \xrightarrow[\psi\hat{\phi}]{} \psi(\phi A, \phi B) \xrightarrow[\hat{\psi}]{} (\psi\phi A, \psi\phi B)$; (3.2)

(iii) *χ^0 is the composite* $I'' \xrightarrow[\psi^0]{} \psi I' \xrightarrow[\psi\phi^0]{} \psi\phi I$. (3.3)

Proof. It is immediate that composition as defined above is associative, with identities $1 = (1, 1, 1)$. What has to be verified is that X satisfies CF 1—CF 3; that is, that the exteriors of the following three diagrams commute:

$$
\begin{tikzcd}
\psi\phi I \arrow[r, "\psi\phi j"] & \psi\phi(A A) \arrow[d, "\psi\hat{\phi}"] \\
\psi I' \arrow[u, "\psi\phi^0"] \arrow[r, "\psi j'"'] & \psi(\phi A, \phi A) \arrow[d, "\hat{\psi}"] \\
I'' \arrow[u, "\psi^0"] \arrow[r, "j''"'] & (\psi\phi A, \psi\phi A)
\end{tikzcd}
\tag{3.4}
$$

$$
\begin{tikzcd}
\psi\phi(I A) \arrow[r, "\psi\hat{\phi}"] & \psi(\phi I, \phi A) \arrow[r, "\hat{\psi}"] \arrow[d, "\psi(\phi^0, 1)"] & (\psi\phi I, \psi\phi A) \arrow[d, "(\psi\phi^0, 1)"] \\
 & \psi(I', \phi A) \arrow[r, "\hat{\psi}"'] & (\psi I', \psi\phi A) \arrow[d, "(\psi^0, 1)"] \\
\psi\phi A \arrow[uu, "\psi\phi i"] \arrow[ur, "\psi i'"'] \arrow[rr, "i''"'] & & (I'', \psi\phi A)
\end{tikzcd}
\tag{3.5}
$$

$$
\begin{tikzcd}
 & & \psi(\phi(A B), \phi(A C)) \arrow[dr, "\hat{\psi}"] \arrow[dd, "\psi(1, \hat{\phi})"] & \\
B C) \arrow[r, "\psi\phi L"] \arrow[dd] & \psi\phi((A B), (A C)) \arrow[ur, "\psi\hat{\phi}"] & & (\psi\phi(A B), \psi\phi(A C)) \arrow[dd, "(1, \psi\hat{\phi})"] \\
 & & \psi(\phi(A B), (\phi A, \phi C)) \arrow[dr, "\hat{\psi}"] & \\
B, \phi C) \arrow[r, "\psi L'"] \arrow[dd] & \psi((\phi A, \phi B), (\phi A, \phi C)) \arrow[ur, "\psi(\hat{\phi}, 1)"'] \arrow[dr, "\hat{\psi}"] & & (\psi\phi(A B), \psi(\phi A, \phi C)) \arrow[dd, "(1, \hat{\psi})"] \\
 & & (\psi(\phi A, \phi B), \psi(\phi A, \phi C)) \arrow[ur, "(\psi\hat{\phi}, 1)"] \arrow[dd, "(1, \hat{\psi})"] & \\
B, \psi\phi C) \arrow[r, "L''"'] & ((\psi\phi A, \psi\phi B), (\psi\phi A, \psi\phi C)) \arrow[dr, "(\hat{\psi}, 1)"'] & & (\psi\phi(A B), (\psi\phi A, \psi\phi C)) \\
 & & (\psi(\phi A, \phi B), (\psi\phi A, \psi\phi C)) \arrow[ur, "(\psi\hat{\phi}, 1)"'] &
\end{tikzcd}
\tag{3.6}
$$

Now in (3.4) one region commutes by ψ of CF1 for Φ, and the other region by CF1 for Ψ; in (3.5) one region commutes by ψ of CF2 for Φ, one by CF2 for Ψ, and the third by the naturality of $\hat{\psi}$; and in (3.6) one region commutes by ψ of CF3 for Φ, one by CF3 for Ψ, two by the naturality of $\hat{\psi}$, and the last region trivially.

Proposition 3.2. *A closed functor $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'_0$ is an isomorphism in the category $\mathscr{C}\ell_0$ if and only if $\phi : \mathscr{V}_0 \to \mathscr{V}'_0$ is an isomorphism of categories, each $\hat{\phi}_{AB}$ is an isomorphism, and ϕ^0 is an isomorphism.*

Proof. If Φ has an inverse $\Psi = (\psi, \hat{\psi}, \psi^0)$, the composites (3.1)—(3.3) are all the identity, and so are the corresponding composites with Φ and Ψ interchanged. It follows at once that ϕ, $\hat{\phi}$, ϕ^0 are all isomorphisms.

If ϕ, $\hat{\phi}$, ϕ^0 are all isomorphisms, define $\psi = \phi^{-1}$ and take for $\hat{\psi}$ and ψ^0 the unique values that render the composites (3.2) and (3.3) equal to the identity; $\hat{\psi}$ is clearly natural, and we have $\Psi\Phi = 1$, but we must show that Ψ satisfies CF1 — CF3.

Consider diagrams (3.4)—(3.6); we know that the exteriors commute and that all the internal regions commute except those that express CF1—CF3 for Ψ. It follows that the latter regions commute also (using, in the case of (3.6), the fact that $\psi\hat{\phi}$ is an isomorphism).

Thus Ψ is a left inverse of Φ. But ψ, $\hat{\psi}$, ψ^0 are all isomorphisms, and so by the same argument Ψ itself has a left inverse. Hence Ψ is a two-sided inverse for Φ.

Proposition 3.3. *In a closed functor $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$, ϕ^0 is uniquely determined when ϕ and $\hat{\phi}$ are given, and ϕ^0 is an isomorphism if each of ϕ, $\hat{\phi}^0$ is.*

Proof. Let $(\phi, \hat{\phi}, \phi^0)$ and $(\phi, \hat{\phi}, \overline{\phi^0})$ both be closed functors $\mathscr{V} \to \mathscr{V}'$. We express the fact that the first satisfies CF1 and the second satisfies CF2, in each case taking $A = I$. Thus

$$j' = \hat{\phi}.\, \phi j_I.\, \phi^0,$$
$$i' = (\overline{\phi^0}, 1).\, \hat{\phi}.\, \phi i_I.$$

Since $j_I = i_I$ by Proposition 2.7, it follows that the composites

$$I' \xrightarrow[j']{} (\phi I, \phi I) \xrightarrow[(\overline{\phi^0}, 1)]{} (I', \phi I),$$

$$I' \xrightarrow[\phi^0]{} \phi I \xrightarrow[i']{} (I', \phi I),$$

are equal; but the first of these is $\iota'\, \overline{\phi^0}$ by Proposition 2.4, while the second is $\iota' \phi^0$ by Proposition 2.6. Since ι' is an isomorphism, we have $\overline{\phi^0} = \phi^0$.

If $\hat{\phi}$ is an isomorphism, so too by CF2 is

$$(\phi^0, 1) : (\phi I, \phi A) \to (I', \phi A).$$

If ϕ also is an isomorphism, we can replace ϕA here by any $X \in \mathscr{V}'_0$. Doing this and applying V', we see that

$$V'(\phi^0, 1) : V'(\phi I, X) \to V'(I', X)$$

is an isomorphism for all X; hence ϕ^0 is an isomorphism.

Proposition 3.4. *Let $\mathscr{V}$ and $\mathscr{V}'$ be closed categories and let $\phi : \mathscr{V}_0 \to \mathscr{V}'_0$ be a functor. Then there is a bijection between morphisms*

$$\phi^0 : I' \to \phi I$$

and natural transformations

$$\phi_0 : V \to V'\phi : \mathscr{V}_0 \to \mathscr{S},$$

given by requiring commutativity in the diagram

$$\begin{tikzcd}
V(IA) \arrow[rr, "\phi"] \arrow[d, "\iota^{-1}"'] & & V'(\phi I, \phi A) \arrow[d, "V'(\phi^0, 1)"] \\
VA \arrow[r, "\phi_0"'] & V'\phi A \arrow[r, "\iota'"'] & V'(I', \phi A)
\end{tikzcd} \tag{3.7}$$

Proof. Since ι and ι' are natural isomorphisms, this is immediate from the representation theorem.

Thus we can use ϕ, $\hat{\phi}$, ϕ_0 instead of ϕ, $\hat{\phi}$, ϕ^0 as the data for a closed functor. We record the form of (3.7) got by evaluating at $f \in V(IA)$:

$$\phi f . \phi^0 = \iota' \phi_0 \iota^{-1} f. \tag{3.8}$$

Taking in particular $A = I$ and $f = 1_I$, we get

$$\phi^0 = \iota' \phi_{0I} \iota^{-1} 1_I. \tag{3.9}$$

Now replace A by (AB) and let $f = \iota g$ where $g \in V(AB)$; since we also have $f = (1, g) j$ by (2.8), (3.8) becomes

$$\phi(1, g) . \phi j . \phi^0 = \iota' \phi_0 g. \tag{3.10}$$

Taking in particular $B = A$ and $g = 1_A$, we get

$$\phi j_A . \phi^0 = \iota' \phi_0 1_A. \tag{3.11}$$

Proposition 3.5. *Axiom* CF1 *is equivalent to the commutativity of the* *diagram*

$$\begin{tikzcd} V(A\,B) \arrow[r, "\phi_0"] \arrow[dr, "\phi"'] & V'\phi(A\,B) \arrow[d, "V'\hat{\phi}"] \\ & V'(\phi A, \phi B) \end{tikzcd} \tag{3.12}$$

Proof. By the representation theorem (3.12) commutes if and only if both legs give the same result when we set $B = A$ and evaluate at 1_A. But $\phi 1 = 1$, and by (3.11)

$$V'\hat{\phi} \, . \, \phi_0 1 = V'\hat{\phi} \, . \, \iota'^{-1}(\phi j \, . \, \phi^0) \, .$$

Since by the naturality of ι' we have

$$V'\hat{\phi} \, . \, \iota'^{-1} = \iota'^{-1} \, . \, V'(1, \hat{\phi}) \, ,$$

the commutativity of (3.12) is equivalent to

$$1 = \iota'^{-1} \, . \, V'(1, \hat{\phi}) \, (\phi j \, . \, \phi^0) \, ,$$

that is,

$$\iota' 1 = V'(1, \hat{\phi}) \, (\phi j \, . \, \phi^0)$$

or

$$j' = \hat{\phi} \, . \, \phi j \, . \, \phi^0 \, ,$$

which is CF1.

Proposition 3.6. *If* $(\chi, \hat{\chi}, \chi^0)$ *is the composite of the closed functors*

$$(\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}' \quad \textit{and} \quad (\psi, \hat{\psi}, \psi^0) : \mathscr{V}' \to \mathscr{V}''$$

then $\chi_0 : V \to V'' \psi \phi$ *is the composite*

$$\chi_0 : \qquad V \xrightarrow[\phi_0]{} V'\phi \xrightarrow[\psi_0 \phi]{} V''\psi\phi \, . \tag{3.13}$$

Proof. We show that if we *define* χ_0 by (3.13) and then use Proposition 3.4 to define χ^0 we get (3.3). By (3.9) we have

$$\begin{aligned} \chi^0 &= \iota'' \, \chi_0 \, \iota^{-1} 1 \\ &= \iota'' \psi_0 \, \phi_0 \, \iota^{-1} 1 \\ &= \iota'' \psi_0 \, \iota'^{-1} \phi^0 \quad \text{by (3.9)} \\ &= \psi \phi^0 \, . \, \psi^0 \quad \text{by (3.8)} \, , \end{aligned}$$

which agrees with (3.3).

We say that a closed functor $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ is *normal* if $V = V'\phi : \mathscr{V}_0 \to \mathscr{S}$ and $\phi_0 = 1 : V \to V'\phi$. From Proposition 3.6 we get at once:

Proposition 3.7. *The identity closed functor, and the composite of normal closed functors, are normal; so is the inverse of a normal closed functor that is an isomorphism.*

In view of Proposition 3.5, we may define a normal closed functor $\Phi : \mathscr{V} \to \mathscr{V}'$ directly, as consisting of a functor $\phi : \mathscr{V}_0 \to \mathscr{V}'_0$ and a natural transformation $\hat{\phi} : \phi(AB) \to (\phi A, \phi B)$, satisfying the axioms

NCF0. $$V = V'\phi : \mathscr{V}_0 \to \mathscr{S};$$

NCF1. $$V'\hat{\phi} : V'\phi(AB) \to V'(\phi A, \phi B)$$

coincides with

$$\phi : V(AB) \to V'(\phi A, \phi B);$$

and also the axioms CF2, CF3, in which ϕ^0 is defined by (3.7) with $\phi_0 = 1$.

Proposition 3.8. *The axioms* CF2 *and* CF3 *for a closed functor*

$$\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$$

are consequences of CF1 *if* V' *is faithful, provided that* $\phi_0 : VA \to V'\phi A$ *is an epimorphism for each* A *(and so in particular if* Φ *is normal).*

Proof. For simplicity we shall give the proof only for the case where Φ is normal; the reader will easily provide the proof of the general case, relying on (3.12) instead of its special case NCF1.

Since V' is faithful the diagrams of CF2 and CF3 commute if their images under V' do so. However V' of CF2 coincides, in view of NCF0 and NCF1, with the diagram (3.7) (with $\phi_0 = 1$) which defines ϕ^0. Again V' of CF3 coincides, in view of NCF0, NCF1, and (2.7), with the diagram asserting the naturality in B of $\hat{\phi}_{AB}$.

Proposition 3.9. *Let* $\mathscr{V}$ *and* $\mathscr{V}'$ *be closed categories with* $V' : \mathscr{V}'_0 \to \mathscr{S}$ *faithful, and for each* $A \in \mathscr{V}_0$ *let* ϕA *be an object of* $\mathscr{V}'_0$ *with* $V'\phi A = VA$. *Then there is a normal closed functor* $\Phi = (\phi, \hat{\phi}) : \mathscr{V} \to \mathscr{V}'$ *with the given value on objects if and only if*

(i) *for each* $f : A \to B$ *in* $\mathscr{V}_0$, *the morphism* $Vf : VA \to VB$ *is* $V'\phi f$ *for some* $\phi f : \phi A \to \phi B$;

(ii) *for each* A, B *in* $\mathscr{V}_0$, *the morphism* $\phi : V(AB) \to V'(\phi A, \phi B)$ *is* $V'\hat{\phi}$ *for some* $\hat{\phi} : \phi(AB) \to (\phi A, \phi B)$;

and if these conditions are satisfied Φ *is unique.*

Proof. The conditions are clearly necessary, and if they are satisfied ϕf and $\hat{\phi}$ are unique by the faithfulness of V'. We further conclude from the faithfulness of V' that ϕ is functorial and that $\hat{\phi}$ is natural (because $V'\hat{\phi} = \phi$ is).

Remark 3.10. In concrete cases of closed functors, as in the proposition below, we often by abuse of language denote a closed functor $(\phi, \hat{\phi}, \phi^0)$ by the letter ϕ.

Proposition 3.11. *If $\mathscr{V}$ is a closed category, the functor $V : \mathscr{V}_0 \to \mathscr{S}$ admits a unique extension to a normal closed functor $(V, \hat{V}, V^0) : \mathscr{V} \to \mathscr{S}$, which we still denote by V. We have*

$$\hat{V}_{AB} = V_{AB} : V(AB) \to (VA, VB), \tag{3.14}$$

and $V^0 : * \to VI$ *is given by*

$$V^0 * = \iota_I^{-1} 1 \tag{3.15}$$

where $\iota_I : VI \to V(II)$. Moreover for any $f \in V(IA)$, the image of $*$ under

$$* \xrightarrow[V^0]{} VI \xrightarrow[Vf]{} VA \quad \textit{is} \quad \iota^{-1} f \in VA; \tag{3.16}$$

and in particular the image of $*$ under

$$* \xrightarrow[V^0]{} VI \xrightarrow[Vj]{} V(AA) \quad \textit{is} \quad 1_A. \tag{3.17}$$

Proof. Clearly $\hat{V}$ is unique by NCF1, and NCF0 and NCF1 are in fact satisfied, which suffices by Proposition 3.8. The equations (3.15) and (3.16) are translations of (3.9) and (3.8), and (3.17) is a special case of (3.16).

Proposition 3.12. *A closed functor $\Phi : \mathscr{V} \to \mathscr{V}'$ is normal if and only if the following diagram of closed functors commutes:*

$$\begin{tikzcd} \mathscr{V} \arrow[rr, "\Phi"] \arrow[dr, "V"'] & & \mathscr{V}' \arrow[dl, "V'"] \\ & \mathscr{S} & \end{tikzcd}$$

Proof. Set $V'\Phi = X = (\chi, \hat{\chi}, \chi^0)$; then by (3.1), (3.2) and (3.13) we have $\chi = V'\phi$, $\chi = V' . V'\hat{\phi}$, and $\chi_0 = \phi_0$. So if $X = V$ we have $\phi_0 = V_0 = 1$ and Φ is normal; while if Φ is normal we have $\chi = V'\phi = V$ by NCF0, and $\hat{\chi} = V'. V'\hat{\phi} = V'\phi = V$ by NCF0 and NCF1, so that $X = V$.

Remark 3.13. We shall refer to $V : \mathscr{V} \to \mathscr{S}$ as the *basic closed functor* associated with $\mathscr{V}$.

4. Closed Natural Transformations

Let $\Phi = (\phi, \hat{\phi}, \phi^0)$, $\Psi = (\psi, \psi, \psi^0)$ be closed functors $\mathscr{V} \to \mathscr{V}'$. A *closed natural transformation*

$$\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$$

consists of a natural transformation

$$\eta : \phi \to \psi : \mathscr{V}_0 \to \mathscr{V}'_0$$

satisfying the following two axioms.

CN 1. The following diagram commutes:

$$\begin{tikzcd} I' \arrow[r, "\phi^0"] \arrow[dr, "\psi^0"'] & \phi I \arrow[d, "\eta_I"] \\ & \psi I \end{tikzcd}$$

CN 2. The following diagram commutes:

$$\begin{tikzcd} \phi(AB) \arrow[r, "\hat{\phi}"] \arrow[dd, "\eta"'] & (\phi A, \phi B) \arrow[d, "(1, \eta)"] \\ & (\phi A, \psi B) \\ \psi(AB) \arrow[r, "\hat{\psi}"'] & (\psi A, \psi B) \arrow[u, "(\eta, 1)"'] \end{tikzcd}$$

Proposition 4.1. *If we define ϕ_0 and ψ_0 as in Proposition 3.4,* CN 1 *is equivalent to the commutativity of the diagram*

$$\begin{tikzcd} V \arrow[r, "\phi_0"] \arrow[dr, "\psi_0"'] & V'\phi \arrow[d, "V'\eta"] \\ & V'\psi \end{tikzcd} \tag{4.1}$$

Proof. We show that if we *define* ψ_0 by (4.1), the ψ^0 that corresponds to it by Proposition 3.4 is that given by CN 1. We have by (3.9)

$$\psi^0 = \iota' \psi_0 \iota^{-1} 1 = \iota' \,.\, V'\eta \,.\, \phi_0 \iota^{-1} 1 \,;$$

but by the naturality of ι' we have

$$\iota' \,.\, V'\eta = V'(1, \eta) \,.\, \iota',$$

and thus

$$\begin{aligned}\psi^0 &= V'(1,\eta)\,\iota'\phi_0\,\iota^{-1}\,1\\ &= V'(1,\eta)\,\phi^0 \quad \text{by (3.9)}\\ &= \eta\,\phi^0,\end{aligned}$$

as required.

Theorem 4.2. *Closed categories, closed functors, and closed natural transformations form a hypercategory $\mathscr{C}\ell$ if we define the composite of $\eta:\Phi\to\Phi'$ and $\zeta:\Phi'\to\Phi''$ to be the composite $\zeta\eta$ of $\eta:\phi\to\phi'$ and $\zeta:\phi'\to\phi''$, and if for $\Psi:\mathscr{V}'\to\mathscr{V}$, $X:\mathscr{W}\to\mathscr{W}'$, and $\eta:\Phi\to\Phi':\mathscr{V}\to\mathscr{W}$ we define $\eta\Psi$ and $X\eta$ to be $\eta\psi$ and $\chi\eta$. Moreover $\eta:\Phi\to\Phi'$ is an isomorphism if and only if $\eta:\phi\to\phi'$ is.*

The proofs are straightforward, and we leave them to the reader.

Proposition 4.3. *The axiom* CN2 *for a closed natural transformation $\eta:\Phi\to\Psi:\mathscr{V}\to\mathscr{V}'$ is a consequence of* CN1 *if V' is faithful and Φ is normal.*

Proof. If Φ is normal we have $\phi_0 = 1$ and (4.1) gives

$$V'\eta = \psi_0\,. \tag{4.2}$$

Since V' is faithful it suffices to show that the image under V' of the diagram CN2 commutes. Using NCF0 and NCF1 for Φ, and (4.2), we may write V' of CN2 in the form

$$\begin{array}{ccc} V(A\,B) & \xrightarrow{\quad\phi\quad} & V'(\phi A, \phi B) \\ & & \big\downarrow V'(1,\eta) \\ \Big\downarrow \psi_0 & & V'(\phi A, \psi B) \\ & & \big\uparrow V'(\eta, 1) \\ V'\psi(A\,B) & \xrightarrow[\quad V'\hat{\psi}\quad]{} & V'(\psi A, \psi B) \end{array}$$

Since $V'\psi \,.\, \psi_0 = \psi$ by (3.12), this is just the diagram that asserts the naturality of η.

Remark 4.4. For the above proposition it would suffice to assume ϕ_0 epimorphic instead of Φ normal.

Proposition 4.5. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a closed functor, the natural transformation $\phi_0 : V \to V'\phi : \mathscr{V}_0 \to \mathscr{S}$ is a closed natural transformation $\phi_0 : V \to V'\Phi : \mathscr{V} \to \mathscr{S}$. Moreover if $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ we have the commutative diagram of closed natural transformations*

$$\begin{tikzcd} V \arrow[r, "\phi_0"] \arrow[dr, "\psi_0"'] & V'\Phi \arrow[d, "V'\eta"] \\ & V'\Psi \end{tikzcd} \tag{4.3}$$

Proof. By Proposition 4.3 we need verify only CN 1. If we write this in the equivalent form (4.1) it becomes

$$\begin{tikzcd} V \arrow[r, "1"] \arrow[dr, "\phi_0"'] & V \arrow[d, "\phi_0"] \\ & V'\phi \end{tikzcd}$$

which commutes trivially. The diagram (4.3) is just a translation of (4.1).

5. Categories Over a Closed Category

Let $\mathscr{V}$ be a closed category. A *category $\mathscr{A}$ over $\mathscr{V}$*, or a *$\mathscr{V}$-category*, consists of the following four data:

(i) a class obj $\mathscr{A}$ of "objects";

(ii) for each $A, B \in \operatorname{obj} \mathscr{A}$, an object $\mathscr{A}(AB)$ of $\mathscr{V}_0$;

(iii) for each $A \in \operatorname{obj} \mathscr{A}$, a morphism

$$j_A : I \to \mathscr{A}(AA)$$

in $\mathscr{V}_0$;

(iv) for each $A, B, C \in \operatorname{obj} \mathscr{A}$, a morphism

$$L^A_{BC} : \mathscr{A}(BC) \to (\mathscr{A}(AB), \mathscr{A}(AC))$$

in $\mathscr{V}_0$.

These data are to satisfy the following three axioms, in which $L^{\mathscr{A}(AB)}$ and $j_{\mathscr{A}(AB)}$ are the L and the j of $\mathscr{V}$, while the other L's and j's are those of $\mathscr{A}$:

VC1. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(BB) \arrow[rr, "L^A"] & & (\mathscr{A}(AB), \mathscr{A}(AB)) \\
& I \arrow[ul, "j_B"] \arrow[ur, "j_{\mathscr{A}(AB)}"'] &
\end{tikzcd}$$

VC2. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(AC) \arrow[r, "L^A"] \arrow[dr, "i"'] & (\mathscr{A}(AA), \mathscr{A}(AC) \arrow[d, "(j, 1)"] \\
& (I, \mathscr{A}(AC))
\end{tikzcd}$$

VC3. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(CD) \arrow[rr, "L^B"] \arrow[d, "L^A"'] & & (\mathscr{A}(BC), \mathscr{A}(BD)) \arrow[dd, "(1, L^A)"] \\
(\mathscr{A}(AC), \mathscr{A}(AD)) \arrow[d, "L^{\mathscr{A}(AB)}"'] & & \\
((\mathscr{A}(AB), \mathscr{A}(AC)), (\mathscr{A}(AB), \mathscr{A}(AD))) \arrow[rr, "(L^A, 1)"'] & & (\mathscr{A}(BC), (\mathscr{A}(AB), \mathscr{A}(AD)))
\end{tikzcd}$$

If $\mathscr{A}$ and $\mathscr{B}$ are $\mathscr{V}$-categories, a $\mathscr{V}$-functor $T : \mathscr{A} \to \mathscr{B}$ consists of the following two data:

(i) a function $T : \text{obj}\, \mathscr{A} \to \text{obj}\, \mathscr{B}$;

(ii) for each $B, C \in \text{obj}\, \mathscr{A}$, a morphism

$$T = T_{BC} : \mathscr{A}(BC) \to \mathscr{B}(TB, TC)$$

in $\mathscr{V}_0$.

These data are to satisfy the following two axioms:

VF1. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(BB) \arrow[rr, "T_{BB}"] & & \mathscr{B}(TB, TB) \\
& I \arrow[ul, "j"] \arrow[ur, "j"'] &
\end{tikzcd}$$

VF2. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(CD) \arrow[rr, "L^B"] \arrow[d, "T_{CD}"'] & & (\mathscr{A}(BC), \mathscr{A}(BD)) \arrow[dd, "{(1, T_{BD})}"] \\
\mathscr{B}(TC, TD) \arrow[d, "L^{TB}"'] & & \\
(\mathscr{B}(TB, TC), \mathscr{B}(TB, TD)) \arrow[rr, "{(T_{BC}, 1)}"'] & & (\mathscr{A}(BC), \mathscr{B}(TB, TD))
\end{tikzcd}$$

Theorem 5.1. *$\mathscr{V}$-categories and $\mathscr{V}$-functors form a "category" $\mathscr{V}_*$ if we define the composite of $T : \mathscr{A} \to \mathscr{B}$ and $S : \mathscr{B} \to \mathscr{C}$ to be $P : \mathscr{A} \to \mathscr{C}$ where*

$$PA = STA \tag{5.1}$$

and P_{AB} is the composite

$$P_{AB} : \mathscr{A}(AB) \xrightarrow[T_{AB}]{} \mathscr{B}(TA, TB) \xrightarrow[S_{TA,\,TB}]{} \mathscr{C}(STA, STB)\,. \tag{5.2}$$

Proof. Clearly composition is associative, with obvious identities $1 : \mathscr{A} \to \mathscr{A}$. We must verify that P satisfies VF1 and VF2, that is, that the exteriors of the following two diagrams commute (See page 446 for diagram (5.4)):

$$\begin{tikzcd}
\mathscr{A}(BB) \arrow[r, "T"] & \mathscr{B}(TB, TB) \arrow[r, "S"] & \mathscr{C}(STB, STB) \\
 & I \arrow[ul, "j"] \arrow[u, "j"'] \arrow[ur, "j"'] &
\end{tikzcd} \tag{5.3}$$

In (5.3) one region commutes by VF1 for T and the other by VF1 for S; in (5.4) one region commutes by VF2 for T, one by VF2 for S, and the third trivially.

Theorem 5.2. *If $\mathscr{V}$ is a closed category we get a $\mathscr{V}$-category, also denoted by $\mathscr{V}$, if we take the objects of $\mathscr{V}$ to be those of $\mathscr{V}_0$, take $\mathscr{V}(AB)$ to be (AB), and take for j and L those of the closed category $\mathscr{V}$. Moreover if $\mathscr{A}$ is any $\mathscr{V}$-category and $A \in \mathscr{A}$, we get a $\mathscr{V}$-functor $L^A : \mathscr{A} \to \mathscr{V}$ if we take $L^A B = \mathscr{A}(AB)$ and $(L^A)_{BC} = L^A_{BC}$.*

Proof. The axioms VC1—VC3 for $\mathscr{V}$ reduce to CC1—CC3, and the axioms VF1 and VF2 for L^A reduce to VC1 and VC3.

Remark. In accordance with the above proposition, "object of $\mathscr{V}$" and "object of $\mathscr{V}_0$" are synonyms; we often write $A \in \mathscr{V}$.

$$
\begin{tikzcd}
\mathscr{A}(CD) \arrow[rrrr, "L"] \arrow[d, "T"'] & & & & (\mathscr{A}(BC), \mathscr{A}(BD)) \arrow[d, "{(1,T)}"] \\
\mathscr{B}(TC, TD) \arrow[rr, "L"] \arrow[d, "S"'] & & (\mathscr{B}(TB, TC), \mathscr{B}(TB, TD)) \arrow[rr, "{(T,1)}"] \arrow[dd, "{(1,S)}"] & & (\mathscr{A}(BC), \mathscr{B}(TB, TD)) \arrow[dd, "{(1,S)}"] \\
\mathscr{C}(STC, STD) \arrow[d, "L"'] & & & & \\
(\mathscr{C}(STB, STC), \mathscr{C}(STB, STD)) \arrow[rr, "{(S,1)}"] & & (\mathscr{B}(TB, TC), \mathscr{C}(STB, STD)) \arrow[rr, "{(T,1)}"] & & (\mathscr{A}(BC), \mathscr{C}(STB, STD))
\end{tikzcd}
\tag{5.4}
$$

Proposition 5.3. *We obtain a $\mathscr{V}$-category $\mathscr{I}$ with a single object $*$ if we take $\mathscr{I}(**) = I$, take $j : I \to \mathscr{I}(**)$ to be $1 : I \to I$, and take $L : \mathscr{I}(**) \to \to (\mathscr{I}(**), \mathscr{I}(**))$ to be $i : I \to (II)$. Moreover if $\mathscr{A}$ is a $\mathscr{V}$-category and $A \in \mathscr{A}$, we get a $\mathscr{V}$-functor $J^A : \mathscr{I} \to \mathscr{A}$ if we set $J^A * = A$ and take*

$J^A : \mathscr{I}(**) \to \mathscr{A}(AA)$ to be $j : I \to \mathscr{A}(AA)$. There are no $\mathscr{V}$-functors $\mathscr{I} \to \mathscr{A}$ other than the J^A.

We leave the verification to the reader.

Proposition 5.4. *An $\mathscr{S}$-category $\mathscr{A}$ may be identified with an ordinary category $\mathscr{A}$ if we identify the image of $j : * \to \mathscr{A}(AA)$ with 1_A and identify $(L^A_{BC} f)g$ with the composite fg, where $f \in \mathscr{A}(BC)$ and $g \in \mathscr{A}(AB)$. An $\mathscr{S}$-functor is then an ordinary functor, and in particular the functor $L^A : \mathscr{A} \to \mathscr{S}$ is then the left represented functor $\mathscr{A}(A\,-)$.*

Proof. The reader will easily verify that VC1—VC3 express the two identity laws and the associative law for composition, while VF1 and VF2 become $T1 = 1$ and $T(fg) = Tf \cdot Tg$.

Remark. By analogy with the above we call the $\mathscr{V}$-functor $L^A : \mathscr{A} \to \mathscr{V}$ a left represented $\mathscr{V}$-functor.

6. The Effect of a Closed Functor

In this section it will be convenient to use j, L for the appropriate data in a $\mathscr{V}$-category $\mathscr{A}$, and j', L' for the corresponding data in *a* $\mathscr{V}'$-category $\mathscr{B}$, etc.

Proposition 6.1. *If $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ is a closed functor and $\mathscr{A}$ is a $\mathscr{V}$-category, the following data define a $\mathscr{V}'$-category $\Phi_* \mathscr{A}$:*

(i) *the objects of $\Phi_* \mathscr{A}$ are those of $\mathscr{A}$;* (6.1)

(ii) $(\Phi_* \mathscr{A})(AB) = \phi\mathscr{A}(AB)$ *(that is, $\phi(\mathscr{A}(AB))$);* (6.2)

(iii) $j' : I' \to \phi\mathscr{A}(AA)$ *is the composite*

$$I' \xrightarrow{\phi^0} \phi I \xrightarrow{\phi j} \phi\mathscr{A}(AA); \tag{6.3}$$

(iv) $L' : \phi\mathscr{A}(BC) \to (\phi\mathscr{A}(AB), \phi\mathscr{A}(AC))$ *is the composite*

$$\phi\mathscr{A}(BC) \xrightarrow{\phi L} \phi(\mathscr{A}(AB), \mathscr{A}(AC)) \xrightarrow{\hat{\phi}} (\phi\mathscr{A}(AB), \phi\mathscr{A}(AC)). \tag{6.4}$$

Proof. The axioms VC1—VC3 for $\Phi_* \mathscr{A}$ assert the commutativity of the exteriors of the following three diagrams:

$$\begin{tikzcd}
\phi\mathscr{A}(BB) \arrow[r, "\phi L"] & \phi(\mathscr{A}(AB), \mathscr{A}(AB)) \arrow[r, "\hat{\phi}"] & (\phi\mathscr{A}(AB), \phi\mathscr{A}(AB)) \\
 & \phi I \arrow[ul, "\phi j"] \arrow[u, "\phi j"'] & I' \arrow[l, "\phi^0"] \arrow[u, "j'"']
\end{tikzcd} \tag{6.5}$$

$$
\begin{tikzcd}
 & \phi(\mathscr{A}(AA), \mathscr{A}(AC)) \arrow[r, "\hat{\phi}"] \arrow[d, "\phi(j,1)"] & (\phi\mathscr{A}(AA), \phi\mathscr{A}(AC)) \arrow[d, "(\phi j, 1)"] \\
\phi\mathscr{A}(AC) \arrow[ur, "\phi L"] \arrow[r, "\phi i"] \arrow[drr, "i'"'] & \phi(I, \mathscr{A}(AC)) \arrow[r, "\hat{\phi}"] & (\phi I, \phi\mathscr{A}(AC)) \arrow[d, "(\phi^0, 1)"] \\
 & & (I', \phi\mathscr{A}(AC))
\end{tikzcd}
\tag{6.6}
$$

In the next diagram, X, Y, Z stand for $\mathscr{A}(AB)$, $\mathscr{A}(AC)$, $\mathscr{A}(AD)$:

$$
\begin{tikzcd}
\phi\mathscr{A}(CD) \arrow[r, "\phi L"] \arrow[dd, "\phi L"'] & \phi(\mathscr{A}(BC), \mathscr{A}(BD)) \arrow[r, "\hat{\phi}"] \arrow[d, "\phi(1, L)"] & (\phi\mathscr{A}(BC), \phi\mathscr{A}(BD)) \arrow[d, "(1, \phi L)"] \\
 & \phi(\mathscr{A}(BC), (XZ)) \arrow[r, "\hat{\phi}"'] & (\phi\mathscr{A}(BC), \phi(XZ)) \arrow[ddd, "(1, \hat{\phi})"] \\
\phi(YZ) \arrow[r, "\phi L"'] \arrow[d, "\hat{\phi}"'] & \phi((XY), (XZ)) \arrow[u, "\phi(L, 1)"'] \arrow[d, "\hat{\phi}"] & \\
(\phi Y, \phi Z) \arrow[d, "L'"'] & (\phi(XY), \phi(XZ)) \arrow[d, "(1, \hat{\phi})"] \arrow[uur, "(\phi L, 1)"'] & \\
((\phi X, \phi Y), (\phi X, \phi Z)) \arrow[r, "(\hat{\phi}, 1)"'] & (\phi(XY), (\phi X, \phi Z)) \arrow[r, "(\phi L, 1)"'] & (\phi\mathscr{A}(BC), (\phi X, \phi Z))
\end{tikzcd}
\tag{6.7}
$$

In (6.5) one region commutes by ϕ of VC1 for $\mathscr{A}$, and the other by CF1; in (6.6) one region commutes by ϕ of VC2 for $\mathscr{A}$, one by CF2, and the third by the naturality of $\hat{\phi}$; in (6.7) one region commutes by ϕ of VC3 for $\mathscr{A}$, one by CF3, two by the naturality of $\hat{\phi}$, and one trivially.

The proofs of the following propositions are similar but rather easier: we leave them to the reader.

Proposition 6.2. *If $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ is a closed functor and*

$T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, the following data define a $\mathscr{V}'$-functor

$$\Phi_* T : \Phi_* \mathscr{A} \to \Phi_* \mathscr{B}:$$

(i) $(\Phi_* T)A = TA;$ (6.8)

(ii) $(\Phi_* T)_{BC} = \phi T_{BC} : \phi \mathscr{A}(BC) \to \phi \mathscr{B}(TB, TC).$ (6.9)

Proposition 6.3. *The assigments $\mathscr{A} \mapsto \Phi_* \mathscr{A}$, $T \mapsto \Phi_* T$ constitute a functor $\Phi_* : \mathscr{V}_* \to \mathscr{V}'_*$ from the "category" of $\mathscr{V}$-categories and $\mathscr{V}$-functors to that of $\mathscr{V}'$-categories and $\mathscr{V}'$-functors.*

Proposition 6.4. *If $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ is a closed natural transformation and $\mathscr{A}$ is a $\mathscr{V}$-category, we obtain a $\mathscr{V}'$-functor $\eta_{*\mathscr{A}} : \Phi_* \mathscr{A} \to \Psi_* \mathscr{A}$ if we set*

(i) $\eta_{*\mathscr{A}} A = A;$ (6.10)

(ii) $(\eta_{*\mathscr{A}})_{BC} = \eta_{\mathscr{A}(BC)} : \phi \mathscr{A}(BC) \to \psi \mathscr{A}(BC).$ (6.11)

Moreover the $\eta_{*\mathscr{A}}$ for $\mathscr{A} \in \mathscr{V}_*$ constitute a natural transformation

$$\eta_* : \Phi_* \to \Psi_* : \mathscr{V}_* \to \mathscr{V}'_*.$$

Theorem 6.5. *The assigments $\mathscr{V} \mapsto \mathscr{V}_*$, $\Phi \mapsto \Phi_*$, $\eta \mapsto \eta_*$ constitute a hyperfunctor $_* : \mathscr{C}\ell \to \mathscr{C}at$ from the "hypercategory" of closed categories to that of categories.*

We now consider the effect of Φ_* on the particular $\mathscr{V}$-category $\mathscr{V}$ and on the particular $\mathscr{V}$-functors $L^A : \mathscr{A} \to \mathscr{V}$.

Theorem 6.6. *If $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ is a closed functor, we obtain a $\mathscr{V}'$-functor $\hat{\Phi} : \Phi_* \mathscr{V} \to \mathscr{V}'$ if we set*

(i) $\hat{\Phi} A = \phi A;$ (6.12)

(ii) $\hat{\Phi}_{BC} = \hat{\phi}_{BC} : \phi(BC) \to (\phi B, \phi C).$ (6.13)

If $\Psi : \mathscr{V}' \to \mathscr{V}''$ is another closed functor, and if $X = \Psi\Phi : \mathscr{V} \to \mathscr{V}''$, then $\hat{X}$ is the composite

$$\Psi_* \Phi_* \mathscr{V} \xrightarrow[\Psi_* \hat{\Phi}]{} \Psi_* \mathscr{V}' \xrightarrow[\hat{\Psi}]{} \mathscr{V}''. \tag{6.14}$$

If $\mathscr{A}$ is a $\mathscr{V}$-category and $A \in \mathscr{A}$, the following diagram of $\mathscr{V}'$-functors commutes:

$$\begin{tikzcd}
\Phi_* \mathscr{A} \arrow[r, "\Phi_* L^A"] \arrow[dr, "L'^A"'] & \Phi_* \mathscr{V} \arrow[d, "\hat{\Phi}"] \\
 & \mathscr{V}'
\end{tikzcd} \tag{6.15}$$

Proof. The axioms VF1 and VF3 for $\hat{\Phi}$ reduce to CF1 and CF3 for Φ. The assertions (6.14) and (6.15) are immediate from (3.1), (3.2), (6.2) and (6.4).

The next proposition does the same for the particular $\mathscr{V}$-category $\mathscr{I}$ and the particular $\mathscr{V}$-functors $J^A : \mathscr{I} \to \mathscr{A}$; we write $\mathscr{I}'$ for the $\mathscr{V}'$-category analogous to $\mathscr{I}$. We leave the proofs to the reader.

Proposition 6.7. *If $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ is a closed functor, we obtain a $\mathscr{V}'$-functor $\Phi^0 : \mathscr{I}' \to \Phi_* \mathscr{I}$ if we set:*

(i) $\Phi^0 * = *$; (6.16)

(ii) $\Phi^0_{**} : \mathscr{I}'(**) \to \phi \mathscr{I}(**)$ *is* $\phi^0 : I' \to \phi I$. (6.17)

If $\Psi : \mathscr{V}' \to \mathscr{V}''$ is another closed functor, and if $X = \Psi\Phi : \mathscr{V} \to \mathscr{V}''$, then X^0 is the composite

$$\mathscr{I}'' \xrightarrow[\Psi^0]{} \Psi_* \mathscr{I}' \xrightarrow[\Psi_* \Phi^0]{} \Psi_* \Phi_* \mathscr{I} \tag{6.18}$$

If $\mathscr{A}$ is a $\mathscr{V}$-category and $A \in \mathscr{A}$, the following diagram of $\mathscr{V}'$-functors commutes:

$$\begin{tikzcd} \mathscr{I}' \arrow[r, "\Phi^0"] \arrow[dr, "J'^A"'] & \Phi_* \mathscr{I} \arrow[d, "\Phi_* J^A"] \\ & \Phi_* \mathscr{A} \end{tikzcd} \tag{6.19}$$

7. The Effect of the Closed Functor $V : \mathscr{V} \to \mathscr{S}$

We apply the results of § 6 to the particular closed functor $V : \mathscr{V} \to \mathscr{S}$. Each $\mathscr{V}$-category $\mathscr{A}$ determines an ordinary category $V_* \mathscr{A}$, with the same objects as $\mathscr{A}$, and with

$$(V_* \mathscr{A})(AB) = V\mathscr{A}(AB). \tag{7.1}$$

The j' of $V_* \mathscr{A}$ is the composite

$$* \xrightarrow[V^0]{} VI \xrightarrow[Vj]{} V\mathscr{A}(AA), \tag{7.2}$$

so that the identity 1_A in $V_* \mathscr{A}$ is the image of $*$ under (7.2). By (3.16) we can express this by:

$$\iota 1_A = j_A \tag{7.3}$$

where

$$\iota : V\mathscr{A}(AA) \to V(I, \mathscr{A}(AA)).$$

Just as (2.4) gives Lemma 2.2, so (7.3) gives:

Lemma 7.1. *For any* $f : \mathscr{A}(AA) \to X$ *in* $\mathscr{V}_0$, *the composite*

$$I \xrightarrow{j} \mathscr{A}(AA) \xrightarrow{f} X$$

is the image of $1 \in V\mathscr{A}(AA)$ *under the composite map*

$$V\mathscr{A}(AA) \xrightarrow{Vf} VX \xrightarrow{\iota} V(IX).$$

The L' of $V_*\mathscr{A}$ is the composite

$$V\mathscr{A}(BC) \xrightarrow{VL^A} V(\mathscr{A}(AB), \mathscr{A}(AC)) \xrightarrow{V} (V\mathscr{A}(AB), V\mathscr{A}(AC)), \quad (7.4)$$

so that the composite in $V_*\mathscr{A}$ of $g \in V\mathscr{A}(AB)$ and $f \in V\mathscr{A}(BC)$ is

$$fg = (V((VL^A)f))g. \quad (7.5)$$

Taking $\mathscr{A}$ to be the $\mathscr{V}$-category $\mathscr{V}$, we have:

Proposition 7.2. $V_*\mathscr{V} = \mathscr{V}_0$.

Proof. $V_*\mathscr{V}$ has the same objects as $\mathscr{V}$, and so the same objects as $\mathscr{V}_0$; and by (7.1) and (2.1),

$$(V_*\mathscr{V})(AB) = V(AB) = \mathscr{V}_0(AB).$$

By (7.3) and (2.4), $V_*\mathscr{V}$ and $\mathscr{V}_0$ have the same identities; by (7.5), (2.6), and (2.2) the composite in $V_*\mathscr{V}$ is given by

$$\begin{aligned} fg &= (V((VL^A)f))g \\ &= (V(1,f))g \\ &= \mathscr{V}_0(1,f)g, \end{aligned}$$

and this is also the composite fg in $\mathscr{V}_0$.

In accordance with this result we denote $V_*\mathscr{A}$ by $\mathscr{A}_0$, for any $\mathscr{V}$-category $\mathscr{A}$; and we re-write (7.1) for reference as

$$\mathscr{A}_0(AB) = V\mathscr{A}(AB). \quad (7.6)$$

Similarly if $T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, we denote $V_* T$ by T_0, so that $T_0 : \mathscr{A}_0 \to \mathscr{B}_0$. The assignment $T \mapsto T_0$ is functorial, and T_0 is given by:

$$T_0 A = TA, \quad (7.7)$$

$$T_0 f = (VT)f \quad \text{for} \quad f \in V\mathscr{A}(BC). \quad (7.8)$$

We call $\mathscr{A}_0$ the underlying category of $\mathscr{A}$, and T_0 the underlying functor of T.

For the underlying functor of $L^A : \mathscr{A} \to \mathscr{V}$ we adopt the special notation $\mathscr{A}(A-) : \mathscr{A}_0 \to \mathscr{V}_0$, so that

$$\mathscr{A}(A-) = V_* L^A. \quad (7.9)$$

The value of $\mathscr{A}(A\,-)$ on the object B is $\mathscr{A}(AB)$, and its value $\mathscr{A}(A\,f)$ on the morphism $f \in V\mathscr{A}(BC)$ is given by

$$\mathscr{A}(A\,f) = (VL^A)\,f\,. \tag{7.10}$$

Comparing this with (2.6), we see that

$$\mathscr{V}(A\,-) = (A\,-)\,. \tag{7.11}$$

Since the functor $\hat{V} : V_*\mathscr{V} \to \mathscr{S}$ is just $V : \mathscr{V}_0 \to \mathscr{S}$, (6.15) becomes the commutative diagram

$$\begin{tikzcd} \mathscr{A}_0 \arrow[r, "\mathscr{A}(A\,-)"] \arrow[dr, "\mathscr{A}_0(A\,-)"'] & \mathscr{V}_0 \arrow[d, "V"] \\ & \mathscr{S} \end{tikzcd} \tag{7.12}$$

In particular for $f \in \mathscr{A}_0(BC)$ we have

$$V\mathscr{A}(A\,f) = \mathscr{A}_0(A\,f)\,. \tag{7.13}$$

As the reader no doubt suspects, we shall later show the existence of a functor $\mathscr{A}(-\,B) : \mathscr{A}_0^* \to \mathscr{V}_0$, forming together with $\mathscr{A}(A\,-)$ a bifunctor

$$\hom \mathscr{A} : \mathscr{A}_0^* \times \mathscr{A}_0 \to \mathscr{V}_0$$

satisfying the analogue of CC0. Until we do we can state the analogue of only one half of Proposition 2.4, namely the commutativity of

$$\begin{tikzcd} I \arrow[r, "j"] \arrow[dr, "\iota f"'] & \mathscr{A}(A\,A) \arrow[d, "\mathscr{A}(A\,f)"] \\ & \mathscr{A}(A\,B) \end{tikzcd} \tag{7.14}$$

where $f \in V\mathscr{A}(AB)$; this follows at once from Lemma 7.1 and (7.13).

Now let $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ be a closed functor. By Proposition 4.5 the natural transformation $\phi_0 : V \to V'\phi$ is a closed natural transformation $\phi_0 : V \to V'\Phi$. By Proposition 6.4 therefore, it induces a natural transformation $\phi_{0*} : V_* \to V'_*\Phi_* : \mathscr{V}_* \to \mathscr{S}_*$. We shall write Φ_0 for ϕ_{0*}; then Proposition 6.4 may be stated for this special case as:

Proposition 7.3. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a closed functor, we have for each*

$\mathscr{V}$-category $\mathscr{A}$ a functor $\Phi_{0\mathscr{A}} : \mathscr{A}_0 \to (\Phi_* \mathscr{A})_0$ given by

(i) $\Phi_{0\mathscr{A}} A = A$; (7.15)

(ii) $\Phi_{0\mathscr{A}} f = \phi_0 f$ *for* $f \in V\mathscr{A}(BC)$, (7.16)

where

$$\phi_0 : V\mathscr{A}(BC) \to V'\phi\mathscr{A}(BC).$$

Moreover if $T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, we have a commutative diagram of functors

$$\begin{tikzcd} \mathscr{A}_0 \arrow[r, "T_0"] \arrow[d, "\Phi_{0\mathscr{A}}"'] & \mathscr{B}_0 \arrow[d, "\Phi_{0\mathscr{B}}"] \\ (\Phi_*\mathscr{A})_0 \arrow[r, "(\Phi_* T)_0"'] & (\Phi_*\mathscr{B})_0 \end{tikzcd} \tag{7.17}$$

If Φ is normal we have $\mathscr{A}_0 = (\Phi_* \mathscr{A})_0$ and $\Phi_{0\mathscr{A}} = 1$.

Consider now the $\mathscr{V}'$-functor $\hat{\Phi} : \Phi_* \mathscr{V} \to \mathscr{V}'$ and its underlying functor $V'_* \hat{\Phi} : (\Phi_* \mathscr{V})_0 \to \mathscr{V}'_0$; the latter is given on objects by $(V'_* \hat{\Phi})\, A = \phi A$, and on morphisms by $(V'_* \hat{\Phi}) f = (V'\hat{\phi}) f$, where

$$V'\hat{\phi} : V'\phi(AB) \to V'(\phi A, \phi B).$$

It follows immediately from (3.12) that we have a commutative diagram of functors

$$\begin{tikzcd} \mathscr{V}_0 \arrow[r, "\Phi_{0\mathscr{V}}"] \arrow[dr, "\phi"'] & (\Phi_*\mathscr{V})_0 \arrow[d, "V'_*\hat{\Phi}"] \\ & \mathscr{V}_0' \end{tikzcd} \tag{7.18}$$

Proposition 7.4. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a closed functor, we have for any $\mathscr{V}$-category $\mathscr{A}$ and any $A \in \mathscr{A}$ a commutative diagram of functors*

$$\begin{tikzcd} \mathscr{A}_0 \arrow[r, "\mathscr{A}(A\, -)"] \arrow[d, "\Phi_{0\mathscr{A}}"'] & \mathscr{V}_0 \arrow[d, "\phi"] \\ (\Phi_*\mathscr{A})_0 \arrow[r, "(\Phi_*\mathscr{A})(A\, -)"'] & \mathscr{V}_0' \end{tikzcd} \tag{7.19}$$

Proof. Consider the diagram:

$$\begin{tikzcd}
\mathscr{A}_0 \arrow[r, "V_* L^A"] \arrow[d, "\Phi_{0\mathscr{A}}"'] & \mathscr{V}_0 \arrow[d, "\Phi_{0\mathscr{V}}"] \arrow[dr, "\phi"] & \\
(\Phi_* \mathscr{A})_0 \arrow[r, "V'_* \Phi_* L^A"'] & (\Phi_* \mathscr{V})_0 \arrow[r, "V'_* \hat{\Phi}"'] & \mathscr{V}_0'
\end{tikzcd}$$

The rectangle commutes by (7.17), and the triangle by (7.18). The top edge is $V_* L^A = \mathscr{A}(A -)$, and the bottom edge is, by (6.15),

$$V'_* L'^A = (\Phi_* \mathscr{A})(A -).$$

Proposition 7.5. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ and $\Psi : \mathscr{V}' \to \mathscr{V}''$ are closed functors with composite $X = \Psi\Phi : \mathscr{V} \to \mathscr{V}''$, we have for any $\mathscr{V}$-category $\mathscr{A}$ a commutative diagram of functors:*

$$\begin{tikzcd}
\mathscr{A}_0 \arrow[r, "\Phi_{0\mathscr{A}}"] \arrow[dr, "X_{0\mathscr{A}}"'] & (\Phi_* \mathscr{A})_0 \arrow[d, "\Psi_{0\Phi_*\mathscr{A}}"] \\
& (\Psi_* \Phi_* \mathscr{A})_0
\end{tikzcd} \tag{7.20}$$

Proof. Immediate from (3.13).

Proposition 7.6. *If $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ is a closed natural transformation we have for any $\mathscr{V}$-category $\mathscr{A}$ a commutative diagram of functors*

$$\begin{tikzcd}
\mathscr{A}_0 \arrow[r, "\Phi_{0\mathscr{A}}"] \arrow[dr, "\Psi_{0\mathscr{A}}"'] & (\Phi_* \mathscr{A})_0 \arrow[d, "V'_* \eta_{*\mathscr{A}}"] \\
& (\Psi_* \mathscr{A})_0
\end{tikzcd} \tag{7.21}$$

Proof. Immediate from Proposition 4.5.

8. $\mathscr{V}$-functors into $\mathscr{V}$

We shall in § 10 define $\mathscr{V}$-natural transformations between $\mathscr{V}$-functors, turning $\mathscr{V}_*$ into a hypercategory; but we cannot do so until we have defined, for a $\mathscr{V}$-category $\mathscr{A}$, the functor $\mathscr{A}(-B) : \mathscr{A}_0^* \to \mathscr{V}_0$. The easiest way to get this and to establish its properties is by using the representation theorem for $\mathscr{V}$-functors, which we shall prove below. To discuss the representation theorem we need $\mathscr{V}$-natural transformations, but only for functors into $\mathscr{V}$; and here there is no difficulty, for we already have the functor $(-B) : \mathscr{V}_0^* \to \mathscr{V}_0$.

For a closed category $\mathscr{V}$ and $\mathscr{V}$-functors $T, S : \mathscr{A} \to \mathscr{V}$, a *$\mathscr{V}$-natural transformation* $\alpha : T \to S$ consists of a family of morphisms $\alpha_A : TA \to SA$ in $\mathscr{V}_0$, indexed by the objects of $\mathscr{A}$, satisfying the axiom:

VN$(\mathscr{V})$. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(A\,B) \arrow[r, "T_{AB}"] \arrow[d, "S_{AB}"'] & (TA, TB) \arrow[d, "{(1, \alpha_B)}"] \\
(SA, SB) \arrow[r, "{(\alpha_A, 1)}"'] & (TA, SB)
\end{tikzcd}$$

Proposition 8.1. *For a fixed $\mathscr{V}$-category $\mathscr{A}$, the $\mathscr{V}$-functors $\mathscr{A} \to \mathscr{V}$ and the $\mathscr{V}$-natural transformations between them form a "category" if we define the composite $\beta\alpha$ of $\alpha : T \to S$ and $\beta : S \to R$ by*

$$(\beta\alpha)_A = \beta_A \alpha_A . \tag{8.1}$$

Moreover $\alpha : T \to S$ is an isomorphism in this category if and only if each α_A is an isomorphism in $\mathscr{V}_0$.

Proof. The composition law (8.1) is associative, with identities 1_T having components 1_{TA}. The axiom VN for $\beta\alpha$ asserts the commutativity of the exterior of the following diagram:

$$\begin{tikzcd}
 & (TA, TB) \arrow[rr, "{(1, \alpha)}"] & & (TA, SB) \arrow[dr, "{(1, \beta)}"] & \\
\mathscr{A}(A\,B) \arrow[ur, "T"] \arrow[rr, "S"] \arrow[dr, "R"'] & & (SA, SB) \arrow[ur, "{(\alpha, 1)}"'] \arrow[dr, "{(1, \beta)}"] & & (TA, RB) \\
 & (RA, RB) \arrow[rr, "{(\beta, 1)}"'] & & (SA, RB) \arrow[ur, "{(\alpha, 1)}"'] &
\end{tikzcd} \tag{8.2}$$

Here one region commutes by VN for α, one by VN for β, and one trivially.

If $\beta = \alpha^{-1}$ then by (8.1) we have $\beta_A = \alpha_A^{-1}$. Conversely if α is $\mathscr{V}$-natural and each α_A is an isomorphism, define β by $\beta_A = \alpha_A^{-1}$. Then in (8.2) the top region, the right region, and the exterior all commute.

Since $(\alpha, 1)$ is an isomorphism, the bottom region commutes also, and this is VN for β.

Proposition 8.2. *Let $Q : \mathscr{C} \to \mathscr{A}$ be a $\mathscr{V}$-functor and $\alpha : T \to S : \mathscr{A} \to \to \mathscr{V}$ a $\mathscr{V}$-natural transformation. Then there is a $\mathscr{V}$-natural transformation $\alpha Q : TQ \to SQ : \mathscr{C} \to \mathscr{V}$ with components*

$$(\alpha Q)_C = \alpha_{QC}. \tag{8.3}$$

Proof. Write the diagram VN for α, with A and B replaced by QC and QD, and compose both legs with $Q_{CD} : \mathscr{C}(CD) \to \mathscr{A}(QC, QD)$. There results the diagram VN for αQ.

Proposition 8.3. *If $T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, the morphisms*

$$T_{BC} : \mathscr{A}(BC) \to \mathscr{B}(TB, TC), \quad C \in \mathscr{A},$$

are the components of a $\mathscr{V}$-natural transformation

$$T_B : L^B \to L^{TB}\, T : \mathscr{A} \to \mathscr{V}.$$

In particular, taking T to be $L^A : \mathscr{A} \to \mathscr{V}$, the morphisms

$$L^A_{BC} : \mathscr{A}(BC) \to (\mathscr{A}(AB), \mathscr{A}(AC)), \quad C \in \mathscr{A},$$

are the components of a $\mathscr{V}$-natural transformation

$$L^A_B : L^A \to L^{\mathscr{A}(AB)}\, L^A : \mathscr{A} \to \mathscr{V}.$$

Proof. The axiom VN for T_B reduces to VF2 for T.

Proposition 8.4. *If $f \in \mathscr{V}_0(AB)$, the morphisms*

$$(f, 1) : (BC) \to (AC), \quad C \in \mathscr{V},$$

are the components of a $\mathscr{V}$-natural transformation

$$L^f : L^B \to L^A : \mathscr{V} \to \mathscr{V}.$$

Proof. VN for L^f asserts the commutativity of the diagram

$$\begin{tikzcd}
(CD) \arrow[r, "L^B"] \arrow[d, "L^A"'] & ((BC)(BD)) \arrow[d, "{(1,(f,1))}"] \\
((AC)(AD)) \arrow[r, "{((f,1),1)}"'] & ((BC)(AD)),
\end{tikzcd}$$

which is precisely the assertion that L^A_{CD} is natural in A (which it is, by hypothesis).

Proposition 8.5. *The morphisms*

$$i_A : A \to (IA)$$

are the components of a $\mathscr{V}$-natural transformation

$$i : 1 \to L^I : \mathscr{V} \to \mathscr{V}.$$

Proof. VN for i is CC4 for $\mathscr{V}$.

We now prove the representation theorem for $\mathscr{V}$-functors:

Theorem 8.6. *Let $T : \mathscr{A} \to \mathscr{V}$ be a $\mathscr{V}$-functor, let $K \in \mathscr{A}$, and denote by $\{p\}$ the class of $\mathscr{V}$-natural transformations*

$$p : L^K \to T : \mathscr{A} \to \mathscr{V},$$

with components

$$p_A : \mathscr{A}(KA) \to TA.$$

Define a map $\Gamma : \{p\} \to V(I, TK)$ by setting Γp equal to the composite

$$\Gamma p :\ I \xrightarrow[j_K]{} \mathscr{A}(KK) \xrightarrow[p_K]{} TK. \tag{8.4}$$

Then Γ is a bijection with inverse $\Omega : V(I, TK) \to \{p\}$ where $\Omega\theta$ is the composite

$$\Omega\theta:\qquad L^K \xrightarrow[T_K]{} L^{TK}T \xrightarrow[L^\theta T]{} L^I T \xrightarrow[i^{-1}T]{} T, \tag{8.5}$$

with components

$$\mathscr{A}(KA) \xrightarrow[T]{} (TK, TA) \xrightarrow[(\theta, 1)]{} (I, TA) \xrightarrow[i^{-1}]{} TA \tag{8.6}$$

Proof. Note that by Propositions 8.1 to 8.5, $\Omega\theta$ defined by (8.5) is indeed a $\mathscr{V}$-natural transformation.

Consider the diagram

$$\begin{tikzcd}
\mathscr{A}(KK) \arrow[rr, "T"] & & (TK, TK) \arrow[rr, "{(\theta, 1)}"] & & (I, TK) \arrow[rr, "i^{-1}"] & & TK \\
& & & I \arrow[ulll, "j"] \arrow[ul, "j"'] \arrow[ur, "\iota\theta"] \arrow[urrr, "\theta"'] & & &
\end{tikzcd}$$

The left region commutes by VF1 for T, the middle region by Proposition 2.4, and the right region by Proposition 2.6. Thus $\Gamma\Omega\theta = \theta$, or $\Gamma\Omega = 1$.

Now let $p \in \{p\}$ and consider the diagram

$$\begin{tikzcd}
\mathscr{A}(KA) \arrow[r, "L^K"] \arrow[d, "T"'] & (\mathscr{A}(KK), \mathscr{A}(KA)) \arrow[r, "(j,1)"] \arrow[d, "(1,p)"] & (I, \mathscr{A}(KA)) \arrow[r, "i^{-1}"] \arrow[d, "(1,p)"] & \mathscr{A}(KA) \arrow[d, "p"] \\
(TK, TA) \arrow[r, "(p_K, 1)"'] & (\mathscr{A}(KK), TA) \arrow[r, "(j_K, 1)"'] & (I, TA) \arrow[r, "i^{-1}"'] & TA
\end{tikzcd}$$

The left region commutes by VN for p, the middle region trivially, and the right region by the naturality of i. The composite $i^{-1}(j, 1)L^K$ along the top edge is 1 by VC2 for $\mathscr{A}$; hence

$$p = i^{-1}(p_K j_K, 1)\, T = \Omega \Gamma p,$$

so that $\Omega\Gamma = 1$.

Corollary 8.7. *In the circumstances of Theorem 8.6 we also have a bijection $\Gamma' : \{p\} \to VTK$ given by*

$$\Gamma' p = (V p_K)\, 1_K \tag{8.7}$$

where

$$V p_K : V\mathscr{A}(KK) \to VTK.$$

Proof. By Lemma 2.2, $\Gamma' p = \iota^{-1} \Gamma p.$ (8.8)

We now consider the effect of a closed functor $\mathscr{V} \to \mathscr{V}'$:

Proposition 8.8. *Let $\Phi = (\phi, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ be a closed functor and let $\alpha : T \to S : \mathscr{A} \to \mathscr{V}$ be a $\mathscr{V}$-natural transformation. Then the morphisms*

$$\phi \alpha_A : \phi TA \to \phi SA$$

are the components of a $\mathscr{V}'$-natural transformation

$$\phi\alpha : \hat{\Phi} . \Phi_* T \to \hat{\Phi} . \Phi_* S : \Phi_* \mathscr{A} \to \mathscr{V}'.$$

Proof. VN for $\phi\alpha$ asserts the commutativity of the exterior of the diagram:

$$\begin{tikzcd}
\phi\mathscr{A}(AB) \arrow[r, "\phi T"] \arrow[d, "\phi S"'] & \phi(TA, TB) \arrow[r, "\hat{\phi}"] \arrow[d, "\phi(1,\alpha)"] & (\phi TA, \phi TB) \arrow[dd, "(1, \phi\alpha)"] \\
\phi(SA, SB) \arrow[r, "\phi(\alpha, 1)"'] \arrow[d, "\hat{\phi}"'] & \phi(TA, SB) \arrow[dr, "\hat{\phi}"] & \\
(\phi SA, \phi SB) \arrow[rr, "(\phi\alpha, 1)"'] & & (\phi TA, \phi SB)
\end{tikzcd}$$

One region commutes by ϕ of VN for α, and two by the naturality of $\hat{\phi}$.

We apply the above proposition to a $\mathscr{V}$-natural transformation

$$p : L^K \to T : \mathscr{A} \to \mathscr{V};$$

note that $\hat{\Phi} . \Phi_* L^K = L'^K$ by (6.15), so that we have

$$\phi p : L'^K \to \hat{\Phi} . \Phi_* T : \Phi_* \mathscr{A} \to \mathscr{V}'.$$

Then:

Proposition 8.9. *Let $\Phi : \mathscr{V} \to \mathscr{V}'$ be a closed functor, $T : \mathscr{A} \to \mathscr{V}$ a $\mathscr{V}$-functor and $K \in \mathscr{A}$. Let $\{p\}$ be the class of $\mathscr{V}$-natural transformations*

$$p : L^K \to T : \mathscr{A} \to \mathscr{V}$$

and let $\{q\}$ be the class of $\mathscr{V}'$-natural transformations

$$q : L'^K \to \hat{\Phi} . \Phi_* T : \Phi_* \mathscr{A} \to \mathscr{V}'.$$

Let $\phi : \{p\} \to \{q\}$ be the map $p \mapsto \phi p$ given by Proposition 8.8, define $\Gamma' : \{p\} \to VTK$ by (8.1), *and $\Delta' : \{q\} \to V'\phi TK$ analogously. Then we have a commutative diagram*

$$\begin{tikzcd} \{p\} \arrow[r, "\phi"] \arrow[d, "\Gamma'"'] & \{q\} \arrow[d, "\Delta'"] \\ VTK \arrow[r, "\phi_{0TK}"'] & V'\phi TK \end{tikzcd} \tag{8.9}$$

Proof. Define $\Delta : \{q\} \to V'(I', \phi TK)$ by the analogue of (8.4); it follows at once from (6.3) that $\Delta \phi p$ is the composite

$$\Delta \phi p: \qquad I' \xrightarrow[\phi^0]{} \phi I \xrightarrow[\phi \Gamma p]{} \phi TK. \tag{8.10}$$

By (3.8) we can write this as $\Delta \phi p = \iota' \phi_0 \iota^{-1} \Gamma p$, which by (8.8) is $\Delta' \phi p = \phi_0 \Gamma' p$; and this is (8.9).

Proposition 8.10. *Let $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ be a closed natural transformation. Then η is also a $\mathscr{V}'$-natural transformation*

$$\eta : \hat{\Phi} \to \hat{\Psi} . \eta_{*\mathscr{V}} : \Phi_* \mathscr{V} \to \mathscr{V}'$$

where the second $\mathscr{V}'$-functor here is the composite

$$\Phi_* \mathscr{V} \xrightarrow[\eta_* \mathscr{V}]{} \Psi_* \mathscr{V} \xrightarrow[\hat{\Psi}]{} \mathscr{V}'.$$

Proof. VN for η reduces to CN2 for η.

Note that with η as in Proposition 8.10 each $\mathscr{V}$-functor $T : \mathscr{A} \to \mathscr{V}$ gives rise to a commutative diagram

$$\begin{tikzcd} \Phi_* \mathscr{A} \arrow[r, "\Phi_* T"] \arrow[d, "\eta_{*\mathscr{A}}"'] & \Phi_* \mathscr{V} \arrow[d, "\eta_{*\mathscr{V}}"] \\ \Psi_* \mathscr{A} \arrow[r, "\Psi_* T"'] & \Psi_* \mathscr{V} \end{tikzcd}$$

so that we have a $\mathscr{V}'$-natural transformation

$$\eta \, . \, \Phi_* \, T : \hat{\Phi} \, . \, \Phi_* \, T \to \hat{\Psi} \, . \, \Psi_* \, T \, . \, \eta_{*\mathscr{A}}.$$

We leave the reader to pursue the relation of this to Proposition 8.9.

Proposition 8.11. *A $\mathscr{V}$-natural transformation $\alpha : T \to S : \mathscr{A} \to \mathscr{V}$ is also a natural transformation $\alpha : T_0 \to S_0 : \mathscr{A}_0 \to \mathscr{V}_0$.*

Proof. Applying V to VN for $\alpha : T \to S$ we get the criterion (1.1) for naturality of $\alpha : T_0 \to S_0$.

9. The Bifunctor hom $\mathscr{A}$

Let $\mathscr{V}$ be a closed category and $\mathscr{A}$ a $\mathscr{V}$-category. For A, $B \in \mathscr{A}$ let $\{p\}$ be the class of $\mathscr{V}$-natural transformations

$$p : L^B \to L^A : \mathscr{A} \to \mathscr{V}$$

with components

$$p_C : \mathscr{A}(BC) \to \mathscr{A}(AC) \, .$$

Since $L^A B = \mathscr{A}(AB)$, we have by Corollary 8.7 a bijection

$$\Gamma' : \{p\} \to V\mathscr{A}(AB)$$

given by

$$\Gamma' p = (V p_B) \, 1_B \, . \tag{9.1}$$

For each $f \in V\mathscr{A}(AB)$ $(= \mathscr{A}_0(AB))$, define

$$L^f : L^B \to L^A$$

by

$$\Gamma' L^f = f \, , \tag{9.2}$$

that is, by

$$(VL^f) \, 1 = f \, . \tag{9.3}$$

By Proposition 8.11, L^f is also a natural transformation

$$L^f : V_* L^B \to V_* L^A ,$$

that is,

$$L^f : \mathscr{A}(B-) \to \mathscr{A}(A-) .$$

Applying V gives another natural transformation

$$VL^f : V\mathscr{A}(B-) \to V\mathscr{A}(A-) ,$$

or by (7.12)

$$VL^f : \mathscr{A}_0(B-) \to \mathscr{A}_0(A-) .$$

Proposition 9.1. $VL^f = \mathscr{A}_0(f, 1) : \mathscr{A}_0(BC) \to \mathscr{A}_0(AC)$.

Proof. By the representation theorem, since VL^f and $\mathscr{A}_0(f, 1)$ are both natural transformations, it suffices to show that, when $C = B$, they have the same value at 1_B. But $\mathscr{A}_0(f, 1)1 = f$, and $(VL^f)1 = f$ by (9.3).

Proposition 9.2. *The assignments $A \mapsto L^A$, $f \mapsto L^f$ constitute a functor from $\mathscr{A}_0^*$ to the "category" of $\mathscr{V}$-functors $\mathscr{A} \to \mathscr{V}$ and $\mathscr{V}$-natural transformations between them.*

Proof. We have to show that

$$L^1 = 1 , \tag{9.4}$$

$$L^{fg} = L^g L^f . \tag{9.5}$$

Since $\Gamma' 1 = (V1)1 = 1$, (9.4) follows since Γ' is a bijection. Now

$$\begin{aligned} \Gamma'(L^g L^f) &= VL^g \,.\, VL^f \,.\, 1 && \text{by (9.1)} \\ &= VL^g \,.\, f && \text{by (9.3)} \\ &= \mathscr{A}_0(g, 1) f && \text{by Proposition 9.1} \\ &= fg ; \end{aligned}$$

so that (9.5) follows since Γ' is a bijection.

Now regarding L^f merely as a natural transformation

$$L^f : \mathscr{A}(B-) \to \mathscr{A}(A-) ,$$

it follows from Proposition 9.2 that the assigments $A \mapsto \mathscr{A}(A-)$, $f \mapsto L^f$ constitute a functor from $\mathscr{A}_0^*$ to the category of functors $\mathscr{A}_0 \to \mathscr{V}_0$ and natural transformations between them. Since, as is well known, a functor into a functor category corresponds to a bifunctor, we have here a bifunctor

$$\hom \mathscr{A} : \mathscr{A}_0^* \times \mathscr{A}_0 \to \mathscr{V}_0 .$$

Its value $\hom \mathscr{A}(AB)$ on objects is $\mathscr{A}(AB)$, and we agree to write $\mathscr{A}(f, g)$ for $\hom \mathscr{A}(f, g)$. The defining conditions of $\hom \mathscr{A}$ may then be written:

$$\hom \mathscr{A}(A -) = \mathscr{A}(A -) \,(= V_* L^A), \tag{9.6}$$

$$\mathscr{A}(f, 1) = L^f : \mathscr{A}(BC) \to \mathscr{A}(AC). \tag{9.7}$$

Proposition 9.3. *The following diagram of functors commutes:*

$$\begin{tikzcd} \mathscr{A}_0^* \times \mathscr{A}_0 \arrow[r, "\hom \mathscr{A}"] \arrow[dr, "\operatorname{Hom} \mathscr{A}_0"'] & \mathscr{V}_0 \arrow[d, "V"] \\ & \mathscr{S} \end{tikzcd} \tag{9.8}$$

Proof. In view of (7.12), we have only to show that $V\mathscr{A}(f, 1) = \mathscr{A}_0(f, 1)$; this follows from (9.7) and Proposition 9.1.

We record the evaluated form of (9.8):

$$V\mathscr{A}(f, g) = \mathscr{A}_0(f, g). \tag{9.9}$$

Note that when $\mathscr{A} = \mathscr{V}$ the L^f defined in Proposition 8.4 clearly satisfies (9.3), so that the notation L^f is consistent. So is the notation $\hom \mathscr{V}$, as we see by comparing (9.6) and (9.7) with (7.11) and Proposition 8.4.

Now that $\mathscr{A}(AB)$ is the value of a functor $\hom \mathscr{A}$ the question can be raised of the naturality of

$$j_A : I \to \mathscr{A}(AA) \quad \text{and} \quad L^A_{BC} : \mathscr{A}(BC) \to (\mathscr{A}(AB), \mathscr{A}(AC)).$$

Similarly if $T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, one can discuss the naturality of $T_{BC} : \mathscr{A}(BC) \to \mathscr{B}(TB, TC)$, meaning of course the naturality of $T_{BC} : \mathscr{A}(BC) \to \mathscr{B}(T_0 B, T_0 C)$; recall that $T_0 A = TA$ by (7.7).

Proposition 9.4. *If $\mathscr{A}$ is a $\mathscr{V}$-category the morphisms*

$$j_A : I \to \mathscr{A}(AA) \quad \textit{and} \quad L^A_{BC} : \mathscr{A}(BC) \to (\mathscr{A}(AB), \mathscr{A}(AC))$$

are natural in every variable; and if $T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor the morphism $T_{BC} : \mathscr{A}(BC) \to \mathscr{B}(TB, TC)$ is natural in both variables.

Proof. For $f \in \mathscr{A}_0(AB) = V\mathscr{A}(AB)$, consider the diagram

$$\begin{tikzcd} I \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, "\iota f"] & \mathscr{A}(AA) \arrow[d, "\mathscr{A}(1, f)"] \\ \mathscr{A}(BB) \arrow[r, "\mathscr{A}(f, 1)"'] & \mathscr{A}(AB) \end{tikzcd} \tag{9.10}$$

The upper triangle commutes by (7.14); so does the lower one by Lemma 7.1 in view of (9.9). Hence the exterior commutes, so that j is natural.

By Propositions 8.3 and 8.11, T_B is a natural transformation

$$\mathscr{A}(B-) \to \mathscr{B}(T_0 B, T_0 -),$$

i.e. T_{BC} is natural in C. To prove it also natural in B is to show, for each $f \in \mathscr{A}_0(AB)$, the commutativity of

$$\begin{tikzcd} \mathscr{A}(BC) \arrow[r, "T_{BC}"] \arrow[d, "\mathscr{A}(f,1)"'] & \mathscr{B}(TB, TC) \arrow[d, "\mathscr{B}(T_0 f, 1)"] \\ \mathscr{A}(AC) \arrow[r, "T_{AC}"'] & \mathscr{B}(TA, TC) \end{tikzcd} \tag{9.11}$$

Now (9.11) is the C-component of the following diagram of $\mathscr{V}$-natural transformations:

$$\begin{tikzcd} L^B \arrow[r, "T_B"] \arrow[d, "L^f"'] & L^{TB}T \arrow[d, "L^{T_0 f}T"] \\ L^A \arrow[r, "T_A"'] & L^{TA}T \end{tikzcd}$$

and so by the representation theorem for $\mathscr{V}$-functors it suffices to apply V to (9.11), put $C = B$, and verify that both legs have the same value at 1_B. Using (7.8) and (9.9), we have

$$V\mathscr{B}(T_0 f, 1) \,.\, VT \,.\, 1 = V\mathscr{B}(T_0 f, 1) \,.\, 1 = T_0 f,$$

$$VT \,.\, V\mathscr{A}(f, 1) \,.\, 1 = VT \,.\, f = T_0 f.$$

Applying the above to the $\mathscr{V}$-functor L^A, we have the naturality in B and in C of L^A_{BC}; it remains to prove its naturality in A, namely the commutativity for $f \in \mathscr{A}_0(DA)$ of

$$\begin{tikzcd} \mathscr{A}(BC) \arrow[r, "L^A"] \arrow[d, "L^D"'] & (\mathscr{A}(AB), \mathscr{A}(AC)) \arrow[d, "(1, \mathscr{A}(f,1))"] \\ (\mathscr{A}(DB), \mathscr{A}(DC)) \arrow[r, "(\mathscr{A}(f,1), 1)"'] & (\mathscr{A}(AB), \mathscr{A}(DC)) \end{tikzcd} \tag{9.12}$$

This does indeed commute, being by (9.7) precisely the axiom VN for L^f. This completes the proof.

The category $\mathscr{A}_0$ and the functor hom $\mathscr{A}$ associated with a $\mathscr{V}$-category $\mathscr{A}$, whose existence we have shown constructively, are characterized by the following uniqueness theorem, in which of course $\mathscr{A}_0$ and hom $\mathscr{A}$ cease temporarily to have the meanings we have given them.

Proposition 9.5. *Let $\mathscr{V}$ be a closed category, and suppose we are given a category $\mathscr{A}_0$, a functor* hom $\mathscr{A} : \mathscr{A}_0^* \times \mathscr{A}_0 \to \mathscr{V}_0$ *(whose values on objects and on morphisms we write as $\mathscr{A}(AB)$ and $\mathscr{A}(f, g)$), a natural transformation $j_A : I \to \mathscr{A}(AA)$, and a natural transformation $L^A_{BC} : \mathscr{A}(BC) \to$ $\to (\mathscr{A}(AB), \mathscr{A}(AC))$, satisfying* VC1, VC2, VC3 *and also* (9.8) *and* (7.3). *Then $\mathscr{A}$ is* a fortiori *a $\mathscr{V}$-category, and we necessarily have:*

(i) $\mathscr{A}_0 = V_* \mathscr{A}$;

(ii) $\mathscr{A}(A -) = V_* L^A$;

(iii) $\mathscr{A}(f, 1) = L^f$.

Proof. From (7.3) we get Lemma 7.1, and use it to write VC1 in the form

$$(VL^A_{BB}) 1_B = 1_{\mathscr{A}(AB)}; \tag{9.13}$$

just as in Proposition 2.3 we show that this is equivalent to (7.10). The proof that $V_* \mathscr{A}$ is $\mathscr{A}_0$ is now exactly similar to that of Proposition 7.2, and then (7.10) may be written $V_* L^A = \mathscr{A}(A -)$. The naturality of L^A_{BC} in A gives (9.12), which shows that $\mathscr{A}(f, 1) : L^A \to L^D$ is $\mathscr{V}$-natural; since $(V\mathscr{A}(f, 1)) 1 = \mathscr{A}_0(f, 1) 1 = f$, we conclude from (9.3) that

$$\mathscr{A}(f, 1) = L^f.$$

Similarly for $\mathscr{V}$-functors:

Proposition 9.6. *Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathscr{V}$-categories, and suppose we are given a functor $T_0 : \mathscr{A}_0 \to \mathscr{B}_0$ and a natural transformation $T_{BC} : \mathscr{A}(BC) \to$ $\to \mathscr{B}(T_0 B, T_0 C)$, such that T satisfies* VF1 *and* VF2 *if we write TA for $T_0 A$. Then T is* a fortiori *a $\mathscr{V}$-functor, and we necessarily have*

$$T_0 = V_* T : \mathscr{A}_0 \to \mathscr{B}_0.$$

Proof. We use Lemma 7.1 to write VF1 for T in the form

$$(VT_{BB}) 1_B = 1_{TB}; \tag{9.14}$$

then by the representation theorem the two natural transformations

$$T_{0BC}, VT_{BC} : V\mathscr{A}(BC) \to V\mathscr{B}(T_0 B, T_0 C)$$

coincide, which completes the proof.

Remark 9.7. It follows from Proposition 9.5 that we could have given an alternative definition of $\mathscr{V}$-category, including the category $\mathscr{A}_0$ and the functor hom $\mathscr{A}$ among the data, insisting on naturality of j and L, and adding (9.8) and (7.3) to VC1—VC3 as axioms. Then j is a superfluous datum, as (7.3) defines it; and VC1 may be expressed without using j in the form (9.13), or equivalently as (7.9) or (7.10). If V is faithful, axioms VC2 and VC3 are then unnecessary, by an argument exactly like that of Proposition 2.10. Indeed when V is faithful the data themselves are somewhat redundant; we leave the reader to formulate an analogue of Proposition 2.11. Similarly we may include T_0 in the definition of a $\mathscr{V}$-functor T, and require T_{BC} to be natural. Then VF1 may be written as (9.14) or (7.8), and VF2 is a consequence if V is faithful.

Proposition 9.8. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a closed functor and $\mathscr{A}$ is a $\mathscr{V}$-category, the following diagram of functors commutes:*

$$\begin{tikzcd} \mathscr{A}_0^* \times \mathscr{A}_0 \arrow[r, "\mathrm{hom}\,\mathscr{A}"] \arrow[d, "\Phi_{0\mathscr{A}}^* \times \Phi_{0\mathscr{A}}"'] & \mathscr{V}_0 \arrow[d, "\phi"] \\ (\Phi_* \mathscr{A})_0^* \times (\Phi_* \mathscr{A})_0 \arrow[r, "\mathrm{hom}\,\Phi_*\mathscr{A}"'] & \mathscr{V}'_0 \end{tikzcd} \tag{9.15}$$

Proof. As we already have Proposition 7.4, it only remains to show that (9.15) commutes when evaluated at morphisms $f \in \mathscr{A}_0^*(BA)$ and $1 \in \mathscr{A}_0(CC)$; that is, that for $f \in \mathscr{A}_0(AB)$ the morphisms

$$\phi \mathscr{A}(f, 1) : \phi \mathscr{A}(BC) \to \phi \mathscr{A}(AC), \tag{9.16}$$

$$(\Phi_* \mathscr{A})(\phi_0 f, 1) : \phi \mathscr{A}(BC) \to \phi \mathscr{A}(AC), \tag{9.17}$$

coincide.

Now by Proposition 8.8, (9.16) is the C-component of the $\mathscr{V}'$-natural transformation

$$\phi L^f : L'^B \to L'^C : \Phi_* \mathscr{A} \to \mathscr{V}',$$

since $\hat{\Phi} \,.\, \Phi_* L^B = L'^B$ by (6.15); while (9.17) is the C-component of the $\mathscr{V}'$-natural transformation

$$L'^{\phi_0 f} : L'^B \to L'^C : \Phi_* \mathscr{A} \to \mathscr{V}'.$$

We have therefore to show that

$$\phi L^f = L'^{\phi_0 f}, \tag{9.18}$$

and by (9.3) it suffices to show that

$$(V'\phi Lf)\,1_B = \phi_0 f. \tag{9.19}$$

By the naturality of ϕ_0 we have a commutative diagram

$$\begin{tikzcd} V\mathscr{A}(BB) \arrow[r, "VLf"] \arrow[d, "\phi_0"'] & V\mathscr{A}(AB) \arrow[d, "\phi_0"] \\ V'\phi\mathscr{A}(BB) \arrow[r, "V'\phi Lf"'] & V'\phi\mathscr{A}(AB) \end{tikzcd}$$

Evaluating both legs at 1_B gives (9.19), since $(VLf)1 = f$ by (9.3), and $\phi_0 1 = 1$ since Φ_0 is a functor.

10. $\mathscr{V}$-natural Transformations

We now define $\mathscr{V}$-natural transformations in general. For a closed category $\mathscr{V}$ and $\mathscr{V}$-functors $T, S : \mathscr{A} \to \mathscr{B}$, a *$\mathscr{V}$-natural transformation* $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ consists of a family of morphisms $\alpha_A : TA \to SA$ in $\mathscr{B}_0$, indexed by the objects of $\mathscr{A}$, satisfying the axiom:

VN. The following diagram commutes:

$$\begin{tikzcd} \mathscr{A}(AB) \arrow[r, "T_{AB}"] \arrow[d, "S_{AB}"'] & \mathscr{B}(TA, TB) \arrow[d, "\mathscr{B}(1, \alpha_B)"] \\ \mathscr{B}(SA, SB) \arrow[r, "\mathscr{B}(\alpha_A, 1)"'] & \mathscr{B}(TA, SB) \end{tikzcd}$$

Proposition 10.1. *A $\mathscr{V}$-natural transformation $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ is also a natural transformation $\alpha : T_0 \to S_0 : \mathscr{A}_0 \to \mathscr{B}_0$. If V is faithful, the naturality of α conversely implies its $\mathscr{V}$-naturality.*

Proof. Both statements follow from the fact that V of the diagram VN is the diagram (1.1) expressing the naturality of α.

Theorem 10.2. *$\mathscr{V}$-categories, $\mathscr{V}$-functors, and $\mathscr{V}$-natural transformations form a "hypercategory" (which we still denote by $\mathscr{V}_*$) if we define the composite of*

(i) $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ *and* $\beta : S \to R : \mathscr{A} \to \mathscr{B}$ *by*

$$(\beta\alpha)_A = \beta_A \alpha_A; \tag{10.1}$$

(ii) $Q : \mathscr{C} \to \mathscr{A}$ *and* $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ *by*

$$(\alpha Q)_C = \alpha_{QC}; \tag{10.2}$$

(iii) $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ *and* $P : \mathscr{B} \to \mathscr{D}$ *by*

$$(P\alpha)_A = P_0 \alpha_A . \tag{10.3}$$

Moreover α *is an isomorphism if and only if each* α_A *is an isomorphism.*

The hypercategory $\mathscr{S}_*$ *is then* $\mathscr{C}at$, *and* $V_* : \mathscr{V}_* \to \mathscr{S}_*$ *is a hyperfunctor if we set*

$$V_* \alpha = \alpha . \tag{10.4}$$

Proof. The proof that $\beta\alpha$ is $\mathscr{V}$-natural, and of the statement about isomorphisms, is a trivial generalization of the proof of Proposition 8.1; similarly the proof of Proposition 8.2 generalizes to show that αQ is $\mathscr{V}$-natural. To show that $P\alpha$ is $\mathscr{V}$-natural is to show the commutativity of the exterior of the diagram

$$\begin{tikzcd}
 & \mathscr{B}(TA, TB) \arrow[rr, "P"] \arrow[dr, "\mathscr{B}(1, \alpha)"] & & \mathscr{D}(PTA, PTB) \arrow[dr, "\mathscr{D}(1, P_0\alpha)"] & \\
\mathscr{A}(A B) \arrow[ur, "T"] \arrow[dr, "S"'] & & \mathscr{B}(TA, SB) \arrow[rr, "P"] & & \mathscr{D}(PTA, PSB) \\
 & \mathscr{B}(SA, SB) \arrow[rr, "P"'] \arrow[ur, "\mathscr{B}(\alpha, 1)"'] & & \mathscr{D}(PSA, PSB) \arrow[ur, "\mathscr{D}(P_0\alpha, 1)"'] &
\end{tikzcd}$$

One region commutes by VN for α, and the other two by the naturality of P guaranteed by Proposition 9.4.

V_* clearly preserves all the above laws of composition, so that HC2—HC4 for $\mathscr{V}_*$ follow from HC2—HC4 for $\mathscr{C}at$. As we already have HC1, this completes the proof.

Proposition 10.3. *If* $\Phi : \mathscr{V} \to \mathscr{V}'$ *is a closed functor and*

$$\alpha : T \to S : \mathscr{A} \to \mathscr{B}$$

is a $\mathscr{V}$*-natural transformation, we get a* $\mathscr{V}'$*-natural transformation*

$$\Phi_* \alpha : \Phi_* T \to \Phi_* S : \Phi_* \mathscr{A} \to \Phi_* \mathscr{B}$$

if we set

$$(\Phi_* \alpha)_A = \Phi_{0\mathscr{B}} \alpha_A \tag{10.5}$$

where

$$\Phi_{0\mathscr{B}} : \mathscr{B}_0 \to (\Phi_* \mathscr{B})_0 .$$

In particular if Φ is normal we have (cf. (10.4))

$$\Phi_* \alpha = \alpha. \tag{10.6}$$

Proof. Writing γ for $\Phi_* \alpha$, VN for γ asserts the commutativity of:

$$\begin{tikzcd} \phi\mathscr{A}(A\,B) \arrow[r, "\phi T"] \arrow[d, "\phi S"'] & \phi\mathscr{B}(TA, TB) \arrow[d, "(\Phi_*\mathscr{B})(1,\gamma)"] \\ \phi\mathscr{B}(SA, SB) \arrow[r, "(\Phi_*\mathscr{B})(\gamma, 1)"'] & \phi\mathscr{B}(TA, SB) \end{tikzcd} \tag{10.7}$$

Since $\gamma = \Phi_{0\mathscr{B}}\alpha$, it follows from (9.15) that $(\Phi_*\mathscr{B})(1,\gamma) = \phi\mathscr{B}(1,\alpha)$ and $(\Phi_*\mathscr{B})(\gamma, 1) = \phi\mathscr{B}(\alpha, 1)$; thus (10.7) does indeed commute, being ϕ of VN for α.

Proposition 10.4. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a closed functor and $\alpha : T \to S : \mathscr{A} \to \mathscr{V}$ is a $\mathscr{V}$-natural transformation, the $\mathscr{V}'$-natural transformation*

$$\phi\alpha : \hat{\Phi}.\,\Phi_*\,T \to \hat{\Phi}.\,\Phi_*\,S : \Phi_*\,\mathscr{A} \to \mathscr{V}'$$

of Proposition 8.8 is given by

$$\phi\alpha = \hat{\Phi}.\Phi_*\,\alpha. \tag{10.8}$$

Proof. By (10.3) and (10.5) the A-component of $\hat{\Phi}.\Phi_*\,\alpha$ is $V'_*\hat{\Phi}.\Phi_{0\mathscr{V}}.\alpha_A$, and this is $\phi\alpha_A$ by (7.18).

Proposition 10.5. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a closed functor, $\Phi_* : \mathscr{V}_* \to \mathscr{V}'_*$ is a hyperfunctor.*

Proof. We have to show that Φ_* respects the composition laws (10.1), (10.2), (10.3). For (10.1) this follows since $\Phi_{0\mathscr{B}}$ is a functor, and for (10.2) it is trivial since $(\Phi_* Q)\,C = QC$ by (6.8). For (10.3), the A-component of $\Phi_*(P\alpha)$ is $\Phi_{0\mathscr{D}}P_0\alpha_A$, which by (7.17) is equal to $(\Phi_* P)_0\Phi_{0\mathscr{B}}\alpha_A$, which is the A-component of $\Phi_*\,P.\Phi_*\,\alpha$.

Proposition 10.6. *If $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ is a closed natural transformation, the $\mathscr{V}'$-functors $\eta_{*\mathscr{A}} : \Phi_*\,\mathscr{A} \to \Psi_*\,\mathscr{A}$ defined by* (6.10) *and* (6.11) *are the components of a hypernatural transformation $\eta_* : \Phi_* \to \Psi_* : \mathscr{V}_* \to \mathscr{V}'_*$.*

Proof. By Proposition 6.4 η_* is natural, i.e. HN 1 is satisfied. We have to verify HN 2, namely the coincidence of

$$\eta_{*\mathscr{B}}.\,\Phi_*\,\alpha : \eta_{*\mathscr{B}}.\,\Phi_*\,T \to \eta_{*\mathscr{B}}.\,\Phi_*\,S : \Phi_*\,\mathscr{A} \to \Psi_*\,\mathscr{B}$$

and

$$\Psi_* \alpha . \eta_{*\mathscr{A}} : \Psi_* T . \eta_{*\mathscr{A}} \to \Psi_* S . \eta_{*\mathscr{A}} : \Phi_* \mathscr{A} \to \Psi_* \mathscr{B},$$

where $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-natural transformation. By (10.2), (10.3), and (10.5), the A-components of $\eta_{*\mathscr{B}} . \Phi_* \alpha$ and of $\Psi_* \alpha . \eta_{*\mathscr{A}}$ are respectively $V'_* \eta_{*\mathscr{B}} . \Phi_{0\mathscr{B}} . \alpha_A$ and $\Psi_{0\mathscr{B}} \alpha_A$, and these are equal by (7.21).

Theorem 10.7. *The assignments $\mathscr{V} \mapsto \mathscr{V}_*$, $\Phi \mapsto \Phi_*$, $\eta \to \eta_*$ constitute a hyperfunctor $_* : \mathscr{C}\ell \to \mathscr{H}yp$ from the "hypercategory" of closed categories to that of hypercategories.*

Proof. As we already have Theorem 6.5, all that remains to be shown is that, if $\Phi : \mathscr{V} \to \mathscr{V}'$ and $\Psi : \mathscr{V}' \to \mathscr{V}''$ are closed functors and $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-natural transformation, we have $(\Psi \Phi)_* \alpha = \Psi_* \Phi_* \alpha$; this is immediate from (7.20).

We now give a form of the representation theorem for $\mathscr{V}$-functors analogous to Theorem 1.1:

Theorem 10.8. *Let $T : \mathscr{A} \to \mathscr{B}$ be a $\mathscr{V}$-functor, let $K \in \mathscr{A}$ and $M \in \mathscr{B}$, and denote by $\{p\}$ the class of $\mathscr{V}$-natural transformations*

$$p : L^K \to L^M T : \mathscr{A} \to \mathscr{V}$$

with components

$$p_A : \mathscr{A}(KA) \to \mathscr{B}(M, TA).$$

Define a map $\Gamma' : \{p\} \to V\mathscr{B}(M, TK)$ by:

$$\Gamma' p = (V p_K)\, 1_K. \tag{10.9}$$

Then Γ' is a bijection with inverse $\Omega' : V\mathscr{B}(M, TK) \to \{p\}$ where $\Omega' \theta$ is the composite

$$\Omega' \theta : \; L^K \xrightarrow[T_K]{} L^{TK} T \xrightarrow[L^\theta T]{} L^M T \tag{10.10}$$

with components

$$\mathscr{A}(KA) \xrightarrow[T]{} \mathscr{B}(TK, TA) \xrightarrow[\mathscr{B}(\theta, 1)]{} \mathscr{B}(M, TA). \tag{10.11}$$

Proof. Γ' is a bijection by Corollary 8.7, and clearly $\Gamma' \Omega' = 1$.

Remark 10.9. In the circumstances of Theorem 10.8 let $\Phi : \mathscr{V} \to \mathscr{V}'$ be a closed functor and let $\{q\}$ be the class of $\mathscr{V}'$-natural transformations

$$q : L'^K \to L'^M . \Phi_* T : \Phi_* \mathscr{A} \to \mathscr{V}'$$

with components

$$q_A : \phi \mathscr{A}(KA) \to \phi \mathscr{B}(M, TA).$$

Define $\Delta' : \{q\} \to V'\phi\mathscr{B}(M, TK)$ analogously to Γ'; then by Proposition 8.9 we have a commutative diagram

$$\begin{tikzcd} \{p\} \arrow[r, "\phi"] \arrow[d, "\Gamma'"'] & \{q\} \arrow[d, "\Delta'"] \\ V\mathscr{B}(M, TK) \arrow[r, "\phi_0"'] & V'\phi\mathscr{B}(M, TK) \end{tikzcd}$$

where $(\phi p)_A = \phi p_A$. In particular if Φ is normal we have $\phi_0 = 1$, and so $\phi : \{p\} \to \{q\}$ is also a bijection. Note especially the case when Φ is $V : \mathscr{V} \to \mathscr{S}$.

Proposition 10.10. *Let $T, S : \mathscr{A} \to \mathscr{B}$ be $\mathscr{V}$-functors and let $\alpha_A : TA \to SA$ be a family of morphisms. Then the following assertions are equivalent:*
(a) *The α_A are the components of a $\mathscr{V}$-natural transformation $\alpha : T \to S$.*
(b) *The morphisms*

$$\mathscr{B}(1, \alpha_B) : \mathscr{B}(TA, TB) \to \mathscr{B}(TA, SB), \quad B \in \mathscr{B},$$

are for each A the components of a $\mathscr{V}$-natural transformation

$$L^{TA}\, T \to L^{TA}\, S.$$

(c) *The composite morphisms*

$$\mathscr{A}(A\,B) \xrightarrow{T} \mathscr{B}(TA, TB) \xrightarrow{\mathscr{B}(1,\alpha)} \mathscr{B}(TA, S\,B), \quad B \in \mathscr{B},$$

are for each A the components of a $\mathscr{V}$-natural transformation

$$L^A \to L^{TA}\, S.$$

Proof. If α is $\mathscr{V}$-natural, $\mathscr{B}(1, \alpha) = L^{TA}\alpha$ is also $\mathscr{V}$-natural, so (a) implies (b). Similarly (b) implies (c) since $T_A : L^A \to L^{TA}\, T$ is $\mathscr{V}$-natural. We have to show that (c) implies (a).

In the diagram VN for α, namely

$$\begin{tikzcd} \mathscr{A}(A\,B) \arrow[r, "T"] \arrow[d, "S"'] & \mathscr{B}(TA, TB) \arrow[d, "\mathscr{B}(1, \alpha)"] \\ \mathscr{B}(SA, SB) \arrow[r, "\mathscr{B}(\alpha, 1)"'] & \mathscr{B}(TA, SB) \end{tikzcd}$$

the top leg is by hypothesis (c) $\mathscr{V}$-natural for each A; but the bottom leg is also $\mathscr{V}$-natural, being the B-component of $L^\alpha S.S_A$. If we put

$B = A$, apply V, and evaluate at 1_A, each leg gives α_A; hence by the representation theorem for $\mathscr{V}$-functors the diagram commutes.

Remark 10.11. By a *representation* of a $\mathscr{V}$-functor $T : \mathscr{A} \to \mathscr{V}$ we mean an object K of $\mathscr{A}$ together with a $\mathscr{V}$-natural isomorphism $p : L^K \to T$; if T admits a representation we say it is *representable*. If $p : L^K \to T$ and $q : L^M \to S$ are representations of $T, S : \mathscr{A} \to \mathscr{V}$, it is clear from the representation theorem that a bijection is set up between $\mathscr{V}$-natural transformations $\alpha : T \to S$ and morphisms $f : M \to K$ by requiring commutativity in the diagram

$$\begin{tikzcd} L^K \arrow[r, "p"] \arrow[d, "L^f"'] & T \arrow[d, "\alpha"] \\ L^M \arrow[r, "q"'] & S \end{tikzcd}$$

In particular if $p : L^K \to T$ and $q : L^M \to T$ are two representations of T then there is a unique isomorphism $f \in V\mathscr{A}(MK)$ giving a commutative diagram

$$\begin{tikzcd} L^K \arrow[r, "p"] \arrow[d, "L^f"'] & T \\ L^M \arrow[ur, "q"'] & \end{tikzcd}$$

Remark 10.12. If $\mathscr{A}$ is a $\mathscr{V}$-category, any morphism $f \in \mathscr{A}_0(AB)$ is a $\mathscr{V}$-natural transformation $f : J^A \to J^B : \mathscr{I} \to \mathscr{A}$; for VN for f is just (9.10). If $T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, it is then consistent with (10.3) to write Tf for $T_0 f$. We shall use both notations, the one for brevity and the other where there is danger of confusion.

Chapter II

Monoidal Closed Categories

1. Monoidal Categories

A *monoidal category* $\mathscr{V} = (\mathscr{V}_0, \otimes, I, r, l, a)$ consists of the following six data:

(i) a category $\mathscr{V}_0$;
(ii) a functor $\otimes : \mathscr{V}_0 \times \mathscr{V}_0 \to \mathscr{V}_0$ (written between its arguments and called the *tensor product* of $\mathscr{V}$);
(iii) an object I of $\mathscr{V}_0$;

(iv) a natural isomorphism $r = r_A : A \otimes I \to A$;
(v) a natural isomorphism $l = l_A : I \otimes A \to A$;
(vi) a natural isomorphism $a = a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.
These data are to satisfy the following five axioms:

MC1. The following diagram commutes:

$$\begin{tikzcd}
(I \otimes A) \otimes B \arrow[rr, "a"] \arrow[dr, "l \otimes 1"'] & & I \otimes (A \otimes B) \arrow[dl, "l"] \\
& A \otimes B &
\end{tikzcd}$$

MC2. The following diagram commutes:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "a"] \arrow[dr, "r \otimes 1"'] & & A \otimes (I \otimes B) \arrow[dl, "1 \otimes l"] \\
& A \otimes B &
\end{tikzcd}$$

MC3. The following diagram commutes:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a"] \arrow[d, "a \otimes 1"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1 \otimes a"']
\end{tikzcd}$$

MC4. The following diagram commutes:

$$\begin{tikzcd}
(A \otimes B) \otimes I \arrow[rr, "a"] \arrow[dr, "r"'] & & A \otimes (B \otimes I) \arrow[dl, "1 \otimes r"] \\
& A \otimes B &
\end{tikzcd}$$

MC5. $$l_I = r_I : I \otimes I \to I.$$

We remark at once that the above axioms are not independent; we have listed them all, and arranged them in the above rather odd order, for later comparison with CC1—CC5. In fact it has been shown (KELLY [*9*]) that:

Proposition 1.1. MC1, MC4, *and* MC5 *are consequences of* MC2 *and* MC3.

Natural isomorphisms such as a, r, l are said to be *coherent* if, roughly speaking, all diagrams made by their use alone (with their inverses, 1, and $\otimes$), such as the diagrams of MC1—MC5, commute. For an exact description of the meaning of coherence, see MAC LANE [*14*], where it is proved that MC1—MC5 imply:

Proposition 1.2. *The isomorphisms a, r, l are coherent.*

Let $\mathscr{V} = (\mathscr{V}_0, \otimes, I, r, l, a)$ and $\mathscr{V}' = (\mathscr{V}'_0, \otimes', I', r', l', a')$ be monoidal categories; we write $\otimes$ for $\otimes'$ when there is no danger of confusion. A *monoidal functor* $\Phi = (\phi, \tilde{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ consists of

(i) a functor $\phi : \mathscr{V}_0 \to \mathscr{V}'_0$;
(ii) a natural transformation $\tilde{\phi} = \tilde{\phi}_{AB} : \phi A \otimes \phi B \to \phi(A \otimes B)$;
(iii) a morphism $\phi^0 : I' \to \phi I$.

These data are to satisfy the following three axioms:

MF1. The following diagram commutes:

$$\begin{tikzcd}
\phi I \otimes \phi A \arrow[r, "\tilde{\phi}"] & \phi(I \otimes A) \arrow[d, "\phi l"] \\
I' \otimes \phi A \arrow[u, "\phi^0 \otimes 1"] \arrow[r, "l'"'] & \phi A
\end{tikzcd}$$

MF2. The following diagram commutes:

$$\begin{tikzcd}
\phi A \otimes \phi I \arrow[r, "\tilde{\phi}"] & \phi(A \otimes I) \arrow[d, "\phi r"] \\
\phi A \otimes I' \arrow[u, "1 \otimes \phi^0"] \arrow[r, "r'"'] & \phi A
\end{tikzcd}$$

MF3. The following diagram commutes:

$$\begin{tikzcd}
\phi((A \otimes B) \otimes C) \arrow[r, "\phi a"] & \phi(A \otimes (B \otimes C)) \\
\phi(A \otimes B) \otimes \phi C \arrow[u, "\tilde{\phi}"] & \phi A \otimes \phi(B \otimes C) \arrow[u, "\tilde{\phi}"'] \\
(\phi A \otimes \phi B) \otimes \phi C \arrow[u, "\tilde{\phi} \otimes 1"] \arrow[r, "a'"'] & \phi A \otimes (\phi B \otimes \phi C) \arrow[u, "1 \otimes \tilde{\phi}"']
\end{tikzcd}$$

Let $\Phi = (\phi, \tilde{\phi}, \phi^0)$ and $\Psi = (\psi, \tilde{\psi}, \psi^0)$ be monoidal functors $\mathscr{V} \to \mathscr{V}'$. A *monoidal natural transformation*

$$\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$$

consists of a natural transformation

$$\eta : \phi \to \psi : \mathscr{V}_0 \to \mathscr{V}'_0$$

satisfying the following two axioms:

MN1. The following diagram commutes:

$$\begin{tikzcd} I' \arrow[r, "\phi^0"] \arrow[dr, "\psi^0"'] & \phi I \arrow[d, "\eta_I"] \\ & \psi I \end{tikzcd}$$

MN2. The following diagram commutes:

$$\begin{tikzcd} \phi A \otimes \phi B \arrow[r, "\tilde{\phi}"] \arrow[d, "\eta \otimes \eta"'] & \phi(A \otimes B) \arrow[d, "\eta"] \\ \psi A \otimes \psi B \arrow[r, "\tilde{\psi}"'] & \psi(A \otimes B) \end{tikzcd}$$

We define the composite of monoidal functors

$$\Phi = (\phi, \tilde{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}' \quad \text{and} \quad \Psi = (\psi, \tilde{\psi}, \psi^0) : \mathscr{V}' \to \mathscr{V}''$$

to be $X = (\chi, \tilde{\chi}, \chi^0) : \mathscr{V} \to \mathscr{V}''$ where

(i) χ is the composite

$$\mathscr{V}_0 \xrightarrow{\varphi} \mathscr{V}'_0 \xrightarrow{\psi} \mathscr{V}''_0; \tag{1.1}$$

(ii) $\tilde{\chi}$ is the composite

$$\psi\phi A \otimes \psi\phi B \xrightarrow{\tilde{\psi}} \psi(\phi A \otimes \phi B) \xrightarrow{\psi\tilde{\varphi}} \psi\phi(A \otimes B); \tag{1.2}$$

(iii) χ^0 is the composite

$$I'' \xrightarrow{\psi^0} \psi I' \xrightarrow{\psi\varphi^0} \psi\phi I. \tag{1.3}$$

We define the composite of monoidal natural transformations $\eta : \Phi \to \Phi' : \mathscr{V} \to \mathscr{W}$ and $\zeta : \Phi' \to \Phi'' : \mathscr{V} \to \mathscr{W}$ to be the composite $\zeta\eta$ of $\eta : \phi \to \phi'$ and $\zeta : \phi' \to \phi''$; and for $\Psi : \mathscr{V}' \to \mathscr{V}$, $\eta : \Phi \to \Phi' : \mathscr{V} \to \mathscr{W}$ and $X : \mathscr{W} \to \mathscr{W}'$ we define $\eta\Psi$ and $X\eta$ to be $\eta\psi$ and $\chi\eta$. We now leave the reader to prove, along the lines of Theorem I.3.1,

Theorem 1.3. *Monoidal categories, monoidal functors, and monoidal natural transformations form with the above rules of composition a "hypercategory" $\mathscr{M}on$. Moreover a monoidal functor $\Phi = (\phi, \tilde{\phi}, \phi^0)$ is an isomorphism if and only if each of ϕ, $\tilde{\phi}$, ϕ^0 is an isomorphism, and a monoidal natural transformation $\eta : \Phi \to \Psi$ is an isomorphism if and only if $\eta : \phi \to \psi$ is.*

2. Monoidal Closed Categories

A *monoidal closed category* (or equally *closed monoidal category*) $\mathscr{V} = ({}^m\mathscr{V}, p, {}^c\mathscr{V})$ consists of the following three data:

(i) a monoidal category ${}^m\mathscr{V} = (\mathscr{V}_0, \otimes, I, r, l, a)$;

(ii) a closed category ${}^c\mathscr{V} = (\mathscr{V}_0, V, \hom\mathscr{V}, I, i, j, L)$ with the same $\mathscr{V}_0$ and I as ${}^m\mathscr{V}$;

(iii) a natural isomorphism $p = p_{ABC} : (A \otimes B, C) \to (A(BC))$.

These data are to satisfy the following four axioms, whose bizarre numeration is for later convenience:

MCC2. The following diagram commutes:

$$\begin{tikzcd}
(I \otimes A, B) \arrow[rr, "p"] & & (I(AB)) \\
& (AB) \arrow[ul, "{(l,1)}"] \arrow[ur, "i"'] &
\end{tikzcd}$$

MCC3. The following diagram commutes:

$$\begin{tikzcd}
((A \otimes B) \otimes C, D) \arrow[r, "p"] & (A \otimes B, (CD)) \arrow[r, "p"] & (A, (B(CD))) \\
(A \otimes (B \otimes C), D) \arrow[u, "{(a,1)}"] \arrow[rr, "p"'] & & (A, (B \otimes C, D)) \arrow[u, "{(1,p)}"']
\end{tikzcd}$$

MCC3′. The following diagram commutes:

$$\begin{tikzcd}
(CD) \arrow[r, "L^{A \otimes B}"] \arrow[d, "L^B"'] & ((A \otimes B, C), (A \otimes B, D)) \arrow[dd, "{(1,p)}"] \\
((BC)(BD)) \arrow[d, "L^A"'] & \\
((A(BC)), (A(BD))) \arrow[r, "{(p,1)}"'] & ((A \otimes B, C), (A(BD)))
\end{tikzcd}$$

MCC4. The following diagram commutes:

$$\begin{tikzcd}
(A \otimes I, B) \arrow[rr, "p"] & & (A(IB)) \\
& (AB) \arrow[ul, "{(r, 1)}"] \arrow[ur, "{(1, i)}"'] &
\end{tikzcd}$$

Remark 2.1. We shall normally denote the monoidal category ${}^m\mathscr{V}$ and the closed category ${}^c\mathscr{V}$ by the same symbol $\mathscr{V}$ as the monoidal closed category, except where we wish to distinguish between the three structures.

If $\mathscr{V}$ and $\mathscr{V}'$ are monoidal closed categories, a *monoidal closed functor* $\Phi : \mathscr{V} \to \mathscr{V}'$ is to be a quadruple $(\phi, \tilde{\phi}, \hat{\phi}, \phi^0)$ where ${}^m\Phi = (\phi, \tilde{\phi}, \phi^0)$ is a monoidal functor ${}^m\mathscr{V} \to {}^m\mathscr{V}'$ and ${}^c\Phi = (\phi, \hat{\phi}, \phi^0)$ is a closed functor ${}^c\mathscr{V} \to {}^c\mathscr{V}'$, and where the following axiom is satisfied:

MCF3. The following diagram commutes:

$$\begin{tikzcd}
\phi(A \otimes B, C) \arrow[r, "\phi p"] \arrow[d, "\hat{\phi}"'] & \phi(A(BC)) \arrow[d, "\hat{\phi}"] \\
(\phi(A \otimes B), \phi C) \arrow[d, "{(\tilde{\phi}, 1)}"'] & (\phi A, \phi(BC)) \arrow[d, "{(1, \hat{\phi})}"] \\
(\phi A \otimes \phi B, \phi C) \arrow[r, "p'"'] & (\phi A, (\phi B, \phi C))
\end{tikzcd}$$

If $\Phi, \Psi : \mathscr{V} \to \mathscr{V}'$ are monoidal closed functors, a *monoidal closed natural transformation* $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ is to be a natural transformation $\eta : \phi \to \psi : \mathscr{V}_0 \to \mathscr{V}'_0$ which is both a monoidal natural transformation $\eta : {}^m\Phi \to {}^m\Psi : {}^m\mathscr{V} \to {}^m\mathscr{V}'$ and a closed natural transformation $\eta : {}^c\Phi \to {}^c\Psi : {}^c\mathscr{V} \to {}^c\mathscr{V}'$.

We define composition of monoidal closed functors by (1.1), (1.2), (1.3), and I(3.2); note that I(3.1) and I(3.3) reproduce (1.1) and (1.3). We define composition of monoidal closed natural transformations, with themselves or with monoidal closed functors, to be their composition as monoidal natural transformations, which is the same as their composition as closed natural transformations. To prove the following theorem requires only the verification that axiom MCF3 survives composition, which we leave to the reader:

Theorem 2.2. *Monoidal closed categories, monoidal closed functors, and monoidal closed natural transformations form a "hypercategory" $\mathscr{MCl}$.*

3. Relations Between the Data

Both the data and the axioms for a monoidal closed category are highly redundant (quite apart from the redundancy noted in Proposition 1.1). To examine the interconnexions we place ourselves in the following *basic situation*: we suppose given a category $\mathscr{V}_0$, functors

$$\otimes : \mathscr{V}_0 \times \mathscr{V}_0 \to \mathscr{V}_0 \quad \text{and} \quad \hom \mathscr{V} : \mathscr{V}_0^* \times \mathscr{V}_0 \to \mathscr{V}_0 ,$$

a natural isomorphism

$$\pi = \pi_{ABC} : \mathscr{V}_0(A \otimes B, C) \to \mathscr{V}_0(A(BC)),$$

and a functor $V : \mathscr{V}_0 \to \mathscr{S}$ satisfying CC0.

The naturality of π gives a commutative diagram

$$\begin{tikzcd}
\mathscr{V}_0(A \otimes B, C) \arrow[r, "\pi"] \arrow[d, "\mathscr{V}_0(f \otimes g, h)"'] & \mathscr{V}_0(A, (BC)) \arrow[d, "\mathscr{V}_0(f, (g, h))"] \\
\mathscr{V}_0(A' \otimes B', C') \arrow[r, "\pi"'] & \mathscr{V}_0(A', (B'C'))
\end{tikzcd}$$

where $f : A' \to A, g : B' \to B, h : C \to C'$. Evaluating this at $x \in \mathscr{V}_0(A \otimes \otimes B, C)$ gives a commutative diagram:

$$\begin{tikzcd}
A' \arrow[r, "\pi(hx(f \otimes g))"] \arrow[d, "f"'] & (B'C') \\
A \arrow[r, "\pi x"'] & (BC) \arrow[u, "(g, h)"']
\end{tikzcd} \tag{3.1}$$

Define natural transformations (natural by Proposition I.1.2)

$$t = t_{BC} : (BC) \otimes B \to C ,$$

$$u = u_{AB} : A \to (B, A \otimes B) ,$$

by

$$t = \pi^{-1} 1_{(BC)} , \tag{3.2}$$

$$u = \pi 1_{A \otimes B} . \tag{3.3}$$

Then for $x : A \otimes B \to C$ and $y : A \to (BC)$ we have commutative diagrams:

$$\begin{tikzcd}
A \arrow[rr, "\pi x"] \arrow[dr, "u"'] & & (BC) \\
& (B, A \otimes B) \arrow[ur, "(1, x)"'] &
\end{tikzcd} \tag{3.4}$$

$$\begin{tikzcd} A \otimes B \arrow[rr, "\pi^{-1}y"] \arrow[dr, "y \otimes 1"'] & & C \\ & (BC) \otimes B \arrow[ur, "t"'] & \end{tikzcd} \tag{3.5}$$

These may be regarded as special cases of (3.1), or as coming from the representation theorem (cf. I(1.4)) applied to π and to π^{-1}. Taking in particular $x = t$ and $y = u$ in (3.4) and (3.5) we get commutative diagrams

$$\begin{tikzcd} (BC) \arrow[rr, "1"] \arrow[dr, "u"'] & & (BC) \\ & (B, (BC) \otimes B) \arrow[ur, "{(1, t)}"'] & \end{tikzcd} \tag{3.6}$$

$$\begin{tikzcd} A \otimes B \arrow[rr, "1"] \arrow[dr, "u \otimes 1"'] & & A \otimes B \\ & (B, A \otimes B) \otimes B \arrow[ur, "t"'] & \end{tikzcd} \tag{3.7}$$

The following lemma allows us to use in a systematic way the fact that π is a natural isomorphism.

Lemma 3.1. *With the basic situation as above, let $\mathscr{W}$ be a category, $P, Q : \mathscr{V}_0 \to \mathscr{W}$ functors, and $B \in \mathscr{V}_0$. Then there is a bijection between natural transformations $\alpha = \alpha_A : PA \to Q(A \otimes B)$ and natural transformations $\beta = \beta_A : P(BA) \to QA$, given by*

(i) *β_A is the composite*

$$P(BA) \xrightarrow[\alpha_{(BA)}]{} Q((BA) \otimes B) \xrightarrow[Qt]{} QA; \tag{3.8}$$

(ii) *α_A is the composite*

$$PA \xrightarrow[Pu]{} P(B, A \otimes B) \xrightarrow[\beta_{A \otimes B}]{} Q(A \otimes B). \tag{3.9}$$

Proof. Consider the diagram

$$\begin{tikzcd} \mathscr{V}_0(A \otimes B, C) \arrow[rr, "\pi"] \arrow[d, "Q"'] & & \mathscr{V}_0(A(BC)) \arrow[d, "P"] \\ \mathscr{W}(Q(A \otimes B), QC) \arrow[dr, "{\mathscr{W}(\alpha, 1)}"'] & & \mathscr{W}(PA, P(BC)) \arrow[dl, "{\mathscr{W}(1, \beta)}"] \\ & \mathscr{W}(PA, QC) & \end{tikzcd} \tag{3.10}$$

In the language of Theorem I.1.1, the left edge of (3.10) is $\Omega\alpha$ and the right edge is $\Omega\beta$ (different Ω's, the second with reversed variance). It follows, since π is an isomorphism, that the commutativity of (3.10) sets up a bijection between α's and β's. Putting $C = A \otimes B$ and evaluating at 1 gives (3.9); putting $A = (BC)$ and evaluating at t gives (3.8).

Note that by evaluating (3.10) at $x \in \mathscr{V}_0(A \otimes B, C)$ we get

$$Qx.\alpha = \beta.P\pi x : PA \to QC. \tag{3.11}$$

Again, since π is a natural isomorphism, the representation theorem shows that the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{V}_0(I \otimes A, B) \arrow[rr, "\pi"] & & \mathscr{V}_0(I(AB)) \\
& \mathscr{V}_0(AB) \arrow[ul, "\mathscr{V}_0(l,1)"] \arrow[ur, "v"'] &
\end{tikzcd} \tag{3.12}$$

sets up a bijection between natural isomorphisms $l: I \otimes A \to A$ and natural isomorphisms

$$v = v_{AB} : \mathscr{V}_0(AB) \to \mathscr{V}_0(I(AB)).$$

Further the representation theorem applied to v shows that there is a bijection between natural transformations v and natural transformations $j = j_A : I \to (AA)$, given by

$$j_A = v_{AA} 1_A. \tag{3.13}$$

Then if $f \in \mathscr{V}_0(AB)$ we have by I(1.4) the commutative diagram (cf. I(2.8))

$$\begin{tikzcd}
I \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, "vf"] & (AA) \arrow[d, "{(1,f)}"] \\
(BB) \arrow[r, "{(f,1)}"'] & (\mathrm{AB})
\end{tikzcd} \tag{3.14}$$

Evaluating (3.12) at 1_A after putting $B = A$ now gives:

$$\pi l = j. \tag{3.15}$$

In the same way commutativity of the diagram

$$\begin{tikzcd}
\mathscr{V}_0(A \otimes I, B) \arrow[rr, "\pi"] & & \mathscr{V}_0(A(IB)) \\
& \mathscr{V}_0(AB) \arrow[ul, "\mathscr{V}_0(r,1)"] \arrow[ur, "\mathscr{V}_0(1,i)"'] &
\end{tikzcd} \tag{3.16}$$

sets up a bijection between natural isomorphisms $r: A \otimes I \to A$ and natural isomorphisms $i: A \to (IA)$. Putting $B = A$ and evaluating at 1 gives:

$$\pi r = i. \tag{3.17}$$

Again, commutativity of the diagram

$$\begin{tikzcd}
\mathscr{V}_0((A \otimes B) \otimes C, D) \arrow[r, "\pi"] & \mathscr{V}_0(A \otimes B, (CD)) \arrow[r, "\pi"] & \mathscr{V}_0(A, (B(CD))) \\
\mathscr{V}_0(A \otimes (B \otimes C), D) \arrow[u, "\mathscr{V}_0(a, 1)"] \arrow[rr, "\pi"'] & & \mathscr{V}_0(A, (B \otimes C, D)) \arrow[u, "\mathscr{V}_0(1, p)"']
\end{tikzcd} \tag{3.18}$$

sets up a bijection between natural isomorphisms

$$a: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

and natural isomorphisms $p: (B \otimes C, D) \to (B(CD))$. Evaluating (3.18) at $x \in \mathscr{V}_0(A \otimes (B \otimes C), D)$ gives a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "\pi\pi(xa)"] \arrow[dr, "\pi x"'] & & (B(CD) \\
& (B \otimes C, D) \arrow[ur, "p"'] &
\end{tikzcd} \tag{3.19}$$

Finally, an application of Lemma 3.1 with $\mathscr{W} = \mathscr{V}_0^*$ (watch the variances!), $P = (-, (BC))$, and $Q = (- \; C)$, gives a bijection between natural transformations $p: (A \otimes B, C) \to (A(BC))$ and natural transformations $L: (AC) \to ((BA)(BC))$, determined by either of the commutative diagrams

$$\begin{tikzcd}
(AC) \arrow[rr, "L"] \arrow[dr, "(t, 1)"'] & & ((BA)(BC)) \\
& ((BA) \otimes B, C) \arrow[ur, "p"'] &
\end{tikzcd} \tag{3.20}$$

$$\begin{tikzcd}
(A \otimes B, C) \arrow[rr, "p"] \arrow[dr, "L"'] & & (A(BC)) \\
& ((B, A \otimes B), (BC)) \arrow[ur, "(u, 1)"'] &
\end{tikzcd} \tag{3.21}$$

If we write (3.11) for this special case, we get for $x : A \otimes B \to C$ a commutative diagram

$$\begin{tikzcd}
(CD) \arrow[r, "{(x, 1)}"] \arrow[d, "L"'] & (A \otimes B, D) \arrow[d, "p"] \\
((BC)(BD)) \arrow[r, "{(\pi x, 1)}"'] & (A(BD))
\end{tikzcd} \tag{3.22}$$

Now suppose that we have besides the basic situation $\mathscr{V}_0$ etc., a second one $\mathscr{V}'_0$ etc.; and that we have a functor $\phi : \mathscr{V}_0 \to \mathscr{V}'_0$. Then since π and π' are both natural isomorphisms, a trivial generalization of the argument of Lemma 3.1 shows that the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{V}_0(A \otimes B, C) \arrow[r, "\pi"] \arrow[d, "\phi"'] & \mathscr{V}_0(A(BC)) \arrow[d, "\phi"] \\
\mathscr{V}'_0(\phi(A \otimes B), \phi C) \arrow[d, "{\mathscr{V}'_0(\tilde{\phi}, 1)}"'] & \mathscr{V}'_0(\phi A, \phi(BC)) \arrow[d, "{\mathscr{V}'_0(1, \hat{\phi})}"] \\
\mathscr{V}'_0(\phi A \otimes \phi B, \phi C) \arrow[r, "\pi'"'] & \mathscr{V}'_0(\phi A, (\phi B, \phi C))
\end{tikzcd} \tag{3.23}$$

sets up a bijection between natural transformations

$$\tilde{\phi} : \phi A \otimes \phi B \to \phi(A \otimes B)$$

and natural transformations

$$\hat{\phi} : \phi(BC) \to (\phi B, \phi C).$$

If we evaluate (3.23) at $x \in \mathscr{V}_0(A \otimes B, C)$, we get a commutative diagram

$$\begin{tikzcd}
\phi A \arrow[r, "\phi \pi x"] \arrow[dr, "{\pi'(\phi x . \tilde{\phi})}"'] & \phi(BC) \arrow[d, "\hat{\phi}"] \\
& (\phi B, \phi C)
\end{tikzcd} \tag{3.24}$$

Proposition 3.2. *If $\mathscr{V}$ is a monoidal closed category, define π and v by*

$$\pi_{ABC} = V p_{ABC}, \tag{3.25}$$

$$v_{AB} = V i_{(AB)} \, (= \iota_{(AB)}); \tag{3.26}$$

then the basic situation obtains and we have (3.12), (3.13), (3.16), (3.18), *and* (3.20). *Moreover if* $\Phi : \mathscr{V} \to \mathscr{V}'$ *is a monoidal closed functor we have* (3.23).

Proof. Applying V to MCC2 and using (3.26) gives (3.12); while (3.13), in view of (3.26), is CC5. Similarly applying V to MCC4 and to MCC3 gives (3.16) and (3.18).

Axiom MCC3′ may be interpreted as an instance of VN, stating: p_{ABC} is the C-component of a $\mathscr{V}$-natural transformation

$$p_{AB} : L^{A \otimes B} \to L^A L^B. \tag{3.27}$$

It follows that (3.20) is a diagram of $\mathscr{V}$-natural transformations

$$\begin{tikzcd}
L^A \arrow[rr, "L^B_A"] \arrow[dr, "L^t"'] & & L^{(BA)} L^B \\
& L^{(BA)\otimes B} \arrow[ur, "p_{(BA),B}"'] &
\end{tikzcd}$$

so that its commutativity will follow if, after putting $C = A$ and applying V, both legs have the same value at 1. But $(VL)\ 1 = 1$ by I (2.5), and $Vp.\ V(t, 1).\ 1 = Vp.t = \pi t = 1$.

If Φ is a monoidal closed functor, diagram (3.23) is the exterior of:

$$\begin{tikzcd}
V(A \otimes B, C) \arrow[rrr, "Vp"] \arrow[dr, "\phi_0"] \arrow[dd, "\phi"'] & & & V(A(BC)) \arrow[dl, "\phi_0"'] \arrow[dd, "\phi"] \\
& V'\phi(A \otimes B, C) \arrow[r, "V'\phi p"] \arrow[dl, "V'\hat{\phi}"] & V'\phi(A(BC)) \arrow[dr, "V'\hat{\phi}"'] & \\
V'(\phi(A \otimes B), \phi C) \arrow[d, "V'(\tilde{\phi}, 1)"'] & & & V'(\phi A, \phi(BC)) \arrow[d, "V'(1, \hat{\phi})"] \\
V'(\phi A \otimes \phi B, \phi C) \arrow[rrr, "V'p'"'] & & & V'(\phi A, (\phi B, \phi C))
\end{tikzcd}$$

Here one region commutes by V' of MCF3, two regions by I(3.11), and one by the naturality of ϕ_0.

4. Relations Between the Axioms

Proposition 4.1. *Suppose that in the basic situation of* § 3 *we have natural isomorphisms* a, l, r, p, v, i *and natural transformations* $j, L,$ *con-*

nected by (3.12), (3.13), (3.16), (3.18), *and* (3.20). *Then the following implications hold between the axioms* MC, MCC, *and* CC:

(i) *in the presence of* CC5, *we have* MC1 $\Leftrightarrow$ $Vp = \pi$ $\Leftrightarrow$ CC1;

(ii) MC2 $\Leftrightarrow$ MCC2 $\Leftrightarrow$ CC2;

(iii) MC3 $\Leftrightarrow$ MCC3 $\Leftrightarrow$ MCC3′ $\Leftrightarrow$ CC3;

(iv) MC4 $\Leftrightarrow$ MCC4 $\Leftrightarrow$ CC4;

(v) CC5 $\Rightarrow$ MC5 *(one way only!)*.

Proof of (i). Note first that Lemma I.2.2, Proposition I.2.3, and Proposition I.2.4 use only CC0 and CC5, and so are available here.

Since π is an isomorphism we get from MC1 an equivalent diagram by applying π twice to each leg. Now

$$\begin{aligned} \pi\pi(la) &= p.\pi l && \text{by } (3.19)\\ &= pj && \text{by } (3.15); \end{aligned}$$

and

$$\begin{aligned} \pi\pi(l \otimes 1) &= \pi\pi(1(l \otimes 1))\\ &= \pi(\pi 1.\, l) && \text{by } (3.1)\\ &= \pi(ul) && \text{by } (3.3)\\ &= (1, u)\pi l && \text{by } (3.1)\\ &= (1, u)j && \text{by } (3.15). \end{aligned}$$

Thus our equivalent diagram to MC1 is

$$\begin{tikzcd} I \arrow[r, "j"] \arrow[d, "j"'] & (A \otimes B, A \otimes B) \arrow[d, "p"] \\ (AA) \arrow[r, "{(1, u)}"'] & (A(B, A \otimes B)) \end{tikzcd}$$

and by Lemma I.2.2 this may be expressed as $(Vp)1 = V(1, u)1$; that is, $(Vp)1 = u$. Since we also have $\pi 1 = u$, and since Vp and π are both natural, the statement $(Vp)1 = u$ is further equivalent to $Vp = \pi$, by the representation theorem.

If $Vp = \pi$, we get by applying V to (3.20), putting $C = A$, and evaluating at 1,

$$\begin{aligned} (VL)1 &= \pi\,.\,V(t, 1)\,.\,1\\ &= \pi t\\ &= 1\,. \end{aligned}$$

Conversely if $(VL)1 = 1$, (3.21) gives

$$\begin{aligned}(Vp)1 &= V(u, 1) \,.\, VL \,.\, 1 \\ &= V(u, 1) \,.\, 1 \\ &= u,\end{aligned}$$

and hence $Vp = \pi$.

Thus $Vp = \pi$ is equivalent to $(VL)1 = 1$, and this is, in the presence of CC0 and CC5, equivalent to CC1 by Proposition I.2.3.

Proof of (ii). Applying π twice to each leg of MC2, we get

$$\begin{aligned}\pi\pi((1 \otimes l)a) &= p.\pi(1 \otimes l) && \text{by (3.19)} \\ &= p(l, 1).\pi 1 && \text{by (3.1)} \\ &= p(l, 1)u,\end{aligned}$$

and

$$\begin{aligned}\pi\pi(r \otimes 1) &= \pi(\pi 1.r) && \text{by (3.1)} \\ &= \pi(ur) \\ &= (1, u).\pi r && \text{by (3.1)} \\ &= (1, u)i && \text{by (3.17)} \\ &= iu && \text{by the naturality of } i.\end{aligned}$$

Thus the equivalent diagram to MC2 is

$$\begin{tikzcd}
(I \otimes B, A \otimes B) \arrow[rrrr, "p"] & & & & (I(B, A \otimes B)) \\
& (B, A \otimes B) \arrow[ul, "(l, 1)"] & & (B, A \otimes B) \arrow[ur, "i"'] & \\
& & A \arrow[ul, "u"] \arrow[ur, "u"'] & &
\end{tikzcd}$$

and this is equivalent to MCC2 by an application of Lemma 3.1.

Now use (3.22) to replace $p(l, 1)$ in MCC2 by $(\pi l, 1)L$, which by (3.15) is $(j, 1)L$; the result is precisely CC2.

Proof of (iii). First note that if we put $D = (A \otimes B) \otimes C$ and $x = a^{-1}$ in (3.19) we get

$$\begin{aligned}p.\pi a^{-1} &= \pi\pi 1 \\ &= \pi u \\ &= (1, u)u \quad \text{by (3.4)};\end{aligned}$$

thus we have

$$\pi a^{-1} = p^{-1}(1, u)\, u\,. \tag{4.1}$$

Now write MC3 in the form

$a^{-1}(1 \otimes a)\, a = a(a^{-1} \otimes 1) : (A \otimes (B \otimes C)) \otimes D \to (A \otimes B) \otimes (C \otimes D)$
and apply π twice to each term. We have

$$\begin{aligned}
\pi\pi(a^{-1}(1 \otimes a)\, a) &= p.\pi(a^{-1}(1 \otimes a)) && \text{by (3.19)}\\
&= p(a, 1).\pi a^{-1} && \text{by (3.1)}\\
&= p(a, 1)\, p^{-1}(1, u)\, u && \text{by (4.1);}
\end{aligned}$$

and

$$\begin{aligned}
\pi\pi(a(a^{-1} \otimes 1)) &= \pi(\pi a. a^{-1}) && \text{by (3.1)}\\
&= (1, \pi a).\pi a^{-1} && \text{by (3.1)}\\
&= (1, \pi a)\, p^{-1}(1, u)\, u && \text{by (4.1)}\\
&= p^{-1}(1, (1, \pi a))\,(1, u)\, u && \\
& && \text{by the naturality of } p^{-1}\\
&= p^{-1}(1, (1, \pi a)\, u)\, u && \\
&= p^{-1}(1, \pi\pi a)\, u && \text{by (3.4)}\\
&= p^{-1}(1, p.\pi 1)\, u && \text{by (3.19)}\\
&= p^{-1}(1, p)\,(1, u)\, u\,. &&
\end{aligned}$$

The equivalent diagram to which we have now reduced MC3 is:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "u"'] & (B, A \otimes B) \arrow[r, "{(1, u)}"] & (B(C \otimes D, (A \otimes B) \otimes (C \otimes D))) \arrow[d, "p^{-1}"] \\
(B, A \otimes B) \arrow[d, "{(1, u)}"'] & & (B \otimes (C \otimes D), (A \otimes B) \otimes (C \otimes D)) \arrow[d, "{(a, 1)}"] \\
(B(C \otimes D, (A \otimes B) \otimes (C \otimes D))) \arrow[d, "{(1, p)}"'] & & ((B \otimes C) \otimes D, (A \otimes B) \otimes (C \otimes D)) \arrow[d, "p"] \\
(B(C(D, (A \otimes B) \otimes (C \otimes D)))) \arrow[rr, "p^{-1}"'] & & (B \otimes C, (D, (A \otimes B) \otimes (C \otimes D)))
\end{tikzcd}$$

and this is equivalent to MCC3 by two applications of Lemma 3.1.

To show that MCC3 $\Leftrightarrow$ MCC3′, we transform both legs of MCC3;

we have

$$\begin{aligned} pp(a,1) &= p(\pi a,1)L && \text{by (3.22)} \\ &= (\pi\pi a,1)LL && \text{by (3.22)} \\ &= (p.\pi 1,1)LL && \text{by (3.19)} \\ &= (u,1)(p,1)LL; \end{aligned}$$

and

$$\begin{aligned} (1,p)p &= (1,p)(u,1)L && \text{by (3.21)} \\ &= (u,1)(1,p)L. \end{aligned}$$

The diagram we now have is equivalent to MCC3′ by an application of Lemma 3.1.

To show that MCC3′ ⇔ CC3, consider the proof of Proposition I.8.1; it makes no use of the properties of closed categories, beyond the fact that hom $\mathscr{V}$ is a bifunctor, and shows purely formally that if two families of morphisms satisfy diagrams of the form VN, so does their composite. We can interpret MCC3′ as VN for p_{AB} and CC3 as VN for L^A_B; and diagrams (3.20) and (3.21) allow us to deduce each of these from the other, if we know that $(t, 1)$ and $(u, 1)$ satisfy VN. This is indeed the case, for the proof in Proposition I.8.4 that $(f, 1)$ satisfies VN uses only the naturality of L.

Proof of (iv). Applying π twice to each leg of MC4 we get

$$\begin{aligned} \pi\pi((1\otimes r)a) &= p.\pi(1\otimes r) && \text{by (3.19)} \\ &= p(r,1).\pi 1 && \text{by (3.1)} \\ &= p(r,1)u, \end{aligned}$$

and

$$\begin{aligned} \pi\pi r &= \pi i && \text{by (3.17)} \\ &= (1,i)u && \text{by (3.4)}; \end{aligned}$$

the resulting diagram is equivalent to MCC4 by an application of Lemma 3.1.

Now use (3.22) to replace $p(r, 1)$ in MCC4 by $(\pi r, 1)L$, which is $(i, 1)L$ by (3.17); the result is precisely CC4.

Proof of (v). The result $j_I = i_I$ of Proposition I.2.7 used only CC0 and CC5, and so is available here. By (3.15) and (3.17) this gives $l_I = r_I$, which is MC5.

Proposition 4.2. *Under the conditions of Proposition 4.1,* MC2 *is a consequence of* MC1, MC3, MC4, *and* MC5.

Proof. From the naturality of l we have a commutative diagram

$$\begin{tikzcd} I \otimes (I \otimes A) \arrow[r, "l"] \arrow[d, "1 \otimes l"'] & I \otimes A \arrow[d, "l"] \\ I \otimes A \arrow[r, "l"'] & A \end{tikzcd}$$

and since l is an isomorphism we have

$$l = 1 \otimes l : I \otimes (I \otimes A) \to I \otimes A . \tag{4.2}$$

Consider the special case of MC2 when A is put equal to I; (4.2) enables us to replace therein $1 \otimes l$ by l, and MC5 to replace $r \otimes 1$ by $l \otimes 1$; thus this special case becomes an instance of MC1, and is therefore available under the present hypotheses.

Now consider the diagram

$$\begin{tikzcd} (I \otimes A, B) \arrow[rr, "p"] \arrow[d, "(1 \otimes l, 1)"'] & & (I(AB)) \arrow[r, "i^{-1}"] \arrow[d, "(1, (l, 1))"] & (AB) \arrow[d, "(l, 1)"] \\ (I \otimes (I \otimes A), B) \arrow[rr, "p"] \arrow[d, "(a, 1)"'] & & (I, (I \otimes A, B)) \arrow[r, "i^{-1}"] \arrow[d, "(1, p)"] & (I \otimes A, B) \arrow[d, "p"] \\ ((I \otimes I) \otimes A, B) \arrow[r, "p"] \arrow[d, "(r^{-1} \otimes 1, 1)"'] & (I \otimes I, (AB)) \arrow[r, "p"] \arrow[dr, "(r^{-1}, 1)"'] & (I, (I(AB))) \arrow[r, "i^{-1}"] \arrow[d, "(1, i^{-1})"] & (I(AB)) \arrow[d, "i^{-1}"] \\ (I \otimes A, B) \arrow[rr, "p"'] & & (I(AB)) \arrow[r, "i^{-1}"'] & (AB) \end{tikzcd}$$

The diagram commutes by MCC3 and MCC4 (which are available here by Proposition 4.1) together with the naturality of p and of i^{-1}. The composite left edge is 1 by the above special case of MC2, so that the composite right edge is also 1. This is MCC2, which is equivalent to MC2 by Proposition 4.1.

Proposition 4.3. *Let $\mathscr{V}$ and $\mathscr{V}'$ be monoidal closed categories, $\phi : \mathscr{V}_0 \to \mathscr{V}'_0$ a functor, $\phi^0 : I' \to \phi I$ a morphism, and $\tilde{\phi}$ and $\hat{\phi}$ natural transformations connected by* (3.23). *Then the axioms* CF, MF, *and* MCF *are* *related by:*

(i) MF 1 $\Leftrightarrow$ CF 1;

(ii) MF 2 $\Leftrightarrow$ CF 2;

(iii) MF 3 $\Leftrightarrow$ MCF 3 $\Leftrightarrow$ CF 3.

Proof of (i). Apply π' to both legs of MF 1. We have

$$\begin{aligned}
\pi'(\phi l \,.\, \tilde{\phi}(\phi^0 \otimes 1)) &= \pi'(\phi l \,.\, \tilde{\phi}) .\, \phi^0 && \text{by} \quad (3.1)\\
&= \hat{\phi} \,.\, \phi \pi l \,.\, \phi^0 && \text{by} \quad (3.24)\\
&= \hat{\phi} \,.\, \phi j \,.\, \phi^0 && \text{by} \quad (3.15),
\end{aligned}$$

and

$$\pi' l' = j' \qquad \text{by} \quad (3.15);$$

the equivalent diagram to which we have reduced MF 1 is now precisely CF 1.

Proof of (ii). Applying π' to MF 2 we get

$$\begin{aligned}
\pi'(\phi r \,.\, \tilde{\phi}(1 \otimes \phi^0)) &= (\phi^0, 1)\, \pi'(\phi r \,.\, \tilde{\phi}) && \text{by} \quad (3.1)\\
&= (\phi^0, 1)\, \hat{\phi} \,.\, \phi \pi r && \text{by} \quad (3.24)\\
&= (\phi^0, 1)\, \hat{\phi} \,.\, \phi i && \text{by} \quad (3.17),
\end{aligned}$$

and

$$\pi' r' = i' \qquad \text{by} \quad (3.17);$$

the resulting diagram is precisely CF 2.

Proof of (iii). Apply π' twice to each leg of MF 3. We get

$$\begin{aligned}
\pi' \pi'(\tilde{\phi}(1 \otimes \tilde{\phi})\, a') &= p' \,.\, \pi'(\tilde{\phi}(1 \otimes \tilde{\phi})) && \text{by} \quad (3.19)\\
&= p'(\tilde{\phi}, 1) \,.\, \pi' \tilde{\phi} && \text{by} \quad (3.1)\\
&= p'(\tilde{\phi}, 1)\, \hat{\phi} \,.\, \phi u && \text{by} \quad (3.24) \ \text{with}\, x = 1;
\end{aligned}$$

and

$$\begin{aligned}
\pi' \pi'(\phi a \,.\, \tilde{\phi}(\tilde{\phi} \otimes 1)) &= \pi'(\pi'(\phi a \,.\, \tilde{\phi}) \,.\, \tilde{\phi}) && \text{by} \quad (3.1)\\
&= \pi'(\hat{\phi} \,.\, \phi \pi a \,.\, \tilde{\phi}) && \text{by} \quad (3.24)\\
&= (1, \hat{\phi}) \,.\, \pi'(\phi \pi a \,.\, \tilde{\phi}) && \text{by} \quad (3.1)\\
&= (1, \hat{\phi})\, \hat{\phi} \,.\, \phi \pi \pi a && \text{by} \quad (3.24)\\
&= (1, \hat{\phi})\, \hat{\phi} \,.\, \phi(p u) && \text{by} \quad (3.19) \ \text{with} \ x = 1\\
&= (1, \hat{\phi})\, \hat{\phi} \,.\, \phi p \,.\, \phi u;
\end{aligned}$$

giving a diagram equivalent to MCF 3 by an application of Lemma 3.1.

To show MCF3 $\Leftrightarrow$ CF3, we use (3.22) to transform the legs of the former. We have

$$\begin{aligned} p'(\tilde{\phi}, 1)\,\hat{\phi} &= (\pi'\tilde{\phi}, 1)\,L\,\hat{\phi} && \text{by (3.22)} \\ &= (\hat{\phi}\,.\,\phi u, 1)\,L\,\hat{\phi} && \text{by (3.24) with } x = 1 \\ &= (\phi u, 1)\,(\hat{\phi}, 1)\,L\,\hat{\phi}; \end{aligned}$$

and

$$\begin{aligned} (1, \hat{\phi})\,\hat{\phi}\,.\,\phi p &= (1, \hat{\phi})\,\hat{\phi}\,.\,\phi(u, 1)\,.\,\phi L && \text{by (3.21)} \\ &= (1, \hat{\phi})\,(\phi u, 1)\,\hat{\phi}\,.\,\phi L && \text{by the naturality of } \hat{\phi} \\ &= (\phi u, 1)\,(1, \hat{\phi})\,\hat{\phi}\,.\,\phi L; \end{aligned}$$

giving a diagram equivalent to CC3 by an application of Lemma 3.1.

Proposition 4.4. *Let $\Phi, \Psi : \mathscr{V} \to \mathscr{V}'$ be monoidal closed functors and $\eta : \phi \to \psi$ a natural transformation. Then* CN1 *is identical with* MN1, *and* CN2 *is equivalent to* MN2.

Proof. We apply π' to both legs of MN2, getting

$$\begin{aligned} \pi'(\eta\,\tilde{\phi}) &= (1, \eta)\,.\,\pi'\,\tilde{\phi} && \text{by (3.1)} \\ &= (1, \eta)\,\hat{\phi}\,.\,\phi u && \text{by (3.24) with } x = 1; \end{aligned}$$

and

$$\begin{aligned} \pi'(\tilde{\psi}(\eta \otimes \eta)) &= (\eta, 1)\,.\,\pi'\,\tilde{\psi}\,.\,\eta && \text{by (3.1)} \\ &= (\eta, 1)\,\hat{\psi}\,.\,\psi u\,.\,\eta && \text{by (3.24) for } \Psi \text{ with } x = 1 \\ &= (\eta, 1)\,\hat{\psi}\,\eta\,.\,\phi u && \text{by the naturality of } \eta; \end{aligned}$$

giving a diagram equivalent to CN2 by an application of Lemma 3.1.

5. The Forgetful Hyperfunctors

We say that a hyperfunctor $\Phi : \mathfrak{A} \to \mathfrak{B}$ is *locally isomorphic* if for each pair of objects $\mathscr{A}, \mathscr{B}$ in $\mathfrak{A}$ the functor $\mathfrak{A}(\mathscr{A}\mathscr{B}) \to \mathfrak{B}(\Phi\mathscr{A}, \Phi\mathscr{B})$ determined by Φ is an isomorphism of categories.

We have forgetful hyperfunctors $\mathscr{MCl} \to \mathscr{Mon}$ and $\mathscr{MCl} \to \mathscr{Cl}$ given by $\mathscr{V} \mapsto {}^m\mathscr{V}$, $\Phi \mapsto {}^m\Phi$, $\eta \mapsto \eta$ and $\mathscr{V} \mapsto {}^c\mathscr{V}$, $\Phi \mapsto {}^c\Phi$, $\eta \mapsto \eta$. From Propositions 3.2, 4.3 and 4.4 we have at once:

Theorem 5.1. *The forgetful hyperfunctors $\mathscr{MCl} \to \mathscr{Mon}$ and $\mathscr{MCl} \to \mathscr{Cl}$ are locally isomorphic.*

It remains to examine which monoidal categories and which closed categories admit enrichment to a monoidal closed category. From Theorem 5.1 we have:

Corollary 5.2. *If a monoidal category* ${}^m\mathscr{V}$ *(resp. a closed category* ${}^c\mathscr{V}$*) admits enrichment to a monoidal closed category* $\mathscr{V}$, *then* $\mathscr{V}$ *is unique to within an isomorphism of the form* $(1, 1, \hat{\phi}, 1)$ *(resp.* $(1, \tilde{\phi}, 1, 1)$*).*

We shall say that a monoidal category (resp. a closed category) is *closed* (resp. *monoidal*) if it admits enrichment to a monoidal closed category.

Theorem 5.3. *A closed category* $\mathscr{V}$ *is monoidal if and only if the* $\mathscr{V}$-*functor* $L^A L^B : \mathscr{V} \to \mathscr{V}$ *is representable for each* $A, B \in \mathscr{V}$. *If representations*

$$p_{AB} : L^{A \otimes B} \to L^A L^B$$

with components

$$p_{ABC} : (A \otimes B, C) \to (A\,(BC))$$

are chosen, there is exactly one monoidal closed structure with the given p.

Proof. The necessity is clear from (3.27). If representations as above are given, there is a unique functor $\otimes$, with the given values $A \otimes B$ on objects, rendering p natural in A and B — it is already natural in C by Proposition 8.11. For the naturality of p means the commutativity of

$$\begin{tikzcd}
L^{A'\otimes B'} \arrow[r, "p_{A'B'}"] \arrow[dd, "L^{f\otimes g}"'] & L^{A'} L^{B'} \arrow[d, "L^f L^{B'}"] \\
 & L^A L^{B'} \arrow[d, "L^A L^g"] \\
L^{A\otimes B} \arrow[r, "p_{AB}"'] & L^A L^B
\end{tikzcd} \tag{5.1}$$

with components

$$\begin{tikzcd}
(A' \otimes B', C) \arrow[r, "p"] \arrow[d, "(f\otimes g, 1)"'] & (A'(B'C)) \arrow[d, "(f, (g, 1))"] \\
(A \otimes B, C) \arrow[r, "p"'] & (A\,(BC)
\end{tikzcd}$$

and by Remark I.10.11, $f \otimes g$ is uniquely determined by (5.1); the functoriality of $\otimes$ is then clear.

Similarly the representation theorem for $\mathscr{V}$-functors gives the existence of unique isomorphisms l, a, r satisfying MCC2, MCC3, MCC4. Defining

π and v by (3.25) and (3.26), we obtain (3.12), (3.16), and (3.18) by applying V to MCC2, MCC3, and MCC4; the naturality of l, a, r now follows by Proposition I.1.2. Also (3.13) is just CC5, and (3.20) follows exactly as in the proof of Proposition 3.2. Proposition 4.1 now ensures the validity of the remaining axioms.

Corollary 5.4. *If a closed category $\mathscr{V}$ is monoidal, so is any closed category isomorphic to $\mathscr{V}$.*

Proof. The representability of $L^A L^B$ is easily seen to survive passage to an isomorph.

The question of which monoidal categories are closed is somewhat more complicated due to the necessity of constructing the functor $V : \mathscr{V}_0 \to \mathscr{S}$; we shall deal first with the case where V is given. By a *normalization* of a monoidal category $\mathscr{V}$ we shall mean a functor $V : \mathscr{V}_0 \to \mathscr{S}$ together with a natural isomorphism $\iota = \iota_A : VA \to \mathscr{V}_0(IA)$; a monoidal category with a given normalization is said to be *normalized.* A monoidal closed category has a canonical normalization given by the V and $\iota = Vi$ it already possesses; any monoidal category admits a normalization, namely $V = \mathscr{V}_0(I\ -)$ and $\iota = 1$, but if $\mathscr{V}$ is also closed this differs in general from the canonical one. A normalized monoidal category shall be said to be closed only if it admits enrichment to a monoidal closed category with the given V and with Vi equal to the given ι.

Theorem 5.5. *A normalized monoidal category $\mathscr{V}$ is closed if and only if the following two conditions are satisfied:*

(i) *the functor $\mathscr{V}_0(- \otimes B, C) : \mathscr{V}_0^* \to \mathscr{S}$ is representable for each $B, C \in \mathscr{V}_0$;*

(ii) *representing objects (BC) and representations*

$$\pi = \pi_{ABC} : \mathscr{V}_0(A \otimes B, C) \to \mathscr{V}_0(A\,(BC))$$

of the above functors may be so chosen that

$$V(BC) = \mathscr{V}_0(BC) \tag{5.2}$$

and

the composite $\mathscr{V}_0(BC) \xrightarrow[\mathscr{V}_0(l,1)]{} \mathscr{V}_0(I \otimes B, C) \xrightarrow[\pi]{} \mathscr{V}_0(I\,(BC))$ *is* $\iota_{(BC)}$. (5.3)

$\mathscr{V}$ then admits a unique monoidal closed structure with Vp equal to the given π.

Proof. The conditions are necessary by Proposition 3.2; suppose they are satisfied.

By the representation theorem, there is a unique way of extending

(BC) to a functor hom $\mathscr{V}$ with respect to which π_{ABC} is natural in B and in C as well as in A.

For $f : B' \to B$ and $g : C \to C'$ we have by the naturality of π and of l a commutative diagram

$$\begin{tikzcd}
\mathscr{V}_0(BC) \arrow[r, "\mathscr{V}_0(l, 1)"] \arrow[d, "\mathscr{V}_0(f, g)"'] & \mathscr{V}_0(I \otimes B, C) \arrow[r, "\pi"] \arrow[d, "\mathscr{V}_0(1 \otimes f, g)"] & \mathscr{V}_0(I(BC)) \arrow[d, "\mathscr{V}_0(1, (f, g))"] \\
\mathscr{V}_0(B'C') \arrow[r, "\mathscr{V}_0(l, 1)"'] & \mathscr{V}_0(I \otimes B', C') \arrow[r, "\pi"'] & \mathscr{V}_0(I(B'C'))
\end{tikzcd}$$

Now by (5.3) and the naturality of ι the left edge of this diagram must also be $V(f, g)$; thus CC0 is satisfied.

Next define v, j, i, p, L by (3.12), (3.13), (3.16), (3.18), and (3.20); these definitions are all forced by Proposition 3.2. By (3.12) and (5.3) we have $v_{BC} = \iota_{(BC)}$; axiom CC5 will follow if we prove $Vi = \iota$.

Since $A \cong (IA)$, to show $Vi_A = \iota_A$ it suffices to show $Vi_{(IA)} = \iota_{(IA)}$. By Proposition I.2.5, which uses only the fact that i is a natural isomorphism, we have $i_{(IA)} = (1, i_A)$. Since $\iota_{(IA)} = v_{IA}$, we have to prove that

$$V(1, i_A) = v_{IA} : \mathscr{V}_0(IA) \to \mathscr{V}_0(I(IA)).$$

By the representation theorem, it suffices to put $A = I$ and evaluate at 1_I; since $V(1, i_I)\, 1 = i_I$, and $v_{II}\, 1 = j_I$ by (3.13), we need $i_I = j_I$. This follows at once from MC5, namely $r_I = l_I$, by (3.15) and (3.17).

From Proposition 4.1 it now follows that the remaining axioms are satisfied and that $Vp = \pi$.

Remark 5.6. A normalized monoidal category may satisfy condition (i) of Theorem 5.5 but admit no π satisfying condition (ii). For instance if $\mathscr{V}_0$ has only a finite number of objects and the $\mathscr{V}_0(BC)$ are all different, it is clearly impossible to satisfy (5.2). However if $\mathscr{V}_0$ is so large that V *admits transport of structure,* which is the case in most of the large categories that occur in nature, condition (ii) can always be satisfied. We say that a functor $V : \mathscr{V}_0 \to \mathscr{S}$ admits transport of structure if, for any $A \in \mathscr{V}_0$, $X \in \mathscr{S}$, and isomorphism $f : VA \to X$, there is a $B \in \mathscr{V}_0$ with $VB = X$ and an isomorphism $g : A \to B$ with $Vg = f$.

Proposition 5.7. *If $\mathscr{V}$ is a normalized monoidal category and $V : \mathscr{V}_0 \to \mathscr{S}$ admits transport of structure, $\mathscr{V}$ is closed if and only if condition* (i) *of Theorem 5.5 is satisfied.*

Proof. First choose representing objects $(BC)'$ and representations

$\pi' : \mathscr{V}_0(A \otimes B, C) \to \mathscr{V}_0(A, (BC)')$. Then we have an isomorphism

$$\mathscr{V}_0(BC) \xrightarrow[\mathscr{V}_0(l,1)]{} \mathscr{V}_0(I \otimes B, C) \xrightarrow[\pi']{} \mathscr{V}_0(I, (BC)') \xrightarrow[\iota^{-1}]{} V(BC)'. \quad (5.4)$$

Now choose for each B, C an object (BC) with $V(BC) = \mathscr{V}_0(BC)$ and an isomorphism $k_{BC} : (BC) \to (BC)'$ with Vk_{BC} equal to (5.4). Define a new representation π as the composite

$$\pi\colon \mathscr{V}_0(A \otimes B, C) \xrightarrow[\pi']{} \mathscr{V}_0(A, (BC)') \xrightarrow[\mathscr{V}_0(1,k^{-1})]{} \mathscr{V}_0(A(BC)). \quad (5.5)$$

We already have (5.2) by our choice of (BC); we show that π satisfies (5.3). Consider the diagram

$$\begin{tikzcd}
\mathscr{V}_0(BC) \arrow[r, "\mathscr{V}_0(l,1)"] \arrow[d, "Vk"'] & \mathscr{V}_0(I \otimes B, C) \arrow[d, "\pi'"] \\
V(BC)' \arrow[r, "\iota"] \arrow[d, "Vk^{-1}"'] & \mathscr{V}_0(I, (BC)') \arrow[d, "\mathscr{V}_0(1,k^{-1})"] \\
V(BC) \arrow[r, "\iota"'] & \mathscr{V}_0(I(BC))
\end{tikzcd}$$

The top region commutes by our choice of k, and the bottom region by the naturality of ι. In view of (5.5) this gives (5.3).

We now consider criteria for an unnormalized monoidal category to be closed.

Theorem 5.8. *A monoidal category $\mathscr{V}$ is closed if and only if the following two conditions are satisfied:*

(i) *the functor $\mathscr{V}_0(- \otimes B, C) : \mathscr{V}_0^* \to \mathscr{S}$ is representable for each $B, C \in \mathscr{V}_0$;*

(ii) *representing objects (BC) and representations*

$$\pi = \pi_{ABC} : \mathscr{V}_0(A \otimes B, C) \to \mathscr{V}_0(A(BC))$$

of the above functors may be so chosen that $\mathscr{V}_0(BC)$ and the composite

$$\mathscr{V}_0(BC) \xrightarrow[\mathscr{V}_0(l,1)]{} \mathscr{V}_0(I \otimes B, C) \xrightarrow[\pi]{} \mathscr{V}_0(I(BC)) \quad (*)$$

depend only on (BC).

With such representations chosen, a monoidal closed structure with Vp equal to the given π is unique except for some indeterminacy in the definition of V.

Proof. We define V and ι on the full subcategory of $\mathscr{V}_0$ determined by objects of the form (BC). We define $V(BC)$ to be $\mathscr{V}_0(BC)$ and $\iota_{(BC)}$ to be (*); these definitions are consistent by condition (ii). Moreover they are forced by Proposition 3.2, which shows that condition (ii) is necessary; we already know that condition (i) is necessary. We take the value of Vf for $f: (BC) \to (DE)$ to be $\iota^{-1} \mathscr{V}_0(1, f)\, \iota$; this is forced if ι is to be natural, and does make V a functor and ι natural.

We have considerable liberty in completing the definitions of V and of ι. For definiteness let us define VA, where A is not of the form (BC), to be $\mathscr{V}_0(IA)$, and define $\iota: VA \to \mathscr{V}_0(IA)$ to be 1; then so define V on morphisms that ι is natural.

We now have (5.2) and (5.3), and Theorem 5.5 gives the desired result. We have only to note that our manner of completing the definitions of V and of ι makes no difference to the forced definitions of j, i, L, p.

Theorem 5.9. *A monoidal category $\mathscr{V}$ possesses an isomorph $\mathscr{V}'$ which is closed if and only if it satisfies condition* (i) *of Theorem 5.8. Moreover if representations $\pi: \mathscr{V}_0(A \otimes B, C) \to \mathscr{V}_0(A(BC))$ are given, there is a canonical way of constructing the monoidal closed category $\mathscr{V}'$.*

Proof. Condition (i) is necessary because it clearly survives passage to an isomorph. Suppose representations π as above are given.

We define a new category $\mathscr{V}_0'$ and an isomorphism $\phi: \mathscr{V}_0 \to \mathscr{V}_0'$. The objects of $\mathscr{V}_0'$ are those of $\mathscr{V}_0$, and ϕ is the identity on objects. We set

$$\mathscr{V}_0'(BC) = \mathscr{V}_0(I, (BC)) \tag{5.6}$$

and define $\phi_{BC}: \mathscr{V}_0(BC) \to \mathscr{V}_0'(BC)$ to be the composite

$$\phi_{BC}: \mathscr{V}_0(BC) \xrightarrow[\mathscr{V}_0(l,1)]{} \mathscr{V}_0(I \otimes B, C) \xrightarrow[\pi]{} \mathscr{V}_0(I(BC)). \tag{5.7}$$

Finally we define composition in $\mathscr{V}_0'$ so that ϕ becomes a functor.

We now use the isomorphism ϕ to transfer to $\mathscr{V}_0'$ the structure of monoidal category on $\mathscr{V}_0$, getting a monoidal category $\mathscr{V}'$ and an isomorphic monoidal functor $\Phi = (\phi, \tilde{\phi}, \phi^0): \mathscr{V} \to \mathscr{V}'$. To be precise, we have $A \otimes' B = A \otimes B, f \otimes' g = \phi(\phi^{-1} f \otimes \phi^{-1} g), I' = I, a' = \phi a$, $l' = \phi l,\ r' = \phi r,\ \tilde{\phi} = 1,\ \phi^0 = 1$.

We next define a normalization of $\mathscr{V}'$. We define

$$V'A = \mathscr{V}_0(IA) \tag{5.8}$$

and define $\iota': V'A \to \mathscr{V}_0'(IA)$ to be

$$\iota' = \phi: \mathscr{V}_0(IA) \to \mathscr{V}_0'(IA); \tag{5.9}$$

then define V' on morphisms so that ι' is natural.

Finally we define representations $\pi': \mathscr{V}_0'(A \otimes B, C) \to \mathscr{V}_0'(A(BC))$ by setting π' equal to the composite

$$\pi': \mathscr{V}_0'(A \otimes B, C) \xrightarrow[\phi^{-1}]{} \mathscr{V}_0(A \otimes B, C) \xrightarrow[\pi]{} \mathscr{V}_0(A(BC)) \xrightarrow[\phi]{} \mathscr{V}_0'(A(BC)); \tag{5.10}$$

this is clearly natural in A.

$\mathscr{V}'$ satisfies (5.2) by (5.6) and (5.8); we show that it also satisfies (5.3). Consider the diagram

$$\begin{tikzcd}
\mathscr{V}_0(BC) \arrow[r, "\mathscr{V}_0(l, 1)"] \arrow[d, "\phi"] & \mathscr{V}_0(I \otimes B, C) \arrow[r, "\pi"] \arrow[d, "\phi"] & \mathscr{V}_0(I(BC)) \arrow[d, "\phi"] \\
\mathscr{V}'_0(BC) \arrow[r, "\mathscr{V}'_0(l', 1)"'] & \mathscr{V}'_0(I \otimes B, C) \arrow[r, "\pi'"'] & \mathscr{V}'_0(I, (BC))
\end{tikzcd}$$

The left region commutes by the naturality of ϕ, since $l' = \phi l$, and the right region by (5.10). The left edge and the top edge are equal isomorphisms by (5.7); hence the bottom edge equals the right edge, which is ι' by (5.9). This is (5.3) for $\mathscr{V}'$, and we now appeal to Theorem 5.5.

We complete this section by describing an economical way of giving a monoidal closed category:

Theorem 5.10. *Suppose given a category $\mathscr{V}_0$, functors $\otimes : \mathscr{V}_0 \times \mathscr{V}_0 \to \mathscr{V}_0$ and* hom $\mathscr{V} : \mathscr{V}_0^* \times \mathscr{V}_0 \to \mathscr{V}_0$, *and a functor $V : \mathscr{V}_0 \to \mathscr{S}$ satisfying* CC0. *Suppose further given an object I of $\mathscr{V}_0$, a natural isomorphism $i : A \to (IA)$, and a natural isomorphism $p : (A \otimes B, C) \to (A(BC))$. Then these data can be completed to give a monoidal closed category if and only if the r and the a defined by* (3.16) *and* (3.18), *with $\pi = Vp$, satisfy* MCC4 *and* MCC3; *and $\mathscr{V}$ is then unique. Moreover if V is faithful the satisfaction of* MCC4 *and* MCC3 *is automatic.*

Proof. The necessity of the conditions follows from Proposition 3.2; moreover if V is faithful MCC4 and MCC3 follow from their images under V, which are (3.16) and (3.18).

If the conditions are satisfied we define v by (3.26), j by (3.13), l by (3.12), and L by (3.20). Since we have forced CC5 by our definition of v, and since we have $Vp = \pi$, it follows from Propositions 4.1 and 4.2 that all the axioms are satisfied.

6. Categories over a Monoidal Closed Category

If $\mathscr{V}$ is a monoidal category, we define a $\mathscr{V}$-category $\mathscr{A}$ to consist of the following four data:

(i) a class obj $\mathscr{A}$ of "objects";

(ii) for each $A, B \in \text{obj}\, \mathscr{A}$, an object $\mathscr{A}(A B)$ of $\mathscr{V}_0$;

(iii) for each $A \in \text{obj}\, \mathscr{A}$, a morphism

$$j_A : I \to \mathscr{A}(A A)$$

in $\mathscr{V}_0$;

(iv) for each A, B, $C \in \operatorname{obj} \mathscr{A}$, a morphism

$$M^B_{AC} : \mathscr{A}(BC) \otimes \mathscr{A}(AB) \to \mathscr{A}(AC)$$

in $\mathscr{V}_0$.

These data are to satisfy the following three axioms:

VC1′. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(BB) \otimes \mathscr{A}(AB) \arrow[r, "M"] & \mathscr{A}(AB) \\
I \otimes \mathscr{A}(AB) \arrow[u, "j \otimes 1"] \arrow[ur, "l"'] &
\end{tikzcd}$$

VC2′. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(AC) \otimes \mathscr{A}(AA) \arrow[r, "M"] & \mathscr{A}(AC) \\
\mathscr{A}(AC) \otimes I \arrow[u, "1 \otimes j"] \arrow[ur, "r"'] &
\end{tikzcd}$$

VC3′. The following diagram commutes:

$$\begin{tikzcd}
(\mathscr{A}(CD) \otimes \mathscr{A}(BC)) \otimes \mathscr{A}(AB) \arrow[rr, "a"] \arrow[d, "M \otimes 1"'] & & \mathscr{A}(CD) \otimes (\mathscr{A}(BC) \otimes \mathscr{A}(AB)) \arrow[d, "1 \otimes M"] \\
\mathscr{A}(BD) \otimes \mathscr{A}(AB) \arrow[dr, "M"'] & & \mathscr{A}(CD) \otimes \mathscr{A}(AC) \arrow[dl, "M"] \\
& \mathscr{A}(AD) &
\end{tikzcd}$$

If $\mathscr{A}$ and $\mathscr{B}$ are $\mathscr{V}$-categories where $\mathscr{V}$ is a monoidal category, a $\mathscr{V}$-functor $T : \mathscr{A} \to \mathscr{B}$ is to consist of the following two data:

(i) a function $T : \operatorname{obj} \mathscr{A} \to \operatorname{obj} \mathscr{B}$;

(ii) for each $B, C \in \operatorname{obj} \mathscr{A}$, a morphism

$$T = T_{BC} : \mathscr{A}(BC) \to \mathscr{B}(TB, TC)$$

in $\mathscr{V}_0$.

These data are to satisfy the following two axioms:

VF 1′. The following diagram commutes:

$$\begin{tikzcd} \mathscr{A}(BB) \arrow[rr, "T_{BB}"] & & \mathscr{B}(TB, TB) \\ & I \arrow[ul, "j"] \arrow[ur, "j"'] & \end{tikzcd}$$

VF 2′. The following diagram commutes:

$$\begin{tikzcd} \mathscr{A}(CD) \otimes \mathscr{A}(BC) \arrow[r, "M"] \arrow[d, "T \otimes T"'] & \mathscr{A}(BD) \arrow[d, "T"] \\ \mathscr{B}(TC, TD) \otimes \mathscr{B}(TB, TC) \arrow[r, "M"'] & \mathscr{B}(TB, TD) \end{tikzcd}$$

We leave the reader to verify:

Proposition 6.1. *If $\mathscr{V}$ is a monoidal category, $\mathscr{V}$-categories and $\mathscr{V}$-functors form a "category" $\mathscr{V}_*$ if we define composition by* I (5.1) *and* I (5.2).

Remark 6.2. If $\mathscr{V}$ is monoidal without being closed, $\mathscr{V}$ itself does not in general have the structure of a $\mathscr{V}$-category. However there is always a $\mathscr{V}$-category $\mathscr{I}$, defined as in Proposition I.5.3 except that in place of L we give $M : \mathscr{I}(**) \otimes \mathscr{I}(**) \to \mathscr{I}(**)$, defining it to be $l_I : I \otimes I \to I$. The rest of Proposition I.5.3 then applies word for word.

We also leave the reader to verify:

Proposition 6.3. *From a monoidal functor $\Phi = (\phi, \tilde{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ we get a functor $\Phi_* : \mathscr{V}_* \to \mathscr{V}'_*$ if we define $\Phi_*\mathscr{A}$ by* I (6.1), I (6.2), I (6.3), *and:*

$$M' : \phi\mathscr{A}(BC) \otimes \phi\mathscr{A}(AB) \to \phi\mathscr{A}(AC)$$

is the composite

$$\phi\mathscr{A}(BC) \otimes \phi\mathscr{A}(AB) \xrightarrow[\tilde{\phi}]{} \phi(\mathscr{A}(BC) \otimes \mathscr{A}(AB)) \xrightarrow[\phi M]{} \phi\mathscr{A}(AC); \quad (6.1)$$

and define $\Phi_* T$ by I (6.8) *and* I (6.9). *Further if $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ is a monoidal natural transformation we get a natural transformation*

$$\eta_* : \Phi_* \to \Psi_* : \mathscr{V}_* \to \mathscr{V}'_*$$

if we define $\eta_{*\mathscr{A}}$ by I (6.10) *and* I (6.11). *Finally the assignments $\mathscr{V} \mapsto \mathscr{V}_*$, $\Phi \mapsto \Phi_*$, $\eta \mapsto \eta_*$ constitute a hyperfunctor $_* : \mathscr{M}on \to \mathscr{C}at$.*

Theorem 6.4. *Let $\mathscr{V}$ be a monoidal closed category. Then the "categories" ${}^m\mathscr{V}_*$ and ${}^c\mathscr{V}_*$ coincide if we identify the ${}^m\mathscr{V}$-category $\mathscr{A} = (\mathrm{obj}\,\mathscr{A}, \mathscr{A}(A\,B), j, M)$ with the ${}^c\mathscr{V}$-category $\mathscr{A} = (\mathrm{obj}\,\mathscr{A}, \mathscr{A}(A\,B), j, L)$ where*

$$L^A_{BC} = \pi M^B_{AC}; \tag{6.2}$$

here π is

$$\pi = Vp : V(\mathscr{A}(BC) \otimes \mathscr{A}(A\,B), \mathscr{A}(A\,C)) \to$$
$$\to V(\mathscr{A}(BC), (\mathscr{A}(A\,B), \mathscr{A}(A\,C)))\,.$$

Moreover if $\Phi : \mathscr{V} \to \mathscr{V}'$ is a monoidal closed functor, ${}^m\Phi_* : {}^m\mathscr{V}_* \to {}^m\mathscr{V}'_*$ coincides with ${}^c\Phi_* : {}^c\mathscr{V}_* \to {}^c\mathscr{V}'_*$, and if $\eta : \Phi \to \Psi$ is a monoidal closed natural transformation, $\eta_* : {}^m\Phi_* \to {}^m\Psi_*$ coincides with $\eta_* : {}^c\Phi_* \to {}^c\Psi_*$.

Proof. We shall prove (i) VC1 $\Leftrightarrow$ VC1′; (ii) VC2 $\Leftrightarrow$ VC2′; (iii) VC3 $\Leftrightarrow$ VC3′; (iv) VF2 $\Leftrightarrow$ VF2′; (v) I(6.4) and (6.1) are related by (6.2). The other matters to be verified are trivial.

Proof of (i). Apply π to both legs of VC1′; we get

$$\begin{aligned} \pi(M(j \otimes 1)) &= \pi M\,.\,j && \text{by} \quad (3.1)\\ &= Lj && \text{by} \quad (6.2); \end{aligned}$$

and

$$\pi l = j \qquad \text{by} \quad (3.15);$$

the resulting diagram equivalent to VC1′ is precisely VC1.

Proof of (ii). Applying π to both legs of VC2′ we get

$$\begin{aligned} \pi(M(1 \otimes j)) &= (j, 1)\,.\,\pi M && \text{by} \quad (3.1)\\ &= (j, 1)\, L && \text{by} \quad (6.2); \end{aligned}$$

and

$$\pi r = i \qquad \text{by} \quad (3.17);$$

the resulting diagram is VC2.

Proof of (iii). Applying π twice to each leg of VC3′ we get

$$\begin{aligned} \pi\pi(M(1 \otimes M)\,a) &= p\,.\,\pi(M(1 \otimes M)) && \text{by} \quad (3.19)\\ &= p(M, 1)\,.\,\pi M && \text{by} \quad (3.1)\\ &= p(M, 1)\, L && \text{by} \quad (6.2)\\ &= (\pi M, 1)\, L L && \text{by} \quad (3.22)\\ &= (L, 1)\, L L && \text{by} \quad (6.2); \end{aligned}$$

and

$$\begin{aligned} \pi\pi(M(M \otimes 1)) &= \pi(\pi M\,.\,M) && \text{by} \quad (3.1)\\ &= \pi(L M) && \text{by} \quad (6.2) \end{aligned}$$

$$
\begin{aligned}
&= (1, L) \,.\, \pi M && \text{by} \quad (3.1)\\
&= (1, L)\, L && \text{by} \quad (6.2);
\end{aligned}
$$

the resulting diagram is precisely VC3.

Proof of (iv). Applying π to each leg of VF2′ we get

$$
\begin{aligned}
\pi(T M) &= (1, T) \,.\, \pi M && \text{by} \quad (3.1)\\
&= (1, T)\, L && \text{by} \quad (6.2);
\end{aligned}
$$

and

$$
\begin{aligned}
\pi(M(T \otimes T)) &= (T, 1) \,.\, \pi M \,.\, T && \text{by} \quad (3.1)\\
&= (T, 1)\, L\, T && \text{by} \quad (6.2);
\end{aligned}
$$

the resulting diagram is VF2.

Proof of (v) We have, with M' defined by (6.1),

$$
\begin{aligned}
\pi' M' &= \pi'(\phi M \,.\, \tilde{\phi}) &&\\
&= \hat{\phi} \,.\, \phi \pi M && \text{by} \quad (3.24)\\
&= \hat{\phi} \,.\, \phi L && \text{by} \quad (6.2)\\
&= L' && \text{by} \quad \mathrm{I}(6.4).
\end{aligned}
$$

Remark 6.5. If $\mathscr{V}$ and $\mathscr{V}'$ are monoidal closed categories we shall, in view of Theorems 5.1 and 6.4, identify a monoidal closed functor $\Phi = (\phi, \tilde{\phi}, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ with the monoidal functor $(\phi, \tilde{\phi}, \phi^0)$ and the closed functor $(\phi, \hat{\phi}, \phi^0)$.

7. The $\mathscr{V}$-functor K^B

Let $\mathscr{V}$ be a monoidal closed category. By Lemma 3.1 the natural transformation

$$p^{-1} : (A(BC)) \to (A \otimes B, C)$$

determines a natural transformation

$$K^B_{AC} : (A\,C) \to (A \otimes B, C \otimes B)$$

connected with p^{-1} by the diagrams

$$
\begin{tikzcd}
(A\,C) \arrow[rr, "K^B"] \arrow[dr, "{(1, u)}"'] & & (A \otimes B, C \otimes B) \\
& (A, (B, C \otimes B)) \arrow[ur, "p^{-1}"'] &
\end{tikzcd}
\qquad (7.1)
$$

$$\begin{tikzcd}
(A(BC)) \arrow[rr, "p^{-1}"] \arrow[dr, "K^B"'] & & (A \otimes B, C) \\
& (A \otimes B, (BC) \otimes B) \arrow[ur, "{(1,t)}"'] &
\end{tikzcd} \tag{7.2}$$

Theorem 7.1. *Let $\mathscr{V}$ be a monoidal closed category. For each $B \in \mathscr{V}$ we obtain a $\mathscr{V}$-functor $K^B : \mathscr{V} \to \mathscr{V}$ if we set $K^B A = A \otimes B$ and $(K^B)_{AC} = K^B_{AC}$, and the underlying functor $V_* K^B : \mathscr{V}_0 \to \mathscr{V}_0$ is $- \otimes B$. Moreover the morphisms $t_{BC} : (BC) \otimes B \to C$ and $u_{CB} : C \to (B, C \otimes B)$ are the C-components of $\mathscr{V}$-natural transformations $t_B : K^B L^B \to 1 : \mathscr{V} \to \mathscr{V}$ and $u_{\cdot B} : 1 \to L^B K^B : \mathscr{V} \to \mathscr{V}$.*

Proof. Applying V to (7.1) and evaluating at $f \in V(AC)$ gives

$$\begin{aligned}
(VK^B) f &= \pi^{-1} V(1, u) f \\
&= \pi^{-1}(u f) \\
&= \pi^{-1} u \,.\, (1 \otimes f) \quad \text{by (3.1)} \\
&= 1 \otimes f \qquad\qquad \text{by (3.3)}.
\end{aligned}$$

Thus $V_* K^B = - \otimes B$, and we have VF1 for K^B in the form $(VK^B) 1 = 1$ (cf. Remark I.9.7). Leave aside for the moment the question of VF2 for K_B.

From (3.20) and (7.2) we get a commutative diagram

$$\begin{tikzcd}
(AC) \arrow[r, "L^B"] \arrow[d, "{(t,1)}"'] & ((BA)(BC)) \arrow[dl, "p^{-1}"] \arrow[d, "K^B"] \\
((BA) \otimes B, C) & ((BA) \otimes B, (BC) \otimes B) \arrow[l, "{(1,t)}"] ;
\end{tikzcd}$$

the exterior of this is VN for t_B. Similarly from (3.21) and (7.1) we get a commutative diagram

$$\begin{tikzcd}
(AC) \arrow[r, "{(1,u)}"] \arrow[d, "K^B"'] & (A(B, C \otimes B)) \\
(A \otimes B, C \otimes B) \arrow[ur, "p"] \arrow[r, "L^B"'] & ((B, A \otimes B), (B, C \otimes B)) \arrow[u, "{(u,1)}"'] ;
\end{tikzcd}$$

the exterior of this is VN for $u_{\cdot B}$.

Since the proofs of the assertions of Theorem I.10.2 make no use of VF2 for the $\mathscr{V}$-functors involved, we can use them here before we have VF2 for K^B. The composite $p^{-1}(1, u)$ in (7.1) is therefore the C-component of a $\mathscr{V}$-natural transformation

$$L^A \xrightarrow[L^A u_{\cdot B}]{} L^A L^B K^B \xrightarrow[p^{-1}_{AB} K^B]{} L^{A\otimes B} K^B,$$

and so this composite, which by (7.1) is K^B_{AC}, satisfies VN; and this is VF2 for K^B.

Proposition 7.2. *If $\mathscr{V}$ is a monoidal closed category and $f \in \mathscr{V}_0(AB)$, the morphisms*

$$1 \otimes f : C \otimes A \to C \otimes B$$

are the C-components of a $\mathscr{V}$-natural transformation

$$K^f : K^A \to K^B.$$

Proof. VN for K^f asserts the commutativity of

$$\begin{tikzcd}
(CD) \arrow[r, "K^A"] \arrow[d, "K^B"'] & (C \otimes A, D \otimes A) \arrow[d, "{(1, 1 \otimes f)}"] \\
(C \otimes B, D \otimes B) \arrow[r, "{(1 \otimes f, 1)}"'] & (C \otimes A, D \otimes B)
\end{tikzcd}$$

and this is just the assertion that K^B_{CD} is natural in B, which it is by its definition (7.1).

The following result shows the relation of t and u to special cases of M (i.e. the M of $\mathscr{V}$ itself) and K:

Proposition 7.3. *In a monoidal closed category we have commutative diagrams:*

$$\begin{tikzcd}
(BC) \otimes B \arrow[r, "t"] \arrow[d, "1 \otimes i"'] & C \arrow[d, "i"] \\
(BC) \otimes (IB) \arrow[r, "M"'] & (IC)
\end{tikzcd} \tag{7.3}$$

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "i"'] & (B, C \otimes B) \arrow[d, "{(l, 1)}"] \\
(IC) \arrow[r, "K"'] & (I \otimes B, C \otimes B)
\end{tikzcd} \tag{7.4}$$

Proof. Applying π to both legs of (7.3) we get

$$\begin{aligned} \pi(it) &= (1, i) \,.\, \pi t && \text{by (3.1)} \\ &= (1, i) && \text{by (3.2)}, \end{aligned}$$

and

$$\begin{aligned} \pi(M(1 \otimes i)) &= (i, 1) \,.\, \pi M && \text{by (3.1)} \\ &= (i, 1) L && \text{by (6.2)}; \end{aligned}$$

thus (7.3) reduces to CC4.

We compose each leg of (7.4) with the isomorphism

$$p : (I \otimes B, C \otimes B) \to (I(B, C \otimes B)),$$

getting

$$\begin{aligned} pKi &= (1, u) i && \text{by (7.1)} \\ &= iu && \text{by the naturality of } i, \end{aligned}$$

and

$$p(l, 1) u = iu \qquad \text{by MCC2};$$

thus (7.4) commutes.

The following is an adjoint form, as it were, of the representation theorem for $\mathscr{V}$-functors:

Proposition 7.4. *Let $\mathscr{V}$ be a monoidal closed category, $\mathscr{A}$ a $\mathscr{V}$-category, $T : \mathscr{A} \to \mathscr{V}$ a $\mathscr{V}$-functor, $A \in \mathscr{A}$ and $B \in \mathscr{V}$. Denote by $\{q\}$ the class of $\mathscr{V}$-natural transformations*

$$q : K^B L^A \to T : \mathscr{A} \to \mathscr{V}$$

with components

$$q_C : \mathscr{A}(AC) \otimes B \to TC.$$

Define a map $\Delta : \{q\} \to V(B, TA)$ by setting Δq equal to the composite

$$\Delta q: \qquad B \xrightarrow{l^{-1}} I \otimes B \xrightarrow{j \otimes 1} \mathscr{A}(AA) \otimes B \xrightarrow{q_A} TA. \tag{7.5}$$

Then Δ is a bijection with inverse Π, where $\Pi\theta$ for $\theta : B \to TA$ is the composite

$$\Pi\theta: \qquad K^B L^A \xrightarrow{K^B T_A} K^B L^{TA} T \xrightarrow{K^B L^\theta T} K^B L^B T \xrightarrow{\iota_B T} T, \tag{7.6}$$

with components

$$\mathscr{A}(AC) \otimes B \xrightarrow{T \otimes 1} (TA, TC) \otimes B \xrightarrow{(\theta, 1) \otimes 1} (B, TC) \otimes B \xrightarrow{t} TC. \tag{7.7}$$

Proof. By (3.4), πq_C is the composite

$$\mathscr{A}(AC) \xrightarrow{u} (B, \mathscr{A}(AC) \otimes B) \xrightarrow{(1, q)} (B, TC)$$

which is the component of a $\mathscr{V}$-natural transformation

$$\pi q: \qquad L^A \xrightarrow[u_{\cdot B}L^A]{} L^B K^B L^A \xrightarrow[L^B q]{} L^B T.$$

Similarly, if $\bar{q}_C : \mathscr{A}(A\,C) \to (B, TC)$ are the components of a $\mathscr{V}$-natural transformation $\bar{q} : L^A \to L^B T$, then $\pi^{-1}\bar{q}_C$, which by (3.5) is the composite

$$\mathscr{A}(A\,C) \otimes B \xrightarrow[\bar{q}\otimes 1]{} (B, TC) \otimes B \xrightarrow[t]{} TC,$$

is the component of a $\mathscr{V}$-natural transformation

$$\pi^{-1}\bar{q}: \qquad K^B L^A \xrightarrow[K^B\bar{q}]{} K^B L^B T \xrightarrow[t_B T]{} T.$$

Thus we have a bijection $\pi : \{q\} \to \{\bar{q}\}$, and so by Theorem I.10.8 we have a bijection

$$\{q\} \xrightarrow[\pi]{} \{\bar{q}\} \xrightarrow[\Gamma']{} V(B, TA),$$

with inverse

$$V(B, TA) \xrightarrow[\Omega']{} \{\bar{q}\} \xrightarrow[\pi^{-1}]{} \{q\}.$$

Comparison of I (10.11), (3.4), and (7.6) shows at once that $\pi^{-1}\Omega' = \Pi$. It remains to show that $\Gamma'\pi = \Delta$.

Evaluating (3.12) at $x \in \mathscr{V}_0(A\,B)$ and using (3.26) gives $\iota x = \pi(xl)$, and (3.4) then gives

$$\iota x = (1, xl)\,u. \tag{7.8}$$

Applying this to Δq gives

$$\begin{aligned} \iota\Delta q &= (1, q)\,(1, j \otimes 1)\,u \\ &= (1, q)\,u j \quad \text{by the naturality of } u \\ &= \pi q . j \qquad \text{by (3.4)} \\ &= \iota(\Gamma'\pi q) \quad \text{by I(10.9) and Lemma I.2.2;} \end{aligned}$$

thus $\Delta q = \Gamma'\pi q$, as required.

If in the above proposition we take $\mathscr{A} = \mathscr{V}$ and $A = I$, and use the isomorphism $i : 1 \to L^I$, we get:

Corollary 7.5. *If $\mathscr{V}$ is a monoidal closed category and $T : \mathscr{V} \to \mathscr{V}$ is a $\mathscr{V}$-functor, there is a bijection between $\mathscr{V}$-natural transformations $q : K^B \to T$ and morphisms $\theta : B \to TI$, where θ is the composite*

$$B \xrightarrow[l^{-1}]{} I \otimes B \xrightarrow[q_I]{} TI. \tag{7.9}$$

In particular, q is determined by q_I.

8. The Underlying Category of a $\mathscr{V}$-category

The closed category $\mathscr{S}$ is monoidal, with the cartesian product $A \times B$ for $A \otimes B$, since $L^A L^B$ admits the representation

$$p : (A \times B, C) \to (A\,(BC))$$

where

$$((pf)x)y = f(x,y), \quad f \in (A \times B, C), \quad x \in A, \quad y \in B. \tag{8.1}$$

We verify at once that a, r, l have their expected values and that M given by (6.2) corresponds to the usual composition law in categories.

Let $\mathscr{V}$ be a normalized monoidal category, and define $V^0 : * \to VI$ by

$$V^0 * = \iota_I^{-1} 1_I; \tag{8.2}$$

then by the naturality of ι (cf. I(3.16)) the image of $*$ under the composite $* \xrightarrow[V^0]{} VI \xrightarrow[Vf]{} VA$ is given by

$$Vf.\,V^0 * = \iota^{-1} f. \tag{8.3}$$

Now define a natural transformation

$$\tilde{V} : VA \times VB \to V(A \otimes B)$$

by the commutative diagram

$$\begin{tikzcd}
VA \times VB \arrow[rr, "\tilde{V}"] \arrow[d, "\iota \times \iota"'] & & V(A \otimes B) \\
\mathscr{V}_0(IA) \times \mathscr{V}_0(IB) \arrow[r, "\otimes"'] & \mathscr{V}_0(I \otimes I, A \otimes B) \arrow[r, "\mathscr{V}_0(l^{-1}, 1)"'] & \mathscr{V}_0(I, A \otimes B) \arrow[u, "\iota^{-1}"']
\end{tikzcd} \tag{8.4}$$

where $\otimes$ is the map sending (f, g) to $f \otimes g$; we record the evaluated form of (8.4) as

$$\iota \tilde{V}(x, y) = (\iota x \otimes \iota y) l_I^{-1}. \tag{8.5}$$

Proposition 8.1. *If $\mathscr{V}$ is a normalized monoidal category, and V^0 and $\tilde{V}$ are defined by* (8.2) *and* (8.4), *the triple* $(V, \tilde{V}, V^0)$ *is a monoidal functor $V : \mathscr{V} \to \mathscr{S}$.*

Proof. Consider for example axiom MF3 for V, which reads:

$$\begin{tikzcd}
V((A \otimes B) \otimes C) \arrow[r, "Va"] & V(A \otimes (B \otimes C)) \\
V(A \otimes B) \times VC \arrow[u, "\tilde{V}"] & VA \times V(B \otimes C) \arrow[u, "\tilde{V}"'] \\
(VA \times VB) \times VC \arrow[u, "\tilde{V} \times 1"] \arrow[r, "a'"'] & VA \times (VB \times VC) \arrow[u, "1 \times \tilde{V}"']
\end{tikzcd}$$

Since both legs are natural and since $V \cong \mathscr{V}_0(I-)$, repeated application of Proposition 7.4 shows that it suffices to put $A = B = C = I$ and verify that both legs have the same value at $((\iota^{-1}1, \iota^{-1}1), \iota^{-1}1)$; this verification is immediate. Similarly we verify MF1 and MF2.

Essentially the following result is given by Bénabou [3]:

Theorem 8.2. *Let $\mathscr{V}$ be a normalized monoidal category and let $\mathscr{A}$ be a $\mathscr{V}$-category. Then we can find in exactly one way a category $\mathscr{A}_0$ and a functor* hom $\mathscr{A}: \mathscr{A}_0^* \times \mathscr{A}_0 \to \mathscr{V}_0$ *such that:*

(i) *$\mathscr{A}_0$ has the same objects as $\mathscr{A}$;*

(ii) hom $\mathscr{A}(AB) = \mathscr{A}(AB)$; (8.6)

(iii) *the following diagram commutes:*

$$\begin{tikzcd} \mathscr{A}_0^* \times \mathscr{A}_0 \arrow[r, "\hom \mathscr{A}"] \arrow[dr, "\mathrm{Hom}\,\mathscr{A}_0"'] & \mathscr{V}_0 \arrow[d, "V"] \\ & \mathscr{S} \end{tikzcd} \tag{8.7}$$

(iv) $j_A = \iota 1_A$, (8.8)

where $\iota: V\mathscr{A}(AA) \to \mathscr{V}_0(I, \mathscr{A}(AA))$;

(v) *M^A_{BC} is natural in A, B, and C, and j_A in A.*

Proof. We first prove the uniqueness. The objects and the morphisms of $\mathscr{A}_0$ are fixed by (8.6) and (8.7); we must have

$$\mathscr{A}_0(AB) = V\mathscr{A}(AB). \tag{8.9}$$

Next, $\iota_{\mathscr{A}(AB)}$ gives a natural isomorphism (using (8.9))

$$\iota: \mathscr{A}_0(AB) \to \mathscr{V}_0(I, \mathscr{A}(AB)).$$

Since $\iota 1 = j$ by (8.8), the representation theorem (applied both to $\mathscr{A}_0(A-)$ and $\mathscr{A}_0(-B)$) shows that for $f \in \mathscr{A}_0(AB)$ we have a commutative diagram

$$\begin{tikzcd} I \arrow[r, "j"] \arrow[d, "j"'] \arrow[dr, "\iota f"] & \mathscr{A}(AA) \arrow[d, "\mathscr{A}(1, f)"] \\ \mathscr{A}(BB) \arrow[r, "\mathscr{A}(f, 1)"'] & \mathscr{A}(AB) \end{tikzcd} \tag{8.10}$$

For $f \in \mathscr{A}_0(BC)$ the following diagram commutes by the naturality of M and by (8.10):

$$\begin{tikzcd} I \otimes \mathscr{A}(AB) \arrow[r, "j \otimes 1"] \arrow[dr, "\iota f \otimes 1"'] & \mathscr{A}(BB) \otimes \mathscr{A}(AB) \arrow[r, "M"] \arrow[d, "\mathscr{A}(1,f) \otimes 1"] & \mathscr{A}(AB) \arrow[d, "\mathscr{A}(1,f)"] \\ & \mathscr{A}(BC) \otimes \mathscr{A}(AB) \arrow[r, "M"'] & \mathscr{A}(AC) \end{tikzcd}$$

Composing both legs of this with $l^{-1}: \mathscr{A}(AB) \to I \otimes \mathscr{A}(AB)$, and using VC1′, we get a commutative diagram

$$\begin{tikzcd} \mathscr{A}(AB) \arrow[r, "\mathscr{A}(1,f)"] \arrow[d, "l^{-1}"'] & \mathscr{A}(AC) \\ I \otimes \mathscr{A}(AB) \arrow[r, "\iota f \otimes 1"'] & \mathscr{A}(BC) \otimes \mathscr{A}(AB) \arrow[u, "M"'] \end{tikzcd} \tag{8.11}$$

This fixes the value of $\mathscr{A}(1, f)$. Similarly we get the following diagram, which fixes the value of $\mathscr{A}(f, 1)$:

$$\begin{tikzcd} \mathscr{A}(CD) \arrow[r, "\mathscr{A}(f,1)"] \arrow[d, "r^{-1}"'] & \mathscr{A}(BD) \\ \mathscr{A}(CD) \otimes I \arrow[r, "1 \otimes \iota f"'] & \mathscr{A}(CD) \otimes \mathscr{A}(BC) \arrow[u, "M"'] \end{tikzcd} \tag{8.12}$$

Thus the functor hom $\mathscr{A}$ is unique. Finally (8.7) gives $\mathscr{A}_0(1, f) = V\mathscr{A}(1, f)$; so that $\mathscr{A}_0(1, f)$ is determined, and with it the law of composition in $\mathscr{A}_0$; for $\mathscr{A}_0(1, f)g = fg$.

We now prove the existence. We take the monoidal functor V of Proposition 8.1 and define

$$\mathscr{A}_0 = V_* \mathscr{A}. \tag{8.13}$$

We then have (i) and (8.9). By the definition I(6.3) of j' (cf. Proposition 6.3), the identity of $\mathscr{A}_0(AA)$ is the image of $*$ under

$$* \xrightarrow[V^0]{} VI \xrightarrow[Vj]{} V\mathscr{A}(AA),$$

so that by (8.3) we have (8.8). The naturality of j now follows from that of ι.

By (6.1) the M of $\mathscr{A}_0$ is the composite

$$V\mathscr{A}(BC) \times V\mathscr{A}(AB) \xrightarrow[\tilde{V}]{} V(\mathscr{A}(BC) \otimes \mathscr{A}(AB)) \xrightarrow[VM]{} V\mathscr{A}(AC);$$

evaluating this at $f \in V\mathscr{A}(BC)$ and $g \in V\mathscr{A}(AB)$, and using (8.5) and the naturality of ι, we find that $\iota(fg) \in \mathscr{V}_0(I, \mathscr{A}(AC))$ is the composite

$$I \xrightarrow[l_I^{-1}]{} I \otimes I \xrightarrow[\iota f \otimes \iota g]{} \mathscr{A}(BC) \otimes \mathscr{A}(AB) \xrightarrow[M]{} \mathscr{A}(AC). \tag{8.14}$$

We now define $\mathscr{A}(1, f)$ by (8.11), and observe at once that by (8.8) and VC1′, $\mathscr{A}(1, 1) = 1$. Similarly we define $\mathscr{A}(f, 1)$ by (8.12). We have yet to prove that these definitions, together with (8.6), give a bifunctor hom $\mathscr{A}$, that (8.7) is satisfied, and that M is natural.

Since we have (8.9), (8.7) will follow if we prove $V\mathscr{A}(1, f) = \mathscr{A}_0(1, f)$ and $V\mathscr{A}(f, 1) = \mathscr{A}_0(f, 1)$; by symmetry we need only prove one of these. We take the first and express it in the evaluated form

$$(V\mathscr{A}(1, f))g = fg, \quad \text{where} \quad g \in \mathscr{A}_0(AB) \quad \text{and} \quad f \in \mathscr{A}_0(BC).$$

By the naturality of ι, the same statement may be expressed:

$$\text{the composite} \quad I \xrightarrow[\iota g]{} \mathscr{A}(AB) \xrightarrow[\mathscr{A}(1,f)]{} \mathscr{A}(AC) \quad \text{is} \quad \iota(fg). \tag{8.15}$$

Consider the diagram

$$\begin{tikzcd}
I \arrow[r, "\iota g"] \arrow[d, "l^{-1}"'] & \mathscr{A}(AB) \arrow[r, "\mathscr{A}(1,f)"] \arrow[d, "l^{-1}"] & \mathscr{A}(AC) \\
I \otimes I \arrow[r, "1 \otimes \iota g"'] & I \otimes \mathscr{A}(AB) \arrow[r, "\iota f \otimes 1"'] & \mathscr{A}(BC) \otimes \mathscr{A}(AB); \arrow[u, "M"']
\end{tikzcd}$$

the left region commutes by the naturality of l, and the right region by (8.11); since by (8.14) the long leg is $\iota(fg)$, we have (8.15).

We now prove part of the naturality of M, namely the commutativity of

$$\begin{tikzcd}
\mathscr{A}(BC) \otimes \mathscr{A}(AB) \arrow[r, "M"] \arrow[d, "\mathscr{A}(1,f) \otimes 1"'] & \mathscr{A}(AC) \arrow[d, "\mathscr{A}(1,f)"] \\
\mathscr{A}(BD) \otimes \mathscr{A}(AB) \arrow[r, "M"'] & \mathscr{A}(AD)
\end{tikzcd} \tag{8.16}$$

where $f : C \to D$. Writing x for ιf and using (8.11), (8.16) is the exterior of the following diagram:

$$\begin{tikzcd}
\mathscr{A}(BC) \otimes \mathscr{A}(AB) \arrow[rr, "M"] \arrow[d, "l^{-1} \otimes 1"'] \arrow[dr, "l^{-1}"] & & \mathscr{A}(AC) \arrow[d, "l^{-1}"] \\
(I \otimes \mathscr{A}(BC)) \otimes \mathscr{A}(AB) \arrow[r, "a"'] \arrow[d, "(x \otimes 1) \otimes 1"'] & I \otimes (\mathscr{A}(BC) \otimes \mathscr{A}(AB)) \arrow[r, "1 \otimes M"'] \arrow[d, "x \otimes 1"] & I \otimes \mathscr{A}(AC) \arrow[d, "x \otimes 1"] \\
(\mathscr{A}(CD) \otimes \mathscr{A}(BC)) \otimes \mathscr{A}(AB) \arrow[r, "a"'] \arrow[d, "M \otimes 1"'] & \mathscr{A}(CD) \otimes (\mathscr{A}(BC) \otimes \mathscr{A}(AB)) \arrow[r, "1 \otimes M"'] & \mathscr{A}(CD) \otimes \mathscr{A}(AC) \arrow[d, "M"] \\
\mathscr{A}(BD) \otimes \mathscr{A}(AB) \arrow[rr, "M"'] & & \mathscr{A}(AD)
\end{tikzcd}$$

One region commutes by VC3′, one by MC1, one by the naturality of l, one by the naturality of a, and one trivially.

Another part of the naturality of M, namely the commutativity of

$$\begin{tikzcd}
\mathscr{A}(CD) \otimes \mathscr{A}(BC) \arrow[r, "M"] \arrow[d, "1 \otimes \mathscr{A}(f, 1)"'] & \mathscr{A}(BD) \arrow[d, "\mathscr{A}(f, 1)"] \\
\mathscr{A}(CD) \otimes \mathscr{A}(AC) \arrow[r, "M"'] & \mathscr{A}(AD)
\end{tikzcd} \tag{8.17}$$

where $f : A \to B$, follows by symmetry. The final part is the commutativity of

$$\begin{tikzcd}
\mathscr{A}(CD) \otimes \mathscr{A}(AB) \arrow[r, "\mathscr{A}(f, 1) \otimes 1"] \arrow[d, "1 \otimes \mathscr{A}(1, f)"'] & \mathscr{A}(BD) \otimes \mathscr{A}(AB) \arrow[d, "M"] \\
\mathscr{A}(CD) \otimes \mathscr{A}(AC) \arrow[r, "M"'] & \mathscr{A}(AD)
\end{tikzcd} \tag{8.18}$$

where $f : B \to C$. Writing x for ιf and using (8.11) and (8.12), (8.18) is the exterior of the following diagram:

$$\begin{tikzcd}
 & \mathscr{A}(CD) \otimes \mathscr{A}(AB) \arrow[dl, "r^{-1} \otimes 1"'] \arrow[dr, "1 \otimes l^{-1}"] & \\
(\mathscr{A}(CD) \otimes I) \otimes \mathscr{A}(AB) \arrow[rr, "a"'] \arrow[d, "(1 \otimes x) \otimes 1"'] & & \mathscr{A}(CD) \otimes (I \otimes \mathscr{A}(AB)) \arrow[d, "1 \otimes (x \otimes 1)"] \\
(\mathscr{A}(CD) \otimes \mathscr{A}(BC)) \otimes \mathscr{A}(AB) \arrow[rr, "a"'] \arrow[d, "M \otimes 1"'] & & \mathscr{A}(CD) \otimes (\mathscr{A}(BC) \otimes \mathscr{A}(AB)) \arrow[d, "1 \otimes M"] \\
\mathscr{A}(BD) \otimes \mathscr{A}(AB) \arrow[r, "M"'] & \mathscr{A}(AD) & \mathscr{A}(CD) \otimes \mathscr{A}(AC) \arrow[l, "M"]
\end{tikzcd}$$

This commutes by MC2, the naturality of a, and VC3$'$.

It now remains to prove that hom $\mathscr{A}$ is a functor; we need to show that $\mathscr{A}(1, f)\mathscr{A}(1, g) = \mathscr{A}(1, fg)$, that $\mathscr{A}(g, 1)\mathscr{A}(f, 1) = \mathscr{A}(fg, 1)$, and that $\mathscr{A}(1, f)\, \mathscr{A}(g, 1) = \mathscr{A}(g, 1)\mathscr{A}(1, f)$. We need not prove the second of these, for it will follow by symmetry when we have proved the first. Consider the diagram, where $g : B \to C$ and $f : C \to D$:

$$\begin{tikzcd}
 & \mathscr{A}(BB) \otimes \mathscr{A}(AB) \arrow[r, "M"] \arrow[d, "\mathscr{A}(1, g) \otimes 1"] & \mathscr{A}(AB) \arrow[d, "\mathscr{A}(1, g)"] \\
I \otimes \mathscr{A}(AB) \arrow[ur, "j \otimes 1"] \arrow[r, "\iota g \otimes 1"'] \arrow[dr, "\iota(fg) \otimes 1"'] & \mathscr{A}(BC) \otimes \mathscr{A}(AB) \arrow[r, "M"] \arrow[d, "\mathscr{A}(1, f) \otimes 1"] & \mathscr{A}(AC) \arrow[d, "\mathscr{A}(1, f)"] \\
 & \mathscr{A}(BD) \otimes \mathscr{A}(AB) \arrow[r, "M"'] & \mathscr{A}(AD)
\end{tikzcd}$$

The triangles commute by (8.15) (using $j = \iota 1$) and the rectangles by (8.16). Since $M(j \otimes 1) = l$ by VC1$'$, one leg is $\mathscr{A}(1, f)\mathscr{A}(1, g)l$; the other leg is $\mathscr{A}(1, fg)l$ by (8.11). Thus $\mathscr{A}(1, f)\mathscr{A}(1, g) = \mathscr{A}(1, fg)$.

Now let $g : A \to B$ and $f : C \to D$, and consider the diagram

$$\begin{tikzcd}
\mathscr{A}(BC) \arrow[r, "l^{-1}"] \arrow[d, "(g, 1)"'] & I \otimes \mathscr{A}(BC) \arrow[r, "\iota f \otimes 1"] \arrow[d, "1 \otimes \mathscr{A}(g, 1)"] & \mathscr{A}(CD) \otimes \mathscr{A}(BC) \arrow[r, "M"] \arrow[d, "1 \otimes \mathscr{A}(g, 1)"] & \mathscr{A}(BD) \arrow[d, "\mathscr{A}(g"] \\
\mathscr{A}(AC) \arrow[r, "l^{-1}"'] & I \otimes \mathscr{A}(AC) \arrow[r, "\iota f \otimes 1"'] & \mathscr{A}(CD) \otimes \mathscr{A}(AC) \arrow[r, "M"'] & \mathscr{A}(AD)
\end{tikzcd}$$

One region commutes by the naturality of l, one trivially, and one by (8.17). The top and the bottom edge are each $\mathscr{A}(1, f)$, by (8.11); hence $\mathscr{A}(g, 1)\mathscr{A}(1, f) = \mathscr{A}(1, f)\mathscr{A}(g, 1)$. This completes the proof.

We leave to the reader the proof of:

Proposition 8.3. *If $\mathscr{V}$ is a normalized monoidal category and $T: \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, there is exactly one functor $T_0: \mathscr{A}_0 \to \mathscr{B}_0$, with $T_0 A = TA$, such that $T_{BC}: \mathscr{A}(BC) \to \mathscr{B}(T_0 B, T_0 C)$ is natural; namely*

$$T_0 = V_* T. \tag{8.19}$$

Remark 8.4. If $\mathscr{V}$ is a normalized monoidal category, it is now possible to define $\mathscr{V}$-natural transformations exactly as in § I.10; the proofs of Proposition I.10.1 and Theorem I.10.2 remain valid word for word.

To discuss the effect of a monoidal functor $\Phi: \mathscr{V} \to \mathscr{V}'$, where $\mathscr{V}$ and $\mathscr{V}'$ are both normalized, we define $\phi_0: V \to V'\phi$ exactly as in Proposition I.3.4, except that we must now write $\mathscr{V}_0(AB)$ and not $V(AB)$, etc. Note that equations I(3.8) and I(3.9) are still valid, and these give I(3.13) and I(4.1).

The analogue of Proposition I.4.5, namely that ϕ_0 is a monoidal natural transformation

$$\phi_0: V \to V'\Phi: \mathscr{V} \to \mathscr{S}, \tag{8.20}$$

is still valid but needs a new proof; we can then write I(4.1) as I(4.3).

To prove (8.20), note that MN1 for ϕ_0 states the commutativity of

$$\begin{tikzcd} * \arrow[r, "V^0"] \arrow[d, "V'^0"'] & VI \arrow[d, "\phi_0"] \\ V'I' \arrow[r, "V'\phi_0"'] & V'\phi I \end{tikzcd}$$

which follows immediately from I(3.9) and (8.3). MN2 for ϕ_0 states the commutativity of

$$\begin{tikzcd} VA \times VB \arrow[rr, "\widetilde{V}"] \arrow[d, "\phi_0 \times \phi_0"'] & & V(A \otimes B) \arrow[d, "\phi_0"] \\ V'\phi A \times V'\phi B \arrow[r, "\widetilde{V}'"'] & V'(\phi A \otimes \phi B) \arrow[r, "V'\widetilde{\phi}"'] & V'\phi(A \otimes B) \end{tikzcd} \tag{8.21}$$

Since $V \cong \mathscr{V}_0(I\,-)$, it follows by repeated application of Proposition 7.4 that it suffices to put $A = B = I$ and verify that both legs of (8.21) have the same value at $(\iota^{-1}1, \iota^{-1}1)$. Using (8.5) and I(3.9), the resulting assertion is the commutativity of

$$\begin{tikzcd}
I' \otimes I' \arrow[rr, "\phi^0 \otimes \phi^0"] & & \phi I \otimes \phi I \arrow[d, "\tilde{\phi}"] \\
I' \arrow[u, "l'^{-1}"] \arrow[r, "\phi^0"'] & \phi I \arrow[r, "\phi l^{-1}"'] & \phi(I \otimes I),
\end{tikzcd}$$

which is immediate from MF1 for Φ and the naturality of l'.

We then have at once the analogue of Propositions I.7.3, I.7.5, and I.7.6; we also have that of Proposition I.9.8, but we need a new proof for this. By symmetry it suffices to prove that

$$\phi\mathscr{A}(1, f) = (\Phi_* \mathscr{A})(1, \phi_0 f)$$

where $f \in \mathscr{A}_0(BC)$; since by I(3.8) (with ιf in place of f) we have $\phi\iota f.\ \phi^0 = \iota'\phi_0 f$, we are led by the definition (8.11) to proving the commutativity of the exterior of:

$$\begin{tikzcd}
\phi\mathscr{A}(AB) \arrow[r, "\phi l^{-1}"] \arrow[d, "l'^{-1}"'] & \phi(I \otimes \mathscr{A}(AB)) \arrow[r, "\phi(\iota f \otimes 1)"] & \phi(\mathscr{A}(BC) \otimes \mathscr{A}(AB)) \arrow[dr, "\phi M"] & \\
 & & & \phi\mathscr{A}(AC) \\
I' \otimes \phi\mathscr{A}(AB) \arrow[r, "\phi^0 \otimes 1"'] & \phi I \otimes \phi\mathscr{A}(AB) \arrow[uu, "\tilde{\phi}"] \arrow[r, "\phi\iota f \otimes 1"'] & \phi\mathscr{A}(BC) \otimes \phi\mathscr{A}(AB) \arrow[uu, "\tilde{\phi}"] \arrow[ur, "M'"'] &
\end{tikzcd}$$

The left region commutes by MF1, the middle region by the naturality of $\tilde{\phi}$, and the right region by (6.1).

We now have at once the analogues of Propositions I.10.3, I.10.5, and I.10.6, and Theorem I.10.7. What we have no analogue of, if $\mathscr{V}$ is not closed, is propositions referring to the $\mathscr{V}$-category $\mathscr{V}$.

Theorem 8.5. *If $\mathscr{V}$ is a monoidal closed category, the monoidal functor $V : \mathscr{V} \to \mathscr{S}$ of Proposition 8.1 coincides with the closed functor $V : \mathscr{V} \to \mathscr{S}$ of Proposition I.3.11. Moreover the category $\mathscr{A}_0$ and the functors* hom $\mathscr{A}$, *T_0, $\Phi_{0\mathscr{A}}$ defined above then coincide with those of Chapter I.*

Proof. Let the unique monoidal closed functor extending $(V, \tilde{V}, V^0)$ be $(V, \tilde{V}, \hat{V}, V^0)$. Since we have the same V^0 as in Proposition I.3.11, and since V_0 depends only on V and V^0, it follows that $(V, \hat{V}, V^0)$ is normal. Hence by NCF1 we have $\hat{V} = V$, as required.

The assertions about $\mathscr{A}_0$, hom $\mathscr{A}$, and T_0 are clear from the uniqueness clauses of Propositions I.9.5 and I.9.6, Theorem 8.2, and Proposition 8.3. The assertion about $\Phi_{0\mathscr{A}}$ is obvious.

Remark 8.6. We have included a definite functor $V : \mathscr{V}_0 \to \mathscr{S}$ as part of the definition of a closed category $\mathscr{V}$, while treating it as an "extra" for a monoidal $\mathscr{V}$, because when $\mathscr{V}$ is closed $\mathscr{V}$ itself is a $\mathscr{V}$-category, and it is most tedious if the $\mathscr{V}_0$ and the hom $\mathscr{V}$ constructed in § I.7 and § I.9 differ from those given as part of the data of $\mathscr{V}$.

Chapter III

Symmetric Monoidal Closed Categories

1. Symmetric Monoidal Categories

A *symmetry* for a monoidal category $\mathscr{V}$ consists of a natural isomorphism $c = c_{AB} : A \otimes B \to B \otimes A$ in $\mathscr{V}_0$, satisfying the following two axioms:

MC6. $c_{BA} c_{AB} = 1 : A \otimes B \to A \otimes B$.

MC7. The following diagram commutes:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a"] \arrow[d, "c \otimes 1"'] & A \otimes (B \otimes C) \arrow[r, "c"] & (B \otimes C) \otimes A \arrow[d, "a"] \\
(B \otimes A) \otimes C \arrow[r, "a"'] & B \otimes (A \otimes C) \arrow[r, "1 \otimes c"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

A monoidal category $\mathscr{V}$ together with a symmetry c for $\mathscr{V}$ is called a *symmetric monoidal category*. Note that a monoidal category $\mathscr{V}$, even a closed one, may admit several distinct symmetries; an example of this is given in § IV.6 below.

We have from MacLane [*14*] and Kelly [*9*]:

Proposition 1.1. *In a symmetric monoidal category $\mathscr{V}$ the natural isomorphisms a, r, l, c are coherent.*

If $\mathscr{V}$ and $\mathscr{V}'$ are symmetric monoidal categories, a monoidal functor $\Phi = (\phi, \tilde{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$ is said to be *symmetric* if the following axiom is satisfied:

MF4. The following diagram commutes:

$$\begin{tikzcd} \phi(A \otimes B) \arrow[r, "\phi c"] & \phi(B \otimes A) \\ \phi A \otimes \phi B \arrow[u, "\tilde{\phi}"] \arrow[r, "c'"'] & \phi B \otimes \phi A \arrow[u, "\tilde{\phi}"'] \end{tikzcd}$$

One easily verifies:

Proposition 1.2. *Composites and inverses of symmetric monoidal functors are symmetric.*

Thus symmetric monoidal categories, symmetric monoidal functors, and monoidal natural transformations (no change in the definition of these last) form a sub-hypercategory $\mathcal{S}\mathcal{M}on$ of $\mathcal{M}on$.

The monoidal closed category $\mathcal{S}$ admits an obvious symmetry $c : A \times B \to B \times A$ given by $c[x, y] = [y, x]$. (In this chapter we shall use square brackets to denote ordered pairs to avoid confusion with our use of $(- -)$ in a closed category.)

Proposition 1.3. *If the symmetric monoidal category $\mathcal{V}$ has a normalization V, ι, the monoidal functor $V : \mathcal{V} \to \mathcal{S}$ is symmetric.*

Proof. To verify the commutativity of

$$\begin{tikzcd} V(A \otimes B) \arrow[r, "Vc"] & V(B \otimes A) \\ VA \times VB \arrow[u, "\tilde{V}"] \arrow[r, "c'"'] & VB \times VA \arrow[u, "\tilde{V}"'] \end{tikzcd} \tag{1.1}$$

it suffices by repeated application of Proposition II. 7.4, since $V \cong \cong \mathcal{V}_0(I -)$, to put $A = B = I$ and show that both legs have the same value at $[\iota^{-1}1, \iota^{-1}1]$. This reduces to showing the commutativity of

$$\begin{tikzcd} I \otimes I \arrow[rr, "c"] & & I \otimes I \\ & I \arrow[ul, "l^{-1}"] \arrow[ur, "l^{-1}"'] & \end{tikzcd}$$

which we have by coherence.

Remark 1.4. In future we shall suppose without explicit mention, wherever the context requires it, that a normalization is chosen for the monoidal category $\mathscr{V}$; if $\mathscr{V}$ is closed the normalization is of course to be the canonical one.

2. Duality for $\mathscr{V}$-categories

If $\mathscr{V}$ is a symmetric monoidal category, $\mathscr{V}$-categories and $\mathscr{V}$-functors are defined as in Chapter II, the symmetry playing no part in these definitions; similarly, given a normalization of $\mathscr{V}$, $\mathscr{V}$-natural transformations are defined as in Chapter II.

Proposition 2.1. *If $\mathscr{V}$ is a symmetric monoidal category and $\mathscr{A}$ is a $\mathscr{V}$-category, the following data define a $\mathscr{V}$-category $\mathscr{A}^*$ called the dual of $\mathscr{A}$:*

(i) $\operatorname{obj} \mathscr{A}^* = \operatorname{obj} \mathscr{A}$; (2.1)

(ii) $\mathscr{A}^*(AB) = \mathscr{A}(BA)$; (2.2)

(iii) $j: I \to \mathscr{A}^*(AA)$ *is* $j: I \to \mathscr{A}(AA)$; (2.3)

(iv) $M: \mathscr{A}^*(BC) \otimes \mathscr{A}^*(AB) \to \mathscr{A}^*(AC)$ *is the composite*

$$\mathscr{A}(CB) \otimes \mathscr{A}(BA) \xrightarrow{c} \mathscr{A}(BA) \otimes \mathscr{A}(CB) \xrightarrow{M} \mathscr{A}(CA). \quad (2.4)$$

Proof. We verify VC3′ for $\mathscr{A}^*$, leaving the reader to verify VC1′ and VC2′. We need the commutativity of the exterior of (see page 515):

The hexagon commutes by coherence, the pentagon by VC3′ for $\mathscr{A}$, and the two quadrangles by the naturality of c.

In the following propositions the absence of a proof indicates that they are straightforward and that their verification is left to the reader.

Proposition 2.2. *If $\mathscr{V}$ is a symmetric monoidal category and $T: \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, the following data define a $\mathscr{V}$-functor $T^*: \mathscr{A}^* \to \mathscr{B}^*$:*

(i) $T^*A = TA$; (2.5)

(ii) $T^*_{BC}: \mathscr{A}^*(BC) \to \mathscr{B}^*(TB, TC)$ *is* $T_{CB}: \mathscr{A}(CB) \to$

$$\to \mathscr{B}(TC, TB). \quad (2.6)$$

Proposition 2.3. *If $\mathscr{V}$ is a symmetric monoidal category, the assignments $\mathscr{A} \mapsto \mathscr{A}^*$, $T \mapsto T^*$ constitute an involutory functor $D: \mathscr{V}_* \to \mathscr{V}_*$.*

Remark 2.4. It will be clear from the context whether $\mathscr{V}_*$ denotes the hypercategory or the underlying category; a notational distinction here would be cumbersome.

Proposition 2.5. *Let $\mathscr{V}$ be a symmetric monoidal category and let $\mathscr{I}$ be the $\mathscr{V}$-category of Remark II.6.2; then $\mathscr{I}^* = \mathscr{I}$ and $J^{A*} = J^A$.*

$$\begin{tikzcd}[column sep=small]
(\mathscr{A}(DC)\otimes\mathscr{A}(CB))\otimes\mathscr{A}(BA) \arrow[rrrr, "a"] \arrow[d, "c\otimes 1"'] & & & & \mathscr{A}(DC)\otimes(\mathscr{A}(CB)\otimes\mathscr{A}(BA)) \arrow[d, "1\otimes c"] \\
(\mathscr{A}(CB)\otimes\mathscr{A}(DC))\otimes\mathscr{A}(BA) \arrow[dr, "c"] \arrow[dd, "M\otimes 1"'] & & & & \mathscr{A}(DC)\otimes(\mathscr{A}(BA)\otimes\mathscr{A}(CB)) \arrow[dl, "c"'] \arrow[dd, "1\otimes M"] \\
& \mathscr{A}(BA)\otimes(\mathscr{A}(CB)\otimes\mathscr{A}(DC)) \arrow[rr, "a^{-1}"] \arrow[dd, "1\otimes M"] & & (\mathscr{A}(BA)\otimes\mathscr{A}(CB))\otimes\mathscr{A}(DC) \arrow[dd, "M\otimes 1"'] & \\
\mathscr{A}(DB)\otimes\mathscr{A}(BA) \arrow[dr, "c"'] & & & & \mathscr{A}(DC)\otimes\mathscr{A}(CA) \arrow[dl, "c"] \\
& \mathscr{A}(BA)\otimes\mathscr{A}(DB) \arrow[dr, "M"'] & & \mathscr{A}(CA)\otimes\mathscr{A}(DC) \arrow[dl, "M"] & \\
& & \mathscr{A}(DA) & &
\end{tikzcd}$$

Proposition 2.6. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor, the functor $\Phi_* : \mathscr{V}_* \to \mathscr{V}'_*$ commutes with D; that is,*

$$(\Phi_* \mathscr{A})^* = \Phi_* \mathscr{A}^*, \tag{2.7}$$

$$(\Phi_* T)^* = \Phi_* T^*. \tag{2.8}$$

Proposition 2.7. *Let $\eta : \Phi \to \Psi : \mathcal{V} \to \mathcal{V}'$ be a monoidal natural transformation between symmetric monoidal functors. Then if $\mathcal{A}$ is a $\mathcal{V}$-category we have*

$$\eta_{*\mathcal{A}^*} = (\eta_{*\mathcal{A}})^* : \Phi_* \mathcal{A}^* \to \Psi_* \mathcal{A}^*. \tag{2.9}$$

Remark 2.8. A $\mathcal{V}$-functor $T : \mathcal{A}^* \to \mathcal{B}$ is sometimes called a *contravariant $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$.*

Applying Proposition 2.6 to the symmetric monoidal functor $V : \mathcal{V} \to \mathcal{S}$ gives:

$$(\mathcal{A}_0)^* = (\mathcal{A}^*)_0, \tag{2.10}$$

$$(T_0)^* = (T^*)_0; \tag{2.11}$$

we therefore write $\mathcal{A}_0^*$, T_0^*; note that for $\mathcal{S}$-categories duality reduces to the classical concept. In the following proposition c is the functor sending $[AB]$ to $[BA]$ and $[f, g]$ to $[g, f]$.

Proposition 2.9. *If $\mathcal{V}$ is a symmetric monoidal category and $\mathcal{A}$ is a $\mathcal{V}$-category, the following diagram of functors commutes:*

$$\begin{tikzcd} \mathcal{A}_0 \times \mathcal{A}_0^* \arrow[r, "\hom \mathcal{A}^*"] \arrow[d, "\mathrm{c}"'] & \mathcal{V}_0 \\ \mathcal{A}_0^* \times \mathcal{A}_0 \arrow[ur, "\hom \mathcal{A}"'] & \end{tikzcd} \tag{2.12}$$

Proof. Commutativity on objects is immediate, and it remains to show that

$$\mathcal{A}^*(1, f) = \mathcal{A}(f, 1), \tag{2.13}$$

$$\mathcal{A}^*(f, 1) = \mathcal{A}(1, f). \tag{2.14}$$

It suffices by symmetry to prove the first of these. Writing x for ιf and $\overline{M}$ for the "M" of $\mathcal{A}^*$, we have by II(8.11) and II(8.12) to prove the commutativity of the exterior of:

$$\begin{tikzcd} & \mathcal{A}(AB) \arrow[dl, "l^{-1}"'] \arrow[dr, "r^{-1}"] & \\ I \otimes \mathcal{A}(AB) \arrow[d, "x \otimes 1"'] & & \mathcal{A}(AB) \otimes I \arrow[ll, "c"'] \arrow[d, "1 \otimes x"] \\ \mathcal{A}(BC) \otimes \mathcal{A}(AB) \arrow[dr, "M"'] & & \mathcal{A}(AB) \otimes \mathcal{A}(BC) \arrow[ll, "c"'] \arrow[dl, "\overline{M}"] \\ & \mathcal{A}(AC) & \end{tikzcd}$$

The top region commutes by coherence, the middle one by the naturality of c, and the bottom one by (2.4).

Proposition 2.10. *Let $\mathscr{V}$ be a symmetric monoidal category and $\alpha : T \to S : \mathscr{A} \to \mathscr{B}$ a $\mathscr{V}$-natural transformation. Then the $\alpha_A : TA \to SA$ are also the components of a $\mathscr{V}$-natural transformation*

$$\alpha^* : S^* \to T^* : \mathscr{A}^* \to \mathscr{B}^*.$$

Proof. In view of Propositions 2.2 and 2.9, VN for α^* is identical with VN for α.

Proposition 2.11. *If $\mathscr{V}$ is a symmetric monoidal category, $D : \mathscr{V}_* \to \mathscr{V}_*$ becomes an involutory hyperfunctor if we set $D\alpha = \alpha^*$; and if $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor, D and Φ_* commute as hyperfunctors; that is,*

$$(\Phi_* \alpha)^* = \Phi_* \alpha^* . \tag{2.15}$$

Proof. Clearly D respects, in a contravariant way, composition of $\mathscr{V}$-natural transformations, with themselves and with $\mathscr{V}$-functors; to be precise we have

$$(\beta \alpha)^* = \alpha^* \beta^* , \tag{2.16}$$

$$(T \alpha)^* = T^* \alpha^* , \tag{2.17}$$

$$(\alpha S)^* = \alpha^* S^* . \tag{2.18}$$

Proposition 2.7 applied to $\phi_0 : V \to V'\Phi : \mathscr{V} \to \mathscr{S}$ gives

$$\Phi_{0\mathscr{B}^*} = (\Phi_{0\mathscr{B}})^* , \tag{2.19}$$

whence (2.15) from I(10.5).

Remark 2.12. Hypercategories will be shown in § IV.2 to be $\mathscr{V}$-categories for a suitable $\mathscr{V}$, namely the category of small categories with an appropriate symmetric monoidal closed structure. The dual of a hypercategory $\mathfrak{A}$ is therefore given by $\mathfrak{A}^*(\mathscr{A}\mathscr{B}) = \mathfrak{A}(\mathscr{B}\mathscr{A})$. However there is another kind of dual given by $\mathfrak{A}^\dagger (\mathscr{A}\mathscr{B}) = \mathfrak{A}(\mathscr{A}\mathscr{B})^*$, the dual of the category $\mathfrak{A}(\mathscr{A}\mathscr{B})$. The type of contravariance exhibited by the hyperfunctor $D : \mathscr{V}_* \to \mathscr{V}_*$ in Proposition 2.11 is that appropriate to this second kind of duality.

3. Tensor Products of $\mathscr{V}$-categories

For a symmetric monoidal category $\mathscr{V}$ we construct by suitable combinations of a, a^{-1}, and c, (the details being irrelevant by coherence), a natural isomorphism

$$m : (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D) ,$$

called the *middle-four interchange.*

Proposition 3.1. *If $\mathscr{V}$ is a symmetric monoidal category and $\mathscr{A}$, $\mathscr{B}$ are $\mathscr{V}$-categories, the following data define a $\mathscr{V}$-category $\mathscr{A} \otimes \mathscr{B}$:*

(i) *the objects of $\mathscr{A} \otimes \mathscr{B}$ are the ordered pairs $[AB]$, $A \in \mathscr{A}$, $B \in \mathscr{B}$;* (3.1)

(ii) $(\mathscr{A} \otimes \mathscr{B})([AB], [A'B']) = \mathscr{A}(AA') \otimes \mathscr{B}(BB')$; (3.2)

(iii) $j : I \to (\mathscr{A} \otimes \mathscr{B})([AB], [AB])$ *is the composite*

$$I \xrightarrow[l^{-1}]{} I \otimes I \xrightarrow[j \otimes j]{} \mathscr{A}(AA) \otimes \mathscr{B}(BB); \tag{3.3}$$

(iv) $M : (\mathscr{A} \otimes \mathscr{B})([A'B'], [A''B'']) \otimes (\mathscr{A} \otimes \mathscr{B})([AB], [A'B']) \to$
$\to (\mathscr{A} \otimes \mathscr{B})([AB], [A''B''])$

is given by the commutative diagram

$$\begin{tikzcd}
(\mathscr{A}(A'A'') \otimes \mathscr{B}(B'B'')) \otimes (\mathscr{A}(AA') \otimes \mathscr{B}(BB')) \arrow[r, "M"] \arrow[d, "m"'] & \mathscr{A}(AA'') \otimes \mathscr{B}(BB'') \\
(\mathscr{A}(A'A'') \otimes \mathscr{A}(AA')) \otimes (\mathscr{B}(B'B'') \otimes \mathscr{B}(BB')) \arrow[ur, "M \otimes M"'] &
\end{tikzcd} \tag{3.4}$$

The proof is straightforward, although the diagrams are hard to fit on a page. The diagram proving VC3′ for $\mathscr{A} \otimes \mathscr{B}$ looks like this:

$$\begin{tikzcd}
\bullet \arrow[rrrr, "a"] \arrow[d, "m \otimes 1"'] & & & & \bullet \arrow[d, "1 \otimes m"] \\
\bullet \arrow[dr, "m"] \arrow[dd, "M \otimes M) \otimes 1"'] & & & & \bullet \arrow[dl, "m"'] \arrow[dd, "1 \otimes (M \otimes"] \\
& \bullet \arrow[rr, "a \otimes a"] \arrow[dd, "(M \otimes 1) \otimes (M \otimes 1)"] & & \bullet \arrow[dd, "(1 \otimes M) \otimes (1 \otimes M)"'] & \\
\bullet \arrow[dr, "m"'] & & & & \bullet \arrow[dl, "m"] \\
& \bullet \arrow[dr, "M \otimes M"'] & & \bullet \arrow[dl, "M \otimes M"] & \\
& & \bullet & &
\end{tikzcd}$$

The hexagon commutes by coherence, the two quadrangles by the naturality of m, and the pentagon by VC3′ for $\mathscr{A}$ and for $\mathscr{B}$. The reader may verify VC1′ and VC2′. Similarly we easily verify:

Proposition 3.2. *If $\mathscr{V}$ is a symmetric monoidal category and $T:\mathscr{A}\to\mathscr{C}$ and $S:\mathscr{B}\to\mathscr{D}$ are $\mathscr{V}$-functors, the following data define a $\mathscr{V}$-functor $T\otimes S:\mathscr{A}\otimes\mathscr{B}\to\mathscr{C}\otimes\mathscr{D}$:*

(i) $(T\otimes S)[A\,B]=[T\,A, S\,B]$; (3.5)

(ii) $(T\otimes S)_{[AB],[A'B']}:(\mathscr{A}\otimes\mathscr{B})([A\,B],[A'\,B'])\to$
$\to(\mathscr{C}\otimes\mathscr{D})([T\,A,S\,B],[T\,A',S\,B'])$ *is*

$$\mathscr{A}(A\,A')\otimes\mathscr{B}(B\,B')\xrightarrow[T_{AA'}\otimes S_{BB'}]{}\mathscr{C}(TA,TA')\otimes\mathscr{D}(S\,B,S\,B'). \qquad (3.6)$$

Proposition 3.3. *If $\mathscr{V}$ is a symmetric monoidal category the assignments $\mathscr{A},\mathscr{B}\mapsto\mathscr{A}\otimes\mathscr{B}$ and $T,S\mapsto T\otimes S$ constitute a functor $\otimes:\mathscr{V}_*\times\mathscr{V}_*\to\to\mathscr{V}_*$.*

We now define $\mathscr{V}$-functors

$$\begin{aligned}
\mathfrak{a}&:(\mathscr{A}\otimes\mathscr{B})\otimes\mathscr{C}\to\mathscr{A}\otimes(\mathscr{B}\otimes\mathscr{C}),\\
\mathfrak{r}&:\mathscr{A}\otimes\mathscr{I}\to\mathscr{A},\\
\mathfrak{l}&:\mathscr{I}\otimes\mathscr{A}\to\mathscr{A},\\
\mathfrak{c}&:\mathscr{A}\otimes\mathscr{B}\to\mathscr{B}\otimes\mathscr{A}.
\end{aligned}$$

For instance, $\mathfrak{a}[[AB]\,C]=[A\,[B\,C]]$, and

$\mathfrak{a}:((\mathscr{A}\otimes\mathscr{B})\otimes\mathscr{C})([[A\,B]\,C],[[A'\,B']\,C'])\to$
$\to(\mathscr{A}\otimes(\mathscr{B}\otimes\mathscr{C}))([A\,[B\,C]],[A'\,[B'\,C']])$

is

$$a:(\mathscr{A}(AA')\otimes\mathscr{B}(B\,B'))\otimes\mathscr{C}(C\,C')\to\mathscr{A}(AA')\otimes(\mathscr{B}(B\,B')\otimes\mathscr{C}(C\,C')).$$

We then easily verify:

Proposition 3.4. *If $\mathscr{V}$ is a symmetric monoidal category, then $\mathfrak{a}$, $\mathfrak{r}$, $\mathfrak{l}$, $\mathfrak{c}$ are coherent natural isomorphisms in $\mathscr{V}_*$, defining on $\mathscr{V}_*$ the structure of a symmetric monoidal "category" $\mathscr{V}_\#$.*

To discuss the effect of a symmetric monoidal functor we need the following lemma, which is an easy consequence of MF3 and MF4:

Lemma 3.5. *For a symmetric monoidal functor $\Phi:\mathscr{V}\to\mathscr{V}'$ we have a commutative diagram:*

$$\begin{tikzcd}
\phi((A\otimes B)\otimes(C\otimes D)) \arrow[r, "\phi m"] & \phi((A\otimes C)\otimes(B\otimes D)) \\
\phi(A\otimes B)\otimes\phi(C\otimes D) \arrow[u, "\tilde{\phi}"] & \phi(A\otimes C)\otimes\phi(B\otimes D) \arrow[u, "\tilde{\phi}"'] \\
(\phi A\otimes\phi B)\otimes(\phi C\otimes\phi D) \arrow[u, "\tilde{\phi}\otimes\tilde{\phi}"] \arrow[r, "m'"'] & (\phi A\otimes\phi C)\otimes(\phi B\otimes\phi D) \arrow[u, "\tilde{\phi}\otimes\tilde{\phi}"']
\end{tikzcd} \qquad (3.7)$$

Now if $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor and $\mathscr{A}, \mathscr{B}$ are $\mathscr{V}$-categories, we define a $\mathscr{V}'$-functor

$$\tilde{\phi}_{\#} : \Phi_* \mathscr{A} \otimes \Phi_* \mathscr{B} \to \Phi_* (\mathscr{A} \otimes \mathscr{B});$$

$\tilde{\phi}_{\#}$ is the identity on objects, and $\tilde{\phi}_{\#[AB][A'B']}$ is

$$\tilde{\phi} : \phi \mathscr{A}(AA') \otimes \phi \mathscr{B}(BB') \to \phi(\mathscr{A}(AA') \otimes \mathscr{B}(BB')). \tag{3.8}$$

Verification that $\tilde{\phi}_{\#}$ is indeed a $\mathscr{V}'$-functor is easy using (3.7). Similarly we define a $\mathscr{V}'$-functor

$$\phi^0_{\#} : \mathscr{I}' \to \Phi_* \mathscr{I};$$

$\phi^0_{\#} * = *$, and $\phi^0_{\#} : \mathscr{I}'(**) \to \phi \mathscr{I}(**)$ is $\phi^0 : I' \to \phi I$.

Proposition 3.6. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor, the triple $(\Phi_*, \tilde{\phi}_{\#}, \phi^0_{\#})$ is a symmetric monoidal functor $\Phi_{\#} : \mathscr{V}_{\#} \to \mathscr{V}'_{\#}$.*

Proposition 3.7. *If $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ is a monoidal natural transformation where Φ and Ψ are symmetric monoidal functors, then*

$$\eta_* : \Phi_* \to \Psi_* : \mathscr{V}_* \to \mathscr{V}'_*$$

is a monoidal natural transformation

$$\eta_* : \Phi_{\#} \to \Psi_{\#} : \mathscr{V}_{\#} \to \mathscr{V}'_{\#}.$$

Proposition 3.8. *The assignments $\mathscr{V} \mapsto \mathscr{V}_{\#}$, $\Phi \mapsto \Phi_{\#}$, $\eta \mapsto \eta_*$, constitute a hyperfunctor from $\mathscr{S}\mathscr{M}on$ to itself.*

Proposition 3.9. *If $\mathscr{V}$ is a symmetric monoidal category we have*

$$(\mathscr{A} \otimes \mathscr{B})^* = \mathscr{A}^* \otimes \mathscr{B}^*, \tag{3.9}$$

$$(T \otimes S)^* = T^* \otimes S^*. \tag{3.10}$$

Proposition 3.10. *If $\mathscr{V}$ is a symmetric monoidal category, D becomes a symmetric monoidal functor $D : \mathscr{V}_{\#} \to \mathscr{V}_{\#}$ if we set $\tilde{D} = 1$, $D^0 = 1$. Moreover if $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor, D commutes with $\Phi_{\#}$.*

We now consider underlying categories. Note that $(\mathscr{A} \otimes \mathscr{B})_0$ is not expressible in terms of $\mathscr{A}_0$ and $\mathscr{B}_0$. Applying Proposition 3.6 to the symmetric monoidal functor $V : \mathscr{V} \to \mathscr{S}$ gives a symmetric monoidal functor $V_{\#} : \mathscr{V}_{\#} \to \mathscr{S}_{\#}$. In particular we have the functor

$$\tilde{V}_{\#} : \mathscr{A}_0 \times \mathscr{B}_0 \to (\mathscr{A} \otimes \mathscr{B})_0;$$

it is the identity on objects and is given on morphisms by

$$\tilde{V}_{\#}[f, g] = \tilde{V}[f, g]. \tag{3.11}$$

Let us introduce for this last morphism the notation

$$f \otimes g = \tilde{V}[f, g]; \tag{3.12}$$

thus if $f : A \to A'$ in $\mathscr{A}_0$ and $g : B \to B'$ in $\mathscr{B}_0$ we have $f \otimes g : [AB] \to$ $\to [A'B']$ in $(\mathscr{A} \otimes \mathscr{B})_0$. (If $\mathscr{V}$ happens to be closed and if $\mathscr{A} = \mathscr{B} = \mathscr{V}$, one must be careful to distinguish this from $f \otimes g : A \otimes B \to A' \otimes B'$ in $\mathscr{V}_0$.)

Since $\tilde{V}_{\#}$ is a functor we have

$$hf \otimes kg = (h \otimes k)(f \otimes g), \tag{3.13}$$

$$1 \otimes 1 = 1. \tag{3.14}$$

From the naturality of $\tilde{V}_{\#}$ we get, for $\mathscr{V}$-functors $T : \mathscr{A} \to \mathscr{C}$ and $S : \mathscr{B} \to \mathscr{D}$,

$$(T \otimes S)_0 (f \otimes g) = T_0 f \otimes S_0 g. \tag{3.15}$$

The fact that $V_{\#}$ satisfies MF1—MF4 can be expressed in terms of the functors $\mathfrak{a}_0 : ((\mathscr{A} \otimes \mathscr{B}) \otimes \mathscr{C})_0 \to (\mathscr{A} \otimes (\mathscr{B} \otimes \mathscr{C}))_0$, etc., as follows:

$$\mathfrak{l}_0 (1 \otimes f) = f; \tag{3.16}$$

$$\mathfrak{r}_0 (f \otimes 1) = f; \tag{3.17}$$

$$\mathfrak{a}_0 ((f \otimes g) \otimes h) = f \otimes (g \otimes h); \tag{3.18}$$

$$\mathfrak{c}_0 (f \otimes g) = g \otimes f. \tag{3.19}$$

Note that the 1 of (3.16) and (3.17) is the 1 of $\mathscr{I}_0$, and is the element $V^0 * = \iota^{-1} 1$ of VI.

Lemma 3.11. *Let $f \in \mathscr{A}_0(BC)$ and $g \in \mathscr{B}_0(YZ)$, so that*

$$f \otimes g \in (\mathscr{A} \otimes \mathscr{B})_0 ([B\,Y], [CZ]).$$

Then $\iota f : I \to \mathscr{A}(BC)$, $\iota g : I \to \mathscr{B}(YZ)$, and $\iota(f \otimes g) : I \to \mathscr{A}(BC) \otimes \mathscr{B}(YZ)$ are connected by the commutative diagram:

$$\begin{tikzcd} I \arrow[r, "\iota(f \otimes g)"] \arrow[d, "l^{-1}"'] & \mathscr{A}(BC) \otimes \mathscr{B}(YZ) \\ I \otimes I \arrow[ur, "\iota f \otimes \iota g"'] & \end{tikzcd} \tag{3.20}$$

Proof. Immediate from II (8.5).

In the following proposition $\mathfrak{m}$ is the middle-four interchange for categories:

Proposition 3.12. *If $\mathscr{V}$ is a symmetric monoidal category and $\mathscr{A}$, $\mathscr{B}$*

are $\mathscr{V}$-categories, the following diagram of functors commutes:

$$\begin{tikzcd}
(\mathscr{A} \otimes \mathscr{B})_0^* \times (\mathscr{A} \otimes \mathscr{B})_0 \arrow[rr, "\hom(\mathscr{A} \otimes \mathscr{B})"] & & \mathscr{V}_0 \\
(\mathscr{A}_0^* \times \mathscr{B}_0^*) \times (\mathscr{A}_0 \times \mathscr{B}_0) \arrow[u, "\tilde{V}_{\#}^* \times \tilde{V}_{\#}"] \arrow[d, "\mathfrak{m}"'] & & \\
(\mathscr{A}_0^* \times \mathscr{A}_0) \times (\mathscr{B}_0^* \times \mathscr{B}_0) \arrow[rr, "\hom \mathscr{A} \times \hom \mathscr{B}"'] & & \mathscr{V}_0 \times \mathscr{V}_0 \arrow[uu, "\otimes"']
\end{tikzcd} \tag{3.21}$$

Proof. Commutativity on objects is immediate, and commutativity on morphisms states:

$$(\mathscr{A} \otimes \mathscr{B})\,(h \otimes k, f \otimes g) = \mathscr{A}(h, f) \otimes \mathscr{B}(k, g)\,. \tag{3.22}$$

It suffices by symmetry to prove:

$$(\mathscr{A} \otimes \mathscr{B})\,(1, f \otimes g) = \mathscr{A}(1, f) \otimes \mathscr{B}(1, g)\,. \tag{3.23}$$

Writing x and y for ιf and ιg, and $\bar{M}$ for the "M" of $\mathscr{A} \otimes \mathscr{B}$, we need by II(8.11) and by (3.20) the commutativity of the exterior of:

$$\begin{tikzcd}
\mathscr{A}(AB) \otimes \mathscr{B}(XY) \arrow[rr, "l^{-1}"] \arrow[d, "l^{-1} \otimes l^{-1}"'] & & I \otimes (\mathscr{A}(AB) \otimes \mathscr{B}(XY)) \arrow[d, "l^{-1} \otimes 1"] \\
(I \otimes \mathscr{A}(AB)) \otimes (I \otimes \mathscr{B}(XY)) \arrow[d, "(x \otimes 1) \otimes (y \otimes 1)"'] & & (I \otimes I) \otimes (\mathscr{A}(AB) \otimes \mathscr{B}(XY)) \arrow[ll, "m"'] \arrow[d, "(x \otimes y) \otimes 1"] \\
(\mathscr{A}(BC) \otimes \mathscr{A}(AB)) \otimes (\mathscr{B}(YZ) \otimes \mathscr{B}(XY)) \arrow[dr, "M \otimes M"'] & & (\mathscr{A}(BC) \otimes \mathscr{B}(YZ)) \otimes (\mathscr{A}(AB) \otimes \mathscr{B}(\arrow[ll, "m"] \arrow[dl, "\overline{M}"] \\
& \mathscr{A}(AC) \otimes \mathscr{B}(XZ) &
\end{tikzcd}$$

The top region commutes by coherence, the middle region by the naturality of m, and the bottom region by (3.4).

Proposition 3.13. *Let $\mathscr{V}$ be a symmetric monoidal category and let $\alpha : T \to S : \mathscr{A} \to \mathscr{C}$ and $\beta : P \to Q : \mathscr{B} \to \mathscr{D}$ be $\mathscr{V}$-natural transformations. Then the $\alpha_A \otimes \beta_B : [TA, PB] \to [SA, QB]$ are the components of a $\mathscr{V}$-natural transformation $\alpha \otimes \beta : T \otimes P \to S \otimes Q : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C} \otimes \mathscr{D}$.*

Proof. In view of (3.22), VN for $\alpha \otimes \beta$ is the tensor product of VN for α and VN for β.

Proposition 3.14. *If $\mathscr{V}$ is a symmetric monoidal category, $\otimes : \mathscr{V}_* \times \times \mathscr{V}_* \to \mathscr{V}_*$ is in fact a hyperfunctor and the natural isomorphisms $\mathfrak{a}$, $\mathfrak{r}$, $\mathfrak{l}$, $\mathfrak{c}$ are in fact hypernatural.*

Proof. To show that $\otimes$ is a hyperfunctor we need to show

$$(\alpha \otimes \beta)(P \otimes Q) = \alpha P \otimes \beta Q, \tag{3.24}$$

$$(T \otimes S)(\alpha \otimes \beta) = T\alpha \otimes S\beta, \tag{3.25}$$

$$(\gamma \otimes \delta)(\alpha \otimes \beta) = \gamma\alpha \otimes \delta\beta, \text{ and } 1 \otimes 1 = 1. \tag{3.26}$$

Of these, (3.24) is trivial, (3.25) is immediate from (3.15), and (3.26) is immediate from (3.13) and (3.14).

The hypernaturality of $\mathfrak{a}$, $\mathfrak{r}$, $\mathfrak{l}$, $\mathfrak{c}$ is immediate from (3.16)—(3.19).

Proposition 3.15. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor, the natural transformation $\tilde{\phi}_{\#} : \Phi_* \mathscr{A} \otimes \Phi_* \mathscr{B} \to \Phi_*(\mathscr{A} \otimes \mathscr{B})$ is in fact hypernatural.*

Proof. Let $\alpha : T \to T' : \mathscr{A} \to \mathscr{C}$ and $\beta : S \to S' : \mathscr{B} \to \mathscr{D}$ be $\mathscr{V}$-natural transformations. By the naturality of $\tilde{\phi}_{\#}$ we have a commutative diagram

$$\begin{tikzcd}
\Phi_*\mathscr{A} \otimes \Phi_*\mathscr{B} \arrow[r, "\tilde{\phi}_{\#}"] \arrow[d, "\Phi_*T \otimes \Phi_*S"'] & \Phi_*(\mathscr{A} \otimes \mathscr{B}) \arrow[d, "\Phi_*(T \otimes S)"] \\
\Phi_*\mathscr{C} \otimes \Phi_*\mathscr{D} \arrow[r, "\tilde{\phi}_{\#}"'] & \Phi_*(\mathscr{C} \otimes \mathscr{D})
\end{tikzcd} \tag{3.27}$$

and a similar diagram for T', S'. The hypernaturality of $\tilde{\phi}_{\#}$ means that we also have

$$\tilde{\phi}_{\#} \cdot (\Phi_* \alpha \otimes \Phi_* \beta) = \Phi_*(\alpha \otimes \beta) \cdot \tilde{\phi}_{\#}; \tag{3.28}$$

in view of the various definitions this follows from II(8.21).

Proposition 3.16. *If $\mathscr{V}$ is a symmetric monoidal category and α and β are $\mathscr{V}$-natural transformations, we have*

$$(\alpha \otimes \beta)^* = \alpha^* \otimes \beta^*. \tag{3.29}$$

4. $\mathscr{V}$-bifunctors

If $\mathscr{V}$ is a symmetric monoidal category, a $\mathscr{V}$-functor $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is often called a *$\mathscr{V}$-bifunctor*. For its value on objects we use the usual notation $T(AB)$ instead of $T[AB]$. Given such a $\mathscr{V}$-bifunctor we define for each $A \in \mathscr{A}$ a $\mathscr{V}$-functor $T(A\,—) : \mathscr{B} \to \mathscr{C}$ as the composite

$$T(A\,—): \qquad \mathscr{B} \xrightarrow{l^{-1}} \mathscr{I} \otimes \mathscr{B} \xrightarrow{J_A \otimes 1} \mathscr{A} \otimes \mathscr{B} \xrightarrow{T} \mathscr{C}. \tag{4.1}$$

Similarly for each $B \in \mathscr{B}$ we define a $\mathscr{V}$-functor $T(—\,B) : \mathscr{A} \to \mathscr{C}$ as the composite

$$T(—\,B): \qquad \mathscr{A} \xrightarrow{r^{-1}} \mathscr{A} \otimes \mathscr{I} \xrightarrow{1 \otimes J_B} \mathscr{A} \otimes \mathscr{B} \xrightarrow{T} \mathscr{C}. \tag{4.2}$$

We call $T(A\,—)$ and $T(—\,B)$ the *partial functors* of T. For their values on objects we have

$$T(A\,—)\,B = T(—\,B)\,A = T(A\,B). \tag{4.3}$$

Proposition 4.1. *If $\mathscr{V}$ is a symmetric monoidal category and $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is a $\mathscr{V}$-functor, the following diagram commutes, and each leg is equal to $T_{[AB][A'B']} : \mathscr{A}(AA') \otimes \mathscr{B}(BB') \to \mathscr{C}(T(AB), T(A'B'))$:*

$$\begin{tikzcd}
\mathscr{A}(A\,A') \otimes \mathscr{B}(BB') \arrow[r, "T(-B') \otimes T(A-)"] \arrow[dd, "c"'] & \mathscr{C}(T(A\,B'), T(A'B')) \otimes \mathscr{C}(T(A\,B), T(A\,B')) \arrow[d, "M"] \\
 & \mathscr{C}(T(A\,B), T(A'B')) \\
\mathscr{B}(BB') \otimes \mathscr{A}(A\,A') \arrow[r, "T(A'-) \otimes T(-B)"'] & \mathscr{C}(T(A'\,B), T(A'\,B')) \otimes \mathscr{C}(T(A\,B), T(A'\,B)) \arrow[u, "M"']
\end{tikzcd} \tag{4.4}$$

Proof. Consider the diagram

$$\begin{tikzcd}[column sep=small]
 & \mathscr{A}(A\,A') \otimes \mathscr{B}(BB') \arrow[dl, "r^{-1} \otimes l^{-1}"'] \arrow[dr, "r^{-1} \otimes l^{-1}"] & \\
(\mathscr{A}(A\,A') \otimes I) \otimes (I \otimes \mathscr{B}(BB')) \arrow[d, "(1 \otimes j) \otimes (j \otimes 1)"'] & & (\mathscr{A}(A\,A') \otimes I) \otimes (I \otimes \mathscr{B}(BB')) \arrow[ll, "m"'] \arrow[d, "(1 \otimes j) \otimes (j \otimes 1)"] \\
(\mathscr{A}(A\,A') \otimes \mathscr{A}(A\,A)) \otimes (\mathscr{B}(B'\,B') \otimes B(BB')) \arrow[d, "M \otimes M"'] & & (\mathscr{A}(A\,A') \otimes \mathscr{B}(B'\,B')) \otimes (\mathscr{A}(A\,A) \otimes \mathscr{B}(B \arrow[ll, "m"'] \arrow[d, "T \otimes T"] \\
\mathscr{A}(A\,A') \otimes \mathscr{B}(BB') \arrow[dr, "T"'] & & \mathscr{C}(T(A\,B'), T(A'\,B')) \otimes \mathscr{C}(T(A\,B), T(A\,B \arrow[dl, "M"] \\
 & \mathscr{C}(T(A\,B), T(A'\,B')) &
\end{tikzcd}$$

The top region commutes by coherence, the middle region by the naturality of m, and the bottom region by VF2′ for T (in view of (3.4)). Thus the exterior commutes. By VC2′ for $\mathscr{A}$ and VC1′ for $\mathscr{B}$, the left leg is just T; by (4.1) and (4.2) the right leg is the upper leg of (4.4).

The proof that the lower leg of (4.4) is also T is entirely similar, using the diagram

$$\begin{tikzcd}
\mathscr{A}(AA') \otimes \mathscr{B}(BB') \arrow[rr, "c"] \arrow[d, "l^{-1} \otimes r^{-1}"'] & & \mathscr{B}(BB') \otimes \mathscr{A}(AA') \arrow[d, "l^{-1} \otimes r^{-1}"] \\
(I \otimes \mathscr{A}(AA')) \otimes (\mathscr{B}(BB') \otimes I) \arrow[d, "(j \otimes 1) \otimes (1 \otimes j)"'] & & (I \otimes \mathscr{B}(BB')) \otimes (\mathscr{A}(AA') \otimes I) \arrow[ll, "m"'] \arrow[d, "(j \otimes 1) \otimes (1 \otimes j)"] \\
(A'A') \otimes \mathscr{A}(AA')) \otimes (\mathscr{B}(BB') \otimes \mathscr{B}(BB)) \arrow[d, "M \otimes M"'] & & (\mathscr{A}(A'A') \otimes \mathscr{B}(BB')) \otimes (\mathscr{A}(AA') \otimes \mathscr{B}(BB)) \arrow[ll, "m"'] \arrow[d, "T \otimes T"] \\
\mathscr{A}(AA') \otimes \mathscr{B}(BB') \arrow[dr, "T"'] & & \mathscr{C}(T(A'B), T(A'B')) \otimes \mathscr{C}(T(AB), T(A'B)) \arrow[dl, "M"] \\
& \mathscr{C}(T(AB), T(A'B')) &
\end{tikzcd}$$

Proposition 4.2. *Let $\mathscr{V}$ be a symmetric monoidal category and let $T(A\,-) : \mathscr{B} \to \mathscr{C}$ and $T(-\,B) : \mathscr{A} \to \mathscr{C}$ be families of $\mathscr{V}$-functors indexed by $A \in \mathscr{A}$ and $B \in \mathscr{B}$ respectively. Suppose that $T(A\,-)\,B = T(-\,B)\,A$, and write $T(A\,B)$ for their common value; and suppose that* (4.4) *commutes. Then there is a unique $\mathscr{V}$-functor $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ of which $T(A\ \ -)$ and $T(-\,B)$ are the partial functors.*

Proof. Define $T : \mathscr{A}(AA') \otimes \mathscr{B}(BB') \to \mathscr{C}(T(A\,B), T(A'\,B'))$ to be the top leg of (4.4); this is forced by Proposition 4.1, which proves the uniqueness.

VF1′ for T requires the commutativity of the exterior of the diagram

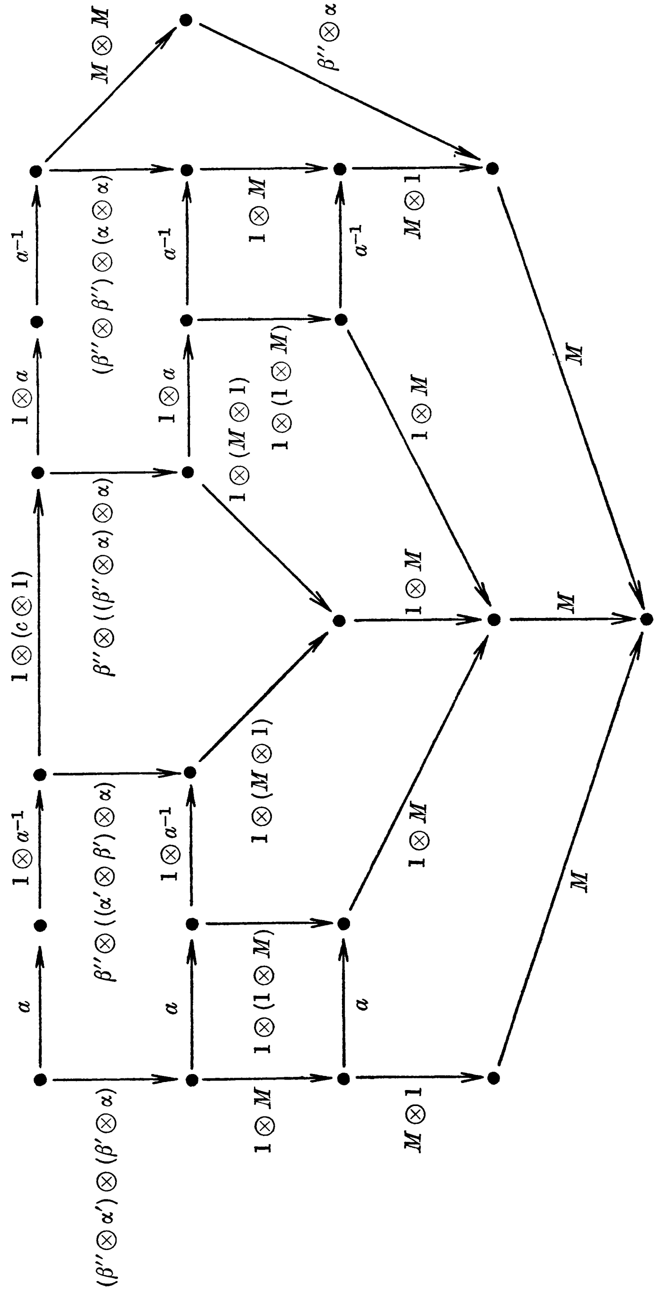
$$\begin{tikzcd}
\mathscr{A}(AA) \otimes \mathscr{B}(BB) \arrow[rr, "T(-B) \otimes T(A-)"] & & \mathscr{C}(T(AB), T(AB)) \otimes \mathscr{C}(T(AB), T(AB)) \arrow[dd, "M"] \\
I \otimes I \arrow[u, "j \otimes j"] \arrow[r, "1 \otimes j"'] & I \otimes \mathscr{C}(T(AB), T(AB)) \arrow[ur, "j \otimes 1"'] \arrow[dr, "l"] & \\
I \arrow[u, "l^{-1}"] \arrow[rr, "j"'] & & \mathscr{C}(T(AB), T(AB))
\end{tikzcd}$$

The top region commutes by VF 1′ for $T(-B)$ and for $T(A-)$, the bottom region by the naturality of l, and the right region by VC 1′ for $\mathscr{C}$.

VF 2′ for T requires the commutativity of the following diagram, in which $\alpha, \alpha', \beta', \beta''$ stand for $T(A-)$, $T(A'-)$, $T(-B')$, $T(-B'')$, and in which $(A'B', A''B'')$ stands for $\mathscr{C}(T(A'B'), T(A''B''))$, etc.:

$$
\begin{tikzcd}
(\mathscr{A}(A'A'')\otimes\mathscr{B}(B'B''))\otimes(\mathscr{A}(AA')\otimes\mathscr{B}(BB')) \arrow[r, "m"] \arrow[d, "(\beta''\otimes\alpha')\otimes(\beta'\otimes\alpha)"'] & (\mathscr{A}(A'A'')\otimes\mathscr{A}(AA'))\otimes(\mathscr{B}(B'B'')\otimes\mathscr{B}(BB')) \arrow[d, "M\otimes M"] \\
((A'B'', A''B'')\otimes(A'B', A'B''))\otimes((AB', A'B')\otimes(AB, AB')) \arrow[d, "M\otimes M"'] & \mathscr{A}(AA'')\otimes\mathscr{B}(BB'') \arrow[d, "\beta''\otimes\alpha"] \\
(A'B', A''B'')\otimes(AB, A'B') \arrow[dr, "M"'] & (AB'', A''B'')\otimes(AB, AB'') \arrow[d, "M"] \\
 & (AB, A''B'')
\end{tikzcd}
$$

Writing m out in terms of a and c, this becomes the exterior of the following diagram, in which we leave the reader to fill in the objects:



The two hexagons at the top commute by the naturality of a. The pentagon in the middle of the top commutes by (4.4) (tensored with two other diagrams that commute trivially). The pentagon at the far right commutes by VF2′ for $T(-B'')$ and for $T(A-)$. The four other pentagons commute by VC3′ for $\mathscr{C}$, and the two quadrangles by the naturality of a.

Finally it is easily verified that T has the given partial functors.

Remark 4.3. We note for later purposes that (4.4) is needed only to get VF2′ for T.

Proposition 4.4. *Let $\mathscr{V}$ be a symmetric monoidal category and let $P:\mathscr{A}'\to\mathscr{A}$, $Q:\mathscr{B}'\to\mathscr{B}$, $N:\mathscr{C}\to\mathscr{C}'$, and $T:\mathscr{A}\otimes\mathscr{B}\to\mathscr{C}$ be $\mathscr{V}$-functors. Define $S:\mathscr{A}'\otimes\mathscr{B}'\to\mathscr{C}'$ to be the composite*

$$\mathscr{A}'\otimes\mathscr{B}'\xrightarrow[P\otimes Q]{}\mathscr{A}\otimes\mathscr{B}\xrightarrow[T]{}\mathscr{C}\xrightarrow[N]{}\mathscr{C}'.$$

Then $S(-B'):\mathscr{A}'\to\mathscr{C}'$ is the composite

$$\mathscr{A}'\xrightarrow[P]{}\mathscr{A}\xrightarrow[T(-,QB')]{}\mathscr{C}\xrightarrow[N]{}\mathscr{C}',$$

with a similar formula for $S(A'-)$.

Proposition 4.5. *If $\mathscr{V}$ is a symmetric monoidal category and $P:\mathscr{A}\to\mathscr{C}$, $Q:\mathscr{B}\to\mathscr{D}$ are $\mathscr{V}$-functors, set $T=P\otimes Q:\mathscr{A}\otimes\mathscr{B}\to\mathscr{C}\otimes\mathscr{D}$. Then $T(-B)$ is the composite*

$$\mathscr{A}\xrightarrow[\mathfrak{r}^{-1}]{}\mathscr{A}\otimes\mathscr{I}\xrightarrow[P\otimes J_{QB}]{}\mathscr{C}\otimes\mathscr{D}.$$

Proposition 4.6. *If $\mathscr{V}$ is a symmetric monoidal category and $P:\mathscr{A}\to\mathscr{B}$ is a $\mathscr{V}$-functor, denote by T the composite $\mathscr{I}\otimes\mathscr{A}\xrightarrow[\mathfrak{l}]{}\mathscr{A}\xrightarrow[P]{}\mathscr{B}$. Then $T(*-)=P$.*

Proposition 4.7. *If $\mathscr{V}$ is a symmetric monoidal category and $P:\mathscr{A}\otimes\mathscr{B}\to\mathscr{C}$ is a $\mathscr{V}$-functor, denote by T the composite $\mathscr{B}\otimes\mathscr{A}\xrightarrow[\mathfrak{c}]{}\mathscr{A}\otimes\mathscr{B}\xrightarrow[P]{}\mathscr{C}$. Then $T(-A)=P(A-)$.*

Proposition 4.8. *If $\mathscr{V}$ is a symmetric monoidal category and $P:\mathscr{A}\otimes(\mathscr{B}\otimes\mathscr{C})\to\mathscr{D}$ is a $\mathscr{V}$-functor, denote by T the composite*

$$(\mathscr{A}\otimes\mathscr{B})\otimes\mathscr{C}\xrightarrow[\mathfrak{a}]{}\mathscr{A}\otimes(\mathscr{B}\otimes\mathscr{C})\xrightarrow[P]{}\mathscr{D},$$

and write S for $T(-C):\mathscr{A}\otimes\mathscr{B}\to\mathscr{D}$. Then $S(-B)=P(-[BC])$.

Remark 4.9. Propositions 4.6—4.8 allow us to identify $(\mathscr{A}\otimes\mathscr{B})\otimes\mathscr{C}$ with $\mathscr{A}\otimes(\mathscr{B}\otimes\mathscr{C})$, etc., and to write for example $T:\mathscr{A}\otimes\mathscr{B}\otimes\mathscr{C}\to\mathscr{D}$. We then write $T(-BC)$ etc. for the partial functors.

If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor, and $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is a $\mathscr{V}$-functor, let us write $\Phi_{**} T : \Phi_* \mathscr{A} \otimes \Phi_* \mathscr{B} \to \Phi_* \mathscr{C}$ for the composite

$$\Phi_{**} T : \qquad \Phi_* \mathscr{A} \otimes \Phi_* \mathscr{B} \xrightarrow[\tilde{\Phi}_{\#}]{} \Phi_* (\mathscr{A} \otimes \mathscr{B}) \xrightarrow[\Phi_* T]{} \Phi_* \mathscr{C} . \tag{4.5}$$

Proposition 4.10. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor and $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is a $\mathscr{V}$-functor, we have*

$$(\Phi_{**} T)(A -) = \Phi_* (T(A -)) : \Phi_* \mathscr{A} \to \Phi_* \mathscr{C} . \tag{4.6}$$

Proposition 4.11. *If $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ is a monoidal natural transformation where Φ, Ψ are symmetric monoidal functors, and if $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is a $\mathscr{V}$-functor, we have a commutative diagram:*

$$\begin{tikzcd}
\Phi_*\mathscr{A} \otimes \Phi_*\mathscr{B} \arrow[r, "\eta_{*\mathscr{A}} \otimes \eta_{*\mathscr{B}}"] \arrow[d, "\Phi_{**}T"'] & \Psi_*\mathscr{A} \otimes \Psi_*\mathscr{B} \arrow[d, "\Psi_{**}T"] \\
\Phi_*\mathscr{C} \arrow[r, "\eta_{*\mathscr{C}}"'] & \Psi_*\mathscr{C}
\end{tikzcd} \tag{4.7}$$

Proof. This follows easily from the naturality of η_* together with Proposition 3.7.

Proposition 4.12. *Let $\mathscr{V}$ be a symmetric monoidal category and T, S : $\mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ be $\mathscr{V}$-functors. Let*

$$\alpha_{AB} : T(A\,B) \to S(A\,B) , \quad A \in \mathscr{A}, \ B \in \mathscr{B} ,$$

be a family of morphisms in $\mathscr{C}_0$. Then the α_{AB} are the components of a $\mathscr{V}$-natural transformation $\alpha : T \to S$ if and only if, for each A, α_{AB} is the B-component of a $\mathscr{V}$-natural transformation $\alpha_A : T(A -) \to S(A -)$ and further, for each B, α_{AB} is the A-component of a $\mathscr{V}$-natural transformation $\alpha_{\cdot B} : T(- B) \to S(- B)$.

Proof. Suppose that $\alpha : T \to S : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is $\mathscr{V}$-natural; then so is $\alpha(J^A \otimes 1)\, \mathfrak{l}^{-1} : T(A -) \to S(A -) : \mathscr{B} \to \mathscr{C}$; and the B-component of $\alpha(J^A \otimes 1)\, \mathfrak{l}^{-1}$ is α_{AB}. Thus α_A is $\mathscr{V}$-natural, and a similar argument shows that $\alpha_{\cdot B}$ is $\mathscr{V}$-natural.

Now suppose that α_A and $\alpha_{\cdot B}$ are $\mathscr{V}$-natural. Using the top leg of (4.4) to express T and S, VN for α becomes the exterior of the following diagram:

$$
\begin{tikzcd}
\mathscr{C}(T(AB'), T(A'B')) \otimes \mathscr{C}(T(AB), T(AB')) \arrow[rr, "M"] \arrow[dr, "\mathscr{C}(1,\alpha)\otimes 1"] & & \mathscr{C}(T(AB), T(A'B')) \arrow[dd, "\mathscr{C}(1,\alpha)"] \\
& \mathscr{C}(T(AB'), S(A'B')) \otimes \mathscr{C}(T(AB), T(AB')) \arrow[dr, "M"] & \\
\mathscr{A}(AA') \otimes \mathscr{B}(BB') \arrow[uu, "T(-B')\otimes T(A-)"] \arrow[dd, "S(-B')\otimes S(A-)"'] \arrow[r, "S(-B')\otimes T(A-)"] & \mathscr{C}(S(AB'), S(A'B')) \otimes \mathscr{C}(T(AB), T(AB')) \arrow[u, "\mathscr{C}(\alpha,1)\otimes 1"] \arrow[d, "1\otimes\mathscr{C}(1,\alpha)"] & \mathscr{C}(T(AB), S(A'B')) \\
& \mathscr{C}(S(AB'), S(A'B')) \otimes \mathscr{C}(T(AB), S(AB')) \arrow[ur, "M"] & \\
\mathscr{C}(S(AB'), S(A'B')) \otimes \mathscr{C}(S(AB), S(AB')) \arrow[rr, "M"] \arrow[ur, "1\otimes\mathscr{C}(\alpha,1)"] & & \mathscr{C}(S(AB), S(A'B')) \arrow[uu, "\mathscr{C}(\alpha,1)"']
\end{tikzcd}
$$

The left regions commute by VN for $\alpha_{\cdot B'}$ and α_A, and the other regions by the naturality of M.

Proposition 4.13. *The map* $\alpha \mapsto \alpha_A$ *of Proposition 4.12 satisfies:*

(i) $(N\alpha)_A = N\alpha_A$ *where* $N : \mathscr{C} \to \mathscr{C}'$; (4.8)

(ii) $(\alpha(P \otimes Q))_A = \alpha_{PA} Q$ *where* $P : \mathscr{A}' \to \mathscr{A}$ *and* $Q : \mathscr{B}' \to \mathscr{B}$; (4.9)

(iii) $(\beta\alpha)_A = \beta_A \alpha_A$ *where* $\beta : S \to U : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$. (4.10)

If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric monoidal functor and $\alpha : T \to S : \mathscr{A} \otimes \otimes \mathscr{B} \to \mathscr{C}$ is a $\mathscr{V}$-natural transformation, define

$$\Phi_{**}\alpha : \Phi_{**} T \to \Phi_{**} S : \Phi_* \mathscr{A} \otimes \Phi_* \mathscr{B} \to \Phi_* \mathscr{C}$$

by

$$\Phi_{**}\alpha = \Phi_* \alpha . \tilde{\phi}_{\#} . \tag{4.11}$$

Then, since $\tilde{\phi}_{\#}$ is the identity on objects, we have

$$(\Phi_{**}\alpha)_{AB} = (\Phi_* \alpha)_{AB} , \tag{4.12}$$

which is $\Phi_{0\mathscr{C}}(\alpha_{AB})$ by I(10.5). Thus we have trivially:

Proposition 4.14. *If* $\Phi : \mathscr{V} \to \mathscr{V}'$ *is a symmetric monoidal functor and* $\alpha : T \to S : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ *is a* $\mathscr{V}$*-natural transformation, we have*

$$(\Phi_{**}\alpha)_A = \Phi_* \alpha_A . \tag{4.13}$$

From the hypernaturality of η_* together with Proposition 3.7, we get:

Proposition 4.15. *If* $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ *is a monoidal natural transformation where* Φ *and* Ψ *are symmetric monoidal functors, and if* $\alpha : T \to S : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ *is a* $\mathscr{V}$*-natural transformation, then*

$$\eta_{*\mathscr{C}} . \Phi_{**}\alpha : \eta_{*\mathscr{C}} . \Phi_{**} T \to \eta_{*\mathscr{C}} . \Phi_{**} S : \Phi_* \mathscr{A} \otimes \Phi_* \mathscr{B} \to \Psi_* \mathscr{C}$$

coincides with

$$\Psi_{**}\alpha . (\eta_{*\mathscr{A}} \otimes \eta_{*\mathscr{B}}) : \Psi_{**} T . (\eta_{*\mathscr{A}} \otimes \eta_{*\mathscr{B}}) \to$$
$$\to \Psi_{**} S . (\eta_{*\mathscr{A}} \otimes \eta_{*\mathscr{B}}) : \Phi_* \mathscr{A} \otimes \Phi_* \mathscr{B} \to \Psi_* \mathscr{C} .$$

Proposition 4.16. *Let* $\Phi : \mathscr{V} \to \mathscr{V}'$ *be a symmetric monoidal functor, and* $\alpha : T \to S : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ *a* $\mathscr{V}$*-natural transformation. We have*

$$T^*(A-) = (T(A-))^* , \tag{4.14}$$

$$\Phi_{**} T^* = (\Phi_{**} T)^* , \tag{4.15}$$

$$\Phi_{**}\alpha^* = (\Phi_{**}\alpha)^* . \tag{4.16}$$

Proposition 4.17. *Let* $\Phi : \mathscr{V} \to \mathscr{V}'$ *be a symmetric monoidal functor, let* $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ *be a* $\mathscr{V}$*-functor, and let* $\alpha : P \to P' : \mathscr{A}' \to \mathscr{A}$ *and* $\beta : Q \to Q' : \mathscr{B}' \to \mathscr{B}$ *be* $\mathscr{V}$*-natural transformations. Then*

$$\Phi_{**}T.(\Phi_* P \otimes \Phi_* Q) = \Phi_* T.\Phi_{**}(P \otimes Q) : \Phi_* \mathscr{A}' \otimes \Phi_* \mathscr{B}' \to \Phi_* \mathscr{C}, \tag{4.17}$$

and

$$\Phi_{**}T.(\Phi_* \alpha \otimes \Phi_* \beta) = \Phi_* T.\Phi_{**}(\alpha \otimes \beta). \tag{4.18}$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
\Phi_*\mathscr{A}' \otimes \Phi_*\mathscr{B}' \arrow[r, "\Phi_* P \otimes \Phi_* Q"] \arrow[d, "\tilde{\phi}_\#"'] & \Phi_*\mathscr{A} \otimes \Phi_*\mathscr{B} \arrow[d, "\tilde{\phi}_\#"] \arrow[dr, "\Phi_{**}T"] & \\
\Phi_*(\mathscr{A}' \otimes \mathscr{B}') \arrow[r, "\Phi_*(P \otimes Q)"'] & \Phi_*(\mathscr{A} \otimes \mathscr{B}) \arrow[r, "\Phi_* T"'] & \Phi_*\mathscr{C}
\end{tikzcd}$$

by the naturality of $\tilde{\phi}_\#$ and the definition of $\Phi_{**}T$; and by the hyper-naturality of $\tilde{\phi}_\#$ we have a similar diagram with α, β in place of P, Q.

Taking the symmetric monoidal functor $V : \mathscr{V} \to \mathscr{S}$ and a $\mathscr{V}$-functor $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$, we write

$$T_{00} = V_{**} T, \tag{4.19}$$

so that $T_{00} : \mathscr{A}_0 \times \mathscr{B}_0 \to \mathscr{C}_0$ is the composite

$$\mathscr{A}_0 \times \mathscr{B}_0 \xrightarrow[\tilde{V}_\#]{} (\mathscr{A} \otimes \mathscr{B})_0 \xrightarrow[T_0]{} \mathscr{C}_0. \tag{4.20}$$

If $\alpha : T \to S : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is a $\mathscr{V}$-natural transformation, we have by (4.12) since V is normal

$$(V_{**}\alpha)_{AB} = \alpha_{AB}; \tag{4.21}$$

we shall in practice identify $V_{**}\alpha$ with α, so that we also have $\alpha : T_{00} \to$ $\to S_{00} : \mathscr{A}_0 \times \mathscr{B}_0 \to \mathscr{C}_0$. Proposition 4.17 gives in this case

$$T_{00}(\alpha_A, \beta_B) = T_0(\alpha_A \otimes \beta_B), \tag{4.22}$$

so that by I(10.3) we have

$$(T(\alpha \otimes \beta))_{AB} = T_{00}(\alpha_A, \beta_B). \tag{4.23}$$

Proposition 4.18. *Let $\mathscr{V}$ be a symmetric monoidal category and let $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ be a $\mathscr{V}$-functor. Then*

$$T(A\,-) : \mathscr{B}(BB') \to \mathscr{C}(T_{00}(AB), T_{00}(AB'))$$

is natural in A; with a similar result for $T(-\,B)$.

Proof. Let $f \in \mathscr{A}_0(AA')$ and compose both legs of (4.4) with

$$\mathscr{B}(BB') \xrightarrow[l^{-1}]{} I \otimes \mathscr{B}(BB') \xrightarrow[if \otimes 1]{} \mathscr{A}(AA') \otimes \mathscr{B}(BB').$$

By the naturality of ι, the composite

$$I \xrightarrow[\iota f]{} \mathscr{A}(AA') \xrightarrow[T(-B')]{} \mathscr{C}(T(AB'), T(A'B'))$$

is $\iota((VT(-B'))f)$, that is, $\iota((T(-B'))_0 f)$, or $\iota T_{00}(1, f)$ by (4.6). Using the naturality of c and II (8.11) and II (8.12), we find we have the diagram

$$\begin{tikzcd}
\mathscr{B}(BB') \arrow[r, "T(A-)"] \arrow[d, "T(A'-)"'] & \mathscr{C}(T(AB), T(AB')) \arrow[d, "\mathscr{C}(1, T_{00}(f, 1))"] \\
\mathscr{C}(T(A'B), T(A'B')) \arrow[r, "\mathscr{C}(T_{00}(f, 1), 1)"'] & \mathscr{C}(T(AB), T(A'B'))\cdot
\end{tikzcd} \tag{4.24}$$

which expresses the naturality in A of $T(A-)$.

Proposition 4.19. *Let $\mathscr{V}$ be a symmetric monoidal category and let $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ be a $\mathscr{V}$-functor. Then for each $f \in \mathscr{A}_0(AA')$, the morphism*

$$T_{00}(f, 1) : T(AB) \to T(A'B)$$

is the B-component of a $\mathscr{V}$-natural transformation

$$T(f, 1) : T(A-) \to T(A'-);$$

with a similar result for $T(1, g)$.

Proof. VN for $T(f, 1)$ is (4.24).

5. Extraordinary $\mathscr{V}$-natural Transformations

Let $\mathscr{V}$ be a symmetric monoidal category. We now introduce for $\mathscr{V}$-categories the extraordinary kinds of natural transformation introduced for ordinary categories in [7].

Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathscr{V}$-categories, $T : \mathscr{A}^* \otimes \mathscr{A} \to \mathscr{B}$ a $\mathscr{V}$-functor, and B a fixed object of $\mathscr{B}$. A family of morphisms in $\mathscr{B}_0$,

$$\gamma_A : B \to T(AA), \quad A \in \mathscr{A},$$

is said to be $\mathscr{V}$-natural if the following axiom is satisfied:

VN′. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(AA') \arrow[r, "T(A-)"] \arrow[d, "T(-A')"'] & \mathscr{B}(T(AA), T(AA')) \arrow[d, "\mathscr{B}(\gamma_A, 1)"] \\
\mathscr{B}(T(A'A'), T(AA')) \arrow[r, "\mathscr{B}(\gamma_{A'}, 1)"'] & \mathscr{B}(B, T(AA'))
\end{tikzcd}$$

Similarly a family of morphisms in $\mathscr{B}_0$,

$$\delta_A : T(AA) \to B, \quad A \in \mathscr{A},$$

is said to be $\mathscr{V}$-natural if the following axiom is satisfied:

VN″. The following diagram commutes:

$$\begin{tikzcd}
\mathscr{A}(AA') \arrow[r, "T(-A)"] \arrow[d, "T(A'-)"'] & \mathscr{B}(T(A'A), T(AA)) \arrow[d, "\mathscr{B}(1, \delta_A)"] \\
\mathscr{B}(T(A'A), T(A'A')) \arrow[r, "\mathscr{B}(1, \delta_{A'})"'] & \mathscr{B}(T(A'A), B)
\end{tikzcd}$$

Now if we have $\mathscr{V}$-functors

$$T : \mathscr{A}_1^* \otimes \mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_p^* \otimes \mathscr{A}_p \otimes \mathscr{C}_1 \otimes \cdots \otimes \mathscr{C}_r \to \mathscr{D},$$

$$S : \mathscr{B}_1^* \otimes \mathscr{B}_1 \otimes \cdots \otimes \mathscr{B}_q^* \otimes \mathscr{B}_q \otimes \mathscr{C}_1 \otimes \cdots \otimes \mathscr{C}_r \to \mathscr{D},$$

and morphisms

$$\alpha_{A_1 \cdots A_p B_1 \cdots B_q C_1 \cdots C_r} : T(A_1 A_1 \cdots A_p A_p C_1 \cdots C_r) \to S(B_1 B_1 \cdots B_q B_q C_1 \cdots C_r),$$

we define α to be $\mathscr{V}$-natural if it is so in each variable $A_1 \cdots C_r$ separately when the others are held fixed. Proposition 4.12 shows that we may group the variables $C_1, \ldots, C_r$ at pleasure; the situation is entirely similar for the other variables, and we leave the reader to adapt the proof of Proposition 4.12 to prove:

Proposition 5.1. *If $\mathscr{V}$ is a symmetric monoidal category, if*

$$T : \mathscr{A}^* \otimes \mathscr{B}^* \otimes \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$$

is a $\mathscr{V}$-functor, and if $C \in \mathscr{C}$, a family of morphisms

$$\alpha_{AB} : T(ABAB) \to C$$

is $\mathscr{V}$-natural in $[AB]$ if and only if it is so in each of A, B separately.

Proposition 5.2. *The rules for composition of extraordinary $\mathscr{V}$-natural transformations are formally identical with those of* [7].

Proof. The considerations of [7] use only the formal properties of diagrams VN, VN′, VN″.

For composition of $\mathscr{V}$-natural transformations with $\mathscr{V}$-functors, we have:

Proposition 5.3. *Let $\mathscr{V}$ be a symmetric monoidal category, let*

$$T : \mathscr{A}^* \otimes \mathscr{A} \to \mathscr{B}, \quad P : \mathscr{C} \to \mathscr{A}, \quad Q : \mathscr{B} \to \mathscr{D}$$

be $\mathscr{V}$-functors, and let $\gamma_A : B \to T(A\,A)$ be $\mathscr{V}$-natural. Then the family of morphisms

$$Q_0 \gamma_{PC} : QB \to QT(PC, PC)$$

is also $\mathscr{V}$-natural, the relevant bifunctor now being

$$QT(P^* \otimes P) : \mathscr{C}^* \otimes \mathscr{C} \to \mathscr{D}.$$

We leave the reader to adapt the proof from that of Theorem I.10.2. There is of course a corresponding result for $\mathscr{V}$-natural transformations of the VN″ type, but it is only the dual of the above and needs no separate proof. Note that by Remark I.10.12 we may write $Q\gamma_{PC}$ instead of $Q_0\gamma_{PC}$; for the name of the family we shall use $Q\gamma P$.

The proof of the following proposition is exactly like that of Proposition I.10.3:

Proposition 5.4. *Let $\Phi : \mathscr{V} \to \mathscr{V}'$ be a symmetric monoidal functor, $T : \mathscr{A}^* \otimes \mathscr{A} \to \mathscr{B}$ a $\mathscr{V}$-functor, and $\gamma_A : B \to T(A\,A)$ a $\mathscr{V}$-natural transformation. Then $\Phi_{0\mathscr{B}}\gamma_A : B \to (\Phi_{**}T)(A\,A)$ is a $\mathscr{V}'$-natural transformation, which we shall write $\Phi_*\gamma$.*

Analogously to Proposition I.10.6 we have (cf. also Proposition 4.11):

Proposition 5.5. *Let $\eta : \Phi \to \Psi : \mathscr{V} \to \mathscr{V}'$ be a monoidal natural transformation where Φ, Ψ are symmetric monoidal functors, and let $\gamma_A : B \to$ $\to T(A\,A)$ be a $\mathscr{V}$-natural transformation, where $T : \mathscr{A}^* \otimes \mathscr{A} \to \mathscr{B}$. Then the $\mathscr{V}'$-natural transformations*

$$\eta_{*\mathscr{B}} \cdot \Phi_* \gamma : B \to \eta_{*\mathscr{B}} \cdot (\Phi_{**} T)(A\,A)$$

and

$$\Psi_*(\gamma\,\eta_{*\mathscr{A}}) : B \to \Psi_{**}T(\eta_{*\mathscr{A}}A, \eta_{*\mathscr{A}}A)$$

coincide.

6. Symmetric Monoidal Closed Categories

A *symmetric monoidal closed category* $\mathscr{V}$ shall mean a monoidal closed category $\mathscr{V}$ with a symmetry as in § 1. If $\mathscr{V}$ and $\mathscr{V}'$ are symmetric monoidal closed categories, a symmetric monoidal functor $\Phi = (\phi, \tilde{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$, identified with the monoidal closed functor $\Phi = (\phi, \tilde{\phi}, \hat{\phi}, \phi^0) : \mathscr{V} \to \mathscr{V}'$, shall be called a *symmetric closed functor*. Then symmetric monoidal closed categories, symmetric closed functors, and closed natural transformations form a sub-hypercategory $\mathscr{SMCl}$ of $\mathscr{MCl}$.

Proposition 6.1. *If V is faithful, the monoidal closed category $\mathscr{V}$ admits at most one symmetry.*

Proof. Let c and $\bar{c}$ be two symmetries. From (1.1) we get

$$Vc.\tilde{V} = V\bar{c}.\tilde{V}.$$

Now apply II(3.24) with $\Phi = V$ and $x = c$; we get, since $\hat{V} = V$,

$$V.V\pi c = V.V\pi\bar{c}.$$

Since V is faithful and π is an isomorphism, we deduce $c = \bar{c}$.

If $\mathscr{V}$ is a symmetric monoidal closed category and $\mathscr{A}$ is a $\mathscr{V}$-category, we have for each $A \in \mathscr{A}$ the $\mathscr{V}$-functor $L^A : \mathscr{A}^* \to \mathscr{V}$. To distinguish this from the L of $\mathscr{A}$ we write it as $R^A : \mathscr{A}^* \to \mathscr{V}$, and treat it as an attribute of $\mathscr{A}$ rather than $\mathscr{A}^*$. We have

$$R^A B = \mathscr{A}(BA), \tag{6.1}$$

and

$$R^A_{BC} : \mathscr{A}^*(BC) \to (\mathscr{A}^*(AB), \mathscr{A}^*(AC));$$

that is,

$$R^A_{BC} : \mathscr{A}(CB) \to (\mathscr{A}(BA), \mathscr{A}(CA)). \tag{6.2}$$

Since the M of $\mathscr{A}^*$ is given by (2.4), we have by II(6.2)

$$R = \pi(Mc). \tag{6.3}$$

It follows that R^A_{BC} in (6.2) is natural in all variables, since π, M, c are. We call R^A the *right represented $\mathscr{V}$-functor.*

The underlying functor $V_* R^A : \mathscr{A}^*_0 \to \mathscr{V}_0$ is given by

$$V_* R^A = \mathscr{A}(-A) : \mathscr{A}^*_0 \to \mathscr{V}_0, \tag{6.4}$$

in view of Proposition 2.9. Translating Proposition I.8.3 we get:

Proposition 6.2. *If $\mathscr{V}$ is a symmetric monoidal closed category and $T : \mathscr{A} \to \mathscr{B}$ is a $\mathscr{V}$-functor, the morphisms*

$$T_{BC} : \mathscr{A}(BC) \to \mathscr{B}(TB, TC), \quad B \in \mathscr{A},$$

are the components of a $\mathscr{V}$-natural transformation

$$T_{\cdot C} : R^C \to R^{TC} T^* : \mathscr{A}^* \to \mathscr{V}.$$

As in Proposition I.8.4, the naturality of R^A in A gives:

Proposition 6.3. *If $\mathscr{V}$ is a symmetric monoidal closed category and $\mathscr{A}$ is a $\mathscr{V}$-category, let $f \in \mathscr{A}_0(AB)$. Then the morphisms*

$$\mathscr{A}(1, f) : \mathscr{A}(CA) \to \mathscr{A}(CB), \quad C \in \mathscr{A},$$

are the components of a $\mathscr{V}$*-natural transformation*

$$R^f : R^A \to R^B : \mathscr{A}^* \to \mathscr{V}.$$

Theorem 6.4. *If* $\mathscr{V}$ *is a symmetric monoidal closed category and* $\mathscr{A}$ *is a* $\mathscr{V}$*-category, the* $\mathscr{V}$*-functors* $R^B : \mathscr{A}^* \to \mathscr{V}$ *and* $L^A : \mathscr{A} \to \mathscr{V}$ *are the partial functors of a* $\mathscr{V}$*-functor*

$$\operatorname{Hom} \mathscr{A} : \mathscr{A}^* \otimes \mathscr{A} \to \mathscr{V}.$$

We defer the proof of this, which consists in verifying (4.4), to § 7. By I(7.9) and (6.4) we have:

$$(\operatorname{Hom} \mathscr{A})_{00} = \hom \mathscr{A} : \mathscr{A}_0^* \times \mathscr{A}_0 \to \mathscr{V}_0. \tag{6.5}$$

Proposition 6.5. *If* $\mathscr{V}$ *is a symmetric monoidal closed category and* $\mathscr{A}$ *is a* $\mathscr{V}$*-category, the following diagram of* $\mathscr{V}$*-functors commutes:*

$$\begin{tikzcd}
\mathscr{A} \otimes \mathscr{A}^* \arrow[r, "\operatorname{Hom} \mathscr{A}^*"] \arrow[d, "c"'] & \mathscr{V} \\
\mathscr{A}^* \otimes \mathscr{A} \arrow[ur, "\operatorname{Hom} \mathscr{A}"'] &
\end{tikzcd} \tag{6.6}$$

Proof. By the uniqueness in Proposition 4.2, it suffices to check that the partial functors agree. This is obvious from Proposition 4.7 and the definition of R^A.

Proposition 6.6. *Let* $\Phi : \mathscr{V} \to \mathscr{V}'$ *be a symmetric closed functor and* $\mathscr{A}$ *a* $\mathscr{V}$*-category. Then*

$$\operatorname{Hom} \Phi_* \mathscr{A} : \Phi_* \mathscr{A}^* \otimes \Phi_* \mathscr{A} \to \mathscr{V}'$$

is the composite

$$\Phi_* \mathscr{A}^* \otimes \Phi_* \mathscr{A} \xrightarrow[\Phi_{**}\operatorname{Hom}\mathscr{A}]{} \Phi_* \mathscr{V} \xrightarrow[\hat{\Phi}]{} \mathscr{V}'.$$

Proof. It suffices to check the partial functors; in view of Proposition 4.10 the result follows from I(6.15) for $\mathscr{A}$ and for $\mathscr{A}^*$.

Now for a symmetric monoidal closed category $\mathscr{V}$ define a natural transformation

$$H^B_{AC} : (A\,C) \to (B \otimes A, B \otimes C)$$

as the composite

$$(A\,C) \xrightarrow[K^B_{AC}]{} (A \otimes B, C \otimes B) \xrightarrow[(c,c)]{} (B \otimes A, B \otimes C); \tag{6.7}$$

see § II.7 for the definition of K.

Proposition 6.7. *For each B in the symmetric monoidal closed category $\mathscr{V}$ we obtain a $\mathscr{V}$-functor $H^B : \mathscr{V} \to \mathscr{V}$ if we set $H^B A = B \otimes A$ and $(H^B)_{AC} = H^B_{AC}$, and the underlying functor $V_* H^B : \mathscr{V}_0 \to \mathscr{V}_0$ is $B \otimes -$.*

Proof. We have, for $f \in \mathscr{V}_0(AC)$,

$$\begin{aligned}(VH^B)f &= V(c,c).\,VK^B.f && \text{by (6.7)}\\ &= V(c,c)\,(1 \otimes f) && \text{by Theorem II.7.1}\\ &= c(1 \otimes f)\,c \\ &= f \otimes 1 && \text{by the naturality of } c.\end{aligned}$$

Thus we have $V_* H^B = B \otimes -$, and we have VF1 for H^B in the form $(VH^B)1 = 1$ (cf. Remark I.9.7).

Axiom VF2 for H^B is axiom VN for H^B_A. Since the proofs of the assertions of Theorem I.10.2 make no use of VF2 for the $\mathscr{V}$-functors involved, we can use them here before we have VF2 for H^B. Since $c^2 = 1$, the definition (6.7) may be written in the form VN to show that

$$c_{CB} : C \otimes B \to B \otimes C$$

is the C-component of a $\mathscr{V}$-natural transformation

$$c_{\cdot B} : K^B \to H^B .$$

The composite (6.7) is therefore the C-component of a $\mathscr{V}$-natural transformation

$$L^A \xrightarrow[K^B_A]{} L^{A\otimes B} K^B \xrightarrow[L^c K^B]{} L^{B\otimes A} K^B \xrightarrow[L^{B\otimes A} c_{\cdot B}]{} L^{B\otimes A} H^B ;$$

thus H^B_{AC} is $\mathscr{V}$-natural in C, which is VF2 for H^B.

From the naturality in B of H^B_{AC}, which is immediate from its definition (6.7), we get just as in Proposition II.7.2:

Proposition 6.8. *If $\mathscr{V}$ is a symmetric monoidal closed category and $f \in \mathscr{V}_0(AB)$, the morphisms*

$$f \otimes 1 : A \otimes C \to B \otimes C , \quad C \in \mathscr{V} ,$$

are the components of a $\mathscr{V}$-natural transformation

$$H^f : H^A \to H^B : \mathscr{V} \to \mathscr{V} .$$

We defer to § 7 the proof of:

Theorem 6.9. *If $\mathscr{V}$ is a symmetric monoidal closed category, the $\mathscr{V}$-functors $K^B : \mathscr{V} \to \mathscr{V}$ and $H^A : \mathscr{V} \to \mathscr{V}$ are the partial functors of a $\mathscr{V}$-*

functor

$$\mathrm{Ten} : \mathscr{V} \otimes \mathscr{V} \to \mathscr{V} .$$

Since $V_* K^B = - \otimes B$ and $V_* H^A = A \otimes -$, we have

$$(\mathrm{Ten})_{00} = \otimes : \mathscr{V}_0 \times \mathscr{V}_0 \to \mathscr{V}_0 . \tag{6.8}$$

Proposition 6.10. *If $\mathscr{V}$ is a symmetric monoidal closed category and $\mathscr{A}$ and $\mathscr{B}$ are $\mathscr{V}$-categories, the following diagram of $\mathscr{V}$-functors commutes:*

$$\begin{tikzcd}
(\mathscr{A}^* \otimes \mathscr{B}^*) \otimes (\mathscr{A} \otimes \mathscr{B}) \arrow[r, "\mathrm{Hom}(\mathscr{A} \otimes \mathscr{B})"] \arrow[d, "\mathfrak{m}"'] & \mathscr{V} \\
(\mathscr{A}^* \otimes \mathscr{A}) \otimes (\mathscr{B}^* \otimes \mathscr{B}) \arrow[r, "\mathrm{Hom}\,\mathscr{A} \otimes \mathrm{Hom}\,\mathscr{B}"'] & \mathscr{V} \otimes \mathscr{V} \arrow[u, "\mathrm{Ten}"']
\end{tikzcd} \tag{6.9}$$

Proof. It suffices to check the partial functors; we must therefore show that the $L^{[AB]}$ of $\mathscr{A} \otimes \mathscr{B}$ is given by the composite

$$\mathscr{A} \otimes \mathscr{B} \xrightarrow[L^A \otimes L^B]{} \mathscr{V} \otimes \mathscr{V} \xrightarrow[\mathrm{Ten}]{} \mathscr{V} ; \tag{6.10}$$

the same result applied to $\mathscr{A}^*$ and $\mathscr{B}^*$ then gives equality of the other partial functors.

Let the partial functors of $L^{[AB]} : \mathscr{A} \otimes \mathscr{B} \to \mathscr{V}$ be $L^{[AB]}(-D) = P : \mathscr{A} \to \mathscr{V}$ and $L^{[AB]}(C-) = Q : \mathscr{B} \to \mathscr{V}$. Then to show that $L^{[AB]}$ is (6.10) is to show that P is the composite

$$\mathscr{A} \xrightarrow[L^A]{} \mathscr{V} \xrightarrow[K^{\mathscr{B}(BD)}]{} \mathscr{V} \tag{6.11}$$

and that Q is the composite

$$\mathscr{B} \xrightarrow[L^B]{} \mathscr{V} \xrightarrow[H^{\mathscr{A}(AC)}]{} \mathscr{V} , \tag{6.12}$$

as we see from Proposition 4.4.

We immediately verify that P agrees with (6.11) and Q with (6.12) on objects. By (4.2), P_{XY} is given by:

$$\mathscr{A}(XY) \xrightarrow[r^{-1}]{} \mathscr{A}(XY) \otimes I \xrightarrow[1 \otimes j]{} \mathscr{A}(XY) \otimes \mathscr{B}(DD) \xrightarrow[L^{[AB]}]{}$$
$$\to (\mathscr{A}(AX) \otimes \mathscr{B}(BD), \mathscr{A}(AY) \otimes \mathscr{B}(BD)) . \tag{6.13}$$

Taking π^{-1} of (6.13), using II(3.1), II(6.2), and (3.4), we get the upper leg of the following diagram:

$$
\begin{tikzcd}
\mathscr{A}(X\,Y) \otimes (\mathscr{A}(A\,X) \otimes \mathscr{B}(B\,D)) \arrow[r, "r^{-1} \otimes 1"] \arrow[d, "a^{-1}"] & (\mathscr{A}(X\,Y) \otimes I) \otimes (\mathscr{A}(A\,X) \otimes \mathscr{B}(B\,D)) \arrow[r, "(1 \otimes j) \otimes 1"] \arrow[d, "m"] & (\mathscr{A}(X\,Y) \otimes \mathscr{B}(D\,D)) \otimes (\mathscr{A}(A\,X) \otimes \mathscr{B}(B\,D)) \arrow[d, "m"] \\
(\mathscr{A}(X\,Y) \otimes \mathscr{A}(A\,X)) \otimes \mathscr{B}(B\,D) \arrow[r, "1 \otimes l^{-1}"'] \arrow[rrd, "M \otimes 1"'] & (\mathscr{A}(X\,Y) \otimes \mathscr{A}(A\,X)) \otimes (I \otimes \mathscr{B}(B\,D)) \arrow[r, "1 \otimes (j \otimes 1)"'] & (\mathscr{A}(X\,Y) \otimes \mathscr{A}(A\,X)) \otimes (\mathscr{B}(D\,D) \otimes \mathscr{B}(B\,D)) \arrow[d, "M \otimes M"] \\
 & & \mathscr{A}(A\,Y) \otimes \mathscr{B}(B\,D)
\end{tikzcd}
\tag{6.14}
$$

Now in (6.14) the top left region commutes by coherence, the top right region by the naturality of m, and the bottom region by VC1′ for $\mathscr{B}$. It follows that we have

$$\pi^{-1} P = (M \otimes 1)\, a^{-1}. \tag{6.15}$$

Thus

$$\begin{aligned} P &= \pi((M \otimes 1)\, a^{-1}) \\ &= p^{-1} . \pi\pi(M \otimes 1) && \text{by II}(3.19) \\ &= p^{-1} . \pi(uM) && \text{by II}(3.1) \text{ with } x = 1 \\ &= p^{-1}(1, u) . \pi M && \text{by II}(3.1) \\ &= KL && \text{by II}(7.1) \text{ and II}(6.2); \end{aligned}$$

which proves that P is (6.11).

In an exactly similar way one proves that

$$\pi^{-1} Q = (1 \otimes M)\, c\, a^{-1}(1 \otimes c)\,; \tag{6.16}$$

by the naturality of c we also have

$$\pi^{-1} Q = c(M \otimes 1)\, a^{-1}(1 \otimes c)\,;$$

II(3.1) then gives

$$\begin{aligned} Q &= (c, c)\, \pi((M \otimes 1)\, a^{-1}) \\ &= (c, c)\, KL && \text{by the calculation above} \\ &= HL && \text{by } (6.7). \end{aligned}$$

This completes the proof.

7. The $\mathscr{V}$-naturality of the Canonical Morphisms

Proposition 7.1. *Let $\mathscr{V}$ be a symmetric monoidal closed category and let $T(A\,-) : \mathscr{B} \to \mathscr{C}$ and $T(-\,B) : \mathscr{A} \to \mathscr{C}$ be families of $\mathscr{V}$-functors indexed by $A \in \mathscr{A}$ and $B \in \mathscr{B}$, with $T(A\,-)B = T(-\,B)A = T(AB)$. Then these are the partial functors of a bifunctor $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ if and only if either of the following diagrams commutes:*

$$\begin{tikzcd}
\mathscr{A}(AA') \arrow[r, "T(-B')"] \arrow[d, "T(-B)"'] & \mathscr{C}(T(AB'), T(A'B')) \arrow[r, "L"] & (\mathscr{C}(T(AB), T(AB')), \mathscr{C}(T(AB), T(A'B'))) \arrow[dd, "(T(A\,-), 1)"] \\
\mathscr{C}(T(AB), T(A'B)) \arrow[d, "R"'] & & \\
(\mathscr{C}(T(A'B), T(A'B')), \mathscr{C}(T(AB), T(A'B'))) \arrow[rr, "(T(A'\,-), 1)"'] & & (\mathscr{B}(BB'), \mathscr{C}(T(AB), T(A'B')))
\end{tikzcd} \tag{7.1}$$

$$
\begin{tikzcd}
\mathscr{B}(B\,B') \arrow[r, "T(A'\,-)"] \arrow[d, "T(A\,-)"'] & \mathscr{C}(T(A'\,B), T(A'\,B')) \arrow[r, "L"] & (\mathscr{C}(T(A\,B), T(A'\,B)), \mathscr{C}(T(A\,B), T(A'\,B'))) \arrow[dd, "(T(-\,B), 1)"] \\
\mathscr{C}(T(A\,B), T(A\,B')) \arrow[d, "R"'] & & \\
(\mathscr{C}(T(A\,B'), T(A'B')), \mathscr{C}(T(A\,B), T(A'\,B'))) \arrow[rr, "(T(-\,B'), 1)"'] & & (\mathscr{A}(A\,A'), \mathscr{C}(T(A\,B), T(A'\,B')))
\end{tikzcd}
\tag{7.2}
$$

Proof. By Propositions 4.1 and 4.2, it suffices to show that each of (7.1), (7.2) is equivalent to (4.4). Applying π to both legs of (4.4), we get

$$
\begin{aligned}
\pi(M(T(-\,B') \otimes T(A\,-))) &= (T(A\,-), 1).\pi M. T(-\,B') && \text{by II}(3.1) \\
&= (T(A\,-), 1)\, L T(-\,B') && \text{by II}(6.2);
\end{aligned}
$$

and

$$
\begin{aligned}
\pi(M(T(A'-) \otimes T(-\,B))\,c) &= \pi(Mc(T(-\,B) \otimes T(A'-))) && \text{by the naturality of } c, \\
&= (T(A'-), 1)\, R T(-\,B) && \text{by II}(3.1) \text{ and } (6.3);
\end{aligned}
$$

thus (4.4) is equivalent to (7.1).

Similarly we get (7.2) if we reverse the direction of the arrow c in (4.4) before applying π.

Corollary 7.2. *If $\mathscr{V}$ is a symmetric monoidal closed category and $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is a $\mathscr{V}$-functor, then*

$$T(A\,-) : \mathscr{B}(BB') \to \mathscr{C}(T(AB), T(AB'))$$

and $$T(-\,B) : \mathscr{A}(A\,A') \to \mathscr{C}(T(AB), T(A'B))$$

are $\mathscr{V}$-natural in A and B respectively, with respect to the bifunctors

$$\mathscr{A}^* \otimes \mathscr{A} \xrightarrow[T(-\,B)^* \otimes T(-\,B')]{} \mathscr{C}^* \otimes \mathscr{C} \xrightarrow[\mathrm{Hom}\,\mathscr{C}]{} \mathscr{V},$$

$$\mathscr{B}^* \otimes \mathscr{B} \xrightarrow[T(A\,-)^* \otimes T(A'\,-)]{} \mathscr{C}^* \otimes \mathscr{C} \xrightarrow[\mathrm{Hom}\,\mathscr{C}]{} \mathscr{V}.$$

Proof. VN′ for $T(A\,-)$ is (7.1), and for $T(-\,B)$ is (7.2).

Remark 7.3. As we noted in Remark 4.3, we needed (4.4), or equivalently (7.1) or (7.2), only to get VF2 for T. Now VF2 for T is not involved in the definitions of extraordinary $\mathscr{V}$-natural transformations in § 5, nor in Propositions 5.2, 5.3, 5.4. We can therefore use all of these before proving Theorem 6.4 and 6.9. Indeed, we shall prove these precisely by appealing to (7.1) and (7.2), stated in terms of $\mathscr{V}$-naturality; the proofs are contained in the following theorem. The bifunctors, or in the first instance their partial functors, with respect to which the stated $\mathscr{V}$-naturality obtains, are obvious compositions of H, K, L, and R.

Theorem 7.4. *If $\mathscr{V}$ is a symmetric monoidal closed category, the morphisms a, r, l, c, p, t, u, H, K are $\mathscr{V}$-natural in every variable. Moreover if $\mathscr{A}$ is a $\mathscr{V}$-category, the morphisms M, L, R, j are $\mathscr{V}$-natural in every variable.*

Remark 7.5. The $\mathscr{V}$-naturality in A of H^A and of L^A establishes Theorems 6.4 and 6.9 by means of Proposition 7.1.

Proof of Theorem 7.4. We first observe that c_{AB} is $\mathscr{V}$-natural in both variables, for since $c^2 = 1$ the definition (6.7) of H may be interpreted as VN for c in either variable. We next prove the $\mathscr{V}$-naturality of t_{BC}; it is $\mathscr{V}$-natural in C by Theorem II.7.1. Consider the diagram:

$$\begin{tikzcd}
(BB')\otimes((B'C)\otimes B) \arrow[rr, "1\otimes c"] \arrow[d, "a^{-1}"'] & & (BB')\otimes(B\otimes(B'C)) \arrow[d, "a^{-1}"] & \\
((BB')\otimes(B'C))\otimes B \arrow[d, "c\otimes 1"'] & & ((BB')\otimes B)\otimes(B'C) \arrow[d, "c"] \arrow[dr, "t\otimes 1"] & \\
((B'C)\otimes(BB'))\otimes B \arrow[rr, "a"] \arrow[d, "M\otimes 1"'] & & (B'C)\otimes((BB')\otimes B) \arrow[d, "1\otimes t"] & B'\otimes(B'C) \arrow[dl, "c"] \\
(BC)\otimes B \arrow[dr, "t"'] & & (B'C)\otimes B' \arrow[dl, "t"] & \\
& C & &
\end{tikzcd} \tag{7.3}$$

The top region commutes by coherence, and the right region by the naturality of c. The bottom region would commute by VC3′ for $\mathscr{V}$ if we had (IB), (IB'), (IC) in place of B, B', C at the extreme right of each object, and M in place of t; it therefore commutes by II(7.3).

Now apply π to each leg of (7.3). We get

$$\begin{aligned}
\pi(tc(t\otimes 1)a^{-1}(1\otimes c)) &= (1,t)\,(c,c)\,\pi((t\otimes 1)a^{-1}) && \text{by II(3.1)}\\
&= (1,t)\,(c,c)\,p^{-1}\pi\pi(t\otimes 1) && \text{by II(3.19)}\\
&= (1,t)\,(c,c)\,p^{-1}\pi(ut) && \text{by II(3.1)}\\
& && \text{with } x=1\\
&= (1,t)\,(c,c)\,p^{-1}(1,u) && \text{by II(3.1)}\\
& && \text{since } \pi t = 1\\
&= (1,t)\,(c,c)\,K && \text{by II(7.1)}\\
&= (1,t)\,H && \text{by (6.7);}
\end{aligned}$$

and

$$\begin{aligned}
\pi(t(M\otimes 1)(c\otimes 1)a^{-1}) &= \pi(\pi^{-1}(Mc).a^{-1}) && \text{by II(3.5)}\\
&= p^{-1}\pi\pi\,(\pi^{-1}(Mc)) && \text{by II(3.19)}\\
&= p^{-1}\pi(Mc) && \\
&= p^{-1}R && \text{by (6.3)}\\
&= (1,t)\,KR && \text{by II(7.2);}
\end{aligned}$$

the resulting diagram is precisely VN″ for the $\mathscr{V}$-naturality of t_{BC} in B. Note that by stopping one line before the end of the above calculation we have

$$(1,t)H = p^{-1}R. \tag{7.4}$$

We now prove the $\mathscr{V}$-naturality of u_{AB}; it is $\mathscr{V}$-natural in A by Theorem II.7.1. For its $\mathscr{V}$-naturality in B, VN′ is the exterior of the diagram: (see page 545):
The top left region commutes by (7.4), and the bottom region by II(3.7). The other regions, reading from left to right, commute (i) by the naturality of H, (ii) trivially, (iii) by the naturality of p, (iv) by II(3.21).

We turn to L, R, and M. Because L^A and R^A are $\mathscr{V}$-functors, L^A_{BC} and R^A_{BC} are $\mathscr{V}$-natural in B and C by Proposition I.8.3 and Proposition 6.2. Since $M = \pi^{-1}L$ by II(6.2), it is by II(3.5) the composite:

$$M^B_{AC}: \mathscr{A}(BC)\otimes\mathscr{A}(AB) \xrightarrow[L\otimes 1]{} (\mathscr{A}(AB),\mathscr{A}(AC))\otimes\mathscr{A}(AB) \xrightarrow[t]{} \mathscr{A}(AC). \tag{7.5}$$

Since t is $\mathscr{V}$-natural in everything and $L\otimes 1$ is $\mathscr{V}$-natural in everything except A, M^B_{AC} is $\mathscr{V}$-natural in B and C. (We are implicitly using Propositions 5.2 and 5.3, and Proposition 6.8. We continue to use these implicitly, as well as I(9.7), Proposition II.7.2, and Proposition 6.3; note that these last and Proposition 6.8 become subsumed under Proposition

$$
\begin{tikzcd}
(B\,C) \arrow[rrr, "R"] \arrow[dddd, "H"'] \arrow[dr, "H"] & & & ((C, A \otimes C), (B, A \otimes C)) \arrow[r, "{(u, 1)}"] & (A\,(B, A \otimes C)) & \\
& ((C, A \otimes C) \otimes B,\ (C, A \otimes C) \otimes C) \arrow[rr, "{(1, t)}"] \arrow[d, "{(u \otimes 1, 1)}"] & & ((C, A \otimes C) \otimes B, A \otimes C) \arrow[u, "p"] \arrow[ddr, "{(u \otimes 1, 1)}"] & & \\
& (A \otimes B, (C, A \otimes C) \otimes C) \arrow[drrr, "{(1, t)}"] & & & & ((B, A \otimes B), (B, A \otimes C)) \arrow[uul, "{(u, 1)}"'] \\
& & & & & \\
(A \otimes B, A \otimes C) \arrow[uur, "{(1, u \otimes 1)}"'] \arrow[rrrr, "1"'] & & & & (A \otimes B, A \otimes C) \arrow[uuuu, "p"] \arrow[uur, "L"'] &
\end{tikzcd}
$$

4.19 only *after* we know that we have bifunctors Hom $\mathscr{A}$ and Ten.) We also have $M = \pi^{-1} R \,.\, c$ by (6.3), so that M is the composite:

$$M^B_{AC} : \mathscr{A}(BC) \otimes \mathscr{A}(AB) \xrightarrow[c]{} \mathscr{A}(AB) \otimes \mathscr{A}(BC) \to$$
$$\xrightarrow[R \otimes 1]{} (\mathscr{A}(BC), \mathscr{A}(AC)) \otimes \mathscr{A}(BC) \xrightarrow[t]{} \mathscr{A}(AC)\,. \tag{7.6}$$

Since t and c are $\mathscr{V}$-natural in everything, and $R \otimes 1$ in everything except C, M^B_{AC} is $\mathscr{V}$-natural in B and A. Thus M is $\mathscr{V}$-natural in all variables. Now $L = \pi M$, and so by II(3.4) we have $L = (1, M)\, u$; thus L is $\mathscr{V}$-natural in all variables since M and u are. Similarly $R = \pi(Mc) = (1, Mc)\, u$ is $\mathscr{V}$-natural in all variables.

p is now $\mathscr{V}$-natural in all variables by II(3.21), then K by II(7.1) (the inverses of *ordinary* $\mathscr{V}$-natural transformations are $\mathscr{V}$-natural by Theorem I.10.2), then H by (6.7).

Since i is $\mathscr{V}$-natural by Proposition I.8.5, it follows from MCC2, MCC3, MCC4 that $(l, 1)$, $(a, 1)$, $(r, 1)$ are $\mathscr{V}$-natural in every variable. The $\mathscr{V}$-naturality of l, a, r themselves now follows by Proposition I.10.10.

Finally we consider j. VN′ for j is the exterior of the following diagram:

$$\begin{tikzcd}
\mathscr{A}(AC) \arrow[r, "L"] \arrow[d, "R"'] \arrow[dr, "i"] & (\mathscr{A}(AA), \mathscr{A}(AC)) \arrow[d, "{(j,1)}"] \\
(\mathscr{A}(CC), \mathscr{A}(AC)) \arrow[r, "{(j,1)}"'] & (I, \mathscr{A}(AC))
\end{tikzcd}$$

The two regions commute by VC2 for $\mathscr{A}$ and for $\mathscr{A}^*$. This completes the proof.

Proposition 7.6. *If $\mathscr{V}$ is a symmetric monoidal closed category and $T : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ is a $\mathscr{V}$-functor, then*

$$T : \mathscr{A}(AA') \otimes \mathscr{B}(BB') \to \mathscr{C}(T(AB), T(A'B'))$$

is $\mathscr{V}$-natural in A, A', B, and B'.

Proof. By Propositions I.8.3 and 6.2, T is $\mathscr{V}$-natural in $[AB]$ and $[A'B']$. Therefore by Proposition 4.12 it is $\mathscr{V}$-natural in A, B, A', B' with respect to the appropriate partial functors. By Proposition 6.10 these are what they should be.

Proposition 7.7. *If $\Phi : \mathscr{V} \to \mathscr{V}'$ is a symmetric closed functor, $\hat{\phi}_{AB}$ and*

$\tilde{\phi}_{AB}$ *are* $\mathscr{V}'$*-natural in all variables. More precisely we have for fixed* A:

$$\hat{\phi}_A : \hat{\Phi} \,.\, \Phi_* L^A \to L'^{\phi A} \,.\, \hat{\Phi} : \Phi_* \mathscr{V} \to \mathscr{V}' ,$$

$$\tilde{\phi}_A : H'^{\phi A} \,.\, \hat{\Phi} \to \hat{\Phi} \,.\, \Phi_* H^A : \Phi_* \mathscr{V} \to \mathscr{V}' ,$$

with similar results for fixed B.

Proof. Since $\hat{\phi}_{AB}$ are the components of the $\mathscr{V}'$-functor $\hat{\Phi}: \Phi_* \mathscr{V} \to \mathscr{V}'$, the result for $\hat{\phi}$ follows by Propositions I.8.3 and 6.2, in view of I(6.15).

Now consider the diagram of MCF3; p', $(1, \hat{\phi})$, and $\hat{\phi}$ are $\mathscr{V}'$-natural in all variables, and so is ϕp by Proposition I.8.8. We deduce the $\mathscr{V}'$-naturality of $(\tilde{\phi}, 1)\, \hat{\phi}$, and so by composition that of $(\tilde{\phi}, 1)\, \hat{\phi} \,.\, \phi H$. Since $\hat{\phi} \,.\, \phi H$ are the components of $\hat{\Phi} \,.\, \Phi_* H^A$, the $\mathscr{V}'$-naturality of $\tilde{\phi}$ follows from Proposition I.10.10.

Lemma 7.8. *Let* $\mathscr{V}$ *be a symmetric monoidal closed category and let* $P, Q : \mathscr{A} \to \mathscr{B}$ *and* $T : \mathscr{A}^* \otimes \mathscr{A} \to \mathscr{B}$ *be* $\mathscr{V}$*-functors. Consider families of morphisms*

$$\alpha_A : P A \to Q A ,$$

$$\gamma_A : B \to T(A A) ,$$

$$\delta_A : T(A A) \to B .$$

Then the $\mathscr{V}$*-naturality of* α, γ, δ *is equivalent to that of the families*

$$\iota \alpha_A : I \to \mathscr{B}(P A, Q A),$$

$$\iota \gamma_A : I \to \mathscr{B}(B, T(A A)),$$

$$\iota \delta_A : I \to \mathscr{B}(T(A A), B) .$$

Proof. We give the proof for α, leaving the others to the reader. Since by I(7.14) $\iota\alpha$ is the composite

$$I \xrightarrow[j]{} \mathscr{B}(P A, P A) \xrightarrow[\mathscr{B}(1,\alpha)]{} \mathscr{B}(P A, Q A) ,$$

the $\mathscr{V}$-naturality of $\iota\alpha$ follows from that of α in view of the $\mathscr{V}$-naturality of j.

Now suppose that $\iota\alpha$ is $\mathscr{V}$-natural. By Proposition I.10.10, to prove the $\mathscr{V}$-naturality of α it suffices to prove the $\mathscr{V}$-naturality in C of the composite

$$\mathscr{A}(A\, C) \xrightarrow[P]{} \mathscr{B}(P A, P C) \xrightarrow[\mathscr{B}(1,\alpha)]{} \mathscr{B}(P A, Q C) ,$$

and since i is $\mathscr{V}$-natural, it suffices to prove the $\mathscr{V}$-naturality in C of the composite

$$\mathscr{A}(A\, C) \xrightarrow[P]{} \mathscr{B}(P A, P C) \xrightarrow[\mathscr{B}(1,\alpha)]{} \mathscr{B}(P A, Q C) \xrightarrow[i]{} (I, \mathscr{B}(P A, Q C)) . \quad (7.7)$$

Now (7.7) is certainly $\mathscr{V}$-natural in A. If we put $A = C$, apply V, and evaluate at 1_C we get $\iota\alpha_C$. Thus by Theorem I.10.8 (with $\mathscr{A}^*$ in place of $\mathscr{A}$!), (7.7) is the composite

$$\mathscr{A}(A\,C) \xrightarrow[P]{} \mathscr{B}(PA, PC) \xrightarrow[R]{} (\mathscr{B}(PC, QC), \mathscr{B}(PA, QC)) \to \\ \xrightarrow[(\iota\alpha, 1)]{} (I, \mathscr{B}(PA, QC)). \tag{7.8}$$

By the $\mathscr{V}$-naturality of $\iota\alpha$, this is $\mathscr{V}$-natural in C, as required.

We now prove the $\mathscr{V}$-analogue of Proposition I.1.2:

Proposition 7.9. *Let $\mathscr{V}$ be a symmetric monoidal closed category and let $T : \mathscr{D} \otimes \mathscr{A} \to \mathscr{B}$, $P : \mathscr{C} \otimes \mathscr{D}^* \to \mathscr{A}$, $Q : \mathscr{C} \to \mathscr{B}$ be $\mathscr{V}$-functors. Let*

$$q_{CDA} : \mathscr{A}(P(C\,D), A) \to \mathscr{B}(Q\,C, T(D\,A))$$

be a family of morphisms, $\mathscr{V}$-natural in A for each C, D; and in the language of Theorem I.10.8 let $\Gamma' q_{CD}$ be

$$\theta_{CD} : Q\,C \to T(D, P(C\,D)).$$

Then q is $\mathscr{V}$-natural in C (resp. D) if and only if θ is.

Proof. q as $\Omega'\theta$ is the composite

$$\mathscr{A}(P(C\,D), A) \xrightarrow[T(D-)]{} \mathscr{B}(T(D, P(C\,D)), T(D\,A)) \xrightarrow[\mathscr{B}(\theta, 1)]{} \mathscr{B}(Q\,C, T(D\,A))$$

and so is $\mathscr{V}$-natural if θ is. Similarly $\iota\theta$ is the composite

$$I \xrightarrow[j]{} \mathscr{A}(P(C\,D), P(C\,D)) \xrightarrow[q]{} \mathscr{B}(Q\,C, T(D, P(C\,D)))$$

and is $\mathscr{V}$-natural if q is. Lemma 7.8 then gives the $\mathscr{V}$-naturality of θ.

Chapter IV

Examples

1. Elementary Examples

We have seen that the category $\mathscr{S}$ of sets admits an obvious structure of symmetric monoidal closed category, and that $\mathscr{S}$-categories are ordinary categories, etc.

The category $\mathscr{P}_0$ of pointed sets has as objects sets A with a distinguished element a_0 and as morphisms maps $f : A \to B$ with $fa_0 = b_0$. If we take for the tensor product the "smash" product $A \mathbin{\mathsf{x}} B$, consisting of the cartesian product $A \times B$ with $A \times b_0 \cup a_0 \times B$ shrunk to a single point, $\mathscr{P}_0$ becomes a symmetric monoidal category $\mathscr{P}$. It is closed, $(A\,B)$ being $\mathscr{P}_0(A\,B)$ with the distinguished element $f_0 : A \to B$ where $f_0 A = b_0$; the basic functor $P : \mathscr{P}_0 \to \mathscr{S}$ is the forgetful functor assigning to each pointed set its underlying set, and I is a set with two points, one of them distinguished. A $\mathscr{P}$-category $\mathscr{A}$ is a pointed category, i.e. one in which each $\mathscr{A}(AB)$ has a distinguished element 0 such that $f0g = 0$; and a $\mathscr{P}$-functor $T: \mathscr{A} \to \mathscr{B}$ is one such that $T\,0 = 0$. Since P is faithful, a $\mathscr{P}$-natural transformation is just a natural transformation.

The category $\mathscr{G}_0$ of abelian groups admits a symmetric monoidal closed structure $\mathscr{G}$ that is too familiar to need description; indeed it is by analogy with the situation here that we use the name "tensor product" for $\otimes$ in any monoidal category. $\mathscr{G}$-categories are just pre-additive categories, and $\mathscr{G}$-functors are additive functors, i.e. those for which $T(f+g) = Tf + Tg$.

In the same way we get the symmetric monoidal closed category $\mathscr{M}K$ of modules over a commutative ring K; $\otimes$ is now $\otimes_K$, I is K, and the morphisms $f: A \to B$ form a K-module (AB). A ring-morphism $L \to K$ induces a symmetric closed functor $\mathscr{M}K \to \mathscr{M}L$, and in particular we have the forgetful closed functor $\mathscr{M}K \to \mathscr{M}\boldsymbol{Z} = \mathscr{G}$. (In future, when only symmetric monoidal closed categories are in question, "closed functor" shall mean "symmetric closed functor" unless the contrary is stated.) We also have forgetful closed functors $\mathscr{G} \to \mathscr{P} \to \mathscr{S}$; all of these are normal, so that the composite $\mathscr{M}K \to \mathscr{G} \to \mathscr{P} \to \mathscr{S}$ is the basic closed functor $\mathscr{M}K \to \mathscr{S}$.

In our definition of closed category we began with an ordinary category $\mathscr{V}_0$ and a functor $V: \mathscr{V}_0 \to \mathscr{S}$. The reader will guess that we might instead lay down a basic symmetric monoidal closed category $\mathscr{W}$ in place of $\mathscr{S}$, and then define a *closed category over* $\mathscr{W}$, starting with a $\mathscr{W}$-category $\mathscr{V}_0$ and a $\mathscr{W}$-functor $V: \mathscr{V}_0 \to \mathscr{W}$, and taking all the data to be $\mathscr{W}$-functors and $\mathscr{W}$-natural transformations. This is the case, and we shall show in a later paper that to give a symmetric monoidal closed category $\mathscr{V}$ *over* $\mathscr{W}$ is the same thing as to give a symmetric monoidal closed category $\overline{\mathscr{V}}$ and a *normal* closed functor $\Phi: \overline{\mathscr{V}} \to \mathscr{W}$; then $\mathscr{V}_0$ is the $\mathscr{W}$-category $\Phi_* \overline{\mathscr{V}}$, and V is $\hat{\Phi}$. Thus $\mathscr{M}K$ may be considered as a closed category over $\mathscr{G}$, or for that matter over $\mathscr{P}$; and many other examples will appear below.

All of the above examples fit into the class of "algebraic categories" in the sense of Lawvere [*11*], also known as "varieties" or as "equational categories". The following considerations are due to Linton [*13*]. (Cf. also Freyd [*8*].)

An algebraic category $\mathscr{K}_0$ comes equipped with a faithful forgetful functor K into $\mathscr{S}$, which has an adjoint F, where FX is the free algebra on the set X. Thus K admits a representation $\iota: KA \to \mathscr{K}_0(IA)$ where I is the free algebra on one generator. Let us call $\mathscr{K}_0$ *commutative* if, for each n-ary operation t of the algebraic theory (we allow n to be any cardinal) and each algebra A, the map $t: (KA)^n \to KA$ is a morphism

$A^n \to A$ in $\mathscr{K}_0$; this means that if s is any m-ary operation of the theory we have, with an obvious notation, commutativity in the diagram

$$\begin{tikzcd} (KA)^{mn} \arrow[r, "s^n"] \arrow[d, "t^m"'] & (KA)^n \arrow[d, "t"] \\ (KA)^m \arrow[r, "s"'] & KA \end{tikzcd} \tag{1.1}$$

It is easy to see that the set $\mathscr{K}_0(A\,B)$ forms a subalgebra $(A\,B)$ of the cartesian power algebra B^{KA}, for all A and B, if and only if $\mathscr{K}_0$ is commutative.

Supposing henceforth $\mathscr{K}_0$ commutative, define (AB) as above; it is clearly a functor, and satisfies CC0. Moreover ι is now a natural isomorphism $i : A \to (IA)$. The set $\mathscr{K}_0(A, (BC))$ may be identified with the set of *bimorphisms* $f : KA \times KB \to KC$, a bimorphism being a map f for which the two partial maps $f(a-)$ and $f(-b)$ are, for each $a \in KA$ and $b \in KB$, morphisms in $\mathscr{K}_0$. It is further clear that there is a bijection between bimorphisms $f : KA \times KB \to KC$ and morphisms $g : A \otimes B \to C$, where $A \otimes B$ is a suitable quotient algebra of $F(KA \times KB)$ (impose upon the latter the relations ensuring "bilinearity"). There results an isomorphism

$$\pi : \mathscr{K}_0(A \otimes B, C) \to \mathscr{K}_0(A\,(B\,C))$$

which is at once seen to be a natural isomorphism of algebras

$$p : (A \otimes B, C) \to (A\,(B\,C))\,.$$

Since K is faithful, we have by Theorem II.5.10 a monoidal closed structure $\mathscr{K}$ on $\mathscr{K}_0$, which is moreover clearly symmetric by the definition of $\otimes$. Such a $\mathscr{K}$ will be called an *algebraic closed category*. A less elementary example is given by Mac Lane's theory of affine modules ([*4*], Chapter XII).

Note that it is not true in a general symmetric monoidal closed category $\mathscr{V}$ that the tensor product is the universal object for bimorphisms; this cannot be the case unless V is faithful, and need not be the case then, as is shown by the example of quasi-topological spaces in § 2 below.

2. Cartesian Closed Categories

Any category $\mathscr{V}_0$ that admits finite products (including the product of *no* objects, i.e. a terminal object I) admits a structure of symmetric monoidal category $\mathscr{V}$ in which $A \otimes B$ is taken to be $A \times B$; for the canonical isomorphisms

$$(A\times B)\times C\cong A\times(B\times C),\ I\times A\cong A,\ A\times I\cong A,\ A\times B\cong B\times A$$

are easily seen to be coherent. Such a symmetric monoidal category is said to be *cartesian*. We are of course supposing that for each A, B a definite product $A\times B$ with its projections is chosen; different choices would replace $\mathscr{V}$ by an isomorph.

If a symmetric monoidal category is given, it is easily seen that its monoidal structure coincides with some cartesian structure if and only if the following two conditions are satisfied:

(i) I is terminal, so that for each A there is a unique $\theta: A\to I$;

(ii) the morphisms $A\otimes B\xrightarrow[1\otimes\theta]{} A\otimes I\xrightarrow[r]{} A$ and $A\otimes B\xrightarrow[\theta\otimes 1]{} I\otimes B\xrightarrow[l]{} B$

are the projections of a product.

In particular a and c are then uniquely determined.

By Theorem II.5.9, a cartesian monoidal category $\mathscr{V}$ is closed (or is so after replacing $\mathscr{V}_0$ by an isomorph) if and only if the functor $-\times B$ has a coadjoint. Thus for $\mathscr{V}$ to be closed it is necessary that $-\times B$ preserve colimits, which places severe restrictions on $\mathscr{V}_0$.

In particular, if $\mathscr{V}_0$ has an initial object O, it is a colimit, and so we must have $O\times B=O$. If $\mathscr{V}_0$ is pointed, i.e. if $O\simeq I$, we then have

$$B\cong I\times B\cong O\times B\cong O,$$

so that every object is initial. Thus the only cartesian closed categories that are pointed are those equivalent to the unit category $\mathscr{I}$ with a single object and a single morphism. The closed category $\mathscr{S}$ is cartesian, but the closed categories $\mathscr{P}$, $\mathscr{G}$, $\mathscr{M}K$ of § 1, being pointed, are not.

A prime example of a cartesian closed category is the category of small categories. Let $\mathscr{C}_0$ be the category with small categories A as its objects and functors $T: A\to B$ as its morphisms, and give it the cartesian monoidal structure $\mathscr{C}$. Then $\mathscr{C}$ is closed, for we get an adjunction $\pi:\mathscr{C}_0(A\times B, C)\cong\mathscr{C}_0(A(BC))$ if we take (BC) to be the functor category whose objects are functors $T: B\to C$ and whose morphisms are natural transformations $\alpha: T\to S$. The basic functor $C:\mathscr{C}_0\to\mathscr{S}$ sends the category A to its set of objects, and I is the category with one object and one morphism.

One easily verifies that $\mathscr{C}$-categories, $\mathscr{C}$-functors, and $\mathscr{C}$-natural transformations are precisely hypercategories, hyperfunctors, and hypernatural transformations. $\mathscr{C}$ itself is a $\mathscr{C}$-category, and hence a hypercategory.

If we ignore considerations of smallness and legitimacy, we can identify $\mathscr{C}$ with $\mathscr{C}at$, which is now a closed "category". The hypercategory $\mathscr{S}_*$ is $\mathscr{C}at$ *qua* hypercategory, while the symmetric monoidal category $\mathcal{S}_{\#}$ (cf. Proposition III.3.4) is $\mathcal{C}at$ *qua* closed category. Then $\mathcal{C}at_*$ is the hypercategory $\mathcal{H}yp$, while the symmetric monoidal category $\mathcal{C}at_{\#}$ is in fact a cartesian closed category which we shall still call $\mathcal{H}yp$. The basic closed functor $C : \mathcal{C}at \to \mathcal{S}$ induces a hyperfunctor $C_* : \mathcal{H}yp \to \mathcal{C}at$ whose effect is to ignore the hypermorphisms; it is in fact a normal closed functor (Proposition III.3.6) and exhibits $\mathcal{H}yp$ as a closed category *over* $\mathcal{C}at$.

If in a hypercategory the morphisms are regarded as objects and the hypermorphisms as morphisms there results a category. In this way we get a hyperfunctor $\mathcal{H}yp \to \mathcal{C}at$, and it is easy to see that it is induced by the closed functor $M = (M, \tilde{M}, M^0) : \mathcal{C}at \to \mathcal{S}$ defined as follows. For any category A, MA is the set of all morphisms $f : X \to X'$ in A. If $g : Y \to Y'$ is an element of MB then $\tilde{M} : MA \times MB \to M(A \times B)$ maps the pair (f, g) to the morphism $(f, g) : (X, X') \to (Y, Y')$ in $A \times B$. $M^0 : * \to MI$ is uniquely defined since I has only one morphism.

Any category can be made into a hypercategory by giving it identities as its only hypermorphisms. The resulting hyperfunctor $\mathcal{C}at \to \mathcal{H}yp$ is induced by a closed functor $D : \mathcal{S} \to \mathcal{C}at$, where DX is the discrete category based on X and $\tilde{D}$, D^0 are suitably defined.

Just as a class has only objects; a category has also morphisms between objects; and a hypercategory has also hypermorphisms between morphisms; so one may define an n-category with morphisms of every type i, $1 \leqq i \leqq n$, a morphism of type i connecting two of type $i - 1$ (cf. EHRESMANN [6]). Then n-categories form a cartesian closed category $\mathcal{C}^n$, and $\mathcal{C}^n_{\#} = \mathcal{C}^{n+1}$; in particular $\mathcal{C}^0 = \mathcal{S}$, $\mathcal{C}^1 = \mathcal{C}at$, $\mathcal{C}^2 = \mathcal{H}yp$. Forgetting the morphisms of type n gives a normal closed functor $\mathcal{C}^n \to \mathcal{C}^{n-1}$, so that $\mathcal{C}^n$ may be regarded as a closed category over $\mathcal{C}^{n-1}$. Note that there are many kinds of contravariance for $\mathcal{C}^n$-functors; for $\mathcal{C}^n$, besides its duality involution D, inherits involutions from the D's of the $\mathcal{C}^i$ with $i < n$ (cf. Remark III.2.12).

Another interesting example of a cartesian closed category is that of simplicial sets (i.e. complete semi-simplicial complexes). This is best viewed as a functor category, and as such will be treated in a later paper.

If $\mathcal{W}_0$ is any category admitting finite products then, although the cartesian monoidal structure on $\mathcal{W}_0$ may not be closed, it may be possible to find a full product-preserving embedding of $\mathcal{W}_0$ into a category $\mathcal{V}_0$ whose cartesian monoidal structure is closed. An example of this is SPANIER's [16] embedding of the category of topological spaces in that of quasi-topological spaces, which is a cartesian closed category. The "compactly defined" hausdorff spaces (sometimes called k-spaces; cf. RONALD BROWN [5]) form a full closed cartesian subcategory. A detailed examination and generalization of Spanier's construction will appear elsewhere.

3. Closed Categories with one Object

Let M be an abelian monoid, written multiplicatively, and let $\mathscr{V}_0$ be the category with a single object I and with M for the monoid $\mathscr{V}_0(II)$ of endomorphisms of I. Define $I \otimes I = I$, and $f \otimes g = fg$ for $f, g \in M$; then $\otimes$ is a functor, and gives $\mathscr{V}_0$ the structure of a symmetric monoidal category $\mathscr{V}$ if we take a, l, r, c all to be 1. $\mathscr{V}$ is in fact closed, with $(II) = I$ and $(f, g) = fg$; it suffices to take 1 for the adjunction π. Then i, L, p all turn out to be 1, and $V : \mathscr{V}_0 \to \mathscr{S}$ is given by $VI = M$ (regarded as a set) and $(Vf)\, g = fg$.

It is an easy exercise to show that any closed category $\mathscr{V}$ with a single object must be isomorphic to that constructed above for some M. If M consists of the identity alone, we obtain the closed category $\mathscr{I}$ with one object and one morphism.

With $\mathscr{V}$ as above let $\mathscr{G}$ be the closed category of abelian groups and consider the (not necessarily symmetric) closed functors

$$\Phi = (\phi, \tilde{\phi}, \phi^0) : \mathscr{V} \to \mathscr{G}.$$

First, ϕI is to be some abelian group A; and to be a functor, ϕ must map the monoid M into the monoid of endomorphisms of A; let us write fa for $(\phi f)a$, where $f \in M$ and $a \in A$. Next we have $\tilde{\phi} : \phi I \otimes \phi I \to$ $\to \phi(I \otimes I)$, that is, $\tilde{\phi} : A \otimes A \to A$; write ab for $\tilde{\phi}(a \otimes b)$. The naturality of $\tilde{\phi}$ is expressed by: $(fa)\,(gb) = (fg)\,(ab)$, for $f, g \in M$ and $a, b \in A$. Finally we have $\phi^0 : \boldsymbol{Z} \to \phi I$; write 1 for $\phi^0 1 \in A$. The axioms MF1—MF3 give $1a = a$, $a1 = a$, and $(ab)\,c = a(bc)$. Thus a closed functor $\Phi : \mathscr{V} \to \mathscr{G}$ is just an algebra over the monoid ring $\boldsymbol{Z}(M)$ of M, and the closed functor Φ is symmetric if and only if this algebra is commutative. One easily verifies that a closed natural transformation $\Phi \to \Psi$ corresponds to a morphism of $\boldsymbol{Z}(M)$-algebras.

One can generalize by considering closed functors $\mathscr{V} \to \mathscr{W}$, where $\mathscr{W}$ is any (not necessarily symmetric monoidal) closed category, and so obtain what we might call a $\mathscr{W}$-algebra over $\boldsymbol{Z}(M)$. Again we may suppose that M is itself a ring, so that $\mathscr{V}_0$ is pre-additive; if we restrict $\Phi : \mathscr{V} \to \mathscr{G}$ to be additive, it corresponds to an M-algebra.

In particular, closed functors $\mathscr{I} \to \mathscr{G}$ correspond to rings, and closed functors $\mathscr{I} \to \mathscr{S}$ to monoids.

4. Ordered Sets

Any full subcategory of $\mathscr{S}$ that is closed under $A \times B$ and (AB) has a cartesian closed structure consistent with that of $\mathscr{S}$. Thus we get the closed category of finite sets; and if n is any integer > 1 or is ∞, we get a still smaller closed category by excluding all sets of cardinal c with $1 < c < n$.

Taking $n = \infty$ gives the closed category of sets with at most one element, a closed category that is at once cartesian and algebraic. For our purposes it is more convenient to replace this category by a skeleton, namely the full subcategory of $\mathscr{S}$ determined by the empty set $\emptyset$ and a fixed one-element set $*$ whose only member is also called $*$. This category admits the cartesian monoidal structure given by $* \times * = *$, $A \times B = \emptyset$ otherwise; but to make it closed we must replace it by an isomorph, which we do by relabelling each of its three morphisms $\emptyset \to \emptyset \to * \to *$ by the same symbol, namely $*$. Then we can take $(* \, \emptyset) = \emptyset$ and $(A\,B) = *$ otherwise, and we get a closed category $\mathscr{T}$ (for "tiny"). $T : \mathscr{T}_0 \to \mathscr{S}$ is given by $T\,\emptyset = \emptyset$ and $T\,* = *$, and thanks to our relabelling we have $T(A\,B) = \mathscr{T}_0(AB)$ as required.

It is clear that a $\mathscr{T}$-category is a category $\mathscr{A}$ in which each $\mathscr{A}(AB) = \emptyset$ or $*$, and that a $\mathscr{T}$-functor is just a functor. In such a category all diagrams commute, and any category in which all diagrams commute becomes a $\mathscr{T}$-category when we relabel all its morphisms with the same symbol $*$. If we write $A < B$ whenever $\mathscr{A}(AB) = *$, we see that a small $\mathscr{T}$-category $\mathscr{A}$ is the same thing as a pre-ordered set, i.e. a set $\mathscr{A}$ with a binary relation $A < B$ satisfying

$$A < B \quad \text{and} \quad B < C \quad \text{imply} \quad A < C,$$
$$A < A;$$

while a $\mathscr{T}$-functor is an order-preserving map.

If $A < B$ and $B < A$ then A and B are isomorphic and we write $A \sim B$; if $A \sim B$ implies $A = B$, the pre-order is an order and the category $\mathscr{A}$ is skeletal. The passage from a pre-ordered set $\mathscr{A}$ to the associated ordered set $\overline{\mathscr{A}}$, consisting in factoring out the equivalence relation $A \sim B$, corresponds to the passage from the category $\mathscr{A}$ to a skeleton $\overline{\mathscr{A}}$.

A monoidal structure on a given pre-ordered set (i.e. small $\mathscr{T}$-category) $\mathscr{V}_0$ is determined by a function $A \otimes B$ and an object I of $\mathscr{V}_0$; the fact that $\otimes$ is a functor is expressed by the condition

$$A < B \quad \text{implies} \quad A \otimes C < B \otimes C \quad \text{and} \quad C \otimes A < C \otimes B, \tag{4.1}$$

while the existence of a, r, l (which are then unique and coherent) is expressed by

$$(A \otimes B) \otimes C \sim A \otimes (B \otimes C), \quad A \otimes I \sim A, \quad I \otimes A \sim A. \tag{4.2}$$

This monoidal category has a normalization given by

$$VA = * \quad \text{if} \quad I < A, \quad VA = \emptyset \quad \text{otherwise}. \tag{4.3}$$

For it to be closed we need a function (BC) satisfying

$$A \otimes B < C \quad \text{if and only if} \quad A < (BC)\,; \tag{4.4}$$

condition (ii) of Theorem II.5.5 is automatically satisfied. The existence of i, L, p now implies

$$A \sim (IA)\,, \tag{4.5}$$

$$(BC) < ((AB)(AC))\,, \tag{4.6}$$

$$(A \otimes B, C) \sim (A(BC))\,. \tag{4.7}$$

Clearly a monoidal structure $\mathscr{V}$ on $\mathscr{V}_0$ induces under passage to the quotient a monoidal structure $\overline{\mathscr{V}}$ on the skeleton $\overline{\mathscr{V}}_0$, which is closed if $\mathscr{V}$ is; we have only to replace $\sim$ by $=$ in the above.

As an example let $\mathscr{V}_0$ be both an ordered set and a group, the two structures being related by

$$A < B \quad \text{implies} \quad AC < BC \quad \text{and} \quad CA < CB\,. \tag{4.8}$$

Then if we take $A \otimes B$ to be AB and I to be 1, (4.1) and (4.2) are satisfied, and we have a monoidal structure, which is symmetric if the group is abelian. This monoidal structure is closed, for (4.4) is satisfied with $(BC) = CB^{-1}$. Note the special case when $\mathscr{V}_0$ is given the trivial (i.e. discrete) order.

A further example, suggested by Lawvere, is the following. Let $\mathscr{V}_0$ be a pre-ordered set with finite products; thus there is a greatest element 1 and there is a greatest lower bound $A \wedge B$ of any two elements A, B. Now suppose that the cartesian monoidal structure $\mathscr{V}$ is closed, and write $B \Rightarrow C$ instead of (BC). Then (4.4) becomes

$$A \wedge B < C \quad \text{if and only if} \quad A < B \Rightarrow C\,, \tag{4.9}$$

while its consequences (4.5)—(4.7) become

$$A \sim 1 \Rightarrow A\,, \tag{4.10}$$

$$B \Rightarrow C < (A \Rightarrow B) \Rightarrow (A \Rightarrow C)\,, \tag{4.11}$$

$$(A \wedge B) \Rightarrow C \sim A \Rightarrow (B \Rightarrow C)\,. \tag{4.12}$$

A pre-ordered set with the above properties is called a *Brouwerian logic*, the motivation being as follows. Let $\mathscr{V}_0$ be the set of all sentences in some given first-order theory or some propositional calculus, classical or intuitionistic, and interpret "$A < B$" as "A entails B", "$A \wedge B$" as "A and B", and "$A \Rightarrow B$" as "A implies B".

Assume now that $\mathscr{V}_0$ has a least element 0, and define *negation* as

$$A^{\#} = A \Rightarrow 0\,. \tag{4.13}$$

Then (4.9) with $C = 0$ gives

$$A \wedge B \sim 0 \quad \text{if and only if} \quad A < B^{\#} \tag{4.14}$$

which gives an alternative definition of $B^{\#}$. Since the functor $A \Rightarrow B$ is contravariant in A, we have

$$A < B \quad \text{implies} \quad B^{\#} < A^{\#}. \tag{4.15}$$

The Brouwerian logic is said to be *classical* if it has a least element 0 and if the negation satisfies

$$A^{\#\#} \sim A. \tag{4.16}$$

Theorem 4.1. (LAWVERE). *For a Brouwerian logic $\mathscr{V}_0$ the following three conditions are equivalent:*

(i) *$\mathscr{V}_0$ is classical.*

(ii) *The dual $\mathscr{V}_0^*$ (i.e. the set $\mathscr{V}_0$ with the order reversed) is also a Brouwerian logic and the two negations are isomorphic.*

(iii) *The ordered set $\overline{\mathscr{V}}_0$ associated to the pre-ordered set $\mathscr{V}_0$ is a Boolean algebra.*

Proof. (i) *implies* **(ii).** By (4.15) and (4.16), $^{\#}$ is an order-reversing involution and so $\mathscr{V}_0$ has least upper bounds given by de Morgan's law

$$A \vee B = (A^{\#} \wedge B^{\#})^{\#}$$

and $\mathscr{V}_0^*$ is also a Brouwerian logic. The dual of (4.14) shows that the negation in $\mathscr{V}_0^*$ is again $^{\#}$.

(ii) *implies* **(iii).** By (4.14) applied to $\mathscr{V}_0^*$, which has the same negation as $\mathscr{V}_0$, we have

$$A \vee B \sim 1 \quad \text{if and only if} \quad B^{\#} < A.$$

Combining this with (4.14) we have

$$A \wedge B \sim 0 \quad \text{and} \quad A \vee B \sim 1 \quad \text{if and only if} \quad B^{\#} \sim A.$$

Since the relation on the left between A and B is symmetric, we have

$$B^{\#} \sim A \quad \text{if and only if} \quad A^{\#} \sim B,$$

that is, $A^{\#\#} \sim A$. From this and (4.15), $^{\#}$ is an order-reversing involution, and so we have de Morgan's law

$$(A \vee B)^{\#} = A^{\#} \wedge B^{\#}.$$

From (4.14) with $A = B^{\#}$ we get

$$B \wedge B^{\#} \sim 0.$$

Because $- \wedge B$ has a coadjoint it commutes with coproducts, giving

$$(A \vee C) \wedge B = (A \wedge B) \vee (C \wedge B).$$

Thus we have a Boolean algebra.
(iii) *implies* **(i)**: trivial.

Corollary 4.2. *In a classical logic we have*

$$B \Rightarrow C \;\sim\; B^{\#} \vee C.$$

Proof. In a Boolean algebra, $A \wedge B < C$ if and only if $A < B^{\#} \vee C$.

The following is an example of a non-classical Brouwerian logic. Let X be a topological space, and let $\mathscr{V}_0$ be the set of open subsets of X ordered by

$$A < B \quad \text{if and only if} \quad \bar{A} \supset B$$

where $\bar{A}$ denotes the closure of A. Then $A \wedge B$ is $A \cup B$, and

$$A \wedge B < C \quad \text{iff} \quad \overline{A \cup B} \supset C \quad \text{iff} \quad \bar{A} \cup \bar{B} \supset C \quad \text{iff} \quad \bar{A} \supset C - \bar{B};$$

thus (4.9) is satisfied with $C - \bar{B}$ for $B \Rightarrow C$. The greatest element 1 is $\emptyset$ and the least element 0 is X; $A^{\#}$ being $A \Rightarrow 0$ is $X - \bar{A}$. Thus $A^{\#\#}$ is the interior of $\bar{A}$, and is in general different from A.

Finally let $\mathscr{V}_0$ be a linearly ordered set with a greatest element 1. Then it is a Brouwerian logic, for $A \wedge B = \min(A, B)$ and we obtain (4.9) if we set $B \Rightarrow C$ equal to 1 if $B < C$ and equal to C otherwise. If $\mathscr{V}_0$ has a least element 0 we find that $A^{\#} = 0$ if $A \neq 0$ while $0^{\#} = 1$. We have $A^{\#\#} = 0$ or 1, so that $\mathscr{V}_0$ is classical if and only if it has either one or two elements, i.e., if and only if it is either $\mathscr{I}$ or $\mathscr{T}$.

5. Modules over Algebras

Let Λ be an algebra over the commutative ring K and let $\mathscr{V}_0$ be the category of two-sided Λ-modules. For $A, B \in \mathscr{V}_0$ define $A \otimes B$ to be $A \otimes_\Lambda B$, made into a two-sided Λ-module by using the left Λ-operation on A and the right Λ-operation on B. With Λ itself as I and the obvious definitions of a, r, l we obtain a monoidal category $\mathscr{V}$ (not in general symmetric) over $\mathscr{M}K$.

This monoidal structure is closed, for we have $\pi : \mathscr{V}_0(A \otimes B, C) \cong \cong \mathscr{V}_0(A(BC))$, where (BC) is the K-module of those K-morphisms $f : B \to C$ satisfying $f(b\lambda) = (fb)\lambda$ for $b \in B$ and $\lambda \in \Lambda$, made into a two-sided Λ-module by setting

$$(\gamma f \lambda) b = \gamma(f(\lambda b)), \quad b \in B, \quad \gamma, \lambda \in \Lambda.$$

Then i and p turn out to have their expected values, and L to correspond to the usual composition. The basic functor $V : \mathscr{V}_0 \to \mathscr{M}K$ takes $A \in \mathscr{V}_0$ to the K-module $\{a \in A \mid \lambda a = a\lambda \text{ for all } \lambda \in \Lambda\}$.

Now suppose that Λ is a Hopf algebra over K, with co-algebra structure given by algebra-morphisms $\varepsilon : \Lambda \to K$, $\eta : \Lambda \to \Lambda \otimes \Lambda$. Let

$\mathscr{W}_0$ be the category of left Λ-modules, and for $A, B \in \mathscr{W}_0$ define $A \otimes B$ to be $A \otimes_K B$, which is at first a $(\Lambda \otimes \Lambda)$-module, and which we make into a Λ-module by pull-back along η. Similarly make the K-module K into a Λ-module by pull-back along ε. Then the a, r, l of $\mathscr{M}K$ are easily verified to be Λ-morphisms, and define on $\mathscr{W}_0$ a monoidal structure $\mathscr{W}$ with K as I. $\mathscr{W}$ is symmetric if and only if the co-algebra structure of Λ is commutative. Define a normalization $W: \mathscr{W}_0 \to \mathscr{M}K$ of $\mathscr{W}$ by setting

$$WA = \{a \in A \mid \lambda a = (\varepsilon \lambda) a \quad \text{for all} \quad \lambda \in \Lambda\};$$

then we have $\iota: WA \cong \mathscr{W}_0(KA)$ where $(\iota a) k = k a$.

Now the right operation of Λ on itself gives to $\Lambda \otimes B$ the structure of a right Λ-module, and this in turn gives to $\mathscr{W}_0(\Lambda \otimes B, C)$ the structure of a left Λ-module which we call (BC). We have an isomorphism of K-modules

$$\mathscr{W}_0((\Lambda \otimes B) \otimes_\Lambda A, C) \cong \mathscr{W}_0(A, \mathscr{W}_0(\Lambda \otimes B, C)) \tag{5.1}$$

where $(\Lambda \otimes B) \otimes_\Lambda A$ gets its Λ-module structure from the left Λ-module structure of $\Lambda \otimes B$. The right member of (5.1) is $\mathscr{W}_0(A(BC))$; we assert that the left member is isomorphic to $\mathscr{W}_0(A \otimes B, C)$. Indeed the isomorphism

$$(\Lambda \otimes_K B) \otimes_\Lambda A \cong (\Lambda \otimes_\Lambda A) \otimes_K B \cong A \otimes B$$

is an isomorphism of $(\Lambda \otimes \Lambda)$-modules, and so *a fortiori* of Λ-modules. Thus, since $\mathscr{W}_0$ admits transport of structure, $\mathscr{W}$ is closed. (It is in fact necessary to replace the above (BC) by an isomorph in order to get actual equality $W(BC) = \mathscr{W}_0(BC)$.)

6. Complexes and Graded Modules

Let K be a commutative ring. A complex over K is a diagram

$$A: \quad \cdots \longrightarrow A_n \xrightarrow{d} A_{n-1} \longrightarrow \cdots$$

in the category of K-modules, satisfying $dd = 0$; and a morphism of complexes is a morphism of diagrams, so that we have a category $\mathscr{C}_0 K$. The category $\mathscr{G}_0 K$ of graded K-modules is the full subcategory determined by the complexes A in which $d = 0$.

In $\mathscr{C}_0 K$ we define a tensor product $A \otimes B$ by

$$(A \otimes B)_n = \sum A_p \otimes B_q, \quad p + q = n;$$
$$d(a \otimes b) = da \otimes b + (-1)^p a \otimes db, \quad a \in A_p, \quad b \in B_q.$$

For I we take the complex, denoted by K, which has $K_0 = K$ and $K_n = 0$ for $n \neq 0$. With the obvious definitions of a, r, l we obtain a monoidal category $\mathscr{C}K$ over $\mathscr{M}K$. We easily verify that $\mathscr{C}K$ is closed,

(BC) being given by

$$(BC)_n = \prod (A_p, B_{n+p}), \quad p \in \mathbf{Z};$$
$$(df)_p a = d(f_p a) + (-1)^{p+1} f_{p-1} da, \quad f \in (AB)_n, \quad a \in A_p.$$

Then p and i turn out to have their expected values, and L and M correspond to the usual composition: for instance the n-component of M, mapping $\sum_{p+q=n} (BC)_p \otimes (AB)_q$ into $(AC)_n$, takes $g \otimes f$ to gf, where $(gf)_r = g_{r+p} f_r$ for $g \in (BC)_q$ and $f \in (AB)_p$. The basic functor $\mathscr{C}_0 K \to \mathscr{M} K$ turns out to be Z_0, the functor sending each complex over K to its K-module of 0-cycles; the morphisms $A \to B$ are clearly the 0-cycles of (AB).

The subcategory $\mathscr{G}_0 K$ is closed under $A \otimes B$ and (AB), and so inherits from $\mathscr{C} K$ the structure of a monoidal closed category $\mathscr{G} K$ over $\mathscr{M} K$; the functor $Z_0 : \mathscr{C}_0 K \to \mathscr{M} K$ when restricted to $\mathscr{G}_0 K$ merely sends each graded K-module A to its 0-component.

We now discuss possible symmetries for $\mathscr{C} K$ and $\mathscr{G} K$. If c is such a symmetry, consider the object $K^p \in \mathscr{G} K$ satisfying $K^p_p = K$, $K^p_q = 0$ if $q \neq p$, and let $1^p \in K^p_p$ be the identity of K. Since

$$K^p \otimes K^q \cong K^{p+q} \cong K^q \otimes K^p,$$

it follows that

$$c(1^p \otimes 1^q) = \varepsilon(p,q)\, 1^q \otimes 1^p \quad \text{where} \quad \varepsilon(p,q) \in K.$$

By naturality it follows that $c : A \otimes B \to B \otimes A$ must be given by

$$c(a \otimes b) = \varepsilon(p,q)\, b \otimes a, \quad a \in A_p, \quad b \in B_q.$$

The conditions MC6, MC7 on c now become

$$\varepsilon(p,q)\,\varepsilon(q,p) = 1,$$
$$\varepsilon(p,q+r) = \varepsilon(p,q)\,\varepsilon(p,r).$$

If we set

$$k = \varepsilon(1,1)$$

we have

$$k^2 = 1$$

and

$$c(a \otimes b) = k^{pq}(b \otimes a), \quad a \in A_p, \quad b \in B_q.$$

For the category $\mathscr{G} K$ there are no further conditions, and thus we have one symmetry for every $k \in K$ with $k^2 = 1$; in particular we can take k to be 1 or -1, getting in one case $c(a \otimes b) = b \otimes a$ and in the other $c(a \otimes b) = (-1)^{pq}(b \otimes a)$; if $K = \mathbf{Z}$ these are the only symmetries for $\mathscr{G} K$. For the category $\mathscr{C} K$ however we still must ensure that c commutes with the differentiation d. Taking $p = q = 1$ we have

$$dc(a \otimes b) = k\,d(b \otimes a) = k\,db \otimes a - k\,b \otimes da\,;$$
$$cd(a \otimes b) = c(da \otimes b) - c(a \otimes db) = b \otimes da - db \otimes a\,.$$

Since this is to hold for all a and b we must have $k = -1$; and for $k = -1$ we do in fact have $dc = c\,d$. Thus $\mathscr{C}K$ has a unique symmetry $c(a \otimes b) = (-1)^{pq}\, b \otimes a$.

We define three closed functors $\Phi, Z, H : \mathscr{C}K \to \mathscr{G}K$, which are symmetric if $\mathscr{G}K$ is given the symmetry with $k = -1$. The functor

$$\Phi : \mathscr{C}_0 K \to \mathscr{G}_0 K$$

forgets the differential structure, so that ΦA is A considered merely as a graded module; $\tilde{\Phi}$ and Φ^0 are the identity; clearly Φ is not normal. The functor Z (resp. H) assigns to each complex A its cycles ZA (resp. its homology HA) regarded as an object of $\mathscr{G}K$. $\tilde{Z}$ and $\hat{Z}$ are the usual natural transformations

$$ZA \otimes ZB \to Z(A \otimes B)\,, \quad Z(AB) \to (ZA, ZB)\,,$$

and similarly for H; Z^0 and H^0 are the identity. It is clear that Z is normal while H is not.

There is a completely different monoidal closed structure $\mathscr{G}'K$ on $\mathscr{G}_0 K$ given by

$$(A \otimes' B)_n = A_n \otimes B_n\,,$$
$$(AB)'_n = (A_n, B_n)\,,$$
$$I'_n = K \quad \text{for all } n;$$

the basic functor $\mathscr{G}_0 K \to \mathscr{M}K$ in this case sends A to $\prod A_n$, and is faithful.

7. Simplicial Complexes

A simplicial complex A is a set together with a family of finite subsets of A called the spanning subsets of A; there are two axioms, namely that every subset of a spanning subset spans, and that every single-point subset spans. A morphism or simplicial map $f : A \to B$ is a map of the set A into the set B such that if T spans in A then fT spans in B. There results a category $\mathscr{K}_0$; there is a faithful forgetful functor $K : \mathscr{K}_0 \to \mathscr{S}$ sending the simplicial complex A to A regarded merely as a set, and K is represented by the simplicial complex I consisting of a single point, which is also the terminal object of $\mathscr{K}_0$.

We shall describe three different structures of closed category on $\mathscr{K}_0$, in each of which K is the basic functor. Two of these are symmetric monoidal; the third is an example of a non-monoidal closed category.

For the first, give $\mathscr{K}_0$ the cartesian monoidal structure $\mathscr{K}$; the product $A \times B$ is the simplicial product, i.e. the product of the underlying sets with $T \subset A \times B$ spanning if and only if its projections in A and in B both span. This monoidal structure is closed, with (AB) consisting of all simplicial maps $f: A \to B$ with a subset $\{f_1, \ldots, f_n\}$ spanning if and only if, for each spanning subset T in A, the set $f_1 T \cup \cdots \cup f_n T$ spans in B.

For the second monoidal structure $\mathscr{K}'$ on $\mathscr{K}_0$, we define $A \otimes' B$ so that it solves the problem of bimorphisms; we take $A \otimes' B$ to be the product $A \times B$ of the underlying sets with, for its spanning sets, those of the forms

$$\begin{aligned} &a \times S, \quad a \in A, \quad S \text{ spans in } B, \\ &T \times b, \quad T \text{ spans in } A, \quad b \in B. \end{aligned}$$

Again I is the identity for $\otimes'$, and $\mathscr{K}'$ is symmetric and closed; $(AB)'$ consists of the simplicial maps $f: A \to B$, with $\{f_1, \ldots, f_n\}$ spanning if and only if, for each $a \in A$, the set $\{f_1 a, \ldots, f_n a\}$ spans in B.

The third closed structure $\mathscr{K}''$ is not monoidal, and we start by defining $(AB)''$ to consist of the simplicial maps $f: A \to B$, with $\{f_1, \ldots, f_n\}$ spanning if and only if either $n = 1$ or $f_1 A \cup \cdots \cup f_n A$ spans in B. We easily verify conditions (i)—(iii) of Proposition I.2.11, so that since K is faithful we have a closed category.

To see that $\mathscr{K}''$ is not monoidal, it suffices to show that the functor $(A\, -)''$ does not preserve products and so cannot have an adjoint. Let s^n denote the complex consisting of $n+1$ points with all subsets spanning, and let $+$ denote the coproduct in $\mathscr{K}_0$, i.e., the disjoint union; we write ps^n for the p-fold coproduct $s^n + \cdots + s^n$. Then

$$\begin{aligned} (2s^0, 2s^0)'' &= 4s_0, \\ (2s^0, s^1)'' &= s^3, \\ (2s^0, 2s^0 \times s^1)'' = (2s^0, 2s^1)'' &= 2s^3 + 8s^0. \end{aligned}$$

References

[1] Bénabou, J., Catégories avec multiplication. C. R. Acad. Sci. Paris **256** 1887—1890, (1963)

[2] —, Algèbre élémentaire dans les catégories avec multiplication. C. R. Acad. Sci. Paris **258** 771—774, (1964)

[3] —, Catégories relatives. C. R. Acad. Sci. Paris **260** 3824—3827, (1965)

[4] Birkhoff, G., and Mac Lane, S., Algebra. Macmillan (1967) (to appear).

[5] Brown, R., Function spaces and product topologies. Quart. J. Math. Oxford Ser. II **15** 238—250, (1964)

[6] Ehresmann, C., Catégories structurées. Ann. Sci. École Norm. Sup. **80** 349—425, (1963)

[7] Eilenberg, S., and Kelly, G. M., A generalization of the functorial calculus. J. Algebra **3** 366–375, (1966)
[8] Freyd, P., Algebra-valued functors in general and tensor products in particular. Colloq. Math. **14** 89–106, (1966)
[9] Kelly, G. M., On Mac Lane's conditions for coherence of natural associativities, commutativities, etc. J. Algebra **1** 397–402, (1964)
[10] — Tensor products in categories. J. Algebra **2** 15–37, (1965)
[11] Lawvere, F. W., Functorial semantics of algebraic theories. (Dissertation, Columbia Univ., 1963).
[12] Linton, F. E. J., Autonomous categories and duality of functors. J. Algebra **2** 315–349, (1965)
[13] — Autonomous equational categories. J. Math. Mech. (to appear).
[14] MacLane, S., Natural associativity and commutativity. Rice Univ. Studies **49** 28–46, (1963)
[15] — Categorical algebra. Bull. Amer. Math. Soc. **71** 40–106, (1965)
[16] Spanier, E., Quasi-topologies. Duke Math. J. **30** 1–14, (1963)
[17] Yoneda, N., On the homology theory of modules. J. Fac. Sci. Univ. Tokyo Sect. I. **7** 193–221, (1954)

Department of Mathematics
Columbia University
New York, N.Y. 10027

Pure Mathematics Department
University of Sydney
Sydney, N. S. W.