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TENSOR PRODUCT OF OPERADS AND ITERATED LOOP SPACES

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An algebraic characterization of an n-fold loop space in terms of its n different 1-fold loop structures is established. This amounts to describing the higher homotopy commutativity for such a space as a strict partial commutativity of the 1-fold loop structures. The tensor product of operads (a special case of the construction for algebraic theories) is ideally suited for this. In particular we show that the operad of little n-cubes C_n is homotopy equivalent to the n-fold tensor product $C_1^{\otimes n}$, i.e., 'tensoring these A_∞-structures yields an iterated loop structure'. This is not true for arbitrary A_∞-operads.

Introduction

The use of the tensor product construction for algebraic theories, along with other ideas from universal algebra, was introduced into iterated loop space theory by Boardman and Vogt [2, 3]. In [4] they specialized to the case of PROPS in order to obtain a consistency statement for their delooping of an infinite loop space. A further restriction yields a tensor product construction for operads.

In this paper the tensor product of operads is used to relate precisely a given n-fold loop space structure on a space Y to its various 1-fold loop space structures in the following sense. Let $\mathscr{C}_n$ denote the operad of little n-cubes and let Y be a connected $\mathscr{C}_n$-space [11]. Then Y is a $\mathscr{C}_1$-space in n distinct ways via the obvious inclusions $\alpha^i : \mathscr{C}_1 \to \mathscr{C}_n$, $1 \le i \le n$. Conversely, if Y is a $\mathscr{C}_1$-space in n distinct ways we can ask when Y is (equivalent to) a $\mathscr{C}_n$-space. This is the case exactly when these $\mathscr{C}_1$-actions pairwise interchange in the sense that each is a morphism with respect to the others.

For a precise statement, let $\mathscr{C}_1^{\otimes n}$ denote the n-fold tensor product of $\mathscr{C}_1$ with itself. The tensor product has a universal property which together with the maps α^i determine a map $\alpha : \mathscr{C}_1^{\otimes n} \to \mathscr{C}_n$. Our main result, Theorem 2.9, is that α is a local Σ-equivalence, i.e. each $\alpha_j : \mathscr{C}_1^{\otimes n}(j) \to \mathscr{C}_n(j)$ is a Σ_j-equivariant homotopy equivalence. This fact is the main ingredient for showing that the n-fold delooping

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constructions of May and Segal agree up to homotopy. The equivalence of the two constructions is a consequence of a general axiomatic characterization of n-fold delooping machines which will appear in a subsequent paper.

Theorem 2.9 is proved by showing the following two statements:

(i) α induces an isomorphism of $\mathscr{C}_1^{\otimes n}$ onto a suboperad $\mathscr{D}_n$ of $\mathscr{C}_n$.

(ii) The inclusion $\mathscr{D}_n \to \mathscr{C}_n$ is a local Σ-equivalence.

The operad $\mathscr{D}_n$ has a simple geometric description from which the second statement easily follows. The proof of the first statement, however, is somewhat involved, so we will give a sketch of it here.

As is pointed out in [4], little is known about the tensor product construction for algebraic theories of which the tensor product of operads is a special case. Thus a direct comparison of $\mathscr{C}_1^{\otimes n}$ and $\mathscr{C}_n$ is not feasible. Instead we exploit certain algebraic properties of $\mathscr{C}_1^{\otimes n}$.

We first look at a familiar example of this when $n = 1$. Any loop space ΩX has a product which is homotopy associative and has a homotopy unit. By passing to the space of Moore loops on X, we obtain a strictly associative product with a strict unit without changing the homotopy type.

Formally, we define a category $\mathscr{T}[\mathbb{R}_+]$ and a functor $R: \mathscr{T} \to \mathscr{T}[\mathbb{R}_+]$ on spaces for which $R\Omega X$ is the space of Moore loops. This functor converts algebraic properties up to homotopy into strict algebraic properties. For example, if Y is any $\mathscr{C}_1$-space, then RY is a topological monoid.

Now given an n-fold loop space $\Omega^n X$, $n > 1$, we must also consider higher homotopy commutativity. The same functor R converts this to a type of strict commutativity, called semi-interchange, on the space of n-fold Moore loops $R\Omega^n X$. Specifically, $R\Omega^n X$ has n different monoid multiplications which are related by the semi-interchange property, and we call this an n-fold monoid structure (Definition 1.9). This is the analogue for n-fold loop spaces of the well known fact that any A_∞-space is equivalent to a topological monoid.

A key point in the proof of Theorem 2.9 is the observation that $RC_1^{\otimes n}X$ is an n-fold monoid, where $C_1^{\otimes n}$ is the associated monad, and that the semi-interchange condition corresponds exactly to the defining property of the tensor product. Thus the notion of n-fold monoid precisely captures the operad structure of $\mathscr{C}_1^{\otimes n}$ (see Proposition 1.11).

In Section 1, we also construct the free n-fold monoid J_nRX (Construction 1.10). This is a generalization of the classical James construction to n-fold loop spaces and should be of some independent interest. We then obtain a commutative diagram

$$\begin{tikzcd} J_nRX \arrow[r, "\tau"] \arrow[dr, "\beta"'] & RC_1^{\otimes n}X \arrow[d, "R\alpha"] \\ & RC_nX \end{tikzcd}$$

in which τ is an isomorphism of n-fold monoids.

In Section 2 we show that β is a closed inclusion with image RD_nX, and it follows that we have a homeomorphism $C_1^{\otimes n}X \to D_nX$ on the monad level. It is now an easy matter to pass back to operads and obtain an isomorphism $\alpha : \mathscr{C}_1^{\otimes n} \to \mathscr{D}_n$. We conclude Section 2 by extending these results to infinite loop spaces.

In addition, there are two appendices. Appendix A contains some additional results on tensor products of operads, including an extension of the main theorem. Appendix B collects a few basic results on cofibrations in $\mathscr{T}[\mathbb{R}_+]$ needed for the proof of Theorem 2.9.

1. Semi-interchangeable monoid structures

The main objective of this section is to prove Proposition 1.11 which relates interchangeable operad actions to semi-interchangeable monoid structures on objects in an appropriate category of spaces $\mathscr{T}(R)$. We begin by describing this and several related categories. The diagram preceeding Proposition 1.11 is a schematic for the results of this section and the reader may find it helpful to refer to it occasionally. We will work in the category $\mathscr{T}$ of compactly generated (weak Hausdorff) spaces with nondegenerate basepoints.

Let $\mathbb{R}_+$ be the non-negative reals with basepoint 0, and write $\mathscr{T}[\mathbb{R}_+]$ for the category of 'spaces over $\mathbb{R}_+$'. The objects are all maps $p: X \to \mathbb{R}_+$ (also written (X, p)) with $p^{-1}(0) = *$, and morphisms are commutative triangles

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \mathbb{R}_+ & \end{tikzcd}$$

A *section* of $p: X \to \mathbb{R}_+$ is a map $\eta: \mathbb{R}_+ \to X$ such that $p \circ \eta = \mathrm{id}$ and η is a cofibration in $\mathscr{T}[\mathbb{R}_+]$; this means $(X, \eta(\mathbb{R}))$ is an NDR pair by a homotopy h with $(ph)(t, x) = p(x)$ for $x \in X$ and $t \in I$. This condition implies that $(p^{-1}(1), \eta(1))$ is an NDR pair which is necessary for Lemma 1.2.

Let $\mathscr{T}^s[\mathbb{R}_+]$ denote the subcategory of $\mathscr{T}[\mathbb{R}_+]$ of objects (X, p, η) with section and maps commuting with sections. These categories are related in the next two lemmas.

Lemma 1.1. (i) *There is a functor $R: \mathscr{T} \to \mathscr{T}[\mathbb{R}_+]$ right adjoint to the forgetful functor $U: \mathscr{T}[\mathbb{R}_+] \to \mathscr{T}$.*

(ii) *The unit $(X, p) \to RU(X, p)$ and counit $URY \to Y$ are homotopy equivalences.*

Proof. For Y any space, let RY be the subspace $\{(y, t) \mid t > 0 \text{ or } y = *\}$ of $Y \times \mathbb{R}_+$ with projection $\pi_2: RY \to \mathbb{R}_+$ given by $\pi_2(y, t) = t$, and define R on maps in the obvious way. (i) and (ii) are easily verified. $\square$

Note that $* \to Y$ induces $\eta_*: \mathbb{R}_+ = R(*) \to RY$, a section of $RY \to \mathbb{R}_+$ by Proposition B.1. Hence we can consider R as a functor $\mathcal{T} \to \mathcal{T}^s[\mathbb{R}_+]$, but it now fails to be right adjoint to U restricted to $\mathcal{T}^s[\mathbb{R}_+]$.

Let $\mathbb{R}_+^*$ denote the multiplicative group of positive reals and let it act on $\mathbb{R}_+$ by multiplication. We define an $\mathbb{R}_+^*$-object in $\mathcal{T}^s[\mathbb{R}_+]$ to be an object (Y, q, η) with an $\mathbb{R}_+^*$-action $\varphi: \mathbb{R}_+^* \times Y \to Y$ for which q and η are equivariant. Let $\mathcal{T}(R)$ denote the subcategory of $\mathcal{T}^s[\mathbb{R}_+]$ consisting of these $\mathbb{R}_+^*$-objects and the equivariant maps.

The following internal characterization of such objects will be useful in several ways. Note that we can regard R as a functor to $\mathcal{T}(R)$, where RX is given the obvious $\mathbb{R}_+^*$-action.

Lemma 1.2. *There is a functor $U_1: \mathcal{T}(R) \to \mathcal{T}$ such that R and U_1 determine an equivalence of categories.*

Proof. This is a consequence of the following two observations:

(i) (Y, q, η) is isomorphic in $\mathcal{T}^s[\mathbb{R}_+]$ to (RX, π_2, η_*) for some space X if and only if there is an $\mathbb{R}_+^*$-action $\varphi: \mathbb{R}_+^* \times Y \to Y$ with q and η equivariant.

(ii) A map g in $\mathcal{T}^s[\mathbb{R}_+]$ of spaces satisfying (i) has $g \cong Rf$ if and only if g is $\mathbb{R}_+^*$-equivariant. In particular, the isomorphism in (i) is of this form.

Proof of (i). Given an action φ define $h: Y \to Rq^{-1}(1)$ by

$$h(y) = \begin{cases} \varphi\left(\dfrac{1}{qy}, y\right) & \text{if } y \neq * , \\ (\eta(1), 0) & \text{if } y = * . \end{cases}$$

We give $q^{-1}(1)$ the basepoint $* = \eta(1)$ which as noted above is non-degenerate. It is easy to see that h is an isomorphism in $\mathcal{T}(R)$ with inverse given by

$$(x, t) \to \begin{cases} \varphi(t, x), & t > 0 , \\ * , & t = 0 . \end{cases}$$

Conversely for any space X, RX has an obvious $\mathbb{R}_+^*$-action for which the projection and section are equivariant, and any isomorphism $h: (Y, q, \eta) \cong (RX, \pi_2, \eta_*)$ gives Y an appropriate $\mathbb{R}_+^*$-action.

Proof of (ii). If $g: (Y_1, q_1, \eta_1) \to (Y_2, q_2, \eta_2)$ is an $\mathbb{R}_+^*$-map of spaces satisfying (i), then we have $g \cong Rf$, where $f: q_1^{-1}(1) \to q_2^{-1}(1)$ is the restriction of g.

Of course we take $U_1 Y = q^{-1}(1)$ and $U_1 g = f$. $\square$

We will also need the notion of $\mathcal{C}$-space in $\mathcal{T}^s[\mathbb{R}_+]$ and the analogue of Lemma 1.2. For any operad $\mathcal{C}$ write $\mathcal{C}[\mathcal{T}]$ for the usual category of $\mathcal{C}$-spaces [11], and denote by $\mathcal{C}[\mathbb{R}_+]$ the category of $\mathcal{C}$-spaces in $\mathcal{T}^s[\mathbb{R}_+]$ defined as follows.

The objects are those (X, p, η) such that X is a $\mathscr{C}$-space and p and η are maps of $\mathscr{C}$-spaces, where $\mathscr{C}$ acts on $\mathbb{R}_+$ by addition. The morphisms are maps in $\mathscr{T}^s[\mathbb{R}_+]$ which are also maps of $\mathscr{C}$-spaces.

The functor $R: \mathscr{T} \to \mathscr{T}^s[\mathbb{R}_+]$ restricts to a functor $R': \mathscr{C}[\mathscr{T}] \to \mathscr{C}[\mathbb{R}_+]$ as follows. If (Z, θ) is a $\mathscr{C}$-space, then the composites

$$\begin{tikzcd} \mathscr{C}(j) \times (RZ)^j \subseteq \mathscr{C}(j) \times Z^j \times \mathbb{R}_+^j \arrow[r, "\theta_j \times \{+\}"] \arrow[dr, "(R'\theta)_j"'] & Z \times \mathbb{R}_+ \\ & RZ \arrow[u, phantom, "\cup\!\mid"] \end{tikzcd}$$

define an action of $\mathscr{C}$ on RZ such that $(RZ, R'\theta)$ is in $\mathscr{C}[\mathbb{R}_+]$.

The analogue of Lemma 1.2 for $\mathscr{C}$-spaces follows from Lemma 1.3 below. Let r be the composite $Y \xrightarrow{h^{-1}} Rq^{-1}(1) \xrightarrow{\text{proj}} q^{-1}(1) \subseteq Y$.

Lemma 1.3. *Let (Y, q, η, φ) be in $\mathscr{T}(R)$ with (Y, θ) in $\mathscr{C}[\mathbb{R}_+]$, so that q and η are $\mathscr{C}$-maps. Suppose θ is also compatible with φ, i.e. $\theta_j(c, \varphi(s, y_1), \ldots, \varphi(s, y_j)) = \varphi(s, \theta_j(y_1, \ldots, y_j))$. The following are equivalent:*

(i) *$\theta = R'\bar{\theta}$ for some $\mathscr{C}$-action $\bar{\theta}$ on $q^{-1}(1)$, i.e.*

$$\begin{tikzcd} \mathscr{C}(j) \times Rq^{-1}(1)^j \arrow[r, "R'\bar{\theta}_j"] \arrow[d, "1 \times h^j"'] & Rq^{-1}(1) \arrow[d, "h"] \\ \mathscr{C}(j) \times Y^j \arrow[r, "\theta_j"] & Y \end{tikzcd}$$

commutes for all j.

(ii) *$\varphi(j, \theta_j(c, y_1, \ldots, y_j)) = \varphi[\sum_{k=1}^{j} qy_k, \theta_j(c, ry_1, \ldots, ry_j)]$ for all j.*

(iii) *the two composites*

$$\begin{tikzcd} \mathscr{C}(j) \times Y^j \arrow[r, shift left, "\theta_j"] \arrow[r, shift right, "\theta_j(1 \times r^j)"'] & Y \arrow[r, "r"] & Y \end{tikzcd}$$

are equal.

Proof. Suppose θ is a $\mathscr{C}$-action satisfying (ii). Define $\bar{\theta}_j$ to be $r \circ \theta_j$ restricted to $\mathscr{C}(j) \times q^{-1}(1)^j$, for $j \geq 1$ and $\bar{\theta}_0(\eta(1)) = \eta(1)$. The condition in (ii) implies $\bar{\theta}$ is a $\mathscr{C}$-action on $q^{-1}(1)$. The remaining verifications are straightforward but tedious. $\square$

The spaces satisfying Lemma 1.3 and the $\mathscr{C}$-maps in $\mathscr{T}(R)$ form a category $\mathscr{C}(R)$, and the functors U_1 of Lemma 1.2 and R' determine functors

$$\begin{tikzcd} \mathscr{C}[\mathscr{T}] \arrow[r, shift left, "R"] & \mathscr{C}(R) . \arrow[l, shift left, "U_1"] \end{tikzcd}$$

This slight abuse of notation should cause no confusion since these functors agree with those of Lemma 1.2 on underlying objects.

Lemma 1.4

$$\mathscr{C}[\mathscr{T}] \underset{U_1}{\overset{R}{\rightleftarrows}} \mathscr{C}(R)$$

is an equivalence of categories.

Proof. The proof of Lemma 1.2 adapts using Lemma 1.3. □

We will be concerned mainly with the case when $\mathscr{C}$ is a tensor product of operads. On the level of operad actions on a space, the following definition describes the defining property of the tensor product:

Definition 1.5. Let $\mathscr{A}$ and $\mathscr{B}$ be operads acting on a space X. We say the two actions *interchange* if the diagrams

$$(*) \qquad \begin{tikzcd} (X^s)^r \arrow[r, equal, "\sigma"] \arrow[drr, "a^r"'] & (X^r)^s \arrow[r, "b^s"] & X^s \arrow[d] \\ & & X^r \arrow[r, "b"] & X \end{tikzcd}$$

commute for all $a \in \mathscr{A}(s)$, $b \in \mathscr{B}(r)$ and all $r, s, \geq 0$. Here a, b etc. denotes action by that element and $\sigma = \sigma_{r,s} \in \Sigma_{rs}$ permutes coordinates according to $\sigma_{r,s}((i-1)s+j) = (j-1)r+i$ for $1 \leq i \leq r$ and $1 \leq j \leq s$.

When $(*)$ commutes for fixed r, s we say that (r, s)-interchange holds; note that this is equivalent to saying (s, r)-interchange holds.

For each s, let $\mathscr{B}$ act 'diagonally' on X^s, i.e. X^s is the s-fold product in the category of $\mathscr{B}$-spaces. Then Definition 1.5 is the requirement that each $a \in \mathscr{A}(s)$ is a morphism of $\mathscr{B}$-spaces, all $r, s \geq 0$, or equivalently each $b \in \mathscr{B}(r)$ is a morphism of $\mathscr{A}$-spaces, all $r, s \geq 0$.

A concrete example of Definition 1.5 is given by the n inclusions $\alpha^i : \mathscr{C}_1 \to \mathscr{C}_n$; if X is a $\mathscr{C}_n$-space, then we show in Example 1.7 below that the induced $\mathscr{C}_1$-actions on X pairwise interchange.

The following result gives a convenient way of verifying interchange in case $\mathscr{A} = \mathscr{B}$:

Proposition 1.6. *Let $\mathscr{A}$ be an operad which satisfies the condition*

(#) *Each $a \in \mathscr{A}(j)$, $j \geq 3$ factors as $a = \gamma(\bar{a}; a_{j_1}, \ldots, a_{j_k})$ for some $\bar{a} \in \mathscr{A}(k)$ and $a_{j_i} \in \mathscr{A}(j_i)$ with $2 \leq k < j$ and $j_i > 0$, all i.*

Let X be a space with two $\mathscr{A}$-actions for which (r, s)-interchange holds when $r, s \leq 2$. Then the two actions interchange.

Proof. By the symmetry of the interchange condition, the obvious induction on pairs (r, s) reduces to the statement: if $r \geq 2$ and (u, v)-interchange holds for $u \leq r$ and $v \leq s$, then $(r+1, s)$-interchange holds.

Now if $a \in \mathscr{A}(r+1)$ and $b \in \mathscr{A}(s)$, we must show $ba^s\sigma_{r+1,s} = ab^{r+1}$. This is a straightforward calculation once a is replaced by its factorization (#). $\square$

Example 1.7. If (X, θ) is a $\mathscr{C}_n$-space, then the $\mathscr{C}_1$-actions induced by $\alpha^i, \alpha^j : \mathscr{C}_1 \to \mathscr{C}_n$ interchange if $i \neq j$.

To see this let $c = \langle c_1, \ldots, c_s \rangle \in \mathscr{C}_1(s)$, $d = \langle d_1, \ldots, d_r \rangle \in \mathscr{C}_1(r)$ and let $\bar{c}, \bar{d}$ denote their images under α^i, α^j respectively. Using the associativity of the action θ, the interchange condition $\bar{c}\bar{d}\sigma_{rs} = \bar{d}\bar{c}$ is implied by $\gamma(\bar{c}; \bar{d}, \ldots, \bar{d}) = \gamma(\bar{d}; \bar{c}, \ldots, \bar{c})$ and this is equivalent to $\bar{c}_k\bar{d}_l = \bar{d}_l\bar{c}_k$ for $1 \leq k \leq s$ and $1 \leq l \leq r$. This is obvious since the linear factors of $\bar{c}_k$ (respectively $\bar{d}_l$) are all the identity except for the ith (respectively jth) and $i \neq j$.

Notice that appeal to Proposition 1.6 in this case would only complicate a trivial argument. However, the proof of Proposition 1.9 below is greatly simplified by the use of Proposition 1.6.

We next discuss briefly the properties of tensor product of operads that we will need. These are existence and the universal property.

Existence is easily deduced from [4] (see [4, discussion after Definition 5.2]) and the well-known relation between PROPS and operads [1, §2.3].

The universal property is most conveniently expressed in terms of May's notion of a *pairing* $\tau : (\mathscr{A}, \mathscr{B}) \to \mathscr{C}$ of operads [13]. This consists of maps $\tau : \mathscr{A}(r) \times \mathscr{B}(s) \to \mathscr{C}(rs)$ for $r, s \geq 0$ such that

(i) If $\mu \in \Sigma_r$ and $\nu \in \Sigma_s$, then

$$\tau(a\mu, b\nu) = \tau(a, b)(\mu \wedge \nu)$$

where $a \in \mathscr{A}(r)$, $b \in \mathscr{B}(s)$ and $\mu \wedge \nu \in \Sigma_{rs}$ is determined by μ and ν.

(ii) If $a_i \in \mathscr{A}(r_i)$, $1 \leq i \leq r$ and $b_j \in \mathscr{B}(s_j)$, $1 \leq j \leq s$, then

$$\gamma\Big(\tau(a, b); \mathop{\times}_{(i,j)} \tau(a_i, b_j)\Big)\omega = \tau\Big(\gamma(a; \mathop{\times}_i a_i), \gamma(b; \mathop{\times}_j b_j)\Big)$$

where ω is an appropriate permutation (see [13, 1.4]).

(iii) $\tau(1, 1) = 1$.

The tensor product $\mathscr{A} \otimes \mathscr{B}$ is universal for such pairings; thus there is a pairing $\tau : (\mathscr{A}, \mathscr{B}) \to \mathscr{A} \otimes \mathscr{B}$ such that any pairing $\sigma : (\mathscr{A}, \mathscr{B}) \to \mathscr{C}$ determines uniquely a map of operads $\bar{\sigma} : \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ for which the following diagrams commute for all r, s:

$$\begin{tikzcd}
\mathscr{A}(r) \times \mathscr{B}(s) \arrow[r, "\tau"] \arrow[dr, "\sigma"'] & (\mathscr{A} \otimes \mathscr{B})(rs) \arrow[d, "\bar{\sigma}"] \\
& \mathscr{C}(rs)
\end{tikzcd}$$

If we delete condition (iii) we obtain the notion of *non-unital pairing*. The universal property in this case differs from the previous one only in that $\bar{\sigma}$ may not preserve units.

Tensor products are commutative and associative up to natural isomorphism [10], so the universal property extends to finitely many factors in the obvious way. Explicitly, the notion of pairing generalizes to that of 'n-linear map' $(\mathscr{A}_1, \ldots, \mathscr{A}_n) \to \mathscr{C}$ for $n \geq 2$, satisfying conditions analogous to those above. $(\mathscr{A}_1, \ldots, \mathscr{A}_n) \to \bigotimes_{i=1}^n \mathscr{A}_i$ is then universal for such n-linear maps.

For example, there is a pairing $\sigma : (\mathscr{C}_1, \mathscr{C}_1) \to \mathscr{C}_2$ given by

$$\sigma(c, d) = \langle c_1 \times d_1, \ldots, c_1 \times d_s, \ldots, c_r \times d_1, \ldots, c_r \times d_s \rangle$$

for $c = \langle c_1, \ldots, c_r \rangle \in \mathscr{C}_1(r)$ and $d = \langle d_1, \ldots, d_s \rangle \in \mathscr{C}_1(s)$. More generally there is an n-linear map $(\mathscr{C}_1, \ldots, \mathscr{C}_1) \to \mathscr{C}_n$, $n \geq 2$, so the universal property provides a map of operads $\alpha : \bigotimes_{i=1}^n \mathscr{C}_1 \to \mathscr{C}_n$.

The following 'local' universal property will also be useful. Note that the n-linear map $\tau : (\mathscr{A}_1, \ldots, \mathscr{A}_n) \to \bigotimes_{i=1}^n \mathscr{A}_1$ restricts to give maps of operads $\tau_j : \mathscr{A}_j \to \bigotimes_{i=1}^n \mathscr{A}_i$ with $\tau_j(a) = \tau(1, \ldots, 1, a, 1, \ldots, 1)$ and a in the jth coordinate. The local universal property now reads as follows:

> If the space X has $\mathscr{A}_i$-actions $1 \leq i \leq n$, such that the $\mathscr{A}_i$- and $\mathscr{A}_j$-actions interchange for all $i \neq j$, then X has a unique $\bigotimes_{i=1}^n \mathscr{A}_i$ action for which each τ_j induces the given $\mathscr{A}_j$-action.

Remark. If $\mathscr{E}X$ is the endomorphism operad of X (see [11]), then the local universal property is just the universality of τ applied to the obvious n-linear map $(\mathscr{A}_1, \ldots, \mathscr{A}_n) \to \mathscr{E}X$.

We denote the n-fold tensor product of $\mathscr{C}_1$ with itself by $\mathscr{C}_1^{\otimes n}$ and its associated monad by $C_1^{\otimes n}$. An object in $\mathscr{C}_1^{\otimes n}[\mathbb{R}_+]$ has an algebraic structure determined by its interchangeable $\mathscr{C}_1$-actions. The nature of this structure is given in the following definition:

Definition 1.8. Let $(X, p, \eta) \in \mathscr{T}^s[\mathbb{R}_+]$ and let $\{\mu_i\}_{i=1}^n$ be monoid multiplications on X (with $*$ as two-sided unit) such that p and η are homomorphisms for each μ_i, where $\mathbb{R}_+$ is considered as a monoid under addition. We say that the $\{\mu_i\}_{i=1}^n$ *semi-interchange* if $\mu_i(\mu_j(x_1, x_2), \mu_j(x_3, x_4)) = \mu_j(\mu_i(x_1, x_3), \mu_i(x_2, x_4))$ whenever

$p(x_1)\cdot p(x_4)=p(x_2)\cdot p(x_3)$ and $i\neq j$. In this case we call X *an n-fold monoid*. A *homomorphism* of n-fold monoids is a map in $\mathcal{T}^s[\mathbb{R}_+]$ which is a homomorphism for ith multiplications, $1\leq i\leq n$.

The following are the most important examples of n-fold monoids:

(i) If X is a space with n interchangeable $\mathscr{C}_1$-actions (e.g. a $\mathscr{C}_n$-space), then RX is an n-fold monoid, (see Proposition 1.9 below).

(ii) The space of n-fold Moore loops $R\Omega^n Y$ is an n-fold monoid.

(iii) The free n-fold monoid on a space X in $\mathcal{T}^s[\mathbb{R}_+]$, (see Construction 1.10 below).

Denote the category of n-fold monoids by $\mathcal{M}_n[\mathbb{R}_+]$, and let $\mathcal{M}_n(R)$ be the subcategory with objects and maps also in $\mathcal{T}(R)$ such that the $\mathbb{R}_+^*$-action φ is compatible with each μ_i, i.e. $\varphi(s,\mu_i(a,b))=\mu_i(\varphi(s,a),\varphi(s,b))$.

The identity functor on $\mathcal{T}^s[\mathbb{R}_+]$ induces a functor $S':\mathscr{C}_1^{\otimes n}[\mathbb{R}_+]\to\mathcal{M}_n[\mathbb{R}_+]$ (Proposition 1.9 below) and its restriction $S:\mathscr{C}_1^{\otimes n}(R)\to\mathcal{M}_n(R)$ is an isomorphism of categories (Proposition 1.11 below). This makes precise the relationship between interchangeable $\mathscr{C}_1^{\otimes n}$-actions and semi-interchangeable monoid structures on objects in $\mathcal{T}(R)$. This is definitely not the case for S' (see the definition of T in Proposition 1.9 below) and is related to the fact that only in $\mathcal{M}_n(R)$ do we have homotopy commutative multiplications. Because of this $\mathscr{C}_1^{\otimes n}[\mathbb{R}_+]$ and $\mathcal{M}_n[\mathbb{R}_+]$ have no particular significance in the present context and should be regarded merely as technical aids for obtaining the results of Proposition 1.11.

The following conventions will be useful here and in Section 2:

Notation. From now on we will write iterated products involving $\{\mu_i\}_{i=1}^n$ without parentheses using the usual conventions. For example, $\mu_i(y_1,y_2)$ becomes $\mu_i y_1 y_2$, associativity is $\mu_i\mu_i y_1 y_2 y_3=\mu_i y_1\mu_i y_2 y_3$ and interchange is $\mu_i\mu_j y_1 y_2\mu_j y_3 y_4=\mu_j\mu_i y_1 y_3\mu_i y_2 y_4$.

We will also write $\theta=\{\theta^i\}_{i=1}^n$ for a $\mathscr{C}_1^{\otimes n}$-action, where θ^i is the $\mathscr{C}_1$-action induced from θ by the ith inclusion $\mathscr{C}_1\to\mathscr{C}_1^{\otimes n}$.

Define elements $c(r_1,\ldots,r_j)$ in $\coprod_{0\leq k\leq j}\mathscr{C}_1(k)$ for $j\geq 2$ and $r_i\in\mathbb{R}_+$, all i, inductively as follows. Let $c(r_1,r_2)=\langle c,d\rangle$ in $\mathscr{C}_1(2)$, where $c(t)=at$ and $d(t)=(1-a)t+a$, $0\leq t\leq 1$, where $a=r_1/(r_1+r_2)$ when $r_1,r_2>0$. Also let $c(r,0)=c(0,r)=1$ in $\mathscr{C}_1(1)$, if $r>0$ and $c(0,0)=0\in\mathscr{C}_1(0)$.

Suppose $c(r_1,\ldots,r_{j-1})$ is defined and let $s=\sum_{i=1}^{j-1}r_i$. Then let

$$c(r_1,\ldots,r_j)=\begin{cases}\gamma(c(s,r_j);c(r_1,\ldots,r_{j-1}),1) & \text{if } s,r_j>0\,,\\ c(0,r_j) & \text{if } s=0\,,\\ c(r_1,\ldots,r_{j-1}) & \text{if } r_j=0\,.\end{cases}$$

Proposition 1.9. *There are functors* $S':\mathscr{C}_1^{\otimes n}[\mathbb{R}_+]\to\mathcal{M}_n[\mathbb{R}_+]$ *and* $T:\mathcal{M}_n(R)\to\mathscr{C}_1^{\otimes n}(R)$ *which are the identity on the underlying objects.*

Proof. S' and T are both induced by the identity on $\mathcal{T}^s[\mathbb{R}_+]$, so we need to show how a $\mathcal{C}_1^{\otimes n}$-action determines an n-fold monoid structure, and conversely.

Given (Y, q, η) with $\mathcal{C}_1^{\otimes n}$-action $\theta = \{\theta^i\}_{i=1}^n$, define

$$\mu_i y_1 y_2 = \begin{cases} \theta_2^i(c(qy_1, qy_2), y_1, y_2) & \text{if } qy_k \neq 0\,, \\ y_1 & \text{if } qy_2 = 0\,, \\ y_2 & \text{if } qy_1 = 0\,. \end{cases}$$

Note that $* = \eta(0) \in Y$ is a two-sided unit for each μ_i. For notational reasons we will restrict our calculations to $Y - *$; the remaining cases are easily checked.

Associativity. Using the associativity of θ^i we have

$$\mu_i y_1 \mu_i y_2 y_3 = \theta_3^i(\gamma(c(r_1, r_2 + r_3); 1, c(r_2, r_3)), y_1, y_2, y_3)$$

and

$$\mu_i \mu_i y_1 y_2 y_3 = \theta_3^i(\gamma(c(r_1 + r_2, r_3); c(r_1, r_2), 1), y_1, y_2, y_3)\,,$$

where $r_k = qy_k$. An easy calculation shows that the two elements of $\mathcal{C}_1(3)$ appearing here are both equal to $c(r_1, r_2, r_3)$.

Semi-interchange. Let $r_k = qy_k$, $1 \le k \le 4$ with $r_1 r_4 = r_2 r_3$ and $i \neq j$. We must show $\mu_j \mu_i y_1 y_3 \mu_i y_2 y_4 = \mu_i \mu_j y_1 y_2 \mu_j y_3 y_4$, which in terms of θ is

$$\begin{aligned} &\theta_2^j(c(r_1 + r_3, r_2 + r_4), \theta_2^i(c(r_1, r_3), y_1, y_3), \theta_2^i(c(r_2, r_4), y_2, y_4)) \\ &\quad = \theta_2^i(c(r_1 + r_2, r_3 + r_4), \theta_2^j(c(r_1, r_2), y_1, y_2), \theta_2^j(c(r_3, r_4), y_3, y_4))\,. \end{aligned}$$

To see this we note first that $r_1 r_4 = r_2 r_3$ is equivalent to

$$\begin{cases} \dfrac{r_1}{r_1 + r_2} = \dfrac{r_3}{r_3 + r_4} = \dfrac{r_1 + r_3}{r_1 + r_2 + r_3 + r_4}\,, \\[2ex] \dfrac{r_1}{r_1 + r_3} = \dfrac{r_2}{r_2 + r_4} = \dfrac{r_1 + r_2}{r_1 + r_2 + r_3 + r_4}\,. \end{cases}$$

Hence $c(r_1 + r_2, r_3 + r_4) = c(r_1, r_3) = c(r_2, r_4)$ and $c(r_1 + r_3, r_2 + r_4) = c(r_1, r_2) = c(r_3, r_4)$. Therefore the equation holds by interchange of θ^i and θ^j.

Since q and η are easily seen to be homomorphisms for each μ_i, the multiplications $\{\mu_i\}_{i=1}^n$ do give Y an n-fold monoid structure.

Conversely, let $\{\mu_i\}_{i=1}^n$ be an n-fold monoid structure on (Y, q, η, φ) in $\mathcal{T}(R)$. We will define a $\mathcal{C}_1^{\otimes n}$-action $\bar{\theta}$ on $X = q^{-1}(1)$ such that $(Y, R'\bar{\theta})$ is an object in $\mathcal{C}_1^{\otimes n}(R)$.

Define $\mathcal{C}_1$-actions $\bar{\theta}^i$ on X by

$$\begin{aligned} \bar{\theta}_j^i(c, x_1, \ldots, x_j) = \mu_i \eta(b_1) \mu_i \varphi(a_1, x_1) \mu_i \eta(b_2 - (a_1 + b_1)) \mu_i \varphi(a_2, x_2) \\ \ldots \mu_i \varphi(a_j, x_j) \eta(1 - (a_j + b_j))\,. \end{aligned}$$

where $c = \langle c_1, \ldots, c_j \rangle$ with $c_k(t) = a_k t + b_k$, $1 \le k \le j$ and $t \in I$.

The verification that $\bar{\theta}^i$ is associative is straightforward but tedious, the key fact needed being $\mu_i\eta(r)\eta(s) = \eta(r+s)$.

It remains to show that $\bar{\theta}^i$ and $\bar{\theta}^j$ interchange for $i \neq j$. The local universal property then gives a $\mathscr{C}_1^{\otimes n}$-action $\bar{\theta}$ on X. By Proposition 1.6 we are reduced to verifying (r, s)-interchange for $r, s \leq 2$. These calculations are quite lengthy and can be found in [6], so they will not be reproduced here. This completes the proof of the proposition. $\square$

As remarked above, S' restricts to an isomorphism of categories (T is its inverse). Recall that $C_1^{\otimes n}X$ is the free $\mathscr{C}_1^{\otimes n}$-space on X and note that we can regard $RC_1^{\otimes n}X$ as the free $\mathscr{C}_1^{\otimes n}$-space over $\mathbb{R}_+$ on RX (at least in $\mathscr{C}_1^{\otimes n}(R)$). Hence we can replace $RC_1^{\otimes n}X$ by the free n-fold monoid on RX. For this to be useful, we must produce a model suitable for our purposes. The following construction gives such a model:

Construction 1.10. Let (X, p, η) be in $\mathcal{T}^s[\mathbb{R}_+]$ and define S_r, the space of *well-formed words* of length r in $\{\mu_i\}_{i=1}^n \amalg X$ as follows:

Let $S_1 = X$ and for $r \geq 2$, $S_r = \{\mu_i\}_{i=1}^n \times \amalg_{1 \leq l < r} (S_l \times S_{r-l})$, where $\{\mu_i\}_{i=1}^n$ is a discrete set of n elements. Let $S_X = \amalg_{r \geq 1} S_r$ and define $\mathbb{R}_+ \xrightarrow{\tilde{\eta}} S_X \xrightarrow{\tilde{p}} \mathbb{R}_+$ by $\tilde{\eta}(t) = \eta(t) \in S_1$ and $\tilde{p}(a) = \Sigma_{k=1}^r p(x_k)$, where $a \in S_r$ and $x_1, \ldots, x_r$ are the length 1 'component words' of a. Now define $J'_nX = S_X/\sim$, where $\sim$ is the equivalence relation generated by the following relations:

(i) $\mu_i a * \sim a \sim \mu_i * a$, $* = \tilde{\eta}(0)$;

(ii) $\mu_i \tilde{\eta}(s)\tilde{\eta}(t) \sim \tilde{\eta}(s+t)$, $s, t \in \mathbb{R}_+$;

(iii) $\mu_i a_1 \mu_i a_2 a_3 \sim \mu_i \mu_i a_1 a_2 a_3$, $a_k \in S_X$;

(iv) $\mu_i \mu_j a_1 a_2 \mu_j a_3 a_4 \sim \mu_j \mu_i a_1 a_3 \mu_i a_2 a_4$

whenever $i \neq j$ and $\tilde{p}(a_1) \cdot \tilde{p}(a_4) = \tilde{p}(a_2) \cdot \tilde{p}(a_3)$.

We give J'_nX the quotient topology. It is then a filtered colimit by compactly generated subspaces, so is itself compactly generated. We thus have an object $(J'_nX, \tilde{p}, \tilde{\eta})$ in $\mathcal{T}^s[\mathbb{R}_+]$ (Proposition B.1), and with the obvious definition of ith multiplication J'_nX becomes an n-fold monoid. It is easily checked that J'_nX has the appropriate universal property, and hence is the free n-fold monoid functor adjoint to the inclusion $\mathcal{M}_n[\mathbb{R}_+] \to \mathcal{T}^s[\mathbb{R}_+]$. We also have the restriction $J_n : \mathcal{T}(R) \to \mathcal{M}_n(R)$ and it is adjoint to $\mathcal{M}_n(R) \to \mathcal{T}(R)$.

Summarizing, we have produced the following diagram to which Proposition 1.11 refers:

$$
\begin{tikzcd}
\mathcal{T}^s[\mathbb{R}_+] \arrow[r, "J'_n"] & \mathcal{M}_n[\mathbb{R}_+] & \mathscr{C}_1^{\otimes n}[\mathbb{R}_+] \arrow[l, "S'"'] \\
\cup & \cup & \cup \\
\mathcal{T}(R) \arrow[r, "J_n"] & \mathcal{M}_n(R) \arrow[r, shift right, "T"'] & \mathscr{C}_1^{\otimes n}(R) \arrow[l, shift right, "S'"'] \\
\mathcal{T} \arrow[u, "R", "\cong"'] \arrow[rr, "C_1^{\otimes n}"] & & \mathscr{C}_1^{\otimes n}[\mathcal{T}] \arrow[u, "R"', "\cong"] \arrow[uuu, bend right=60, "R'"']
\end{tikzcd}
$$

Proposition 1.11. (i) *S is an isomorphism of categories.*

(ii) *There is a natural isomorphism of functors*

$$\tau : J_nR \to SRC_1^{\otimes n} .$$

In particular identifying $\mathcal{M}_n(R)$ and $\mathcal{C}_1^{\otimes n}(R)$ via S (recall that $S = \mathrm{id}$ on underlying objects), we have that $\tau X : J_nRX \cong RC_1^{\otimes n}X$ is an isomorphism of n-fold monoids and of $\mathcal{C}_1^{\otimes n}$-spaces over $\mathbb{R}_+$.

Proof. (i) Since S is the identity on underlying objects and maps, we need only check that n-fold monoid structures correspond to $\mathcal{C}_1^{\otimes n}$-actions over $\mathbb{R}_+$. This is easily verified for the composite $S \circ T$. Conversely, for (Y, θ) in $\mathcal{C}_1^{\otimes n}(R)$ with $h : Rq^{-1}(1) \cong Y$, $\theta = \{\theta^i\}_{i=1}^n$ determines an n-fold monoid structure on Y which in turn gives $\bar{\theta} = \{\bar{\theta}^i\}_{i=1}^n$, a $\mathcal{C}_1^{\otimes n}$-action on $q^{-1}(1)$ as in the proof of Proposition 1.9. We need to show $R'\bar{\theta}^i = \theta^i$ via h, i.e. $h(R'\bar{\theta}^i_j) = \theta^i_j(1 \times h^j)$, all i, j. Now θ satisfies the conditions of Lemma 1.3, so each θ^i does also. Hence we obtain actions $\tilde{\theta}^i$ on $q^{-1}(1)$ such that $R'\tilde{\theta}^i = \theta^i$. A somewhat tedious calculation shows $\tilde{\theta}^i = \bar{\theta}^i$, so (i) follows.

(ii) If $\eta : X \to C_1^{\otimes n}X$ is the unit of the monad $C_1^{\otimes n}$, then $R\eta$ induces $\tau X : J_n'RX \to S'RC_1^{\otimes n}X$ by the universal property of J_n'. Now τX is clearly $\mathbb{R}_+^*$-equivariant, hence is a map in $\mathcal{M}_n(R)$, and so we have $\tau X : J_nRX \to SRC_1^{\otimes n}X$.

Now since $J_nRX \cong R\tilde{p}^{-1}(1)$ with $\tilde{p}^{-1}(1)$ in $\mathcal{C}_1^{\otimes n}[\mathcal{T}]$, the map $g : X \to \tilde{p}^{-1}(1)$ by $g(x) = (x, 1)$ induces a map $\bar{g} : C_1^{\otimes n}X \to \tilde{p}^{-1}(1)$. It now follows from universal properties that $R\bar{g}$ is inverse to τX. This completes the proof of the proposition $\square$

At this point we have the commutative diagram of the introduction. The advantage of this is that the combinatorial structure of J_nRX is relatively easy to work with (as opposed to $C_1^{\otimes n}X$).

2. Decomposition in $\mathcal{C}_n$

We begin by discussing the notion of decomposable elements in $\mathcal{C}_n$; these form a suboperad denoted by $\mathcal{D}_n$. We then give some elementary properties of decomposables which are used to show that $\mathcal{D}_n \to \mathcal{C}_n$ is a local Σ-equivalence and also that $\beta : J_nRX \to RC_nX$ is a closed inclusion with image RD_nX.

First, recall from Section 1 the elements $c(r, s)$ in $\mathcal{C}_1(2)$ for r, s positive real numbers. Let $c^i(r, s)$ denote the image under the ith inclusion $\alpha^i : \mathcal{C}_1 \to \mathcal{C}_n$, $1 \leq i \leq n$.

Definition 2.1. Let $n \geq 2$, $c \in \mathcal{C}_n(j)$ and $1 \leq i \leq n$. Call c *i-decomposable* if $j = 0, 1$ or if $j \geq 2$ and $c = \gamma(c(r, s); c_1, c_2)$ for some $r, s > 0$ and $c_k \in \mathcal{C}_n(j_k)$ with $j_k > 0$, $k = 1, 2$. We also write $c = c_1 \cup_t c_2$, where $t = r/r + s$.

If there is a sequence of i-decompositions for various i, say $i_1, \ldots, i_{j-1}$, such that $c = c_1 \cup_t c_2$ is an i_1-decomposition, c_1 or c_2 is i_2-decomposable, and so on to i_{j-1}, then c is called *decomposable*; otherwise it is *indecomposable*.

Visually, c is i-decomposable if we can insert a codimension 1 hyperplane L in I^n orthogonal to the i-axis which does not meet the interior of any component cube of c, and each 'side' of L in I^n contains at least one component cube. We shall occasionally use the notation $L_t = \{(r_1, \ldots, r_n) \in I^n \mid r_i = t\}$, $0 < t < 1$. Thus in Definition 2.1 with $t = r/r + s$, L_t is a decomposing hyperplane for $c = c_1 \cup_t c_2$.

We note the following elementary facts about decomposable elements:

(a) If $j \leq 3$, then all elements of $\mathscr{C}_n(j)$ are decomposable.

(b) If $j \geq 4$, there are indecomposable configurations in $\mathscr{C}_n(j)$.

(c) The decomposable elements in $\mathscr{C}_n$ form a suboperad $\mathscr{D}_n$.

(a) and (c) are obvious from the definition. For (b), note that Fig. 1 is indecomposable in $\mathscr{C}_2(4)$. For $j > 4$, fill in the empty cube arbitrarily and for $n > 2$, use the inclusion $\mathscr{C}_2 \to \mathscr{C}_n$.

We remark that $\mathscr{D}_n$ can also be described as the suboperad of $\mathscr{C}_n$ generated by the image of $(\mathscr{C}_1, \ldots, \mathscr{C}_1) \to \mathscr{C}_n$, the n-linear map of Section 1.

We next show that $(\mathscr{C}_n(j), \mathscr{D}_n(j))$ is a Σ_j-equivariant DR pair, all $j \geq 0$.

Let $\tilde{\mathscr{C}}_n$ denote the *extended n-cubes operad*: it consists of little n-cubes $c = \langle c_1, \ldots, c_j \rangle$ whose component cubes c_i may be degenerate, i.e. some of the linear factors of c_i might be constant. There is an obvious inclusion $\mathscr{C}_n \to \tilde{\mathscr{C}}_n$.

Let $H: I \times \mathscr{C}_n(j) \to \tilde{\mathscr{C}}_n(j)$ be the homotopy which shrinks each little n-cube to its center points. Specifically, define $g: I \times I \to I$ by $g(s, t) = g_s(t) = (1 - s)t + s/2$ and let $g_s^n: I^n \to I^n$ be the n-fold product. Now taking $H(s, c) = \langle c_1 \circ g_s^n, \ldots, c_j \circ g_s^n \rangle$ we see that $H(0, c) = c$, $H(1, c) = c(\frac{1}{2}, \ldots, \frac{1}{2})$ and $H(s, c) \in \mathscr{C}_n(j)$ for $s < 1$.

The following lemma shows that $u: \mathscr{C}_n(j) \to I$ given by $u(c) = \inf\{s \mid H(s, c) \in \mathscr{D}_n(j)\}$ is well defined:

Lemma 2.2. (i) *There is* $s \in [0, 1)$ *such that* $H(s, c) \in \mathscr{D}_n(j)$.

(ii) $H(u(c), c) \in \mathscr{D}_n(j)$, *hence* $u(c) = 0$ *if and only if* $c \in \mathscr{D}_n(j)$. $\square$

Both of these are visually obvious, so we omit proofs.

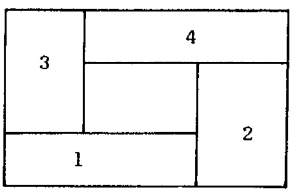

Fig. 1.

Define $h: I \times \mathscr{C}(j) \to \mathscr{C}(j)$ by $h(s, c) = H(s \cdot u(c), c)$. $(\mathscr{C}_n(j), \mathscr{D}_n(j))$ is then a Σ_j-equivariant DR pair via (h, u), so we have shown

Proposition 2.3. *$\mathscr{D}_n \to \mathscr{C}_n$ is a local Σ-equivalence.* $\square$

For the remainder of this section X will always denote a countably infinite discrete space. We will need the following lemma in order to pass from associated monads back to operads.

Lemma 2.4. *Let $\psi: \mathscr{C} \to \mathscr{D}$ be a morphism of operads.*

(i) *The quotient map*

$$\pi: \coprod_{j \le 0} \mathscr{C}(j) \times_{\Sigma_j} X^j \to CX$$

has a section ρ which is a closed inclusion.

(ii) *If $\psi: CX \to DX$ is a homeomorphism, then $\psi: \mathscr{C} \to \mathscr{D}$ is an isomorphism of operads.*

(iii) *If each $\psi_j: \mathscr{C}(j) \to \mathscr{D}(j)$ is a closed inclusion, then so is $\psi: CX \to DX$*

Proof. Let $A_j = \mathscr{C}(j) \times_{\Sigma_j} X^j$ and $B_j = \mathscr{D}(j) \times_{\Sigma_j} X^j$.

(i) For $a \in CX$, there is a unique representative $(c, y) \in A_k$ such that k is minimal. Let $\rho(a) = (c, y)$. ρ is continuous because X is discrete, and a closed inclusion since $\pi \circ \rho = \mathrm{id}$.

(ii) We first observe that an operad can be recovered from its associated monad as follows. Let $X = \{0, 1, 2, \ldots\}$, a discrete space with basepoint 0. The map $f_j: \mathscr{C}(j) \to \coprod_{k \ge 0} A_k$, $j > 0$, given by $f_j(c) = (c, 1, 2, \ldots, j)$ is a closed inclusion with $\operatorname{im} f_j \subseteq \operatorname{im} \rho$. Hence we can identify $\mathscr{C}(j)$ with $\bar{\mathscr{C}}(j) = (\rho^{-1} f_j)(\mathscr{C}(j))$ in CX as Σ_j-spaces by requiring $\rho^{-1} f_j$ to be Σ_j-equivariant. Explicitly, the action is given by $[c, 1, \ldots, j] \cdot \sigma = [c\sigma, 1, \ldots, j]$. ($\mathscr{C}(0)$ is identified with $[*, 0]$.) If γ is the composition in $\mathscr{C}$, then $\bar{\gamma}$ in $\bar{\mathscr{C}}$ is defined by $\bar{\gamma}(\bar{c}; \bar{c}_1, \ldots, \bar{c}_j) = \overline{\gamma(c; c_1, \ldots, c_j)}$, where $\bar{d} = [d, 1, \ldots, j]$.

One now checks that $\rho^{-1} f: \mathscr{C} \to \bar{\mathscr{C}}$ is an isomorphism of operads and that $\psi: CX \to DX$ restricts to a homeomorphism $\psi_j: \bar{\mathscr{C}}(j) \to \bar{\mathscr{D}}(j)$, for all j.

(iii) In the diagram

$$\begin{tikzcd}
\coprod_{j \ge 0} A_j \arrow[r, "\tilde{\psi}"] \arrow[d, shift right, "\pi"'] & \coprod_{j \ge 0} B_j \arrow[d, shift right, "\pi'"'] \\
CX \arrow[r, "\psi"] \arrow[u, shift right, bend right, "\rho"'] & DX \arrow[u, shift right, bend right, "\rho'"']
\end{tikzcd}$$

we have $\rho' \circ \psi = \tilde{\psi} \circ \rho$, so we need each $A_j \to B_j$ a closed inclusion and this follows from Proposition B.3. $\square$

From Proposition 2.3, Lemma 2.4 and Proposition B.1 we conclude that $RD_nX \to RC_nX$ is a closed inclusion. Since the image of the n-linear map $(\mathscr{C}_1, \ldots, \mathscr{C}_1) \to \mathscr{C}_n$ is contained in $\mathscr{D}_n$ we can regard $R\alpha$ as a map $RC_1^{\otimes n}X \to RD_nX$. Thus we can regard β as a map $J_nRX \to RD_nX$ which we now show to be a homeomorphism.

In the next three lemmas we take advantage of the natural filtration of J_nRX by word length. Since the three proofs are similar we will give details for Lemma 2.6 only.

Let π and ρ be the maps of Lemma 2.4 for $\mathscr{C} = \mathscr{D}_n$ and let $\hat{\beta} = R\rho \circ \beta$. If $a \in J_nRX$ we define its *length* $\lambda(a)$ to be the smallest length of a representative word.

Lemma 2.5. *Let $\hat{\beta}(a) = (c, x, t)$ with $\lambda(a) \geq 2$, and suppose $c = c_1 \cup_s c_2$ is an i-decomposition, i.e. $c = \gamma(c^i(r_1, r_2); c_1, c_2)$ with $s = r_1/r_1 + r_2$. Then there exist a_1, a_2 in J_nRX such that $a = \mu_i a_1 a_2$ and $\hat{\beta}(a_k) = (c_k, x_k, r_k)$ for some $x_k \in X^{j_k}$, $k = 1, 2$.* $\square$

Lemma 2.6. *$\hat{\beta} = R\rho \circ \beta$ is injective, hence β is also injective.*

Proof. $\hat{\beta}$ is clearly injective on $RX \subseteq J_nRX$, so we can restrict to classes of length ≥ 2. By the preceding lemma it suffices to show $\hat{\beta}(\mu_i a_1 a_2) = \hat{\beta}(\mu_i b_1 b_2)$ implies $\mu_i a_1 a_2 = \mu_i b_1 b_2$. Let $\hat{\beta}(a_k) = (c_k, x_k, t_k)$ and $\hat{\beta}(b_k) = (d_k, y_k, s_k)$, $k = 1, 2$. Then $\gamma(c^i(t_1, t_2); c_1, c_2) = \gamma(c^i(s_1, s_2); d_1, d_2)$ with $s_1 + s_2 = t_1 + t_2$.

If $s_1 = t_1$, then $\hat{\beta}(a_k) = \hat{\beta}(b_k)$, so $a_k = b_k$, $k = 1, 2$ by induction on length.

Now suppose $s_1 > t_1$. Then $\gamma(c^i(t_1, t_2); c_1, c_2)$ is i-decomposable by L_{s_1}, so that there exist a_1', a_2' such that $\mu_i a_1 a_2 = \mu_i a_1' a_2'$ and $\hat{\beta}(a_k') = (c_k', x_k', t_k')$ $k = 1, 2$ with $t_1'/(t_1' + t_2') = s_1/(s_1 + s_2)$. Hence $\hat{\beta}(u_i a_1' a_2') = \hat{\beta}(\mu_i b_1 b_2)$ with $t_1' = s_1$, so the previous case applies.

The case $s_1 < t_1$ is similar. $\square$

Lemma 2.7. im $\beta = RD_nX$. $\square$

Lemma 2.8. *$\beta : J_nRX \to RD_nX$ is a homeomorphism.*

Proof. Let $x_0 = *$, x_1, $x_2, \ldots$ be the distinct points of X and write $X_j = \{x_0, x_1, \ldots, x_j\}$. We will show each $J_nRX_j \to RD_nX_j$ is a homeomorphism and this will imply that β is a homeomorphism as follows.

Note that $X = \operatorname{colim} X_j$ and each (X_j, X_{j-1}) is NDR. Now J_n and R preserve colimits since they are left adjoints, and D_n preserves NDR colimits by [11, 2.6], so the conclusion follows.

We now show $\beta : J_nRX_j \to RD_nX_j$ is a homeomorphism, $j \geq 1$. Applying R to the standard filtration of D_nX_j we obtain a filtered space $RD_nX_j = \bigcup_{r \geq 0} F_r$. Let $G_r = \beta^{-1}(F_r)$ and let ρ be the map of Lemma 2.4 for D_nX_j. Now $(R\rho)(F_r -$

$F_{r-1}) = \mathscr{D}_n(r) \times_{\Sigma_r} (X-*)^r \times \mathbb{R}_+^*$, $r \geq 1$, and this subset is both open and closed, so the same is true for $F_r - F_{r-1}$ and $G_r - G_{r-1}$. From this and induction on r it will suffice to show that $\beta_r : G_r - G_{r-1} \to F_r - F_{r-1}$ is a homeomorphism, $r \geq 1$.

We will describe an increasing sequence of compact sets $\{U_m\}_{m\geq 1}$ in $G_r - G_{r-1}$ such that the corresponding sequence $\{T_m = \beta U_m\}_{m \geq 1}$ has $F_r - F_{r-1} = \operatorname{colim} T_m$. Since $F_r - F_{r-1}$ and $G_r - G_{r-1}$ are Hausdorff, this shows that β_r^{-1} is continuous.

First recall the map $\pi : \coprod_{k\geq 1} S_k \to J_n RX_j$ of Construction 1.10, where S_k is the space of words of length k. Let k, l, m be integers with $k, m \geq 1$ and $0 \leq l \leq k$. Let $S_k(l, m) \subseteq S_k$ consist of those words of length k with precisely l length one component words (x, t) having $x \neq *$, and such that each parameter t is in the closed interval $[0, m]$, or if $x \neq *$, then $t \in [1/m, m]$. It is easy to see that the sets $S_k(l, m)$ are compact.

The following three statements define the sets U_m, T_m and imply $F_r - F_{r-1} = \operatorname{colim} T_m$. Let $W_k = (\beta\pi)(S_k)$, $k \geq 1$.

(1) For each $r \geq 1$, there is k_r such that $F_r \subseteq W_{k_r}$.

(2) Let $U_m = \pi(S_{k_r}(r, m))$ and $T_m = \beta(U_m)$, $m \geq 1$. Then $F_r - F_{r-1} = \bigcup_{m\geq 1} T_m$.

(3) $\{T_m\}_{m\geq 1}$ is cofinal in the collection of all compact sets in $F_r - F_{r-1}$.

These statements are verified in [6]. This completes the proof. □

Theorem 2.9. (i) $\alpha : \mathscr{C}_1^{\otimes n} \to \mathscr{C}_n$ *is a local* Σ*-equivalence.*

(ii) $\alpha : C_1^{\otimes n} Y \to C_n Y$ *is a homotopy equivalence for any space* Y.

Proof. By Lemma 2.8, $R\alpha : RC_1^{\otimes n} X \to RD_n X$ is a homeomorphism. Since it is also $\mathbb{R}_+^*$-equivariant, its restriction $\alpha : C_1^{\otimes n} X \to D_n X$ is a homeomorphism and thus $\alpha : \mathscr{C}_1^{\otimes n} \to \mathscr{D}_n$ is an isomorphism of operads by Lemma 2.4. The first statement now follows by Proposition 2.3.

The second statement is a consequence of the first and [12, A.2]. □

The following result is the appropriate generalization for $n \geq 2$ of [7, 6.12].

Corollary 2.10. *The natural of map of n-folds monoids*

$$J_n RY \xrightarrow{\lambda} R\Omega^n \Sigma^n Y$$

is a group completion (in homology), for $n \geq 2$.

Proof. We have a commutative diagram of H-maps

$$\begin{tikzcd} J_n RY \arrow[r, "\beta"] \arrow[dr, "\lambda"'] & RC_n Y \arrow[d, "R\gamma"] \\ & R\Omega^n \Sigma^n Y \end{tikzcd}$$

where λ is induced by $R\eta: RY \to R\Omega\Sigma Y$ with η and γ the usual natural transformations. γ is known to be a group completion [5, 9] and β is a homotopy equivalence by Theorem 2.9, hence λ is a group completion by Proposition B.2. $\square$

We conclude this section with the observation that there are similar results for infinite loop spaces. First let $\mathscr{C}_\infty = \operatorname{colim} \mathscr{C}_n$ and $\mathscr{C}_1^\infty = \operatorname{colim} \mathscr{C}_1^{\otimes n} \cong \operatorname{colim} \mathscr{D}_n = \mathscr{D}_\infty$, the colimits taken with respect to the inclusions $\mathscr{C}_n \to \mathscr{C}_{n+1}$ and their restrictions to $\mathscr{C}_1^{\otimes n} \cong \mathscr{D}_n$. Similarly we let $J_\infty X = \operatorname{colim} J_n X$.

Theorem 2.11. *The inclusion $\mathscr{C}_1^\infty \to \mathscr{C}_\infty$ is a local Σ-equivalence, hence $\mathscr{C}_1^\infty$ is an E_∞-operad.* $\square$

Definition 2.12. (X, p, η) in $\mathscr{T}^s[\mathbb{R}_+]$ is an *E_∞-monoid* if it has pairwise semi-interchangeable monoid multiplications $\{\mu_i\}_{i=1}^\infty$ such that p and η are homomorphisms for each μ_i.

Porposition 2.13. *There is an isomorphism of categories $\mathscr{M}_\infty(R) \cong \mathscr{C}_1^\infty(R)$ and an isomorphism of E_∞-monoids $J_\infty RY \cong RC_1^\infty Y$, for any space Y.* $\square$

Proposition 2.14. *The natural map of E_∞-monoids*

$$J_\infty RX \longrightarrow RQX$$

is a group completion, where $QX = \operatorname{colim} \Omega^n \Sigma^n X$. $\square$

Proposition 2.15. *Any infinite loop space X is equivalent (as infinite loop spaces) to an E_∞-monoid.*

Proof. Using two-sided bar constructions we obtain a C_1^∞-space Y and an (infinite loop) equivalence $X \simeq Y$. By Proposition 2.13, RY is an E_∞-monoid and the projection $RY \to Y$ is clearly a C_1^∞-map. $\square$

Appendix A

Suppose f and g are maps of operads with each a local Σ-equivalence, i.e. the jth component map is a Σ_j-equivariant homotopy equivalence. Then $f \otimes g$ may or may not be an equivalence, as the examples in Propositions A.3 and A.2 show. The result of Proposition A.2 follows easily from Theorem 2.9 and Proposition A.1(i)(c), while Proposition A.3 is proved using a notion of homotopy of operads (Definition A.5) that respects the functoriality of the tensor product. These examples are obtained using the following standard operads.

Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ be the operads with $\mathcal{M}(j)=\Sigma_j$, $\mathcal{N}(j)=*$, all j and $\mathcal{P}(1)=*$, $\mathcal{P}(j)=\phi$, $j>1$. Thus an $\mathcal{M}$-space is a monoid, an $\mathcal{N}$-space a commutative monoid and a $\mathcal{P}$-space is just a space.

Regard the unit interval I as a monoid under $st=\min\{s,t\}$, and define an operad $\mathcal{I}$ with $\mathcal{I}(j)=I$, $j>0$ (Σ_j acts trivially) and with composition determined by the monoid product. Recall from [11] the whiskering $\mathcal{C}'$ of an operad $\mathcal{C}$ with $\mathcal{C}'(j)=\mathcal{C}(j)$, $j\neq 1$ and $\mathcal{C}'(1)=\mathcal{C}(1)\vee I$, where 0 in I is identified to the unit e in $\mathcal{C}(1)$. Here we are using the above product on I to form $\mathcal{C}'$ rather than the usual one used in [11] (cf. Remark A.4).

Proposition A.1. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be operads.*

(i) *There are natural isomorphisms of operads*

(a) $\mathcal{C}\otimes\mathcal{P}\cong\mathcal{C}$;

(b) $\mathcal{A}\otimes\mathcal{B}\cong\mathcal{A}\times\mathcal{B}$, *if* $\mathcal{A}(j)=\phi=\mathcal{B}(j)$, $j>1$.

Moreover, the monoid $\mathcal{A}(1)$ is abelian if and only if the product γ determines a map of operads $\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ (equivalently an internal pairing $(\mathcal{A},\mathcal{A})\to\mathcal{A}$).

(c) $\mathcal{M}^{\otimes n}\cong\mathcal{N}$, *if* $n\geq 2$.

(ii) *There is a natural homotopy equivalence of operads*

$$\bar{\sigma}:\mathcal{C}\otimes\mathcal{P}'\to\mathcal{C}\times\mathcal{I}.$$

Proof. (ii) will be proved later. We prove (i). The identity map of $\mathcal{C}$ determines a pairing $(\mathcal{C},\mathcal{P})\to\mathcal{C}$ which is obviously universal, so (a) follows.

Given the conditions on $\mathcal{A}$ and $\mathcal{B}$ and any map $\tau:(\mathcal{A},\mathcal{B})\to\mathcal{A}\otimes\mathcal{B}$, requiring that τ be a pairing is the same as requiring that $\tau:\mathcal{A}(1)\times\mathcal{B}(1)\to(\mathcal{A}\otimes\mathcal{B})(1)$ be a monoid homomorphism, i.e. a map of operads. In particular, the identity map of $\mathcal{A}\times\mathcal{B}$ regarded as a pairing induces a map $\mathcal{A}\otimes\mathcal{B}\to\mathcal{A}\times\mathcal{B}$ which is inverse to τ if τ is the universal pairing.

For the second statement, write ab for $\gamma(a;b)$ and note that the statement that γ is a pairing is the equation $(ab)(cd)=(ac)(bd)$ which is equivalent to $\mathcal{A}(1)$ being abelian. This proves (b).

The third isomorpism follows from a general result for algebraic theories [4], i.e. if $\mathcal{A}$ is the theory of monoids and $\mathcal{C}\mathcal{A}$ the theory of commutative monoids, then there is an isomorphism of theories $\mathcal{A}^{\otimes n}\cong\mathcal{C}\mathcal{A}$, for $n\geq 2$.

Proposition A.2. *If $\varepsilon:\mathcal{C}_1\to\mathcal{M}$ is the usual augmentation* [11] *(an equivalence), then $\varepsilon^{\otimes n}:\mathcal{C}_1^{\otimes n}\to\mathcal{M}^{\otimes n}$ is not an equivalence, for $n\geq 2$.*

Proof. This follows from Proposition A.1(c) and Theorem 2.9, since in the commutative diagram

$$\begin{tikzcd} \mathscr{C}_1^{\otimes n} \arrow[r] \arrow[d, "\simeq"'] & \mathscr{M}^{\otimes n} \arrow[d, "\cong"] \\ \mathscr{C}_n \arrow[r] & \mathscr{N} \end{tikzcd}$$

the bottom map is only an $(n-2)$-equivalence. $\square$

Proposition A.3. *Let $\mathscr{A}_i^{(m)}$ denote $\mathscr{A}_i$ whiskered m times, for $1 \le i \le n$. Then the natural map of operads $\bigotimes_{i=1}^n \mathscr{A}_i^{(m)} \to \bigotimes_{i=1}^n \mathscr{A}_i$ is an equivalence.*

Remark A.4. If we take each $\mathscr{A}_i = \mathscr{C}_1$ in Proposition A.3, we obtain a mild extension of Theorem 2.9. This result turns out to be useful in comparing the n-fold delooping constructions of May and Segal. One reason we have chosen minimum as the product on I is that it is required for this application of Proposition A.3.

Additionally, we would like the collection of identity maps on $\mathscr{C}(r) \times I$ to determine a pairing $\sigma : (\mathscr{C}, \mathscr{P}') \to \mathscr{C} \times \mathscr{I}$ inducing the map $\bar{\sigma} : \mathscr{C} \otimes \mathscr{P}' \to \mathscr{C} \times \mathscr{I}$ of Proposition A.1. It is easy to see that σ is a pairing with minimum but not with the usual product on I. Although we do not really need the map $\bar{\sigma}$, it does make the following definition more plausible:

Definition A.5. A *homotopy of operads* is a map of operads $H : \mathscr{C} \otimes \mathscr{P}' \to \mathscr{D}$.

Remark. This is a reasonable notion of homotopy in view of the following observations which are easily verified. Let $\tau : (\mathscr{C}, \mathscr{P}') \to \mathscr{C} \otimes \mathscr{P}'$ be the universal pairing and σ, $\bar{\sigma}$ as in Remark A.4.

$$\begin{tikzcd} & \mathscr{C} \otimes \mathscr{P}' \arrow[dr, "H"] \arrow[dd, "\bar{\sigma}"'] & \\ (\mathscr{C}, \mathscr{P}') \arrow[ur, "\tau"] \arrow[dr, "\sigma"'] & & \mathscr{D} \\ & \mathscr{C} \times \mathscr{I} \arrow[ur, "F"'] & \end{tikzcd}$$

(1) A map H as in the diagram determines a map $F = H \circ \tau \circ \sigma^{-1}$ and conversely a map F determines a map $H = F \circ \bar{\sigma}$. If H is a map of operads, then F is generally not a map of operads.

(2) The component maps $H_t, F_t : \mathscr{C} \to \mathscr{D}$ are equal, $0 \le t \le 1$, where $H_t = H \circ \tau(-, t)$. Moreover each H_t preserves composition and H_1 also preserves units.

(3) The map $\bar{\sigma}$ is a homotopy equivalence (Proposition A.1).

Definition A.6. A map of operads $f : \mathscr{A} \to \mathscr{B}$ is *split* if there is a map $g : \mathscr{B} \to \mathscr{A}$ preserving composition (but possibly not units) and a map of operads

$H:\mathcal{A}\otimes\mathcal{P}'\to\mathcal{A}$ such that $f\circ g=\mathrm{id}$ and H is a homotopy from $g\circ f$ to the identity of $\mathcal{A}$.

The following proposition shows that these notions are appropriate for the tensor product functor on operads:

Proposition A.7. *If $f_i:\mathcal{A}_i\to\mathcal{B}_i$ are split maps of operads, $1\le i\le n$, then $\bigotimes_{i=1}^{n} f_i$ is also split, hence an equivalence.*

Proof. Let g_i split f_i with homotopy $H_i:\mathcal{A}_i\otimes\mathcal{P}'\to\mathcal{A}_i$, for $1\le i\le n$, and write $f=\bigotimes f_i$, $g=\bigotimes g_i$ (possibly $g(1)\neq 1$). The diagonal map $\Delta:\mathcal{P}'(1)\to\mathcal{P}'(1)^n$ is a homomorphism of abelian monoids and by Proposition A.1(b) can be regarded as a map of operads $\mathcal{P}'\to(\mathcal{P}')^n\cong(\mathcal{P}')^{\otimes n}$. Define a homotopy H as the composite

$$\begin{aligned}\left(\bigotimes_{i=1}^{n}\mathcal{A}_i\right)\otimes\mathcal{P}'&\xrightarrow{\mathrm{id}\otimes\Delta}\left(\bigotimes_{i=1}^{n}\mathcal{A}_i\right)\otimes\left(\bigotimes_{i=1}^{n}\mathcal{P}'\right)\\&\cong\bigotimes_{i=1}^{n}\left(\mathcal{A}_i\otimes\mathcal{P}'\right)\xrightarrow{\otimes H_i}\bigotimes_{i=1}^{n}\mathcal{A}_i\,.\end{aligned}$$

The functoriality of the tensor product now implies $f\circ g=\mathrm{id}$ and $H:g\circ f\simeq\mathrm{id}$. □

It remains to prove Propositions A.3 and A.1(ii) which are easy consequences of Proposition A.7 and the following lemma:

Lemma A.8. *The natural retraction $\rho:\mathcal{A}'\to\mathcal{A}$ is split for any operad $\mathcal{A}$.*

Proof. The inclusion $i:\mathcal{A}\to\mathcal{A}'$ preserves composition (but not units), and $\rho\circ i=\mathrm{id}$. A homotopy $H:\mathcal{A}'\otimes\mathcal{P}'\to\mathcal{A}'$ is induced by the pairing $h:(\mathcal{A}',\mathcal{P}')\to\mathcal{A}'$ defined by

$$h(a,t)=\begin{cases}a & \text{if } a\in\mathcal{A}\,,\\ a\cdot t & \text{if } a\in I\,.\end{cases}$$

Then $H_0=i\circ\rho$ and $H_1=\mathrm{id}$, so ρ is split. □

Proof of Proposition A.1(ii). Since $\mathrm{id}:\mathcal{C}\to\mathcal{C}$ is split, Proposition A.7 and Lemma A.8 imply that the top map in the following diagram is an equivalence:

$$\begin{array}{ccc}\mathcal{C}\otimes\mathcal{P}' & \xrightarrow{\mathrm{id}\otimes\rho} & \mathcal{C}\otimes\mathcal{P}\\ {\scriptstyle\bar\sigma}\downarrow & & \downarrow{\scriptstyle\cong}\\ \mathcal{C}\times\mathcal{J} & \xrightarrow{\simeq} & \mathcal{C}\end{array}$$

□

Proof of Proposition A.3. The map of the proposition factors as $\otimes \mathscr{A}_i^{(m)} \to \otimes \mathscr{A}_i^{(m-1)} \to \cdots \to \otimes \mathscr{A}_i' \to \otimes \mathscr{A}_i$ and Proposition A.7 and Lemma A.8 apply directly to each map. □

Appendix B

Proposition B.1. (i) *$R: \mathscr{T} \to \mathscr{T}[\mathbb{R}_+]$ preserves closed inclusions.*

(ii) *If $X \in \mathscr{T}$, then $(RX, \eta(\mathbb{R}_+))$ and $(RX, \eta(0))$ are NDR pairs such that (RX, π_2, η) is in $\mathscr{T}[\mathbb{R}_+]$, where $\eta = R(* \to X)$. Moreover, R preserves colimits of cofibrations.*

(iii) *If $(X, p, \eta) \in \mathscr{T}^s[\mathbb{R}_+]$, then $(J_n'X, \tilde{\eta}(\mathbb{R}_+))$ and $(J_n'X, \tilde{\eta}(0))$ are NDR pairs such that $(J_n'X, \tilde{p}, \tilde{\eta})$ is in $\mathscr{T}^s[\mathbb{R}_+]$.*

Proof. (i) First recall that the functor $k: \omega H \to \mathscr{T}$ which assigns a compactly generated space to a weak Hausdorff space preserves closed inclusions.

If $\iota: A \to X$ is a closed inclusion, then so is $\iota \times 1: A \times \mathbb{R}_+ \to X \times \mathbb{R}_+$. Hence the restriction $R\iota: RA \to RX$ is a closed inclusion in the relative topology and (applying k) also in the compactly generated subspace topology.

(ii) Let (X, A) be NDR via (h, u). Define $I \times RX \xrightarrow{g} RX \xrightarrow{v} I$ by $v(x, t) = u(x)$ and $g(s, (x, t)) = (h(s, x), t)$. Then (RX, RA) is NDR via (g, v) and g is height preserving. Taking $A = *$ gives $(RX, \eta(\mathbb{R}_+))$ is NDR which also implies that $(RX, \eta(0))$ is NDR, since $(\eta(\mathbb{R}_+), \eta(0))$ is.

Since R is a left adjoint it preserves colimits of closed inclusions by (i), and we have just seen that it preserves NDR pairs, hence preserves colimits of cofibrations.

(iii) Let $(X, \eta(\mathbb{R}_+))$ be NDR via (h, u). Define $I \times \coprod_{r\geq 1} S_r \xrightarrow{g} \coprod_{r \geq 1} S_r \xrightarrow{v} I$ inductively as follows. Take $v = u$ on $S_1 = X$ and if $v(a)$ and $v(b)$ are defined, let $v(\mu_i ab) = \max\{v(a), v(b)\}$. Also let $g(s, x) = h(s, x)$ for $x \in S_1$ and $g(s, \mu_i ab) = \mu_i g(s, a) g(s, b)$ whenever $g(s, a)$ and $g(s, b)$ are defined. These maps induce $I \times J_n'X \xrightarrow{g} J_n'X \xrightarrow{v} I$ such that $(J_n'X, \tilde{\eta}(\mathbb{R}_+))$ is NDR and g is height preserving. As in (ii), it follows that $(J_n'X, \tilde{\eta}(0))$ is also NDR. □

Proposition B.2. *$R: \mathscr{T} \to \mathscr{T}[\mathbb{R}_+]$ preserves group completions.*

Proof. We take group completion in the sense of [12, 1.3]. If X is an admissible H-space, then so is RX under coordinate-wise multiplication, and the projection $\varepsilon: RX \to X$ is an equivalence of H-spaces. Hence $\pi_0 \varepsilon: \pi_0 RX \to \pi_0 X$ is an isomorphism of monoids, and so induces an isomorphism of localizations

$$\begin{tikzcd}
(H_* RX)[\pi_0 RX]^{-1} \arrow[r, "\bar{\varepsilon}_*", "\cong"'] \arrow[d, "(\overline{Rf})_*"'] & (H_* X)[\pi_0 X]^{-1} \arrow[d, "\bar{f}_*"] \\
H_* RY \arrow[r, "\varepsilon_*", "\cong"'] & H_* Y
\end{tikzcd}$$

and so Rf is a group completion if f is. □

Proposition B.3. *Given a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ Z \arrow[r, "g"] & W \end{tikzcd}$$

with p and q quotient maps. Then if f is a closed inclusion with g injective and $q^{-1}(g(Z)) \subseteq f(X)$, g is also a closed inclusion.

Proof. See [8, A.7.2]. □

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