On categories of slices

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Abstract

In this paper we give an algebraic description of the category of n-slices for an arbitrary group G, in the sense of Hill-Hopkins-Ravenel. Specifically, given a finite group G and an integer n, we construct an explicit G-spectrum W (called an *isotropic slice n-sphere*) with the following properties: (i) the n-slice of a G-spectrum X is equivalent to the data of a certain quotient of the Mackey functor $[W, X]$ as a module over the endomorphism Green functor $[W, W]$; (ii) the category of n-slices is equivalent to the full subcategory of right modules over $[W, W]$ for which a certain restriction map is injective. We use this theorem to recover the known results on categories of slices to date, and exhibit the utility of our description in several new examples. We go further and show that the Green functors $[W, W]$ for certain slice n-spheres have a special property (they are *geometrically split*) which reduces the amount of data necessary to specify a $[W, W]$-module. This step is purely algebraic and may be of independent interest.

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Introduction

The stable homotopy category has a standard filtration $\{\tau_{\geq n}\mathsf{Sp}\}$ by its subcategories of n-connective spectra for $n \in \mathbb{Z}$. We have a good computational handle on this filtration for the following reasons:

(i) We can easily *build* objects in $\tau_{\geq n}\mathsf{Sp}$: all n-connective objects are obtained from a wedge of copies of S^n by iteratively attaching cells of dimension $k \geq n$.

(ii) We can *compute* when an object Y is n-truncated: we need to check that the homotopy groups of Y vanish above dimension n.

(iii) We can *compute* when an object X is in $\tau_{\geq n}\mathsf{Sp}$: we need to check that the homotopy groups of X vanish below dimension n.

(iv) The category of n-connective and n-truncated objects has a completely *algebraic* description: it is equivalent to the category of abelian groups via the Eilenberg-MacLane functor. Moreover, knowledge of the nth Postnikov layer of a spectrum X is equivalent to knowledge of $\pi_n X$.

Motivated by the Schubert cell structures on complex Grassmanians, Dan Dugger [Dug05] defined the slice filtration in C_2-equivariant homotopy theory to study Atiyah's **R**eal K-theory [Ati66]. This filtration was later generalized to a filtration of G-equivariant stable homotopy theory for any finite group G and used to great effect by Hill-Hopkins-Ravenel [HHR16] in their solution to the Kervaire invariant one problem.

Associated to the slice filtration on the category Sp^G of G-spectra are the notions of *slice n-connective* and *slice n-truncated* spectra. (The reader may review the relevant definitions below in Definition 1.46.) A G-spectrum which is both slice n-connective and slice n-truncated is called an *n-slice*. The relevant features of the slice filtration are as follows:

(i) By design, we can easily *build* slice n-connective G-spectra.

(ii) By definition, we can *compute* when a G-spectrum is slice n-truncated by computing certain homotopy classes of maps in from representation spheres.

(iii) Thanks to a recent result of Hill-Yarnall [HY17], it is possible to *compute* when a G-spectrum is slice n-connective: we need to check certain connectivity conditions on each of its geometric fixed point spectra.

The purpose of this paper is to give an analogue of the property (iv) of the Postnikov filtration, in complete generality. That is, we provide an algebraic description of the layers of the slice filtration together with a replacement for the functor π_n in this context.

As a way to establish notation and provide motivation for our approach, we begin by reviewing the completely understood case of $G = C_2$. Recall that, given G-spectra X and Y we may form a Mackey functor $[X, Y]$ whose value on finite G-sets is given by:

$$\mathsf{Fin}_G \ni T \mapsto [T_+ \wedge X, Y]^G.$$

When $X = S^V$ is a representation sphere associated to a virtual representation, V, then we define the Vth *homotopy Mackey functor* by

$$\underline{\pi}_V X := [\underline{S^V, X}].$$

Finally, we denote by Slice_n the category of n-slices, by $P_n^n X$ the n-slice of a G-spectrum X, and by $\mathsf{Mack}(G, \mathsf{Ab})$ the category of Mackey functors valued in abelian groups.

Theorem ([HHR16], [Hil11]). *Let $G = C_2$, and let ρ denote the regular representation.*

(a) The functor

$$\underline{\pi}_{n\rho-1} : \mathsf{Slice}_{2n-1} \longrightarrow \mathsf{Mack}(C_2, \mathsf{Ab})$$

is an equivalence of categories.

(b) *The functor*

$$\underline{\pi}_{n\rho} : \mathsf{Slice}_{2n} \longrightarrow \mathsf{Mack}(C_2, \mathsf{Ab})$$

is fully faithful. The essential image consists of those Mackey functors $\underline{M}$ such that the restriction map

$$\mathrm{res} : \underline{M}(*) \longrightarrow \underline{M}(C_2)$$

is injective.

(c) *The slices of a G-spectrum X are determined by the formulae:*

$$\underline{\pi}_{n\rho-1} P_{2n-1}^{2n-1} X = \underline{\pi}_{n\rho-1} X.$$

$$\underline{\pi}_{n\rho} P_{2n}^{2n} X = \frac{\underline{\pi}_{n\rho} X}{\ker(\mathrm{res})}.$$

One might hope that, in general, one may compute n-slices directly in terms of a single $RO(G)$-graded homotopy Mackey functor. Unfortunately, n-slices need not be $RO(G)$-graded suspensions of Eilenberg-MacLane spectra in general (Counterexample 3.12). Instead, we will need to probe G-spectra by objects more general than representation spheres. This brings us to the key definition of the paper.

Definition. An **isotropic slice n-sphere** is a compact G-spectrum W with the property that, for every subgroup $H \subseteq G$, the geometric fixed point spectrum $W^{\Phi H}$ is equivalent to a nonzero, finite wedge of spheres of dimension $\lfloor n/|H| \rfloor$.

Remark. The appearance of the floor function here is inspired by the theorem of [HY17] characterizing the slice filtration in terms of connectivity conditions on geometric fixed points. We will review that theorem below in §0 and generalize it in §1.3.

Example. For any group G, $S^{n\rho}$ is an isotropic slice $n|G|$-sphere and $S^{n\rho-1}$ is an isotropic slice $(n|G|-1)$-sphere.

Example. For any group G, the cofiber of the collapse map $G_+ \to S^0$ is a slice 1-sphere. It is not equivalent to a representation sphere or an induced representation sphere unless $|G| = 2$.

Next, we will need a way to state a generalization of the injectivity condition in part (b) above.

Definition. A subgroup $H \subseteq G$ is called an n-**jump** if the inequality

$$\left\lfloor \frac{n+1}{|H|} \right\rfloor > \left\lfloor \frac{n}{|H|} \right\rfloor$$

holds. We denote by Jump_n the set of conjugacy classes of n-jumps.

It is not so obvious that isotropic slice n-spheres exist for arbitrary G and n, but indeed they do (Proposition 2.12).

Finally, we remark that, for any G-spectrum X, the Mackey functor

$$\underline{\mathrm{End}}(X) := [X, X]$$

admits the canonical structure of a Green functor (Definition 2.40) under composition. Moreover, given another G-spectrum Y, the Mackey functor $[X, Y]$ is naturally a right module over $\underline{\mathrm{End}}(X)$ via precomposition.

Now we can state a version of our first main result (Theorem 2.35).

Theorem. *Let W be an isotropic slice n-sphere. Define a G-set*

$$T^{jump} := \coprod_{[H] \in \mathrm{Jump}_n} G/H.$$

(a) *The functor*

$$[W, -] : \mathsf{Slice}_n \longrightarrow \mathsf{RMod}_{\underline{\mathrm{End}}(W)}$$

is fully faithful. The essential image consists of those $\underline{\mathrm{End}}(W)$-*modules* $\underline{M}$ *with the property that, for every* G-*set* T, *the restriction map associated to the projection* $T^{jump} \times T \to T$,

$$\underline{M}(T) \longrightarrow \underline{M}(T^{jump} \times T)$$

is injective.

(b) *Let*

$$L^{inj} : \mathsf{RMod}_{\underline{\mathrm{End}}(W)} \longrightarrow \mathsf{RMod}_{\underline{\mathrm{End}}(W)}$$

denote the localization functor which enforces the injectivity constraint in (a). Then the n-slice of a G-*spectrum* X *is determined by the formula*

$$\underline{[W, P_n^n X]} = L^{inj}\underline{[W, X]}.$$

This result is enough for many applications. For example, it is straightforward to deduce from it the previous known results on categories of slices (see §3). Nevertheless, the results proved in the body of the paper are stronger in several respects:

- Using a construction of MacPherson-Vilonen and some special features of the Green functor $\underline{\mathrm{End}}(W)$, we provide a simpler description (Theorem 2.82) of the category $\mathsf{RMod}_{\underline{\mathrm{End}}(W)}$ which cuts down on the computation necessary to determine an n-slice.

- We do not restrict ourselves to the standard slice filtration, but allow more general filtrations. Thus, the user can adapt to more situations of interest.

- All of the results are proven in the setting of parameterized stable homotopy theory. One reason is to allow for more flexible inductive techniques. It also means the results apply to other settings, such as Goodwillie calculus [Gla16].

We now give a summary of each of the sections.

Section 0. We include this section for the equivariant homotopy theorist eager to find ready-to-use definitions and statements without the need to unravel too much notation or contend with the level of generality used in the body of the paper. In this section, we survey all of the results of the paper in a form specialized to the case of the standard slice filtration on G-spectra. We also provide enough of a sketch of the proofs that an expert may reconstruct the details. After reading this section, a reader versed in equivariant homotopy theory should be able to fruitfully skip to §3 and understand the examples presented therein without contemplating the words 'parameterized ∞-category' or 'inductive orbital category'.

Section 1. In this section we set up the formal backdrop for our work. We review and develop gluing techniques, inductive techniques, and a sort of 'six-functor' formalism for manipulating homotopy theories parameterized over certain bases which we call *inductive orbital categories* (Definition 1.10). We define a notion of slice filtration in this context and prove some elementary results about these. As a quick application of this formalism, we produce a characterization of slice filtrations which, as a corollary, provides a streamlined proof of the main result in [HY17]. The framework of parameterized homotopy theory we use is due to Barwick-Dotto-Glasman-Nardin-Shah. They have obtained most of the results in this section independently, using slightly different terminology. To the best of the author's knowledge, however, the remaining sections, including the main theorem and description of slices, is new.

Section 2. This section contains the bulk of the work. We begin by introducing slice spheres (Definition 2.3) which generalize (induced) representation spheres. The transition from the category of n-slices, Slice_n, to our final algebraic description takes several steps. First, using slice spheres, or more generally *testing*

subcategories of slice spheres, as the collection of free algebras for a Lawvere theory, we prove that Slice_n is a localization of the category of models for this Lawvere theory (Theorem 2.29). After showing that slice spheres exist in sufficient supply, we prove this using a 'many-object' variant of a classical theorem dating back to Freyd and Gabriel. The key step is to establish that equivalences are detected by the testing subcategory.

The next step moves from the Lawvere theory to a category of modules over a Green functor. This is entirely algebraic, and the argument is essentielly a parameterized version of the aforementioned result of Freyd-Gabriel. From here, we prove that Slice_n is a localization of the category $\mathsf{RMod}_{\underline{\mathrm{End}}(W)}$ where W is an isotropic slice n-sphere (Definition 2.3, Theorem 2.35).

Next, we distill a special feature of the Green functor $\underline{\mathrm{End}}(W)$ which shows that its structure is strongly controlled by the endomorphisms of its geometric fixed points $\mathrm{End}(W^{\Phi T})$ for each orbit T. We call Green functors that share this property *geometrically split* (Definition 2.48) and digress to prove a purely algebraic result about the structure of modules over geometrically split Green functors. While this work is purely algebraic, it is not trivial. For example, one must contend with the combinatorics of the Burnside category (Proposition 2.64).

By definition, a slice n-sphere W has the property that $W^{\Phi T}$ is a finite wedge of spheres, so the work ultimately reduces to understanding modules over the matrix rings $M_n(\mathbb{Z})$. Of course, these rings are Morita equivalent to $\mathbb{Z}$. This observation leads us to formulate a description of Slice_n that does not require computing $\underline{\mathrm{End}}(W)$ or understanding its action $[W, X]$. Instead, one need only understand the homology of the wedge of spheres $W^{\Phi T}$ as a module over the group $\mathrm{Aut}(T)$ of automorphisms of the orbit T. In practice, this is much easier. Our main general result identifies Slice_n with a localization of the category of *twisted Mackey functors* (Definition 2.73, Theorem 2.82).

Section 3. A general theory is no good without examples. We show that the machinery in the first two sections can be made put to use in cases of interest. First, we show in §3.1 how to recover the known results on slices of G-spectra to date, i.e. the cases of $(n|G| - \epsilon)$-slices where $\epsilon = 0, 1, 2$ ([HHR16; Ull12]). In §3.2 we carry out our theory in the case $G = C_p$. The categories of slices for C_p were previously determined by [HY17] in a slightly different form, and we show how to recover their description from ours. Finally, in §3.3 we move on to a new example. This is the first case where we see slices that are *not $RO(G)$-graded* suspensions of Eilenberg-MacLane spectra, and so are not amenable to previous methods of attack.

Notations and conventions.

- If G is a group we denote by ρ_G (or just ρ, if G is understood) the real, regular representation of G.

- If X and Y are objects in a model category (or, more generally, a relative category) we will use $\mathrm{map}(X, Y)$ to denote the *derived* mapping space between X and Y.

- We will use $[X, Y]$ to denote the set of maps in a homotopy category, with no further decoration if it is clear where these objects live. So, for example, if X and Y are G-spectra, then $[X, Y]$ is the set of maps between X and Y in the homotopy category of G-spectra, not the set of maps between the underlying spectra in the homotopy category of spectra.

- We say that a spectrum X is n-**connective** if $\pi_k(X) = 0$ for all $k < n$.

- We say a spectrum X is n-**truncated** if $\pi_k(X) = 0$ for all $k > n$.

- Since W is being used a lot, we will write $\mathrm{Aut}(G/H)$ for the automorphisms of the G-set G/H (i.e. the Weyl group) instead of WH or $W_G H$.

- If $\mathcal{C}$ and K are ∞-categories, we denote by $\mathsf{Psh}_{\mathcal{C}}(K)$ the ∞-category of $\mathcal{C}$-valued presheaves. If $\mathcal{C}$ happens to equivalent to a 1-category, for example if $\mathcal{C} = \mathsf{Set}$ is the category of sets, then there is a natural equivalence $\mathsf{Psh}_{\mathcal{C}}(K) \cong \mathsf{Psh}_{\mathcal{C}}(hK)$ where hK denotes the homotopy category of K.

0 An overview in the equivariant case

This section contains a survey of our main results and definitions in the case of G-spectra. We assume the reader is familiar with the basic definitions of equivariant stable homotopy theory. A nice overview can be found in §1-§3 of [HHR16]. We begin by reviewing the definition of the slice filtration.

Definition 0.1.   • A G-spectrum X is **slice n-connective** if it belongs to the full subcategory of Sp^G generated under extensions and homotopy colimits by the objects

$$G/H_+ \wedge S^{k\rho_H - \epsilon}, \quad \epsilon = 0, 1 \text{ and } k|H| - \epsilon \geq n.$$

We denote the homotopy theory of slice n-connective G-spectra by $\mathsf{Sp}^G_{\geq n}$. We will often write $X \geq n$ to indicate that X is slice n-connective.

- A G-spectrum Y is **slice n-truncated** if, for every $X \geq n+1$, the mapping space $\mathrm{map}(X, Y)$ is weakly contractible. We denote the homotopy theory of slice n-truncated G-spectra by $\mathsf{Sp}^G_{\leq n}$ and indicate that Y is slice n-truncated by writing $Y \leq n$.

- We denote by $P_n : \mathsf{Sp}^G \to \mathsf{Sp}^G_{\geq n}$ a right adjoint to the inclusion, and by $P^n : \mathsf{Sp}^G \to \mathsf{Sp}^G_{\leq n}$ a left adjoint to the inclusion.

- We say that $A \in \mathsf{Sp}^G$ is an n-**slice** if $A \leq n$ and $A \geq n$. We denote the category of n-slices by Slice_n.

Remark 0.2. If A and B are n-slices, then the mapping space $\mathrm{map}(A, B)$ is discrete, which justifies the use of the word *category* of n-slices.

Remark 0.3. If G is trivial, then $X \geq n$ (resp. $Y \leq n$) if and only if X is n-connective (resp. n-truncated) in the classical sense. Hence the slice filtration is the usual Postnikov filtration.

The slice filtration for non-trivial groups is *not* the filtration associated to a t-structure. Specifically, we have an inclusion

$$\Sigma\left(\mathsf{Sp}^G_{\geq n}\right) \subseteq \mathsf{Sp}^G_{\geq n+1}$$

which is usually strict. Nevertheless, each subcategory $\mathsf{Sp}^G_{\geq n}$ is the collection of connective objects for a t-structure on Sp^G. We denote the heart of this t-structure by $\heartsuit_n$. The following is elementary from the inclusion above:

Lemma 0.4. *There is an inclusion* $\mathsf{Slice}_n \longrightarrow \heartsuit_n$ *which exhibits the source as an accessible localization of the target, i.e. it admits an accessible left adjoint.*

While the heart of a t-structure is always an abelian category, the same is not true in general for Slice_n. Our results on Slice_n are obtained by first identifying $\heartsuit_n$ with a more algebraic category, and then identifying the localization that yields Slice_n.

When the slice filtration was first introduced, one had to exhibit slice n-connectivity by an explicit construction. Recently, a much simpler characterization of slice connectivity was given by Hill-Yarnall [HY17].

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Theorem 0.5 (Hill-Yarnall). *Let X be a G-spectrum. Then $X \geq n$ if and only if, for all $H \subseteq G$, the spectrum $X^{\Phi H}$ is $(\lfloor n/|H| \rfloor)$-connective.*

This motivates the following definition.

Definition 0.6. Let W be a compact G-spectrum.

- W is a **slice sphere** if, for every $H \subseteq G$, the spectrum $W^{\Phi H}$ is a finite (possibly trivial) wedge of spheres.

- W is a **slice n-sphere** if, for every $H \subseteq G$, the spectrum $W^{\Phi H}$ is a finite (possibly trivial) wedge of spheres of dimension $\lfloor n/|H| \rfloor$.

- W is an **isotropic slice n-sphere** if it is a slice n-sphere and $W^{\Phi H}$ is nonzero for each $H \subseteq G$.

To motivate the next steps, let us imagine that we are trying to identify the heart of Sp^G with its standard t-structure, i.e. the t-structure for which X is 0-connective if the (genuine) fixed points X^H are 0-connective for all $H \subseteq G$. So we would like to understand G-spectra A such that A is 0-connective and 0-truncated.

Now, by definition, a map $X \to Y$ is an equivalence if and only if $X^H \to Y^H$ is an equivalence for all $H \subseteq G$. We know that S^0 detects equivalences between 0-connective and 0-truncated spectra. The restriction-induction adjunction implies that equivalences between 0-connective, 0-truncated G-spectra are detected by the subcategory $\{G/H_+ \wedge S^0\}_{H \subseteq G}$, and hence also by the subcategory $\{T_+ \wedge S^0\}$ where T ranges over all finite G-sets.

Thus, the restricted Yoneda embedding:

$$\underline{\pi}_0 : \left(\mathsf{Sp}^G\right)^{\heartsuit} \longrightarrow \mathsf{Psh}^{\times}_{\mathsf{Set}}(\{T_+ \wedge S^0\})$$

is conservative, where $\mathsf{Psh}^{\times}_{\mathsf{Set}}$ denotes the category of product-preserving presheaves of sets. Using the fact that $[T_+ \wedge S^0, Y] = 0$ when Y is 1-connective, and in particular that

$$[T_+ \wedge S^0, Y] = [T_+ \wedge S^0, \tau_{\leq 0} Y],$$

one shows that $\underline{\pi}_0$ preserves cokernels. The Barr-Beck theorem implies $\underline{\pi}_0$ is monadic, and the same equality above applied to $Y = G/H_+ \wedge S^0$, shows that the monad in question is the identity functor. We conclude that the assignment $A \mapsto \{[T_+ \wedge S^0, A]\}$ yields an equivalence of categories:

$$\left(\mathsf{Sp}^G\right)^{\heartsuit} \cong \mathsf{Psh}^{\times}_{\mathsf{Set}}(\{T_+ \wedge S^0\}).$$

This is already a sort of algebraic description: categories of product-preserving presheaves are known as *models for Lawvere theories* and behave like categories of algebraic objects. Our understanding of a Lawvere theory is only as good as our understanding of the (maps between) *free* objects, which in this case are the objects $\{T_+ \wedge S^0\}$.

Theorem 0.7 (Segal, tom Dieck, Lewis-May-Steinberger). *The category $\{T_+ \wedge S^0\}$ is equivalent to the Burnside category of finite G-sets. Hence, from the definition of Mackey functors,*

$$\mathsf{Psh}^{\times}_{\mathsf{Set}}(\{T_+ \wedge S^0\}) \cong \mathsf{Mack}(G; \mathsf{Ab}).$$

Our first algebraic description of the category of n-slices follows this outline closely. First, we require a replacement for the category $\{T_+ \wedge S^0\}$ of test objects.

Definition 0.8. We say that a subcategory Test_n of the category of slice n-spheres is a **testing subcategory** if

(i) For all finite G-sets T and $W \in \mathsf{Test}_n$, the G-spectrum $T_+ \wedge W$ is also in Test_n.;

(ii) For every $H \subseteq G$, there is some $W \in \mathsf{Test}_n$ such that $W^{\Phi H}$ is nonzero.

It is not obvious from the definition that testing subcategories exist. To show that they do, it suffices to produce a single isotropic slice n-sphere.

Proposition 0.9. *(Proposition 2.12) For any G and n, there exists an isotropic slice n-sphere.*

Proof sketch. Begin with the sphere S^k where $k = \lfloor n/|G| \rfloor$ and then inductively attach induced spheres to bump up the dimension of each geometric fixed point spectrum without messing up the work you've already done. In order to ensure that the geometric fixed points remain wedges of spheres, one must choose attaching maps which split upon taking geometric fixed points. The simplest way to achieve this in general is to use the counit $G/H_+ \wedge X \to X$. $\qquad\square$

Now we may define an analog of the functor $\underline{\pi}_0$:

Definition 0.10. If Test_n is a testing subcategory, define the category of **model n-slices** by

$$\mathsf{Model}_n := \mathsf{Psh}_{\mathsf{Set}}^{\times}(\mathsf{Test}_n).$$

Let $\hat{\pi}_n$ denote the restricted Yoneda embedding:

$$\hat{\pi}_n : \mathrm{hSp}^G \longrightarrow \mathsf{Model}_n.$$

Theorem 0.11 (Theorem 2.29). *(i) The restriction of $\hat{\pi}_n$ to $\heartsuit_n$ yields an equivalence of categories*

$$\hat{\pi}_n : \heartsuit_n \xrightarrow{\cong} \mathsf{Model}_n.$$

(ii) Under this equivalence, the category Slice_n corresponds to the full subcategory of Model_n spanned by presheaves F with the property that, for all $W \in \mathsf{Test}_n$, the projection $T_+^{jump} \wedge W \to W$ produces an injective map
$$F(W) \longrightarrow F(T_+^{jump} \wedge W).$$

Proof sketch. Part (i) is established following the same outline as in the computation of the heart of the standard t-structure above. The main step is to prove that the functor is conservative, and this is done by standard induction arguments using isotropy separation.

For part (ii), let L^{inj} denote the monad on Model_n which enforces the injectivity constraint in the statement. Then the main computation is that, for any G-spectrum X, $L^{inj}\hat{\pi}_n X \cong \hat{\pi}_n P^n X$. The key yet elementary fact used here is the following: while it is *not* true that $\hat{\pi}_n$ vanishes on all slice $(n+1)$-connective spectra, it is nevertheless the case that $\hat{\pi}_n \Sigma X = 0$ when $X \geq n$. (Proposition). This allows us to control the fact that $\hat{\pi}_n$ is not quite exact on Slice_n. $\qquad\square$

This brings the category of n-slices into the realm of algebra. However, as before, our understanding of Model_n is only as good as our understanding of the testing subcategory. For certain examples, it is possible to analyze this category directly.

Example 0.12. If $n = k|G|$, then $S^{k\rho}$ generates a testing subcategory equivalent to the category $\{T_+ \wedge S^0\}$, since smashing with $S^{k\rho}$ is invertible. Thus,

$$\heartsuit_{k|G|} \cong \mathsf{Mack}(G; \mathsf{Ab})$$

and, unwinding the definitions, we recover the equivalence [HHR16; Hil11]:

$$\mathsf{Slice}_{k|G|} \cong \{\text{Mackey functors with injective restriction maps}\}.$$

In general, however, it is convenient to have an alternative description. We observe that we already understand $\{T_+ \wedge S^0\}$ and attempt to understand Test_n in terms of what we know, so as not to reinvent the wheel. We restrict to the case of testing subcategories generated by a single isotropic slice n-sphere, W, for ease.

In this case, $\hat{\pi}_n X$ is essentially the data of:

- the Mackey functor $[\underline{W, X}]$, and

- its interaction with maps $T_+ \wedge W \to U_+ \wedge W$ for G-sets T, U.

The name for such a thing is a module over a Green functor. So we show, very formally, that

Theorem 0.13 (Theorem 2.35). *(i) The functor $[\underline{W, -}]$ induces an equivalence of categories*

$$[\underline{W, -}] : \heartsuit_n \xrightarrow{\cong} \mathsf{RMod}_{\underline{\mathrm{End}}(W)}.$$

(ii) Under this equivalence, the category Slice_n corresponds to the full subcategory of $\mathsf{RMod}_{\underline{\mathrm{End}}(W)}$ spanned by those modules $\underline{M}$ such that the restriction map

$$\underline{M}(T) \longrightarrow \underline{M}(T^{jump} \times T)$$

is injective for all finite G-sets T.

Proof sketch. In the theory of ordinary abelian categories, one can recognize categories of modules as those abelian categories that admit a compact, projective generator. A similar result is true in the context of G-categories to recognize G-categories of modules over a Green functor, and the claim follows. $\square$

Again, for some applications, this result suffices.

Example 0.14. If $n = k|G|$, we get yet another proof that $\mathsf{Slice}_{k|G|}$ is the category of Mackey functors with injective restriction maps. Indeed, since $S^{k\rho}$ is invertible,

$$\underline{\mathrm{End}}(S^{k\rho}) \cong \underline{\mathrm{End}}(S^0) \cong \underline{A}.$$

But the Burnside Mackey functor $\underline{A}$ is the unit in the category of Mackey functors, so a right $\underline{A}$-module is just a Mackey functor.

Remark 0.15. This argument applies more generally whenever one can find an *invertible* isotropic slice n-sphere. In this case, the category of n-slices will be equivalent, via the corresponding (Pic-graded) homotopy Mackey functor, to the category of Mackey functors satisfying certain injectivity conditions on their restriction maps.

Now we come to the final simplification which will ultimately remove the need to compute the action of $\underline{\mathrm{End}}(W)$ entirely. A (right) module $\underline{M}$ over a Green functor $\underline{R}$ consists of, in particular, a collection of $\underline{R}(G/H)$-modules, $\underline{M}(G/H)$, for each $H \subseteq G$. However, much of the description of this action is redundant due to the Frobenius relation dictating the action of a transfer:

$$m \cdot \mathrm{tr}(r) = \mathrm{tr}(\mathrm{res}(m) \cdot r).$$

In general, we may not be able to untangle the transferred ring elements from those not in the image of the transfer, but occasionally we are lucky. We give a name to this situation.

If $\underline{R}$ is a Green functor then, for each $H \subseteq G$ we can form the quotient:

$$\underline{R}(G/H) \longrightarrow \frac{\underline{R}(G/H)}{\langle \mathrm{tr}_K^H \underline{R}(G/K) | K \text{ subconjugate to } H \rangle} =: \underline{R}^{\Phi H}.$$

Definition 0.16. We say that $\underline{R}$ is **geometrically split** if the map $\underline{R}(G/H) \to \underline{R}^{\Phi H}$ admits an $\operatorname{Aut}(G/H)$-equivariant ring section.

If we choose splittings for a geometrically split Mackey functor $\underline{R}$, then a right $\underline{R}$-module gives rise to a sequence of $\underline{R}^{\Phi H}$-modules $\underline{M}(G/H)$ with a compatible action of $\operatorname{Aut}(G/H)$. The precise structure remaining is explained in Definition 2.51 below. We denote the resulting category by $\mathsf{RMod}_{\underline{R}^{\Phi}}$.

We then have the following piece of algebra:

Theorem 0.17 (Theorem 2.53)**.** *With notation as above, the forgetful functor*

$$\mathsf{RMod}_{\underline{R}} \longrightarrow \mathsf{RMod}_{\underline{R}^{\Phi}}$$

is an equivalence of categories.

Proof sketch. We prove this using the main theorem of [FP04] on comparisons of abelian category recollements. The key step is showing that $\mathsf{Rmod}_{\underline{R}}$ admits a pre-hereditary recollement, in the language of [FP04], and proving this fact requires actually digging into the structure of the Burnside category. (Proposition 2.64). $\qquad\square$

To apply this to our situation we need to identify $\underline{\operatorname{End}}(W)^{\Phi H}$.

Proposition 0.18. *There is an isotropic slice n-sphere W with the property that, for every $H \subseteq G$, the following conditions are satisfied:*

(a) The natural map

$$[W, W]^H \longrightarrow \operatorname{End}(W^{\Phi H})$$

admits an $\operatorname{Aut}(G/H)$-equivariant ring section.

(b) The natural map

$$\underline{\operatorname{End}}(W)^{\Phi H} \longrightarrow \operatorname{End}(W^{\Phi H})$$

is an isomorphism.

(c) Let $J_H = \pi_{\lfloor n/|H| \rfloor} W^{\Phi H}$ as an $\operatorname{Aut}(G/H)$-module and left $\operatorname{End}(W^{\Phi H})$-module. Then the map

$$\operatorname{End}(W^{\Phi H}) \longrightarrow \operatorname{End}_{\mathbb{Z}}(J_H)$$

is an isomorphism of $\operatorname{Aut}(G/H)$-modules.

Proof sketch. For part (ii), we use isotropy separation and some connectivity arguments. Part (iii) is elementary because $W^{\Phi H}$ is a finite wedge of spheres of dimension $\lfloor n/|H| \rfloor$. Both of these statements are true for every isotropic slice n-sphere. Part (i) relies on an inductive argument applied to a specific construction of an isotropic slice n-sphere. $\qquad\square$

Remark 0.19. We believe part (i) of this proposition holds for *every* isotropic slice n-sphere, but we have not tried to prove this.

By definition, we know that $W^{\Phi H}$ is a finite, nonzero wedge of spheres. In particular, J_H is a free abelian group. Morita theory, slightly generalized to account for the action of $\operatorname{Aut}(G/H)$, then yields an equivalence

$$\mathsf{RMod}_{\operatorname{End}(W^{\Phi H})\text{-}\operatorname{Aut}(G/H)} \cong \mathsf{Mod}_{\operatorname{Aut}(G/H)}$$

Explicitly the equivalence is given by the functors:

$$\mathsf{RMod}_{\operatorname{End}(W^{\Phi H})\text{-}\operatorname{Aut}(G/H)} \ni N \mapsto N \otimes_{\operatorname{End}(W^{\Phi H})} J_H$$

$$\mathsf{Mod}_{\operatorname{Aut}(G/H)} \ni M \mapsto M \otimes J_H^*$$

where tensor products are given the diagonal action and $J_H^* := \mathrm{Hom}_{\mathbb{Z}}(J_H, \mathbb{Z})$.

Combining this equivalence with the definition of $\mathsf{RMod}_{\underline{R}^\Phi}$ we get an equivalent category whose objects consist of the data $\{M_{(G/H)}\}$ of a collection of $\mathrm{Aut}(G/H)$-modules together with maps between them after tensoring with iterations of the J_K and their duals, satisfying various properties. We call this category the category of **twisted Mackey functors**, denoted TwMack_n, and the precise definition is contained in Definition 2.73 below.

In order to apply this algebra to homotopy theory, it will be helpful to do some unraveling. If we choose a (non-equivariant) $\mathbb{Z}$-summand of J_H, then this produces an idempotent in $\mathrm{End}(J_H)$. We can carry this across a chosen splitting to an idempotent in $[W, W]^H$, and from there split off an H-equivariant summand:

$$W_{(G/H)} \to W \to W_{(G/H)}.$$

If X is a G-spectrum, then $[W_{(G/H)}, X]^H$ still has an $\mathrm{Aut}(G/H)$-action coming from the one on (the restriction of) X. These modules are essentially the objects $M_{(G/H)}$ described above.[1]

If K is subconjugate to H, then composing the inclusions and retractions gives K-equivariant maps:

$$W_{(G/K)} \to W \to W_{(G/H)},$$

$$W_{(G/H)} \to W \to W_{(G/K)}.$$

Using these, we get analogues of restriction and transfer maps:

$$[W_{(G/H)}, X]^H \xrightarrow{\mathrm{res}} [W_{(G/H)}, X]^K \longrightarrow [W_{(G/K)}, X]^K,$$

$$[W_{(G/K)}, X]^K \longrightarrow [W_{(G/H)}, X]^K \xrightarrow{\mathrm{tr}} [W_{(G/H)}, X]^H.$$

The relations these maps satisfy satisfy are somewhat more involved than their Mackey functor cousins, but they are about as manageable in practice. The main point of the algebra above is that these maps are the only data necessary to describe the associated $\underline{\mathrm{End}}(W)$-module, and hence determine the n-slice of X.

We summarize this discussion in the following paraphrased theorem:

Theorem 0.20 (Theorem 2.82). *(i) The procedure above yields an equivalence of categories:*

$$\heartsuit_n \cong \mathsf{TwMack}_n.$$

(ii) Under this equivalence, the category Slice_n corresponds to the full subcategory of TwMack_n spanned by those objects $\{M_{(G/H)}\}$ satisfying an explicit injectivity constraint on their restriction maps.

The upshot is the following procedure for computing the n-slice of a G-spectrum X:

Step 1. Find or construct an isotropic slice n-sphere, W.

Step 2. Compute the $\mathrm{Aut}(G/H)$ action on each $W^{\Phi H}$ to determine the modules J_H.

Step 3. Choose a summand of J_H and determine the corresponding H-equivariant summand $W_{(G/H)}$ of W.

Step 4. Compute the groups $[W_{(G/H)}, X]^H$ and the 'twisted' restrictions and transfers between them.

Step 5. Make the requisite (direct sum of) restriction maps injective.

At this point the reader should be prepared for the examples in §3.

_____

[1] One may have to alter the action to account for how the action on J_H intertwines the chosen idempotent. In our examples, we will usually have $W_{(G/H)} = \downarrow_H W$, in which case we take the usual action on homotopy groups, or we'll have $\downarrow_H W = W^{\Phi H}$, in which case we may indeed take the trivial action on the summand.

1 Filtrations on stratified categories

A common strategy for proving statements about Sp^G is to induct over the poset of subgroups using geometric fixed points, and eventually reduce to a statement about non-equivariant spectra. We can axiomatize the structure necessary to make arguments like this and arrive at the notion of a stratified homotopy theory, which we review in §1.1.

In the situation of Sp^G we actually have *two* methods for reducing to non-equivariant considerations: geometric fixed points and genuine fixed points. Again, we can distill the requisite properties into a definition, that of a homotopy theory of Mackey functors on an inductive orbital category (Definitions 1.27 and 1.10). We discuss this in §1.2.

In §1.3, we define slice filtrations (Definition 1.46) associated to dimension functions for homotopy theories of Mackey functors and explore some of their elementary properties. We give a recognition theorem (Theorem 1.57) for comparing a given filtration to the slice filtration associated to a dimension function. As a corollary, we obtain a streamlined proof of the theorem of Hill-Yarnall characterizing the original slice filtration in terms of geometric fixed points (Corollary 1.58).

We pause now to collect some justifications for our chosen level of generality:

- When making inductive arguments about G-spectra, one is often led to consider homotopy theories associated to families of subgroups of G. These homotopy theories are usually not equivalent to Sp^H for any group H.

- There are several homotopy theories of Mackey functors that do not fall under the direct purview of equivariant homotopy theory, e.g. the homotopy theory of cyclotomic spectra [BG16b; BM13] and of n-excisive functors [Gla16]. Since it does not require extra work, it seems prudent to develop the theory in a way that applies to these examples.

- Even when restricting to Sp^G, it is convenient to consider filtrations other than the standard slice filtration. For example, the *regular* slice filtration was used to great effect by Ullman [Ull12] in his thesis. We allow for yet further variants, so that one may choose a filtration suited to the application at hand.

Finally, as we mentioned in the introduction, most of the results in §1.3 were obtained independently by Barwick-Dotto-Glasman-Nardin-Shah. We will try to indicate the major overlaps where they occur.

1.1 Review of recollements and stratifications

There are many situations where we study objects of a homotopy theory $\mathcal{C}$ by breaking it up into two pieces. Here are some examples.

- Let σ denote the sign representation of C_2. If X is a C_2-spectrum, then X sits in a cofiber sequence

$$S(\infty\sigma)_+ \wedge X \longrightarrow X \longrightarrow S^{\infty\sigma} \wedge X$$

called the *isotropy separation sequence*. The first term has the property that it is built out free C_2-spectra, in the sense that it has a filteration with associated graded a wedge of copies of suspensions of $C_{2+} \wedge X$. The last term has the property that its underlying spectrum vanishes, so all its information is contained in its fixed points.

Alternatively, we can recover X from the homotopy Cartesian square:

$$
\begin{array}{ccc}
X & \longrightarrow & S^{\infty\sigma} \wedge X \\
\downarrow & & \downarrow \\
F(S(\infty\sigma)_+, X) & \longrightarrow & S^{\infty\sigma} \wedge F(S(\infty\sigma)_+, X)
\end{array}
$$

which is sometimes called the *Tate fracture square*. The bottom piece of this square can be studied entirely in terms of the local system on BC_2 which underlies X, and the left vertical map is a sort of completion while the right horizontal map is a sort of localization.

- Let X be a space, $j : U \hookrightarrow X$ an open embedding with closed complement $i : Y \hookrightarrow X$, and denote by $\mathsf{Shv}(X; \mathsf{Sp})$ the homotopy theory of sheaves of spectra on X. Then the restriction functor $j^* : \mathsf{Shv}(X; \mathsf{Sp}) \to \mathsf{Shv}(U; \mathsf{Sp})$ admits a left adjoint $j_!$ (extension by zero) and every sheaf decomposes into a natural cofiber sequence:

$$j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F}.$$

The first term is supported on U, and the last term is set-theoretically supported on Y (in the sense that it is annihilated by j^*).

Alternatively, we can recover $\mathcal{F}$ from the homotopy Cartesian square:

$$
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & i_* i^* \mathcal{F} \\
\downarrow & & \downarrow \\
j_* j^* \mathcal{F} & \longrightarrow & i_* i^* j_* j^* \mathcal{F}
\end{array}
$$

The right vertical map provides gluing data and is a generalization of a *clutching function* from the classical study of vector bundles.

- If M is a complex of abelian groups, then there is a cofiber sequence in the derived category $\mathsf{D}(\mathbb{Z})$:

$$\Gamma_p M \longrightarrow M \longrightarrow M\left[\frac{1}{p}\right]$$

where $\Gamma_p M$ has the property that every element of $H_k(\Gamma_p M)$ is annihilated by a power of p, and $M[1/p]$ has the property that p acts invertibly.

Alternatively, we can recover M from the arithmetic fracture square:

$$
\begin{array}{ccc}
M & \longrightarrow & M\left[\frac{1}{p}\right] \\
\downarrow & & \downarrow \\
\hat{L}M & \longrightarrow & (\hat{L}M)\left[\frac{1}{p}\right]
\end{array}
$$

Here, $\hat{L}M$ denotes the derived functor of p-completion, which plays a prominent role in $K(1)$-local homotopy theory.

We collect some common features of these examples into a definition. It is the evident adaptation of the Grothendieck school's notion of a recollement to our setting.

Definition 1.1. ([Lur16, A.8.1]) Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ denote full subcategories. We say that $\mathcal{C}$ is a **recollement** of $\mathcal{C}_0$ and $\mathcal{C}_1$ if the following conditions are satisfied:

(a) The subcategories $\mathcal{C}_0$ and $\mathcal{C}_1$ are closed under equivalence.

(b) The inclusion functors $\mathcal{C}_i \hookrightarrow \mathcal{C}$ admit left adjoints L_i.

(c) The functors L_0 and L_1 are left exact.

(d) The functor L_1 carries every object of $\mathcal{C}_0$ to a final object of $\mathcal{C}_1$.

(e) The functors L_0 and L_1 are jointly conservative.

We will need a slight generalization of this definition where $\mathcal{C}$ is glued together from more than just two subcategories. Recall that an **interval** in a poset P is a subset $I \subseteq \mathsf{P}$ such that if $x, y \in I$ and $x < z < y$, then $z \in I$.

Definition 1.2 ([Gla15]). Let $\mathcal{C}$ be an ∞-category which admits finite limits, and let P be a poset. Then a **P-stratification** of $\mathcal{C}$ is a map of posets

$$\mathfrak{S} : \{\text{intervals in } \mathsf{P}\} \longrightarrow \{\text{reflective subcategories of } \mathcal{C}\}$$

such that:

- $\mathfrak{S}(\mathsf{P}) = \mathcal{C}$,

- $\mathfrak{S}(\varnothing) \subseteq \mathcal{C}$ is the full subcategory of final objects,

- whenever an interval $I \subseteq \mathsf{P}$ is decomposed as $I = I_0 \amalg I_1$ in such a way that no element of I_0 is strictly greater than any element of I_1, then $\mathfrak{S}(I)$ is a recollement of $\mathfrak{S}(I_0)$ and $\mathfrak{S}(I_1)$.

These definitions are meant to generalize the fracture squares that appeared in our motivating examples. In order to get the entire package, cofiber sequences and all, we need to move to a stable setting. In this case, there is yet another characterization of stratifications.

Theorem 1.3. *Let $\mathcal{C}$ be a stable ∞-category and let*

$$\mathfrak{S}_0 : \{\text{downward closed intervals in } \mathsf{P}\} \longrightarrow \{\text{full subcategories of } \mathcal{C} \text{ closed under equivalences}\}$$

be a map of posets. The following conditions are equivalent:

(i) Each $\mathfrak{S}_0(I)$ is reflective and coreflective; i.e. the inclusion admits both left and right adjoints, and $\mathfrak{S}_0(\mathsf{P}) = \mathcal{C}$.

(ii) The function $\mathfrak{S}_0$ extends to a P-stratification of $\mathcal{C}$.

If either of these conditions are satisfied then, when I is downward closed and J is its upwardly closed complement, we may identify $\mathfrak{S}(J)$ with the full subcategory of $\mathcal{C}$ spanned by those objects Y such that the mapping space $\mathrm{map}_{\mathcal{C}}(X, Y)$ is contractible for each $X \in \mathfrak{S}(I)$.

Proof. The proof in Barwick-Glasman [BG16a, Lemma 3] works just as well in the case of a poset. $\square$

When $\mathsf{P} = \Delta^1$ it will be useful to set some notation down for the morass of adjoints in play. The reader is encouraged to keep in mind the example where X is a space, $j : U \hookrightarrow X$ is an open embedding, and $i : Y = X - U \hookrightarrow X$ is the closed complement. Then $\mathcal{C} = \mathsf{Shv}(X; \mathsf{Sp})$, $\mathcal{C}_0 = \mathsf{Shv}(Y; \mathsf{Sp})$, and $\mathcal{C}_1 = \mathsf{Shv}(U; \mathsf{Sp})$.

Remark 1.4. In the case where $\mathcal{C}$ is stable and a recollement of $\mathcal{C}_0$ and $\mathcal{C}_1$, we have the following diagram where each arrow is left adjoint to the arrow below it:

$$\mathcal{C}_1 \underset{\underset{j_*}{\xrightarrow{\hspace{1cm}}}}{\overset{\overset{j_!}{\xrightarrow{\hspace{1cm}}}}{\underset{j^*}{\xleftarrow{\hspace{1cm}}}}} \mathcal{C} \underset{\underset{i^!}{\xrightarrow{\hspace{1cm}}}}{\overset{\overset{i^*}{\xrightarrow{\hspace{1cm}}}}{\underset{i_*}{\xleftarrow{\hspace{1cm}}}}} \mathcal{C}_0$$

With this notation, $\mathcal{C}$ is the oplax limit [GHN15, 2.8] of the exact functor $i^* j_* : \mathcal{C}_1 \to \mathcal{C}_0$. The ∞-category $\mathcal{C}_1$ is now embedded into $\mathcal{C}$ in two different ways as a full subcategory: first as a reflective subcategory via j_*, as in the definition of a recollement, but also as a coreflective subcategory via $j_!$. The essential images are characterized as the two different orthogonal complements of $\mathcal{C}_0$. We will denote the image of j_* by $\mathcal{C}_1^\wedge$ and the image of $j_!$ by $\mathcal{C}_1^\vee$. The notation is supposed to suggest that the reflective subcategory contains *complete*

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objects while the coreflective subcategory contains *nilpotent* objects. An explicit equivalence between these categories is obtained by the two inverse functors:

$$j_*j^* : \mathcal{C}_1^\vee \longrightarrow \mathcal{C}_1^\wedge$$

$$j_!j^* : \mathcal{C}_1^\wedge \longrightarrow \mathcal{C}_1^\vee$$

See Barwick-Glasman [BG16a] for details.

Remark 1.5. A stratification of $\mathcal{C}$ gives rise to a lax functor $\mathsf{P}^{op} \to \mathsf{Cat}_\infty$ recording the atomic localizations, and $\mathcal{C}$ can be recovered as the *op*lax limit of this lax diagram. In fact, this process yields an equivalence between the homotopy theory of stratified ∞-categories and the homotopy theory of lax functors $\mathsf{P}^{op} \to \mathsf{Cat}_\infty$ (i.e. locally cocartesian fibrations over P^{op}). Making this precise would take us too far afield, but for a version of this reconstruction theorem without the language of lax functors see [Gla15, 3.18]. The idea is that a lax functor out of P^{op} is the same data as an ordinary functor out of the *relaxation* of P^{op}, which is modeled on the poset of nonempty subsets of P^{op}.

1.2 Mackey functors

In the homotopy theory of G-spectra there are two a priori unrelated inductive approaches to understanding a G-spectrum X. On the one hand, we can reduce questions about X to questions about its geometric fixed points, $X^{\Phi H}$. This is the point of view that motivated the previous section. On the other hand, we can reduce questions about X to questions about its *genuine* fixed points, X^H. At a key point below (namely in our construction of isotropic slice spheres) we will utilize the interplay between these two approaches.

First, however, we develop a general setting where these two sorts of inductive tools- genuine and geometric fixed points- can both be defined.

Remark 1.6. Many of the definitions and examples in the beginning of this section are pulled directly from Glasman [Gla15], with the notable exception of Definition 1.10.

Definition 1.7. An **epiorbital category** is an essentially finite category $\mathcal{O}$ satisfying the following conditions:

- Every morphism in $\mathcal{O}$ is an epimorphism.

- $\mathcal{O}$ admits pushouts and coequalizers.

Define a relation on the set of isomorphism classes of objects in $\mathcal{O}$ by $[X] \geq [Y]$ if $\mathrm{Hom}(X, Y)$ is nonempty. It is easy to check that this forms a poset, which we denote $\mathsf{P}_\mathcal{O}$.

Given an essentially small ∞-category $\mathcal{C}$, we will denote by $\mathsf{Fin}_\mathcal{C}$ the ∞-category obtained by freely adjoining finite coproducts. An explicit model can be obtained as the full subcategory of $\mathsf{Psh}(\mathcal{C})$ spanned by finite coproducts of representable functors. We remark that, in the case when $\mathcal{C}$ is an ordinary category, it doesn't matter if we use the ∞-category of presheaves of spaces, or the ordinary category of presheaves of sets in this construction.

Definition 1.8. An **orbital ∞-category** is an essentially small ∞-category $\mathcal{O}$ such that $\mathsf{Fin}_\mathcal{O}$ admits pullbacks. We will often call elements of $\mathcal{O}$ **orbits**.

Proposition 1.9. *([Gla15, 2.14]) Every epiorbital category is orbital.*

Unfortunately, neither of these two levels of generality is quite right for what we need. We will prove most of our theorems by induction on the size of the poset of an epiorbital category, but unfortunately sometimes the inductive procedure takes us outside the realm of epiorbital categories. We will take a middle ground, and propose the following.

Definition 1.10. An **inductive orbital category** is an essentially finite, orbital, (discrete) category $\mathcal{O}$ with the property that every endomorphism is an isomorphism. Notice that the isomorphism classes of objects again form a poset, $\mathsf{P}_{\mathcal{O}}$.

Warning 1.11. In this definition we use 'essentially finite' in the sense of *ordinary* category theory. Our inductive orbital categories will generally *not* be finite as ∞-categories, i.e. they will not usually have only finitely many non-degenerate simplices.

Remark 1.12. In fact, the condition that the isomorphism classes of objects form a poset under the relation

$$[X] \geq [Y] \iff \operatorname{Hom}(X,Y) \neq \varnothing$$

is equivalent to the condition that every endomorphism is an isomorphism. Both of these, in turn, are equivalent to the condition that the category admits a conservative map to a poset. Categories in which every endomorphism is an isomorphism are sometimes called EI-categories in the literature.

Remark 1.13. Barwick-Dotto-Glasman-Nardin-Shah define and study the more general but related notion of a *perfect orbital ∞-category* which is likely the correct setting for the sorts of inductive arguments used below. All of our results should hold in this generality with minimal change.

We choose this definition because of the following closure properties.

Lemma 1.14. *Let $\mathcal{O}$ be an inductive orbital category.*

(i) If $I \subseteq \mathsf{P}_{\mathcal{O}}$ is an interval, then the corresponding full subcategory $\mathcal{O}_I \subseteq \mathcal{O}$ is an inductive orbital category.

(ii) It $T \in \mathsf{Fin}_{\mathcal{O}}$, then $\mathcal{O}_{/T}$ is an inductive orbital category.

Proof. In both cases, it is clear that every endomorphism is still an isomorphism, and that the categories are still discrete, so we just need to check that these categories are orbital. For (ii) this is immediate since $\mathsf{Fin}_{\mathcal{O}_{/T}} \cong (\mathsf{Fin}_{\mathcal{O}})_{/T}$. So we are left with (i). To that end, consider a pullback square in $\mathsf{Fin}_{\mathcal{O}}$

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

where $U, V, V' \in \mathcal{O}_I$. We can write U' as a finite coproduct $U' = \coprod_{\alpha \in A} S_\alpha$ where each $S_\alpha \in \mathcal{O}$. Define $U'' \in \mathcal{O}_I$ to be the coproduct over the subset $B \subseteq A$ of β with $S_\beta \in \mathcal{O}_I$. We claim that the diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is a pullback square in $\mathcal{O}_I$. Indeed, since each S_i maps to $V' \in \mathcal{O}_I$, we know that $[S_i] \geq p$ for some $p \in I$. Since I is an interval, the only way that $[S_i]$ could fail to be in I is if every element of I is not greater than $[S_i]$. So, for every $W \in \mathsf{Fin}_{\mathcal{O}_I}$, $\operatorname{Hom}(W, S_i)$ is empty when $S_i \notin \mathcal{O}_I$. The claim now follows from the string of isomorphisms, for $W \in \mathcal{O}_I$,

$$\operatorname{Hom}(W, U') = \operatorname{Hom}\Big(W, \coprod_\alpha S_\alpha\Big) = \coprod_{\alpha \in A} \operatorname{Hom}(W, S_\alpha) = \coprod_{\beta \in B} \operatorname{Hom}(W, S_\beta) = \operatorname{Hom}(W, U'').$$

$\square$

We now recall the examples of interest.

Example 1.15. If G is a finite group, then the category $\mathcal{O}_G$ of non-empty transitive G-sets, i.e. orbits, is an epiorbital category. The partial order on isomorphism classes is opposite to the poset of conjugacy classes of subgroups. That is:

$$[G/H] \geq [G/K] \iff H \text{ is subconjugate to } K.$$

More generally, any subgroup $H \subseteq G$ yields a full subcategory $\mathcal{O}_{G/H} \subseteq \mathcal{O}_G$ of those orbits with stabilizers that contain H up to conjugacy (i.e. the full subcategory corresponding to the downward closed interval $\{[T] \leq [G/H]\}$). This is also an epiorbital category, and when N is normal the two possible interpretations of the symbol $\mathcal{O}_{G/N}$ agree.

Example 1.16. Crucially, if $\mathcal{F}$ is a family of subgroups of G closed under conjugation and passage to subgroups, then the full subcategory $\mathcal{O}_{\mathcal{F}} \subseteq \mathcal{O}_G$ of transitive G-sets with stabilizers in $\mathcal{F}$ is an inductive orbital category. It is *not* an epiorbital category in general.

Example 1.17. The category $\mathsf{Surj}_{\leq n}$ of finite sets of cardinality at most n and surjective maps between them is an epiorbital category.

Example 1.18. If $\mathcal{G}$ is an ∞-groupoid, then it is an orbital ∞-category. Unfortunately, even when $\mathcal{G}$ is finite and discrete, it is not epiorbital because coequalizers do not exist except in trivial cases. In this case, however, it is an inductive orbital category.

Recall that the twisted arrow category $\mathsf{TwArr}(\mathcal{C})$ of an ∞-category $\mathcal{C}$ is a specific model of a left fibration

$$\mathsf{TwArr}(\mathcal{C}) \longrightarrow \mathcal{C}^{op} \times \mathcal{C}$$

classifying the functor

$$\mathrm{map}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Spaces}$$

Warning 1.19. Different authors have different conventions for the twisted arrow category. For example, ours agrees with Mac Lane, and with Barwick et. al., but is *dual* to the fibration used by Lurie and by Gepner-Haugseng-Nikolaus.

Example 1.20. When $\mathcal{C} = \Delta^n$, the twisted arrow category is the poset of intervals in $[n]$, ordered by inclusion.

Definition 1.21 (Barwick [Bar14]). For any orbital ∞-category, $\mathcal{O}$, define a simplicial set $\mathsf{A}^{\mathit{eff}}(\mathcal{O})$ by declaring the n-simplices to be the set of functors $F : \mathsf{TwArr}(\Delta^n)^{op} \to \mathsf{Fin}_{\mathcal{O}}$ such that, for all $0 \leq i \leq j \leq k \leq \ell \leq n$, the square

$$\begin{array}{ccc} F_{i\ell} & \longrightarrow & F_{ik} \\ \downarrow & & \downarrow \\ F_{j\ell} & \longrightarrow & F_{jk} \end{array}$$

is a pullback. We call $\mathsf{A}^{\mathit{eff}}(\mathcal{O})$ the **effective Burnside ∞-category of** $\mathcal{O}$.

Remark 1.22. Even though $\mathcal{O}$ is a discrete category, $\mathsf{A}^{\mathit{eff}}(\mathcal{O})$ will not be discrete in general. Instead, in this case $\mathsf{A}^{\mathit{eff}}(\mathcal{O})$ will be a $(2,1)$-category. The homotopy category $\mathsf{hA}^{\mathit{eff}}(\mathcal{O})$ has a more familiar description: the objects are objects of $\mathsf{Fin}_{\mathcal{O}}$, the morphisms are *isomorphism classes* of spans, and composition is given by pullback.

We will need a condition on the targets of our Mackey functors.

Definition 1.23. Let $\mathcal{C}$ be an ∞-category which admits finite products and finite coproducts. Suppose moreover that $\mathcal{C}$ is pointed, i.e. it admits an object 0 which is both initial and final.

- We say that $\mathcal{C}$ is **semi-additive** if for all $X, Y \in \mathcal{C}$, the canonical map

$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix} : X \amalg Y \longrightarrow X \times Y$$

  is an equivalence. In this case we denote finite coproducts and products by $X \oplus Y$ and refer to them as direct sums.

- We say that $\mathcal{C}$ is **additive** if it is semi-additive and each of the resulting commutative monoids $[X, Y]$ have inverses.

Example 1.24. Any ordinary abelian category, viewed as an ∞-category, is additive.

Example 1.25. Any stable ∞-category is additive.

Example 1.26. By Barwick [Bar14, 4.3], the ∞-category $\mathsf{A}^{\mathit{eff}}(\mathcal{O})$ is semi-additive.

Definition 1.27. [Bar14] If $\mathcal{C}$ is a semi-additive ∞-category and $\mathcal{O}$ is an orbital ∞-category, we denote by $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$ the ∞-category of functors $\mathsf{A}^{\mathit{eff}}(\mathcal{O}) \to \mathcal{C}$ which preserve finite direct sums.
  When $\mathcal{C} = \mathsf{Sp}$ we will denote this ∞-category more simply by $\mathsf{Sp}^{\mathcal{O}}$.

Example 1.28. If $\mathcal{G}$ is a connected groupoid, then $\mathsf{Mack}(\mathcal{G}; \mathcal{C})$ is canonically equivalent to $\mathsf{Fun}(\mathcal{G}, \mathcal{C})$ [Gla15, 2.27].

Example 1.29. For a finite group, G, the ∞-category $\mathsf{Mack}(\mathcal{O}_G; \mathsf{Sp})$ of spectral Mackey functors is equivalent to the ∞-category underlying the model category of orthogonal G-spectra (cf. Nardin or Guilloiu-May [Nar16; GM11]). This justifies the notation Sp^G.

Example 1.30. The ∞-category $\mathsf{Mack}(\mathsf{Surj}_{\leq n}; \mathcal{C})$ is equivalent to the ∞-category of n-excisive functors from Sp to $\mathcal{C}$. This is proven by Glasman in [Gla16], and in unpublished work of Dwyer-Rezk.

Let $I \subseteq \mathsf{P}_{\mathcal{O}}$ denote a downward closed interval, and denote by $\mathsf{Mack}(\mathcal{O}; \mathcal{C})_{\Phi I}$ the full subcategory of $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$ spanned by those functors which take each element in the complement of I to a zero object in $\mathcal{C}$.

Theorem 1.31 ([Gla15]). *If $\mathcal{O}$ is an inductive orbital category, and $\mathcal{C}$ is stable, then the stable ∞-category $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$ admits a canonical $\mathsf{P}_{\mathcal{O}}$-stratification which, for downward closed intervals, takes the form:*

$$\mathfrak{S}_0(I) := \mathsf{Mack}(\mathcal{O}; \mathcal{C})_{\Phi I}.$$

Proof. Either observe that the proof given by Glasman [Gla15] for epiorbital categories works verbatim for inductive orbital categories, or else apply Proposition 3.13 of [Gla15] to each downward closed I separately to deduce that $\mathfrak{S}_0(I)$ is the reflective and coreflective piece of a recollement. $\square$

Warning 1.32. This theorem is false if $\mathcal{C}$ is only assumed to be additive. The issue is that the left adjoint to the inclusion $\mathsf{Mack}(\mathcal{O}; \mathcal{C})_{\Phi I} \subseteq \mathsf{Mack}(\mathcal{O}; \mathcal{C})$ is not exact in general. If $\mathcal{C}$ is abelian, then a related fact is true using the theory of stratifications of abelian categories (which is not the same as the notion of stratification we are using.) Unfortunately, recollements in the theory of abelian categories are less well-behaved than their ∞-categorical counterparts: it is not possible, in general, to recover a stratified abelian category from its atomic localizations, even in the case of a recollement.

We will identify the strata of this stratification momentarily. But first we take some time to introduce a lot of notation.

Notation 1.33. We now collect together our conventions on the various functors that show up in the theory of Mackey functors. We wouldn't make such a fuss, but we will use essentially all of these at some point.
  Below, $\mathcal{O}$ will denote an inductive orbital category unless otherwise specified. All Mackey functors will take values in a fixed presentable, semi-additive ∞-category $\mathcal{C}$ which we suppress (temporarily breaking our convention) to avoid yet more clutter.

(a) If $\mathcal{F} \subseteq \mathsf{P}_{\mathcal{O}}$ is *upward closed*, then let $j_{\mathcal{F}} : \mathcal{O}_{\mathcal{F}} \hookrightarrow \mathcal{O}$ denote the inclusion of the full subcategory spanned by objects whose isomorphism class lies in $\mathcal{F}$. The functor $j_{\mathcal{F}}^{\amalg}$ preserves pullbacks, so we get adjoint functors:

$$(j_{\mathcal{F}})_! : \mathsf{Mack}(\mathcal{O}_{\mathcal{F}}) \rightleftarrows \mathsf{Mack}(\mathcal{O}) : (j_{\mathcal{F}})^*$$

$$(j_{\mathcal{F}})^* : \mathsf{Mack}(\mathcal{O}) \rightleftarrows \mathsf{Mack}(\mathcal{O}_{\mathcal{F}}) : (j_{\mathcal{F}})_*$$

given by left Kan extension, restriction, and right Kan extension, respectively.

(b) If $\widetilde{\mathcal{F}} \subseteq \mathsf{P}_{\mathcal{O}}$ is *downward closed*, then let $\psi_{\widetilde{\mathcal{F}}} : \mathcal{O}_{\widetilde{\mathcal{F}}} \hookrightarrow \mathcal{O}$ denote the inclusion of the evident full subcategory. The associated embedding $\psi_{\widetilde{\mathcal{F}}}^{\amalg}$ admits a right adjoint $i_{\widetilde{\mathcal{F}}}$. We then get the following adjoint pairs:

$$(i_{\widetilde{\mathcal{F}}})^* : \mathsf{Mack}(\mathcal{O}) \rightleftarrows \mathsf{Mack}(\mathcal{O}_{\widetilde{\mathcal{F}}}) : (i_{\widetilde{\mathcal{F}}})_*$$

$$(i_{\widetilde{\mathcal{F}}})_* : \mathsf{Mack}(\mathcal{O}_{\widetilde{\mathcal{F}}}) \rightleftarrows \mathsf{Mack}(\mathcal{O}) : (i_{\widetilde{\mathcal{F}}})^!$$

We note that, perhaps confusingly, $(i_{\widetilde{\mathcal{F}}})^*$ is given by left Kan extension.

If $\mathcal{F}$ is the upward closed complement of $\widetilde{\mathcal{F}}$ we will sometimes abuse notation and denote by $\Phi^{\mathcal{F}} X$ *either* $(i_{\widetilde{\mathcal{F}}})^*$ or $(i_{\widetilde{\mathcal{F}}})_* i_{\widetilde{\mathcal{F}}}^*$ when we believe there is no chance of confusion.

When $\widetilde{\mathcal{F}} = (-\infty, T]$ is the set of all $p \le [T]$ for some $T \in \mathcal{O}$, then we denote $i_{\widetilde{\mathcal{F}}}$ by i_T. We will sometimes denote the value $(i_T)^* X(T)$ by $X^{\Phi T} \in \mathcal{C}$.[2]

(c) In the event that $\psi_{\widetilde{\mathcal{F}}}^{\amalg}$ preserves pullbacks, we get even more:

$$(\psi_{\widetilde{\mathcal{F}}})_! : \mathsf{Mack}(\mathcal{O}_{\widetilde{\mathcal{F}}}) \rightleftarrows \mathsf{Mack}(\mathcal{O}) : (\psi_{\widetilde{\mathcal{F}}})^*$$

$$(\psi_{\widetilde{\mathcal{F}}})^* : \mathsf{Mack}(\mathcal{O}) \rightleftarrows \mathsf{Mack}(\mathcal{O}_{\widetilde{\mathcal{F}}}) : (\psi_{\widetilde{\mathcal{F}}})_*$$

(d) Given an object $T \in \mathsf{Fin}_{\mathcal{O}} \subseteq \mathsf{Psh}(\mathcal{O})$, form the category $\mathcal{O}_{/T}$ of pairs (x, f) where $x \in \mathcal{O}$ and $f \in T(x)$. This is also an inductive orbital category, and the map $\mathcal{O}_{/T} \to \mathcal{O}$ induces a restriction map $\mathrm{res}_T : \mathsf{Mack}(\mathcal{O}) \to \mathsf{Mack}(\mathcal{O}_{/T})$. The restriction admits both a left and right adjoint (given by left and right Kan extension respectively).

$$\mathrm{ind}_T : \mathsf{Mack}(\mathcal{O}_{/T}) \rightleftarrows \mathsf{Mack}(\mathcal{O}) : \mathrm{res}_T$$

$$\mathrm{res}_T : \mathsf{Mack}(\mathcal{O}) \rightleftarrows \mathsf{Mack}(\mathcal{O}_{/T}) : \mathrm{coind}_T$$

A key feature of Mackey functors with values in an additive ∞-category (or, more generally, a semiadditive ∞-category) is that the canonical map $\mathrm{ind}_T \to \mathrm{coind}_T$ is an equivalence. We will often abbreviate (co)induction and restriction by $\uparrow_T$ and $\downarrow_T$, possibly decorated further when there is ambiguity.

Remark 1.34. It will be very useful in inductive arguments to note that the poset $\mathsf{P}_{\mathcal{O}_{/T}} \subseteq \mathsf{P}_{\mathcal{O}}$ is strictly smaller unless T contains a representative of each minimal object as a retract. If $T = \coprod_i T_i$ for $T_i \in \mathcal{O}$, then $\mathsf{P}_{\mathcal{O}_{/T}} = \cup_i \{p \le T_i\}$. Beware, however, that $\mathcal{O}_{/T}$ is *not* the same as $\mathcal{O}_{\cup_i \{p \le T_i\}}$, using the notation in (b) above. The latter is the essential image of the former under the projection $\mathcal{O}_{/T} \to \mathcal{O}$, but the projection is not full in general.

Remark 1.35. The condition that $\psi_{\widetilde{\mathcal{F}}}^{\amalg}$ preserve pullbacks is satisfied in the following important cases:

(i) whenever $\widetilde{\mathcal{F}}$ is a set consisting of minimal elements in $\mathsf{P}_{\mathcal{O}}$,

---

[2]This leads to an unfortunate clash with the standard equivariant notation, but we don't know of a way to avoid it.

(ii) when $\mathcal{O} = \mathcal{O}_G$ and $\widetilde{\mathcal{F}} \subseteq \mathsf{P}_{\mathcal{O}_G}$ is an arbitrary downwardly closed subset.

We will use (i) frequently.

It is not true in general that $\psi_{\widetilde{\mathcal{F}}}^{\amalg}$ preserves pullbacks. For example, this fails in the case $\mathcal{O}_{\widetilde{\mathcal{F}}} = \mathsf{Surj}_{\leq n} \subseteq \mathsf{Surj}_{\leq n+1}$ when $n > 1$.

Definition 1.36. With notation as above, we will refer to the essential image of $(j_{\mathcal{F}})_!$ as the **subcategory of $\mathcal{F}$-nilpotent** objects, the essential image of $(j_{\mathcal{F}})_*$ as the **subcategory of $\mathcal{F}$-complete** objects, and the essential image of $(i_{\widetilde{\mathcal{F}}})_*$ as the **subcategory of $\widetilde{\mathcal{F}}$-geometric** objects. We denote these in the following way:

$$\mathsf{Mack}(\mathcal{O}; \mathcal{C})^{\mathcal{F}-\mathrm{nil}} = \text{subcategory of } \mathcal{F}\text{-nilpotent objects},$$

$$\mathsf{Mack}(\mathcal{O}; \mathcal{C})^{\mathcal{F}-\mathrm{cpl}} = \text{subcategory of } \mathcal{F}\text{-complete objects},$$

$$\mathsf{Mack}(\mathcal{O}; \mathcal{C})_{\Phi\widetilde{\mathcal{F}}} = \text{subcategory of } \widetilde{\mathcal{F}}\text{-geometric objects}.$$

The following is straightfoward from the definitions and the non-trivial [Gla15, 2.27].

Lemma 1.37. *Let $I \subseteq \mathcal{O}$ be an interval, and write it as $I = \mathcal{F} \cap \widetilde{\mathcal{F}}$ where $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are the smallest upward closed and downward closed intervals, respectively, containing I. In the stratification of $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$ determined by Theorem 1.31,*

$$\mathfrak{S}(I) = \mathsf{Mack}(\mathcal{O}; \mathcal{C})^{\mathcal{F}-\mathrm{cpl}} \cap \mathsf{Mack}(\mathcal{O}; \mathcal{C})_{\Phi\widetilde{\mathcal{F}}}.$$

When $I = \{T\}$ for $T \in \mathcal{O}$ we can identify this intersection with $\mathrm{Fun}(\mathrm{Aut}(T), \mathcal{C})$.

To make the notation more memorable, we instantiate each symbol in the example that the reader likely cares about.

Example 1.38 (Equivariant spectra). Let G be a finite group. Note that $\mathsf{P}_{\mathcal{O}_G} = \mathsf{Sub}_G^{op}$ is the poset of conjugacy classes of subgroups of G ordered by reverse inclusion.

- An upward closed subset of Sub_G^{op} is just a *family of subgroups* in the sense of, e.g., tom Dieck [Die79, 7.2]. So there is a universal G-space $\mathrm{E}\mathcal{F}$ for the family, characterized by the property that

$$(\mathrm{E}\mathcal{F})^H \cong \begin{cases} * & H \in \mathcal{F} \\ \varnothing & H \notin \mathcal{F} \end{cases}.$$

  We then have identifications:

$$(j_{\mathcal{F}})_! = \mathrm{E}\mathcal{F}_+ \wedge (-),$$

$$(j_{\mathcal{F}})_* = F(\mathrm{E}\mathcal{F}_+, -).$$

- A downward closed subset of Sub_G^{op} is the complement of a family $\mathcal{F}$ of subgroups. We can then form the G-space $\widetilde{\mathrm{E}\mathcal{F}}$ as the cofiber:

$$\mathrm{E}\mathcal{F}_+ \to S^0 \to \widetilde{\mathrm{E}\mathcal{F}}.$$

  The various functors in 1.33(b) are given classically by:

$$(i_{\widetilde{\mathcal{F}}})_* = \widetilde{\mathrm{E}\mathcal{F}} \wedge (-)$$

$$(i_{\widetilde{\mathcal{F}}})^* = \left(\widetilde{\mathrm{E}\mathcal{F}} \wedge (-)\right)^{\mathcal{F}} = \Phi^{\mathcal{F}}(-)$$

$$(i_{\widetilde{\mathcal{F}}})^! = F(\widetilde{\mathrm{E}\mathcal{F}}, -)^{\mathcal{F}}$$

$$(\psi_{\widetilde{\mathcal{F}}})^* = (-)^{\mathcal{F}}$$

Where $(-)^{\mathcal{F}}$ is the Lewis-May categorical fixed point functor [Lew+86, I.3]. The functors

$$(\psi_{\widetilde{\mathcal{F}}})_*, \ (\psi_{\widetilde{\mathcal{F}}})_!$$

give two different ways of taking an object with some amount of symmetries, and adding more. The latter is probably more familiar, and corresponds, in the case when $\mathcal{F}$ is the set of subgroups subconjugate to H, to the process of taking a $(N_G H/H)$-spectrum, regarding it as an $N_G H$ spectrum, and then inducing up to G.

- An object $T \in \mathsf{Fin}_{\mathcal{O}_G}$ is just a finite G-set, so we will feel no guilt denoting the category instead by Fin_G from now on. If $T = G/H$, then $\mathcal{O}_{/T}$ is equivalent to the orbit category $\mathcal{O}_H$. (Co)induction and restriction are as you'd expect. The asserted equivalence between in induction and coinduction is a special case of the Wirthmüller isomorphism.

We end this section by recording some properties and relations between these functors for later use.

Lemma 1.39. *Fix $T \in \mathsf{Fin}_{\mathcal{O}}$ and let $\widetilde{\mathcal{F}} \subseteq \mathsf{P}_{\mathcal{O}}$ be a downward closed family. Let $\widetilde{\mathcal{F}}_T = \widetilde{\mathcal{F}} \cap \mathsf{P}_{\mathcal{O}_{/T}}$. Then there are essentially canonical commutative diagrams:*

$$
\begin{array}{ccc}
\mathsf{Mack}(\mathcal{O}_{/T}) & \xrightarrow{\ \mathrm{ind}_T\ } & \mathsf{Mack}(\mathcal{O}) \\
{\scriptstyle (i_{\widetilde{\mathcal{F}}_T})^*}\Big\downarrow & & \Big\downarrow{\scriptstyle (i_{\widetilde{\mathcal{F}}})^*} \\
\mathsf{Mack}(\mathcal{O}_{\widetilde{\mathcal{F}}_T}) & \xrightarrow[\ \mathrm{ind}_T\ ]{} & \mathsf{Mack}(\mathcal{O}_{\widetilde{\mathcal{F}}})
\end{array}
$$

$$
\begin{array}{ccc}
\mathsf{Mack}(\mathcal{O}) & \xrightarrow{\ \mathrm{res}_T\ } & \mathsf{Mack}(\mathcal{O}_{/T}) \\
{\scriptstyle (i_{\widetilde{\mathcal{F}}})^*}\Big\downarrow & & \Big\downarrow{\scriptstyle (i_{\widetilde{\mathcal{F}}_T})^*} \\
\mathsf{Mack}(\mathcal{O}_{\widetilde{\mathcal{F}}}) & \xrightarrow[\ \mathrm{res}_T\ ]{} & \mathsf{Mack}(\mathcal{O}_{\widetilde{\mathcal{F}}_T})
\end{array}
$$

In particular, if $[T] \notin \widetilde{\mathcal{F}}$, then $(i_{\widetilde{\mathcal{F}}})^*\mathrm{ind}_T = 0$.

The next proposition is a generalization of the fact that, in equivariant homotopy theory, $E\mathcal{F}_+ \wedge X$ is built out of inductions of restrictions of X to subgroups in the family $\mathcal{F}$.

Proposition 1.40. *Let $\mathcal{F} \subseteq \mathsf{P}_{\mathcal{O}}$ be upward closed. Let $T = \coprod_{S \in \mathcal{O}_{\mathcal{F}}} S \in \mathsf{Fin}_{\mathcal{O}_{\mathcal{F}}} \subseteq \mathsf{Fin}_{\mathcal{O}}$. Then there is a functor*

$$\mathbf{L}j_! : \mathsf{Mack}(\mathcal{O}_{\mathcal{F}}) \to \mathsf{Fun}(\Delta^{op}, \mathsf{Mack}(\mathcal{O}))$$

with the following properties:

(i) for each $n \geq 0$, we have $(\mathbf{L}j_!)_n \cong (\mathrm{ind}_T \circ \mathrm{res}_T)^{\circ n+1}$,

(ii) there is a natural equivalence of functors

$$\operatorname*{colim}_{\Delta^{op}} \mathbf{L}j_! \cong j_!.$$

Proof. We note that $(\mathcal{O}_{\mathcal{F}})_{/T} = \mathcal{O}_{/T}$ since there are no maps from smaller objects to larger objects. This means that the target of the restriction functors associated to T agree.

To avoid ambiguity, we will temporarily denote by ind'_T the induction functor with target $\mathsf{Mack}(\mathcal{O}_{\mathcal{F}})$ and ind_T the induction functor with target $\mathsf{Mack}(\mathcal{O})$. The endofunctor

$$\mathrm{ind}'_T \circ \mathrm{res}_T : \mathsf{Mack}(\mathcal{O}_{\mathcal{F}}) \to \mathsf{Mack}(\mathcal{O}_{\mathcal{F}})$$

admits a canonical structure of a monad, and so we can form the bar construction $\mathsf{Bar} : \mathsf{Mack}(\mathcal{O}_{\mathcal{F}}) \to \mathsf{Fun}(\Delta^{op}, \mathsf{Mack}(\mathcal{O}_{\mathcal{F}}))$ [Lur16, 4.4.2.7]. We define $\mathbf{L}j_!$ as $j_! \circ \mathsf{Bar}$. To verify (i), note that there is an equivalence $j_! \circ \mathrm{ind}'_T \cong \mathrm{ind}_T$ since a composite of left Kan extensions is a left Kan extension of the composite. To verify (ii) it suffices, since $j_!$ preserves colimits, to check that $\mathrm{colim}_{\Delta^{op}} \mathsf{Bar} \cong \mathrm{id}$. This follows from the fact that the adjunction $\mathrm{ind}'_T \dashv \mathrm{res}_T$ is monadic for our choice of T. Indeed, res_T preserves all colimits, so we need only check that $\mathrm{res}_T : \mathsf{Mack}(\mathcal{O}_{\mathcal{F}}) \to \mathsf{Mack}(\mathcal{O}_{/T})$ is conservative. But equivalences of Mackey functors are detected objectwise, and every object in $\mathcal{O}_{\mathcal{F}}$ is accounted for in T. $\qquad\square$

1.3 Slice filtrations and basic properties

In this section we develop a generalization of the slice filtration suitable for stratified homotopy theories.

Definition 1.41. A **filtration** of a stable ∞-category $\mathcal{C}$ is a sequence of full subcategories

$$\cdots \mathcal{C}_{\geq n} \subseteq \mathcal{C}_{\geq n-1} \subseteq \cdots \subseteq \mathcal{C}$$

such that each $\mathcal{C}_{\geq n}$ is coreflective in $\mathcal{C}$ and closed under extensions. We say the filtration is **separated** if $\bigcap \mathcal{C}_{\geq n}$ is trivial. We say that the filtration is **compatible with suspension** if $\Sigma\mathcal{C}_{\geq n} \subseteq \mathcal{C}_{\geq n+1}$. Objects $X \in \mathcal{C}_{\geq n}$ will be called n-**connective** and we'll indicate this property by writing $X \geq n$. We will call a filtration **presentable** if each of the $\mathcal{C}_{\geq n}$ and $\mathcal{C}$ are presentable.

If $\mathcal{C}$ and $\mathcal{C}'$ are equipped with filtrations and $F : \mathcal{C} \to \mathcal{C}'$ is a functor we will say that F is **filtration preserving** if $F(\mathcal{C}_{\geq n}) \subseteq \mathcal{C}'_{\geq n}$.

Example 1.42. If $\mathcal{C}$ has a t-structure then the sequence of subcategories $\{\tau_{\geq n}\mathcal{C}\}$ is a filtration on $\mathcal{C}$ compatible with suspensions. If $\mathcal{C}$ admits countable products and $\tau_{\geq 0}\mathcal{C}$ is stable under these, then separability of the filtration is equivalent to left completeness of the t-structure [Lur16, 1.2.1.19]. If $\mathcal{C}$ is presentable, then the filtration is presentable if and only if the t-structure is accessible in the sense of [Lur16, 1.4.4.12].

Not every filtration arises from a t-structure, but every presentable filtration gives rise to a *sequence* of t-structures.

Definition 1.43. Let $\{\mathcal{C}_{\geq n}\}$ be a presentable filtration on a presentable, stable ∞-category $\mathcal{C}$. Then each subcategory $\mathcal{C}_{\geq n}$ determines an accessible t-structure with $\mathcal{C}_{\geq n}$ as the subcategory of 0-connective objects for that t-structure. The heart is a Grothendieck abelian category [Lur16, 1.3.5.23] which we denote by $\mathcal{C}^{\heartsuit_n}$. If $\mathcal{C}$ is understood, we will abbreviate this to $\heartsuit_n$. We denote the truncation functors associated to the nth t-structure by $\tau^{(n)}_{\leq k}$ and $\tau^{(n)}_{\geq k}$ for $k \in \mathbb{Z}$.

We offer the following generalization of a perversity suited to our examples.

Definition 1.44. Let P be a poset. A **dimension function** for P is a function:

$$\nu : \mathbb{Z} \times \mathsf{P} \longrightarrow \mathbb{Z}$$

such that for any $p \in \mathsf{P}$, $\nu(-, p) : \mathbb{Z} \to \mathbb{Z}$ is weakly increasing and surjective. We say that $p \in \mathsf{P}$ is an n-**jump** if $\nu(n+1, p) > \nu(n, p)$, otherwise we say that p is an n-**rest**. We say that ν **jumps at** n if every $p \in \mathsf{P}$ is an n-jump.

Remark 1.45. Barwick-Dotto-Glasman-Nardin-Shah study the almost identical notion of a *generalized perversity* in their forthcoming work.

Definition 1.46. Suppose given a presentable, stable, P-stratified ∞-category $\mathcal{C}$, a dimension function ν for P, and separated, presentable filtrations on the strata $\mathcal{C}_p$ for each $p \in \mathsf{P}$, compatible with suspension. Denote by L_p the localization $L_p : \mathcal{C} \to \mathcal{C}_p$. Then define the ν-**slice filtration** on $\mathcal{C}$ by declaring $X \geq n$ if and only if $L_p X \geq \nu(n, p)$ for all $p \in \mathsf{P}$. Attached to this filtration we will use the following terminology:

- We will say X is **slice n-connective** and write $X \geq n$ if $L_p X \geq \nu(n, p)$ for all $p \in \mathsf{P}$. The full subcategory of slice n-connective objects is denoted ${}^\nu \mathcal{C}_{\geq n}$ or just $\mathcal{C}_{\geq n}$ if ν is understood.

- We denote the right adjoint to the inclusion $\mathcal{C}_{\geq n} \subseteq \mathcal{C}$ by P_n and call $P_n X$ the **slice n-connective cover of X**.

- We will say Y is **slice n-truncated** and write $Y \leq n$ if, for every $X \geq n + 1$, the mapping space $\mathrm{map}_\mathcal{C}(X, Y)$ is contractible. We denote the full subcategory of slice n-truncated objects by ${}^\nu \mathcal{C}_{\leq n}$ or $\mathcal{C}_{\leq n}$ if ν is understood.

- We denote the cofiber of $P_{n+1} \to \mathrm{id}$ by P^n and call $P^n X$ the n**th slice section of X** or the **slice n-stage of X**.

- We will say A is an n**-slice** if $A \leq n$ and $A \geq n$. We denote the full subcategory of n-slices by Slice_n, and will further decorate this symbol if either ν or $\mathcal{C}$ is unclear from the context.

- There is a canonical equivalence $P^n P_n \cong P_n P^n$ and we denote either of these functors by P^n_n. We call $P^n_n X$ the n**-slice of X**.

We record a few basic consequences of the definition before turning to examples.

Lemma 1.47. *The ν-slice filtration is indeed a filtration. As such, it is separated and compatible with suspension.*

Proof. Each $\mathcal{C}_{\geq n}$ is closed under colimits, extensions, and equivalence since L_p preserves colimits. Since we've assumed $\mathcal{C}$ is presentable, this provides the right adjoint. That the filtration is separated follows from the joint conservativity of the functors L_p together with the fact that ν is weakly increasing and surjective. Finally, in order for ν to be both weakly increasing and surjective, we must have $\nu(n + 1, p) \leq \nu(n, p) + 1$. This, together with compatibility with suspension on each stratum, completes the proof. $\square$

Lemma 1.48. *If ν jumps at n, then $\Sigma \mathcal{C}_{\geq n} = \mathcal{C}_{\geq n+1}$.*

Lemma 1.49. *The functor P^n is a left adjoint to the inclusion $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$.*

Proof. This is classical from the theory of Bousfield localizations, but we recall the proof here since it displays where we use the presentability hypotheses. The subcategory $\mathcal{C}_{\leq n}$ is evidently closed under limits. From our presentability conditions, we also see that $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ is accessible. It follows that the inclusion admits a left adjoint $\tilde{P}^n$ [Lur09, 5.5.2.9], and we need to identify it with P^n. To that end, define $\tilde{P}_{n+1}$ as the fiber of the unit $\mathrm{id} \to \tilde{P}^n$. For any $X \in \mathcal{C}$ we have a fiber sequence

$$\tilde{P}_{n+1} X \to X \to \tilde{P}^n X.$$

So for any $W \geq n + 1$, we get a fibration:

$$\mathrm{map}(W, \tilde{P}_{n+1} X) \to \mathrm{map}(W, X) \to \mathrm{map}(W, \tilde{P}^n X).$$

The last term vanishes by definition of slice coconnectivity, so the first map is an equivalence. It follows that $\tilde{P}_{n+1} X \geq n + 1$ and that $\tilde{P}_{n+1}$ is a right adjoint to the inclusion $\mathcal{C}_{\geq n+1} \subseteq \mathcal{C}$. The result follows. $\square$

Proposition 1.50. *For each n, Slice_n is a reflective subcategory of $\heartsuit_n$. In particular, Slice_n is an ordinary, presentable, additive category. If ν jumps at n, then $\mathsf{Slice}_n = \heartsuit_n$ and is thus Grothendieck abelian.*

Proof. The inclusion $\Sigma \mathcal{C}_{\geq n} \subseteq \mathcal{C}_{\geq n+1}$ yields an inclusion

$$\mathsf{Slice}_n \subseteq \heartsuit_n,$$

and P^n provides the desired left adjoint. The last claim is immediate from the definition of jump. $\square$

Example 1.51 (Perverse t-structures). If X is a space equipped with a finite stratification $X = \coprod_{s \in S} X_s$, then the ∞-category of S-constructible sheaves (valued in the derived category of abelian groups, say) $\mathsf{D}_{S-cstr}(X)$ is an S-stratified ∞-category. Here we consider S as a poset by declaring $s \le s'$ if the closure of $X_{s'}$ contains X_s. If $p : S \to \mathbb{Z}$ is a function (the *perversity*), then we can define a perverse t-structure on $\mathsf{D}_{S-cstr}(X)$ as the slice filtration associated to the dimension function $\nu(n, s) = n + p(s)$. This filtration is a t-structure because ν jumps at every $n \in \mathbb{Z}$.

The homotopy theory of Mackey functors with values in $\mathcal{C}$ has the feature that all of its strata are presheaves valued in $\mathcal{C}$. Thus, a filtration on $\mathcal{C}$ determines a filtration on all the strata in a canonical way. This provides us with the most important class of examples for our work.

Example 1.52. Let $\mathcal{O}$ be an inductive orbital category and $\mathcal{C}$ a presentable, stable ∞-category equipped with a presentable, separated filtration τ, and ν a dimension function on $\mathsf{P}_{\mathcal{O}}$. Then the ν**-slice filtration on** $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$ is defined by

$$X \ge n \iff \text{ for all } T \in \mathcal{O}, \ X^{\Phi T} \in \tau_{\ge \nu(n, T)} \mathcal{C}.$$

Convention 1.53.   • For the remainder of this section, unless otherwise stated, $\mathcal{C}$ will denote a stable, presentable, ∞-category equipped with a presentable, separated filtration.

   • For the remainder of the paper, the homotopy theory Sp will be equipped with its standard t-structure filtration unless otherwise specified. With this convention, there is an unambiguous ν-slice filtration on $\mathsf{Sp}^{\mathcal{O}}$ for any inductive orbital category $\mathcal{O}$ and dimension function ν.

Definition 1.54. If $\mathcal{O}$ is an inductive orbital category, then an $\mathcal{O}$**-parameterized filtration** on $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$ is the data of a filtration, for every $T \in \mathsf{Fin}_{\mathcal{O}}$, on $\mathsf{Mack}(\mathcal{O}_{/T}; \mathcal{C})$ with the following properties:

   • induction and restriction preserve filtration,

   • if $T = \coprod_i T_i$, then the filtration on $\mathsf{Mack}(\mathcal{O}_{/T}; \mathcal{C})$ is identified with the product filtration under the canonical equivalence
   $$\mathsf{Mack}(\mathcal{O}_{/T}; \mathcal{C}) \xrightarrow{\cong} \prod_i \mathsf{Mack}(\mathcal{O}_{T_i}; \mathcal{C}).$$

Remark 1.55. An $\mathcal{O}$-parameterized filtration determines and is determined by a family of filtrations, one on $\mathsf{Mack}(\mathcal{O}_{/T}; \mathcal{C})$ for each $T \in \mathcal{O}$, for which induction and restriction between orbits preserve filtration.

The following is immediate from the definitions, and Lemma 1.39.

Proposition 1.56. *Let $\mathcal{O}$ be an inductive orbital category, and suppose we have a dimension function ν on $\mathsf{P}_{\mathcal{O}}$. For any $T \in \mathsf{Fin}_{\mathcal{O}}$, denote by ν_T the restriction of ν to $\mathsf{P}_{\mathcal{O}_{/T}}$. Then:*

   (i) The family of ν_T-slice filtrations is an $\mathcal{O}$-parameterized filtration on $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$.

   (ii) Restriction and induction preserve slice coconnectivity and take n-slices to n-slices.

   (iii) Restriction and induction preserve coconnectivty for the t-structures associated to the slice filtration, and take elements of $\heartsuit_n$ to elements of $\heartsuit_n$.

Now we prove a recognition theorem that allows one to identify a given parameterized filtration with the ν-slice filtration.

Theorem 1.57. *Suppose $\{F_{\ge n}\}$ is a parameterized filtration of $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$. Then it agrees with the ν-slice filtration on $\mathsf{Mack}(\mathcal{O}; \mathcal{C})$ if and only if the following two conditions are satisfied:*

   (i) for all n, $F_{\ge n} \subseteq {}^{\nu}\mathsf{Mack}(\mathcal{O}; \mathcal{C})_{\ge n}$,

(ii) for every $T \in \mathcal{O}$, the class of objects $\{X^{\Phi T} \in \mathcal{C}_{\geq \nu(n,T)} | X \in F_{\geq n}\}$ generates $\mathcal{C}_{\geq \nu(n,T)}$ under colimits, extensions, and equivalences.

Proof. Using the functors $(i_T)_*$, it is straightforward to see that (ii) is satisfied for the ν-slice filtration, and (i) is tautological. So we prove the other direction.

We proceed by induction on the size of $\mathsf{P}_\mathcal{O}$. So let $T \in \mathsf{P}_\mathcal{O}$ be a minimal element, with associated functors (i^*, i_*), and let $(j_!, j^*)$ be the functors associated to the upward closed complement of T.

Since, by assumption (i), $F_{\geq n} \subseteq {}^\nu\mathsf{Mack}(\mathcal{O}; \mathcal{C})_{\geq n}$, we need to show that any $X \geq n$ belongs to $F_{\geq n}$. To that end, consider the cofiber sequence

$$j_! j^* X \to X \to i_* i^* X.$$

It suffices to show that both $j_! j^* X$ and $i_* i^* X$ belong to $F_{\geq n}$. That $j_! j^* X$ lies in $F_{\geq n}$ follows from the fact that $\{F_{\geq n}\}$ is a parameterized filtration, together with the induction hypothesis and Proposition 1.40.

To complete the proof, it suffices, by the definition of the slice filtration for Mackey functors, to show that if $Y \in \mathsf{Mack}(\mathcal{O}_{\{T\}}; \mathcal{C})_{\geq n}$, then $i_* Y \in F_{\geq n}$.

To that end, recall that, since $\mathcal{O}_{\{T\}}$ consists of a single object and every endomorphism in an inductive orbital category is an isomorphism, $\mathsf{Mack}(\mathcal{O}_{\{T\}}; \mathcal{C}) \cong \mathsf{Fun}(\mathrm{Aut}(T), \mathcal{C})$. Under this equivalence,

$$\mathsf{Mack}(\mathcal{O}_{\{T\}}; \mathcal{C})_{\geq n} \cong \mathsf{Fun}(\mathrm{Aut}(T), \mathcal{C}_{\geq \nu(n,T)})$$

Let $e : * \to \mathrm{Aut}(T)$ denote the inclusion of the identity. Then the subcategory $\mathsf{Fun}(\mathrm{Aut}(T), \mathcal{C}_{\geq \nu(n,T)})$ is generated under extensions, equivalence, and colimits by the essential image of the left Kan extension

$$e_! : \mathcal{C}_{\geq \nu(n,T)} = \mathsf{Fun}(*, \mathcal{C}_{\geq \nu(n,T)}) \to \mathsf{Fun}(\mathrm{Aut}(T), \mathcal{C}_{\geq \nu(n,T)}),$$

also known as $\mathrm{Aut}(T)_+ \wedge (-)$. By our assumption (ii), this subcategory is also generated by elements of the form $e_! X^{\Phi T}$ for $X \in F_{\geq n}$.

Let A denote the class of $Y \in \mathsf{Mack}(\mathcal{O}_{\{T\}}; \mathcal{C})$ such that $i_* Y \in F_{\geq n}$. Then A is closed under extensions, equivalence, and colimits. So it suffices by our assumption (ii) to show that A contains $e_! Z^{\Phi T}$ for every $Z \in F_{\geq n}$. Write Z as an extension

$$j_! j^* Z \to Z \to i_* i^* Z.$$

By the induction hypothesis and the same argument as before, $j_! j^* Z \in F_{\geq n}$, and we have assumed $Z \in F_{\geq n}$. Therefore $i^* i_* Z \in F_{\geq n}$, and hence $Z \in A$, which completes the proof. $\square$

Corollary 1.58 (Hill-Yarnall). • *The original slice filtration on Sp^G agrees with the filtration associated to the dimension function*

$$\nu_{sl}(n, H) = \left\lfloor \frac{n}{|H|} \right\rfloor.$$

• *The regular slice filtration on Sp^G agrees with the filtration associated to the dimension function*

$$\nu_{reg}(n, H) = \left\lceil \frac{n}{|H|} \right\rceil.$$

Proof. We give the proof for the original slice filtration, the proof in the regular case is much the same. Let $F_{\geq n}\mathsf{Sp}^G$ denote the subcategory of spectra which are slice $(n-1)$-positive in the sense of [HHR16, 4.8]. It is elementary to check that this filtration is compatible with restrictions and induction [HHR16, 4.13], so this defines a parameterized filtration on Sp^G. To verify condition (i) it suffices to show that

$$(G_+ \wedge_K S^{m\rho_K - \epsilon})^{\Phi H} \geq \left\lfloor \frac{n}{|H|} \right\rfloor$$

whenever $\epsilon = 0, 1$ and $m|K| - \epsilon \geq n$, and this computation is routine using the double-coset formula.

To verify (ii), let $m = \lfloor n/|H| \rfloor$ and notice that $G_+ \wedge_H S^{(m+1)\rho_H - 1}$ is in $F_{\geq n}\mathsf{Sp}^G$ and has H-geometric fixed points a finite wedge of copies of S^m. The result follows since S^m generates $\mathsf{Sp}_{\geq m}$ under colimits, extensions, and equivalences. $\square$

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We close with an analog of [HY17, 2.10] which, in some cases, allows us to use the values of the Mackey functor instead of its geometric fixed points.

Lemma 1.59. *Let ν be a dimension function and fix $n \in \mathbb{Z}$. Suppose that $\nu(n, -) : \mathsf{P}_{\mathcal{O}} \to \mathbb{Z}$ is order-preserving. Then*

$$X \geq n \iff \text{ for all } T \in \mathcal{O}, \ X(T) \in \mathcal{C}_{\geq \nu(n,T)}.$$

Proof. Let $T \in \mathcal{O}$ be minimal and consider the usual inductive set-up:

$$j_! j^* X \to X \to i_* i^* X.$$

Ordinary connectivity behaves well under induction and restriction, so the inductive hypothesis tells us that either assumption on X leads to a $j^* X$ which is slice n-connective and satisfies $X(S) = j^* X(S) \in \mathcal{C}_{\geq \nu(n,S)}$ for all $S \neq T$. In particular, each $j^* X(S)$ is in $\mathcal{C}_{\geq \nu(n,T)}$ by our assumption on $\nu(n, -)$. It follows from (1.40) that $j_! j^* X$ is both slice n-connective and satisfies $j_! j^* X(T) \in \mathcal{C}_{\geq \nu(n,T)}$. But now both conditions are closed under extensions and cofibers, and $X^{\Phi T} = i_* i^* X(T)$, so the proof is complete. $\square$

2 Categories of slices

We saw in the last section that every slice filtration gives rise to a sequence of Grothendieck abelian categories $\heartsuit_n$ and distinguished reflective subcategories $\mathsf{Slice}_n \subseteq \heartsuit_n$. Every Grothendieck abelian category $\mathcal{A}$ is a left exact localization of a category of right modules over a ring (Gabriel-Popescu [PG64]). The ring in question is the ring of endomorphisms of a chosen generator. Often it is convenient to use a family of generators instead of a single one, and there is a mild generalization of the theorem in this case.

Theorem 2.1 (Gabriel-Popescu [PG64], Kuhn [Kuh94]). *Let $\mathcal{A}$ be a Grothendieck abelian category and $\mathcal{A}_0 \subseteq \mathcal{A}$ an essentially small subcategory closed under finite direct sums. Suppose that, for every object $a \in \mathcal{A}$, there is a set of objects $\{x_\alpha\}$, with each $x_\alpha \subseteq \mathcal{A}_0$, and an epimorphism $\bigoplus_\alpha x_\alpha \to a$. Then:*

(i) The restricted Yoneda embedding to the category of finite product preserving functors

$$G : \mathcal{A} \longrightarrow \mathsf{Psh}^{\times}(\mathcal{A}_0, \mathsf{Set})$$

is fully faithful.

(ii) The functor G admits an exact left adjoint, F.

The following corollary was actually known before the above theorem, and can be found in Freyd's thesis [Fre60] and a paper of Gabriel [Gab62], at least in the case of a single generator.

Corollary 2.2 (Freyd-Gabriel). *In the situation of the above theorem, if every object of $\mathcal{A}_0$ is compact and projective in $\mathcal{A}$, then G is an equivalence of categories.*

Thus, in order to understand $\heartsuit_n$ and its localization Slice_n, we should search for generators. In the case of the standard t-structure on Sp we know that S^n is sufficient, and π_n identifies $\mathsf{Sp}_{\geq n} \cap \mathsf{Sp}_{\leq n}$ with the category of abelian groups in this way. In fact, we could have used a finite wedge of copies of S^n just as well.

From the definition of the slice filtration, we are lead to consider spectra which are finite wedges of spheres on each stratum. Since we hope to apply the corollary above, we restrict attention to *compact* spectra with this property, and thus arrive at the notion of a slice sphere.

In §2.1-2.2 we define slice n-spheres and study the elementary properties of the functors they corepresent (called slice homotopy). Perhaps the only parts that require a bit of care are Proposition 2.12 (showing that slice n-spheres exist in sufficient supply) and Proposition 2.28 (which identifies the slice homotopy group of $P^n X$ in terms of the slice homotopy group of X.) In §2.3 we check the hypotheses of the result of Freyd-Gabriel and conclude that $\heartsuit_n$ is the category of models for an explicit Lawvere theory, and Slice_n is a specified localization thereof.

This bridge to algebra made, we reformulate this Lawvere theory in terms of modules over a Green functor in §2.4. The technique is to use a parameterized version of an argument going back to Gabriel and Freyd, which may be of independent interest.

Finally, we digress in §2.5 to analyze the structure of modules over Green functors with an additional condition that allows us to remove redundant data in describing a module. We show in §2.6 that this condition is satisfied in the case of interest, and this leads to our final description of $\heartsuit_n$ and Slice_n as categories of *twisted Mackey functors*.

2.1 Slice spheres

Definition 2.3. Let ν be a dimension function for $\mathsf{P}_{\mathcal{O}}$. We say that an object $W \in \mathsf{Sp}^{\mathcal{O}}$ is a **slice sphere** if

(i) W is compact,

(ii) for each $T \in \mathcal{O}$, the spectrum $W^{\Phi T}$ is equivalent to a finite wedge of spheres.

We say that a slice sphere W is **homogeneous** if the spheres appearing in the decomposition of $W^{\Phi T}$ are all of the same dimension. We say that an object $W \in \mathsf{Sp}^{\mathcal{O}}$ is a **slice n-sphere** if it is a slice sphere satisfying the further requirement that the spheres in condition (ii) have dimension $\nu(n, T)$. We denote the *homotopy category* of slice n-spheres by Sph_n.

Finally, we define an **isotropic slice n-sphere** to be a slice n-sphere $W \in \mathsf{Sp}^{\mathcal{O}}$ such that, for each $T \in \mathcal{O}$, $W^{\Phi T}$ is *nonzero*.

Warning 2.4. The property of being a slice n-sphere depends on the dimension function ν.

Warning 2.5. A slice sphere need not be a slice n-sphere for any n.

Example 2.6. Any representation sphere is a slice sphere in Sp^G.

Example 2.7. For any group G and subgroup $H \subseteq G$, the cofiber of the map $\nabla : G/H_+ \to S^0$ is a slice sphere. When H is trivial, $\mathrm{cof}(\nabla)$ is an example of a slice 1-sphere for the original slice filtration and a slice 2-sphere for the regular slice filtration. When $|G| > 2$, this cofiber is not equivalent to a wedge of representation spheres.

Remark 2.8. The desire to stick to *compact* objects is motivated both by computations and to have a good algebraic theory. This is also the reason why it is difficult to develop the theory for an arbitrary stratified, stable ∞-category. The trouble is visible already in the case of a recollement. The pushforward functor i_* has no reason to preserve compact objects, and in fact does not in our main example of interest. The pushforward $j_!$ does preserve compact objects, but the ones created this way necessarily vanish on the closed locus. A related disappointment is that one cannot test compactness on strata.

In the case of spectral Mackey functors, however, we have an alternative way to produce examples: using the functors of the form $(\psi_{\widetilde{\mathcal{F}}})_!$. For formal reasons, these functors *do* preserve compact objects, and can be used, in particular, to build a sufficient number of slice spheres.

Remark 2.9. Since slice n-spheres are evidently slice n-connective, the natural map

$$[W, W'] \longrightarrow [\tau_{\leq 0}^{(n)} W, \tau_{\leq 0}^{(n)} W']$$

is an isomorphism. Thus, we may identify Sph_n with a full subcategory of $\heartsuit_n$.

Definition 2.10. A **testing subcategory for n-slices** is a full subcategory $\mathsf{Test}_n \subseteq \mathsf{Sph}_n$ such that, for any $T \in \mathsf{Fin}_{\mathcal{O}}$:

- $\mathrm{ind}_T \mathrm{res}_T(\mathsf{Test}_n) \subseteq \mathsf{Test}_n$, and

- if $T \in \mathcal{O}$, then there exists some $W \in \mathsf{Test}_n$ with $W^{\Phi T}$ equivalent to a *nonzero*, finite wedge of copies of $S^{\nu(n,T)}$.

One way of producing a testing subcategory is to generate one from an isotropic slice n-sphere, which is immediate from the definitions.

Lemma 2.11. *If W is an isotropic n-sphere, let $\mathsf{Test}\langle W \rangle$ be the smallest subcategory of Sph_n containing W and closed under $\mathrm{ind}_T\mathrm{res}_T$, for all $T \in \mathsf{Fin}_{\mathcal{O}}$. Then $\mathsf{Test}\langle W \rangle$ is a testing subcategory.*

At the moment, it is not at all clear that testing subcategories even exist. Luckily, the definition is not vacuous.

Proposition 2.12. Sph_n *is itself a testing subcategory for n-slices. More precisely, there exists an isotropic n-sphere in Sph_n.*

Proof. We begin with a few reductions. First, by (2.11), it suffices to construct a single isotropic n-sphere. Second, suppose $\{T_1, ..., T_k\}$ is a set containing a representative for each minimal element in $\mathsf{P}_{\mathcal{O}}$. If $W_i \in \mathsf{Sp}^{\mathcal{O}/T_i}$ is an isotropic n-sphere, for each $1 \leq i \leq k$, then $\bigvee_i \mathrm{ind}_{T_i} W_i$ is an isotropic n-sphere in $\mathsf{Sp}^{\mathcal{O}}$. So we may assume, without loss of generality, that $\mathcal{O}$ has a terminal object. We may also assume, by induction, that the result holds for posets smaller than $|\mathsf{P}_{\mathcal{O}}|$. Now we proceed.

Let $T \in \mathcal{O}$ be terminal. Recall that we have a functor

$$\left(\psi_{\{T\}}\right)_! : \mathsf{Mack}(\mathcal{O}_{\{T\}}) \cong \mathsf{Sp} \to \mathsf{Mack}(\mathcal{O}) = \mathsf{Sp}^{\mathcal{O}}$$

given by left Kan extension. Define $X_0 := \left(\psi_{\{T\}}\right)_! S^{\nu(n,T)}$. Then X_0 is a slice sphere with the property that $(X_0)^{\Phi T'} = S^{\nu(n,T)}$ for every $T' \in \mathcal{O}$. Our goal is to modify X_0 until it becomes an isotropic slice n-sphere.

To that end, choose a conservative, surjective map of posets $\mathsf{P}_{\mathcal{O}} \to [m]$ for some $m \geq 0$, and an ordering on each fiber. Then list the elements of $\mathsf{P}_{\mathcal{O}}$ in dictionary order: $\mathsf{P}_{\mathcal{O}} = \{p_0, p_1, ..., p_N\}$, where $N + 1 = |\mathsf{P}_{\mathcal{O}}|$. We'll choose the ordering on the fiber over 0 so that $p_0 = T$.

We propose to inductively build X_k with the property that X_k is a homogeneous slice sphere and, for $i \leq k$, $(X_k)^{\Phi p_i}$ is a nonzero wedge of spheres of the form $S^{\nu(n,p_i)}$. So suppose we have such an X_k for $k < N$, then we describe how to build X_{k+1}. Since X_k is a homogeneous slice sphere, we know that $(X_k)^{\Phi p_{k+1}}$ is a (possibly trivial) finite wedge of spheres all of the same dimension, say $(X_k)^{\Phi p_{k+1}} \cong \bigvee S^m$. To finish the proof, we need only treat each of the following three cases.

- Case 1: $(X_k)^{\Phi p_{k+1}} = 0$. In this case, let $A \in \mathsf{Sp}^{\mathcal{O}/p_{k+1}}$ be an isotropic n sphere for the restricted slice filtration, which exists by the induction, and take $X_{k+1} = X_k \vee \mathrm{ind}_{p_{k+1}} A$.

- Case 2: $(X_k)^{\Phi p_{k+1}} \neq 0$ and $m \leq \nu(n, p_{k+1})$. In this case, let $r = \nu(n, p_{k+1}) - m \geq 0$ and $X_k := Y_{k,0}$. Given $Y_{k,i}$ for $i < r$, define $Y_{k,i+1}$ as the cofiber of the counit:

$$\mathrm{ind}_{p_{k+1}}\mathrm{res}_{p_{k+1}} Y_{k,i} \to Y_{k,i}.$$

  This construction does not affect geometric fixed points at p_j for $j \leq k$, by the construction of our ordering, and it modifies all other geometric fixed points by replacing the previous wedge of spheres with a new wedge of spheres of one higher dimension (or doing nothing if the geometric fixed points were trivial.) Indeed, on geometric fixed points, the unit of induction-restriction admits a section, and a summand of a wedge of spheres is still a wedge of spheres. Thus, $X_{k+1} := Y_{k,r}$ does the trick.

- Case 3: $(X_k)^{\Phi p_{k+1}} \neq 0$ and $m > \nu(n, p_{k+1})$. This is exactly as before, except we define $Y_{k,i+1}$ as the *fiber* of the unit

$$Y_{k,i} \to \mathrm{coind}_{p_{k+1}}\mathrm{res}_{p_{k+1}} Y_{k,i}.$$

  Again, we check that this is still a wedge of spheres of appropriate dimension using the fact that the geometric fixed points of the unit for the coinduction-restriction adjunction admits a retraction.

$\square$

Remark 2.13. Isotropic slice spheres with fewer cells may be constructed by modifying the above construction to be more efficient. Specifically, one may replace the use of $\mathrm{ind}_T\mathrm{res}_T X \to X$ with any map $\mathrm{ind}_T Y \to X$ which becomes split upon restriction to T. We suspect that all isotropic slice spheres arise from this more general procedure, but have not tried to prove it.

For later use, we record an evident but useful observation.

Lemma 2.14. *Fix $T \in \mathsf{Fin}_{\mathcal{O}}$ and a testing subcategory $\mathsf{Test}_n(\mathcal{O}) \subseteq \mathsf{Sph}_n(\mathcal{O})$. Then $\mathrm{res}_T(\mathsf{Test}_n(\mathcal{O}))$ is a testing subcategory for n-slices in $\mathsf{Sp}^{\mathcal{O}/T}$.*

2.2 Slice homotopy groups

Having defined the appropriate notion of spheres in our context, we now develop the resulting analogue of homotopy groups and their basic properties.

Definition 2.15. Fix a testing subcategory $\mathsf{Test}_n \subseteq \mathsf{Sph}_n$. Then the restricted Yoneda embedding defines a functor:

$$\hat{\pi}_n : \mathsf{Sp}^{\mathcal{O}} \longrightarrow \mathsf{Psh}^{\times}_{\mathsf{Set}}(\mathsf{Test}_n).$$

to the category of product-preserving presheaves on Test_n. We will call the target of $\hat{\pi}_n$ the category of n-**models** and denote it by Model_n. We call $\hat{\pi}_n X$ the n-**th slice homotopy group** of X.

Remark 2.16. Since Test_n is additive, there is a canonical equivalence

$$\mathsf{Model}_n \cong \mathsf{Psh}^{\oplus}_{\mathsf{Ab}}(\mathsf{Test}_n)$$

with the category of additive presheaves. We will often move back and forth between the equivalent interpretations of Model_n for convenience. For example, with this definition, it is clear that Model_n is abelian.

The name 'slice homotopy group' is slightly abusive since $\hat{\pi}_n X$ is really a diagram of groups. We don't think this will cause confusion.

Warning 2.17. Both the category Model_n and the functor $\hat{\pi}_n$ depend on the dimension function ν and the choice of testing subcategory.

The following lemma is evident and will often be used without comment.

Lemma 2.18. *The functor $\hat{\pi}_n$ is homological, i.e. $\hat{\pi}_n$ sends cofiber sequences to exact sequences.*

Warning 2.19. Despite the notation, it is very much *not* the case that $\hat{\pi}_n(\Sigma X)$ is the same as $\hat{\pi}_{n-1}X$ in general.

Proposition 2.20. *For any testing subcategory $\mathsf{Test}_n \subseteq \mathsf{Sph}_n$, the functor*

$$\hat{\pi}_n : \heartsuit_n \to \mathsf{Model}_n$$

admits a left adjoint, denoted $H : \mathsf{Model}_n \to \heartsuit_n$. It is determined by the property that, if $W \in \mathsf{Test}_n$, then $H[-, W] = \tau^{(n)}_{\leq 0} W$. Composing with P^n yields a functor

$$P^n H : \mathsf{Model}_n \longrightarrow \mathsf{Slice}_n$$

left adjoint to the restriction $\hat{\pi}_n|_{\mathsf{Slice}_n}$.

Proof. The inclusion $\heartsuit_n \subseteq \mathsf{Sp}^{\mathcal{O}}_{\geq n}$ preserves limits (since it admits $\tau^{(n)}_{\leq 0}$ as a left adjoint), and so does the Yoneda embedding and restriction, whence $\hat{\pi}_n$ preserves limits. Since each object of Test_n is compact, by definition, the functor $\hat{\pi}_n$ also preserves filtered colimits. The result now follows from the adjoint functor theorem, since the source and target are presentable categories. The formula for H follows from checking that $\tau^{(n)}_{\leq 0} W$ corepresents the expected functor on $\heartsuit_n$. $\square$

Warning 2.21. The functor $\hat{\pi}_n$ is generally not right exact as a functor with domain Slice_n.

We will find much use out of the following elementary vanishing conditions for slice homotopy groups.

Proposition 2.22. *(a) If $X \geq n$, then, for any slice n-sphere W, $[W, \Sigma X] = 0$.*

(b) If $T \in \mathcal{O}$ is minimal and an n-jump, and $X \in \mathsf{Sp}_{\geq \nu(n+1,T)}$, then $[W, i_* X] = 0$ for any slice n-sphere W.

Proof. First we prove (a). By induction on the size of $\mathsf{P}_{\mathcal{O}}$, and standard arguments, we're reduced to checking that $\hat{\pi}_n i_* i^* \Sigma X = 0$ where (i^*, i_*) is the adjoint pair associated to a minimal element $T \in \mathcal{O}$. But $\hat{\pi}_n (i_* i^* \Sigma X)(W) = [W^{\Phi T}, (\Sigma X)^{\Phi T}]$ by adjunction. Since W is an n-slice sphere, $W^{\Phi T}$ is either 0 or a wedge of copies of $S^{\nu(n,T)}$. On the other hand, $\Sigma X^{\Phi T}$ is $(\nu(n,T)+1)$-connective since $X \geq n$, so in either case we get zero. The proof of (b) is similar and easier. $\square$

Proposition 2.23. *If $Y \leq n$, let $T^{\mathrm{jump}} \in \mathsf{Fin}_{\mathcal{O}}$ be the coproduct over all the n-jumps for ν, in $\mathcal{O}$. Then, for any $T \in \mathsf{Fin}_{\mathcal{O}}$ with T^{jump} as a summand, the map*

$$\hat{\pi}_n A(W) \to \hat{\pi}_n A(\mathrm{ind}_T \mathrm{res}_T W)$$

induced by the counit $\mathrm{ind}_T \mathrm{res}_T W \to W$ is injective.

Proof. The assumptions precisely imply that the cofiber of the counit map is slice $(n+1)$-connective. $\square$

Definition 2.24. Let $\mathsf{Slice}_n^{alg} \subseteq \mathsf{Model}_n$ denote the full subcategory of functors which satisfy the conclusion of (2.23). We temporarily call this the **category of algebraic n-slices**.

Warning 2.25. The category Slice_n^{alg} depends on the choice of testing subcategory. We justify this abuse of notation by Theorem 2.29, below, which implies that changing the testing subcategory yields an equivalent category of algebraic n-slices.

From Proposition 2.23 we get:

Corollary 2.26. *The functor $\hat{\pi}_n : \mathsf{Slice}_n \to \mathsf{Model}_n$ factors through Slice_n^{alg}.*

Given an additive presheaf on Test_n, it is easy to change it into an algebraic n-slice.

Lemma 2.27. *The inclusion $\mathsf{Slice}_n^{alg} \subseteq \mathsf{Model}_n$ admits a left adjoint L^{inj} described explicitly as*

$$L^{inj} \pi(W) = \frac{\pi(W)}{\ker : \pi(W) \to \pi(\mathrm{ind}_T \mathrm{res}_T W)}$$

where $T = T^{\mathrm{jump}}$ as in (2.23).

Next, we give some preliminary evidence for the strong relationship between n-slices and algebraic n-slices.

Proposition 2.28. *For $C \in \mathsf{Sp}^{\mathcal{O}}$, the localization map $C \to P^n C$ induces an isomorphism*

$$L^{inj} \hat{\pi}_n C \xrightarrow{\cong} \hat{\pi}_n P^n C.$$

Proof. By Proposition 2.23, $\hat{\pi}_n P^n C$ belongs to Slice_n^{alg}. By Lemma 2.27, we get a commutative diagram:

$$
\begin{array}{ccc}
 & \hat{\pi}_n C & \\
 {\scriptstyle g} \swarrow & & \searrow {\scriptstyle f} \\
L^{inj} \hat{\pi}_n C & \xrightarrow{\quad h \quad} & \hat{\pi}_n P^n C
\end{array}
$$

Since g is surjective, we can prove that h is an isomorphism by showing that f is surjective and $\ker(f) \subseteq \ker(g)$ (the other inclusion is implied by commutativity of the diagram.)

The obstruction to the surjectivity of f lives in $\hat{\pi}_n \Sigma P_{n+1} C$, but this group vanishes by Proposition 2.22(a), so f is surjective.

Now suppose $W \in \mathsf{Test}_n$ and $W \to C \to P^n C$ is null. Then we have a factorization $W \to P_{n+1} C \to C \to P^n C$. Let $T = T^{\mathrm{jump}}$. Then it suffices to show that the composite

$$\mathrm{ind}_T \mathrm{res}_T W \to W \to P_{n+1} C$$

is null. Equivalently, that $\mathrm{res}_T W \to \mathrm{res}_T P_{n+1} C$ is null. But, by the definition of T^{jump} and the slice filtration, together with the fact that restrictions preserve slice connective covers, we conclude that $\mathrm{res}_T P_{n+1} C \in \Sigma \mathsf{Sp}^{\mathcal{O}/T}_{\geq n}$. Since $\mathrm{res}_T W$ is a slice n-sphere, we conclude the vanishing by Proposition 2.22(a), which completes the proof. $\qquad\square$

2.3 Slices as models for a Lawvere theory

We are now ready for our first algebraic description of slices.

Theorem 2.29. *The functors $\hat{\pi}_n$ and H yield an equivalence of adjoint pairs:*

To prove this, we will need the following classical bit of category theory.

Proposition 2.30 (Freyd, Gabriel). *Let $\mathcal{A}$ be a Grothendieck abelian category and suppose $\mathcal{A}_0 \subseteq \mathcal{A}$ is a full subcategory closed under finite direct sums and satisfying the following properties:*

(i) *Every object of $\mathcal{A}_0$ is compact.*

(ii) *Every object of $\mathcal{A}_0$ is projective.*

(iii) *The restricted Yoneda embedding*

$$G : \mathcal{A} \longrightarrow \mathsf{Psh}^{\times}_{\mathsf{Set}}(\mathcal{A}_0)$$

is conservative.

Then G is an equivalence of categories.

Proof. By (i) and (ii), G preserves all colimits. Since the target of G is generated by representables, G is essentially surjective. On the other hand, we have a functor $L : \mathsf{Psh}^{\times}_{\mathsf{Set}}(\mathcal{A}_0) \longrightarrow \mathcal{A}$ induced from the inclusion $\mathcal{A}_0 \subseteq \mathcal{A}$ by left Kan extension. There is a natural map $LG \to \mathrm{id}$ and we will be done if we can check it is an isomorphism. But G is conservative by assumption, so we need only check that the map

$$GLG \to G$$

is an isomorphism. Now, GL is a colimit preserving endofunctor of $\mathsf{Psh}^{\times}_{\mathsf{Set}}(\mathcal{A}_0)$ which is the identity on $\mathcal{A}_0$, so it is canonically equivalent to the identity. This completes the proof. $\qquad\square$

Remark 2.31. Suppose $\mathcal{A}$ is any category with equalizers, coproducts, and with the property that monic-epimorphisms are isomorphisms. If $\mathcal{A}_0 \subseteq \mathcal{A}$ is a full subcategory, then the following are equivalent:

(i) Every object $x \in \mathcal{A}$ admits an epimorphism $\coprod_\beta a_\beta \to x$ where $a_\beta \in \mathcal{A}_0$.

(ii) The restricted Yoneda embedding $\mathcal{A} \longrightarrow \mathsf{Psh}_{\mathsf{Set}}(\mathcal{A}_0)$ is faithful.

(iii) The restricted Yoneda embedding $\mathcal{A} \longrightarrow \mathsf{Psh}_{\mathsf{Set}}(\mathcal{A}_0)$ is conservative.

In the literature, people use the phrase '$\mathcal{A}_0$ is a family of generators for $\mathcal{A}$' inconsistently to mean any of these, even in situations where they are *not* all equivalent. (Actually, the first two conditions are always equivalent. It is the equivalence with the third condition which is transient.)

Proof of Theorem 2.29. Recall that we have already shown (Proposition 2.28) that $\hat{\pi}_n P^n C = L^{inj}\hat{\pi}_n C$ for any C. Thus, if the top two arrows in the diagram are inverse equivalences, then the two localizations are necessarily equivalent. Now, by Remark 2.9, we may view Test_n as a full subcategory of $\heartsuit_n$, and each slice n-sphere is compact. We now argue that they are also projective. Indeed, suppose that $f : A \to B$ is a map between elements of $\heartsuit_n$. Then form the cofiber sequence in $\mathsf{Sp}^\mathcal{O}$:

$$A \to B \to C.$$

The cokernel of f in $\heartsuit_n$ is given by $\tau_{\leq 0}^{(n)} C$. Suppose f is surjective so that $\tau_{\leq 0}^{(n)} C = 0$. Then we have $C \in \tau_{\geq 1}^{(n)}\mathsf{Sp}^\mathcal{O}$ so that C is the suspension of something slice n-connective. It follows from Proposition 2.22 that $\hat{\pi}_n C = 0$ and by Lemma 2.18 that $\hat{\pi}_n A \to \hat{\pi}_n B$ is surjective. So we've shown that for all $W \in \mathsf{Test}_n$, that $[W, -]$ preserves surjections, and hence each W is projective.

By the previous proposition, it now suffices to check that $\hat{\pi}_n$ is conservative on $\heartsuit_n$. This we prove below. $\square$

Proposition 2.32. *The functor* $\hat{\pi}_n : \heartsuit_n \to \mathsf{Model}_n$ *is conservative.*

Warning 2.33. The statement is obviously false on the larger domain $\mathsf{Sp}^\mathcal{O}$.

Proof. We proceed by induction on the order of $\mathsf{P}_\mathcal{O}$. So let $T \in \mathcal{O}$ be a minimal element with upward closed complement $\mathcal{F}$, and let (i^*, i_*) and $(j_!, j^*)$ be the usual adjoint pairs associated to this situation.

Assume that $f : A \to B$ is a map of objects in $\heartsuit_n$ which induces an isomorphism on $\hat{\pi}_n$ and form the diagram:

$$
\begin{array}{ccccc}
j_! j^* A & \longrightarrow & A & \longrightarrow & i_* i^* A \\
\downarrow & & \downarrow & & \downarrow \\
j_! j^* B & \longrightarrow & B & \longrightarrow & i_* i^* B
\end{array}
$$

It suffices to prove that the left and right vertical maps are equivalences.

- Our induction hypothesis applies to $\mathcal{O}_{/T'}$ and the restricted testing subcategory (2.14) for any $T' \in \mathcal{F}$ since, for such T', $\mathsf{P}_{\mathcal{O}_{/T'}}$ is strictly smaller than $\mathsf{P}_\mathcal{O}$ (1.34). This, together with Proposition 1.40, implies that $j_! j^*(f)$ is an equivalence.

- By the definition of testing subcategory, there is some $W \in \mathsf{Test}_n$ with $W^{\Phi T}$ a wedge of copies of $S^{\nu(n,T)}$. We have a map of fiber sequences:

$$
\begin{array}{ccccc}
\mathrm{map}(W, A) & \longrightarrow & \mathrm{map}(W, i_* i^* A) & \longrightarrow & \mathrm{map}(W, \Sigma j_! j^* A) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{map}(W, B) & \longrightarrow & \mathrm{map}(W, i_* i^* B) & \longrightarrow & \mathrm{map}(W, \Sigma j_! j^* B)
\end{array}
$$

The last vertical map is an equivalence because we have already shown $j_! j^*$ is an equivalence. We know that $[\Sigma^k W, A] = [\Sigma^k W, B] = 0$ for $k > 0$ because $A, B \in \tau_{\leq 0}^{(n)} \mathsf{Sp}^{\mathcal{O}}$, and $\Sigma^k W \in \tau_{\geq k}^{(n)} \mathsf{Sp}^{\mathcal{O}}$ as W is slice n-connective. We also know $[W, A] = \hat{\pi}_n A(W) \to \hat{\pi}_n B(\overline{W}) = [W, B]$ is an isomorphism by assumption. Thus, the first vertical map is an equivalence. We deduce that the middle vertical map is an equivalence.

- Finally, let $e : * \to \operatorname{Aut}(T)$ be the inclusion of the identity. Since $S^{\nu(n,T)}$ is a retract of $W^{\Phi T}$, we conclude that $e_! S^{\nu(n,T)}$ is a retract of $i^* W$. (Here it is important that we are working stably so that the norm $e_! \to e_*$ is an equivalence.) Equivalences are closed under retracts, so we conclude from the previous bullet and adjunction that

$$\operatorname{map}(S^{\nu(n,T)}, A^{\Phi T}) \to \operatorname{map}(S^{\nu(n,T)}, B^{\Phi T})$$

is a weak equivalence. But $A^{\Phi T}$ and $B^{\Phi T}$ are $\nu(n, T)$-connective by the definition of the slice filtration and our assumption that $A, B \geq n$, so we deduce that $i^* f$ is an equivalence and the proof is complete.

$\square$

Example 2.34. Let $\mathsf{A}(G)$ denote the full subcategory of Sp^G containing S^0 and closed under smashing with finite G-sets. It is a consequence of the tom Dieck splitting that $\mathrm{h}\mathsf{A}(G)$ is equivalent to the classical Burnside category [Lew+86, V.9.6]. So if one finds a representation sphere S^V which is also an isotropic slice n-sphere, then we get an equivalence of categories:

$$\mathrm{h}\mathsf{A}(G) \xrightarrow{\;S^V \wedge (-)\;} \mathsf{Test}\langle S^V \rangle \;,$$

and hence an equivalence:

$$\mathsf{Mack}(G; \mathsf{Ab}) \xrightarrow{\;\cong\;} \mathsf{Model}_n \cong \heartsuit_n.$$

Unwinding the definitions we learn that in this case the n-slice of a spectrum is determined by a quotient of its Vth homotopy Mackey functor.

This works more generally for any isotropic slice n-sphere in the Picard group, and more generally still for any inductive orbital category in place of $\mathcal{O}_G$.

2.4 Slices as modules over a Green functor

In the case when Test_n is generated by an isotropic slice n-sphere W, the functor $\hat{\pi}_n X$ records the following data:

- For each $T \in \mathsf{Fin}_{\mathcal{O}}$, the homotopy groups $[\downarrow_T W, \downarrow_T X] = [\uparrow_T \downarrow_T W, X]$.

- For each map $\uparrow_T \downarrow_T W \to \uparrow_{T'} \downarrow_{T'} W$, the induced map

$$[\uparrow_{T'} \downarrow_{T'} W, X] \to [\uparrow_T \downarrow_T W, X].$$

In this section we identify the above data with the $\underline{\operatorname{End}}(W)$-module structure on the homotopy Mackey functor $\underline{[W, X]}$. The notations and notions in this theorem will be defined in the body of the section, but we state it now in any case:

Theorem 2.35. *The functor* $\underline{[W, -]}$ *yields an equivalence of adjoint pairs:*

$$\begin{array}{ccc} \heartsuit_n & \xrightarrow[\cong]{[W,-]} & \mathsf{RMod}_{\underline{\mathrm{End}}(W)} \\ \uparrow \downarrow P^n & & \uparrow \downarrow L^{inj} \\ \mathsf{Slice}_n & \xrightarrow[{[W,-]}]{\cong} & \mathsf{RMod}^{\mathrm{loc}}_{\underline{\mathrm{End}}(W)} \end{array}$$

This identification is straightforward, but is conceptually pleasing and serves as a stepping stone to our simplification at the end of §2. The reader is encouraged to use this section as a quick reminder of the definitions of Green functors and modules over them, and then proceed to §2.5.

We begin by reviewing the symmetric monoidal structure on $\mathsf{Mack}(\mathcal{O}; \mathsf{Ab})$. Classically, one begins with a symmetric monoidal structure on $\mathrm{hA}^{eff}(G)$ (or $\mathrm{hA}(G)$) which arises from the product of finite G-sets. Unfortunately, the categories $\mathsf{Fin}_\mathcal{O}$ need not admit products in general. For example, the categories $\mathsf{Fin}_{\mathcal{O}_\mathcal{F}}$ associated to a family of subgroups of a group G usually do not have terminal objects.

Remark 2.36. The author actually does not know of an example of an inductive orbital category where $\mathsf{Fin}_\mathcal{O}$ does not admit *nonempty* finite products. If no such example exists, the discussion of the symmetric monoidal structure on $\mathsf{Mack}(\mathcal{O}, \mathsf{Ab})$ below could be simplified somewhat.

Nevertheless, the presheaf that a product would represent can always be defined, and this puts a *pro*monoidal structure on $\mathrm{hA}^{eff}(\mathcal{O})$ which we now describe.

Definition 2.37. Given $U, V \in \mathrm{hA}^{eff}(\mathcal{O})$, define a presheaf of sets $(U \times V) : \mathrm{hA}^{eff}(\mathcal{O})^{op} \longrightarrow \mathsf{Set}$ by

$$(U \times V)(T) := \{\text{triples } (S \to T, S \to U, S \to V)\}/\sim$$

Here the maps are in $\mathsf{Fin}_\mathcal{O}$ and two triples are equivalent if there is an isomorphism $S \xrightarrow{\cong} S'$ commuting with all the specified maps. Functoriality comes from pullback and composition. This construction produces a functor

$$\times : \mathrm{hA}^{eff}(\mathcal{O}) \times \mathrm{hA}^{eff}(\mathcal{O}) \longrightarrow \mathsf{Psh}_\mathsf{Set}(\mathrm{hA}^{eff}(\mathcal{O}))$$

(i.e. a profunctor $\mathrm{hA}^{eff}(\mathcal{O})^{\times 2} \to \mathrm{hA}^{eff}(\mathcal{O})$.)

Definition 2.38. The **box product** of abelian group valued Mackey functors $\underline{M}$ and $\underline{N}$ on $\mathcal{O}$ is defined by left Kan extension and restriction via the diagram:

$$\begin{array}{ccccc} \mathrm{hA}^{eff}(\mathcal{O}) \times \mathrm{hA}^{eff}(\mathcal{O}) & \xrightarrow{(\underline{M},\underline{N})} & \mathsf{Ab} \times \mathsf{Ab} & \xrightarrow{\otimes} & \mathsf{Ab} \\ \downarrow \times & & & & \\ \mathsf{Psh}_\mathsf{Set}(\mathrm{hA}^{eff}(\mathcal{O})) & & \xrightarrow{\times_!} & & \\ \uparrow & & & & \\ \mathrm{hA}^{eff}(\mathcal{O}) & & \xrightarrow{\underline{M}\square\underline{N}} & & \end{array}$$

This gives $\mathsf{Mack}(\mathcal{O}; \mathsf{Ab})$ the structure of a symmetric monoidal category. For a reference in much greater generality than we need, see [BGS15].

Definition 2.39. The **Burnside Mackey functor**, $\underline{A}$, is the unit for the symmetric monoidal structure on $\mathsf{Mack}(\mathcal{O}; \mathsf{Ab})$ defined above. Explicitly, it is given by

$$T \mapsto \text{Grothendieck group of the maximal subgroupoid of } (\mathsf{Fin}_\mathcal{O})_{/T} \text{ with respect to } \amalg$$

Functoriality is given by pullback and composition.

Definition 2.40. A **Green functor** is an associative algebra in $\mathsf{Mack}(\mathcal{O}; \mathsf{Ab})$ with respect to the box product. Given a Green functor $\underline{R}$ we let $\mathsf{RMod}_{\underline{R}}$ denote the **category of right modules** over the associative algebra $\underline{R}$.

Example 2.41. For any $X, Y \in \mathsf{Sp}^{\mathcal{O}}$ the assignment

$$\mathsf{Fin}_{\mathcal{O}} \ni T \mapsto [\downarrow_T X, \downarrow_T Y]$$

extends to an abelian group valued Mackey functor on $\mathcal{O}$. In the case $X = Y$, composition endows this Mackey functor with the structure of a Green functor, the **endomorphism Green functor**, denoted $\underline{\mathrm{End}}(X)$. For any Y, the Mackey functor $[X, Y]$ is a right $\underline{\mathrm{End}}(X)$-module in a canonical way.

Next we'll need the condition that corresponds to the localization P^n.

Definition 2.42. Let W be an isotropic slice n-sphere and let $T^{\mathrm{jump}} \in \mathsf{Fin}_{\mathcal{O}}$ denote the coproduct of all (isomorphism classes) of n-jumps. We say that a right $\underline{\mathrm{End}}(W)$-module $\underline{M}$ is **slice local** if, for all $T \in \mathsf{Fin}_{\mathcal{O}}$, the map

$$\underline{M}(T) \longrightarrow \underline{M}(T^{\mathrm{jump}} \times T)$$

induced by the projection $T^{\mathrm{jump}} \times T \to T$ is injective.

With these definitions in hand, we can prove the theorem.

Proof of Theorem 2.35. Given $T \in \mathsf{Fin}_{\mathcal{O}}$ denote by $\underline{\mathrm{End}}(W)_T$ the Mackey functor

$$\mathsf{Fin}_{\mathcal{O}} \ni U \mapsto [\uparrow_U \downarrow_U W, \uparrow_T \downarrow_T W].$$

Note that evaluation at T gives an isomorphism:

$$\mathrm{Hom}_{\underline{\mathrm{End}}(W)}(\underline{\mathrm{End}}(W)_T, \underline{M}) \xrightarrow{\cong} \underline{M}(T). \quad (*)$$

So each module $\underline{\mathrm{End}}(W)_T$ is compact and projective, and together they detect isomorphisms. Let $\mathcal{A}_0$ denote the full subcategory of $\mathsf{RMod}_{\underline{\mathrm{End}}(W)}$ spanned by the objects $\underline{\mathrm{End}}(W)_T$. Then $\mathcal{A}_0$ satisfies the hypotheses of Proposition 2.30. Thus, the restricted Yoneda embedding

$$\mathsf{RMod}_{\underline{\mathrm{End}}(W)} \longrightarrow \mathsf{Psh}^{\times}_{\mathsf{Set}}(\mathcal{A}_0)$$

is an equivalence of categories. On the other hand, we have a functor

$$\mathsf{Test}\langle W \rangle \longrightarrow \mathcal{A}_0$$

given by $\uparrow_T \downarrow_T W \mapsto \underline{\mathrm{End}}(W)_T$. This is an equivalence in view of the formula $(*)$ above together with the induction-restriction adjunction.

So we have a commutative diagram

$$
\begin{array}{ccc}
 & \heartsuit_n & \\
{\scriptstyle [W,-]} \swarrow & & \searrow {\scriptstyle \hat{\pi}_n} \\
\mathsf{RMod}_{\underline{\mathrm{End}}(W)} \underset{\simeq}{\longrightarrow} & \mathsf{Psh}^{\times}_{\mathsf{Set}}(\mathcal{A}_0) \underset{\simeq}{\longrightarrow} & \mathsf{Model}_n
\end{array}
$$

By Theorem 2.29 we know that $\hat{\pi}_n$ is an equivalence, hence so is $[W, -]$. The identification of Slice_n as $\mathsf{RMod}^{\mathrm{loc}}_{\underline{\mathrm{End}}(W)}$ is immediate from the definitions and the corresponding statement in Theorem 2.29. $\qquad\square$

Remark 2.43. In this extended remark we place the elementary result in this section into the broader setting of parameterized category theory. We will assume the reader is familiar with the notions of parameterized category theory found in [Bar+16; Nar16]. The reader unfamiliar or uninterested in these ideas can safely skip this remark.

First notice that, given a Green functor $\underline{R}$, we can define an $\mathcal{O}$-category $\underline{\mathsf{RMod}}_{\underline{R}}$ by the assignment

$$\mathcal{O} \ni T \mapsto \mathsf{RMod}_{\mathrm{res}_T \underline{R}}.$$

This $\mathcal{O}$-category is $\mathcal{O}$-semiadditive in the sense of [Nar16, 5.3]. A priori, homomorphisms in an $\mathcal{O}$-category form an object in $\underline{\mathsf{Set}}_{\mathcal{O}}$, but $\mathcal{O}$-semiadditivity canonically promotes these to elements in $\underline{\mathsf{Mack}}(\mathcal{O}; \mathsf{CMon})$, i.e. Mackey functors valued in commutative monoids. In fact, each of the commutative monoids in question are group-like, so we'll call such an $\mathcal{O}$-category '$\mathcal{O}$-additive.' Finally, each of the fibers is abelian, and in this case we'll say that the $\mathcal{O}$-category is $\mathcal{O}$-abelian.

Now, in general, given a cocartesian section $C : \mathcal{O}^{op} \to \mathcal{A}$ of an $\mathcal{O}$-abelian category $\mathcal{A}$, we get an $\mathcal{O}$-functor

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(C, -) : \mathcal{A} \longrightarrow \underline{\mathsf{Mack}}(\mathcal{O}; \mathsf{Ab})$$

Using this functor we have natural candidates for the notions of (i) $\mathcal{O}$-compact, (ii) $\mathcal{O}$-projective, and (iii) being an $\mathcal{O}$-generator. The key point is to use $\mathcal{O}$-indexed colimits in place of ordinary colimits for each definition. Now the proof of Proposition 2.30 carries over essentially verbatim to prove:

- Suppose $\mathcal{A}$ is an $\mathcal{O}$-presentable, $\mathcal{O}$-abelian $\mathcal{O}$-category with an $\mathcal{O}$-compact, $\mathcal{O}$-projective, $\mathcal{O}$-generator C. Then the $\mathcal{O}$-functor
  $$\underline{\mathrm{Hom}}(C, -) : \mathcal{A} \longrightarrow \underline{\mathsf{RMod}}_{\mathrm{End}(C)}$$
  is an equivalence of $\mathcal{O}$-categories.

The evident generalization to a family of $\mathcal{O}$-generators also applies, as does the analogue of the Gabriel-Kuhn-Popescu theorem. All of the proofs are straightforward adaptations of the classical ones, once you've pinned down the proper definitions (as above).

2.5 Digression: Modules over geometrically split Green functors

The Green functor $\underline{\mathrm{End}}(W)$ for an isotropic slice n-sphere W has several special features which simplifies its category of modules. We single out one of these in this section and study the resulting bit of algebra.

Definition 2.44. Given an abelian group valued Mackey functor $\underline{M}$ and an orbit $T \in \mathcal{O}$. For maps of orbits $U \to T$ denote the associated transfer by

$$\mathrm{tr}_U^T : \underline{M}(U) \to \underline{M}(T).$$

Then we define an abelian group $\underline{M}^{\Phi T}$ by:

$$\underline{M}^{\Phi T} := \frac{\underline{M}(T)}{\langle \mathrm{im}(\mathrm{tr}_U^T) \rangle_{U \to T}}.$$

Remark 2.45. The action of $\mathrm{Aut}(T)$ on $\underline{M}(T)$ descends to an action on $\underline{M}^{\Phi T}$.

Remark 2.46. If $\underline{R}$ is a Green functor, then $\underline{R}^{\Phi T}$ is a ring because the submodule we quotient by is an idea. Similarly, if $\underline{M}$ is a right $\underline{R}$-module, then $\underline{M}^{\Phi T}$ is naturally a right $\underline{R}^{\Phi T}$-module.

Example 2.47. We will prove below (Lemma 2.68) that if W is an isotropic slice n-sphere, then $\underline{\mathrm{End}}(W)^{\Phi T} = \mathrm{End}(W^{\Phi T})$.

If one unwinds the definition of a module over a Green functor, one finds the formula:

$$m \cdot \mathrm{tr}_U^T(r) = \mathrm{tr}_U^T(\mathrm{res}_U^T(m) \cdot r).$$

Thus, the action of transferred elements is somewhat redundant. Of course, it is not usually possible to systematically express an element $r \in \underline{R}(T)$ as the sum of transferred elements and an element not in the image of any transfer. We will show below (Lemma 2.69) that this *does* happen our case of interest. This leads us to the following definition.

Definition 2.48. A Green functor $\underline{R}$ is called **geometrically splittable** if, for all $T \in \mathcal{O}$, the ring map

$$\underline{R}(T) \longrightarrow \underline{R}^{\Phi T}$$

admits an $\mathrm{Aut}(T)$-equivariant section. A **geometrically split** Green functor is a geometrically splittable Green functor $\underline{R}$ equipped with chosen splittings $s_T : \underline{R}^{\Phi T} \longrightarrow \underline{R}(T)$ as above.

Example 2.49. The Burnside Mackey functor (2.39) $\underline{A}$ is geometrically split. Indeed, $\underline{A}^{\Phi T} = \mathbb{Z}$ with trivial $\mathrm{Aut}(T)$-action, so there is a *unique* splitting of the augmentation $\underline{A}(T) \to \mathbb{Z}$.

If $\underline{M}$ is a module over a geometrically split Green functor, then restriction along s_T gives each $\underline{M}(T)$ the structure of a module over $\underline{R}^{\Phi T}$. Enumerating the remaining structure on $\underline{M}$ visible to the rings $\underline{R}^{\Phi T}$, one is lead to define the category in Definition 2.51 below. We will need a little notation before making this definition, however.

Definition 2.50. If M is an abelian group and S is a set, we denote by $M \otimes S$ and M^S the abelian groups $M \otimes \mathbb{Z}\{S\}$ and $\mathrm{Hom}(\mathbb{Z}\{S\}, M)$, respectively. There is a canonical map $M \otimes S \to M^S$ given by sending $m \otimes s$ to the function with value m at s and zero otherwise; thus we may view elements of $M \otimes S$ as elements in M^S.

If a finite group H acts on M and S, define

$$\mathrm{trace} : M \otimes S \longrightarrow M^S$$

by $m \otimes s \mapsto \sum_{h \in H} hm \otimes hs$. This induces a map by the same name:

$$\mathrm{trace} : (M \otimes S)_H \longrightarrow \left(M^S\right)^H.$$

Definition 2.51. Let $\underline{R}$ be a geometrically split Green functor. We define the category of **(right)** $\underline{R}^{\Phi}$-**modules**, $\mathsf{RMod}_{\underline{R}^{\Phi}}$, to consist of objects $\underline{M}$ which consist of the following data:

(i) For each $[T] \in \mathsf{P}_{\mathcal{O}}$ an $\underline{R}^{\Phi T}$-$\mathrm{Aut}(T)$-module $\underline{M}(T)$;

(ii) (Restrictions and transfers) For each pair $[T] \geq [T']$, maps of $\underline{R}^{\Phi T'}$-$\mathrm{Aut}(T')$-modules:

$$(\underline{M}(T) \otimes \mathrm{Hom}(T, T'))_{\mathrm{Aut}(T)} \longrightarrow \underline{M}(T')$$

$$\underline{M}(T') \longrightarrow \left(\underline{M}(T)^{\mathrm{Hom}(T,T')}\right)^{\mathrm{Aut}(T)}$$

of $\underline{R}^{\Phi T'}$-$\mathrm{Aut}(T')$-modules.

subject to the following conditions:

• (Double-coset formula) For each pair, $[T] \geq [T']$, the diagram

commutes;

- (Composition of restrictions and transfers) For each triple $[T_0] \geq [T_1] \geq [T_2]$, the diagrams

$$
\begin{array}{ccc}
\left[(\underline{M}(T_0) \otimes \mathrm{Hom}(T_0,T_1))_{\mathrm{Aut}(T_0)} \otimes \mathrm{Hom}(T_1,T_2)\right]_{\mathrm{Aut}(T_1)} & \longrightarrow & (\underline{M}(T_1) \otimes \mathrm{Hom}(T_1,T_2))_{\mathrm{Aut}(T_1)} \\
\cong \downarrow & & \downarrow \\
(\underline{M}(T_0) \otimes \mathrm{Hom}(T_0,T_2))_{\mathrm{Aut}(T_0)} & \longrightarrow & \underline{M}(T_2)
\end{array}
$$

$$
\begin{array}{ccc}
\underline{M}(T_2) & \longrightarrow & \left(\underline{M}(T_0)^{\mathrm{Hom}(T_0,T_2)}\right)^{\mathrm{Aut}(T_0)} \\
\downarrow & & \downarrow \\
\left(\underline{M}(T_1)^{\mathrm{Hom}(T_1,T_2)}\right)^{\mathrm{Aut}(T_1)} & \longrightarrow & \left[\left(\left(\underline{M}(T_0)^{\mathrm{Hom}(T_0,T_1)}\right)^{\mathrm{Aut}(T_0)}\right)^{\mathrm{Hom}(T_1,T_2)}\right]^{\mathrm{Aut}(T_1)}
\end{array}
$$

commute.

Remark 2.52. Each object $\underline{M}$ is, in particular, a Mackey functor. The definition above just keeps track of the interaction with the $\underline{R}^{\Phi T}$-module structures.

By design, there is a forgetful functor:

$$
\mathsf{RMod}_{\underline{R}} \longrightarrow \mathsf{RMod}_{\underline{R}^{\Phi}}.
$$

Theorem 2.53. *The forgetful functor*

$$
\mathsf{RMod}_{\underline{R}} \longrightarrow \mathsf{RMod}_{\underline{R}^{\Phi}}
$$

is an equivalence.

As usual, the proof relies on induction over the poset $\mathsf{P}_{\mathcal{O}}$. To set up this induction, we'll need to extend the functors $((j_{\mathcal{F}})_!, j_{\mathcal{F}}^*, (j_{\mathcal{F}})_*)$ and $(i_{\widetilde{\mathcal{F}}}^*, (i_{\widetilde{\mathcal{F}}})_*, (i_{\widetilde{\mathcal{F}}})^!)$ to the setting of modules over Green functors.

We begin by recalling a few definitions.

Definition 2.54. [BR70; Lur09] Suppose we are given a diagram of categories

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\widetilde{G}} & \mathcal{C} \\
\psi^* \downarrow & & \downarrow \phi^* \\
\mathcal{D}' & \xrightarrow{G} & \mathcal{D}
\end{array}
$$

commuting up to a specified natural isomorphism $\eta : \phi^* \circ \widetilde{G} \xrightarrow{\cong} G \circ \psi^*$. We say that the diagram is **left adjointable** or satisfies the **left Beck-Chevalley condition** if G and $\widetilde{G}$ admit left adjoints F and $\widetilde{F}$ and the exchange transformation

$$
F \circ \phi^* \to F \circ \phi^* \circ \widetilde{G} \circ \widetilde{F} \overset{\eta}{\cong} F \circ G \circ \psi^* \circ \widetilde{F} \to \psi^* \circ \widetilde{F}
$$

is an isomorphism. We say the diagram is **right adjointable** if the functors G and $\widetilde{G}$ admit right adjoints H and $\widetilde{H}$ and the exchange transformation

$$
\psi^* \circ \widetilde{H} \to H \circ G \circ \psi^* \circ \widetilde{H} \overset{\eta^{-1}}{\cong} H \circ \phi^* \circ \widetilde{G} \circ \widetilde{H} \to H \circ \phi^*
$$

is an isomorphism.

Remark 2.55. A square is right adjointable if and only if the square becomes left adjointable upon applying the functor $(-)^{op} : \mathsf{Cat} \to \mathsf{Cat}$.

The following lemma is elementary and left to the reader.

Lemma 2.56. *Suppose we have a square of functors:*

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\widetilde{G}} & \mathcal{C} \\
{\scriptstyle \psi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\
\mathcal{D}' & \xrightarrow{G} & \mathcal{D}
\end{array}
$$

commuting up to a specified natural isomorphism. Suppose further that G and $\widetilde{G}$ admit left adjoints F and $\widetilde{F}$, respectively and that ψ^* and ϕ^* admit right adjoints ψ_* and ϕ_*, respectively. Then the above square is left adjointable if and only if the square

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\psi^*} & \mathcal{D}' \\
{\scriptstyle \widetilde{G}}\downarrow & & \downarrow{\scriptstyle G} \\
\mathcal{C} & \xrightarrow{\phi^*} & \mathcal{D}
\end{array}
$$

is right adjointable.

Construction 2.57. Let $\mathcal{F} \subseteq \mathsf{P}_{\mathcal{O}}$ be upward closed with downward closed complement $\widetilde{\mathcal{F}}$ and suppose $\underline{R}$ is a Green functor. Denote by $\underline{R}_{\mathcal{F}}$ the restriction of the Green functor to $\mathcal{O}_{\mathcal{F}}$ and by $\Phi^{\mathcal{F}}\underline{R}$ the Green functor on $\mathcal{O}_{\widetilde{\mathcal{F}}}$ defined by $\left(i_{\widetilde{\mathcal{F}}}\right)^*\underline{R}$ (as in Notation 1.33).

Restriction defines a functor:

$$j^* : \mathsf{RMod}_{\underline{R}} \longrightarrow \mathsf{RMod}_{\underline{R}_{\mathcal{F}}}.$$

Extension by zero defines a functor:

$$i_* : \mathsf{RMod}_{\Phi^{\mathcal{F}}\underline{R}} \longrightarrow \mathsf{RMod}_{\underline{R}}.$$

Repeated use of the adjoint functor theorem produces a string of adjoints:

$$
\mathsf{RMod}_{\underline{R}_{\mathcal{F}}} \underset{\xrightarrow{j_*}}{\overset{\xrightarrow{j_!}}{\underset{\xleftarrow{j^*}}{}}} \mathsf{RMod}_{\underline{R}} \underset{\xrightarrow{i^!}}{\overset{\xrightarrow{i^*}}{\underset{\xleftarrow{i_*}}{}}} \mathsf{RMod}_{\Phi^{\mathcal{F}}\underline{R}}
$$

Any $\underline{R}$-module $\underline{M}$ has an underlying Mackey functor, and a priori it is not clear how the functors above compare to the similarly named functors applied to the underlying Mackey functor. Luckily, the two notions agree.

Proposition 2.58. *The formation of the functors $(j_!, j^*, j_*)$ and $(i^*, i_*, i^!)$ on modules commutes with passage to underlying Mackey functors.*

Proof. The forgetful functor

$$\phi : \mathsf{RMod}_{\underline{R}} \longrightarrow \mathsf{Mack}(\mathcal{O}; \mathsf{Ab})$$

admits a left adjoint, given by $(-)\square \underline{R}$, and a right adjoint, given by $\underline{\mathrm{Hom}}(R, -)$. The formation of j^* commutes with both of these, and the formation of i^* and i_* commutes with the first of these. The result now follows by repeated application of Lemma 2.56. $\square$

Warning 2.59. The category $\mathsf{RMod}_{\underline{R}}$ is *not* a recollement of $\mathsf{RMod}_{\underline{R}_{\mathcal{F}}}$ and $\mathsf{RMod}_{\underline{R}^{\Phi\widetilde{F}}}$ in the sense of Definition 1.1, in general, because i^* fails to be left exact in general. Instead, it is an example of a *recollement of abelian categories* as defined, for example, in [FP04], or in Definition 2.60 below.

We would like to make an inductive argument by showing that, in the geometrically split case, the theorem is true for $\mathcal{O}_{\mathcal{F}}$ and for $\mathcal{O}_{\widetilde{\mathcal{F}}}$ and then conclude the result for $\mathcal{O}$. Unfortunately, recollements of abelian categories are less well-behaved than their ∞-categorical cousins, and it is not true in general that a map of recollements is an equivalence if it is so on each piece of the recollement (see [FP04, 2.2]). However, this equivalence criterion is true under further hypotheses on the recollement.

Definition 2.60. [FP04] Suppose we have a collection of additive functors between abelian categories

$$\mathcal{A}_1 \underset{\underset{j_*}{\overset{j^*}{\longleftarrow}}}{\overset{j_!}{\longrightarrow}} \mathcal{A} \underset{\underset{i^!}{\overset{i_*}{\longleftarrow}}}{\overset{i^*}{\longleftarrow}} \mathcal{A}_0$$

where each functor is left adjoint to the functor below it. We say this presents a **recollement of abelian categories** if the following additional conditions are satisfied:

(i) The functors $j_!, j_*$, and i_* are fully faithful,

(ii) The functor i_* has essential image precisely those objects $a \in \mathcal{A}$ such that $j^* a = 0$.

If moreover each category has enough projectives, we say that the recollement of abelian categories is **pre-hereditary** if, for any projective $V \in \mathcal{A}_0$, we have $(L_2 i^*)(i_* V) = 0$.

We will also require one additional functor which only exists when $\underline{R}$ is geometrically split.

Construction 2.61. Suppose $\underline{R}$ is geometrically split, and $\widetilde{\mathcal{F}} \subseteq \mathsf{P}_{\mathcal{O}}$ is a set of minimal elements. We define a functor

$$r : \mathsf{RMod}_{\underline{R}} \longrightarrow \mathsf{RMod}_{\Phi^{\mathcal{F}} \underline{R}}$$

as follows. For an $\underline{R}$-module $\underline{M}$, the underlying Mackey functor of $r\underline{M}$ is the restriction $\psi_{\widetilde{\mathcal{F}}}^* \underline{M}$ of $\underline{M}$ to $\mathcal{O}_{\widetilde{\mathcal{F}}}$. Since $\mathsf{P}_{\mathcal{O}_{\widetilde{\mathcal{F}}}}$ is a collection of pairwise incomparable elements, the Green functor $\Phi^{\mathcal{F}} \underline{R}$ amounts to the data of the rings $\underline{R}^{\Phi T}$ with $\mathrm{Aut}(T)$ action as $[T]$ ranges over the elements of $\widetilde{\mathcal{F}}$. A module is just a module over each of these rings with compatible $\mathrm{Aut}(T)$-action. In our case, we use the $\underline{R}^{\Phi T}$-$\mathrm{Aut}(T)$-module structure on $(r\underline{M})(T) = \underline{M}(T)$ defined by restricting the $\underline{R}(T)$-module structure along s_T.

Example 2.62. In the case $\underline{R} = \underline{A}$ is the Burnside Mackey functor, $r = \psi_{\widetilde{\mathcal{F}}}^*$ is just given by restriction.

Lemma 2.63. *The functor r defined above is exact and gives a retract of i_*. Moreover, r admits a left adjoint, $r_!$, which lifts the functor $(\psi_{\widetilde{\mathcal{F}}})_!$ on underlying Mackey functors.*

Proof. Exactness can be checked pointwise, where it is clear. The $\underline{R}(T)$-module structure on $i_* \underline{M}(T)$ automatically factors through the quotient to an $\underline{R}^{\Phi T}$-module structure, since the source of the relevant transfer maps on $\underline{M}$ is zero by the definition if i_*. The claim that r is a retraction now follows from the fact that the composite

$$\underline{R}^{\Phi T} \xrightarrow{s_T} \underline{R}(T) \longrightarrow \underline{R}^{\Phi T}$$

is the identity, by assumption. $\qquad\square$

Proposition 2.64. *Let $\underline{R}$ be a geometrically split Green functor, $\widetilde{\mathcal{F}} \subseteq \mathsf{P}_{\mathcal{O}}$ a collection of minimal elements with upward closed complement $\mathcal{F}$. Then the string of adjoints from Construction 2.57 presents a pre-hereditary recollement of abelian categories.*

Proof. The fact that we have a recollement of abelian categories is straightforward, so we prove that the recollement is pre-hereditary. We claim that the following sequence of functors $\mathsf{RMod}_{\Phi^{\mathcal{F}} \underline{R}} \longrightarrow \mathsf{RMod}_{\underline{R}}$ is exact:

$$0 \to j_! j^* r_! \to r_! \to i_* \to 0. \quad (*)$$

(Here $r_! \to i_*$ is adjoint to $\mathrm{id} \cong ri_*$.) Suppose for a moment we have established this exactness. Then we can mimic the argument in [FP04, 8.5] to establish the pre-hereditary condition. To elaborate, we first apply i^* to obtain an exact sequence

$$(L_2 i^*)r_! \to (L_2 i^*)i_* \to (L_1 i^*)j_! j^* r_!$$

and then evaluate on a projective $V \in \mathsf{RMod}_{\Phi^{\mathcal{F}}\underline{R}}$. The first term vanishes because $r_!$ preserves projectives, being the left adjoint of an exact functor. The last term vanishes because the composite $(L_1 i^*)j_!$ always vanishes in a recollement. (Indeed, this follows formally from the fact that $j_!$ preserves projectives together with the identity $i^* j_! = 0$.) Thus the middle term vanishes, which was to be shown.

So we are left with checking the exactness of $(*)$. Note that for any recollement of abelian categories, we have an exact sequence

$$j_! j^* \to \mathrm{id} \to i_* i^* \to 0.$$

Since $r_!$ is right exact, the sequence

$$j_! j^* r_! \to r_! \to i_* i^* r_! \to 0$$

is exact. But $i^* r_! \cong \mathrm{id}$ since it is adjoint to $ri_* \cong \mathrm{id}$, so we learn that $(*)$ is exact except possibly at the first nontrivial term. In other words, it suffices to show that

$$j_! j^* r_! \to r_!$$

is injective.

This is detected on underlying Mackey functors, so we need only check the injectivity of

$$j_! j^* (\psi_{\widetilde{\mathfrak{F}}})_! \to (\psi_{\widetilde{\mathfrak{F}}})_!$$

We do this by evaluating on each $T \in \mathcal{O}$. If $T \notin \widetilde{\mathfrak{F}}$, then i^* will evaluate to zero, and it is always the case that $j_! j^* \to \mathrm{id}$ is an equivalence for such T.

So we are left with showing that, for any $\underline{M} \in \mathsf{Mack}(\mathcal{O}_{\widetilde{\mathfrak{F}}}; \mathsf{Ab})$, and any $T \in \widetilde{\mathfrak{F}}$, the map

$$(j_! j^* (\psi_{\mathcal{F}})_! \underline{M})(T) \to ((\psi_{\widetilde{\mathfrak{F}}})_! \underline{M})(T)$$

is injective. In fact, we will show that it admits a natural retract. To construct this retract, we will need to unpack the left Kan extensions taking place on each side. We set-up some temporary notation to handle this.

- We will denote morphisms in effective Burnside categories by $\rightsquigarrow$ to remind the reader that they are represented by spans.

- Let K denote the category whose objects are strings $V \rightsquigarrow U \rightsquigarrow T$ where $V \in \mathsf{Fin}_{\mathcal{O}_{\widetilde{\mathfrak{F}}}}$, $U \in \mathsf{Fin}_{\mathcal{O}_{\mathcal{F}}}$, and the morphisms take place in $\mathsf{hA}^{\mathit{eff}}(\mathcal{O})$. The arrows are commutative diagrams:

where f is a morphism in $\mathsf{hA}^{\mathit{eff}}(\mathcal{O}_{\widetilde{\mathfrak{F}}})$ and g is a morphism in $\mathsf{hA}^{\mathit{eff}}(\mathcal{O}_{\mathcal{F}})$.

- Let K' denote the category whose objects are arrows $V \rightsquigarrow T$ in $\mathsf{hA}^{eff}(\mathcal{O})$ where $V \in \mathsf{Fin}_{\mathcal{O}_{\widetilde{\mathcal{F}}}}$. Morphisms are commutative diagrams:

$$V \xrightarrow{\quad f \quad} V'$$
$$\searrow \qquad \swarrow$$
$$T$$

where f belongs to $\mathsf{hA}^{eff}(\mathcal{O}_{\widetilde{\mathcal{F}}})$.

Composition provides a functor $K \to K'$. From the definitions of $j_!, j^*$, and $(\psi_{\widetilde{\mathcal{F}}})_!$ we get:

$$
\begin{array}{ccc}
\mathrm{colim}_K \underline{M}(V) & \longrightarrow & \mathrm{colim}_{K'} \underline{M}(V) \\
\cong \downarrow & & \cong \downarrow \\
(j_! j^*(\psi_{\widetilde{\mathcal{F}}})_! \underline{M})(T) & \longrightarrow & ((\psi_{\widetilde{\mathcal{F}}})_! \underline{M})(T)
\end{array}
$$

We now decompose K' into two pieces.

- Let K'_0 denote the full subcategory of K' spanned by objects of the form $V \leftarrow S \to T$ where $S \in \mathsf{Fin}_{\mathcal{O}_{\widetilde{\mathcal{F}}}}$. Note that, since $\widetilde{\mathcal{F}}$ consists of minimal elements, this forces $[S] = [V] = [T]$.

- Let K'_1 denote the full subcategory spanned by objects of the form $V \leftarrow S \to T$ where $S \in \mathsf{Fin}_{\mathcal{O}_{\mathcal{F}}}$.

We have a functor $K'_1 \to K$ given by

$$(V \leftarrow S \to T) \mapsto (V \leftarrow S = S = S \to T)$$

and hence a natural factorization:

$$
\begin{array}{ccc}
 & & \mathrm{colim}_{K'_1} \underline{M}(V) \\
 & \nearrow & \downarrow \\
\mathrm{colim}_K \underline{M}(V) & \longrightarrow & \mathrm{colim}_{K'} \underline{M}(V)
\end{array}
$$

We are thus reduced to proving the following two claims:

(i) The category K' decomposes as a disjoint union $K' = K'_0 \coprod K'_1$ so that the right vertical map above has a natural splitting.

(ii) The functor $K'_1 \to K$ is final, so that the diagonal arrow above is an isomorphism.

The statement (i) follows from the fact that the morphisms in K' between $(V \rightsquigarrow T)$ and $(T \rightsquigarrow T)$ (where the latter lies in K'_0) involve a morphism *in* $\mathsf{Fin}_{\mathcal{O}_{\widetilde{\mathcal{F}}}}$ $V \rightsquigarrow T$ or $T \rightsquigarrow V$. In order to make the resulting diagram commute, we conclude that V must be of the form $V \leftarrow S \to T$ where $S \in \mathsf{Fin}_{\mathcal{O}_{\widetilde{\mathcal{F}}}}$, and hence $(V \rightsquigarrow T) \notin K'_1$.

The statement (ii) follows from the fact that $K'_1 \to K$ is right adjoint to the map $K \to K'_1$. $\qquad \square$

We now have a good understanding of the left hand side of Theorem 2.53, so we turn to the right hand side. The category $\mathsf{RMod}_{\underline{R}^\Phi}$ is built by iterated application of a procedure due to Macpherson-Vilonen [MV86].

Definition 2.65 ([MV86]). Let $\mathcal{U}$ and $\mathcal{Z}$ be abelian categories, and let $\xi : F \to G$ be a natural transformation between two additive functors $F, G : \mathcal{U} \to \mathcal{Z}$. Define a category $\mathcal{A}(\xi)$ as follows:

- Objects consist of pairs $U \in \mathcal{U}$ and $Z \in \mathcal{Z}$ equipped with a factorization:

$$FU \xrightarrow{\quad \xi \quad} GU$$

$$\searrow \qquad \nearrow$$

$$Z$$

  - Morphisms are maps of pairs $(U, Z) \to (U', Z')$ commuting with the chosen factorizations of ξ.

We say that $\mathcal{A}(\xi)$ is the **MacPherson-Vilonen recollement** associated to ξ.

Construction 2.66. Let $\widetilde{\mathcal{F}} \subseteq \mathsf{P}_{\mathcal{O}}$ be the set of minimal elements, with upward closed complement $\mathcal{F}$, and let $\underline{R}$ be geometrically split, let i^* and j^* be the usual functors associated to this situation. Define $\mathcal{U} := \mathsf{RMod}_{j^*\underline{R}^\Phi}$. Define $\mathcal{Z} := \mathsf{RMod}_{i^*\underline{R}^\Phi}$. Define $F, G : \mathcal{U} \to \mathcal{Z}$ for $T' \in \widetilde{\mathcal{F}}$ by

$$(F\underline{M})(T') = \bigoplus_{[T] > [T']} (\underline{M}(T) \otimes \mathrm{Hom}(T, T'))_{\mathrm{Aut}(T)}$$

$$(G\underline{M})(T') = \prod_{[T] > [T']} \left(\underline{M}(T)^{\mathrm{Hom}(T,T')} \right)^{\mathrm{Aut}(T)}$$

and use the trace to define a natural transformation $\xi : F \to G$.

By design, we get the following proposition:

Proposition 2.67. *With notation as above, the functor* $\mathsf{RMod}_{\underline{R}^\Phi} \to \mathcal{A}(\xi)$ *is an equivalence of categories.*

Proof of Theorem 2.53. We induct on the size of $\mathsf{P}_{\mathcal{O}}$. If $\mathsf{P}_{\mathcal{O}}$ is discrete, the theorem is clear. For the inductive step, let $\widetilde{\mathcal{F}} \subset \mathsf{P}_{\mathcal{O}}$ be the set of minimal elements in $\mathsf{P}_{\mathcal{O}}$. Then the forgetful functor

$$\mathsf{RMod}_{\underline{R}} \longrightarrow \mathsf{RMod}_{\underline{R}^\Phi}$$

respects the recollement data on source and target. It is an equivalence on each stratum by the induction hypothesis. By Proposition 2.64 and [FP04, Prop. 8.6] combined with Proposition 2.67, we know that both the source and target of the forgetful functor are pre-hereditary. Now Theorem 8.4 in loc. cit. implies that the forgetful functor is an equivalence of categories, which was to be shown. $\square$

2.6 Slices as twisted Mackey functors

In order to apply the results from the previous section, we first need to compute $\underline{\mathrm{End}}(W)^{\Phi T}$ for W an isotropic slice n-sphere, and show that $\underline{\mathrm{End}}(W)$ is geometrically splittable, at least for some choice of W. These statements follow by combining the next two lemmas.

Lemma 2.68. *Let W be an isotropic slice n-sphere, and $T \in \mathcal{O}$ be arbitrary. Then taking geometric fixed points of endomorphisms gives an* $\mathrm{Aut}(T)$*-equivariant isomorphism*

$$\underline{\mathrm{End}}(W)^{\Phi T} \longrightarrow \mathrm{End}(W^{\Phi T}).$$

Proof. The map is $\mathrm{Aut}(T)$-equivariant by functoriality of geometric fixed points, so we may as well replace $\mathcal{O}$ by $\mathcal{O}_{/T}$ so that T is terminal in $\mathcal{O}$. This leads to a recollement situation for $\widetilde{\mathcal{F}} = \{T\}$ and a fiber sequence

$$j_! j^* W \longrightarrow W \longrightarrow i_* W^{\Phi T}$$

First, let $f : W^{\Phi T} \longrightarrow W^{\Phi T}$ be an endomorphism, and consider the diagram of solid arrows:

$$
\begin{array}{ccc}
W & \longrightarrow & i_* W^{\Phi T} \\
\big\downarrow & & \big\downarrow {\scriptstyle i_* f} \\
W & \longrightarrow i_* W^{\Phi T} & \longrightarrow \Sigma j_! j^* W
\end{array}
$$

The composite $W \to \Sigma j_! j^* W$ is null by Proposition 2.22, so the dotted arrow exists. Thus,

$$[W, W] \to [W^{\Phi T}, W^{\Phi T}]$$

is surjective.

Now suppose $f : W \to W$ is an endomorphism such that $f^{\Phi T} = 0$. That is, we have a diagram of solid arrows:

$$
\begin{array}{ccccc}
 & & W & \longrightarrow & i_* W^{\Phi T} \\
 & \nearrow & \big\downarrow {\scriptstyle f} & & \big\downarrow {\scriptstyle 0} \\
j_! j^* W & \longrightarrow & W & \longrightarrow & i_* W^{\Phi T}
\end{array}
$$

The dotted arrow exists since the composite $W \to W \to i_* W^{\Phi T}$ is null by commutativity of the diagram. Thus f factors as a composite:

$$W \to j_! j^* W \to W.$$

Let $S = \coprod_{U \in \mathcal{O} - \{T\}} U \in \mathsf{Fin}_{\mathcal{O}}$. By Proposition 1.40, there is a natural equivalence

$$\operatorname*{hocolim}_{\Delta^{op}} (\mathrm{ind}_S \circ \mathrm{res}_S)^{\circ (n+1)} j^* W \cong j_! j^* W.$$

In particular, there is a map $\mathrm{ind}_S \mathrm{res}_S j^* W \to j_! j^* W$ and the cofiber has an associated graded which is a wedge of nontrivial suspensions of slice n-connective objects. By Proposition 1.40 again, we deduce that f factors as

$$W \xrightarrow{g} \mathrm{ind}_S \mathrm{res}_S W \xrightarrow{\varepsilon} W,$$

where the second map is the counit of the adjunction. Now, by the definition of the unit of an adjunction, we may further factor f as a composite:

$$W \xrightarrow{\eta} \mathrm{ind}_S \mathrm{res}_S W \xrightarrow{\mathrm{ind}_S \tilde{g}} \mathrm{ind}_S \mathrm{res}_S W \xrightarrow{\varepsilon} W$$

where $\tilde{g} : \mathrm{res}_S W \to \mathrm{res}_S W$ is the adjoint of g. Breaking S into its component orbits, we learn that f is a sum of maps each transferred up from $U \in \mathcal{O} - \{T\}$.

Putting it all together, we've shown that if $f^{\Phi T} = 0$ then f lies in the transfer ideal in $\underline{\mathrm{End}}(W)(T)$. The other inclusion always holds, so the result follows. $\qquad\square$

Lemma 2.69. *There exists an isotropic slice n-sphere W such that, for any $T \in \mathcal{O}$, the map*

$$[\downarrow_T W, \downarrow_T W] \to [W^{\Phi T}, W^{\Phi T}]$$

admits an $\mathrm{Aut}(T)$-*equivariant ring section.*

Proof. Let's temporarily call an $\mathcal{O}$-spectrum *good* if it satisfies the conclusion of the lemma. Revisiting the proof of we see that it is enough to prove the following closure properties for the class of good $\mathcal{O}$-spectra.

(i) If $U \in \mathcal{O}$ is minimal, and $X \in \mathsf{Sp}^{\mathcal{O}/U}$ is good, then so is $\mathrm{ind}_U X \in \mathsf{Sp}^{\mathcal{O}}$.

(ii) If X is good and $U \in \mathcal{O}$ is arbitrary, then the cofiber of $\uparrow_U \downarrow_U X \to X$ is also good.

(iii) If X is good and $U \in \mathcal{O}$ is arbitrary, then the fiber of $X \to \uparrow_U \downarrow_U X$ is also good.

Claim (i) follows from the more general observation that if X is good then so is $U_+ \wedge X$ for any set U because taking geometric fixed points commutes with colimits.

To prove (ii), denote the cofiber by Y. We need to find, for each T, an $\mathrm{Aut}(T)$-equivariant ring section of $[\downarrow_T Y, \downarrow_T Y] \to [Y^{\Phi T}, Y^{\Phi T}]$. The argument depends on the relationship between T and U.

- If $[U] \leq [T]$, then the map
$$\downarrow_T \uparrow_U \downarrow_U X \to \downarrow_T X$$
  has a functorial splitting, so $\downarrow_T Y$ is an $\mathrm{Aut}(T)$-equivariant summand of $\downarrow_T \uparrow_U \downarrow_U X$ and the conclusion follows.

- If $[U]$ and $[T]$ are incomparable, then $\downarrow_T \uparrow_U = 0$ and the conclusion is vacuously satisfied.

- If $[T] \leq [U]$ then we have
$$\downarrow_T \uparrow_U \downarrow_U = \uparrow_U^T \downarrow_U^T \downarrow_T$$
  Naturality of the counit of an adjunction, together with (homotopical) functoriality of the cofiber provides us with maps
$$\mathrm{map}\left(\downarrow_T X, \downarrow_T X\right) \longrightarrow \mathrm{map}\left(\uparrow_U^T \downarrow_U^T \downarrow_T X \to \downarrow_T X, \uparrow_U^T \downarrow_U^T \downarrow_T X \to \downarrow_T X\right) \longrightarrow \mathrm{map}(\downarrow_T Y, \downarrow_T Y)$$

Taking the composite on π_0 yields:
$$[\downarrow_T X, \downarrow_T X] \to [\downarrow_T Y, \downarrow_T Y]$$

and naturality ensures that the $\mathrm{Aut}(T)$ action and ring structure (from composition) are preserved. Finally, in this case $Y^{\Phi T} = X^{\Phi T}$ so precomposing with the assumed splitting $[X^{\Phi T}, X^{\Phi T}] \to [\downarrow_T X, \downarrow_T X]$ gives the result.

Claim (iii) is proved in the same way as claim (ii). $\qquad \square$

Remark 2.70. This proof can be modified to treat the isotropic slice spheres constructed via the method described in Remark 2.13. As a corollary of our hunch in that remark, we guess that *every* isotropic slice n-sphere satisfies the conclusion of the preceding lemma. We have not tried to prove this.

Remark 2.71. If $W^{\Phi T}$ is a single sphere, the splitting trivially exists at T. If T is maximal, then the splitting also always exists because $\downarrow_T W = W^{\Phi T}$. Combining these observations we learn that if W is a slice n-sphere with the property that $W^{\Phi T}$ is a single sphere for non-maximal T, then W satisfies the conclusion of the above lemma. This is enough to cover the examples in the next section, for example.

These results, combined with those of §2.5 and Theorem 2.35, now reduce the study of Slice_n to the study of Mackey functors equipped with compatible $\mathrm{Aut}(T)$-equivariant actions of the rings $\mathrm{End}(W^{\Phi T})$. Since $W^{\Phi T}$ is a wedge of spheres of a single dimension, this endomorphism ring is abstractly equivalent to the matrix ring $M_k(\mathbb{Z})$. Moreover, the action of $\mathrm{Aut}(T)$ comes from an action on $\mathbb{Z}^{\oplus k}$.

We now apply some Morita theory to our problem.

Lemma 2.72. *Let J be a finitely generated, free abelian group with an action of a finite group G. Let $R = \mathrm{End}(J)$ with its induced left action of G by conjugation. Denote by J^* the right R-module $\mathrm{Hom}_{\mathbb{Z}}(J, \mathbb{Z})$ with its left, G-action by conjugation (which intertwines the R-module structure). Then the functors:*

$$\mathsf{RMod}_{R\text{-}G} \longrightarrow \mathsf{Mod}_G$$

$$N \mapsto N \otimes_R J$$

$$\mathsf{Mod}_G \longrightarrow \mathsf{RMod}_{R\text{-}G}$$

$$M \mapsto M \otimes J^*$$

are inverse equivalences of categories.

Proof. This is immediate from classical Morita theory once one observes that the unit and counit of the adjunction on underlying modules respect the prescribed G-actions. But, of course, they were defined so that this is the case. □

Definition 2.73. The category of n-**twisted Mackey functors** associated to an isotropic slice n-sphere W with chosen splittings s_T is the category whose objects consist of the following data:

- For each $T \in \mathcal{O}$, an abelian group $M_{(T)}$ with an action of $\mathrm{Aut}(T)$;

- The structure of an object in $\mathsf{RMod}_{\underline{\mathrm{End}}(W)^\Phi}$ on the collection $\{M_{(T)} \otimes J_T^*\}$.

Here $J_T = \pi_{\nu(n,T)} W^{\Phi T}$. We denote by $\mathsf{TwMack}_n^{\mathrm{loc}}$ the full subcategory spanned by those n-twisted Mackey functors with the property that, under the equivalence of Theorem 2.53, the associated $\underline{\mathrm{End}}(W)$-module is slice local in the sense of Definition 2.42.

Remark 2.74. We can express the condition that an n-twisted Mackey functor be slice local directly as follows. For each $T \in \mathcal{O}$ write $T \times T^{jump} = \coprod U_\alpha$ in $\mathsf{Fin}_\mathcal{O}$ as a coproduct of orbits. Then an n-twisted Mackey functor $M_{(-)}$ is slice local if and only if, for every $T \in \mathcal{O}$, the sum of restriction maps

$$M_{(T)} \otimes J_T^* \longrightarrow \bigoplus_\alpha M_{(U_\alpha)} \otimes J_{U_\alpha}^*$$

induced by the projection $T \times T^{jump} \to T$, is injective.

Remark 2.75. Notice that there are canonical isomorphisms

$$\mathrm{End}(W^{\Phi T}) \cong \mathrm{End}(J_T)$$

$$[W^{\Phi T}, S^{\nu(n,T)}] \cong J_T^*$$

of $\mathrm{Aut}(T)$-modules, given by assigning to a map between wedges of spheres its behavior on $\pi_{\nu(n,T)}$.

Remark 2.76. We will often modify this definition somewhat by noting that giving an $\mathrm{End}(W^{\Phi T'})$-module map

$$M_{(T)} \otimes J_T^* \to M_{(T')} \otimes J_{T'}$$

is equivalent to giving a map of abelian groups

$$M_{(T)} \otimes J_T^* \otimes_{\mathrm{End}(W^{\Phi T'})} J_{T'} \to M_{(T')}.$$

A similar observation applies to the restriction maps. Working out the relations between the maps presented this way is more easily done in practice than in theory, as we will see in the next section.

Construction 2.77. Let W be an isotropic slice n-sphere with a prescribed splitting $s_T : \mathrm{End}(W^{\Phi T}) \to \mathrm{End}(\downarrow_T W)$. Choose an $S^{\nu(n,T)}$ summand of the spectrum $W^{\Phi T}$ with corresponding idempotent $e \in \mathrm{End}(W^{\Phi T})$ and retraction

$$\mathrm{pr} : W^{\Phi T} \to S^{\nu(n,T)}.$$

Let $W_{(T)}$ be the summand of $\downarrow_T W$ obtained from the image of e in $\mathrm{End}(\downarrow_T W)$. Now define a map, for any $X \in \mathsf{Sp}^\mathcal{O}$,

$$[W_{(T)}, \downarrow_T X] \otimes [W^{\Phi T}, S^{\nu(n,T)}] \longrightarrow [\downarrow_T W, \downarrow_T X].$$

Given an element $f \otimes g$ write g as $\mathrm{pr} A$ where $A : W^{\Phi T} \to W^{\Phi T}$ is an endomorphism. Then

$$f \otimes g \mapsto (\downarrow_T W \xrightarrow{s_T(A)} \downarrow_T W \to W_{(T)} \xrightarrow{f} \downarrow_T X).$$

Lemma 2.78. *The map constructed in (2.77) is an $\mathrm{End}(W^{\Phi T})$-isomorphism.*

Proof. That this map is an isomorphism is a general fact that belongs to Morita theory: if $e \in \mathrm{End}(J)$ is a full idempotent corresponding to a $\mathbb{Z}$ summand of J, then there is a canonical isomorphism of abelian groups $M \cdot e \cong M \otimes_{\mathrm{End}(J)} J$. In our case, $[\downarrow_T W, \downarrow_T X] \cdot e \cong [W_{(T)}, \downarrow_T X]$ and the map in Construction 2.77 is precisely the composite

$$M \cdot e \otimes J^* \cong M \otimes_{\mathrm{End}(J)} J \otimes J^* \cong M$$

in our setting. $\qquad\qquad\square$

Warning 2.79. The source of the map in 2.77 does not usually have an obvious $\mathrm{Aut}(T)$-action, and hence must inherit one from the target. It is possible to compute what this action must be in terms of the action on J, the chosen idempotent, and the action on $[W_{(T)}, \downarrow_T X]$ designated by placing the trivial action on $W_{(T)}$. However, this is another procedure more easily carried out in practice than in theory.

Definition 2.80. Let W be an isotropic slice n-sphere with chosen splittings s_T, and summands $W_{(T)}$ of each $\downarrow_T W$ arising from an $S^{\nu(n,T)}$ summand of $W^{\Phi T}$. Then define

$$\hat{\pi}_n : \mathsf{Sp}^{\mathcal{O}} \longrightarrow \mathsf{TwMack}_n$$

by the assignment

$$X \mapsto \{[W_{(T)}, \downarrow_T X]\}$$

equipped with the natural $\underline{\mathrm{End}}(W)^{\Phi}$-module structure on the collection

$$\{[W_{(T)}, \downarrow_T X] \otimes [W^{\Phi T}, S^{\nu(n,T)}]\} \cong \{[\downarrow_T W, \downarrow_T X]\}.$$

Remark 2.81. Notice that $\hat{\pi}_n$ as defined above contains essentially the same data as the previously defined $\hat{\pi}_n$ for the case of the testing subcategory generated by the $W_{(T)}$. We hope this justifies our recycling of the notation.

Finally, combining Theorem 2.53 and Theorem 2.35 we conclude:

Theorem 2.82. *The functor $\hat{\pi}_n$ yields an equivalence of adjoint pairs:*

3 Examples and special cases

We now apply the general theory of §2 to several examples. We begin in §3.1 by collecting together the known results for G-spectra in general. Then, in §3.2 we compare our classification of slices for C_p with the one due to [HY17]. Finally, in §3.3 we apply our machinery to a new example: the case of C_4. The slices for C_4 are not all $RO(C_4)$-graded suspensions of Eilenberg-MacLane spectra. As a result, the previous methods for studying slices fail in this case and something like theory we've developed is necessary.

3.1 G-spectra

In this section we will restrict attention to the original slice filtration on G-spectra, which we recall is the one associated to the dimension function

$$\nu(n, H) = \left\lfloor \frac{n}{|H|} \right\rfloor.$$

Remark 3.1. Statements about the regular slice filtration may be recovered from the equality

$$\left\lceil \frac{n+1}{|H|} \right\rceil - 1 = \left\lfloor \frac{n}{|H|} \right\rfloor.$$

That is, regular n-slices are the same as original $(n-1)$-slices. We warn the reader that this does *not* mean that the regular n-slice of a spectrum X is the same as its original $(n-1)$-slice.

To test the effectiveness of our general theory, we show how to recover the previously known results about the slice filtration which hold for an arbitrary group G. Of course, in many cases the original proof is simpler or morally the same as the one given here, this is only meant to be a proof of concept.

Theorem 3.2.   *(i) ([HHR16; Hil11]) The functor $\Sigma^{k\rho}$ yields an equivalence*

$$\Sigma^{k\rho} : \mathsf{Slice}_n \xrightarrow{\cong} \mathsf{Slice}_{n+k|G|}.$$

(ii) ([HHR16; Hil11]) The category of $(k|G|-1)$-slices is equivalent via the functor

$$\underline{M} \mapsto \Sigma^{k\rho-1} H\underline{M}.$$

to the category of Mackey functors.

(iii) ([HHR16; Hil11]) The category of $k|G|$-slices is equivalent via the functor

$$\underline{M} \mapsto \Sigma^{k\rho} H\underline{M}$$

to the category of Mackey functors all of whose restriction maps are injective.

(iv) [Ull13] The category of $(k|G|-2)$-slices is equivalent via the functor

$$\underline{M} \mapsto \Sigma^{k\rho-2} H\underline{M}$$

to the category of Mackey functors all of whose transfer maps are surjective.

(v) [HY17] Fix $n, k \in \mathbb{Z}$. Let V be a virtual representation with the property that, for all $H \subseteq G$,

$$\dim(V^H) + \left\lfloor \frac{n}{|H|} \right\rfloor \geq \left\lfloor \frac{n+k}{|H|} \right\rfloor. \quad (*)$$

Then smashing with S^V gives a functor

$$\Sigma^V : \mathsf{Sp}^G_{\geq n} \longrightarrow \mathsf{Sp}^G_{\geq n+k}.$$

This functor is an equivalence if and only if equality holds in $(*)$.

Proof. Part (v) implies part (i) and both follow immediately from the formula $(S^V \wedge X)^{\Phi H} \cong (S^{V^H} \wedge X^{\Phi H})$. Using this, the statements in (ii)-(iv) follow from the special case when $k = 0$. We treat each in turn.

(ii) The spectrum S^{-1} is an isotropic slice (-1)-sphere and (-1) is a jump for the dimension function. The result now follows from, e.g, Theorem 2.35.

(iii) The spectrum S^0 is an isotropic slice 0-sphere and the only 0-jump is the trivial subgroup. So, by Theorem 2.35, 0-slices are the full subcategory of the category of Mackey functors spanned by those $\underline{M}$ such that the restriction

$$\underline{M}(G/H) \to \underline{M}(G \times G/H) = \underline{M}(G)^{\oplus|G/H|}$$

is injective for all $H \subseteq G$. But this restriction map is given by the usual restriction $\underline{M}(G/H) \to \underline{M}(G)$ followed by the diagonal, so it is injective exactly when the usual restriction is injective. Since any restriction map followed by restriction to $\underline{M}(G)$ must be injective, we conclude that all restriction maps are injective. The result follows.

(iv) If A is a (-2)-slice, then $\Sigma^2 A \geq 0$ and so A is (-2)-connective in the usual sense by Lemma 1.59. On the other hand, $G/H_+ \wedge S^n \geq -1$ for all $n \geq -1$ by inspection of the floor function, so that $\underline{\pi}_n A = 0$ for $n \geq -1$. We conclude that $A \cong \Sigma^{-2} H\underline{M}$ for some Mackey functor $\underline{M}$. We claim that $\Sigma^{-2} H\underline{M}$ is a (-2)-slice if and only if $\underline{M}$ has surjective transfer maps.

Define W_H by the cofiber sequence

$$S^{-2} \xrightarrow{\text{tr}} G/H_+ \wedge S^{-2} \longrightarrow W_H.$$

Then W_H is a slice (-2)-sphere. Indeed, the underlying cofiber sequence splits so that $W_H^{\Phi e}$ is a wedge of copies of S^{-2}, while, for $K \subsetneq G$, the middle term vanishes and $W_H^{\Phi K}$ is a (possibly vanishing) wedge of copies of S^{-1}. This is as prescribed by the floor function $\lfloor -2/|H| \rfloor$.

But now Proposition 2.22 implies that $[\Sigma^{-1} W_H, X] = 0$ for any $X \geq -2$ and any subgroup $H \subseteq G$. This forces the transfers in $\underline{\pi}_{-2} X$ to be surjective by inspection of the long exact sequence associated to the defining cofiber sequence for W_H. We conclude that the condition on Mackey functors is necessary.

Now suppose that $\underline{M}$ is a Mackey functor. If $W \geq -1$ then $\Sigma^2 W \geq 1$ and, in particular, is 1-connective by Lemma 1.59. So $[\Sigma^2 W, H\underline{M}] = [W, \Sigma^{-2} H\underline{M}] = 0$. Thus, we always have $\Sigma^{-2} H\underline{M} \leq -2$. If moreover $\underline{M}$ has surjective transfer maps, we need to show that $\Sigma^{-2} H\underline{M}$ is slice (-2)-connective. To that end, consider the diagram in Sp^H for $|H| \neq 1$, where we use the usual recollement functors on Sp^H associated to $\widetilde{\mathcal{F}} = \{H/H\}$:

$$\Sigma^{-2} j_! j^* \downarrow_H H\underline{M} \longrightarrow \Sigma^{-2} \downarrow_H H\underline{M} \longrightarrow \Sigma^{-2} i_* i^* \downarrow_H H\underline{M} \longrightarrow \Sigma^{-1} j_! j^* \downarrow_H H\underline{M}$$

$$\Sigma^{-2} \uparrow_1^H \downarrow_1^G H\underline{M}$$

(Notice that if H is the trivial subgroup, the top left object is zero and the vertical arrow does not exist because we are using the fact that $\{1\} \subset H$ is a proper subgroup to define that map.) Applying π_{-2}^H we get:

$$\pi_0^H(j_! j^* \downarrow_H H\underline{M}) \xrightarrow{f} \underline{M}(G/H) \longrightarrow \pi_{-2}\left(\Sigma^{-2} H\underline{M}\right)^{\Phi H} \longrightarrow 0$$

$$\underline{M}(G)$$

Since the diagonal arrow is surjective by assumption, so is f, and hence $\pi_{-2}\left(\Sigma^{-2} H\underline{M}\right)^{\Phi H} = 0$ for all nontrivial subgroups $H \subseteq G$. It follows that $(\Sigma^{-2} H\underline{M})^{\Phi H} \geq -1 = \lfloor (-2)/|H| \rfloor$ when $|H| \neq 1$ so that $\Sigma^{-2} H\underline{M} \geq -2$, which was to be shown.

$\square$

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Remark 3.3. Though it is the case that Slice_{-2} is a localization of the category of Mackey functors, it is not true in general that $\heartsuit_{-2}$ is equivalent to the category of Mackey functors, as we will see in the next section when $G = C_p$.

The above theorem is more than enough to recover the known description of slices for the group $G = C_2$.

Corollary 3.4 ([HHR16; Hil11]). *Let $G = C_2$.*

(i) *The functor*
$$\underline{\pi}_{n\rho-1} : \mathsf{Slice}_{2n-1} \longrightarrow \mathsf{Mack}(C_2, \mathsf{Ab})$$
is an equivalence of categories.

(ii) *The functor*
$$\underline{\pi}_{n\rho} : \mathsf{Slice}_{2n} \longrightarrow \mathsf{Mack}(C_2, \mathsf{Ab})$$
is fully faithful. The essential image consists of those Mackey functors $\underline{M}$ such that the restriction map
$$\mathrm{res} : \underline{M}(*) \longrightarrow \underline{M}(C_2)$$
is injective.

(iii) *The slices of a G-spectrum X are determined by the formulae:*
$$\underline{\pi}_{n\rho-1} P_{2n-1}^{2n-1} X = \underline{\pi}_{n\rho-1} X.$$
$$\underline{\pi}_{n\rho} P_{2n}^{2n} X = \frac{\underline{\pi}_{n\rho} X}{\ker(\mathrm{res})}.$$

3.2 C_p-spectra

By the results in §3.1, we can already deduce a description of the all the categories of slices for C_p-spectra. We find, as in [HY17], that a description is possible purely in terms of $RO(C_p)$-graded homotopy Mackey functors. In this section we will employ the following notation:

- We fix an odd prime p.

- We fix a generator γ of C_p.

- We denote by λ the 2-dimensional real representation of C_p where γ acts by rotation through the angle $2\pi/p$.

- Given a Mackey functor $\underline{M}$ for C_p valued in abelian groups, we denote by $\mathrm{tr}(\underline{M})$ the sub-Mackey functor generated under the transfer by $\underline{M}(C_p)$. Equivalently, $\mathrm{tr}(\underline{M})$ is defined by the exact sequence:
$$0 \to \mathrm{tr}(\underline{M}) \to \underline{M} \to \Phi^{C_p}\underline{M} \to 0.$$

Theorem 3.5 (Hill-Yarnall [HY17]). (i) *The functor*
$$\underline{\pi}_{n\rho-1} : \mathsf{Slice}_{pn-1} \longrightarrow \mathsf{Mack}(C_p, \mathsf{Ab})$$
is an equivalence of categories for all $n \in \mathbb{Z}$.

(ii) *The functor*
$$\underline{\pi}_{n\rho+k\lambda} : \mathsf{Slice}_{pn+2k} \longrightarrow \mathsf{Mack}(C_p, \mathsf{Ab})$$
is fully faithful for all $n \in \mathbb{Z}$ and $0 \leq k \leq \dfrac{p-3}{2}$. The essential image is spanned by those Mackey functors all of whose restriction maps are injective.

(iii) *The functor*

$$\underline{\pi}_{n\rho+k\lambda-1} : \mathsf{Slice}_{pn+2k-1} \longrightarrow \mathsf{Mack}(C_p, \mathsf{Ab})$$

is fully faithful for all $n \in \mathbb{Z}$ and $1 \leq k \leq \dfrac{p-1}{2}$. The essential image is spanned by those Mackey functors all of whose transfer maps are surjective.

(iv) *Given a C_p-spectrum X, its slices are determined by the formulae:*

$$\underline{\pi}_{n\rho-1} P^{pn-1}_{pn-1} X = \underline{\pi}_{n\rho-1} X.$$

$$\underline{\pi}_{n\rho+k\lambda} P^{pn+2k}_{pn+2k} X = \frac{\underline{\pi}_{n\rho+k\lambda} X}{\ker(\mathrm{res})}, \quad 0 \leq k \leq \frac{p-3}{2}.$$

$$\underline{\pi}_{n\rho+k\lambda-1} P^{pn+2k-1}_{pn+2k-1} X = \mathrm{tr}\left(\underline{\pi}_{n\rho+k\lambda-1} X\right), \quad 1 \leq k \leq \frac{p-1}{2}.$$

Proof. This follows from the Hill-Yarnall result on periodicity (seen above as Theorem 3.2(v)) applied to the representation λ, together with parts (ii)-(iv) of that same theorem. $\square$

In this section we give a different take on this result. We find that, even though Slice_1 is a localization of the category of Mackey functors, it is not the case that $\heartsuit_1$ is equivalent to the category of Mackey functors. We then describe explicitly how to move back and forth between the two different descriptions of Slice_1.

We begin by recalling an example of an isotropic slice 1-sphere.

Definition 3.6. Let $S^{\lambda/2}$ denote the cofiber of the fold map

$$C_{p+} \to S^0.$$

This is an isotropic slice 1-sphere by inspection.

The reason for the name is the following lemma.

Lemma 3.7. *There is a cofiber sequence*

$$C_{p+} \wedge S^1 \to S^{\lambda/2} \to S^{\lambda}.$$

Proof. Let $S(\lambda)$ denote the unit sphere in the representation λ. Let $\mathrm{sk}_0 S(\lambda) = \{z : z^p = 1\} \cong C_p$. And notice that we have a cofiber sequence

$$C_{p+} \to C_{p+} \to S(\lambda)_+$$

corresponding to attaching the 1-cell $C_p \times I$. Now use the cofiber sequence

$$S(\lambda)_+ \hookrightarrow D(\lambda)_+ \to S^{\lambda}$$

to induce a cell structure on S^{λ}. The attaching maps for this cell structure show that $\mathrm{sk}_1 S^{\lambda} = S^{\lambda/2}$ (after taking suspension spectra) and produce the desired cofiber sequence above. $\square$

From §2 we know that we must study the Weyl group action on the geometric fixed points. Luckily, there aren't many subgroups of C_p.

Lemma 3.8. *Let J denote the augmentation ideal in $\mathbb{Z}[C_p]$. Then there is a canonical C_p-equivariant isomorphism*

$$\pi_1(\downarrow_1 S^{\lambda/2}) \overset{\cong}{\longrightarrow} J.$$

Proof. Apply π_1 to the cofiber sequence:

$$S^0 \to \downarrow_1 S^{\lambda/2} \longrightarrow \downarrow_1 C_{p+} \wedge S^1 \longrightarrow S^1.$$

$\square$

Proposition 3.9. *The category* TwMack_1 *is equivalent to the category whose objects consist of the following data:*

- *An abelian group* $M_{(C_p/C_p)}$,

- *A* C_p-*module* $M_{(C_p)}$,

- *Maps of abelian groups:*
$$R : M_{(C_p/C_p)} \to M_{(C_p)}$$
$$T : M_{(C_p)} \to M_{(C_p/C_p)}$$

subject to the conditions:

- $T((1 + \cdots + \gamma^{p-1})x) = 0$,

- $(1 + \cdots + \gamma^{p-1})R(x) = 0$,

- $TR(x) = (1 - \gamma)x$.

Proof. By definition, an object of TwMack_1 consists of

- An abelian group $M_{(C_p/C_p)}$,

- A C_p-module $M_{(C_p)}$,

- A commutative diagram:

$$\left(M_{(C_p)} \otimes J^*\right)_{C_p} \xrightarrow{\quad \text{trace} \quad} \left(M_{(C_p)} \otimes J^*\right)^{C_p}$$

with maps T' and R' to and from $M_{(C_p/C_p)}$.

We compute the top two pieces of this diagram in more explicit terms. Consider the exact sequence dual to the one defining J:
$$0 \to \mathbb{Z} \to \mathbb{Z}[C_p] \to J^* \to 0$$

The first map is given by $1 \mapsto (1 + \cdots + \gamma^{p-1})$. Since this is split exact as a sequence of abelian groups, we get an exact sequence:
$$0 \to M_{(C_p)} \to \mathbb{Z}[C_p] \otimes M_{(C_p)} \to M_{(C_p)} \otimes J^* \to 0.$$

Now apply C_p coinvariants to get

$$M_{(C_p)}/(1 - \gamma) \overset{(1 + \cdots + \gamma^{p-1})}{\longrightarrow} M_{(C_p)} \to (M_{(C_p)} \otimes J^*)_{C_p} \to 0.$$

The map $\mathbb{Z}[C_p] \to J$ given by $1 \mapsto (1 - \gamma)$ induces an isomorphism $J^* \cong J$, and a similar argument with the defining exact sequence for J yields

$$\left(M_{(C_p)} \otimes J^*\right)^{C_p} \cong \ker\left((1 + \cdots + \gamma^{p-1}) : M_{(C_p)} \to M_{(C_p)}^{C_p}\right).$$

Tracing through the identifications transforms the trace map into $(1 - \gamma)$, and the result is proved. $\qquad\square$

For definiteness, we choose the $\mathbb{Z}$-summand of J corresponding to the element $(1 - \gamma)$ in the basis $(1 - \gamma), (\gamma - \gamma^2), ..., (\gamma^{p-2} - \gamma^{p-1})$. This produces a specific inclusion and retraction:

$$S^1 \to\downarrow_1 S^{\lambda/2} \to S^1$$

and hence natural transformations:

$$R : [S^{\lambda/2}, -] \to [S^1, \downarrow_1 (-)],$$
$$T : [S^1, \downarrow_1 (-)] \to [S^{\lambda/2}, -].$$

Combining the previous result with Theorem 2.82 gives:

Theorem 3.10. *The assignment $\hat{\pi}_1$ given by*

$$X \mapsto \quad R\left(\begin{array}{c} [S^{\lambda/2}, X] \\ \Big\updownarrow \\ [S^1, \downarrow_1 X] \end{array}\right) T$$

gives an equivalence of categories:

$$\hat{\pi}_1 : \heartsuit_1 \xrightarrow{\cong} \mathsf{TwMack}_1.$$

Under this equivalence, the subcategory Slice_1 corresponds to precisely those objects of TwMack_1 for which the map R is injective.

We now describe how to move back and forth between this description and that of Hill-Yarnall.

Proposition 3.11. *Let X be a C_p-spectrum. Then there is a natural isomorphism*

$$\mathrm{im}\left(\mathrm{tr} : \pi_1(\downarrow_1 X) \to \pi_{\lambda-1}(X)\right) \cong \mathrm{cok}\left(R : [S^{\lambda/2}, X] \to \pi_1(\downarrow_1 X)\right).$$

In particular, the 1-twisted Mackey functor associated to X determines $\mathrm{tr}(\underline{\pi}_{\lambda-1} X)$ and vice-versa.

Proof. This follows from the exact sequence associated to the cofiber sequence:

$$S^{\lambda-1} \to C_{p+} \wedge S^1 \to S^{\lambda/2} \to S^\lambda.$$

$\square$

3.3 C_4-spectra

In this section we will see some phenomena not covered by previous techniques. To orient the reader, we begin with a counterexample to the statement that every slice is an $RO(G)$-graded suspension of an Eilenberg-MacLane spectrum.

Counterexample 3.12. Let $\underline{M}$ denote the Mackey functor for C_2 which is a copy of $\mathbb{Z}$ concentrated at $[C_2/C_2]$. Let σ denote the sign representation of C_2. Then define

$$A := \uparrow^{C_4} \Sigma H\mathbb{Z} \vee \uparrow^{C_4}_{C_2} \Sigma^\sigma H\underline{M}.$$

Since slices are preserved under induction, A is a 1-slice for C_4. Now suppose V is a virtual representation of C_4 with the property that $\Sigma^{-V} A$ is an Eilenberg-MacLane spectrum. The collection of Eilenberg-MacLane spectra is closed under retracts and restriction, from which we conclude that the virtual dimension of V is 1 and that $\downarrow_{C_2} V$ makes

$$\Sigma^{\sigma - \downarrow_{C_2} V} H\underline{M}$$

an Eilenberg-MacLane spectrum. Since $\underline{M}$ is concentrated on $[C_2/C_2]$, it only sees the fixed points of the representations we suspend by. That is, we may conclude that $\Sigma^{-a} H\underline{M}$ is an Eilenberg-MacLane spectrum, where a is the (virtual) dimension of the fixed points of V. This spectrum is nonzero, and an Eilenberg-MacLane spectrum, which means $\pi_0 \neq 0$. This forces $a = 0$. There aren't many representations of C_2 with underlying dimension 1 and fixed point dimension 0, so we conclude that $\downarrow_{C_2} V$ is equivalent to the sign representation σ. But there is no virtual real representation of C_4 which restricts to the regular representation of C_2, so no such V exists.

We now proceed with the program from §2 to study slices for C_4.

Construction 3.13. Let $S(\lambda)$ denote the unit sphere in the representation λ. Then it has a cell structure with $\mathrm{sk}_0 S(\lambda) = \{z : z^4 = 1\}$, and $\mathrm{sk}_1 S(\lambda) = S(\lambda)$ obtained by attaching a $(C_4 \times D^1)$-cell. We get an induced cell structure on $\Sigma^\infty(D(\lambda)/S(\lambda)) \cong S^\lambda$ and define $S^{\lambda/2}$ as the 1-skeleton. Notice that this construction produces cofiber sequences:

$$C_{4+} \to S^0 \to S^{\lambda/2},$$

$$C_{4+} \wedge S^1 \to S^{\lambda/2} \to S^\lambda.$$

We have already noted that the cofiber $\mathrm{cof}(G_+ \to S^0)$ is always an isotropic slice 1-sphere. Indeed, all of the geometric fixed points for proper subgroups are just S^0 (with trivial Weyl group action), while the underlying spectrum is a wedge of $(|G| - 1)$ copies of S^1.

The following is proved exactly as in Lemma 3.8.

Lemma 3.14. *There is a canonical C_4-equivariant isomorphism*

$$\pi_1(\downarrow_1 S^{\lambda/2}) \cong J := \ker\left(\mathbb{Z}[C_4] \xrightarrow{\varepsilon} \mathbb{Z}\right)$$

Now we can give a concrete description of TwMack_1.

Proposition 3.15. *The category TwMack_1 is equivalent to the category whose objects consist of the data of the diagram of abelian groups:*

$$
\begin{array}{c}
M_{(C_4/C_4)} \\
R^{C_4}_{C_2} \left(\right) \Big\uparrow T^{C_4}_{C_2} \\
M_{(C_4/C_2)} \\
R^{C_2}_{1} \left(\right) \Big\uparrow T^{C_2}_{1} \\
M_{(C_4)}
\end{array}
$$

Where $M_{(C_4)}$ is a C_4-module, $M_{(C_4/C_2)}$ is a C_4/C_2-module, and the maps are additive and subject to the following relations:

- *(Group action and restrictions)*

$$(1 + \gamma^2) \cdot R^{C_2}_1 = 0$$

$$(1 - \gamma^2) \cdot R^{C_4}_{C_2} = 0$$

$$(1 + \gamma + \gamma^2 + \gamma^3) \cdot R^{C_2}_1 R^{C_4}_{C_2} = 0$$

- *(Group action and transfers)*

$$T^{C_2}_1 \circ (1 + \gamma^2) = 0$$

$$T^{C_4}_{C_2} \circ (1 - \gamma^2) = 0$$

$$T^{C_4}_{C_2} T^{C_2}_1 \circ (1 + \gamma + \gamma^2 + \gamma^3) = 0$$

- *(Double coset formulae)*

$$R_1^{C_2} T_1^{C_2} = (1 - \gamma)$$

$$R_{C_2}^{C_4} T_{C_2}^{C_4} = (1 + \gamma)$$

$$R_1^{C_2} R_{C_2}^{C_4} T_{C_2}^{C_4} T_1^{C_2} = (1 - \gamma)$$

Under this equivalence, the category of 1*-slices is equivalent to the subcategory of those diagrams for which the map*

$$M_{(C_4/C_4)} \overset{(R_{C_2}^{C_4}, R_1^{C_4})}{\longrightarrow} M_{(C_4/C_2)} \oplus M_{(C_4)}$$

is injective.

Proof. Arguing as in the case of C_p, one establishes isomorphisms:

$$(M \otimes J^*)_{C_2} \cong \uparrow^{C_4/C_2} (M/(1 + \gamma^2)), \qquad (M \otimes J^*)^{C_2} \cong \uparrow^{C_4/C_2} (\ker((1 + \gamma^2) : M \to M).$$

$$(M \otimes J^*)_{C_4} \cong M/(1 + \gamma + \gamma^2 + \gamma^3), \qquad (M \otimes J^*)^{C_4} \cong \ker((1 + \gamma + \gamma^2 + \gamma^3) : M \to M)$$

and does a diagram chase. □

Now we establish the link with homotopy theory.

Lemma 3.16. *There is a C_2-equivalence*

$$\downarrow_{C_2} S^{\lambda/2} \cong S^\tau \vee \uparrow_1^{C_2} S^1.$$

Proof. As a C_2-set, $C_{4+} = (C_2 \amalg C_2)_+$. It follows that the cofiber $\operatorname{cof}(C_{2+} \to S^0) \cong S^\tau$ is a summand, and the remaining piece is the cofiber $\operatorname{cof}(C_{2+} \to 0)$ which is $\uparrow_1^{C_2} S^1$. □

While this splitting does not behave nicely with respect to the (C_4/C_2)-action, it does tell us what groups one needs to compute. As in the previous section, the element $(1 - \gamma) \in J$ together with the standard basis $(\gamma^i - \gamma^{i+1})$ of J gives a splitting:

$$S^1 \to \downarrow_1 S^{\lambda/2} \to S^1.$$

We get the following theorem as a corollary of our main results:

Theorem 3.17. *The assignment*

$$
X \mapsto
\begin{array}{c}
[S^{\lambda/2}, X] \\
R_{C_2}^{C_4} \left(\right) \uparrow T_{C_2}^{C_4} \\
[S^\tau \vee \uparrow^{C_2} S^1, \downarrow_{C_2} X] \\
R_1^{C_2} \left(\right) \uparrow T_1^{C_2} \\
[S^1, \downarrow_{C_4} X]
\end{array}
$$

gives an equivalence of categories

$$\heartsuit_1 \overset{\cong}{\longrightarrow} \mathsf{TwMack}_1.$$

Under this equivalence, the 1*-slice of a spectrum is computed by forcing $R_1^{C_2} \oplus R_{C_2}^{C_4}$ to be injective.*

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