# DUGGER

Note Title 10/29/2010

$G = C_{2^n}$ $X$ a $G$-spectrum

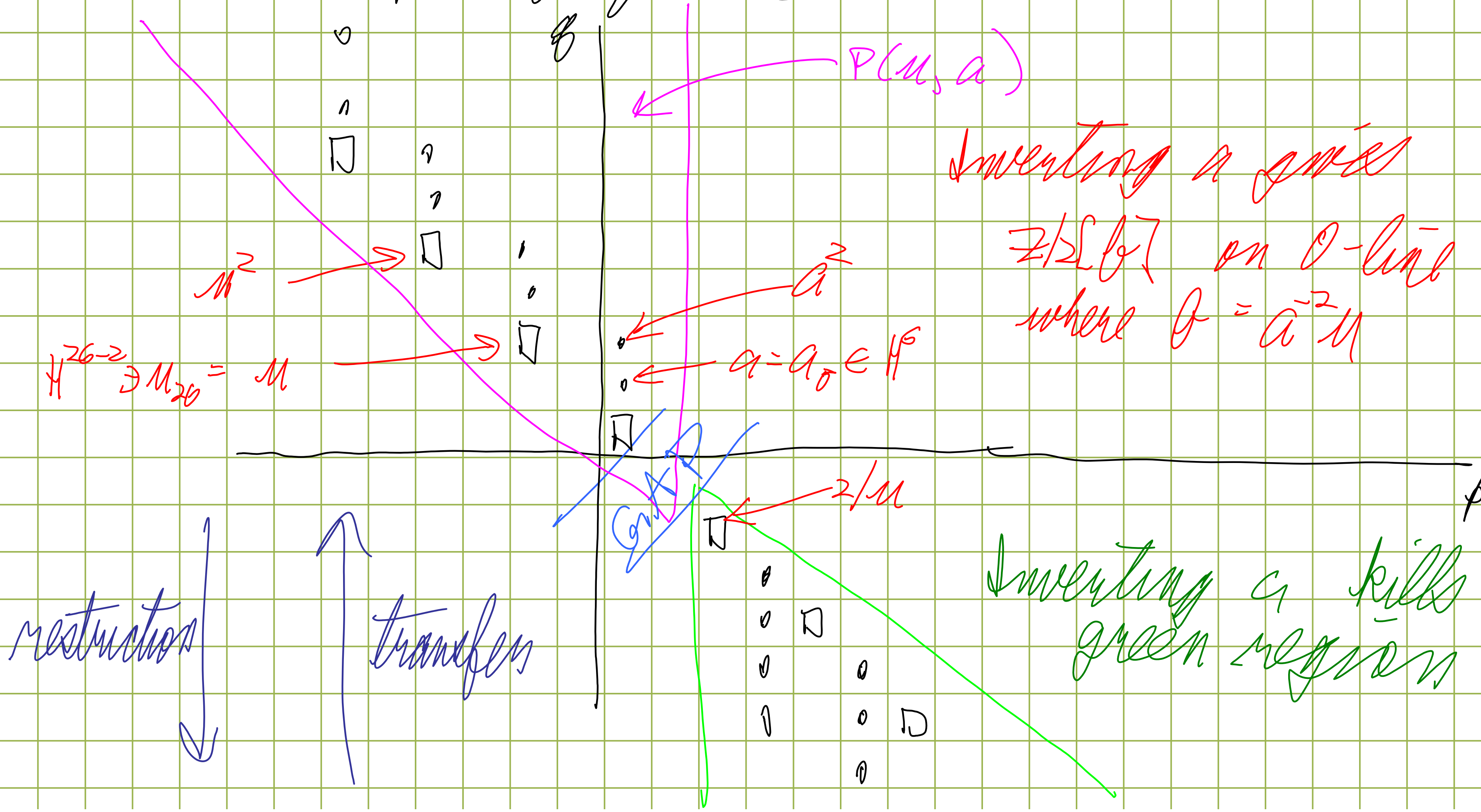
$$\begin{tikzcd} P_2^2 X \arrow[r] & P^2 X \arrow[d] \\ P_1^1 X \arrow[r] & P^1 X \arrow[d] \\ P_0^0 X \arrow[r] & P^0 X \arrow[d] \\ & \vdots \end{tikzcd}$$

slice tower

<u>Slice Thm</u> For $X = MU^{((G))}$, odd slices are contractible and even slices have the form $\hat{W} \wedge H\underline{\mathbb{Z}}$ where $\hat{W}$ is a wedge of regular isotropic slice cells.

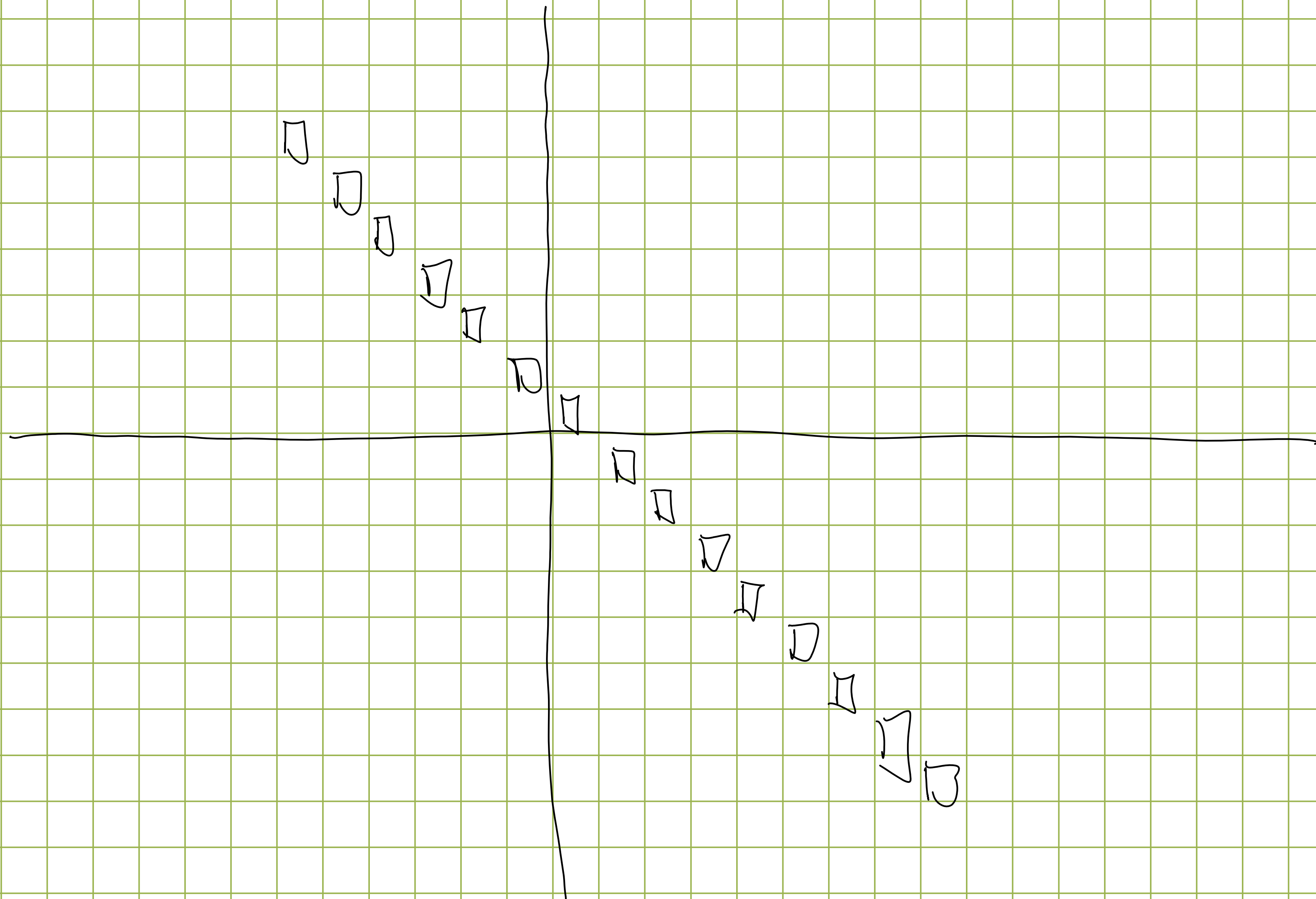
Goal: Prove that Reduction Thm implies Slice Thm.

For $G = C_2$ picture of $H\underline{\mathbb{Z}}^{\rho+*\sigma}(pt)$



Picture of $H\underline{\mathbb{Z}}^{p+q\sigma}(C_2)$

$C_{2+} \wedge S^V \simeq C_{2+} \wedge S^{|V|}$

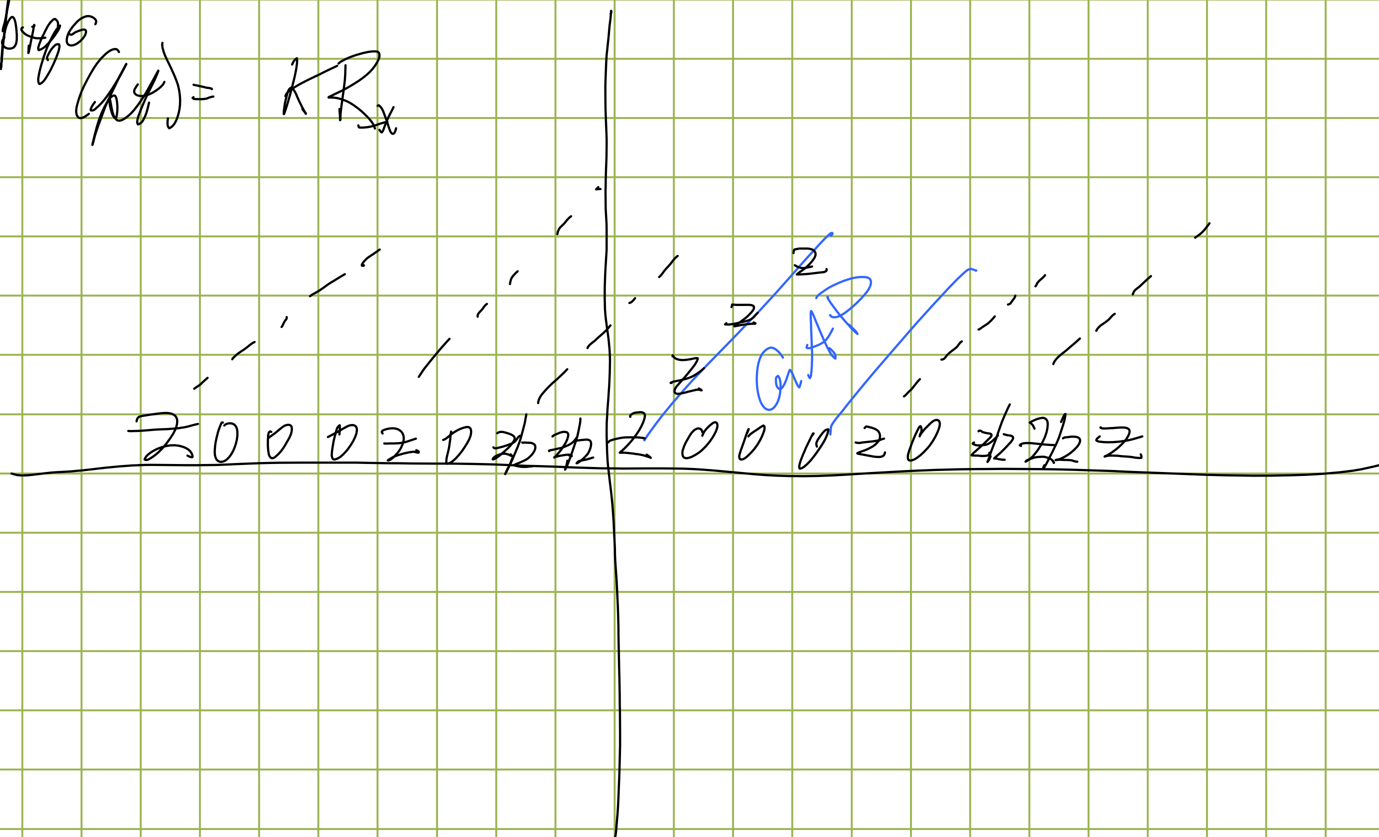


## Slice cells

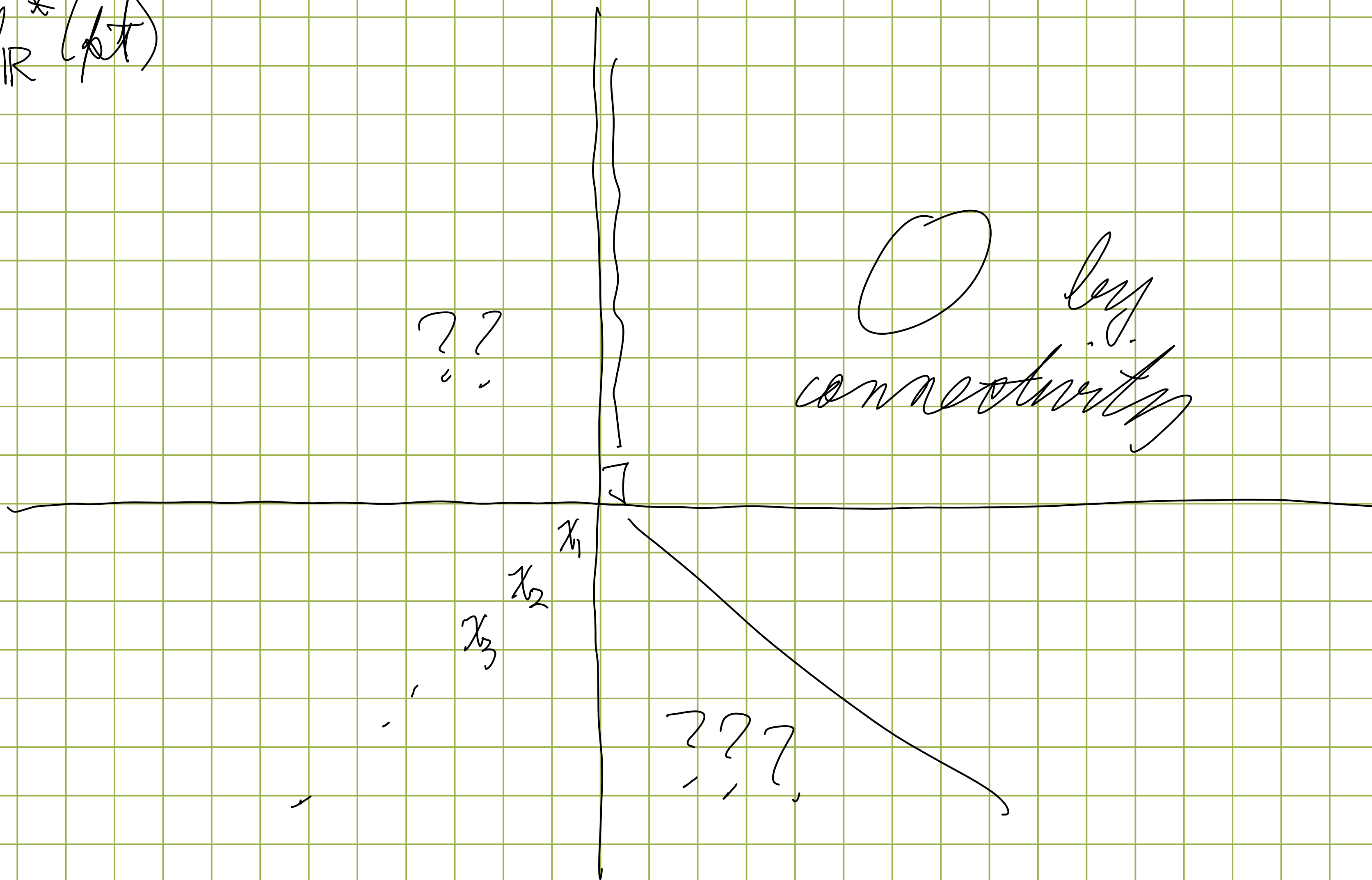
$S^{nP_2} = S^{n(1+\sigma)}$ $\qquad \Sigma^{-1} S^{nP}$

$C_{24} \wedge S^n$ not isotropic

$KR^{p+q\sigma}(pt) = KR_x$



$MV_{\mathbb{R}}^*(pt)$



## Silly fact

$W_i = \mathrm{Fiber}\,(X \longrightarrow P^{i-1}X)$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & W_2X \arrow[r] \arrow[d] \arrow[ddl] & P_2^2X \\
 & W_1X \arrow[r] \arrow[d] \arrow[dl] & P_1^1X \\
X & W_0X \arrow[r] \arrow[l] & P_0^0X
\end{tikzcd}$$

Whitehead tower

$$\begin{tikzcd}
W_{i+1}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & P^iX \arrow[d] \\
W_iX \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & P^{i-1}X \arrow[d] \\
P_i^iX \arrow[r] & * \arrow[r] & \Sigma P_i^iX
\end{tikzcd}$$

Non-equiv topology: $\pi_* MU = \mathbb{Z}[x_1, x_2, \cdots]$ $|x_i| = 2i$

Whitehead tower of connective covers

$$\begin{tikzcd} W_0 MU \arrow[d] & W_2 MU \arrow[l] \arrow[d] & W_4 MU \arrow[l] \arrow[d] & W_6 MU \arrow[l] & \cdots \arrow[l] \\ H\mathbb{Z} & \Sigma^2 H\mathbb{Z} & \Sigma^4 H(\mathbb{Z}\oplus\mathbb{Z}) & \cdots \end{tikzcd}$$

Pretend

① $\mathbb{Z}[x_1, x_2 \cdots] \longrightarrow \pi_* MU$

② We know nothing about $H\mathbb{Z}$ and EM-spectra

Define $A = \mathbb{S}[x_1, x_2, \cdots] = \bigvee_m S[m]$ monomials in $\mathbb{Z}[x_1, x_2 \cdots]$

There are maps $S[m] \wedge S[n] \longrightarrow S[m \cdot n]$

WARNING This is not comm because

e.g.

$$\begin{tikzcd}
S^2 \wedge S^2 \arrow[r, "\simeq"] \arrow[d, "\tau"'] & S^4 \\
S^2 \wedge S^2 \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

diagram does not commute

① There is a ring map $A \longrightarrow MU$

② Let $M_{2d} = \bigvee_{\substack{m \text{ with} \\ |m| \geq 2d}} S[m] \hookrightarrow A$ ideal

This gives a filtration on $A$.

and $M_{2d} / M_{2d+2}$ is wedge of spheres

$=: \hat{W}_{2d}$

$$M_{2d+2} \to M_{2d} \to \hat{W}_{2d}$$ cofib of spectra

but not of $A$-modules.

but it is a htpy cofib seq of $A$-modules.

Now smash with $MU$. Let $K_{2d} = MU \wedge_A M_{2d}$

$$K_{2d} / K_{2d+2} = MU \wedge_A \hat{W}_{2d} = MU \wedge_A (S^0 \wedge_{S^0} \hat{W}_{2d})$$

$$= (MU \wedge_A S^0) \wedge \hat{W}_{2d}$$

This is formal.

## Main point

$$MU \wedge_A S^0 \simeq MU/(x_1, x_2, \dots) \simeq H\mathbb{Z}$$

Now we will do this equivly.

Have produced $\bar{r}_k : S^{k\rho_2} \longrightarrow i_1^* MU^{((G))}$
$C_2$-map

$$G \cdot \bar{r}_k : G_+ \wedge_{C_2} S^{k\rho_2} \longrightarrow MU^{((G))}$$

Define $A = \$[G\cdot \bar{r}_1, G\cdot \bar{r}_2, \dots]$

$$= N_{C_2}^{G}(\$[\bar{r}_1]) \wedge N_{C_2}^{G}(\$[r_2]) \wedge \cdots$$

$$A \simeq \bigvee_f S^{\boxed{\#}\, \Gamma \mathrm{stab}(f)}$$

$f : J \to \mathbb{N}_0$

$$J = (G/C_2) \amalg (G/C_2) \cdots$$

$f$ identifies a monomial

$\mathbb{N}_0 := \{0, 1, 2, \ldots\}$

The # is the dim of monomial

**Def** $M_{2d} = \bigvee_{f} S^{\square}\mathrm{Pstab}(f)$

$\leftarrow$ where dim of sphere $\geq 2d$

$$M_{2d}/M_{2d+2} = \bigvee_{\dim = 2d} S^{\square}$$

$$K_{2d} = MU^{((G))} \wedge_A M_{2d}$$

These form a tower $K_\bullet$ as before

with layers look like $\left(MU^{((G))} \wedge_A S^0\right) \wedge \hat{W}_{2d}$

**Reduction Thm** $MU^{((G))} \wedge_A S^0 \simeq H\underline{\mathbb{Z}}$

$$
\begin{tikzcd}
\vdots \arrow[d, dashed] & \\
K_4 \arrow[d] & \\
K_2 \arrow[r] \arrow[d] & \left(MU^{((G))} \wedge_A S^0\right) \wedge \hat{W}_2 \\
MU^{((G))} = K_0 \arrow[r] & \left(MU^{((G))} \wedge_A S^0\right) \wedge \hat{W}_0
\end{tikzcd}
$$

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
K_2/K_4 \arrow[r] & MU^{((G))}/K_4 \\
K_0/K_2 \arrow[r] & MU^{((G))}/K_2 \arrow[d] \\
 & \ast
\end{tikzcd}$$

$$\begin{tikzcd}
\vdots \arrow[d] \\
P^2 MU^{((G))} \arrow[d] \\
P^0 MU^{((G))}
\end{tikzcd}$$

Will show that towers are the same

SEE PAGE 18

**Def** ① $X \geq n$ $\iff$ X can be built via
on $X^{\geq n+1}$ hocolims from slice cells
of dim $\geq n$

② $X \leq n$ $\iff$ $X \xrightarrow{\simeq} P^{n+1} X$ which means
on $X^{\leq n-1}$ $\nexists$ maps into X from
slice cells of dim $\geq n$.

**Lemma** ① $K_{2d} \geq 2d$

② $MU^{((G))} / K_{2d+2} \leq 2$ (uses Red Thm)

Proof: Let $\mathcal{B} = \left\{ B \,\middle|\, \begin{array}{l} B \text{ is an } A\text{-module and} \\ B \wedge_A M_{2d} > 2d \end{array} \right\}$

Then $G/H_+ \wedge A \in \mathcal{B}$
and $\mathcal{B}$ is closed under hocolim

$\mathcal{B} \ni$ any connective $A$-module
e.g. $MU^{((G))}$

This proves (1)

(2) Induction by Reduction Thm $\quad MU^{((G))}/K_2 = H\underline{\mathbb{Z}} \simeq P^0 H\underline{\mathbb{Z}}$

For inductive step

$$\begin{tikzcd}
H\underline{\mathbb{Z}} \wedge \widehat{W}_{2d} \arrow[r] \arrow[d, "\simeq"] & MU/K_{2d+2} \arrow[r] & MU/K_{2d} \arrow[d, "\simeq"] \\
P^{2d}(H\underline{\mathbb{Z}} \wedge W_{2d}) & & P^{2d}(MU/K_{2d})
\end{tikzcd}$$

Therefore $\quad MU/K_{2d+2} \simeq P^{2d}(-)$

$$\begin{tikzcd}
 & \vdots \arrow[d, phantom] & \\
 & K_4 \arrow[d] & \\
 & K_2 \arrow[d] \arrow[r] & X \wedge W^2 \\
MU^{((G))} \arrow[r] & K_0 \arrow[r] & X \wedge W^0
\end{tikzcd}$$

where $X = MU^{((G))} \wedge_A S^0$

$$\begin{tikzcd}
K_{2d+2} \arrow[d] \arrow[rr, bend left] & & \\
MU^{((G))} \arrow[rr] \arrow[d] & & P^{2d} MU^{((G))} \arrow[dll, dashed, shift left] \\
MU^{((G))}/K_{2d+2} \arrow[urr, dashed] & &
\end{tikzcd}$$

This shows the two towers on page 14 are the same. SLICE THM follows.