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EQUIVARIANT HOMOTOPY CLASSIFICATION *

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1. Introduction

1.1. *Summary.* Given a topological group G and a set $\{G_a\}_{a\in A}$ of subgroups of G, it is useful (see [2]) to consider the equivalence relation on the class of G-spaces (i.e., topological spaces with a continuous left G-action) generated by the equivariant maps which restrict to weak homotopy equivalences on the fixed point sets of the G_a $(a\in A)$. Our aim in this paper is to *classify, up to this equivalence relation, those G-spaces X, for which the fixed point sets X^a of the G_a $(a\in A)$ have the (weak) homotopy type of a given set of CW-complexes T^a $(a\in A)$.*

We obtain such a classification by showing its equivalence to a similar *classification of certain simplicial diagrams of simplicial sets*, which in turn can be derived from the *general classification results for ordinary diagrams of simplicial sets* of [8]. We also show that our classification result admits a considerable simplification in the commonly occurring case that the subgroups G_a $(a\in A)$ are *normal* or, more generally, *rigid* (2.2). This is not surprising in view of the fact that [12] equivariant homotopy theory with respect to a set of rigid subgroups is considerably simpler than general equivariant homotopy theory and is in fact equivalent to a relative version of the theory of fibrations, indexed by a partial order.

1.2. *Organization of the paper.* After giving a more detailed description of the *classification results* mentioned above (in Section 2), we formulate (in Section 3) a slightly different, though essentially equivalent, *equivariant homotopy classification*

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problem and show (in Section 4) that the latter is equivalent to a similar *classification problem for certain simplicial diagrams of simplicial sets.* In Section 5 we state a general classification theorem (5.2) for simplicial diagrams of simplicial sets which, for so-called *locally grouplike* diagrams, admits a considerable simplification (6.2). Translation of these results into equivariant language then immediately yields the classification results mentioned above. In Sections 7 and 8 we prove Theorem 5.2 by showing that the classification problem for simplicial diagrams of simplicial sets can be reduced, by means of a *flattening* process, to the classification problem for ordinary diagrams of simplicial sets which was solved in [8].

In an appendix (Section 9) we discuss a multi-variable version of the *bar construction*, which is slightly different from the ones considered by May [13] and Meyer [14]. Given a sequence $M_0, \ldots, M_n$ of simplicial monoids and a sequence $X_1, \ldots, X_n$ of simplicial sets with a commuting right action of M_{i-1} and left action of M_i on each X_i, it produces a simplicial set $B(M_0, X_1, M_1, \ldots, M_n)$. If the simplicial monoids are *grouplike* (1.3(ii)), then $B(M_0, x_1, M_1, \ldots, M_n)$ has the weak homotopy type of the total complex of a fibration with as base the product of the classifying complexes (1.3(v)) $BM_0 \times \cdots \times BM_n$ and as fibre the product $X_1 \times \cdots \times X_n$.

1.3. *Notation, terminology, etc.* This paper is essentially an application as well as a generalization of [8] and [11] and we will therefore freely use the notation, terminology and results of [7], [8] and [11]. In particular:

(i) We won't distinguish in notation between a *small category* and its *nerve*, nor between a *small simplicial category* and the *diagonal of its nerve.*

(ii) If $\mathbf{C}$ is a simplicial category, then we denote, for every two objects $X, Y \in \mathbf{C}$, the resulting *function complex* by $\hom(X, Y)$ and write $\operatorname{haut} X \subset \hom(X, X)$ for the *maximal simplicial submonoid which is grouplike* (i.e., $\pi_0 \operatorname{haut} X$ is a group). We sometimes add a superscript to hom to remove possible ambiguity or merely to remind the reader which simplicial category we are working in.

(iii) *No separation axioms* will be assumed, though our results remain valid if, for instance, all topological spaces and topological groups are assumed to be *compactly generated* (*but not necessarily Hausdorff*) or *singularly generated* (i.e., the topology is the identification topology obtained from the realization of the singular complex).

(iv) The category $\mathbf{M}$ of G-spaces comes with an obvious *simplicial structure* in which, for every two objects $X, Y \in \mathbf{M}$, the function complex $\hom(X, Y)$ is the simplicial set which has as n-simplices the maps $X \times |\Delta[n]| \to Y \in \mathbf{M}$ (where $|\Delta[n]|$ denotes the realization of the standard n-simplex $\Delta[n]$, with the trivial G-action). Moreover the left G-action on the *left coset space* G/G_a of a subgroup $G_a \subset G$, turns G/G_a into an object of the category $\mathbf{M}$ and, for every object $X \in \mathbf{M}$, the simplicial set $\hom(G/G_a, X)$ is naturally isomorphic to the singular complex $\operatorname{Sing} X^a$ of the fixed point set X^a of G_a.

(v) If M is a simplicial monoid, then we denote by BM the *classifying complex* which has as its n-simplices the n-tuples of n-simplices of M. This is the same as the diagonal of the nerve of the corresponding simplicial category with a single object.

2. The main result

2.1. *Preliminaries.* In this section we give a more detailed description of the equivariant classification result mentioned above. As the difference between the rigid and the non-rigid situation already shows up in the case of a single subgroup and as the two subgroup case already exhibits all the essential features of the general case, we first consider in detail the cases that the set $\{G_a\}_{a\in A}$ of subgroups of G consists of only one or two subgroups.

We will assume throughout (and in view of 3.4 this is not real restriction) that the set $\{G_a\}_{a\in A}$ be *reduced*, i.e., no two of the G_a $(a\in A)$ are *conjugate* or even (3.6) *homotopy conjugate* (which in the rigid case means that no two of the G_a $(a\in A)$ are each conjugate to a subgroup of the other). *The desired classification then is achieved by the set of components of a function complex* $\hom^{\mathbf{D}}(H^c, U^f)$ between suitable diagrams of simplicial sets, in which

(i) the indexing category **D** depends only on the topological group G and the reduced set $\{G_a)_{a\in A}$ of subgroups,

(ii) H^c is any (see [7]) cofibrant **D**-diagram weakly equivalent to H, where

(iii) H is a **D**-diagram of simplicial sets which depends only on G and the G_a $(a\in A)$, and

(iv) U^f is any (see [7]) fibrant **D**-diagram weakly equivalent to U, where

(v) U is a **D**-diagram of simplicial sets which depends only on the CW-complexes T^a $(a\in A)$.

We start with discussing the notion of

2.2. *Rigid subgroups.* Given a topological group G and a subgroup $G_a\subset G$, denote by $N'G_a\subset G$ the *subnormalizer*, i.e., the topological submonoid which consists of the elements $h\in G$ such that $h^{-1}G_ah\subset G_a$. Then G_a is called a *rigid* subgroup of G, if the quotient monoid $N'G_a/G_a$ is grouplike, i.e., $\pi_0(N'G/G_a)$ is a group. This is, for instance, the case if $N'G_a$ is already a group and therefore coincides with the normalizer of G_a (e.g. if G_a is normal in G). In fact, if G is discrete, then G_a is a rigid subgroup of G iff $N'G_a$ is a group, i.e., $h^{-1}G_ah\subset G_a$ implies $h^{-1}G_ah=G_a$.

As for every subgroup $G_a\subset G$, the fixed point set $(G/G_a)^a$ is homeomorphic to $N'G_a/G_a$, it follows readily from 1.3 that G_a *is a rigid subgroup of* G *iff* $\operatorname{haut} G/G_a=\hom(G/G_a, G/G_a)$.

Now we are ready to discuss

2.3. *The case of a single subgroup* $G_a\subset G$. If G_a is a rigid subgroup of G, one takes **D** trivial and

$$H=B\operatorname{haut} G/G_a \quad\text{and}\quad U=B\operatorname{haut} T^a.$$

This is what one would expect as (see Section 4) the equivariant homotopy theory in question is equivalent to the homotopy theory of simplicial sets with a

$\hom(G/G_a, G/G_a) = \text{haut}\, G/G_a$ action, which in turn is well known to be equivalent to the homotopy theory of fibrations over $B\,\text{haut}\, G/G_a$ (see [3] for a treatment of the special case in which $\text{haut}\, G/G_a$ is a simplicial group).

If G_a is not only rigid, but even normal in G, then $N'G_a = G$ and hence $\text{haut}\, G/G_a \approx \text{Sing}\, G/G_a$ and the classification is well known.

If G_a is *not* a rigid subgroup of G, then (see Section 4) the resulting equivariant homotopy theory is still equivalent to the homotopy theory of simplicial sets with a $\hom(G/G_a, G/G_a)$ action. However, as $\hom(G/G_a, G/G_a)$ is not grouplike, the latter is no longer equivalent to a homotopy theory of fibrations and one can thus no longer expect as simple a classification as in the rigid case. In fact, in this case of a single non-rigid subgroup the classification result is essentially as complex as in the general case of an arbitrary set of (not necessarily rigid) subgroups (see 2.5).

Next we consider

2.4. *The case of two subgroups* $G_a, G_b \subset G$. If neither of these is conjugate to a subgroup of the other, then the classification is the product of the two single subgroup classifications with respect to G_a and G_b respectively. One may thus assume that one of these subgroups, say G_b, is conjugate to a subgroup of G_a.

If both G_a and G_b are rigid subgroups of G, then one takes for H and U the diagrams

$$B\,\text{haut}\, G/G_b \leftarrow B(\text{haut}\, G/G_b, \hom(G/G_b, G/G_a), \text{haut}\, G/G_a) \rightarrow B\,\text{haut}\, G/G_a,$$

$$B\,\text{haut}\, T^b \leftarrow B(\text{haut}\, T^b, \hom(T^b, T^a), \text{haut}\, T^a) \rightarrow B\,\text{haut}\, T^a$$

in which the middle entries are bar constructions (see Section 9) and the maps are the obvious ones. If moreover G_a and G_b are normal in G, then G_b is actually a proper subgroup of G_a and one can take for H the somewhat simpler weakly equivalent diagram

$$B\,\text{Sing}\, G/G_b \xleftarrow{\text{id}} B\,\text{Sing}\, G/G_b \xrightarrow{\text{proj}} B\,\text{Sing}\, G/G_a.$$

If G_a and G_b are not both rigid subgroups of G, then the same comments apply as at the end of 2.3.

Finally there is

2.5. *The case of an arbitrary reduced set* $\{G_a\}_{a\in A}$ *of subgroups of* G. If all the G_a $(a \in A)$ are rigid subgroups of G, then one takes, for the indexing category $\mathbf{D}$, the category which has as objects the finite ordered sets $(a_0, \ldots, a_n)$ of distinct elements of A such that each $G_{a_i}(1 \le i \le n)$ is conjugate to a (proper) subgroup of $G_{a_{i-1}}$, and which has as maps the 'deletions' and one takes for H and U the diagrams which assign to an object $(a_0, \ldots, a_n) \in \mathbf{D}$ the bar constructions (see Section 9)

$$B(\text{haut } G/G_{a_n}, \hom(G/G_{a_n} G/G_{a_{n-1}}), \text{haut } G/G_{a_{n-1}}, \dots, \text{haut } G/G_{a_0}),$$

$$B(\text{haut } T^{a_n}, \hom(T^{a_n}, T^{a_{n-1}}), \text{haut } T^{a_{n-1}}, \dots, \text{haut } T^{a_0}).$$

If moreover the subgroups G_a $(a \in A)$ are all normal in G, then the objects of this indexing category **D** are just the finite sequences $(a_0, \dots, a_n)$ of elements of A such that each G_{a_i} $(1 \le i \le n)$ is a proper subgroup of $G_{a_{i-1}}$. In this case one can take for H the somewhat simpler weakly equivalent diagram which sends an object $(a_0, \dots, a_n) \in \mathbf{D}$ to the simplicial set $B\,\text{haut } G/G_{a_n}$.

In the *general case* (i.e., if at least one of the G_a is *not* a rigid subgroup of G) one needs the larger indexing category **D**, which has as objects the finite ordered sets $(a_0, \dots, a_n)$ of (not necessarily distinct) elements of A such that each G/G_{a_i} $(1 \le i \le n)$ is conjugate to a subgroup of $G_{a_{i-1}}$, and which has as maps the 'deletions and/or repetitions'. (Note that even in this case of a single subgroup this indexing category is non-trivial and in fact isomorphic to Δ^{op}). For H one then takes the diagram which assigns to an object $(a_0, \dots, a_n) \in \mathbf{D}$ the product

$$\hom(G/G_{a_n}, G/G_{a_{n-1}}) \times \cdots \times \hom(G/G_{a_1}, G/G_{a_0})$$

while $U(a_0, \dots, a_n)$ is, as above, the bar construction

$$B(\text{haut } T^{a_n}, \hom(T^{a_n}, T^{a_{n-1}}), \text{haut } T^{a_{n-1}}, \dots, \text{haut } T^{a_0}).$$

3. Equivariant homotopy classification

We formulate (in 3.2) an equivariant homotopy classification problem, which is slightly different from the one mentioned in the introduction, though (3.3) essentially equivalent to it. The topological case will be treated in detail; the simplicial case is very similar (3.5).

We start with recalling from [11] the existence of

3.1. *Model categories for equivariant homotopy.* Let G be an arbitrary but fixed topological group. Let **M** be the category of topological spaces with a left G-action and let $\{G_a\}_{a \in A}$ be a set of subgroups of G. Then it follows immediately from 1.3 and [11, 1.2] that **M** *admits a closed simplicial model category structure in which the simplicial structure is as in* 1.3 *and in which a map* $X \to Y \in \mathbf{M}$ *is a weak equivalence or a fibration iff, for every* $a \in A$, *the induced map of fixed point sets* $X^a \to Y^a$ *is a weak homotopy equivalence or a Serre fibration.*

Next we discuss the

3.2. *Classification problem.* This is, given an object $X \in \mathbf{M}$ (3.1), to classify the weak equivalence classes of the *0-conjugates* of X, where two *G-spaces* Y_1 and Y_2 are called *0-conjugate* if, for every $a \in A$, the fixed point sets Y_1^a and Y_2^a (or equivalently the simplicial sets $\hom(G/G_a, Y_1)$ and $\hom(G/G_a, Y_2)$) are weakly homotopy equivalent.

To do this one forms the *0-classification complex* c^0X of X, i.e., the nerve of the subcategory of **M**, which consists of all weak equivalences between 0-conjugates of X, and notes that [8, §2]:

(i) *there is an obvious* 1-1 *correspondence between the components of* c^0X *and the weak equivalence classes of the* 0*-conjugates of* X*, and*

(ii) *For every* 0*-conjugate* Y *of* X*, the corresponding* (*see* (i)) *component of* c^0X *is weakly equivalent to a classifying complex for the self weak equivalences of* Y in the sense of [8, 2.3], i.e., if, in the closed simplicial model category structure of 3.1, the G-space Y' is fibrant and weakly equivalent to Y, the classifying complex B haut Y' of the simplicial monoid haut Y' of self weak equivalences of Y', is weakly equivalent to the component of c^0X which (see (i)) corresponds to Y.

This solves our classification problem in a trivial manner. More effective solutions, i.e., various descriptions of the homotopy type of c^0X in terms of more acessible homotopy types will be obtained in Sections 5 and 6.

3.3. Remark. As in [8, 14], the techniques of [9] and the present paper actually give a little more than we have described above; they give not just a classification of the 0-conjugates of a given G-space, but also a classification of the *realizations* of a given set of homotopy types for the fixed point sets. We have suppressed this point to avoid making the exposition more involved.

3.4. Remark. Given a set $\{G_a\}_{a\in A}$ of subgroups of a topological group G, and a subset $B\subset A$, the resulting set $\{G_b\}_{b\in B}$ of subgroups of G sometimes gives rise to the same classification problem:

If, for instance, for every element $a\in A$, there is an element $b\in B$ such that the subgroups G_a and G_b are *conjugate* (or equivalently the coset spaces G/G_a and G/G_b are isomorphic G-spaces), then the sets $\{G_a\}_{a\in A}$ and $\{G_b\}_{b\in B}$ give rise to identical model category structures on **M** and thus *the same classification problem.*

Similarly, if for every $a\in A$, there is a $b\in B$ such that the subgroups G_a and G_b are *homotopy conjugate* (i.e., G/G_a and G/G_b are isomorphic in $\pi_0\mathbf{M}$), then the sets $\{G_a\}_{a\in A}$ and $\{G_b\}_{b\in B}$ give rise to the same notion of weak equivalence and thus also *to the same classification problem.* Note that *two rigid* (2.1) *subgroups* $G_a, G_b\subset G$ *are homotopy conjugate iff each is conjugate to a subgroup of the other.*

We end with a brief comment on

3.5. *The simplicial case.* If G is a simplicial group, **M** the category of simplicial sets with a left G-action and $\{G_a\}_{a\in A}$ a set of simplicial subgroups of G, then the obvious simplicial analogues of the above results hold. We leave the details to the reader.

4. Reduction to simplicial diagrams of simplicial sets

As a first step towards solving the equivariant classification problem we show here that it is equivalent to a similar problem for

4.1. *Simplicial diagrams of simplicial sets.* Let $\mathbf{C}$ be a small simplicial category (with in each dimension the same objects), let $\mathbf{S}$ be the category of simplicial sets and let $\mathbf{S}^{\mathbf{C}}$ denote the category of $\mathbf{C}$-diagrams of simplicial sets, i.e., the category which has as objects the simplicial functors $\mathbf{C}\to\mathbf{S}$ and as maps the natural transformations between them. Then $\mathbf{S}^{\mathbf{C}}$ admits an obvious *simplicial structure* in which, for every two objects $X, Y\in\mathbf{S}^{\mathbf{C}}$, the function complex $\hom(X, Y)$ is the simplicial set which has as n-simplices the maps $X\times\Delta[n]\to Y\in\mathbf{S}^{\mathbf{C}}$, and one has [11, 1.3]:

4.2. Proposition. *The category $\mathbf{S}^{\mathbf{C}}$ admits a closed simplicial model category structure in which the simplicial structure is as in* 4.1 *and in which a map $X\to Y\in\mathbf{S}^{\mathbf{C}}$ is a weak equivalence or a fibration iff, for every object $C\in\mathbf{C}$, the induced map $XC\to YC\in\mathbf{S}$ is so.*

Now we can formulate

4.3. *The* 0*-classification problem for simplicial diagrams of simplicial sets.* This is, given an object $X\in\mathbf{S}^{\mathbf{C}}$, to classify the weak equivalence classes of the 0*-conjugates* of X, where two objects $Y_1, Y_2\in\mathbf{S}^{\mathbf{C}}$ are called 0*-conjugate* if, for every object $C\in\mathbf{C}$, the simplicial sets Y_1C and Y_2C are weakly (homotopy) equivalent.

As in 3.2 this can be done by means of the 0*-classification complex* c^0X of X, i.e., the nerve of the subcategory of $\mathbf{S}^{\mathbf{C}}$ which consists of all weak equivalences between 0*-conjugates* of X. Again [8, §2] implies that

(i) *there is an obvious* 1–1 *correspondence between the components of c^0X and the weak equivalence classes of the* 0*-conjugates of X, and*

(ii) for every 0*-conjugate* Y of X, *the* (see (i)) *corresponding component of c^0X is weakly equivalent to a classifying complex for the self weak equivalences of Y,* in the sense of [8, 2.3].

It thus remains to show that the equivariant classification problem (3.2) can be reduced to the above one. To do this let $\mathbf{O}\subset\mathbf{M}$ be the *orbit category*, i.e, the full simplicial subcategory spanned by the objects G/G_a $(a\in A)$. Then the functor

$$\hom(\mathbf{O}, -):\mathbf{M}\to\mathbf{S}^{\mathbf{O}^{\mathrm{op}}}$$

clearly preserves weak equivalences and thus induces, for every object $X\in\mathbf{M}$, a map $c^0X\to c^0\hom(\mathbf{O}, X)$. The desired result now follows from [11, 1.6] which implies that the functor $\hom(\mathbf{O}, -)$ induces an equivalence of homotopy theories and that, as a consequence

4.4. Proposition. *For every object* $X \in \mathbf{M}$, *the functor* $\hom(\mathbf{O}, -)$ *induces a weak equivalence* $c^0 X \sim c^0 \hom(\mathbf{O}, X)$.

4.5. Remark. If $\mathbf{C}$ is a small simplicial category and $X \in \mathbf{S}^{\mathbf{C}}$ an object, then one can also consider the problem of classifying the weak equivalence classes of the *conjugates* of X, where two objects $Y_1, Y_2 \in \mathbf{S}^{\mathbf{C}}$ are called *conjugate* if, for every integer $n \geq 0$ and every functor $J : \mathbf{n} \to \mathbf{C}$ (where $\mathbf{n}$ denotes the category which has the integers $0, \ldots, n$ as objects and which has exactly one map $i \to j$ whenever $i \leq j$), the induced $\mathbf{n}$-diagrams $J^* Y_1$ and $J^* Y_2$ are weakly equivalent [8, 1.3]. This is solved by means of the *classification complex* cX of X, which is the nerve of the subcategory of $\mathbf{S}^{\mathbf{C}}$ which consists of all weak equivalences between conjugates of X. Clearly *the 0-classification complex of X is a disjoint union of such classification complexes* (of suitable 0-conjugates of X).

4.6. Remark. If $\mathbf{C}$ is an ordinary small category (i.e., a small simplicial category in which all function complexes are discrete), then the closed simplicial model category structure of 4.1 reduces to the one considered in [7] and [8] and the classification problems of 4.2 and 4.5 reduce to the ones which were solved in [8].

5. A general 0-classification result

We now state a general 0-classification result for simplicial diagrams of simplicial sets, which expresses the homotopy type of the 0-classification complex in terms of more accessible homotopy types and which, together with 3.3 and 4.4, readily implies the general equivariant classification described in 2.5. We assume throughout that the simplicial indexing category $\mathbf{C}$ be *reduced* (5.1); in view of [10] this is no real restriction.

5.1. *Preliminaries.* Given a small simplicial category $\mathbf{C}$ which is *reduced* (i.e., the only isomorphisms in $\pi_0 \mathbf{C}$ are automorphisms) and given a diagram $X \in \mathbf{S}^{\mathbf{C}}$, consider, for every integer $n \geq 0$ and every functor $J : \mathbf{n} \to \mathbf{C}$ (4.5), the induced $\mathbf{n}$-diagram $J^* X$ and note that the *0-classification complexes* $c^0 J^* X$ *depend only on X and on the composition*

$$\mathbf{n} \xrightarrow{J} \mathbf{C} \xrightarrow{\text{proj}} \pi.\mathbf{C},$$

where $\pi.\mathbf{C}$ denotes the (ordinary) category obtained from $\mathbf{C}$ by identifying two maps whenever they have the same domain and range. Hence they give rise to a *$d\pi.\mathbf{C}$-diagram* $c^0_{d\pi.\mathbf{C}} X$, where $d\pi.\mathbf{C}$ denotes the *division* of the category $\pi.\mathbf{C}$ [8, §3]. For every integer $k \geq 0$, let $q_k : d\mathbf{C}_k \to d\pi.\mathbf{C}$ be the projection. Then we denote by $(q_* \downarrow -)$ the $d\pi.\mathbf{C}$-diagram which assigns to an object $I \in d\pi.\mathbf{C}$, the diagonal of the bisimplicial set which in dimension $(*, k)$ consists of the nerve of the over category

$q_k \downarrow I$. If for a simplicial set Y, the symbol Y^{f} stands for a *fibrant* simplicial set which is naturally weakly equivalent to Y [8, 1.5(ii)], then one has

5.2. Theorem. *Let* $\mathbf{C}$ *be a reduced small simplicial category and let* $X \in \mathbf{S}^{\mathbf{C}}$. *Then the 0-classification complex* $c^0 X$ *is naturally weakly equivalent to the function complex*

$$\hom^{d\pi.\mathbf{C}}((q_* \downarrow -), (c^0_{d\pi.\mathbf{C}} X)^{\mathrm{f}}).$$

The proof is quite technical and will be postponed until Section 8.

The usefulness of this theorem is due to the accessibility of the $d\pi.\mathbf{C}$-diagrams involved:

5.3. *Accessibility of* $c^0_{d\pi.\mathbf{C}} X$. The diagram $c^0_{d\pi.\mathbf{C}} X$ of 0-classification complexes is weakly equivalent to the *$d\pi.\mathbf{C}$-diagram of bar constructions* (see Section 9) and natural maps between them which assigns to a sequence $C_0 \to \cdots \to C_n$ of maps in $d\pi.\mathbf{C}$, the bar construction (9.3)

$$B(\operatorname{haut} XC_0^{\mathrm{f}}, \hom(XC_0^{\mathrm{f}}, XC_1^{\mathrm{f}}), \operatorname{haut} XC_1^{\mathrm{f}}, \ldots, \operatorname{haut} XC_n^{\mathrm{f}}).$$

5.4. *Accessibility of* $(q_* \downarrow -)$. The $d\pi.\mathbf{C}$-diagram $(q_* \downarrow -)$ depends only on $\mathbf{C}$ and is weakly equivalent to the rather simple $d\pi.\mathbf{C}$-diagram q_*^{-1}, which assigns to a sequence $C_0 \to \cdots \to C_n$ of maps in $\pi.\mathbf{C}$, the product of function complexes in $\mathbf{C}$

$$\hom(C_0, C_1) \times \cdots \times \hom(C_{n-1}, C_n).$$

In fact it is not difficult to see that the obvious map $(q_* \downarrow -) \to q_*^{-1} \in \mathbf{S}^{d\pi.\mathbf{C}}$ is a weak equivalence.

We end with applying all this to

5.5. *The general equivariant case.* Let G be a topological group, let $\mathbf{M}$ be the category of left G-spaces, let $X \in \mathbf{M}$ and let $\{G_a\}_{a \in A}$ be a reduced (2.1) set of subgroups of G. Then *the resulting orbit category* $\mathbf{O} \subset \mathbf{M}$ (i.e., the full simplicial subcategory spanned by the left coset spaces G/G_a $(a \in A)$) *is reduced* (5.1) and 4.4 and 5.2 now immediately imply that *the 0-classification complex* $c^0 X$ *is weakly equivalent to the function complex*

$$\hom^{d\pi.\mathbf{O}^{\mathrm{op}}}((q_* \downarrow -), (c^0_{d\pi.\mathbf{O}^{\mathrm{op}}} \hom(\mathbf{O}, X))^{\mathrm{f}}).$$

Together with 3.3, 5.3 and 5.4 this yields the *general equivariant classification* result of 2.5.

6. 0-classification for grouplike diagrams

If the indexing category $\mathbf{C}$ is *locally grouplike* (6.1), then the general 0-classification Theorem 5.2 can be considerably simplified and the corresponding equivariant result is just the rigid equivariant classification described in 2.3–5.

We start with a brief discussion of

6.1. *Locally grouplike categories.* A simplicial category $\mathbf{C}$ will be called *locally grouplike* if, for every object $C \in \mathbf{C}$, one has

$$\text{haut } C = \hom(C, C)$$

This definition readily implies:

(i) *A locally grouplike simplicial category with only one object is just a grouplike simplicial monoid.*

(ii) *Let G be a topological group,* $\mathbf{M}$ *the category of left G-spaces and* $\{G_a\}_{a \in A}$ *a set of subgroups of G. Then the resulting orbit category* (*Section* 4) $\mathbf{O} \subset \mathbf{M}$ *is locally grouplike iff the* G_a $(a \in A)$ *are all rigid subgroups of G.*

(iii) *If* $\mathbf{C}$ *is locally grouplike and reduced* (5.1), *then* $\pi.\mathbf{C}$ *is retract free* [8, 4.3] *and hence the projection* $s: d\pi.\mathbf{C} \to sd\pi.\mathbf{C}$ (where $sd\pi.\mathbf{C}$ denotes the subdivision of the category $d\pi.\mathbf{C}$ [7, §5]) *has an obvious cross section* $t: sd\pi.\mathbf{C} \to d\pi.\mathbf{C}$.

Now we can formulate, using the notation of 5.1:

6.2. Theorem. *Let* $\mathbf{C}$ *be a small simplicial category which is locally grouplike and reduced and let* $X \in \mathbf{S}^{\mathbf{C}}$. *Then the 0-classification complex* c^0X *is, in a natural manner, weakly equivalent to the function complex*

$$\hom^{sd\pi.\mathbf{C}}((sq_*\downarrow -), (t^* c^0_{d\pi.\mathbf{C}} X)^{\mathrm{f}}).$$

Proof. The function complexes

$$\hom^{d\pi.\mathbf{C}}((q_* \to -), (c^0_{d\pi.\mathbf{C}} X)^{\mathrm{f}}) \quad \text{and} \quad \hom^{d\pi.\mathbf{C}}((q_* \to -), (s^* t^* c^0_{d\pi.\mathbf{C}} X)^{\mathrm{f}})$$

are weakly equivariant because, as is easy to verify, the $d\pi.\mathbf{C}$-diagrams $c^0_{d\pi.\mathbf{C}} X$ and $s^* t^* c^0_{d\pi.\mathbf{C}} X$ are weakly equivalent. The desired result now follows readily from Theorem 5.2 and [8, 9.5].

Theorem 6.2 is a considerable improvement on Theorem 5.2 as the indexing category $sd\pi.\mathbf{C}$ is much smaller than $d\pi.\mathbf{C}$ (even if $\mathbf{C}$ has only one object) and because one still has the

6.3. *Accessibility of* $(sq_*\downarrow -)$. *The cofibrant* $sd\pi.\mathbf{C}$*-diagram* $(sq_*\downarrow -)$ *is weakly equivalent to the diagram of bar constructions and natural maps between them, which assigns to each sequence* $C_0 \to \cdots \to C_n$ *of non-identity maps in* $\pi.\mathbf{C}$, *the bar construction* (9.2)

$$B(\text{haut } C_0, \hom(C_0, C_1), \text{haut } C_1, \ldots, \text{haut } C_n).$$

Because $\mathbf{C}$ is locally grouplike, the homotopy types of these bar constructions are indeed quite accessible (9.1(vii)).

To prove this one first notes the existence of an obvious weak equivalence $(sq_*\downarrow -)\to(sq_*)^{-1}\in \mathbf{S}^{sd\pi.\mathbf{C}}$, where $(sq_*)^{-1}$ assigns to an object $(I:\mathbf{n}\to\pi.\mathbf{C})\in sd\pi.\mathbf{C}$, the diagonal of the bisimplicial set, which in dimension $(*,k)$ consists of the nerve of the subcategory $(sq_k)^{-1}I\subset d\mathbf{C}_k$. As $(sq_k)^{-1}I$ can be considered as an $(n+1)$-simplicial set, its nerve is naturally weakly equivalent to its diagonal which is just the (see 9.4) classical bar construction on the pull back functor $I^*\mathbf{C}_k\to\mathbf{n}$. The desired result now follows as in 9.6.

6.4. Example. Let $\mathbf{C}$ be a *simplicial group* (i.e., $\mathbf{C}$ is a simplicial category with only one object C and all maps in $\mathbf{C}$ are invertible) and let $X\in\mathbf{S}^{\mathbf{C}}$ be fibrant. Then 6.2 and 6.3 imply that *the 0-classification complex c^0X (which in this case coincides with the classification complex cX (4.5)) is weakly equivalent to the ordinary function complex* $\hom(\mathbf{C},(B\,\mathrm{haut}\,XC)^{\mathrm{f}})$. In view of [3, §2] this implies the second classification result mentioned in [8, 1.2].

Now we get our application to

6.5. *The rigid equivariant case.* Let G be a topological group, let $\mathbf{M}$ be the category of left G-spaces, let $X\in\mathbf{M}$ and let $\{G_a\}_{a\in A}$ be a reduced (2.1) set of rigid subgroups of G. Then 4.4, together with the above results, implies that *the resulting orbit category $\mathbf{O}\subset\mathbf{M}$ is locally grouplike and reduced* and that *0-classification complex c^0X is weakly equivalent to the function complex*

$$\hom^{sd\pi.\mathbf{O}^{\mathrm{op}}}((sq_*\downarrow -),(t^*c^0_{d\pi.\mathbf{O}^{\mathrm{op}}}\hom(\mathbf{O},X))^{\mathrm{f}}).$$

The *rigid equivariant classification* results of 2.3–5 now follow from 3.3 and 6.3.

We end with

6.6. *The normal equivariant case.* The normal equivariant classification results of 2.3–5 follow immediately from the rigid ones (6.5) and (9.1(iv)).

7. Flattening simplicial diagrams of simplicial sets

In preparation for the proof of Theorem 5.2 we show here that the classification problems for simplicial diagrams of simplicial sets can be reduced to similar problems for ordinary diagrams of simplicial sets, which were solved in [8]. To do this we need

7.1. *The flattening of a simplicial category.* The *flattening* of a small simplicial category $\mathbf{C}$ will be the category $b\mathbf{C}$ which has as objects the pairs (C,n), where $C\in\mathbf{C}$ is an object and n is an integer ≥ 0, and which has as maps $(C_1,n_1)\to(C_2,n_2)$ the pairs (c, f), where f is a simplicial operator (i.e., map of Δ^{op}) from dimension n_1 to dimension n_2 and c is a map $c: C_1 \to C_2 \in \mathbf{C}_{n_2}$.

7.2. Remark. If one considers $\mathbf{C}$ as a functor $\mathbf{C}: \Delta^{op} \to \mathbf{Cat}$ (the category of small categories), then $b\mathbf{C}$ is just the *one-sided Grothendieck construction* $* \otimes_{\Delta^{op}} \mathbf{C}$ of [8, 9.1]. It then follows readily from [8, 9.4] and [1, Ch. XII, 4.3 and 5.1] that *the diagonal of the nerve of* $\mathbf{C}$ *is, in a natural manner, weakly equivalent to the nerve of* $b\mathbf{C}$.

A straightforward calculation now yields

7.3. Proposition. *Let* $\mathbf{C}$ *be a small simplicial category. Then there is a pair of adjoint functors*

$$\beta : \mathbf{S}^{b\mathbf{C}} \leftrightarrow \mathbf{S}^{\mathbf{C}} : b$$

where $b: \mathbf{S}^{\mathbf{S}} \to \mathbf{S}^{b\mathbf{C}}$ *is the flattening functor which assigns to every pair of objects* $Y \in \mathbf{S}^{\mathbf{C}}$ *and* $(C, n) \in b\mathbf{C}$, *the simplicial set* $\hom(\Delta[n], YC)$ *and* $\beta: \mathbf{S}^{b\mathbf{C}} \to \mathbf{S}^{\mathbf{C}}$ *is the fattening functor which assigns to every pair of objects* $X \in \mathbf{S}^{b\mathbf{S}}$ *and* $C \in \mathbf{C}$, *the diagonal of* $X(C, -)$. *Moreover*

(i) *both functors preserve weak equivalences,*

(ii) *for every object* $Y \in \mathbf{S}^{\mathbf{C}}$, *the adjunction map* $\beta b Y \to Y$ *is a weak equivalence, and*

(iii) *for every object* $X \in \mathbf{S}^{b\mathbf{C}}$ *which has the property that, for every object* $C \in \mathbf{C}$ *and simplicial operator* f, *the map* $X(i_C, f)$ *is a weak equivalence, the adjunction map* $X \to b\beta X$ *is a weak equivalence.*

7.4. Remark. Proposition 7.3 asserts that *the homotopy theory of* $\mathbf{C}$*-diagrams is equivalent to the homotopy theory of certain* (but not all) $b\mathbf{C}$*-diagrams.* It follows that, for every object $Y \in \mathbf{S}^{\mathbf{C}}$, *the 0-classification complex* $c^0 Y$ *is weakly equivalent to a 'disjoint summand' of* $c^0 bY$. Moreover (see 4.5 and 4.6) *the flattening functor does induce a weak equivalence* $cY \sim cbY$ *of classification complexes.*

8. Proof of Theorem 5.2

Let $h: d_0\mathbf{C} \to d\pi.\mathbf{C}$ and $r_k: d\mathbf{C}_k \to d\pi_0\mathbf{C}$ $(k \geq 0)$ be the projections. Then it is not difficult to verify that [10, §3] $(q_* \downarrow -)$ is weakly equivalent to the homotopy pushdown $\underset{\to}{h}_*(r_* \downarrow -)$ and hence Theorem 5.2 follows from [10, §3] and

8.1. Theorem. *Let* $\mathbf{C}$ *be a small simplicial category and let* $X \in \mathbf{S}^{\mathbf{C}}$. *Then the 0-classification complex* $c^0 X$ *is weakly equivalent to the function complex*

$$\hom^{d\pi_0\mathbf{C}}((r_* \downarrow -), (h^* c^0_{d\pi.\mathbf{C}} X)^{\mathrm{f}}).$$

For every integer $n \geq 0$ *and functor* $J: \mathbf{n} \to \mathbf{C}$, *the classification complex* cJ^*X (4.5) *depends only on* X *and the composition*

$$\mathbf{n} \xrightarrow{J} \mathbf{C} \xrightarrow{\text{proj}} \pi_0 \mathbf{C}.$$

Hence they give rise to a $d\pi_0\mathbf{C}$-diagram $c_{d\pi_0\mathbf{C}}X$. Theorem 8.1 is now an easy consequence of 4.5 and

8.2. Theorem. *Let* $\mathbf{C}$ *be a small simplicial category and let* $X \in \mathbf{S}^{\mathbf{C}}$. *Then the classification complex* cX *is weakly equivalent to the function complex*

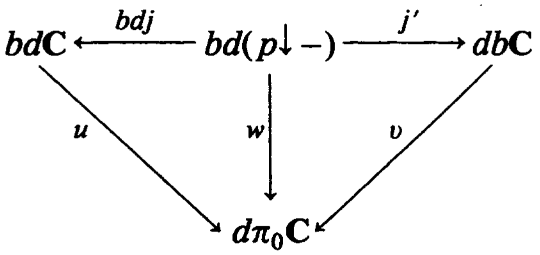
$$\hom^{d\pi_0\mathbf{C}}((r_* \downarrow -), (c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}})$$

It thus remains to prove Theorem 8.2.

To do this, let $bd\mathbf{C}$ be the result of applying the Grothendieck construction of 7.2 to the 'simplicial category' $d\mathbf{C}$ (which is the functor $\Delta^{\mathrm{op}} \to \mathbf{Cat}$ obtained by applying the division functor d dimensionwise to the simplicial category $\mathbf{C}$) and let $u: bd\mathbf{C} \to d\pi_0 C$ and $v: db\mathbf{C} \to d\pi_0\mathbf{C}$ be the projections. Then clearly (7.3) $c_{db\mathbf{C}}X = v^* c_{d\pi_0\mathbf{C}}X$ and hence (7.4 and [8, 3.4]) cX is weakly equivalent to $\underleftarrow{\mathrm{holim}}^{db\mathbf{C}}(v^* c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}}$. Furthermore (7.2) the $d\pi_0\mathbf{C}$-diagram $(r_* \downarrow -)$ is weakly equivalent to the (free and hence also cofibrant) $d\pi_0\mathbf{C}$-diagram $(u \downarrow -)$ and thus [8, 9.1 and 9.5] $\hom((r_* \downarrow -), (c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}})$ is weakly equivalent to $\underleftarrow{\mathrm{holim}}^{bd\mathbf{C}}(u^* c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}})$. It remains to prove the weak equivalence of

$$\underleftarrow{\mathrm{holim}}^{bd\mathbf{C}}(u^* c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}} \quad \text{and} \quad \underleftarrow{\mathrm{holim}}^{db\mathbf{C}}(v^* c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}}.$$

To do this consider the commutative diagram



in which

(i) $p: b\mathbf{C} \to \Delta^{\mathrm{op}}$ denotes the projection,

(ii) $j: (p\downarrow -) \to \mathbf{C}$ is the obvious simplicial functor, and

(iii) j' is induced by the functor which assigns to every object $\mathbf{n} \in \Delta^{\mathrm{op}}$ the forgetful functor $p \downarrow \mathbf{n} \to b\mathbf{C}$.

A straightforward calculation then yields that j' is L-cofinal and thus left cofinal [8, §6] and that therefore

$$\underleftarrow{\mathrm{holim}}^{db\mathbf{C}}(v^* c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}} \quad \text{and} \quad \underleftarrow{\mathrm{holim}}^{bd(p\downarrow -)}(w^* c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}}$$

are weakly equivalent. Furthermore, let $s: d\pi_0\mathbf{C} \to sd\pi_0\mathbf{C}$ and $\bar{w}: d(p\downarrow -) \to d\pi_0\mathbf{C}$ be the projections. Then it is not difficult to verify that, for every object $\mathbf{k}\in\Delta^{\mathrm{op}}$, the functor $j:(p\downarrow\mathbf{k})\to\mathbf{C}_k$ is L-cofinal and that therefore [8, 6.14], for every object $E\in sd\pi_0\mathbf{C}$, the functor j induces weak equivalences $(s\bar{w}_k)^{-1}E\to(s\bar{u}_k)^{-1}E$ and $(s\bar{w}_k\downarrow E)\to(s\bar{u}_k\downarrow E)$ and hence (7.2) a weak equivalence of cofibrant $sd\pi_0\mathbf{C}$-diagrams $(sw\downarrow -)\to(su\downarrow -)$. Finally [8, §6] the $d\pi_0\mathbf{C}$-diagram $s^*\underline{s}_*c_{d\pi_0\mathbf{C}}X$ is weakly equivalent to $c_{d\pi_0\mathbf{C}}X$ and it now follows from [8, 9.1 and 9.5] that j induces a weak equivalence between

$$\underleftarrow{\mathrm{holim}}^{bd\mathbf{C}}(u^*c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}} \quad\text{and}\quad \underleftarrow{\mathrm{holim}}^{bd(p\downarrow -)}(w^*c_{d\pi_0\mathbf{C}}X)^{\mathrm{f}}.$$

9. A generalization of the bar construction

9.1. *Introduction.* Given an integer $n\geq 0$, a sequence $M_0,\dots,M_n$ of simplicial monoids and a sequence $X_1,\dots,X_n$ of simplicial sets with a commuting right action of M_{i-1} and left action of M_i on each X_i, we describe a *bar construction* $B(M_0,X_1,M_1,\dots,M_n)$, which is a simplicial set with all of the following properties (which will be formulated more precisely in 9.2)

(i) *If $n=0$, then $B(M_0)$ is the classifying complex BM_0* (1.3(v)) (see 9.2(i)).

(ii) *The bar construction is natural in $M_0,X_1,M_1,\dots,M_n$* (see 9.2(ii)).

(iii) *The bar construction preserves weak equivalences* (9.2(vi)).

(iv) *The bar construction is natural in* $\mathbf{n}$ (9.2(iii)).

(v) *If $X_n=M_n$, with the obvious left M_n-action, then the projection*

$$B(M_0,X_1,M_1,\dots,M_n)\to B(M_0,X_1,M_1,\dots,M_{n-1})$$

is a weak equivalence (9.2(iv)).

(iv) *If $M_0,\dots,M_n$ are simplicial groups, then the projection*

$$B(M_0,X_1M_1,\dots,M_n)\to BM_0\times\cdots\times BM_n$$

is a fibration with $X_1\times\cdots\times X_n$ as fibre (9.2(v)).

(vii) *If the simplicial monoids $M_0,\dots,M_n$ are locally grouplike, then the fibre $X_1\times\cdots\times X_n$ of the projection $B(M_0,X_1,M_1,\dots,M_n)\to BM_0\times\cdots\times BM_n$ is also its homotopy fibre* (9.2(vii)).

9.2. *The bar construction.* Given $M_0,\dots,M_n$ and $X_1,\dots,X_n$ as above, define the *bar construction* $B(M_0,X_1,M_1,\dots,M_n)$ as the diagonal of the bisimplicial set, which in dimension $(*,k)$ consists of the nerve of the category which has as objects the n-tuples $(x_1,\dots,x_n)$, where each x_i is a k-simplex of X_i and as maps $(x_1,\dots,x_n)\to(x_1',\dots,x_n')$ between such objects the $(n+1)$-tuples $(m_0,\dots,m_n)$ such that each m_i is a k-simplex of M_i and $x_i'm_{i-1}=m_ix_i$ for $1\leq i\leq n$.

It is sometimes convenient to work with a slightly more general notion: Given a small simplicial category $\mathbf{C}$ and a functor $g:\mathbf{C}\to\mathbf{n}$ (4.5), let $B(\mathbf{C}/\mathbf{n})$ denote the diagonal of the bisimplicial set, which in dimension $(*,k)$ consists of the nerve of the category of k-dimensional cross sections of g (which has as objects the cross sections of $g_k : \mathbf{C}_k \to \mathbf{n}$ and as maps the natural transformations between them). If each of the subcategories $g^{-1}(i) \subset \mathbf{C}$ $(0 \leq i \leq n)$ contains exactly one object C_i, then $B(\mathbf{C}/\mathbf{n})$ is isomorphic to the bar construction $B(M_0, X_1, M_1, \dots, M_n)$ where each $M_i = \hom(C_i, C_i)$ and each $X_i = \hom(C_{i-1}, C_i)$. For this reason we will also call $B(\mathbf{C}/\mathbf{n})$ a *bar construction*. (Conversely, given $M_0, \dots, M_n$ and $X_1, \dots, X_n$ as above, it is easy to construct a simplicial category $\mathbf{C}$ together with a functor $\mathbf{C} \to \mathbf{n}$ such that $B(\mathbf{C}/\mathbf{n}) = B(M_0, X_1, M_1, \dots, M_n)$.)

This definition immediately implies:

(i) *If* $n = 0$, *then* $B(\mathbf{C}/\mathbf{0})$ *is just the diagonal of the nerve of* $\mathbf{C}$.

(ii) $B(\mathbf{C}/\mathbf{n})$ *is natural in* $\mathbf{C}$ i.e., a commutative diagram

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C}' \\ & \searrow \quad \swarrow & \\ & \mathbf{n} & \end{array}$$

induces a map $B(\mathbf{C}/\mathbf{n}) \to B(\mathbf{C}'/\mathbf{n}) \in \mathbf{S}$.

(iii) $B(\mathbf{C}/\mathbf{n})$ *is natural in* $\mathbf{n}$, i.e., a pullback diagram

$$\begin{array}{ccc} J^*\mathbf{C} & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow \\ \mathbf{k} & \xrightarrow{J} & \mathbf{n} \end{array}$$

gives rise to a map $B(\mathbf{C}/\mathbf{n}) \to B(J^*\mathbf{C}/\mathbf{k}) \in \mathbf{S}$.

Moreover, rather straightforward calculations yield

(iv) *Let* $J : \mathbf{n-1} \to \mathbf{n}$ *be the functor given by* $Ji = i$ *for* $0 \leq i \leq n-1$ *and assume that, for every integer* $k \leq 0$, *the inclusion* $g_k^{-1}(n) \to \mathbf{C}_k$ *has a left adjoint. Then the projection* $B(\mathbf{C}/\mathbf{n}) \to B(J^*\mathbf{C}/\mathbf{n-1})$ *is a weak equivalence.*

(v) *Suppose that each* $g^{-1}(i)$ *is a simplicial groupoid, let* $C_i \in g^{-1}(i)$ *be an object* $(0 \leq i \leq n)$, *let* $M_i = \hom(C_i, C_i)$ *and let* $X_i = \hom(C_{i-1}, C_i)$. *Then the projection* $B(M_0, X_i, \dots, Mn) \to BM_0 \times \cdots \times BM_n$ *is a fibration with* $X_1 \times \cdots \times X_n$ *as fibre.*

More difficult to prove is:

(iv) Let

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{h} & \mathbf{C}' \\ & \searrow \quad \swarrow & \\ & \mathbf{n} & \end{array}$$

be a commutative diagram such that h is a weak equivalence, i.e., [6, §2] $\pi_0 h$ *is an equivalence of categories and h induces, for every two objects* $C_1, C_2 \in \mathbf{C}$, *a weak equivalence* $\hom(C_1, C_2) \to \hom(hC_1, hC_2)$. *Then h induces a weak equivalence of* *bar constructions* $B(\mathbf{C}/\mathbf{n})\to B(\mathbf{C}'/\mathbf{n})$. This is clear if h is an equivalence of simplicial categories and [5, 5.7] if h is 1-1 and onto on objects. The general case now follows readily from the existence (in the notation of [4]) of the commutative diagram

$$\begin{array}{ccccc} \mathbf{C} & \longleftarrow & F_*\mathbf{C} & \longrightarrow & F_*\mathbf{C}[F_*\mathbf{W}^{-1}] \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}' & \longleftarrow & F_*\mathbf{C}' & \longrightarrow & F_*\mathbf{C}'[FW'^{-1}] \end{array}$$

in which $\mathbf{W}\subset\mathbf{C}$ and $\mathbf{W}'\subset\mathbf{C}'$ are the maximal full subcategories such that $\pi_0\mathbf{W}$ and $\pi_0\mathbf{W}'$ are groupoids and in which the horizontal maps are 1-1 and onto on objects as well as weak equivalences [4, 2.6 and 5.1] and the map on the right is an equivalence of simplicial categories. Moreover combination of this argument with (v) yields

(vii) *Let each* $\pi_0 g^{-1}(i)$ *be a groupoid, let* $C_i\in g^{-1}(i)$ *be an object* $(0\le i\le n)$, *let* $M_i=\hom(C_i,C_i)$ *and let* $X_i=\hom(C_{i-1},C_i)$. *Then the fibre* $X_1\times\cdots\times X_n$ *of the projection* $B(M_0,X_1,M_1,\ldots,M_n)\to BM_0\times\cdots\times BM_n$ *is also its homotopy fibre.*

In general it is difficult to describe the homotopy fibre of the projection $B(\mathbf{C}/\mathbf{n})\to Bg^{-1}(0)\times\cdots\times Bg^{-1}(n)$. However a nice solution exists in the following useful

9.3. *Special case.* Let n be an integer ≥ 0 and $Y\in\mathbf{S}^{\mathbf{n}}$. Then clearly *the 0-classification complex of Y is of the form* $c^0Y=B(\mathbf{C}/\mathbf{n})$ for a functor $g:\mathbf{C}\to\mathbf{n}$ such that each $g^{-1}(i)$ is the ordinary category of all weak equivalences between simplicial sets weakly equivalent to $Y(i)$, which is clearly not a groupoid. Moreover (in the notation of 5.1) c^0Y *is weakly equivalent to the bar construction*

$$B(\operatorname{haut} Y(0)^{\mathrm f}, \hom(Y(0)^{\mathrm f}, Y(1)^{\mathrm f}), \operatorname{haut} Y(1)^{\mathrm f},\ldots,\operatorname{haut} Y(n)^{\mathrm f})$$

in a manner which is natural in Y as well as in $\mathbf{n}$.

To prove this let $\mathbf{S}_*$ denote the simplicial category of simplicial sets and $\mathbf{s}_*^{\mathrm f}\subset\mathbf{S}_*$ the full simplicial subcategory spanned by the fibrant objects, and note that their k-dimensional categories come with obvious notions of weak equivalences. The diagam Y therefore gives rise to corresponding 0-classification complexes c_k^0Y and $c_k^0Y^{\mathrm f}$ and one readily verifies that the obvious maps $c^0Y\to c_k^0Y$ and $c_k^0Y^{\mathrm f}\to c_k^0Y$ and hence the resulting maps $c^0Y\to\operatorname{diag} c_*^0Y$ and $\operatorname{diag} c_*^0Y^{\mathrm f}\to\operatorname{diag} c_*^0Y$ are weak equivalences [6, 7.3]. Moreover $\operatorname{diag} c_*^0Y^{\mathrm f}=B(C_*^{\mathrm f}/n)$ for an obvious function $g_*^{\mathrm f}:\mathbf{C}_*^{\mathrm f}\to\mathbf{n}$ and the desired result now follows from the fact that the inclusion $\mathbf{C}^{\mathrm f}\subset\mathbf{C}_*^{\mathrm f}$ of the full simplicial category spanned by the objects $Y(i)^{\mathrm f}$ $(0\le i\le n)$, is a weak equivalence.

9.4. Remark. Given a sequence $M_0,\ldots,M_n$ of simplicial monoids and a sequence $X_1,\ldots,X_n$ of simplicial sets with a commuting right action of M_{i-1} and left action of M_i on each X_i, one can also consider the *'classical' bar construction*, i.e., the simplicial set which has as k-simplices the ordered $(kn+k+n)$-tuples of k-simplices

$$(m_0^1, \ldots, m_0^k, x_1, m_1^1, \ldots, m_1^k, \ldots, m_n^1, \ldots, m_n^k)$$

where each $m_i^j \in M_i$ and each $x_i \in X_i$ and it is not difficult to verify the properties listed in 9.1, except that the naturality in $\mathbf{n}$ only holds with respect to non-degenerate (i.e., 1-1) functors. Moreover this classical bar construction is *weakly equivalent* to the one considered above, in a manner which is natural in $M_0, X_1, M_1, \ldots, M_n$ as well as in $\mathbf{n}$ (with respect to non-degenerate functors only, of course). One can prove this using [5, 9.5]. However, in the important case that the M_i are locally grouplike this follows easily from the properties listed in 9.1 and the fact that both bar constructions are isomorphic whenever the M_i are actually simplicial groups.

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