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Topological Hochschild homology of ring functors and exact categories

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Abstract

In analogy with Hochschild–Mitchell homology for linear categories topological Hochschild and cyclic homology (THH and TC) are defined for *ring functors* on a category $\mathscr{C}$. Fundamental properties of THH and TC are proven and some examples are analyzed. A special case of a ring functor on an exact category $\mathfrak{C}$ is treated separately, and is compared with algebraic K-theory via a Dennis–Bökstedt trace map. Calling THH and TC applied to these ring functors simply $THH(\mathfrak{C})$ and $TC(\mathfrak{C})$, we get that the iteration of Waldhausen's S construction yields spectra $\{THH(S^{(n)}\mathfrak{C})\}$ and $\{TC(S^{(n)}\mathfrak{C})\}$, and the maps from K-theory become maps of spectra. If $\mathfrak{C}$ is split exact, the THH and TC spectra are Ω-spectra. The inclusion by degeneracies $THH_0(S^{(n)}\mathfrak{C}) \subseteq THH(S^{(n)}\mathfrak{C})$ is a stable equivalence, and it is shown how this leads to a weak resolution theorem for THH. If $\mathscr{P}_A$ is the category of finitely generated projective modules over a unital and associative ring A, we get that $THH(A) \xrightarrow{\simeq} THH(\mathscr{P}_A)$ and $TC(A) \xrightarrow{\simeq} TC(\mathscr{P}_A)$.

The Dennis trace map from algebraic K-theory of a ring generated an interest for Hochschild homology from a K-theoretic point of view. Bökstedt defined in [2] a factorization of the Dennis trace through a theory reminiscent of Hochschild homology, but with a much richer structure. He also made calculations on examples which gave new K-theoretic information, and called the new theory *topological Hochschild homology* (THH). The factorization had earlier been suggested by Goodwillie, and it is known to coincide with the linearization of K-theory in the sense of calculus of functors (see [7], and also the program of R. Schwänzl, R. Staffeldt and F. Waldhausen).

As defined by Bökstedt, topological Hochschild homology comes equiped with a cyclic action. This cyclic action plays an important role, e.g. in the work of Bökstedt et al. [3] on the K-theory analogue of the Novikov conjecture. In fact, the map from

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K-theory factors through the fixed points of the various operations on THH arising from this cyclic structure, and comparison with K-theory should take this into account.

Algebraic K-theory is defined in terms of exact categories (or more generally by categories with cofibrations and weak equivalences), and in this paper we adopt the point of view that THH should be defined on this level. This has both advantages and disadvantages. The most obvious disadvantage is that the theory becomes more involved. One of the advantages is that the additivity of the category can be used constructively on the theory itself. Another advantage is that the map from K-theory is very transparent. Pirashivili and Waldhausen [12] give a model for THH of rings which is defined in terms of the category of finitely generated projective modules over the ring. A variant of this model proved itself useful in [7] where both the advantages mentioned above were crucial tools. This only disadvantage of these constructions is that they admit no interesting cyclic action.

The present paper aims at giving a model for THH which combines the categorical outlook with the presence of the cyclic actions.

This paper is somewhat encyclopedic, but is meant to contain the results needed for the computations on exact categories in Chapter 2 (all but one of the general calculations are needed at some point in this example), as well as a series of intended future applications. Most notably, McCarthy has already proven that the result of Goodwillie of nilpotent extensions is true when reinterpreted properly. The setting should also be general enough to allow for applications. directed towards algebraic K-theory of spaces, and so the "linear" viewpoint of [7] cannot be adopted.

Thus the first goal should be: Define a topological Hochschild homology which meets the following requirements,

(1) It should agree with the definition of Bökstedt if given the same input.

(2) It should be defined in terms of categories and generally be equipped with an appropriate cyclic structure. Rationally it should agree with a Hochschild-type homology.

(3) It should make sense to "mix" K-theory and topological Hochschild homology, and the iteration of the S construction should yield a spectrum corresponding to the K-theory spectrum.

(4) It should respect "Morita equivalence". In particular, if A is a ring and $\mathscr{P}_A$ is the category of finitely generated projective A modules, then the theory applied to $\mathscr{P}_A$ should correspond to the theory applied to A itself.

(5) There should be a simple map from algebraic K-theory mapping into the fixed points of the cyclic actions, and agreeing with the cyclotomic trace. The map should be transparent enough to allow simple comparisons.

Most of the paper is devoted to showing that we indeed have a well defined theory satisfying these demands.

In order to do so, we introduce the concept of a *ring functor* on a category $\mathscr{C}$ which is roughly an FSP (Functor with Smash Product) with several objects (namely the ones in $\mathscr{C}$). When this is done, the construction of THH meeting (1) and (2) is straightforward (up to a choice or two).

To demonstrate that THH obeys the remaining demands we must prove a number of standard theorems. For instance, THH must commute with products, behave well under formation of matrices ("Morita equivalence") and operations similar to the S construction ("triangular matrices"). This is proved directly by means of displaying actual homotopies between suitably simplified models.

Just as for FSPs, our model possesses a "Frobenius action" (see 1.5), and we define the topological cyclic homology TC (as in [3]), as the inverse homotopy limit under the inclusion of fixed points and the Frobenius maps.

The example treated in Chapter 2 is that of exact categories. More precisely: suppose $\mathfrak{C}$ is an exact category. Then we may consider the ring functor on $\mathfrak{C}$ which sends a pair $(a, b) \in \mathfrak{C}^0 \times \mathfrak{C}$ and a pointed simplicial set X to $\mathfrak{C}(a,b) \otimes \mathbf{Z}[X]/\mathbf{Z}[*]$ where $\mathbf{Z}[X]$ is the free simplicial abelian group on X. This makes THH and TC into functors from the category of exact categories. More generally, if $S\mathfrak{C}$ is the S construction of Waldhausen [17], we may apply THH or TC to $S\mathfrak{C}$. These theories are theoretically better behaved than simply applying the functor directly to the categories. In this case various simplifications are possible. For one thing we show directly that the models used in [7] agree with the present definitions. We note that, in accordance with the general idea, some typical K-theoretic properties such as confinality are obeyed by THH. Another point worth mentioning is that the model for THH of an exact category is actually dependent upon the choice of exact sequences, but that if we apply THH to the underlying additive category (with just s plit exact sequences) then this theory agrees with the definition of the homology of a category as in [1]. The proof that $THH(A) \simeq THH(\mathscr{P}_A)$ could have been varied by the use of a construction which associates an FSP to any ring functor in a manner keeping THH unchanged.

Summarizing, the main results of Chapter 2 are

(1) There is a simple map from K-theory of an exact category $\mathfrak{C}$ into $\Omega THH(S\mathfrak{C})$, mapping into the fixed points of both the cyclic actions and the Frobenius maps, and hence there is a lifting to $\Omega TC(S\mathfrak{C})$. This forms a map of spectra upon iterating the S construction.

(2) For split exact categories $THH(\mathfrak{C})$ and $\Omega THH(S\mathfrak{C})$ are equivalent (likewise for TC).

(3) For additive categories the homology of the category $\mathfrak{C}$ itself, $H_k(\mathfrak{C}, \mathfrak{C})$, is isomorphic to $\pi_k THH(\mathfrak{C})$.

(4) The model for THH in [7] is equivalent to the present via a simple map.

(5) For a ring A we have an equivalence $THH(A) \simeq THH(\mathscr{P}_A)$ (likewise for TC).

(6) A weak version of the resolution theorem is valid for THH.

1. Topological Hochschild homology of ring functors on a category

1.0. Introduction

Analogously to Bökstedt's definition of a functor with smash product (FSP) we introduce the notion of a ring functor on a category. Roughly the difference is the same as the difference between a ring with unit and a linear category. That is, a ring functor on a category with only one object is an FSP. We will also need the notion of a module over a ring functor. Important examples will be given in the following section. One may also define the topological Hochschild homology (THH) for ring functors and the remainder of the chapter is dedicated to establishing the basic properties of THH. It turns out that most of what you can do for FSPs you can do for ring functors. The main interest in this generalization is that the comparison with K-theory becomes simpler. For instance in the case of rings, the point is that instead of working with the group completion of spaces of matrices mapping by devious routes into the topological Hochschild homology of the ring, we may now consider a simple map into the topological Hochschild homology of the projective modules, or even better into a mixing of THH and algebraic K-theory. The goal of this chapter is to prove that the range of such maps exists and is well behaved. We will come back to the question whether the theory is comparable (hint: it is!) with the classical case in the text chapter.

The model we present comes with all the usual cyclic structure, and will take pains to show when this structure is preserved. We do this in a somewhat unusual way by restricting our attention to simpler models for THH not containing degeneracies. This requires some care, but the gain in having simple model outweighs the disadvantages as there is essentially only a single trick (more precisely: Lemma 1.5.12) you have to apply repeatedly.

The main results in this chapter are:

(1) For any ring functor A and A-bimodule P the $THH(A,P)$ is a well defined object, with cyclic structure if $P = A$.

(2) Rationally $THH(A,P)$ is equivalent to an additive cyclic nerve.

(3) THH is well behaved under natural isomorphisms and equivalences, respects direct limits, stable equivalences and preserves products.

(4) Morita equivalence is true for THH.

(5) THH may be calculated by means of THH of an FSP, but **not** functorially.

As to the last point it should be noted that the FSP in question rarely will be of a sort which invites closer analysis.

1.0.1. Some language. By a *linear category* we mean a category whose morphism sets are abelian groups and whose composition is bilinear, or what is often called a pre-additive category. Given any category $\mathscr{C}$ we may from the free linear category on $\mathscr{C}$, which we will call $\mathbf{Z}\mathscr{C}$, with the same objects as $\mathscr{C}$, but with morphism sets

$\mathbf{Z}\mathscr{C}(a,b) = \mathbf{Z}[\mathscr{C}(a,b)]$: the free abelian group on the set $\mathscr{C}(a,b)$ of morphisms from a to b in $\mathscr{C}$. An *additive category* is a linear category with a zero object and finite products.

Given two categories $\mathscr{C}$ and $\mathscr{D}$ we will let $\mathscr{D}^{\mathscr{C}}$ denote the category of functors from $\mathscr{C}$ to $\mathscr{D}$ and natural transformations. Let $\mathscr{C}$ be a category. We will write $\mathscr{C}^{\mathrm{o}}$ for the opposite category; i.e. the category with the same objects as $\mathscr{C}$ but with all arrows reversed. We will write $\mathscr{C}(-,-)$ for the functor from $\mathscr{C}^{\mathrm{o}} \times \mathscr{C}$ assigning to each pair of objects a and b in $\mathscr{C}$ the set of morphisms from a to b. For convenience in these notes we will often write $T_0\mathscr{C}$ for $\mathscr{C}^{\mathrm{o}} \times \mathscr{C}$. Let $T_1\mathscr{C}$ be the full subcategory of $T_0\mathscr{C} \times T_0\mathscr{C}$ with objects of the form $((a,b),(c,a))$ and let composition be the functor $T_1\mathscr{C} \to T_0\mathscr{C}$ induced by $((a,b),(c,a)) \mapsto (a,b) \circ (c,a) = (c,b)$. In view of the convention we will often write objects in $T_0\mathscr{C} = \mathscr{C}^{\mathrm{o}} \times \mathscr{C}$ by single letters like f and think of them as arrows. More generally we let $T_k\mathscr{C}$ be the full subcategory of $(\mathscr{C}^{\mathrm{o}} \times \mathscr{C})^{k+1}$ with objects on the form $((c_1,c_0),(c_2,c_1),\ldots,(c_{k+1},c_k))$ (called composable $k+1$-tuples). This notation is inspired by the twisted arrow category $T\mathscr{C}$ whose objects are the morphisms of $\mathscr{C}$ and where an arrow from $f: a \to b$ to $g: c \to d$ is a commutative diagram

$$\begin{array}{ccc} a & \longleftarrow & c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ b & \longrightarrow & d \end{array}$$

The forgetful functor $T\mathscr{C} \to T_0\mathscr{C}$ sends a morphism $f: a \to b$ to (a,b). The definitions of ring functors and modules are based on $T_0\mathscr{C}$, but could equally well (perhaps better) have used $T\mathscr{C}$; however at current all important examples are covered by the present definition.

1.0.2. Simplicial objects. Let Δ be a category of standard ordered finite sets $[n] = \{0 < 1 < \cdots < n\}$ and monotone maps. A simplicial object in a category $\mathscr{C}$ is a functor from Δ^{o} to $\mathscr{C}$. The category of simplicial objects in $\mathscr{C}$ will be denoted $s\mathscr{C}$. We will let $s_*\mathscr{E}ns$ (resp. $fs_*\mathscr{E}ns$) denote the category of pointed simplicial sets (resp. finite pointed simplicial sets). $\Delta(n)$ denotes the pointed simplicial set $\{[q] \mapsto \Delta([q],[n])\}$ pointed at the zero map. As models for the spheres we will choose $S^n = \bigwedge_{n\text{-fold}} S^1$ where $S^1 = \Delta(1)/\partial\Delta(1)$. Unless otherwise stated, by "$\Omega^n X$" we will mean $\mathrm{sing}|X|^{|S^n|}$ where $|\ |$ is the realization and "sing" is the singular complex of a topological space. Given any simplicial set X we will write $\mathbf{Z}[X]$ for the simplicial abelian group which in degree q is $\mathbf{Z}[X_q]$, the free abelian group on X_q. If X is pointed, $\tilde{\mathbf{Z}}[X]$ denotes the quotient $\mathbf{Z}[X]/\mathbf{Z}[*]$. We will say that a pointed simplicial set X is n-connected if $\pi_q(X) = 0$ for all $q \le n$, and if $n \ge 0$ that a map $f: X \to Y$ is n-connected if the homotopy fiber is $n-1$ connected and $\pi_0(X) \to \pi_0(Y)$ is surjective. A pointed simplicial set X is n-reduced if $X_q = *$ for all $q \le n$.

1.0.3. Simplicial categories. A simplicial category is an object of the functor category from Δ^{o} to some category of categories. A simplicial functor is a natural transformation between two simplicial categories. If a pointed simplicial category $\mathscr{C}$ has a representation of the functor $s_*\mathscr{E}ns(X,\mathscr{C}(c,-))$, say a natural equivalence $\mathscr{C}(X\otimes c,-)\cong s_*\mathscr{E}ns(X,\mathscr{C}(c,-))$ where $c\in\mathscr{C}$ and $X\in s_*\mathscr{E}ns$, we will say that $\mathscr{C}$ has products with pointed simplicial sets. Likewise we say that $\mathscr{C}$ has products with finite pointed simplicial sets if $s_*\mathscr{E}ns(X,\mathscr{C}(c,-))$ has a representation for every $X\in fs_*\mathscr{E}ns$.

Suppose that the pointed simplicial categories $\mathscr{C}$ and $\mathscr{D}$ have products with (finite) pointed simplicial sets. Then any pointed simplicial functor $F':\mathscr{C}\to\mathscr{D}$ has associated to it a natural transformation $\lambda_{X,c}:X\otimes F(c)\to F(X\otimes c)$ induced by the identity via

$$\begin{array}{ccc}
id\in\mathscr{C}(X\otimes c,X\otimes c) & \xrightarrow{\cong} & s_*\mathscr{E}ns(X,\mathscr{C}(c,X\otimes c)) \\
 & & \downarrow \\
\lambda_{X,c}\in\mathscr{D}(X\otimes F(c),F(X\otimes c)) & \xleftarrow{\cong} & s_*\mathscr{E}ns(X,\mathscr{D}(F(c),F(X\otimes c)))
\end{array}$$

In particular case of the pointed simplicial category $s_*\mathscr{E}ns^{\mathscr{B}}$ where $\mathscr{B}$ is any category we have a product defined for any $G\in s_*\mathscr{E}ns^{\mathscr{B}}$ and $X\in s_*\mathscr{E}ns$ by sending $b\in\mathscr{B}$ to $(X\wedge G)(b)=X\wedge G(b)$.

1.1. Ring functors on a category

In analogy with the definitions of an FSP (see [2], [3], [8] or [12]) we now define ring functors on a category.

1.1.1. Definition. For $\mathscr{C}$ a small category, let $L\mathscr{C}$ be the full subcategory of the category of pointed simplicial functors $F:fs_*\mathscr{E}ns\to s_*\mathscr{E}ns^{T_0\mathscr{C}}$ such that if X is n-connected and $f=(a,b)\in T_0\mathscr{C}=\mathscr{C}^{o}\times\mathscr{C}$

(1) $F(X)(f)$ is n-connected,

(2) the map $S^1\wedge F(X)(f)\to F(S^1\wedge X)(f)$ induced from the simplicial structure is $2n-c$ connected for some number c not depending on X.

We will denote the natural transformation due to the simplicial structure discussed above by

$$\lambda_{X,Y}:X\wedge F(Y)\to F(X\wedge Y)$$

where $X\wedge F(Y)$ sends $f\in T_0\mathscr{C}$ to $X\wedge F(Y)(f)$.

Remark. One may weaken condition (1) to require that $F(X)(a,b)$ is $n-d^F(a)+c^F(b)$-connected, where $d^F(a)$ and $c^F(b)$ are non-negative numbers depending only on a and b. d^F may be thought of as dimension and c^F as connectivity. This weakening is useful e.g. if $\mathscr{C}$ is some category of spaces.

A spectrum (often called pre-spectrum) is a sequence of spaces X^m with maps $S^1\wedge X^m\xrightarrow{f^m}X^{m+1}$, and an Ω-spectrum is a spectrum where the adjoints of the structure maps yield homotopy equivalences $X^m \simeq \Omega X^{m+1}$. A map of spectra is a stable weak equivalence if it induces an isomorphism on all homotopy groups (as spectra). We may consider F as a functor from $T_0\mathscr{C}$ to spectra via $\underline{\underline{F}}(f) = \{m \mapsto F^m(f) = F(S^m)(f)\}$ with structure maps λ_{S^1,S^m}.

A morphism $\phi: F \to G$ in $L\mathscr{C}$ is called a stable equivalence if it induces a stable weak equivalence $\underline{\underline{F}}(f) \to \underline{\underline{G}}(f)$ for each $f \in T_0\mathscr{C}$.

1.1.2. Definition (*Ring functors*). Let $\mathscr{C}$ be a small category. A ring functor on $\mathscr{C}$ is an object $A \in L\mathscr{C}$ together with a natural transformation which we will call the multiplication

$$\mu_{X,Y}(f,g): A(X)(f) \wedge A(Y)(g) \to A(X \wedge Y)(f \circ g)$$

for every composable pair $(f,g) \in T_1\mathscr{C}$, such that multiplication μ is strictly associative. More precisely

$$\begin{array}{ccc} A(X)(f)\wedge A(Y)(g)\wedge A(Z)(h) & \xrightarrow{\mu\wedge id} & A(X\wedge Y)(f\circ g)\wedge A(Z)(h) \\ \downarrow{\scriptstyle id\wedge\mu} & & \downarrow{\scriptstyle \mu} \\ A(X)(f)\wedge A(Y\wedge Z)(g\circ h) & \xrightarrow{\mu} & A(X\wedge Y\wedge Z)(f\circ g\circ h) \end{array}$$

commutes for all $X, Y, Z \in fs_*\mathscr{E}ns$ and composable triples (f, g, h).

To be entirely clear: μ is a natural transformation from the composite

$$fs_*\mathscr{E}ns \times fs_*\mathscr{E}ns \xrightarrow{A\times A} s_*\mathscr{E}ns^{T_0\mathscr{C}} \times s_*\mathscr{E}ns^{T_0\mathscr{C}} \xrightarrow{\wedge^{\Rightarrow}} s_*\mathscr{E}ns^{T_1\mathscr{C}}$$

to

$$fs_*\mathscr{E}ns \times fs_*\mathscr{E}ns \xrightarrow{\wedge} s_*\mathscr{E}ns \xrightarrow{A} s_*\mathscr{E}ns^{T_0\mathscr{C}} \to s_*\mathscr{E}ns^{T_1\mathscr{C}}$$

where the latter functor is induced by the composition $T_1\mathscr{C} \to T_0\mathscr{C}$.

There is a particularly important ring functor

$$S: fs_*\mathscr{E}ns \to s_*\mathscr{E}ns^{T_0\mathscr{C}}$$

namely the functor sending $X \in s_*\mathscr{E}ns$ to the constant functor $f \in T_0\mathscr{C} \mapsto X$. The multiplication is simply the identity. We will call this ring functor the identity ring functor S. One may think of this ring functor as the analogue of the ring of integers.

If $\mathscr{C}$ is a category, we consider the associated discrete subcategory $d\mathscr{C} \subseteq \mathscr{C}$ consisting of all objects in $\mathscr{C}$, but only the identity morphisms. There is a functor $D: d\mathscr{C} \to T_0\mathscr{C}$ given by the diagonal. Hence there is a functor $s_*\mathscr{E}ns^{T_0\mathscr{C}} \xrightarrow{D^*} s_*\mathscr{E}ns^{d\mathscr{C}}$.

1.1.3. Definition (*Unital ring functors*). Let A be a ring functor on $\mathscr{C}$, and let as above S be the identity ring functor on $\mathscr{C}$. We say that A is a ring functor with unit on $\mathscr{C}$, or simply that A is unital, if there is a natural transformation $\mathbf{1}$ from $D^* \circ S$ to $D^* \circ A$ such that for every $X, Y \in fs_*\mathscr{E}ns$ and $f = (a, c) \in T_0\mathscr{C}$

$$\begin{array}{ccc}
X \wedge Y & \xrightarrow{\mathbf{1}_X(c)\wedge \mathbf{1}_Y(c)} & A(X)(id_c)\wedge A(Y)(id_c) \\
 & \searrow^{\mathbf{1}_{X\wedge Y}(c)} & \downarrow{\scriptstyle \mu_{X,Y}(id_c,\, id_c)} \\
 & & A(X\wedge Y)(id_c)
\end{array}$$

$$\begin{array}{ccc}
X \wedge A(Y)(f) & \xrightarrow{\mathbf{1}_X(c)\wedge id} & A(X)(id_c)\wedge A(Y)(f) \\
 & \searrow^{\lambda_{X,Y}(f)} & \downarrow{\scriptstyle \mu_{X,Y}(id_c,\, f)} \\
 & & A(X\wedge Y)(f)
\end{array}$$

commute, and likewise for $\mu \circ (id \wedge \mathbf{1})$ up to a switch of factors.

The reader may wonder why one has to restrict to the discrete subcategory in order to define the unit, but the reason is natural enough: only when range and target coincides should there be a distinguished "identity morphism".

In particular the identity ring functor S on $\mathscr{C}$ is unital.

1.1.4. Change of underlying category. Given a small category $\mathscr{C}$ we let $\mathscr{F}\mathscr{C}$ (resp. $\mathscr{F}\mathscr{C}^u$) be the category with objects ring functors (resp. unital ring functors) on $\mathscr{C}$ and morphisms transformations in $L\mathscr{C}$ compatible with the multiplicative structure (and unit).

If $\phi : \mathscr{C} \to \mathscr{D}$ is a functor we define $\phi^* : \mathscr{F}\mathscr{D} \to \mathscr{F}\mathscr{C}$ by composition. That is, if A is a ring functor on $\mathscr{D}$ we let $\phi^* A$ be the composite of $A : fs_*\mathscr{E}ns \to s_*\mathscr{E}ns^{\mathscr{D}^o \times \mathscr{D}}$ and $\phi^* : s_*\mathscr{E}ns^{\mathscr{D}^o \times \mathscr{D}} \to s_*\mathscr{E}ns^{\mathscr{C}^o \times \mathscr{C}}$. This clearly is also well defined in the unital case.

Isomorphic functors $\phi \cong \psi : \mathscr{C} \to \mathscr{D}$ induced isomorphic functors $\phi^* \cong \psi^*$. It is worthwhile to spell this out explicitly: assume $\eta : \phi \to \psi$ is an isomorphism. Then

$$A(X)(\eta(a)^{-1}, \eta(b)) : A(X)(\phi(a), \phi(b)) \to A(X)(\psi(a), \psi(b))$$

defines the desired isomorphism between $\phi^* A$ and $\psi^* A$. As this isomorphism is defined by isomorphisms in $\mathscr{D}$ it carries over to morphisms between ring functors and we get that ϕ^* and ψ^* are isomorphic as functors.

Perhaps a note as to how to define ring functors over the twisted arrow category is appropriate here. The definition of ring functors then appears exactly as above, exchanging $T_0\mathscr{C} = \mathscr{C}^o \times \mathscr{C}$ with $T\mathscr{C}$ everywhere (the exposition was originally intended this way). As to the unital case, use the functor $d\mathscr{C} \to T\mathscr{C}$ sending an object to its identity map, and an identity map to the corresponding square with all sides the identity. Over $T\mathscr{C}$, a unital ring functor will then require a natural transformation from the identity functor under restriction to $d\mathscr{C}$.

1.1.5. Modules of ring functors

1.1.6. Definitions (*Modules*). Let $\mathscr{C}$ be a category, A be a ring functor on $\mathscr{C}$ and $P \in L\mathscr{C}$. A left module structure on P is a natural transformation

$$\ell_{X,Y}(f,g): A(X)(f) \wedge P(Y)(g) \to P(X \wedge Y)(f \circ g)$$

for any composable pair $(f,g) \in T_1\mathscr{C}$, such that

$$\begin{array}{ccc} A(X)(f)\wedge A(Y)(g)\wedge P(Z)(h) & \xrightarrow{\mu\wedge id} & A(X\wedge Y)(f\circ g)\wedge P(Z)(h) \\ \downarrow{\scriptstyle id\wedge\ell} & & \downarrow{\scriptstyle \ell} \\ A(X)(f)\wedge P(Y\wedge Z)(g\circ h) & \xrightarrow{\ell} & P(X\wedge Y\wedge Z)(f\circ g\circ h) \end{array}$$

commutes. If A is unital, we define a unital left module structure on P as a left module structure on P such that

$$\begin{array}{ccc} X\wedge P(Y)(f) & \xrightarrow{1_X(c)\wedge id} & A(X)(id_c)\wedge P(Y)(f) \\ & \searrow^{\lambda_{X,Y}(f)} & \downarrow{\scriptstyle \ell_{X,Y}(id_c,f)} \\ & & P(X\wedge Y)(f) \end{array}$$

commutes. A structure of a (unital) right module is defined similarly.

A bimodule structure on P (resp. unital bimodule structure in the case A is unital) consists of compatible left and right module structures on P (resp. compatible unital left and right module structures); i.e., for all composable triples (f,g,h) the following diagram commutes:

$$\begin{array}{ccc} A(X)(f)\wedge P(Y)(g)\wedge A(Z)(h) & \xrightarrow{id\wedge r} & A(X)(f)\wedge P(X\wedge Y)(g\circ h) \\ \downarrow{\scriptstyle r\ell\wedge id} & & \downarrow{\scriptstyle \ell} \\ A(X\wedge Y)(f\circ g)\wedge A(Z)(h) & \xrightarrow{r} & P(X\wedge Y\wedge Z)(f\circ g\circ h) \end{array}$$

We will refer to such P together with this structure as an A bimodule.

Note that all $T \in L\mathscr{C}$ are automatically unital S bimodules due to the simplicial structure.

In the case of the extension where the connectivity assumption (1) in the definition of $L\mathscr{C}$ was weakened we will in addition require of an A module P that for all $a \in \mathscr{C}$ we have that $d^P(a) \le d^A(a)$. The extension to the twisted arrow category is straightforward. If $\phi : \mathscr{D} \to \mathscr{C}$ is any functor we define the ϕ^*A module ϕ^*P in the same manner as ϕ^*A with the obvious actions.

1.1.7. Notation. If A is a ring functor on some category $\mathscr{C}$ we will often write $A^X(c,d)$ instead of $A(X)(c,d)$ when the emphasis is rather on the category than on the simplicial set. In particular, we write $A^n(c,d)$ for $A(S^n)(c,d)$. Similarly, if P is an A module we write $P^X(c,d)$ for $P(X)(c,d)$ and $P^n(c,d)$ for $P(S^n)(c,d)$.

1.2. Examples

We now list some useful ring functors. Sections 1.2.1–1.2.6 contain the most important ones. After that things develop into more of a bestiary, listed here only to have a convenient place to refer back to as the various constructions are needed later on. Many of the examples have analogues for bimodules, but as the list is already long, we leave that to the interested reader.

From the examples below one might be tempted to call ring functors on a category "FSPs with several objects" in analogy with "linear categories are rings with several objects" [11]. Note, however, that a ring functor on $\mathscr{C}$ is not determined by its underlying category. For instance, a simplicial category gives rise to (at least) two ring functors on $\mathscr{C}$ via examples 1.2.2 and 1.2.3.

1.2.1. FSPs. Any FSP A gives rise to a "constant" ring functor on a given category $\mathscr{C}$ via $A(X)(a,b) = A(X)$. If $\mathscr{C}$ is the trivial category, the notions of FSPs and unital ring functors on $\mathscr{C}$ naturally coincide. Note that if A is a unital ring functor on a one point category it is simply an FSP with extra structure. In fact, the natural examples of FSPs come with such an extra structure (see Examples 1.2.2 and 1.2.3 restricted to one point categories). In these notes we call ring functors on one point categories simply FSPs whether they have units or not.

1.2.2. The half smash ring functor on a simplicial category. Let $\mathscr{C}$ be a simplicial category. Then $X \mapsto \mathscr{C}(-,-)_+ \wedge X$ is a unital ring functor. The multiplication is induced by composition via

$$(\mathscr{C}(a,b)_+ \wedge X) \wedge (\mathscr{C}(c,a)_+ \wedge Y)$$
$$\cong (\mathscr{C}(a,b) \times \mathscr{C}(c,a))_+ \wedge (X \wedge Y) \to \mathscr{C}(c,b)_+ \wedge (X \wedge Y)$$

and the unit by the inclusion $X \xrightarrow{x \mapsto id_a \wedge x} \mathscr{C}(a,a)_+ \wedge X$. Later we will show that the homotopy type of the topological Hochschild homology of this ring functor is simply the stabilization of the cyclic nerve of $\mathscr{C}$. When M is a simplicial monoid, regarded as a category with only one object in each dimension, we recover the FSP of [2] associated to the monoid.

1.2.3. The linear ring functor on a linear category. Let $\mathscr{C}$ be a (simplicial) linear category. Then

$$X \mapsto X \mapsto \mathscr{C}(-,-) \otimes_Z \tilde{Z}[X]$$

is a unital ring functor on $\mathscr{C}$ where $\tilde{\mathbf{Z}}[X] = \mathbf{Z}[X]/\mathbf{Z}[*]$. The multiplication is given by sending smash to tensor followed by composition:

$$(\mathscr{C}(a,b) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X]) \wedge (\mathscr{C}(c,a) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[Y])$$
$$\to (\mathscr{C}(a,b) \otimes \mathscr{C}(c,a)) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X \wedge Y] \to \mathscr{C}(c,b) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X \wedge Y]$$

and the unit by the inclusion $X \xrightarrow{x \mapsto id_a \otimes 1 \cdot x} \mathscr{C}(a,a) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X]$.

The above construction extends to all categories $\mathscr{C}$ by passing to the free linear category $\mathbf{Z}\mathscr{C}$, or more directly consider the ring functor on $\mathscr{C}$ (which also is a ring functor on $\mathbf{Z}\mathscr{C}$)

$$X \mapsto \mathbf{Z}\mathscr{C}(-,-) \otimes \tilde{\mathbf{Z}}[X].$$

Note that this may also be obtained from 1.2.2 as $\mathbf{Z}\mathscr{C}(a,b) \otimes \tilde{\mathbf{Z}}[X] \cong \tilde{\mathbf{Z}}[\mathscr{C}(a,b)_+ \wedge X]$. If R is a ring with unit considered as a linear category with only one object then we recover the definition of the topological Hochschild theory of a ring given in [2].

1.2.4. FSPs associated to general ring functors. The incidence FSP. Given a ring functor A on a category $\mathscr{C}$ it determines an FSP $[A]$ given by

$$[A](X) = \prod_{a \in \mathscr{C}} \bigvee_{b \in \mathscr{C}} A^X(a,b).$$

The multiplication is given by matrix multiplication:

$$\begin{array}{c}
[A](X) \wedge [A](Y) = (\prod_{a\in\mathscr{C}} \bigvee_{b\in\mathscr{C}} A^X(a,b)) \wedge (\prod_{c\in\mathscr{C}} \bigvee_{d\in\mathscr{C}} A^Y(c,d)) \\
\downarrow \\
\prod_{c\in\mathscr{C}} \bigvee_{d\in\mathscr{C}} ((\prod_{a\in\mathscr{C}} \bigvee_{b\in\mathscr{C}} A^X(a,b)) \wedge A^Y(c,d)) \\
\downarrow {\scriptstyle \prod\bigvee(pr_d \wedge id)} \\
\prod_{c\in\mathscr{C}} \bigvee_{d\in\mathscr{C}} (\bigvee_{b\in\mathscr{C}} A^X(d,b) \wedge A^Y(c,d)) \\
\downarrow {\scriptstyle \prod\bigvee\bigvee\mu} \\
\prod_{c\in\mathscr{C}} \bigvee_{d\in\mathscr{C}} (\bigvee_{b\in\mathscr{C}} A^{X\wedge Y}(c,b)) \\
\downarrow {\scriptstyle \text{fold over } d} \\
\prod_{c\in\mathscr{C}} \bigvee_{b\in\mathscr{C}} A^{X\wedge Y}(c,b) = [A]\,(X \wedge Y)
\end{array}$$

If A has unit $[A]$ is unital and the unit is given by the diagonal

$$X \xrightarrow{\text{the diagonal}} \prod_{c \in \mathscr{C}} X \xrightarrow{\Pi 1_X(c)} \prod_{c \in \mathscr{C}} A^X(c,c) \subseteq [A](X).$$

There is another FSP associated to A of particular interest, namely $[A]_\vee$ given by

$$[A]_\vee(X) = \bigvee_{(a,b)\in\mathscr{C}^2} A^X(a,b)$$

and multiplication similar to (but slightly easier than) $[A]$. The drawback of this FSP is that it has no unit, but it still is much easier to work with than $[A]$ as it is equipped with a "sum" (fold). Whenever $\mathscr{C}$ is equivalent to the direct limit of its finite subcategories the inclusion $[A]_\vee \to [A]$ is a stable equivalence by the Blakers–Massey triad connectivity theorem.

In the special case where A is a ring functor with values in simplicial abelian groups with bilinear multiplication it determines an FSP $[A]_\oplus$ given by

$$[A]_\oplus(X) = \bigoplus_{(a,b)\in\mathscr{C}^2} A^X(a,b).$$

The multiplication is given by

$$\begin{aligned}
[A]_\oplus(X)\wedge[A]_\oplus(Y) &= \bigoplus_{(a,b)\in\mathscr{C}^2} A^X(a,b)\wedge \bigoplus_{(c,d)\in\mathscr{C}^2} A^Y(c,d)\\
&\longrightarrow \bigoplus_{(a,b)\in\mathscr{C}^2} A^X(a,b)\otimes \bigoplus_{(c,d)\in\mathscr{C}^2} A^Y(c,d)\\
&\longrightarrow \bigoplus_{(a,b,c)\in\mathscr{C}^3} A^X(a,b)\otimes A^Y(c,a)\\
&\xrightarrow{\mu} \bigoplus_{(a,b,c)\in\mathscr{C}^3} AX\wedge Y(c,b) \xrightarrow{\text{sum}} \bigoplus_{(c,b)\in\mathscr{C}^2} A^{X\wedge Y}(c,b)
\end{aligned}$$

where the second arrow sends $A^X(a,b)\otimes A^Y(c,d)$ to the basepoint if $d \neq a$. An important difference between $[A]$ and $[A]_\oplus$ is that if A is unital then $[A]_\oplus$ is unital only if $\mathscr{C}$ is finite (i.e. has only finitely many objects). The unit is still given by the diagonal

$$X \xrightarrow{\text{the diagonal}} \times_{c\in\mathscr{C}} X \xrightarrow{\times 1_{X(c)}} \times_{c\in\mathscr{C}} A^X(c,c) \subseteq [A]_\oplus(X).$$

There is a variant of $[A]_\oplus$ where we allow all row finite matrices. This is more analogous to $[A]$ above and in this case there is a unital stable equivalence from $[A]$. On the other hand the map $[A]_\vee \to [A]_\oplus$ is also a stable equivalence, but this has the disadvantage of not being unital even for finite categories (with more than one object).

1.2.5. The product and sum of two ring functors. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be small categories. Given $A_1 \in \mathscr{F}\mathscr{C}_1$ and $A_2 \in \mathscr{F}\mathscr{C}_2$ we may from their product $A_1 \times A_2 \in \mathscr{F}(\mathscr{C}_1 \times \mathscr{C}_2)$ by

$$(A_1\times A_2)^X((c^1,c^2),(d^1,d^2)) = A_1^X(c^1,d^1)\times A_2^X(c^2,d^2)$$

with componentwise multiplication. If both A_1 and A_2 are unital so is $A_1 \times A_2$ and the unit map is induced by the diagonal

$$X \to X\times X \xrightarrow{1_{X(c^1)}\times 1_{X(c^2)}} A_1^X(c^1,c^1)\times A_2^X(c^2,c^2).$$

When $\mathscr{C} = \mathscr{C}_1 = \mathscr{C}_2$ we may of course compose this with the diagonal diag: $\mathscr{C} \to \mathscr{C}\times\mathscr{C}$ to obtain the new ring functor on $\mathscr{C}$, $\operatorname{diag}^*(A_1\times A_2)$, which we will call the internal product.

Similarly we may define a sum of $A_1 \in \mathscr{F}\mathscr{C}_1$ and $A_2 \in \mathscr{F}\mathscr{C}_2$ by

$$(A_1 \vee A_2)^X((c^1,c^2),(d^1,d^2)) = A_1^X(c^1,d^1) \vee A_2^X(c^2,d^2)$$

with componentwise multiplication. Even when both A_1 and A_2 were unital, this generally has no unit. However, the map $A_1 \vee A_2 \to A_1 \times A_2$ induced by the inclusion is compatible with the multiplicative structure. The considerations concerning the diagonal when $\mathscr{C} = \mathscr{C}_1 = \mathscr{C}_2$ again applies here to give us the sum within ring functors on $\mathscr{C}$.

1.2.6. The $n \times n$ matrix of a ring functor. Let $\mathscr{C}$ be a small category, and let $\boldsymbol{n}$ denote the set $\{1,\dots,n\}$. Consider the n-fold product $\mathscr{C}^n$, and let $pr_k : \mathscr{C}^n \to \mathscr{C}$ be the kth projection. We define the $n \times n$ matrix M_nA of A to be the ring functor on $\mathscr{C}^n$ given by

$$(M_nA)^X(B,C) = \prod_{r \in \boldsymbol{n}} \bigvee_{s \in \boldsymbol{n}} A^X(pr_rB, pr_sC)$$

for $X \in fs_*\mathscr{E}ns$ and $B, C \in \mathscr{C}^n$. The multiplication is given by matrix multiplication, i.e.

$$\begin{array}{c}
(\prod_{r\in \boldsymbol{n}}\bigvee_{s\in \boldsymbol{n}} A^X(pr_rC,pr_sD)) \wedge (\prod_{t\in \boldsymbol{n}}\bigvee_{u\in \boldsymbol{n}} A^Y(pr_t\,B,pr_u\,C)) \\
\downarrow \\
\prod_{t\in \boldsymbol{n}}\bigvee_{u\in \boldsymbol{n}}((\prod_{r\in \boldsymbol{n}}\bigvee_{s\in \boldsymbol{n}} A^X(pr_rC,pr_sD)) \wedge A^Y(pr_t\,B,pr_u\,C)) \\
\Big\downarrow{\scriptstyle \prod\bigvee(u\text{th projection} \wedge id)} \\
\prod_{t\in \boldsymbol{n}}\bigvee_{u\in \boldsymbol{n}}(\bigvee_{s\in \boldsymbol{n}} A^X(pr_uC,pr_sD) \wedge A^X(pr_tB,pr_uC)) \\
\Big\downarrow{\scriptstyle \prod\bigvee\mu} \\
\prod_{t\in \boldsymbol{n}}\bigvee_{u\in \boldsymbol{n}}\bigvee_{s\in \boldsymbol{n}} A^{X\wedge Y}(pr_tB,pr_sD) \\
\Big\downarrow{\scriptstyle \text{fold}} \\
\prod_{t\in \boldsymbol{n}}\bigvee_{s\in \boldsymbol{n}} A^{X\wedge Y}(pr_tB,pr_sD) = (M_nA)^{X\wedge Y}(B,D)
\end{array}$$

where $B, C, D \in \mathscr{C}^n$ and $X, Y \in fs_*\mathscr{E}ns$. If A has a unit so has M_nA via

$$X \xrightarrow{\text{the diagonal}} \prod_{r\in \boldsymbol{n}} \xrightarrow{\Pi 1_X(pr_r(C))} \prod_{r\in \boldsymbol{n}} A^X(pr_r(C), pr_r(C)) \subseteq (M_nA)^X(C,C).$$

We also have the non-unital variant of the matrices using only sums, namely $(M_nA)_\vee$ given by

$$(M_nA)^X_\vee(B,C) = \bigvee_{(r,s)\in \boldsymbol{n}^2} A^X(pr_rB, pr_sC)$$

and multiplication similar to (but easier than) M_nA. The inclusion $(M_nA)_\vee \to M_nA$ is a stable equivalence by Blakers–Massey and condition (1) in Definition 1.1.

In the case where the ring functor has values in simplicial abelian groups with bilinear multiplication, we get matrices defined with $\oplus$ instead of $\vee$. Again this construction is unital if the original ring functor was. Furthermore there is a map compatible with the multiplicative structure from the matrices above to these additive matrices.

1.2.7. Upper triangular matrices. Let $A \in \mathscr{F}\mathscr{C}$. The upper triangular $n \times n$ matrix T_nA is defined to be the subspace of the matrices given by

$$(T_nA)^X(B,C) = \prod_{r \in \mathbf{n}} \bigvee_{s \leq r \in \mathbf{n}} A^X(pr_rB, pr_sC)$$

with induced multiplication. This is again unital if A is. There is also a stably equivalent version without unit using only wedges $(T_nA)_\vee \subseteq T_nA$ defined as for the matrices. In the linear case we may also define the upper triangular matrices with the sum. The same remarks apply to these upper triangular matrices as given for the full matrices.

1.2.8. Making the spectrum an Ω spectrum. Given a ring functor A on a category $\mathscr{C}$ we may associate to it the ring functor on $\mathscr{C}$, denoted A_Ω, given by

$$A_\Omega^X(a,b) = \lim_{k\to\infty} \Omega^k A^{S^k \wedge X}(a,b)$$

and product induced by

$$\Omega^k A^{S^k\wedge X}(a,b) \wedge \Omega^l A^{S^l \wedge Y}(c,a) \to \Omega^{k+l}(A^{S^k\wedge X}(a,b) \wedge A^{S^l\wedge Y}(c,a))$$

$$\xrightarrow{\Omega^{k+l}\mu} \Omega^{k+l} A^{S^k \wedge S^l \wedge X \wedge Y}(c,b).$$

A and A_Ω are stably equivalent ring functors. Furthermore, if A has a unit so does A_Ω via

$$X \to \lim_{k\to\infty} \Omega^k(S^k \wedge X) \xrightarrow{\lim \Omega^k 1_{S^k\wedge X}(c)} A_\Omega^X(c,c).$$

1.2.9. The associated Eilenberg–MacLane ring functor. If A is a ring functor on $\mathscr{C}$ we let A_E be the ring functor on $\mathscr{C}$ given by

$$A_E^X(a,b) = \prod_{i=0}^{\infty} \pi_i(\underline{A}(a,b)) \otimes \tilde{Z}[S^i \wedge X].$$

The multiplication is given by

$$\begin{array}{c}
A_{\mathrm{E}}^{X}(a,b)\wedge A_{\mathrm{E}}^{Y}(c,a)\\
\downarrow\\
\prod_{i,j=0}^{\infty}\pi_i(\underline{\underline{A}}(a,b))\otimes\pi_j(\underline{\underline{A}}(c,a))\otimes\tilde{\mathbf{Z}}[S^{i+j}\wedge X\wedge Y]\\
\downarrow\\
\prod_{p=0}^{\infty}\bigoplus_{i+j=p}\pi_{i+j}(\underline{\underline{A}}(c,b))\otimes\tilde{\mathbf{Z}}[S^{i+j}\wedge X\wedge Y]\\
\downarrow\\
\prod_{p=0}^{\infty}\pi_p(\underline{\underline{A}}(c,b))\otimes\tilde{\mathbf{Z}}[S^{p}\wedge X\wedge Y]=A_{\mathrm{E}}^{X\wedge Y}(c,b)
\end{array}$$

where the first map simply maps smash of products to products of smashes, the second rearranges the terms and uses the product to induce one on the homotopy groups of the spectrum and the last one simply sums up. If A is unital so is A_{E} via

$$X\xrightarrow{[1_{S^i(c)}]\otimes incl}\pi_0(\underline{\underline{A}}(c,c))\otimes\tilde{\mathbf{Z}}[X]\to A_{\mathrm{E}}^{X}(c,c).$$

If the Postnikov invariants of the $A^X(a,b)$ are all zero we may choose a homotopy equivalence $A^X(a,b)\xrightarrow{\simeq}A_{\mathrm{E}}^X(a,b)$.

1.2.10. The disjoint union. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two categories and let $\mathscr{C}_1\coprod\mathscr{C}_2$ denote their disjoint union. If A_j are ring functors on $\mathscr{C}_j$, $j=1,2$, we form the ring functor $A_1\coprod A_2$ on $\mathscr{C}_1\coprod\mathscr{C}_2$ given by

$$(A_1\coprod A_2)^X(a,b)=\begin{cases}A_1^X(a,b) & \text{if both } a,b\in\mathscr{C}_1,\\ A_2^X(a,b) & \text{if both } a,b\in\mathscr{C}_2,\\ * & \text{otherwise}.\end{cases}$$

The multiplication is given by

$$(A_1\coprod A_2)^X(a,b)\wedge(A_1\coprod A_2)^Y(c,a)\to(A_1\coprod A_2)^X(c,a)$$

mapping into the basepoint if not all a,b,c are in a common component and otherwise using the appropriate multiplication. If both A_1 and A_2 are unital, so is $A_1\coprod A_2$.

1.3. The topological Hochschild homology of a ring functor

In this section we will define the topological Hochschild homology of a ring functor on a category $\mathscr{C}$ in analogy with Bökstedt's definition [2] (see also [6]). As defined, this will be a presimplicial object. In the unitial case this will in fact be a simplicial object, but it will for computational reasons be worthwhile to consider the general case. If we choose as our coefficient system (bimodule) the ring functor itself, this will be a (pre-) cyclic object. It will be of importance to us later to keep close track of these structures, so unfortunately this forces us to introduce some more language.

1.3.1. Presimplicial objects. As before, let Δ be the category of standard ordered finite sets $[n] = \{0 < 1 < \cdots < n\}$ and monotone maps. Consider the subcategory $\Delta_m \subseteq \Delta$ with only injective maps. We call a functor from Δ_m^o to some category $\mathscr{C}$ a presimplicial object in $\mathscr{C}$. In [4] presimplicial sets are called Δ-sets and in an earlier version of these notes they were called semi-simplicial. Natural transformations of such are called presimplicial maps. Note that all simplicial objects are by restriction to Δ_m presimplicial objects. Given a presimplicial object X in a category with finite sums, we may form a simplicial object $\tilde{X}$ "by adjoining degeneracies" in the following way: let

$$\tilde{X}_q = \coprod_{\substack{p \le q \\ \text{surjective } \phi \in \Delta([q],[p])}} X_p.$$

Given $\psi \in \Delta([n],[q])$ and σ in the ϕ summand of $\tilde{X}_q$, factor $\phi\psi = \eta\varepsilon$ canonically where ε is a surjective and η an injective map. Then $\psi^*(\sigma) = \eta^*(\sigma)$ in the ε summand. In fact $X \mapsto \tilde{X}$ forms a left adjoint to the forgetful functor.

1.3.2. Precyclic objects. Let Λ (resp. Λ_m) be the smallest subcategory of $\mathscr{E}ns$ containing Δ (resp. Δ_m) and for each $[n]$ the extra morphism $\tau_n : [n] \to [n]$ given by cyclic permutation. A cyclic object in some category $\mathscr{C}$ is a functor from Λ^o to $\mathscr{C}$, and a cyclic map is a natural transformation between two such functors. Similarly a precyclic object is a functor from Λ_m^o and a precyclic map is a natural transformation between two precyclic objects. In an earlier version of these notes precyclic objects were called semi-cyclic.

1.3.3. The category I. Let I be the category of standard finite sets $\{1, \ldots, n\}$, $n \ge 0$ (if $n = 0$ this is the empty set) and injections. When there is no possibility of confusion we make no notational distinction between the sets and their cardinality. This category has more structure than the category of natural numbers N considered as the subcategory with only standard inclusions. Most importantly $\{p \mapsto I^{p+1}\}$ forms a cyclic category with structure maps $\partial_i : I^{p+1} \to I^p$, $\sigma_i : I^p \to I^{p+1}$ and $\tau : I^p \to I^p$ given by

$$\partial_i(x_0, \ldots, x_p) = \begin{cases} (x_0, \ldots, x_i \sqcup x_{i+1}, \ldots, x_p) & \text{for } 0 \le i < p, \\ (x_p \sqcup x_0, \ldots, x_{p-1}) & \text{for } i = p \end{cases}$$

where $\sqcup$ means concatenation, $\sigma_i(x_0, \ldots, x_p) = (x_0, \ldots, x_i, 0, x_{i+1}, \ldots, x_p)$ and, $\tau(x_0, \ldots, x_p) = (x_p, x_0, \ldots, x_{p-1})$.

1.3.4. The topological Hochschild homology of a ring functor. We are now ready for the definition of our object of study. Bökstedt [3] defined the topological Hochschild homology of an FSP (see also [3], [8] and [12]). Analogous to the extension from

Hochschild homology of a ring to the additive cyclic nerve of a linear category we will now define the homology "THH" of a ring functor. One should note that in Examples 1.2.2 and 1.2.3 we recover Bökstedt's topological Hochschild homology of a simplicial monoid and a ring in case the categories in question have merely one object. More generally, if A is any FSP considered as a ring functor over a one point category, then the present definition of $THH(A)$ is the same as Bökstedt's.

1.3.5. Notation. If $\boldsymbol{x} = (x_0, \ldots, x_p) \in I^{p+1}$ we will set

$$V(A,P)(\boldsymbol{x}) = \bigvee_{(c_0,\ldots,c_p)\in\mathscr{C}^{p+1}} P^{x_0}(c_0,c_p) \wedge A^{x_1}(c_1,c_0) \wedge \cdots \wedge A^{x_p}(c_p,c_{p-1})$$

and write simply $V(A)(\boldsymbol{x})$ for $V(A,A)(\boldsymbol{x})$.

1.3.6. Definition (*The topological Hochschild homology*). Let A be a ring functor on a small category $\mathscr{C}$ and let P be a A bimodule. We define $THH(A,P): \Delta_m^o \to s_*\mathscr{E}ns$ to be the presimplicial object given by

$$THH_p(A,P) = \underset{\boldsymbol{x}\in I^{p+1}}{\underrightarrow{\mathrm{holim}}}\ \Omega^{\sqcup \boldsymbol{x}} V(A,P)(\boldsymbol{x})$$

where $\boldsymbol{x} = (x_0, \ldots, x_p)$ and $\bigsqcup \boldsymbol{x} = x_0 \sqcup x_1 \sqcup \cdots \sqcup x_p$.

Face maps $d_i: \Omega^{\sqcup \boldsymbol{x}} V(A,P)(\boldsymbol{x}) \to \Omega^{\sqcup \partial_i \boldsymbol{x}} V(A,P)(\partial_i \boldsymbol{x})$ are defined as follows: d_0 is induced by

$$\ell: P^{x_0}(c_0,c_p) \wedge A^{x_1}(c_1,c_0) \to P^{x_0 \sqcup x_1}(c_1,c_p),$$

d_i for $0 < i < p$ is induced by

$$\mu: A^{x_i}(c_i,c_{i-1}) \wedge A^{x_{i+1}}(c_{i+1},c_i) \to A^{x_i \sqcup x_{i+1}}(c_{i+1},c_{i-1})$$

and d_p is induced by

$$r: A^{x_p}(c_p,c_{p-1}) \wedge P^{x_0}(c_0,c_p) \to P^{x_p \sqcup x_0}(c_0,c_{p-1}).$$

Note that in the case where $P = A$, the presimplicial object $THH(A) = THH(A, A)$ is in fact a precyclic object via $t: \Omega^{\sqcup \boldsymbol{x}} V(A)(\tau \boldsymbol{x})$ induced by cyclic permutation. In the case where A has a unit and P is unital $THH(A,P)$ is a simplicial object: the degeneracies are defined by using $1_{S^0}(id_{c_i})$ to insert a factor $A_0(c_i,c_i)$ in the ith slot of $V(A,P)(\sigma_i \boldsymbol{x})$. Again, if $P = A$ then $THH(A)$ is a cyclic object.

This colimit system is good in the sense of [2, 1.5], i.e. thanks to the goodness of the index category I and the connectivity assumptions, THH can be approximated in each degree by terms $\Omega^{\sqcup \boldsymbol{x}} V(A,P)(\boldsymbol{x})$ provided each coordinate in $\boldsymbol{x}$ is big.

Note. In the case where the connectivity hypothesis on the ring functor and bimodule is weakened (see Section 1.1), we have to filter this object. One way to do this is filtering by d^A: for each $d \in N$ set $V^d(A,P)(x)$ to be the wedge over objects with $d^A \leq d$ and let

$$THH_p(A,P) = \underset{x \in I^{p+1}}{\underrightarrow{\text{holim}}} \lim_{d \to \infty} \Omega^{\sqcup x} V^d(A,P)(x).$$

The structure maps respect the filtering and so this defines a (pre-) simplicial object as above, with cyclic structure if $P = A$.

There are several ways of turning THH into a spectrum. The most elementary is the following. For any $X \in s_*\mathscr{E}ns$ let $THH(A,P;X)$ be defined as the diagonal of

$$THH_p(A,P;X_q) = \underset{x \in I^{p+1}}{\underrightarrow{\text{holim}}} \Omega^{\sqcup x}(X_q \wedge V(A,P)(x)).$$

This defines a spectrum by setting

$$\underline{\underline{THH}}(A,P;X) = \{m \mapsto THH(A,P;S^m \wedge X)\}.$$

As before, where appropriate we simplify notation to $\underline{\underline{THH}}(A,P) = \underline{\underline{THH}}(A,P;S^0)$, $\underline{\underline{THH}}(A,X) = \underline{\underline{THH}}(A,A;X)$ and $\underline{\underline{THH}}(A) = \underline{\underline{THH}}(A;S^0)$. If X is a cyclic space $THH(A;X)$ may again be considered as a (pre-) cyclic object under the diagonal action.

One should note that $THH(A,P)$ only depends on the values of A and P on the objects of $\mathscr{C}$. More precisely, if $d\mathscr{C}$ is the subcategory with all objects, but only identity morphisms, then A (resp. P) may be considered as a ring functor over $d\mathscr{C}$, say dA (resp. dA-bimodule, say dP), by composition with $s_*\mathscr{E}ns^{\mathscr{C}^o \times \mathscr{C}} \xrightarrow{(incl.)^*} s_*\mathscr{E}ns^{d\mathscr{C}^o \times d\mathscr{C}}$. Then $THH(dA,dP) = THH(A,P)$.

1.4. Calculations on examples

In the following subsections we will establish the most basic properties of the THH construction on some particularly useful examples. First we will look at its rational homotopy type. Then we will show that THH of the ring functor on the category $\mathscr{C}$ given by $X \mapsto \mathscr{C}(-,-)_+ \wedge X$ has the stable homotopy type of the cyclic nerve of $\mathscr{C}$ (see below). Finally we will obtain a simpler description in the case where the bimodule is linear. This description will later be used in Chapter 2 to give a simple description in the case of the linear ring functor associated to an additive category.

1.4.1. The additive cyclic nerve. Recall the definition of the additive cyclic nerve of a small (simplicial) linear category $\mathfrak{C}$ with coefficients in a $\mathfrak{C}$ bimodule $M: \mathfrak{C}^o \otimes \mathfrak{C} \to s\mathscr{A}b$.

$$CN_q(\mathfrak{C},M) = \bigoplus_{(c_0,\ldots,c_q) \in \mathscr{C}^{q+1}} M(c_0,c_q) \otimes_Z \mathfrak{C}(c_1,c_0) \otimes_Z \cdots \otimes_Z \mathfrak{C}(c_q,c_{q-1})$$

with face and degeneracy operations as for THH. Note that if $M = \mathfrak{C}(-,-)$ this is a cyclic object.

One example of the additive cyclic nerve is the following. Let $\mathscr{C}$ be any small category and $M:\mathscr{C}^{\mathrm{o}}\times\mathscr{C}\to\mathscr{A}b$ any functor. Then $CN.(\mathbf{Z}\mathscr{C},M)$ is the standard resolution of Mitchell (see [1, 11]) for the homology $\mathrm{Tor}^{\mathscr{C}^{\mathrm{o}}\otimes\mathscr{C}}(\mathbf{Z}\mathscr{C},M^*)$ (meaning Tor in the abelian category of all functors from $\mathscr{C}^{\mathrm{o}}\times\mathscr{C}$ to $\mathscr{A}b$ of $\mathbf{Z}\mathscr{C}(-,-)$ and $M^*:(\mathscr{C}^{\mathrm{o}}\times\mathscr{C})^{\mathrm{o}}\cong\mathscr{C}^{\mathrm{o}}\times\mathscr{C}\xrightarrow{M}\mathscr{A}b$). We will in a later section show that when $\mathscr{C}$ is additive this homology actually coincides with the homotopy of THH of the ring functor $X\mapsto\mathscr{C}(-,-)\otimes\mathbf{Z}[X]/\mathbf{Z}[*]$.

In this context the proposition below should be compared with the statement that topological Hochschild theory of a ring rationally coincides with ordinary Hochschild homology of the ring.

Given an $A\in\mathscr{F}\mathscr{C}$, recall from 1.2.9 the definition of A_{E}, the "Eilenberg–MacLane" ring functor. Similarly we may for any A bimodule P associate to it an A_{E} bimodule by

$$P_{\mathrm{E}}^X(a,b)=\prod_{i=0}^{\infty}\pi_i(\underline{P}(a,b))\otimes\tilde{\mathbf{Z}}[S^i\wedge X].$$

1.4.2. Definition (*The linear category associated to a ring functor*). Given any ring functor with unit A on a category $\mathscr{C}$ we may associate to it a simplicial linear category $\mathfrak{C}_A$ in the following way. We let the objects be the same as for $\mathscr{C}$, but with morphism sets $\mathfrak{C}_A(a,b)=A_{\mathrm{E}}^0(a,b)$ $(=A_{\mathrm{E}}^{S^0}(a,b)$, in the notation adopted in 1.1.7). Similarly if P is any unital A bimodule we may form a bimodule $M_P:\mathfrak{C}_A^{\mathrm{o}}\otimes\mathfrak{C}\to s\mathscr{A}b$ given by $M_P(a,b)=P_{\mathrm{E}}^0(a,b)$.

There is a map from $THH(A_{\mathrm{E}},P_{\mathrm{E}})$ to $CN(\mathfrak{C}_A,M_P)$ given as follows: Send smashes of simplicial abelian groups to tensors, and send the wedge to the sum.

$$\begin{array}{c}\underset{x\in I^{q+1}}{\underrightarrow{\mathrm{holim}}}\,\Omega^{\sqcup x}(\bigvee P_E^{x_0}(c_0,c_q)\wedge A_E^{x_1}(c_1,c_0)\wedge\ldots\wedge A_E^{x_q}(c_q,c_{q-1}))\\ \big\downarrow\\ \underset{x\in I^{q+1}}{\underrightarrow{\mathrm{holim}}}\,\Omega^{\sqcup x}(\bigoplus P_E^{x_0}(c_0,c_q)\otimes A_E^{x_1}(c_1,c_0)\otimes\ldots\otimes A_E^{x_q}(c_q,c_{q-1}))\end{array}$$

As $\underline{A}_{\mathrm{E}}$ and $\underline{P}_{\mathrm{E}}$ are Eilenberg–MacLane spectra, the latter is isomorphic to

$$\underset{x\in I^{q+1}}{\underrightarrow{\mathrm{holim}}}\ \Omega^{\sqcup x}\tilde{\mathbf{Z}}[S^{\sqcup x}]\otimes\bigoplus_{(c_0,\ldots,c_q)\in\mathscr{C}^{q+1}}P_{\mathrm{E}}^0(c_0,c_q)\otimes A_{\mathrm{E}}^0(c_1,c_0)\otimes\cdots\otimes A_{\mathrm{E}}^0(c_q,c_{q-1})$$

which again maps by a homotopy equivalence to

$$\bigoplus_{(c_0,\ldots,c_q)\in\mathscr{C}^{q+1}}P_{\mathrm{E}}^0(c_0,c_q)\otimes A_{\mathrm{E}}^0(c_1,c_0)\otimes\cdots\otimes A_{\mathrm{E}}^0(c_q,c_{q-1})=CN_q(\mathfrak{C}_A,M_P).$$

This map is compatible with the simplicial structure. In the case $P=A$ it is compatible with the cyclic structure as well.

1.4.3. Proposition (Rational computation of THH). *Let A be a ring functor with unit on $\mathscr{C}$ and P a unital A bimodule, and let $\mathfrak{C}_A$ and M_P be associated linear category and bimodule. Then $\boldsymbol{Q}_\infty THH(A,P) \to \boldsymbol{Q}_\infty CN(\mathfrak{C}_A, M_P)$ is a homotopy equivalence.*

Proof. Recall from 1.2.8 that given any $A \in \mathscr{F}\mathscr{C}$ we associated to it a stably equivalent $A_\Omega / \mathscr{F}\mathscr{C}$ whose spectrum was actually an Ω-spectrum. By the same process we may associate an A_Ω bimodule P_Ω to any A bimodule P. In Section 1.6.10 we will show that stably equivalent ring functors and bimodules have equivalent THHs, so $THH(A,P) \xrightarrow{\simeq} THH(A_\Omega, P_\Omega)$. Now, the rationalization of any H-space has vanishing Postnikov invariants, so we get homotopy equivalences $\boldsymbol{Q}_\infty A_\Omega^X(a,b) \xrightarrow{\simeq} \boldsymbol{Q}_\infty A_{\mathrm{E}}^X(a,b)$ and $\boldsymbol{Q}_\infty P_\Omega^X(a,b) \xrightarrow{\simeq} \boldsymbol{Q}_\infty P_{\mathrm{E}}^X(a,b)$ where, using the notation of [5], $\boldsymbol{Q}_\infty$ means rationalization. Rationalization commutes with smash, infinite wedges of simply connected spaces and loops of fibrant nilpotent spaces by [5, Ch. V, 4.6 and 5.1], and for any bisimplicial set of horizontal rationalization induces a rationalization of the diagonal. By Bökstedt's approximation lemma [2, 1.5] this means that

$$\boldsymbol{Q}_\infty THH(A,P) \xrightarrow{\simeq} \boldsymbol{Q}_\infty THH(A_{\mathrm{E}}, P_{\mathrm{E}}).$$

So to prove the proposition it is enough to show that

$$THH(A_{\mathrm{E}}, P_{\mathrm{E}}) \to CN(\mathfrak{C}_A, M_P)$$

is a rational homotopy equivalence. Replace A_{E} and P_{E} by the stably equivalent ring functor and bimodule given by replacing the product by the wedge. Then $THH_p(A_{\mathrm{E}}, P_{\mathrm{E}})$ is homotopy equivalent to $\mathrm{holim}_{\to}\, \Omega^{\sqcup x} W(x)$ where $W(x)$ denotes

$$\bigvee_{(c_0,\dots,c_p)\in\mathscr{C}^{p+1}} \; \bigvee_{(i_0,\dots,i_p)\in N_0^{p+1}} \pi_{i_0}(\underline{P}(c_0,c_p)) \otimes \tilde{\boldsymbol{Z}}[S^{i_0+x_0}]$$

$$\wedge\, \pi_{i_1}(\underline{A}(c_1,c_0)) \otimes \tilde{\boldsymbol{Z}}[S^{i_1+x_1}] \wedge \cdots \wedge \pi_{i_p}(\underline{A}(c_p,c_{p-1})) \otimes \tilde{\boldsymbol{Z}}[S^{i_p+x_p}]$$

where N_0 denotes the set of natural numbers including zero. For each $(c_0,\dots,c_n) \in \mathscr{C}^{n+1}$, $(i_0,\dots,i_p) \in N_0^{p+1}$ and $\boldsymbol{x} = (x_0,\dots,x_n) \in I^{n+1}$ consider the simplicial sets

$$X = \pi_{i_0}(\underline{P}(c_0,c_p)) \otimes \tilde{\boldsymbol{Z}}[S^{i_0+x_0}] \wedge \pi_{i_1}(\underline{A}(c_1,c_0)) \otimes \tilde{\boldsymbol{Z}}[S^{i_1+x_1}]$$

$$\wedge \cdots \wedge \pi_{i_p}(\underline{A}(c_p,c_{p-1})) \otimes \tilde{\boldsymbol{Z}}[S^{i_p+x_p}]$$

and

$$Y = \pi_{i_0}(\underline{P}(c_0,c_p)) \otimes \tilde{\boldsymbol{Z}}[S^{i_0+x_0}] \otimes \pi_{i_1}(\underline{A}(c_1,c_0)) \otimes \tilde{\boldsymbol{Z}}[S^{i_1+x_1}]$$

$$\otimes \cdots \otimes \pi_{i_p}(\underline{A}(c_p,c_{p-1})) \otimes \tilde{\boldsymbol{Z}}[S^{i_p+x_p}].$$

Both spaces have the rational homology groups

$$H_q(\ ;\boldsymbol{Q}) = \begin{cases} 0 & \text{if } 0 < q < \Sigma, \\ \pi_{i_0}(\underline{\underline{P}}(c_0, c_p)) \otimes \cdots \otimes \pi_{i_p}(\underline{\underline{A}}(c_p, c_{p-1})) \otimes \boldsymbol{Q} & \text{if } q = \Sigma, \\ 0 & \text{if } \Sigma < q < 2\Sigma, \end{cases}$$

where $\Sigma = \sum_{k=0}^{p} (i_k + x_k)$. The map

$$\bigvee_{(c_0,\dots,c_n)\in\mathscr{C}^{n+1}} \bigvee_{(i_0,\dots,i_p)\in N_0^{p+1}} Y \to \bigoplus_{(c_0,\dots,c_n)\in\mathscr{C}^{n+1}} \bigoplus_{(i_0,\dots,i_p)\in N_0^{p+1}} Y$$

is $2((\sqcup \boldsymbol{x})) - 1$ connected by an argument considering finite index sets and using Blakers–Massey, and finally taking the limit. Consequently the map

$$\bigvee_{(c_0,\dots,c_n)\in\mathscr{C}^{n+1}} \bigvee_{(i_0,\dots,i_p)\in N_0^{p+1}} X \to \bigoplus_{(c_0,\dots,c_n)\in\mathscr{C}^{n+1}} \bigoplus_{(i_0,\dots,i_p)\in N_0^{p+1}} Y \cong CN_p(\mathfrak{C}_A, M_P) \otimes_{\boldsymbol{Z}} \tilde{\boldsymbol{Z}}[S^{\sqcup x}]$$

induces an isomorphism on $H_q(-,\boldsymbol{Q})$ for $q < 2(\sqcup \boldsymbol{x})$. Looping $\sqcup \boldsymbol{x}$ times we get that

$$\boldsymbol{Q}_\infty \Omega^{\sqcup x} V(A_{\mathrm{E}}, P_{\mathrm{E}})(\boldsymbol{x}) \to CN_n(\mathfrak{C}_A, M_P) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$$

is $(\sqcup \boldsymbol{x}) - 1$ connected for every $\boldsymbol{x}$. The result then follows. □

1.4.4. The topological Hochschild homology of $X \mapsto \mathscr{C}(-,-)_+ \wedge X$ is stably the cyclic nerve of $\mathscr{C}$. As a simple extension of the computation of THH of a monoid we consider THH of the half smash ring functor of Example 1.2.2.

1.4.5. Definition (*The cyclic nerve of a category*). Given a category $\mathscr{C}$, the cyclic nerve of $\mathscr{C}$ is the simplicial object $N^{\mathrm{cy}}(\mathscr{C})$ given by

$$N_p^{\mathrm{cy}}(\mathscr{C}) = \{\text{circular diagrams } c_p \leftarrow c_0 \leftarrow \cdots \leftarrow c_{p-1} \leftarrow c_p \text{ in } \mathscr{C}\}$$

with face, degeneracies and cyclic operator as in Hochschild homology.

For any (simplicial) category $\mathscr{C}$ let us call the ring functor given by $X \mapsto \mathscr{C}(-,-)_+ \wedge X$ simply $\mathscr{C}_\ltimes$. Furthermore for any space X we let $Q(X) = \lim_{k\to\infty} \Omega^k(S^k \wedge X)$.

1.4.6. Proposition. *Let $\mathscr{C}$ be any simplicial category. Then*

$$THH(\mathscr{C}_\ltimes) \simeq Q(N^{\mathrm{cy}}(\mathscr{C})_+).$$

Proof. Consider the bisimplicial object X_{**} given by

$$X_{pq} = \underset{\boldsymbol{x}\in I^{p+1}}{\underrightarrow{\mathrm{holim}}}\ \Omega^{\sqcup x}(S^{\sqcup x} \wedge (N_q^{\mathrm{cy}}(\mathscr{C}))_+)$$

$THH(C_{\ltimes})$ is isomorphic to the diagonal diag X by the following rewriting:

$$THH_p(\mathscr{C}_{\ltimes}) \cong \underset{x\in I^{p+1}}{\underrightarrow{\text{holim}}}\ \Omega^{\sqcup x} \bigvee_{(c_0,\ldots,c_p)\in\mathscr{C}^{p+1}} \mathscr{C}(c_0,c_p)_+ \wedge \mathscr{C}(c_1,c_0)_+$$

$$\wedge \cdots \wedge \mathscr{C}(c_p,c_{p-1})_+ \wedge S^{\sqcup x}$$

$$\cong \underset{x\in I^{p+1}}{\underrightarrow{\text{holim}}}\ \Omega^{\sqcup x} S^{\sqcup x} \bigvee_{(c_0,\ldots,c_p)\in\mathscr{C}^{p+1}} (\mathscr{C}(c_0,c_p)\times\mathscr{C}(c_1,c_0)$$

$$\times \cdots \times \mathscr{C}(c_p,c_{p-1}))_+$$

$$= \underset{x\in I^{p+1}}{\underrightarrow{\text{holim}}}\ \Omega^{\sqcup x} S^{\sqcup x} \wedge \Bigg(\coprod_{(c_0,\ldots,c_p)\in\mathscr{C}^{p+1}} \mathscr{C}(c_0,c_p)\times\mathscr{C}(c_1,c_0)$$

$$\times \cdots \times \mathscr{C}(c_p,c_{p-1})\Bigg)_+$$

$$= \underset{x\in I^{p+1}}{\underrightarrow{\text{holim}}}\ \Omega^{\sqcup x} S^{\sqcup x} \wedge (N_p^{cy}(\mathscr{C}))_+ .$$

The horizontal face maps $X_{p,*} \to X_{p-1,*}$ are all homotopy equivalence, and so the inclusion $Q(N^{cy}(\mathscr{C})_+) = X_{0,*} \subseteq X_{**}$ is also a homotopy equivalences. □

Of course this could be generalized to a larger class of ring functors and bimodules, in the same vein as the result above.

1.4.7. ***THH* with coefficients in bilinear bimodules may be modelled as a simplicial abelian group.** Recall that *if Y is a simplicial abelian group we may define the loop space on Y to be the simplicial abelian group $\Omega Y = s_*\mathscr{E}ns(S^1, Y)$ where $s_*\mathscr{E}ns(-, Y)$ is given the usual abelian group structure. Furthermore, we may for any functor from a small category to the category of simplicial abelian groups, say $Y:\mathscr{C}\to s\mathscr{A}b$, define the homotopy colimit $\underrightarrow{\text{holim}}_{\mathscr{C}}\, Y$ by using sums instead of wedges in the simplicial replacement lemma (see [5, Chapter XII, Section 5]).*

Let A be a ring functor on $\mathscr{C}$, and let P be a bilinear A-module. More precisely: we assume that $P \in L\mathscr{C}$ factors through

$$s\mathscr{A}b^{\mathscr{C}^o\times\mathscr{C}} \to s_*\mathscr{E}ns^{\mathscr{C}^o\times\mathscr{C}}$$

and that the action of A is distributive over addition, i.e. the two obvious maps

$$(P^X(a,b)\times P^X(a,b))\wedge A^Y(c,a) \to P^{X\wedge Y}(c,b)$$

are equal. Then we can rewrite $THH(A,P)$ in the following fashion.

1.4.8. Proposition. *Let A and P be as above, then THH(A, P) is weakly equivalent to the (pre-) simplicial abelian group*

$$\left\{p \mapsto \underset{\overrightarrow{x \in I^{p+1}}}{\operatorname{holim}}\ \Omega^{\sqcup x} \bigoplus_{(c_0,\dots,c_p)\in\mathscr{C}^{p+1}} P^{x_0}(c_0,c_p)\otimes\tilde{\mathbf{Z}}[A^{x_1}(c_1,c_0)] \otimes \cdots \otimes \tilde{\mathbf{Z}}[A^{x_p}(c_p,c_{p-1})]\right\}.$$

Proof. By Lemma 1.6.9 we may assume that $\mathscr{C}$ only has a finite number of objects, and hence it is possible to choose a common constant c for the connectivity of the structure maps for A and P (see Definition 1.1.1(2)).

Let $\mathbf{x} = (x_0, \dots, x_p) \in I^{p+1}$. Then by the Freudenthal suspension theorem

$$\begin{array}{c} P^{x_0}(c_0,c_p)\wedge A^{x_1}(c_1,c_0)\wedge\cdots\wedge A^{x_p}(c_p,c_{p-1}) \\ \downarrow \\ \lim_{N\to\infty}\Omega^N(S^N\wedge P^{x_0}(c_0,c_p)\wedge A^{x_1}(c_1,c_0)\wedge\cdots\wedge A^{x_p}(c_p,c_{p-1})) \end{array}$$

is $(\sqcup\mathbf{x}) - 2p - 1$ connected. Now, by the requirement on P as an element of $L\mathscr{C}$

$$S^k\wedge P^n(c_0,c_p) \cong S^{k-1}\wedge(S^1\wedge P^n(c_0,c_p)) \to S^{k-1}\wedge P^{n+1}(c_0,c_p)$$

is $k + 2n - 3 - c$ connected, and so $S^N\wedge P^{x_0}(c_0,c_p) \to P^{x_0+N}(c_0,c_p)$ is $2x_0 + N - 3 - c$ connected. Thus

$$\begin{array}{c} \lim_{N\to\infty}\Omega^N(S^N\wedge P^{x_0}(c_0,c_p)\wedge A^{x_1}(c_1,c_0)\wedge\cdots\wedge A^{x_p}(c_p,c_{p-1})) \\ \downarrow \\ \lim_{N\to\infty}\Omega^N(P^{x_0+N}(c_0,c_p)\wedge A^{x_1}(c_1,c_0)\wedge\cdots\wedge A^{x_p}(c_p,c_{p-1})) \end{array}$$

is $x_0 + (\sqcup\mathbf{x}) - p - 3 - c$ connected. But the latter formula is, by the description of homology by means of spectra, just the reduced homology of $A^{x_1}(c_1,c_0)\wedge\cdots\wedge A^{x_p}(c_p,c_{p-1})$ with coefficients in $P^{x_0}(c_0,c_p)$. This is hence weakly equivalent to

$$P^{x_0}(c_0,c_p)\otimes\tilde{\mathbf{Z}}[A^{x_1}(c_1,c_0)\wedge\cdots\wedge A^{x_p}(c_p,c_{p-1})].$$

Now, this is $\sqcup\mathbf{x} - 1 - p$ connected, so by Blakers–Massey

$$\bigvee_{(c_0,\dots,c_p)\in\mathscr{C}^{p+1}} P^{x_0}(c_0,c_p)\otimes\tilde{\mathbf{Z}}[A^{x_1}(c_1,c_0)\wedge\cdots\wedge A^{x_p}(c_p,c_{p-1})]$$
$$\subseteq \bigoplus_{(c_0,\dots,c_p)\in\mathscr{C}^{p+1}} P^{x_0}(c_0,c_p)\otimes\tilde{\mathbf{Z}}[A^{x_1}(c_1,c_0)\wedge\cdots\wedge A^{x_p}(c_p,c_{p-1})]$$

is $2(\sqcup\mathbf{x}) - 2p - 3$ connected.

Collecting all maps we get that

$$\begin{array}{c}
\Omega^{\sqcup x} \bigvee_{(c_0,\dots,c_p)\in\mathscr{C}^{p+1}} P^{x_0}(c_0,c_p)\wedge A^{x_1}(c_1,c_0)\wedge\dots\wedge A^{x_p}(c_p,c_{p-1}) \\
\downarrow \\
\Omega^{\sqcup x}\bigoplus_{(c_0,\dots,c_p)\in\mathscr{C}^{p+1}} P^{x_0}(c_0,c_p)\otimes\tilde{\mathbf{Z}}[A^{x_1}(c_1,c_0)\wedge\dots\wedge A^{x_p}(c_p,c_{p-1})] \\
\| \\
\Omega^{\sqcup x}\bigoplus_{(c_0,\dots,c_p)\in\mathscr{C}^{p+1}} P^{x_0}(c_0,c_p)\oplus\tilde{\mathbf{Z}}[A^{x_1}(c_1,c_0)]\otimes\dots\otimes\tilde{\mathbf{Z}}[A^{x_p}(c_p,c_{p-1})]
\end{array}$$

is $min(x_0 - p - 3 - c, (\sqcup x) - 2p - 3)$ connected (where we have used the identity $\tilde{\mathbf{Z}}[X\wedge Y]\cong\tilde{\mathbf{Z}}[X]\otimes\tilde{\mathbf{Z}}[Y]$ in the last statement, and again loops of simplicial abelian groups are chosen as above). The map on homotopy colimits (as sets) is consequently a weak equivalence. Note that in representing the latter homotopy colimit it does not matter, up to weak equivalence, whether we use wedges or sums in the simplicial replacement functor. This follows by Blakers–Massey and the fact that the connectivity increases with the cardinality of $\sqcup x$. Letting p vary and going to the diagonal we get the stated result. □

1.5. Cyclic structure and topological cyclic homology

The examples of calculations for *THH* above are somewhat deceiving, since a major importance of the *THH* construction lies in the fact that we obtain a cyclic structure when the ring functor itself serves as the bimodule. The computations in 1.4.3, 1.4.6 and 1.4.8 do not preserve this cyclic structure, but in general we will try to keep this structure as long as possible. Going carefully through the proofs in the rational and the $X\mapsto\mathscr{C}(-,-)_+\wedge X$ examples we may in fact recover the cyclic structure. We will not do so here, but for general computations it will be essential that we carry with us this information.

1.5.1. Definition. Let sd_r be the edgewise subdivision functor (see [3] or 1.5.5), and note that the definition extends to precyclic objects to give a precyclic object with a natural C_r action. We say that a precyclic map $f: X\to Y$ is a C-equivalence if $sd_r f^{C_r}$ is a weak equivalence for all r.

If f, in addition to being a C-equivalence, is actually a cyclic map, this implies that $|f|$ is a C_r equivariant homotopy equivalence for every r. This is true since for all $C_s\subseteq C_r$

$$\begin{array}{ccc}
|X|^{C_s} & \cong & |sd_s X^{C_s}| \\
{\scriptstyle |f|^{C_s}}\downarrow & & \downarrow{\scriptstyle |sd_s f^{C_s}|}\,\simeq \\
|Y|^{C_s} & \cong & |sd_s Y^{C_s}|
\end{array}$$

commutes giving that $|f|^{C_s}$ is a homotopy equivalence, and so $|f|$ is a C_r equivariant homotopy equivalence by the equivariant Whitehead theorem [16].

Let A be a ring functor on a small category $\mathscr{C}$ and let $r \in N$. In dimension $q-1$ we have

$$(sd_r THH(A)^{C_r})_{q-1} = (THH_{rq-1}(A))^{C_r} = \Big(\underset{\overrightarrow{x \in I^{rq}}}{\text{holim}}\ \Omega^{\sqcup x} V(A)(x)\Big)^{C_r}$$

$$= \underset{\overrightarrow{x \in I^{q}}}{\text{holim}}\ (\Omega^{\sqcup (x^{\sqcup r})} V(A)(x^{\sqcup r}))^{C_r}$$

where $x^{\sqcup r} \in I^{qr}$ is the image of the r-fold diagonal of $x \in I^q$ (which are the only fixed points under the C_r action on I^{rq}).

Note that $|(S^{x_1} \wedge \cdots \wedge S^{x_q})^{\wedge r}|^{C_r} \cong |S^{x_1} \wedge \cdots \wedge S^{x_q}|$ and $|V(A)(x^{\sqcup r})|^{C_r} \cong |V(A)(x)|$. By restricting the C_r equivariant maps from $|(S^{x_1} \wedge \cdots \wedge S^{x_q})^{\wedge r})|$ to $|V(A)(x^{\sqcup r})|$ to the fixed point sets of the C_r action we get a map to

$$\underset{\overrightarrow{x \in I^{q}}}{\text{holim}}\ \Omega^x V(A)(x) = THH_{q-1} A.$$

This map extends to a simplicial map denoted ϕ_r. Likewise we may define a map

$$(sd_{sr} THH(A))^{C_{sr}} \to (sd_s THH(A))^{C_s}$$

also denoted ϕ_r.

1.5.2. Definition. The map constructed above is the Frobenius map

$$\phi_r : sd_{sr} THH(A)^{C_{sr}} \to sd_s THH(A)^{C_s}.$$

Note that in view of the results in [9], the map ϕ_r should definitely not be called the Frobenius map. In fact, the inclusion of fixed points is very much closer to the classical Frobenius map on the Witt vectors. However, in this paper we will stick to the notation in [3] etc.

Let $\mathscr{F}$ be the category with objects the natural numbers, and a morphism $f_{p,q} : paq \to a$ for every triple $a, p, q \in N$ subject to $f_{p,q} \circ f_{r,s} = f_{pr,qs}$. If A is a unital ring functor there is a functor T from $\mathscr{F}$ to spaces sending $a \in ob\mathscr{F}$ to $|sd_a THH(A)^{C_a}|$ and $f_{p,q}$ to $|\phi_p| \circ i_q$ where $i_q : |sd_{aq} THH(A)^{C_{aq}}| \cong |THH(A)|^{C_{aq}} \subseteq |THH(A)|^{C_a} \cong |sd_a THH(A)^{C_a}|$ is the inclusion.

1.5.3. Definition. Let A be a unital ring functor and $T : \mathscr{F} \to$ spaces the functor above. Then

$$TC(A) = \underset{\overleftarrow{\mathscr{F}}}{\text{holim}}\ T$$

is called the topological cyclic homology of A.

A map $A \to B$ of unital ring functors which gives a C-equivalence $THH(A) \to THH(B)$ clearly induces a homotopy equivalence $TC(A) \to TC(B)$. This is one of the motivations for the care taken with the cyclic structure in the next section. Another approach would be that of Goodwillie [8] who characterized the fibre of the Frobenius map $sd_{sr}THH(A)^{C_{sr}} \to sd_s THH(A)^{C_s}$ as $\lim_{m\to\infty} \Omega^m(THH(A; S^m)_{hC_{rs}})$ up to natural equivalence. His proof passes over to our case by replacing $W = X \wedge (A(S^{x_0}) \wedge \cdots \wedge A(S^{x_j}))^{\wedge sr}$ with $X \wedge V(A)(x^{\sqcup rs})$ everywhere. Thus we get that any map $A \to B$ inducing a weak equivalence $THH(A; X) \to THH(B; X)$ for every finite pointed simplicial set X induces a weak equivalence on TC.

However, in proving the theorems we are concerned with in this chapter, giving the appropriate weak equivalences is often not much simpler than giving the C-equivalences, and we will stick to this more direct route. We conclude this section with some notions useful at that end. References for unproven statements are [14] for general presimplicial properties, [10] appendix for precyclic properties, and [3] for subdivision and general cyclic properties.

1.5.4. Presimplicial sets. Given two presimplicial maps $f, g: X \to Y$ we say that they are prehomotopic (earlier: "semi-homotopic") if there exists a presimplicial map $H: H \times \Delta(1) \to Y$ such that $f(x) = H(x,(1,\dots,1))$ and $g(x) = H(x,(0,\dots,0))$ for any $x \in X_q$ where $(1,\dots,1)$ and $(0,\dots,0)$ in $\Delta(1)_q = \Delta([q],[1])$ are the maps sending everything to 1 and 0 respectively. Similarly a prehomotopy equivalence is a presimplicial map $f: X \to Y$ such that there exists a map $g: Y \to X$ such that both the composites gf and fg are prehomotopic to the identity. The realization of a presimplicial set X is defined just as for a simplicial set:

$$\|X\| = \coprod_q |\Delta(q)| \times X_q / (\theta_* \sigma, x) \sim (\sigma, \theta^* x)$$

for $\theta \in \Delta_m([p],[q])$, $\sigma \in |\Delta(p)|$ and $x \in X_q$.

For simplicial spaces this is the same as the thick realization of Segal [15]. Recall the construction $X \mapsto \tilde{X}$ for adjoining degeneracies to a simplicial set described in 1.3.1. The spaces $|\tilde{X}|$ and $\|X\|$ are homeomorphic, and if X already is a simplicial set then the canonical map $\tilde{X} \to X$ is a weak equivalence. We may even display a section: if $x \in X_q$ there is a unique surjective map $\phi \in \Delta([q],[p])$ and $y \in X_p$ such that $x = X \circ \phi(y)$. We then send x to y in the ϕth coordinate. We say that two presimplicial maps are weakly homotopic if their realizations are homotopic, and that a map is a weak equivalence if the realization is a homotopy equivalence. If X is a simplicial set or a Kan presimplicial set [14] then two maps $X \to Y$ are weakly homotopic if they are prehomotopic, and if Y is a Kan as well these notions coincide.

Let G be a group and X a presimplicial set on which G acts presimplicially. Then $\tilde{X}$ naturally becomes a simplicial G set. If X already were a simplicial G set then $\tilde{X} \to X$ induces a G equivariant homotopy equivalence $|\tilde{X}| \to |X|$.

1.5.5. Edgewise subdivision. Let $r \in N$. Consider the functor $sd_r : \Delta \to \Delta$ given by sending an object $[q]$ to

$$\overbrace{[q] \sqcup [q] \sqcup \cdots \sqcup [q]}^{r \text{ times}} = [(q+1)r - 1]$$

and a morphism $\phi : [q] \to [p]$ to

$$\phi \sqcup \cdots \sqcup \phi : [q] \sqcup \cdots \sqcup [q] \to [p] \sqcup \cdots \sqcup [p]$$

(i.e. $sd_r\phi(a(q+1)+b) = a(p+1) + \phi(b)$ for $0 \le a < r$ and $0 \le b \le q$). If X is a simplicial object we define sd_rX to be $X \circ sd_r$. If X is a simplicial set there is a homeomorphism of realizations $D_r : |sd_rX| \xrightarrow{\cong} |X|$ (see [3]). Note that sd_r restricts to a functor $\Delta_m \to \Delta_m$, and so can we define sd_rX for any presimplicial object.

The realization of a cyclic set comes naturally with a circle action, and we will be concerned with the action of the finite subgroups. Using the edgewise subdivision, this action can be described simplicially. Let $C_r \subseteq S^1$ denote the subgroup of the circle with r elements. Given a (pre-) cyclic set X, the rth edgewise subdivision has a (pre-) simplicial C_r action given by $t_{(p+1)r-1}^{p+1} : X_{(p+1)r-1} = (sd_rX)_p \to (sd_rX)_p$. In the cyclic situation we get that $|sd_rX^{C_r}| \cong |X|^{C_r}$ where X is some cyclic set.

1.5.6. Special homotopies. Let $\mathcal{I}$ be the groupoid on two objects, i.e., is the category with two objects, say 0 and 1, and two non-identity isomorphisms $0 \to 1$ and $1 \to 0$. Recall that the cyclic nerve of a category $\mathscr{C}$ is a cyclic set, and that the C_r fixed points $sd_rN^{\mathrm{cy}}(\mathscr{C})^{C_r}$ can naturally be identified with $N^{\mathrm{cy}}(\mathscr{C})$. This fact, together with the fact that $\Delta(1) = N(0 \to 1)$ is a subset of $N^{\mathrm{cy}}(\mathcal{I})$ makes the following definition useful.

1.5.7. Definition (*Special homotopies* [10]). Let $f, g : X \to Y$ be two precyclic maps. We say that f and g are specially homotopic if there is a precyclic map

$$H : X \times N^{\mathrm{cy}}(\mathcal{I}) \to Y$$

such that $f(x) = H(x, (1 = 1 = \cdots = 1))$ and $g(x) = H(x, (0 = 0 = \cdots = 0))$ for any $x \in X_q$. We will say that f is a special homotopy equivalence if there exists a precyclic map $\bar{f} : Y \to X$ such that both $f \circ \bar{f}$ and $\bar{f}$ are specially homotopic to the identity. In this case we will say that X and Y are specially homotopy equivalent, and that $\bar{f}$ is a special homotopy inverse.

1.5.8. Lemma (McCarthy [10]). *If $f, g : X \to Y$ are specially homotopic maps and X a cyclic set, then for all $r > 0$, $sd_rf^{C_r}$ and $sd_rg^{C_r} : sd_rX^{C_r} \to sd_rY^{C_r}$ are prehomotopic (and hence weakly homotopic). If f has a special homotopy inverse and both X and Y are cyclic sets, then $sd_rf^{C_r}$ is a prehomotopy equivalence, and thus f is a C-equivalence.*

Proof. We have by the observation $sd_rN^{\mathrm{cy}}(\mathcal{I})^{C_r} = N^{\mathrm{cy}}(\mathcal{I})$ above, that

$$sd_r(X \times N^{\mathrm{cy}}(\mathcal{I}))^{C_r} = sd_rX^{C_r} \times N^{\mathrm{cy}}(\mathcal{I})$$

and so by restriction to $\Delta(1)$ we get a prehomotopy

$$sd_r X^{C_r} \times \Delta(1) \to sd_r X^{C_r}$$

from $sd_r f^{C_r}$ to $sd_r g^{C_r}$. The last statement now follows. □

1.5.9. Presimplicial simplicial sets. Quite often the lack of degeneracies in a presimplicial simplicial set causes no trouble if we have degreewise equivalences to a bisimplicial set. Below follow some useful lemmas. The last is especially important as it tells us how to translate special homotopy equivalences of precyclic simplicial sets to C-equivalences of cyclic simplicial sets.

1.5.10. Lemma. *If $X \to Y \in s_*\mathscr{E}ns^{\Delta_m^o}$ is a map of presimplicial pointed simplicial sets such that $X_q \xrightarrow{\simeq} Y_q \in s_*\mathscr{E}ns$ is a weak equivalence for every q then $X \to Y$ is a weak equivalence.*

Proof. Note that even if we adjoin degeneracies to both X and Y we still have a degreewise weak equivalence

$$\tilde{X}_q = \coprod_{\substack{p \le q \\ \text{surjective } \phi \in \Delta([q],[p])}} X_p \xrightarrow{\simeq} \coprod_{\substack{p \le q \\ \text{surjective } \phi \in \Delta([q],[p])}} Y_p = \tilde{Y}_p .$$

This means (see [5] XII, 4.2]) that $\operatorname{diag} \tilde{X} \to \operatorname{diag} \tilde{Y}$ (diagonals of bisimplicial sets) is a weak equivalence and so $|\tilde{X}| \xrightarrow{\simeq} |\tilde{Y}|$ is a homotopy equivalence. □

1.5.11. Lemma. *Let $f, g : V \to W \in s_*\mathscr{E}ns^{\Delta_m^o}$ be two prehomotopic maps of presimplicial objects. Assume there is a degreewise weak equivalence $h : V \to X$ where X is a simplicial object (bisimplicial set). Then $|\tilde{f}|$ and $|\tilde{g}|$ are homotopic.*

Proof. Let $H : V \times \Delta[1] \to W$ be the prehomotopy. Adjoining degeneracies this is a map $\tilde{H} : \widetilde{V \times \Delta}[1] \to \tilde{W}$ which upon restriction to $\tilde{V} \vee \tilde{V} = \widetilde{V \vee V} \subseteq \widetilde{V \times \Delta}[1]$ is simply $\tilde{f} \vee \tilde{g}$. As $h : V_q \to X_q$ is a weak equivalence for each q, we get that $h \times id : V_q \times \Delta([q],[1]) \to X_q \times \Delta([q],[1])$ is a weak equivalence, and so by Lemma 1.5.10 we get that

$$\widetilde{V \times \Delta}[1] \xrightarrow{\widetilde{h \times id}} \widetilde{X \times \Delta}[1] \to X \times \Delta[1] \to \tilde{X} \times \Delta[1] \leftarrow \tilde{V} \times \Delta[1]$$

is a weak equivalence. Let $|\widetilde{V \times \Delta}[1]| \xrightarrow{\simeq} |\tilde{V}| \times |\Delta[1]|$ be any map representing this weak equivalence. There is a lifting of this map which upon restriction to $|\tilde{V}| \vee |\tilde{V}| \rightarrowtail |\tilde{V}| \times |\Delta[1]|$ is the inclusion (factor the map into a trivial cofibration followed by a trivial fibration. The cofibration splits, and as $|\tilde{V}| \vee |\tilde{V}| \rightarrowtail |\tilde{V}| \times |\Delta[1]|$ is a cofibration we may lift past the trivial fibration). Hence this lifting composed with $|\tilde{H}|$ is the desired homotopy. □

1.5.12. Proposition. *Assume that we are in the situation depicted in the diagram in* $s_*\mathscr{E}ns^{\Lambda^o_\infty}$ *below*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ V & \xrightarrow{g} & W \end{array}$$

where $X \xrightarrow{f} Y$ *is actually a cyclic pointed simplicial set map.*

Suppose for all $r, q > 0$ *the vertical maps induced degreewise weak equivalences* $V^{C_r}_{rq-1} \to X^{C_r}_{rq-1}$ *and* $W^{C_r}_{rq-1} \to Y^{C_r}_{rq-1}$ *and that the bottom horizontal map is a special homotopy equivalence. Then f is a C-equivalence, and so for all r, the realization* $|f|$ *is a* C_r *equivariant homotopy equivalence.*

Proof. The hypotheses assure that $(sd_r V^{C_r})_{q-1} = V^{C_r}_{rq-1} \xrightarrow{\simeq} X^{C_r}_{rq-1} = (sd_r X^{C_r})_{q-1}$ is a weak equivalence. In particular, by the lemma above, it shows that any prehomotopy $sd_r V^{C_r} \times \Delta(1) \to sd_r V^{C_r}$ induces a homotopy on realization. The same considerations apply to W and Y. If $\bar{g}$ is the special homotopy inverse to g let $H: V \times N^{cy}(\mathscr{I}) \to V$ be the special homotopy form the identity to $\bar{g} \circ g$. Then $sd_r(V \times N^{cy}(\mathscr{I}))^{C_r} = sd_r V^{C_r} \times N^{cy}(\mathscr{I})$, and so $sd_r H^{C_r}$ is a special homotopy (and in particular a prehomotopy) from the identity to $sd_r(\bar{g} \circ g)^{C_r}$. Likewise for the other composition. By Lemma 1.5.11 we have that $sd_r V^{C_r} \to sd_r W^{C_r}$ is a weak equivalence for all r and hence we are done. □

1.6. Basic properties

In this section we will establish some basic properties for the *THH* construction of Section 1.3. We first will show that *THH* behaves well under natural isomorphisms of the underlying category, and that it commutes with the direct limit of finite full subcategories. Then we show that stably equivalent ring functors have the same *THH*. Some of these facts we have already used. The matrix ring functor was defined in 1.2.6 and we display the presence of a "trace" map in order to show Morita equivalence. Lastly we will show how *THH* may be calculated as Bökstedt's topological Hochschild homology of an FSP. Throughout this section we will be using ring functors without units as a computational tool, and the reader is referred to the end of the previous section for notation pertaining to presimplicial objects. In particular, remember that prehomotopic maps between simplicial sets give rise to homotopic maps on realization, and that specially homotopic maps between cyclic sets give rise to C_r equivariant homotopy equivalences on realization for all r. Even though the procedure chosen is perhaps a bit non-standard, we hope that the reader will appreciate the fact that so many of the following questions can be taken care of by repeating the same few tricks over and over again. This approach will, among other things, allow us to give a direct proof of Morita equivalence without reference to the standard bisimplicial space mapping by weak equivalences to both THH of the ring functor and the matrices.

1.6.1. Behavior of THH under natural isomorphisms and equivalences. We start out by noting that THH is insensitive to natural isomorphisms of functors. We say that two ring functors on $\mathscr{C}$, say A_0 and A_1, are isomorphic if there are natural isomorphisms of functors $A_0 \to A_1$ and $A_1 \to A_0$ compatible with the multiplicate structure (and unit if there is one). Likewise for (bi)modules. Up to natural isomorphism THH obviously does not see the difference. As a particular example consider the following:

If $\phi: \mathscr{D} \to \mathscr{C}$ in any functor and if A is a ring functor on $\mathscr{C}$ and P an A bimodule we let ϕ^*A (resp. ϕ^*P) denote the composite of A (resp. P) with the functor $s_*\mathscr{E}ns^{\mathscr{C}^o \times \mathscr{C}} \to s_*\mathscr{E}ns^{\mathscr{D}^o \times \mathscr{D}}$ induced by ϕ. This is a ring functor on $\mathscr{D}$ (with unit if A had one) and ϕ^*P is a (unital) ϕ^*A bimodule. We have a map $\bar{\phi}: THH(\phi^*A, \phi^*P) \to THH(A, P)$ given by sending the $(d_0, \ldots, d_p) \in \mathscr{D}^{p+1}$ summand of

$$\bigvee_{(d_0, \ldots, d_p) \in \mathscr{D}^{p+1}} P^{x_0}(\phi(d_0), \phi(d_p)) \wedge A^{x_1}(\phi(d_1), \phi(d_0)) \wedge \cdots \wedge A^{x_p}(\phi(d_p), \phi(d_{p-1}))$$

onto the $(\phi(d_0), \ldots, \phi(d_p)) \in \mathscr{C}^{p+1}$ summand of

$$\bigvee_{(c_0, \ldots, c_p) \in \mathscr{C}^{p+1}} P^{x_0}(c_0, c_p) \wedge A^{x_1}(c_1, c_0) \wedge \cdots \wedge A^{x_p}(c_p, c_{p-1}).$$

1.6.2. Lemma. *Let A be a ring functor on $\mathscr{C}$ and P an A bimodule. If ϕ_0 and ϕ_1 are naturally isomorphic functors from $\mathscr{D}$ to $\mathscr{C}$ then $THH(\phi_0^*A, \phi_0^*P)$ and $THH(\phi_1^*A, \phi_1^*P)$ are isomorphic.*

Proof. Let $\eta: \phi_0 \to \phi_1$ be the natural isomorphism. Then

$$A^X(\eta_a^{-1}, \eta_b): A^X(\phi_0(a), \phi_0(b)) \to A^X(\phi_1(a), \phi_1(b))$$

(resp. $P^X(\eta_a^{-1}, \eta_b)$) induces a natural isomorphism between the functors ϕ_0^*A and ϕ_1^*A (resp. ϕ_0^*P and ϕ_1^*P) compatible with the multiplication (and unit if A has unit and P is unital). □

1.6.3. Corollary. *If $P = A$ in the above lemma we have an isomorphism of cyclic objects.*

However, one should be aware of the following subtle point. Although $THH(\phi_0^*A, \phi_0^*P)$ and $THH(\phi_1^*A, \phi_1^*P)$ are isomorphic, say by $(\eta^{-1}, \eta)^*$, the two maps

$$THH(\phi_0^*A, \phi_0^*P) \xrightarrow{\bar{\phi}_0} THH(A, P)$$

and

$$THH(\phi_0^*A, \phi_0^*P) \underset{\cong}{\xrightarrow{(\eta^{-1}, \eta)^*}} THH(\phi_1^*A, \phi_1^*P) \xrightarrow{\bar{\phi}_1} THH(A, P)$$

are not in general equal. However we have that

1.6.4. Lemma. *The maps $\bar{\phi}_0$ and $\bar{\phi}_1 \circ (\eta^{-1}, \eta)^*$ are prehomotopic.*

Proof. We will define a map

$$H : THH(\phi_0^* A, \phi_0^* P) \times N^{cy}(\mathscr{I}) \to THH(A,P)$$

which upon restriction to $\Delta(1) \subseteq N^{cy}(\mathscr{I})$ is a prehomotopy from $\bar{\phi}_0$ to $\bar{\phi}_1 \circ (\eta^{-1}, \eta)^*$. Let $\alpha = (i_p \leftarrow i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_p) \in N_p^{cy}(\mathscr{I})$. Then $H_\alpha : THH_p(\phi_0^* A, \phi_0^* P) \to THH_p(A,P)$ is given by (notation will be explained below)

$$\underset{x \in I^{p+1}}{\underrightarrow{\operatorname{holim}}}\ \Omega^{\sqcup x} \bigvee_{\phi_{i_0}, \ldots, \phi_{i_p}} P^{x_0}(\eta_{d_0}^{-i_0}, \eta_{d_p}^{i_p}) \wedge A^{x_1}(\eta_{d_1}^{-i_1}, \eta_{d_0}^{i_0}) \wedge \cdots \wedge A^{x_p}(\eta_{d_p}^{-i_p}, \eta_{d_{p-1}}^{i_{p-1}}).$$

Here $\bigvee_{\phi_{i_0}, \ldots, \phi_{i_p}}$ signifies that, say the $(d_0, \ldots, d_p) \in \mathscr{D}^{p+1}$ summand, is sent onto the $(\phi_{i_0}(d_0), \ldots, \phi_{i_p}(d_p)) \in \mathscr{C}^{p+1}$ summand. $\eta_{d_k}^{\pm i_k}$ simply means $\eta_{d_k}^{-1} : \phi_1(d_k) \to \phi_0(d_k)$, $id_{\phi_0(d_k)}$ or $\eta_{d_k} : \phi_0(d_k) \leftarrow \phi_1(d_k)$ according to the value of $\pm i_k$. This assembles to give the desired prehomotopy. □

1.6.5. Corollary. *If $P = A$ in the lemma above, the prehomotopy of the proof yields that $\bar{\phi}_0$ and $\bar{\phi}_1 \circ (\eta^{-1}, \eta)^*$ are specially homotopic.*

1.6.6. Proposition. *Let $\phi : \mathscr{D} \to \mathscr{C}$ be an equivalence of categories, A a ring functor on $\mathscr{C}$ and P an A bimodule. Then*

$$\bar{\phi} : THH(\phi^* A, \phi^* P) \to THH(A,P)$$

is a prehomotopy equivalence. If $P = A$ this is a special homotopy equivalence, and so if A is unital this is a C-equivalence.

Proof. Let $\psi : \mathscr{C} \to \mathscr{D}$ be an inverse. Then

$$THH(A,P) \cong THH(\psi^* \phi^* A, \psi^* \phi^* P) \xrightarrow{\bar{\phi} \circ \bar{\psi}} THH(A,P)$$

and

$$THH(\phi^* A, \phi^* P) \xrightarrow{\bar{\phi}} THH(A,P) \cong THH(\psi^* \phi^* A, \psi^* \phi^* P) \xrightarrow{\bar{\psi}} THH(\phi^* A, \phi^* P)$$

are both prehomotopic (specially homotopic if $P = A$) to the identity. □

In particular, this means that we can define THH uniquely up to special homotopy for all categories equivalent to small ones.

1.6.7. Example. Let $\phi : \mathscr{D} \to \mathscr{C}$ be an equivalence, and consider the ring functors on $\mathscr{C}$ of the type of Examples 1.2.2 and 1.2.3. More precisely, let M be a subcategory of the category of sets containing the image of the morphism set functor $\mathscr{C}(-,-) : \mathscr{C}^o \times \mathscr{C} \to M \subseteq \mathscr{E}ns$. Assume our ring functor factors as

$$fs_* \mathscr{E}ns \xrightarrow{F} s_* \mathscr{E}ns^M \xrightarrow{\mathscr{C}(-,-)^*} s_* \mathscr{E}ns^{\mathscr{C}^o \times \mathscr{C}}.$$

By the proposition above get that $THH(\phi^*(\mathscr{C}(-,-)^*)\circ F)$ and $THH((\mathscr{C}(-,-)^*)\circ F)$ are specially homotopic. On the other hand, as ϕ is an equivalence of categories it induces a natural isomorphism between $(\mathscr{D}(-,-)^*)\circ F$ and $\phi^*(\mathscr{C}(-,-)^*)\circ F$, and so

$$THH((\mathscr{D}(-,-)^*)\circ F)\cong THH(\phi^*(\mathscr{C}(-,-)^*)\circ F)\xrightarrow{\simeq} THH((\mathscr{C}(-,-)^*)\circ F)$$

is a special homotopy equivalence.

This will be important in Chapter 2 where we consider THH as a functor from exact categories via the linear ring functor of Example 1.2.3, and so we get that THH sends natural equivalences to C-equivalences.

1.6.8. Topological Hochschild homology commutes with direct limits. Let $\mathscr{C}$ be a small category and let J be a directed set of subcategories of $\mathscr{C}$ with the property that for any object $c\in\mathscr{C}$ there is a $j\in J$ such that $c\in j$. We then say that J is a *saturated* directed set in $\mathscr{C}$. By abuse of notation we will use the same letter for an element in J and its inclusion into $\mathscr{C}$.

1.6.9. Lemma. *Let J be a saturated directed set in $\mathscr{C}$, A a ring functor on $\mathscr{C}$ and P an A bimodule. Then the map*

$$\varinjlim_{j\in J} THH(j^*A, j^*P)\to THH(A,P)$$

is an isomorphism.

Proof. The colimit commutes with the homotopy colimit, so for every $x\in I^{p+1}$ we consider the map

$$\varinjlim_{j\in J} \Omega^{\sqcup x}V(j^*A, j^*P)(x)\to \Omega^{\sqcup x}V(A,P)(x).$$

A map from $|S^{\sqcup x}|$ to $|V(A,P)(x)|$ has compact image, and hence must stay within a finite number of summands. As J is saturated and directed, there is a $j\in J$ such that the image is contained in summands indexed by elements in j. This gives the desired result. $\square$

If $j:\mathscr{D}\subseteq\mathscr{C}$ is any subcategory we will often write $A|_{\mathscr{D}}$ (resp. $P|_{\mathscr{D}}$) for j^*A (resp. j^*P), the restriction to $\mathscr{D}$. Taking J to be a set of finite subsets (i.e. discrete subcategories) of $ob\mathscr{C}$ such that any object in $\mathscr{C}$ is eventually contained in some $j\in J$, the lemma above tells us that $THH(A,P)$ only depends on the values on finite discrete subcategories $THH(A|_j, P_j)$.

1.6.10. Stably equivalent ring functors. We have already used the following. Let $\phi: A_1\to A_2$ be a stable equivalence (see 1.1.1 for definition) of ring functors on $\mathscr{C}$, and $P_1\to P_2$ a stable equivalence of bimodules (meaning that P_1 is an A_1 bimodule and

P_2 is an A_2 bimodule (and hence an A_1 bimodule via ϕ) and that there is a map $P_1 \to P_2$ compatible with the bimodule structures over A_1 such that $\underline{P}_1 \to \underline{P}_2$ is a stable equivalence.) Then we have

1.6.11. Lemma (*THH* is invariant under stable equivalence). *The stable equivalence* $(A_1 \to A_2, P_1 \to P_2)$ *induces a weak equivalence* $THH(A_1, P_1) \xrightarrow{\simeq} THH(A_2, P_2)$. *If* $P_1 = A_1$ and $P_2 = A_2$ *this is a C-equivalence.*

Proof. We immediately get that as the cardinality of $x \in I^{p+1}$ gets bigger, the connectivity of

$$\Omega^{\sqcup x} V(A_1, P_1)(x) \to \Omega^{\sqcup x} V(A_2, P_2)(x)$$

grows to infinity. Hence $THH_p(A_1, P_1) \to THH_p(A_2, P_2)$ is a weak equivalence for all p. The first statement then follows by Lemma 1.5.10.

In the case where the ring functors themselves serve as bimodules, we will show that $sd_r(THH(A_1))_q^{C_r} \to sd_r(THH(A_2))_q^{C_r}$ is a weak equivalence for all q and the result follows as above.

$$(sd_r THH(A_k)^{C_r})_{q-1} = (THH_{rq-1}(A_k))^{C_r}$$
$$= \left(\underrightarrow{\mathrm{holim}}_{x \in I^{rq}} \Omega^{\sqcup x} V(A_k)(x) \right)^{C_r} = \underrightarrow{\mathrm{holim}}_{x \in I^q} (\Omega^{\sqcup (x^{\sqcup r})} V(A_k)(x^{\sqcup r}))^{C_r}$$

where $x^{\sqcup r} \in I^{qr}$ is the image of the r-fold diagonal of $x \in I^q$ (which are the only fixed points under the C_r action on I^{rq}). Now, for any $C_s \subseteq C_r$, letting $t = r/s$

$$V(A_k)(x^{\sqcup r})^{C_s} = \bigvee_{(c_1, \dots, c_{-t_q}) \in \mathscr{C}^{-t_q}} ((A_k^{x_1}(c_1, c_{-t_q}) \wedge \cdots \wedge A_k^{x_q}(c_q, c_{q-1})$$
$$\wedge (A_k^{x_1}(c_{q+1}, c_q) \wedge \cdots \wedge A_k^{x_q}(c_{2q}, c_{2q-1}) \wedge \cdots$$
$$\wedge (A_k^{x_1}(c_{(t-1)q+1}, c_{(t-1)q}) \wedge \cdots \wedge A_k^{x_q}(c_{-t_q}, c_{tq-1}))^{\wedge_s})^{C_s}$$
$$\cong V(A_k)(x^{\sqcup t}).$$

By Lemma 1.6.9 we may assume that $\mathscr{C}$ is finite, so to have some concrete numbers to work with, say that $A_1^n(a,b) \to A_2^n(a,b)$ is $2n - c$ connected for some constant c (depending on neither a, b nor n). Then $V(A_1)(x^{\sqcup r})^{C_s} \to V(A_2)(x^{\sqcup r})^{C_s}$ is $t \sum (2x_i - c)$-connected, and hence by Lemma 3.11 of [3]

$$(\Omega^{\sqcup (x^{\sqcup r})} V(A_1)(x^{\sqcup r}))^{C_s} \to (\Omega^{\sqcup (x^{\sqcup r})} V(A_2)(x^{\sqcup r}))^{C_s}$$

is $t \sum (x_i - c)$ connected. In particular, if $r = s$ this is $\sum (x_i - c)$-connected and by Lemma 1.5.10 we are done. □

1.6.12. Preservation of products. In this subsection assume we are given two small categories $\mathscr{C}_1$ and $\mathscr{C}_2$ and assume that $A_1 \in \mathscr{F}\mathscr{C}_1$ and $A_2 \in \mathscr{F}\mathscr{C}_2$ and that P_1 and P_2 are A_1 and A_2 bimodules. For the time being we will not assume the presence of a unit. Recall the definitions of $A_1 \times A_2$, $A_1 \vee A_2$ and $A_1 \coprod A_2$ (see 1.2.5 and 1.2.10). Similarly we can define the bimodules $P_1 \times P_2$, $P_1 \vee P_2$ and $P_1 \coprod P_2$. In the unital case, the first and last bimodules will be unital as well, but the wedge will not.

Let $x = (x_0, \dots, x_p) \in I^{p+1}$. Note that $V(A_1 \coprod A_2, P_1 \coprod P_2)(x) = V(A_1, P_1)(x) \vee V(A_2, P_2)(x)$ (the only summands surviving in $V(A_1 \coprod A_2, P_1 \coprod P_2)(x)$ are the ones where all $c_0, \dots, c_p$ lie in the same component of $\mathscr{C}_1 \coprod \mathscr{C}_2$). Hence the inclusion $V(A_1, P_1)(x) \vee V(A_2, P_2)(x) \subseteq V(A_1, P_1)(x) \times V(A_2, P_2)(x)$ induces a map $j: THH(A_1 \coprod A_2, P_1 \coprod P_2) \to THH(A_1, P_1) \times THH(A_2, P_2)$.

1.6.13. Lemma. *Let $A_1 \in \mathscr{F}\mathscr{C}_1$ and $A_2 \in \mathscr{F}\mathscr{C}_2$ and let P_1 and P_2 be A_1 and A_2 bimodules. Then*

$$j: THH(A_1 \coprod A_2, P_1 \coprod P_2) \to THH(A_1, P_1) \times THH(A_2, P_2)$$

is a weak equivalence. If $P_1 = A_1$ and $P_2 = A_2$ it induces a C-equivalence, and hence in the unital case $|j|$ is a C_r equivariant homotopy equivalence for all r.

Proof. Blakers–Massey guarantees that

$$V(A_1, P_1)(x) \vee V(A_2, P_2)(x) \subseteq V(A_1, P_1)(x) \times V(A_2, P_2)(x)$$

is $2(\sqcup x) - 2p - 3$ connected, and so j is a weak equivalence.

For the $P_k = A_k$ case consider for all $q > 0$ and subgroups $C_r \subseteq C_q$ the map

$$j_{q-1}^{C_r}: THH_q(A_1 \coprod A_2)^{C_r} \to (THH_{q-1}(A_1) \times THH_{q-1}(A_2))^{C_r}.$$

Letting $p = q/r$ this is just the map:

$$\underset{\overrightarrow{x \in I^p}}{\text{holim}} \, (\Omega^{\sqcup x^{\sqcup r}} V(A_1)(x^{\sqcup r}) \vee V(A_2)(x^{\sqcup r}))^{C_r}$$

$$\to \underset{\overrightarrow{x \in I^p}}{\text{holim}} \, (\Omega^{\sqcup x^{\sqcup r}} V(A_1)(x^{\sqcup r}) \times V(A_2)(x^{\sqcup r}))^{C_r}$$

Now,

$$(V(A_1)(x^{\sqcup r}) \vee V(A_2)(x^{\sqcup r}))^{C_r} \cong V(A_1)(x) \vee V(A_2)(x)$$

and likewise for the product, so

$$(V(A_1)(x^{\sqcup r}) \vee V(A_2)(x^{\sqcup r}))^{C_r} \to (V(A_1)(x^{\sqcup r}) \times V(A_2)(x^{\sqcup r}))^{C_r}$$

is $2(\sqcup x) - 2p - 3$ connected (and similarly for the subgroups of C_r), and so by [3, Lemma 3.11] the map $(\Omega^{\sqcup x^{\sqcup r}} V(A_1)(x^{\sqcup r}) \vee V(A_2)(x^{\sqcup r}))^{C_r} \to (\Omega^{\sqcup x^{\sqcup r}} V(A_1)(x^{\sqcup r}) \times V(A_2)(x^{\sqcup r}))^{C_r}$ is $(\sqcup x) - 2p - 3$ connected.

Hence it follows that $sd_r j_p^{C_r} = j_{q-1}^{C_r}$ is a weak equivalences for all $q > 0$ and $C_r \subseteq C_q$. By Lemma 1.5.10 and the equivariant Whitehead lemma, the result follows. □

1.6.14. Notation. From now we will need precise notation for handling the various indexations for $V(A, P)(x)$. If E is some set, we let $\{E\}_p$ denote the set of functions

$\mathbf{Z}/(p+1)\mathbf{Z} \to E$. If $e \in \{E\}_p$ we write e_i for $e(i)$. If we write $\{\mathscr{C}\}_p$ where $\mathscr{C}$ is some small category, we shall mean $\{ob\mathscr{C}\}_p$. We shall choose indexation such that the face maps (and cyclic permutations if $P = A$) on $V(A,P)(x)$ corresponds to the cyclic actions on $\{E\}_p$. Thus for instance, using that smash is distributive over wedge up to natural isomorphism, we get that $V(A_1 \vee A_2, P_1 \vee P_2)(x)$ could be written out as

$$\bigvee_{\{(c^1,c^2,r)\in\mathscr{C}^1\times\mathscr{C}^2\times\{1,2\}\}_p} P^{x_0}_{r_0}(c_0^{r_0},c_p^{r_0})\wedge A^{x_1}_{r_1}(c_1^{r_1},c_0^{r_1})\wedge\cdots\wedge A^{x_p}_{r_p}(c_p^{r_p},c_{p-1}^{r_p}).$$

The natural map $f_x: V(A_1 \times A_2, P_1 \times P_2)(x) \to V(A_1,P_1)(x) \times V(A_2,P_2)(x)$ restricts to a projection $g_x: V(A_1 \vee A_2, P_1 \vee P_2)(x) \to V(A_1,P_1)(x) \vee V(A_2,P_2)(x)$ which we may give a section as follows. In the notation above g_x is given by sending the (c^1,c^2,r)th summand to the basepoint if r is not constant, and onto the rth summand if r is either $\equiv 1$ or $\equiv 2$. Choose some fixed but arbitrary object $(a,b) \in \mathscr{C}_1 \times \mathscr{C}_2$ and define a section $i_x: V(A_1,P_1)(x) \vee V(A_2,P_2)(x) \to V(A_1 \vee A_2, P_1 \vee P_2)(x)$ by sending the $(c_0^1,\ldots,c_p^1) \in \mathscr{C}_1^{p+1} \subseteq (\mathscr{C}_1 \coprod \mathscr{C}_2)^{p+1}$ onto the $(c^1,b,1)$ summand, and likewise for the other summands. This defines a simplicial map $i: THH(A_1 \coprod A_2, P_1 \coprod P_2)(x) \to THH(A_1 \vee A_2, P_1 \vee P_2)$. We use this to prove:

1.6.15. Proposition (Preservation of products). *Let $A_1 \in \mathscr{F}\mathscr{C}_1^u$ and $A_2 \in \mathscr{F}\mathscr{C}_2^u$ and let P_1 and P_2 be unital bimodules. Then*

$$THH(A_1 \times A_2, P_1 \times P_2) \xrightarrow{f} THH(A_1,P_1) \times THH(A_2,P_2)$$

is a weak equivalence. If $P_1 = A_1$ and $P_2 = A_2$, then f is a C-equivalence, and so $|f|$ is a C_r equivariant homotopy equivalence for every $r > 0$.

Proof. Let $X = THH(A_1 \times A_2, P_1 \times P_2)$, $Y = THH(A_1,P_1) \times THH(A_2,P_2)$, $V = THH(A_1 \vee A_2, P_1 \vee P_2)$ and $W = THH(A_1 \coprod A_2, P_1 \coprod P_2)$, and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow{\scriptstyle j} \\ V & \xrightarrow{g} & W \end{array}$$

We want to show that all maps are weak equivalences, and that if $P_k = A_k$ we are in the situation of Lemma 1.5.12.

By Blakers–Massey the product and wedge ring functors and bimodules are stably homotopic so by Lemma 1.6.11 the map $V \to X$ is a degreewise weak equivalence, and if $P_k = A_k$ (for $k = 1$ and 2) a degreewise weak equivalence on $sd_r(-)^{C_r}$.

As to the lower horizontal map, we have defined a section i of g, so $g \circ i = id_W$ and we define a map $H: V \times N^{cy}(\mathscr{I}) \to V$ which upon restriction to $\Delta(1) \subseteq N^{cy}(\mathscr{I})$ is a prehomotopy from the identity to $i \circ g$. If $\alpha = (i_p \leftarrow i_0 \leftarrow i_1 \leftarrow \cdots i_{p-1} \leftarrow i_p) \in N_p^{cy}(\mathscr{I})$

let ϕ_α be the self-map of the indexing set $\{(\mathscr{C}_1 \times \mathscr{C}_2) \times \{1,2\}\}_p$ given by

$$\phi_\alpha(c^1,c^2,r)(k) = \begin{cases} (c_k^1, b, 1) & \text{if } r_k = 1 \text{ and } i_k = 0, \\ (a, c_k^2, 2) & \text{if } r_k = 2 \text{ and } i_k = 0, \\ (c_k^1, c_k^2, r_k) & \text{if } i_k = 1. \end{cases}$$

Then we define $H_{\alpha,x}: V(A_1 \vee A_2, P_1 \vee P_2)(x) \to V(A_1 \vee A_2, P_1 \vee P_2)(x)$ by sending y in the (c^1,c^2,r) summand to

$$H_{\alpha,x}(y) = \begin{cases} y & \text{in the } \phi_\alpha(c^1,c^2,r) \text{ summand if } i_k = 0 \Rightarrow r_k = r_{k+1}, \\ * & \text{otherwise.} \end{cases}$$

Applying $\Omega^{\sqcup x}$ and going to the limit this defines, as p varies, a prehomotopy when restricting to $\Delta(1) \subseteq N^{\mathrm{cy}}(\mathscr{I})$ and if $P_k = A_k$ a special homotopy (to see this notice that $d_j y = *$ if $r_j \neq r_{j+1}$). Hence $V \to W$ is a prehomotopy equivalence and if $P_k = A_k$ a special homotopy equivalence. Together with the above lemma we have now proven that all maps are weak equivalences, and collecting everything we have proven for the $P_k = A_k$ part we see that we are in the situation of Lemma 1.5.12 and the result follows. □

Comment. After such a lengthy proof it is worthwhile to extract the ideology used. The following should not be considered as a part of this section (or even this paper), but is meant as an aid for the interested reader.

Let $P = A$ for simplicity. As advertised many times, the crucial step in this approach was that we exchanged the original ring functors with more accessible ones without unit. The accessibility lies in the fact that, in the notation above, we were able to give a "simple" description of a splitting of the map $V \to W$, and furthermore, we could give an explicit homotopy

$$V \times N^{\mathrm{cy}}(\mathscr{I}) \to V.$$

The latter was built on the following idea which we shall have occasion to use many times. Think of an element $\alpha = (i_p \leftarrow i_0 \leftarrow \cdots \leftarrow i_p) \in N_p^{\mathrm{cy}}(\mathscr{I})$ as a string of actions on $A^{x_0}(c_0, c_p) \wedge \cdots \wedge A^{x_p}(c_p, c_{p-1})$ testing compatibility between two pairs. By this we mean the following: "if i_k equals zero, test whether $A^{x_{k+1}}(c_{k+1}, c_k)$ and $A^{x_k}(c_k, c_{k-1})$ always multiply trivially. If it does, send everything to the basepoint, but otherwise leave it as it is." This is a presimplicial (even precyclic) gadget, for taking any other face map but the kth, we still have the same potential question (moved appropriately); and if we use the kth face map, the knowledge of whether the question was asked disappears, but this becomes irrelevant for if it would have been answered affirmatively we are sent to the basepoint in either case.

Thus if the difference between two self-maps on $V(A)(x)$ can be expressed by some criterion of "non-compatibility" implying trivial multiplication, we are able to bridge them by a homotopy, as examplified with $A = A_1 \vee A_2$ in the proof. Here this criterion was that the index did not stay within one of the categories (and hence forcing some face map to be trivial). Of course, there is the additional problem with putting the final answer in the right summand, but this should be regarded as technical. For an example not containing this extra difficulty see below. However, there is one point related to this that we should be aware of. The exchange of wedges for products allowed us to use the natural distributivity of smash over wedges, making it possible to write out $V(A)(x)$ as a wedge over an indexing set. It is important that we make this indexation compatible with the precyclic actions. More precisely: the index sets are maps from $\mathbf{Z}/(p+1)\mathbf{Z}$ and so cyclic sets themselves, and the face maps and cyclic operators of $V(A)(x)$ should send summands onto each other in accordance with the operations on the indexation. We see that this is the case with the indexation $\{\mathscr{C}^1 \times \mathscr{C}^2 \times \{1,2\}\}_p$ of $V(A_1 \vee A_2)(x)$. If we have an element y in the (c^1, c^2, r) summand then $d_j y$ naturally lies in the $d_j(c^1, c^2, r)$ summand (note that if $r_j \neq r_{j+1}$ it is sent to the basepoint).

We also have to check that the reindexation ϕ is a precyclic map (in the sense that $d_j\phi_\alpha = \phi_{d_j\alpha} d_j$ and likewise for the cyclic operator). Now it is easy to see that

$$d_j H_{\alpha,x}(y) = \begin{cases} d_j y & \text{in the } d_j\phi_\alpha(c^1,c^2,r) \text{ summand if for } 0 \le k \le p, \\ & i_k = 0 \Rightarrow r_k = r_{k+1}, \\ * & \text{otherwise.} \end{cases}$$

and

$$H_{d_j\alpha,\, d_j x}(d_j y) = \begin{cases} d_j y & \text{in the } \phi_{d_j\alpha} d_j(c^1,c^2,r) \text{ summand if for} \\ & 0 \le k \neq j \le p,\ i_k = 0 \Rightarrow r_k = r_{k+1}, \\ * & \text{otherwise.} \end{cases}$$

are equal:

i_j	$r_j = r_{j+1}$	$H_{d_j\alpha, d_j x}(d_j y) = d_j H_{\alpha,x}(y)$	
0	Yes	Yes	(Automatic)
0	No	Yes	(both $= *$)
1	Yes	Yes	(No $r_j = r_{j+1}$ condition
1	No	Yes	on $H_{\alpha,x}(y)$ either)

All this testing is implicit in the text proper, and will not be carried through from now on.

It is perhaps noteworthy that a similar, but somewhat easier proof yields that if we had used the internal product of ring functors we would have had the same statement. More precisely: if $\mathscr{C} = \mathscr{C}_1 = \mathscr{C}_2$ in the above situation, we define the internal product of A_1 and A_2 to be $\mathrm{diag}^*(A_1 \times A_2)$ where $\mathrm{diag}: \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ is the diagonal inclusion. There is a map $f \circ \overline{\mathrm{diag}}: THH(\mathrm{diag}^*(A_1 \times A_2), \mathrm{diag}^*(P_1 \times P_2)) \to THH(A_1, P_1) \times THH(A_2, P_2)$, and laziness dictates that we call it simply f (for the definition of $\bar{\phi}$ see 1.6.1). We still get that

$$THH(\mathrm{diag}^*(A_1 \vee A_2), \mathrm{diag}^*(P_1 \vee P_2)) \to THH(\mathrm{diag}^*(A_1 \times A_2), \mathrm{diag}^*(P_1 \times P_2))$$

is induced by stable equivalences, and what one must show is that

$$THH(\mathrm{diag}^*(A_1 \vee A_2), \mathrm{diag}^*(P_1 \vee P_2)) \quad \text{and} \quad THH((A_1 \amalg A_2),(P_1 \amalg P_2))$$

are prehomotopy equivalent (specially homotopy equivalent if $P_k = A_k$).

So writing this out, we get that $V(\mathrm{diag}^*(A_1 \times A_2), \mathrm{diag}^*(P_1 \times P_2))(x)$ equals

$$\bigvee_{(c,r)\in\{\mathscr{C}\times\{1,2\}\}_p} P_{r_0}^{x_0}(c_0,c_n)\wedge A_{r_1}^{x_1}(c_1,c_0)\wedge\cdots\wedge A_{r_p}^{x_p}(c_p,c_{p-1}).$$

Let $i_x: V(A_1 \amalg A_2, P_1 \amalg P_2) = V(A_1,P_1)(x) \vee V(A_2,P_2)(x) \to V(\mathrm{diag}^*(A_1 \vee A_2), \mathrm{diag}^*(P_1 \vee P_2))(x)$ be the inclusion and $g_x: V(A_1 \vee A_2, P_1 \vee P_2)(x) = V(A_1,P_1)(x) \vee V(A_2,P_2)(x)$ be the splitting given by sending all summands but the ones with constant r's to the basepoint. This induces a map

$$g: THH(\mathrm{diag}^*(A_1 \vee A_2), \mathrm{diag}^*(P_1 \vee P_2)) \to THH(A_1 \amalg A_2, P_1 \amalg P_2)$$

(which is simply $g \circ \overline{\mathrm{diag}}$ in the in the notation of 1.6.14) with section i

1.6.16. Lemma (Preservation of internal product). *Let A_1 and A_2 be two ring functors on $\mathscr{C}$ with unit and let P_1 and P_2 be unital bimodules. Then*

$$THH(\mathrm{diag}^*(A_1 \times A_2), \mathrm{diag}^*(P_1 \times P_2)) \xrightarrow{f} THH(A_1,P_1) \times THH(A_2,P_2)$$

is a weak equivalence. If $P_1 = A_1$ and $P_2 = A_2$ then it is a C-equivalence.

Proof. The only thing remaining to demonstrate is that we have a map

$$H: THH(\mathrm{diag}^*(A_1 \vee A_2), \mathrm{diag}^*(P_1 \vee P_2)) \times N^{cy}(\mathscr{I})$$
$$\to THH(\mathrm{diag}^*(A_1 \vee A_2), \mathrm{diag}^*(P_1 \vee P_2))$$

which upon restriction to $\Delta(1) \subseteq N^{cy}(\mathscr{I})$ is a prehomotopy from the identity to $i \circ g$ and which in the $P_k = A_k$ case is a special homotopy. If $\alpha = (i_p \leftarrow i_0 \leftarrow i_1 \leftarrow \cdots i_{p-1} \leftarrow i_p) \in N_p^{cy}(\mathscr{I})$ and y is in the $(c,r) \in \{\mathscr{C} \times \{1,2\}\}_p$ summand we define

$$H_{\alpha,x}(y) = \begin{cases} y & \text{if } i_k = 0 \Rightarrow r_k = r_{k+1}, \\ * & \text{otherwise.} \end{cases}$$

Applying $\Omega^{\sqcup x}$ and going to the limit, as p varies this defines a prehomotopy and if $P_k = A_k$ a special homotopy. □

1.6.17. Morita Equivalence. Let A be a ring functor on $\mathscr{C}$ and P an A bimodule. Recall from Section 1.2.6 the definition of M_nA and $(M_nA)_\vee$. In the same fashion M_nP (resp. $(M_nP)_\vee$) is the M_nA (resp. $(M_nA)_\vee$) bimodule given by

$$(M_nP)^X(C,D) = \prod_{r\in n}\bigvee_{s\in n} P^X(pr_rC, pr_sD)$$

$$(\text{resp. } (M_nP)_\vee^X(C,D) = \bigvee_{(r,s)\in n^2} P^X(pr_rC, pr_sD))$$

with the obvious M_nA (resp. $(M_nA)_\vee$) action. The inclusions $(M_nA)_\vee \to M_nA$ and $(M_nP)_\vee \to M_nP$ are stable equivalences, so for all practical purposes we may just as well work with the simpler wedge construction.

Choose some arbitrary but fixed element $a \in \mathscr{C}$. Let $in: \mathscr{C} \to \mathscr{C}^n$ be the inclusion into the first coordinate given by sending an arrow f in $\mathscr{C}$ to $(f, id_a, \dots, id_a)$ (it is not important that we use the identity on a, any idempotent would do, in particular if we have some sort of zero, we may use this instead). We have a map of ring functors on $\mathscr{C}$ $\eta: A \to in^*(M_nA)_\vee$ by the inclusion

$$A^X(c,d) \to A^X(c,d) \vee \bigvee_{n-1} A^X(c,a) \vee \bigvee_{n-1} A^X(a,d) \vee \bigvee_{(n-1)^2} A^X(a,a)$$

$$= (M_nA)_\vee^X(in(c), in(d))$$

η has a splitting $\varepsilon: in^*(M_nA)_\vee \to A$ by the corresponding projection. Likewise for P. This induces

$$i: THH(A,P) \xrightarrow{\eta} THH(in^*(M_nA)_\vee, in^*(M_nP)_\vee) \xrightarrow{in} THH((M_nA)_\vee, (M_nP)_\vee).$$

Likewise we have inclusions of A into in^*M_nA and P into in^*M_nP, and these are unital if A and P are. These inclusions factor through ε followed by the stable equivalence induced from including the wedges into the products. Let $j: THH(A,P) \to THH(M_nA, M_nP)$ denote the induced map.

The advantage is i is that there is an easy "trace" map back, which we will now define. First write out $V((M_nA)_\vee, (M_nP)_\vee)(x)$:

$$\bigvee_{(r,s,C) \in \{n^2 \times \mathscr{C}^n\}_p} P^{x_0}(pr_{r_0}C_0, pr_{s_p}C_p) \wedge \cdots \wedge A^{x_p}(pr_{r_p}C_p, pr_{s_{p-1}}C_{p-1})$$

(see 1.6.14 for notation). Thus the inclusion above is given by sending the $(c_0, \dots, c_p)$ summand of $V(A,P)(x)$ onto the (r,s,C) summand with $C_k = (c_k, a, \dots, a)$ and $r = s = 1$.

Then the trace map is defined by sending y in the (r,s,C) summand to

$$Tr_x(y) = \begin{cases} * & \text{if } r_k \neq s_k \text{ for some } k, \\ y & \text{in the } \{k \mapsto pr_{r_k}C_k\} \text{ summand if } r_k = s_k \text{ for all } k. \end{cases}$$

This defines a map $Tr: THH((M_nA)_\vee, (M_nP)_\vee) \to THH(A,P)$, with $Tr \circ i = id$.

1.6.18. Proposition (Morita equivalence). *Let A be a unital ring functor on $\mathscr{C}$, P a unital A bimodule and $n \in N$. The inclusions defined above yield weak equivalences $i: THH(A,P) \to THH(M_nA, M_nP)$. If $P = A$ this is a C-equivalence.*

Proof. The framework of the proof is exactly as for the proof of preservation of product. Let $X = V = THH(A,P)$, $Y = THH(M_nA, M_nP)$, and $W = THH((M_nA)_\vee \times, (M_nP)_\vee)$, and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ \| & & \uparrow \\ V & \xrightarrow{i} & W \end{array}$$

We want to show that all maps are weak equivalences, and that if $P = A$ we are in the situation of Lemma 1.5.12.

Now, $Tr \circ i = id$, and we will define a presimplicial homotopy (special if $P = A$) from $i \circ Tr$ to the identity. For each $\alpha = (i_p \leftarrow i_0 \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_p) \in N_p^{cy}(\mathscr{I})$ we define a map $H_{\alpha,x}: V(M_nA, M_nP)(x) \to V(M_nA, M_nP)(x)$ as follows. Let ϕ_α be the self-map of the indexing set $\{\mathbf{n}^2 \times \mathscr{C}\}_p = \mathscr{E}ns(\mathbf{Z}/(p+1)\mathbf{Z}, \mathbf{n}^2 \times \mathscr{C})$ given by

$$\phi_\alpha(r,s,C)(k) = \begin{cases} (1, 1, in \circ pr_{r_k} C_k) & \text{if } i_k = 0, \\ (r_k, s_k, C_k) & \text{if } i_k = 1. \end{cases}$$

Now, if y is in the (r,s,C) summand

$$H_{\alpha,x}(y) = \begin{cases} y \text{ in the } \phi_\alpha(r,s,C) \text{ summand} & \text{if } i_k = 0 \Rightarrow r_k = s_k \text{ for all } 0 \le k \le p, \\ * & \text{otherwise.} \end{cases}$$

This assembles to a presimplicial homotopy from $i \circ Tr$ to the identity. Note that if $P = A$ this actually defines a special homotopy. □

1.6.19. Upper triangular matrices. In the previous subsection we saw that the trace gave rise to an equivalence between the THH of a ring functor and its matrices. If we restrict our attention to upper triangular matrices we note that the only sequences giving non-zero contribution to the generalized trace are in fact the ones deriving from diagonal elements. This is so because once one is outside the diagonal, the next element in the generalized trace must be on the other side of the diagonal. This observation may make the the proposition below more plausible, and will be apparent in the proof itself.

Let $A \in \mathscr{F}\mathscr{C}$ and let P be a bimodule. Recall from Section 1.2.7 the definition of T_nA and $(T_nA)_\vee$. In the same fashion T_nP (resp. $(T_nP)_\vee$) is the T_nA (resp. $(T_nA)_\vee$) bimodule given by

$$(T_nP)^X(C,D) = \prod_{r \in \mathbf{n}} \bigvee_{s \le r \in \mathbf{n}} P^X(pr_rD, pr_sD)$$

$$\left(\text{resp. } (T_nP)_\vee^X(C,D) = \bigvee_{s \le r \in \mathbf{n}^2} P^X(pr_rC, pr_sD)\right)$$

with the obvious T_nA (resp. $(T_nA)_\vee$) action. The inclusions $(T_nA)_\vee \to T_nA$ and $(T_nP)_\vee \to T_nP$ are stable equivalences. Indeed in this setting have a diagram of ring functors on $\mathscr{C}^n$

$$\begin{array}{ccccc} \prod_{k\in n} A & \xrightarrow{i} & T_n A & \xrightarrow{p} & \prod_{k\in n} A \\ \uparrow\wr & & \uparrow\wr & & \uparrow\wr \\ \bigvee_{k\in n} A & \xrightarrow{i} & (T_n A)_\vee & \xrightarrow{p} & \bigvee_{k\in n} A \end{array}$$

where i is given by the inclusion of the diagonal, and p by collapsing of off-diagonal summands. This is possible as no off-diagonal elements may multiply to give a diagonal element. We define the same maps on the bimodules giving us split injections

$$THH\left(\prod_{k\in n} A, \prod_{k\in n} P\right) \xrightarrow{i} THH(T_nA, T_nP)$$

$$THH\left(\bigvee_{k\in n} A, \bigvee_{k\in n} P\right) \xrightarrow{i} THH((T_nA)_\vee, (T_nP)_\vee).$$

Using the trace map again, we will show that the wedge models are specially homotopy equivalent. Note that even though both the inclusion and projection are defined if we use the product representations, the homotopy defined in the proof will not (be defined).

1.6.20. Proposition. *Let A be a ring functor with unit on $\mathscr{C}$, P a unital A bimodule, and $n \in N$. Then the inclusion $i: THH(\prod_{k\in n} A, \prod_{k\in n} P) \to THH(T_nA, T_nP)$ is split weak equivalence. If $P = A$ then this is a C-equivalence.*

Proof. Now let $X = THH(\prod_n A, \prod_n P)$, $Y = THH(T_nA, T_nP)$, $V = THH(\bigvee_n A, \bigvee_n P)$ and $W = THH((T_nA)_\vee, (T_nP)_\vee)$, and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \uparrow & & \uparrow \\ V & \xrightarrow{i} & W \end{array}$$

We want to show that all maps are weak equivalences, and that if $P = A$ we are in the situation of Lemma 1.5.12.

By Blakers–Massey the ring functors and bimodules defined by product or wedge are stably homotopy equivalent so by Lemma 1.6.11 both vertical maps are degreewise weak equivalences, and if $P = A$ degreewise weak equivalences on $sd_r(-)^{C_r}$.

As to the lower horizontal map, the trace defines a special homotopy inverse to i as follows. We may write out $V((T_nA)_\vee, (T_nP)_\vee)(x)$ (resp. $V(\vee_n A, \vee_n P)(x)$) as

$$\bigvee_{\substack{(r,s,C)\in\{n\times n\times\mathscr{C}^n\}_p \\ r_i\geq s_{i-1}}} P^{x_0}(pr_{r_0}C_0, pr_{s_p}C_p)\wedge\cdots\wedge A^{x_p}(pr_{r_p}C_p, pr_{s_{p-1}}C_{p-1})$$

$$\left(\text{resp.}\ \bigvee_{(r,C)\in\{n\times\mathscr{C}^n\}_p} P^{x_0}(pr_{r_0}C_0, pr_{r_0}C_p)\wedge\cdots\wedge A^{x_p}(pr_{r_p}C_p, pr_{r_p}C_{p-1})\right)$$

Note that the condition $r_i = s_i$ for all $i\in \mathbf{Z}/(p+1)\mathbf{Z}$ of the trace definition in the previous section becomes $r_i = s_j$ for all i, j when we include the additional requirement that $r_i \geq s_{i-1}$ for all i. Thus we define $Tr\colon THH((T_nA)_\vee, (T_nP)_\vee) \to THH(\vee_n A, \vee_n P)$ by setting Tr_x to be the map sending y in the (r,s,C) coordinate to be

$$Tr_x(y) = \begin{cases} y \text{ in the } (r,C) \text{ summand} & \text{if } \textit{all } r_i \text{ and } s_j \text{ are equal,} \\ * & \text{otherwise.} \end{cases}$$

Thus we get for y in the (r,s,C) summand and z in the (r,C) summand that

$$i_x\circ Tr_x(y) = \begin{cases} y \text{ in the } (r,s,C) \text{ summand} & \text{if } \textit{all } r_i \text{ and } s_j \text{ are equal,} \\ * & \text{otherwise.} \end{cases}$$

$$Tr_x\circ i_x(y) = \begin{cases} z \text{ in the } (r,C) \text{ summand} & \text{if } \textit{all } r_i \text{ and } r_j \text{ are equal,} \\ * & \text{otherwise.} \end{cases}$$

If $\alpha = (i_p \leftarrow i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_p) \in N_p^{cy}(\mathscr{I})$ let

$$H_{\alpha,x}\colon V((T_nA)_\vee, (T_nP)_\vee)(x) \to V((T_nA)_\vee, (T_nP)_\vee)(x)$$

be defined by sending y in the (r,s,C) summand to

$$H_{\alpha,x}(y) = \begin{cases} y \text{ in the } (r,s,C) \text{ summand} & \text{if } i_k = 0 \Rightarrow r_k = s_k, \\ * & \text{otherwise.} \end{cases}$$

Likewise we define

$$G_{\alpha,x}\colon V((T_nA)_\vee, (T_nP)_\vee)(x) \to V((T_nA)_\vee, (T_nP)_\vee)(x)$$

by sending z in the (r,C) summand to

$$G_{\alpha,x}(z) = \begin{cases} y_r \text{ in the } (r,C) \text{ summand} & \text{if } i_k = 0 \Rightarrow r_k = r_{k+1}, \\ * & \text{otherwise.} \end{cases}$$

This defines maps $H\colon V\times N^{cy}(\mathscr{I}) \to V$ and $G\colon W\times N^{cy}(\mathscr{I}) \to W$. When restricting to $\Delta(1) \subseteq N^{cy}(\mathscr{I})$ we get that H is a prehomotopy from id_V to $i\circ Tr$ and G a prehomotopy from id_W to $Tr\circ i$. When $P = A$ both are special homotopies.

Hence $V \xrightarrow{i} W$ is a prehomotopy equivalence. If $P_k = A_k$, $V \xrightarrow{i} W$ is a special homotopy equivalence. So all maps in the diagram are weak equivalences, and in case $P = A$ we see that we are in the situation of Lemma 1.5.12 and the result follows. □

1.6.21 THH of any ring functor can be calculated by means of an FSP. Given an $A \in \mathscr{F}\mathscr{C}$, recall the definition in 1.2.4 of FSPs $[A]$ and $[A]_\vee$. In the same manner we may define for any A bimodule P the $[A]$ (resp. $[A]_\vee$) bimodule $[P]$ (resp. $[P]_\vee$) given by

$$[P](X) = \prod_{a \in \mathscr{C}} \bigvee_{b \in \mathscr{C}} P^X(a,b) \quad \left(\text{resp. } [P]_\vee(X) = \bigvee_{(a,b) \in \mathscr{C}^2} P^X(a,b)\right)$$

with the obvious actions. Note that the inclusions $[A]_\vee \subseteq [A]$ and $[P]_\vee \subseteq [P]$ are not in general stable equivalences. This however is no serious problem, as we have seen in 1.6.10 that THH may be calculated as the limit of the THHs obtained by restricting to finite subcategories. Here Blakers–Massey guarantees us stable equivalence, so from now on we may assume that our category has only a finite set of objects.

Now, if $\phi : \mathscr{C} \to *$ is the functor to the trivial category, we have maps of ring functors on $\mathscr{C}$ $A \to \phi^*[A]_\vee$ and $A \to \phi^*[A]$ given by the inclusion $A^X(a,b) \subseteq [A]^X_\vee$ and composition with $[A]_\vee \to [A]$. The map $A \to \phi^*[A]$ is unital if A is. The same maps on bimodules gives us maps

$$\begin{array}{ccc}
THH(A,P) & \xrightarrow{\ i\ } & THH([A],[P]) \\
\Vert & & \uparrow \\
THH(A,P) & \xrightarrow{\ j\ } & THH([A]_\vee,[P]_\vee)
\end{array}$$

For each $x \in I^{p+1}$, let $j_x : V(A,P)(x) \to V([A]_\vee,[P]_\vee)(x)$ be the map corresponding to j. We define a splitting in analogy with the trace (each matrix has only one entry, so this makes sense) $Tr : THH([A]_\vee,[P]_\vee) \to THH(A,P)$ given by $Tr_x : V([A]_\vee, [P]_\vee)(x) \to V(A,P)(x)$ where

$$p_x(y) = \begin{cases} y & \text{if } y \in im\, j_x \\ * & \text{otherwise.} \end{cases}$$

This is a well defined presimplicial map! (precyclic if $P = A$), and to see that it is perhaps best to write out $V([A]_\vee,[P]_\vee)(x)$ as follows, again using the natural distributivity between smash and wedge:

$$V([A]_\vee,[P]_\vee)(x) \cong \bigvee_{(a,b) \in \{\mathscr{C}^2\}_p} P^{x_0}(a_0,b_p) \wedge \bigwedge_{1 \le i \le p} A^{x_i}(a_i,b_{i-1}).$$

Then j_x sends the $c = (c_0,\ldots,c_p)$ summand onto the (c,c) summand, and Tr sends the (a,b) summand onto the basepoint if $a \ne b$ and onto the $(a_0,\ldots,a_p)$ summand if $a = b$. Now, $Tr \circ j = id$, and we define a prehomotopy from $j \circ Tr$ to the identify to get

1.6.22. Lemma. *Let A be a ring functor on $\mathscr{C}$ with unit, and let P be a unital A bimodule. Then the inclusion*

$$i : THH(A,P) \subseteq THH([A],[P])$$

is a weak homotopy equivalence, and if $P = A$ a C-equivalence.

Proof. Let $X = V = THH(A,P)$, $Y = THH([A],[P])$ and $W = THH([A]_\vee,[P]_\vee)$, and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \| & & \uparrow \\ V & \xrightarrow{j} & W \end{array}$$

We want to show that all maps are weak equivalences, and that if $P = A$ we are in the situation of Lemma 1.5.12.

The right vertical map is induced by a stable equivalence, and so by Lemma 1.6.10 the requirements are satisfied.

Regarding the lower horizontal map, we know that it is split by Tr, $Tr \circ j = id_V$, and we define a prehomotopy (special if $P = A$)

$$H: THH([A]_\vee,[P]_\vee) \times N^{cy}(\mathscr{I}) \to THH([A]_\vee,[P]_\vee)$$

from $j \circ Tr$ to the identity as follows. If $\alpha = (i_p \leftarrow i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_p) \in N_p^{cy}(\mathscr{I})$ we define a map $H_{\alpha,x}: V([A]_\vee,[P]_\vee)(x) \to V([A]_\vee,[P]_\vee)(x)$ by sending $y \in P^{x_0}(a_0,b_p) \wedge A^{x_1}(a_1,b_0) \wedge \cdots \wedge A^{x_p}(a_p,b_{p-1})$ to

$$H_{\alpha,x}(y) = \begin{cases} y & \text{if for } 0 \le k \le p,\ i_k = 0 \Rightarrow a_k = b_k, \\ * & \text{otherwise.} \end{cases}$$

Applying $\Omega^{\sqcup x}$ and the homotopy colimit this defines the prehomotopy as p varies by restricting to $\Delta(1) \subseteq N^{cy}(\mathscr{I})$. We note that we get a special homotopy if $P = A$. □

2. THH of exact categories and algebraic K-theory

2.0. Introduction

In this chapter we specialize to the case of the linear ring functor given by $X \mapsto \mathfrak{C}(-,-) \otimes \tilde{Z}[X]$ where $\mathfrak{C}$ is an additive category. In this case it is possible to say much more about the topological Hochschild homology. In particular the homotopy type may be recognized as the homology of the category itself. Most results will be derived from the mixing of THH and Waldhausen's S construction, and we prove for split exact categories $\mathfrak{C}$, that the inclusion $|THH(\mathfrak{C})| \to \Omega|THH(S\mathfrak{C})|$ is a C_r equivariant homotopy equivalence for every r. This means that one could choose to study $\Omega THH(S-)$ instead for split exact categories, and it may be argued that this perhaps is the right theory even in the more general cases.

The main interest in this theory is as a target for a map from K-theory. For any exact category $\mathfrak{C}$ we have an inclusion of simplicial objects

$$obS\mathfrak{C} \to THH_0(S\mathfrak{C}) \to THH(S\mathfrak{C}).$$

The latter is the inclusion by degeneracies, and the former is given by sending $c \in obS\mathfrak{C}$ to $id_c \in S\mathfrak{C}(c,c) \subseteq THH_0(S\mathfrak{C})$. The obviously maps into the fixed points of both the cyclic actions and the Frobenius maps, and so defines a map

$$|obS\mathfrak{C}| \to TC(S\mathfrak{C})$$

analogous to a delooping of the cyclotomic trace [3]. As K-theory depends on the choice of split exact sequences, it is not unreasonable to allow for this in the target. Thus the choice of the model incorporating the S construction should not be considered undesirable. In fact, with this definition we immediately get a spectrum level map from K-theory by applying the S construction repeatedly. Anyway, it is clear that this definition is closer to K-theory at the same time as it has attractive properties inherited from K-theory we do not find in general in the simpler definition. For the special case of the K-theory of a ring we shall see that the above map agrees with earlier definitions. More generally, if $\mathfrak{C}$ is any exact category, these maps agree with the Dennis trace when composing with the maps into Hochschild homology.

In the relative situation of a nilpotent extension of a ring, the second author has shown that the map to topological cyclic homology is an equivalence after completion at a prime, thus extending the rational computations of Goodwillie.

As before, we will in this chapter write out the statements in terms of THH only, and leave it to the reader to supply the accompanying statements for TC when necessary.

Roughly, Section 2.1 treats the part of the theory compatible with the cyclic action, whereas the results in Section 2.2 have no cyclic analogues (and thus no counterpart for TC). The results still have interest in relation to the map from K-theory. Corollary 2.2.4 says that the last map in the definition of the map from K-theory, namely the map by degeneracies suitably stabilized is a homotopy equivalence. More precisely the lower right horizontal map in

$$\begin{array}{ccccc}
K(\mathfrak{C}) = \Omega S\mathfrak{C} & \longrightarrow & \Omega THH_0(S\mathfrak{C}) & \longrightarrow & \Omega THH(S\mathfrak{C}) \\
\downarrow \simeq & & \downarrow & & \downarrow \simeq \\
\lim_k \Omega^k S^{(k)}\mathfrak{C} & \longrightarrow & \lim_{k\to\infty} \Omega^k THH_0(S^{(k)}\mathfrak{C}) & \xrightarrow{\simeq} & \lim_{k\to\infty} \Omega^k THH(S^{(k)}\mathfrak{C})
\end{array}$$

is a homotopy equivalence. Thus from the non-cyclic point of view we may choose $\lim_{k\to\infty} \Omega^k THH_0(S^{(k)}\mathfrak{C})$ to be our model for topological Hochschild homology. From the point of algebraic K-theory this is a sufficiently simple model to be comparable with K-theory. In fact it was this model which was used to show that stable K-theory is equal to topological Hochschild homology for simplicial rings in [7]. The proof offered here is a simplification of the original argument [7, Theorem 2.6] extended to the present generality. Furthermore, comparison with K-theory makes it possible to translate theorems on K-theory which are sufficiently nice on some endomorphism categories (see 2.3.1). As an example we prove a resolution theorem.

2.0.1. K-theory of exact categories. A *category with cofibrations* [17] is a pointed category $\mathscr{C}$ together with a cubcategory $co\mathscr{C}$ satisfying the following axioms:

(1) $co\mathscr{C}$ contains all isomorphism in $\mathscr{C}$.

(2) $co\mathscr{C}$ contains all maps from the base point.

(3) If $a \to b \in co\mathscr{C}$ and $a \to c \in \mathscr{C}$, then the pushout $c \coprod_a b$ exists and $c \to c \coprod_a b$ is in $co\mathscr{C}$.

The morphisms in $co\mathscr{C}$ will be called cofibrations and typically be represented by a feathered arrow $\rightarrowtail$. A pointed functor between categories with cofibrations is called *exact* if it preserves cofibrations and the pushout diagrams of axiom (3).

In particular an exact category is a category with cofibration by choosing a zero object and letting the cofibrations to be the admissible monomorphisms.

If $\mathscr{C}$ is a category with cofibrations, a subcategory $w\mathscr{C}$ is *a category of weak equivalences in* $\mathscr{C}$ if $w\mathscr{C}$ contains all isomorphisms, and if for every commutative diagram

$$\begin{array}{ccccc} b & \leftarrowtail & a & \longrightarrow & c \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ b' & \leftarrowtail & a' & \longrightarrow & c' \end{array}$$

where the horizontal maps to the left are cofibrations and where all vertical arrows are in $w\mathscr{C}$, the induced map $b \coprod_a c \to b' \coprod_{a'} c'$ is also in $w\mathscr{C}$. In this case we call $\mathscr{C}$ a category with weak equivalences and cofibrations. The morphisms in $w\mathscr{C}$ are naturally called weak equivalences.

If $\mathscr{C}$ is a category with cofibrations we have many different choices of weak equivalences. The minimal choice is just the isomorphisms, and the maximal is to allow all morphisms as weak equivalences.

Given a category with cofibrations $\mathscr{C}$, we define a simplicial category with cofibrations $S\mathscr{C}$ as follows. Let $[n]$ now denote the ordered set $(0 < 1 < \cdots < n)$ considered as a category, and let $Ar[n]$ be the corresponding arrow category. We let $S_n\mathscr{C}$ be the category of functors $A : Ar[n] \to \mathscr{C}$ having the property that for every j $A(j = j) = 0$ and such that if $i \le j \le k$ then $A(i \le j) \to A(i \le k)$ is a cofibration and

$$\begin{array}{ccc} A(i \le j) & \longrightarrow & A(i \le k) \\ \downarrow & & \downarrow \\ A(j = j) & \longrightarrow & A(j \le k) \end{array}$$

is a pushout. Thus to give an object in $S_n\mathscr{C}$ is the same as to give a sequence of cofibrations

$$c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n$$

together with a choice for each $i \le j$ of "quotients" c_j/c_i representing $0 \coprod_{c_i} c_j$ (for $i = j$ there is no choice but 0). The cofibrations in $S_n\mathscr{C}$ are natural transformations with values in $co\mathscr{C}$, and if $\mathscr{C}$ has a category of weak equivalences $w\mathscr{C}$ so has $S_n\mathscr{C}$ by choosing $wS_n\mathscr{C}$ to be the natural transformations with values in $w\mathscr{C}$.

The K-theory of a category with cofibrations and weak equivalences $\mathscr{C}$ is then defined to be $\Omega|NwS\mathscr{C}|$ where N is the nerve applied degreewise on the simplicial category $\{p \mapsto wS_p\mathscr{C}\}$. In particular, if $\mathscr{C}$ is an exact category we choose the admissible monomorphisms to be cofibrations and the isomorphisms as weak equivalences. This agrees with the definition of Quillen, and we note that $S\mathscr{C}$ becomes a simplicial exact category with the expected structure. As a last point it should be noted that if $i\mathscr{C} \subseteq \mathscr{C}$ is the isomorphisms, then $obS\mathscr{C} = N_0 iS\mathscr{C} \subset NiS\mathscr{C}$ is a homotopy equivalence. When there is no danger of confusion we will simply write $S\mathscr{C}$ for the simplicial set $obS\mathscr{C}$.

2.0.2. Constructions on linear categories and relations with ring functors. Let $\mathfrak{C}$ be a linear category. Then we defined the linear ring functor on $\mathfrak{C}$ to be given by $X \mapsto \mathfrak{C}(-,-) \otimes \tilde{\mathbf{Z}}[X]$ and we denoted this ring functor again $\mathfrak{C}$. For any ring functor we defined the matrix, product, upper triangular matrix of the ring functor. We may also define similar constructions on the category itself, and we will show that as far as topological Hochschild homology concerns, the two approaches are equivalent. Firstly we see that the linear ring functor on a product of linear categories is isomorphic to the product of the respective linear ring functors. We will consequently make no notational distinctions between these ring functors. As to the matrices, let $m_n\mathfrak{C}$ be the matrix category of $\mathfrak{C}$, i.e. its objects are n tuples of objects in $\mathfrak{C}$ and morphisms are $n \times n$ matrices under matrix multiplication. Then

$$\begin{aligned}(m_n\mathfrak{C})^X(A,B) &= \Big(\bigoplus_{1 \le r,s \le n} \mathfrak{C}(pr_rA, pr_sB)\Big) \otimes \tilde{Z}[X] \\ &\cong \bigoplus_{1 \le r,s \le n} (\mathfrak{C}(pr_rA, pr_sB) \otimes \tilde{Z}[X]) = (M_n\mathfrak{C})^X_\oplus(A,B)\end{aligned}$$

and this isomorphism is compatible with composition. Similarly we define the upper triangular matrix category on $\mathfrak{C}$, here denoted $t_n\mathfrak{C}$, and get that

$$(t_n\mathfrak{C})^X(A,B) \cong (T_n\mathfrak{C})^X_\oplus(A,B).$$

As $\mathfrak{C}^n$, $m_n\mathfrak{C}$ and $t_n\mathfrak{C}$ have the same objects, this means that $THH(m_n\mathfrak{C}) = THH(M_n\mathfrak{C})_\oplus)$ and $THH(t_n\mathfrak{C}) = THH((T_n\mathfrak{C})_\oplus)$, and as the inclusions $M_n\mathfrak{C} \to (M_n\mathfrak{C})_\oplus$ and $T_n\mathfrak{C} \to (T_n\mathfrak{C})_\oplus$ are stable equivalences we have that the induced maps $THH(M_n\mathfrak{C}) \to THH(m_n\mathfrak{C})$ and $THH(T_n\mathfrak{C}) \to THH(t_n\mathfrak{C})$ are C-equivalences, and hence C_r homotopy equivalences for all r on the realizations.

Given any small exact category $\mathfrak{C}$, Waldhausen's S-construction yields a simplicial additive category $S\mathfrak{C}$, and we will study THH of the ring functors on $S_p\mathfrak{C}$ given by $X \mapsto S_p\mathfrak{C}(-,-) \otimes \tilde{Z}[X]$. Call this ring functor for simplicity just $S_p\mathfrak{C}$. This forms a simplicial object $THH(S_p\mathfrak{C})$ which we will simply call $THH(S\mathfrak{C})$.

2.0.3. Immediate properties of $THH(S-)$. We will now specialize, and think of THH as a functor from some category of exact categories to pointed cyclic sets. The methods in [10, Sections 3.4–3.6] apply to any such functor provided it sends the trivial category to a point and respects finite products plus some π_*-Kan condition. The latter point is no problem here as we know by 1.4.7 that THH is degreewise equivalent to a simplicial abelian group. Thus we have for free:

2.0.4. Proposition (Additivity). *Let $\mathfrak{C}$ be an exact category. Then*

$$THH(SS_2\mathfrak{C}) \xrightarrow{d_0 \times d_2} THH(S\mathfrak{C}) \times THH(S\mathfrak{C})$$

is a special homotopy equivalence.

2.0.5. Proposition (Long exact sequences). *Let $F : \mathfrak{C} \to \mathfrak{D}$ be an exact functor. Then*

$$THH(S\mathfrak{D}) \to THH(SS(F : \mathfrak{C} \to \mathfrak{D})) \to THH(SS\mathfrak{C})$$

is a special fiber sequence (meaning a fibre sequence of cyclic sets such that the leftmost term is specially homotopy equivalent to the actual homotopy fibre).

2.0.6. Proposition (Delooping theorem). *Let $\mathfrak{C}$ be an exact category. Then*

$$THH(S\mathfrak{C}) \to \Omega THH(SS\mathfrak{C})$$

induces an C_r equivariant homotopy equivalence on the realization for all r.

Thus we have a new spectrum $\{k \mapsto THH(S^k\mathfrak{C})\}$ which is an Ω spectrum after the first term. Just as in the linear case, we shall see that the delooping theorem holds with $\mathfrak{C}$ in place of $S\mathfrak{C}$ in the case where $\mathfrak{C}$ is split exact.

2.0.7. The simplicial theory in an additive category. The following is a collection of special properties of the morphism functor on the additive category and its behaviour on simplicial objects. All results are formal and the reader may safely skip this subsection if he feels familiar with the subject. Let $\mathfrak{C}$ be an additive category. Then the simplicial objects in $\mathfrak{C}$, $s\mathfrak{C}$, form a closed simplicial model category over the pointed simplicial sets. Most importantly it has products with finite pointed simplicial sets: given $X \in fs_*\mathscr{E}ns$ and $c \in s\mathfrak{C}$ then their product is $c \otimes \tilde{Z}[X] = \{p \mapsto \oplus_{\sigma \in X_p - *} c_p\}$. The simplicial homomorphism group, denoted $s\mathfrak{C}(a, b)$, is the simplicial group which in degree p consists of the simplicial $\mathfrak{C}$ maps $a \otimes \tilde{Z}[\Delta[p]_+] \to b$. The canonical isomorphism

$$\phi : s\mathfrak{C}(a \otimes \tilde{Z}[X], b) \cong s_*\mathscr{E}ns(X, s\mathfrak{C}(a, b))$$

is given by $\phi(f)(x \wedge y)(\alpha \otimes z) = f(\alpha \otimes z^*(x \wedge y))$ where $f \in s\mathfrak{C}(a \otimes \tilde{Z}[X], b)_p$, $x \in X_q$, $y \in (\Delta[p]_+)_q$, $\alpha \in a_r$ and $z \in (\Delta[q]_+)_r$ with inverse given by $\phi^{-1}(g)$ $(\beta \otimes (x \wedge y) = g(x \wedge y)(\beta \otimes id)$ where $g \in s_*\mathscr{E}ns(X, s\mathfrak{C}(a, b))_p$, $\beta \in a_q$ and id is the identity $[q] = [q] \in \Delta[q]$.

The inclusion

$$s\mathfrak{C}(a, b) \to s\mathfrak{C}(a \otimes \tilde{Z}[X], b \otimes \tilde{Z}[X])$$

given by $f \mapsto f \otimes id$ is compatible with composition, and in the special case $X = S^n$ it is a homotopy equivalence as seen by induction as follows. It is enough to consider the case $X = S^1$. The statement then is equivalent to showing that $s_* \mathscr{E}ns(S^1, -)$ and $- \otimes \tilde{\boldsymbol{Z}}[S^1]$ are loop and suspension (bar construction) on simplicial abelian groups, and that the inclusion into the loops of the suspension is a homotopy equivalence, which is shown in [13, II, 6.4]. That the map is as given follows from the diagram with short exact rows

$$\begin{array}{ccccc} G & \xrightarrow{i_1} & G \otimes \tilde{\boldsymbol{Z}}[\Delta[1]] & \xrightarrow{id \otimes p} & G \otimes \tilde{\boldsymbol{Z}}[S^1] \\ \downarrow u & & \downarrow v & & \| \\ s_*\mathscr{E}ns(S^1, G \otimes \tilde{\boldsymbol{Z}}[S^1]) & \longrightarrow & s_*\mathscr{E}ns(\Delta[1], G \otimes \tilde{\boldsymbol{Z}}[S^1]) & \xrightarrow{ev_1} & G \otimes \tilde{\boldsymbol{Z}}[S^1] \end{array}$$

where G is any simplicial abelian group. Here i_1 is the obvious inclusion, p the projection $\Delta[1] \to S^1$ and ev_1 is the evaluation at $1 \in \Delta[1]$. The maps u and v are defined as follows. Let $h: \Delta[1] \times \Delta[1] \to \Delta[1]$ be the construction sending a pair onto the one with the most zeros. Then if $g \in G_p$ and $\sigma \in \Delta[1]_p$ we define $v(g, \sigma)$: $\Delta[1] \wedge \Delta[p]_+ \to G \otimes \tilde{\boldsymbol{Z}}[S^1]$ to be the map sending $\tau \wedge \omega$ to $\omega^* g \otimes p \circ h(\sigma, \tau)$. We see that this makes the right square commute, and induces $u(g)(\tau \wedge \omega) = \omega^* g \otimes \tau$. Setting $G = s\mathfrak{C}(a,b)$ and composing with the canonical isomorphism, we see that the map coincides with the inclusion above.

An important special feature is that

$$s\mathfrak{C}(a,b) \otimes \tilde{\boldsymbol{Z}}[X] \cong s\mathfrak{C}(a, b \otimes \tilde{\boldsymbol{Z}}[X])$$

as a sum (=product!) of morphisms is the same as a morphism to the sum.

If $a \in \mathfrak{C}$ considered as a constant simplicial object, and $b \in s\mathfrak{C}$ then $\mathfrak{C}(a,b) = \{p \mapsto \mathfrak{C}(a, b_p)\}$ may naturally be identified with $s\mathfrak{C}(a,b)$, so in particular if $a, b \in \mathfrak{C}$ and $X \in fs_*\mathscr{E}ns$ then $\mathfrak{C}(a,b) \otimes \tilde{\boldsymbol{Z}}[X] \cong s\mathfrak{C}(a, b \otimes \tilde{\boldsymbol{Z}}[X])$. All in all we get that the inclusion

$$\mathfrak{C}(a,b) \otimes \tilde{\boldsymbol{Z}}[X] \to s\mathfrak{C}(a \otimes \tilde{\boldsymbol{Z}}[S^n], b \otimes \tilde{\boldsymbol{Z}}[X \wedge S^n])$$

is a homotopy equivalence compatible with the compositions.

2.0.8. Ring functors on non-linear categories. We end this section by taking a quick and incomplete look at how to extend the idea of the linear ring functor to other categories with weak equivalences and cofibrations. Note that a category with cofibrations $\mathscr{C}$ has finite sums. For a finite set X and an object $c \in \mathscr{C}$ let $\bigvee_X c$ denote the sum (over the basepoint 0) with one copy of c for each element in X. That means that we can define a product with finite pointed simplicial sets by setting for $c \in s\mathscr{C}$

and $X \in s_*\mathscr{E}ns$

$$(c \otimes X) = \left\{ p \mapsto \bigvee_{X_p - *} c_p \right\}$$

(the sum of one copy of c_p for each non-basepoint of X_p plus a copy of the base-point). We then define the functor

$$F : fs_*\mathscr{E}ns \to s_*\mathscr{E}ns^{\mathscr{C}^o \times \mathscr{C}}$$

by

$$F^X(a,b) = s\mathscr{C}(a, b \otimes X)$$

or some fibrant rectification of this in case we have trouble with homotopy liftings. This functor satisfies all the required commutative diagrams for being a ring functor on $\mathscr{C}$. The only problem is the connectivity requirements. So we have to require that for n-connected X and any $a, b \in \mathscr{C}$:

(1) $s\mathscr{C}(a, b \otimes X)$ is n connected (or more generally $n - d(a) + c(b)$ connected),

(2) the map $S^1 \wedge s\mathscr{C}(a, b \otimes X) \to s\mathscr{C}(a, b \otimes (S^1 \wedge X)) \to s\mathscr{C}(a, b \otimes S^1 \wedge X)$ is $2n - c$ connected for some c not depending on X.

In the exact case, tensor product moves outside the morphisms, and so the connectivity requirements are trivially met. However, in general there seems to be trouble. An important case where we still are able to meet the connectivity requirements is the following:

2.0.9. The space case. Let $\mathscr{C} = R_f(X)$; the category of spaces with a given space X as a retract, and only finitely many cells outside X. For any $Y \in \mathscr{C}$ and finite pointed simplicial set V we define $Y \otimes V$ to be the cofibre of $Y \times |*| \cup X \times |V| \to Y \times |V|$ (which is the realization of the construction above). Let $s\mathscr{C}(-,-)$ denote the simplicial mapping space of spaces over X, and define our ring functor on $\mathscr{C}$ to be

$$F^V(Y, Z) = s\mathscr{C}(Y, Z \otimes V).$$

This meets the relaxed connectedness conditions if we set $d^F(Y)$ to be the dimension of Y over X and $c^F(Z)$ to be the connectivity of $X \to Z$.

2.1. Cofinality, split exact categories and Morita equivalence for rings

In this section we prove two theorems with a distinctive K-theoretic flavor. The first shows that the cofinality result of K-theory is true in topological Hochschild homology. The proof consists of simply displaying a special homotopy. The second shows that if $\mathfrak{C}$ is a split exact category, then the map $\mathfrak{C} \to \Omega S\mathfrak{C}$ induces C_r equivariant homotopy equivalences on THH. In general, by the delooping theorem we have a spectrum $n \mapsto THH(S^{(n)}\mathfrak{C})$ (where $S^{(n)}(\mathfrak{C})$ is the S construction applied n times to $\mathfrak{C}$) which is an Ω-spectrum after the first term. The proposition then tells us that for split exact category this spectrum is an Ω-spectrum at all places. This is in analogy with the matrix spectrum which was used to define the Γ space structure in [2]. We end the section with a discussion of the map from K-theory and show that it agrees with the one defined in [2].

Let $\mathfrak{C} \subseteq \mathfrak{D}$ be a full inclusion of additive categories. We say that $\mathfrak{C}$ is cofinal in $\mathfrak{D}$ if for any $d \in \mathfrak{D}$ there exists some $d' \in \mathfrak{D}$ such that $d \oplus d' \in \mathfrak{C}$.

2.1.1. Lemma (Cofinality). *Let $j: \mathfrak{C} \subset \mathfrak{D}$ be an inclusion of a cofinal subcategory into an additive category. Then*

$$THH(\mathfrak{C}) \to THH(\mathfrak{D})$$

is a special homotopy equivalence.

Proof. For each $d \in \mathfrak{D}$ choose a $c(d) \in \mathfrak{C}$ such that d is a summand in $c(d)$, and if d actually is in $\mathfrak{C}$, choose $c(d) = d$. Let $d \xrightarrow{i(d)} c(d) \xrightarrow{p(d)} d$ be the chosen inclusion and projection into and from the sum. Then

$$\bigvee_{(d_0,\dots,d_p)\in\mathfrak{D}^{p+1}} \mathfrak{D}(p(d_0), i(d_p)) \otimes \tilde{Z}[S^{x_0}] \wedge \cdots \wedge \mathfrak{D}(p(d_p), i(d_{p-1})) \otimes \tilde{Z}[S^{x_p}]$$

is a map $V(\mathfrak{D})(x) \to V(\mathfrak{C})(x)$. This map is compatible with the cyclic operations and hence defines a map $D(p,i): THH(\mathfrak{D}) \to THH(\mathfrak{C})$. Obviously $D(p,i) \circ THH(j)$ is the identity on $THH(\mathfrak{C})$ and we will show that the other composite is specially homotopic to the identity. The desired special homotopy can be expressed as follows. Let $\alpha = (i_p \leftarrow i_0 \leftarrow \cdots \leftarrow i_{p-1} \leftarrow i_p) \in N^{cy}(\mathscr{I})$ and let $p(d)^{i_k}$ (resp. $i(d)^{i_k}$) be $p(d)$ (resp. $i(d)$) if $i_k = 1$ and the identity on d otherwise. Then the desired special homotopy $THH(\mathfrak{D}) \times N^{cy}(\mathscr{I}) \to THH(\mathfrak{D})$ is

$$\underset{x\in I^{p+1}}{\underrightarrow{\operatorname{holim}}}\ \Omega^{\sqcup x} \bigvee_{(d_0,\dots,d_p)\in\mathfrak{D}^{p+1}} \mathfrak{D}(p(d_0)^{i_0}, i(d_p)^{i_p}) \otimes \tilde{Z}[S^{x_0}]$$

$$\wedge \cdots \wedge \mathfrak{D}(p(d_p)^{i_p}, i(d_{p-1})^{i_{p-1}}) \otimes \tilde{Z}[S^{x_p}]. \qquad \square$$

2.1.2. A delooping in the split exact case. An exact category $\mathfrak{C}$ is called *split* if all exact sequences split. The category of finitely generated projective modules is an example of a split category, and generally, any additive category may be regarded as an exact category by considering only the split exact sequences.

2.1.3. Proposition. *Let $\mathfrak{C}$ be a split exact category. Then there is a C_r equivariant homotopy equivalence*

$$|THH(\mathfrak{C})| \xrightarrow{\simeq} \Omega|THH(S\mathfrak{C})|$$

Proof. As $\mathfrak{C}$ is split we have by [10] that for each n the functor $t_n\mathfrak{C} \to S_n\mathfrak{C}$ given by sending $(c_1, \dots, c_n)$ to $c_1 \rightarrowtail c_1 \oplus c_2 \rightarrowtail \cdots \rightarrowtail c_1 \oplus \cdots \oplus c_n$ is an equivalence. Let $S_n\mathfrak{C} \to \mathfrak{C}^n$ be the functor sending $c_1 \rightarrowtail c_2 \rightarrowtail \cdots \rightarrowtail c_n$ (with choices of quotients) to

$(c_1, c_2/c_1, \ldots, c_n/c_{n-1})$. The linear ring functor associated to the $\mathfrak{C}^n$ and the n fold product of the linear ring functor associated to $\mathfrak{C}$ are isomorphic, and both interpretations will again be denoted $\mathfrak{C}^n$.

Consider the commutative diagram

$$\begin{array}{ccccc}
THH(S_n\mathfrak{C}) & \longrightarrow & THH(\mathfrak{C}^n) & \xrightarrow[\simeq]{\text{Prop. 1.6.15}} & (THH(\mathfrak{C}))^n \\
\text{Example 1.6.7}\ \big\uparrow \simeq & & \text{Prop. 1.6.20}\ \big\uparrow \simeq & & \\
THH(t_n\mathfrak{C}) & \xrightarrow[\simeq]{2.0.2} & THH(T_n\mathfrak{C}) & &
\end{array}$$

The labelled arrows are C-equivalences for the given reasons (and hence induce C_r equivariant homotopy equivalences for all r on the realizations). Given a simplicial object X, let PX be the path space with $PX_n = X_{n+1}$ and face and degeneracy maps shifted up by one. The sequence $\mathfrak{C} \to PS\mathfrak{C} \to S\mathfrak{C}$ yields a diagram for each n:

$$\begin{array}{ccccc}
THH(\mathfrak{C}) & \longrightarrow & THH(PS_n\mathfrak{C}) & \longrightarrow & THH(S_n\mathfrak{C}) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
THH(\mathfrak{C}) & \longrightarrow & (THH(\mathfrak{C}))^{n+1} & \longrightarrow & (THH(\mathfrak{C}))^n
\end{array}$$

The vertical maps are special homotopy equivalences and the lower sequence is the trivial fibration. The bisimplicial sets involved satisfy the π_*-Kan condition (see [4]) as they are related by termwise (ordinary) equivalences to bisimplicial abelian groups by the results in 1.4.7. Thus we know that the total of the upper row is a cyclic fibre sequence. Furthermore $PS\mathfrak{C}$ is contractible by exact functors, so $THH(PS\mathfrak{C})$ is contractible as a cyclic space and the result follows. □

The above proof breaks down in the cases where exact sequences which do not split are allowed. It may still be argued that $\Omega\,|THH(S\mathfrak{C})|$ is a better behaved theory than $THH(\mathfrak{C})$ in view of the properties listed in Section 2.0. It would be interesting to do calculations on concrete examples to gauge the difference between these theories. In Section 2.3.4 we will give a simple example.

By forgetting structure, any exact category $\mathfrak{C}$ is an additive category, and choosing the split monomorphisms to be cofibrations we define the K-theory of any additive category. We write $\bar{S}\mathfrak{C}$ for the S construction applied to the underlying additive (split exact) category. Thus any exact category give rise to two spectra, namely $\{n \mapsto THH(\bar{S}^{(n)}\mathfrak{C})\}$ and $\{n \mapsto THH(S^{(n)}\mathfrak{C})\}$. The former is always an Ω-spectrum, and is the translation of the matrix spectrum of [2], whereas the other is dependent upon the exact sequences in the category.

2.1.4. The projective modules over a ring. Let A be an associative ring with unit and let $\mathscr{P}_A$ be the category of finitely generated projective A modules. A itself may be considered as a full subcategory whose only object is the rank one module A, and each element $a \in A$ identified with the homomorphism given by multiplication by a.

The classical Morita equivalence now follows from cofinality. One should note that given two Morita equivalent rings considered as categories with only one object, their homologies do not coincide. In particular, the homology of the ring considered as a category with one object is different from the homology of the category of finitely generated projective modules (which is a good thing as $H_n(A, A) = HH_n(\mathbf{Z}[A], A)$ is *not* topological Hochschild homology).

2.1.5. Proposition (Morita equivalence for rings). *Let A be an associative ring with unit. Then the inclusion $A \subseteq \mathscr{P}_A$ given by sending the same object of A to the rank one module induces a special homotopy equivalence*

$$THH(A) \xrightarrow{\simeq} THH(\mathscr{P}_A).$$

Proof. Let $\mathscr{F}_A$ be the category of finitely generated free modules, and let $\mathscr{F}_A^k$ be the subcategory of free modules of rank less than or equal to k. Then the inclusion $m_k A \to \mathscr{F}_A^k$, given by regarding $m_k A$ as the subcategory with only object the rank k module is an equivalence of categories. Consider the diagram where the limit is taken with respect to inclusion by zeros:

$$\begin{array}{ccccc}
THH(A) & \longrightarrow & THH(\mathscr{P}_A) & \underset{\simeq}{\xleftarrow{\text{Prop. 2.1.1}}} & THH(\mathscr{F}_A) \\
{\scriptstyle \text{Prop. 1.6.18}}\downarrow{\scriptstyle \simeq} & & & & {\scriptstyle \text{Lemma 1.6.9}}\uparrow{\scriptstyle \simeq} \\
\lim_{k\to\infty} THH(M_k A) & \underset{\simeq}{\xrightarrow{2.0.2}} & \lim_{k\to\infty} THH(m_k A) & \underset{\simeq}{\xrightarrow{\text{Example. 1.6.7}}} & \lim_{k\to\infty} THH(\mathscr{F}_A^k)
\end{array}$$

The maps are C-equivalences for the given reasons and the result follows. □

2.1.6. The map from K-theory and agreement with earlier definitions in the ring case. As remarked earlier, the raison d'etre for topological Hochschild homology is that it is the target for a map from algebraic K-theory. In our setting this map is extremely simple, and we show that it agrees with the existing definition in the case of rings. We note that the image of this map consists of fixed points under both the cyclic action and the Frobenius map and so the map factors through topological cyclic homology.

Consider the map $D: ob\mathfrak{C} \to THH_0(\mathfrak{C}) \to THH(\mathfrak{C})$ given by sending an object $c \in ob\mathfrak{C}$ to the corresponding identity morphism in

$$\mathfrak{C}(c,c) \subseteq \underset{\substack{\longrightarrow \\ x \in I}}{\operatorname{holim}}\ \Omega^x \bigvee_{c \in \mathfrak{C}} \mathfrak{C}(c,c) \otimes \tilde{\mathbf{Z}}[S^x] = THH_0(\mathfrak{C})$$

followed by the inclusion by degeneracies. This immediately gives the map from algebraic K-theory by using $S\mathfrak{C}$ as our exact category:

$$\Omega|SD| : \Omega|obS\mathfrak{C}| \to \Omega|THH(S\mathfrak{C})|.$$

One should note that the observation that this map induces a map of spectra is trivial, for the commutativity of the diagram

$$\begin{array}{ccc} |obS^{(n)}\mathfrak{C}| & \xrightarrow{|S^{(n)}D|} & |THH(S^{(n)}\mathfrak{C})| \\ \downarrow & & \downarrow \\ \Omega|obS^{(n+1)}\mathfrak{C}| & \xrightarrow{\Omega|S^{(n+1)}D|} & \Omega|THH(S^{(n+1)}\mathfrak{C})| \end{array}$$

follows from the definition of D applied to the sequences

$$\begin{array}{ccccc} S^{(n)}\mathfrak{C} & \longrightarrow & PS^{(n+1)}\mathfrak{C} & \longrightarrow & S^{(n+1)}\mathfrak{C} \\ {\scriptstyle D}\downarrow & & {\scriptstyle PSD}\downarrow & & {\scriptstyle SD}\downarrow \\ THH(S^{(n)}\mathfrak{C}) & \longrightarrow & THH(PS^{(n+1)}\mathfrak{C}) & \longrightarrow & THH(S^{(n+1)}\mathfrak{C}) \end{array}$$

For any additive category let $i\mathfrak{C}$ be the subcategory of isomorphisms, and let $Ni\mathfrak{C}$ denote the nerve of $i\mathfrak{C}$. The inclusion $ob\mathfrak{C} = N_0 i\mathfrak{C} \to Ni\mathfrak{C}$ is a homotopy equivalence. There is a map

$$Ni\mathfrak{C} \to N^{cy}(\mathfrak{C}) \to THH(\mathfrak{C})$$

given by sending an object

$$c_0 \underset{\cong}{\xleftarrow{\alpha_1}} c_1 \underset{\cong}{\xleftarrow{\alpha_2}} \cdots \underset{\cong}{\xleftarrow{\alpha_p}} c_p \in N_p i\mathfrak{C}$$

to

$$c_p \xleftarrow{(\prod_{i=1}^{p}\alpha_i)^{-1}} c_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_p} c_p \in N^{cy}(\mathfrak{C})$$

which is sent to THH as a map from an appropriate smash product of S^0 with itself. Now, D factors through these maps

$$\begin{array}{ccc} ob\mathfrak{C} & \xrightarrow{\sim} & Ni\mathfrak{C} \\ \downarrow & & \downarrow \\ THH_0(\mathfrak{C}) & \longrightarrow & THH(\mathfrak{C}) \end{array}$$

The cyclotomic trace from K-theory of a ring to topological Hochschild homology of a ring A is defined in [3] to be the plus construction on the realization of the map

$$\varinjlim_k Ni(m_k A) \to \varinjlim_k THH(m_k A)$$

(or more precisely, in [3] it is the map of Γ-spaces arising from this map, and sent onto the limit of the fixed points of the finite actions). If we compose with the equivalences $\lim_{k\to\infty} THH(m_k A) \xrightarrow{\simeq} THH(\mathscr{P}_A) \xrightarrow{\simeq} \Omega THH(S\mathscr{P}_A)$ we see that we have a factorization

$$\begin{array}{ccc} |\varinjlim_k Ni(m_k A)| & \longrightarrow & |\varinjlim_k THH(m_k A)| \\ \downarrow & & \downarrow \simeq \\ \Omega|NiS\mathscr{P}_A| & \longrightarrow & \Omega|THH(S\mathscr{P}_A)| \end{array}$$

By the universality of the plus construction with respect to mappings into spaces with fundamental groups void of perfect subgroups the lower map must realize the map defined by the plus construction. So composing with the equivalence $\Omega obS\mathfrak{C} \xrightarrow{\simeq} \Omega NiS\mathfrak{C}$ we get that

$$\Omega SD : \Omega S\mathscr{P}_A \to \Omega THH(S\mathscr{P}_A)$$

agrees with Bökstedt's trace. Inverting $\Omega THH(S\mathscr{P}_A) \xleftarrow{\simeq} THH(\mathscr{P}_A) \xleftarrow{\simeq} THH(A)$ we may regard this as a map $\Omega obS\mathscr{P}_A \to THH(A)$.

As earlier observed, $SD : obS\mathfrak{C} \to THH(S\mathfrak{C})$ maps into the fixed points of both the cyclic action and the Frobenius maps. This is so because in each degree an element $c \in S\mathfrak{C}$ is mapped onto the point represented by an appropriate wedge of S^0's mapping onto $c = c = \cdots = c \in S\mathfrak{C}(c,c) \wedge \cdots \wedge S\mathfrak{C}(c,c)$. Thus we have a lifting to the topological cyclic homology agreeing with the one defined in [3] after completion at a prime.

2.1.7. Weak equivalences. So far, the only weak equivalences we have considered are the isomorphisms. However, at this level, the introduction of weak equivalences poses no new problems. Let $w\mathfrak{C}$ be a subcategory of weak equivalences in an exact category $\mathfrak{C}$ (for instance homotopy equivalence between simplicial modules of some sort). Let $NwS\mathfrak{C}$ be the bisimplicial category with objects elements in the ordinary nerve of $wS\mathfrak{C}$ and morphisms natural transformations in $S\mathfrak{C}$ (note that *all* morphisms in $S\mathfrak{C}$ are allowed, not just the weak equivalences). This becomes a new exact category. Thus, using the linear ring functor construction again we define the topological Hochschild homology of $\mathfrak{C}$ with respect to $w\mathfrak{C}$ as $THH(NwS\mathfrak{C})$. This is compatible with the earlier definition in the case of isomorphisms as the proof of Waldhausen ([17, 1.4.1], extended to allow for morphisms) that $S\mathfrak{C} \simeq NiS\mathfrak{C}$ only required a simplicial homotopy of exact categories and hence induces a homotopy equivalence of cyclic objects $THH(S\mathfrak{C}) \simeq THH(NiS\mathfrak{C})$. Thus, if $\mathfrak{C}$ is an exact category, one may equally well map K-theory of the isomorphisms to THH via

$$obNiS\mathfrak{C} \to THH_0(NiS\mathfrak{C}) \subseteq THH(NiS\mathfrak{C})$$

which is compatible with the homotopy equivalence $S\mathfrak{C} \subseteq NiS\mathfrak{C}$.

More generally, if $w\mathfrak{C}$ is any subcategory of weak equivalences in an exact category we can define the map from the algebraic K-theory $\Omega|obNwS\mathfrak{C}|$ to $\Omega|THH(NwS\mathfrak{C})|$ by the same map as before (inclusion by the identity followed by the degeneracies). In the case where $w\mathfrak{C}$ are the isomorphisms, this brings nothing new, but if it is something else $\Omega|THH(NwS\mathfrak{C})|$ is a new theory, potentially reflecting the algebraic K-theory better than just $THH(\mathfrak{C})$. We have not needed the result, and thus not carried through the calculations, but it seems reasonable to believe that for exact categories where the admissible monomorphisms split up to weak equivalence, an analysis of the proof of the equivalence of the S construction and the Γ category machine of Segal would prove a delooping theorem for this construction. One outcome would then be that in these cases this definition would agree with earlier ones on simplicial rings and that the trace map could have been constructed directly.

2.2. A non-cyclic reduction in the case of the K-theory of an exact category and comparison with the homology of categories

In this section we will give a simpler, but not cyclic, description of the topological Hochschild homology of an exact category. This reduction, together with the treatment of the split exact categories in the previous section, gives that for additive categories THH may be reduced to the ordinary homology in the sense of Baues–Wirsching. The reduction is a parallel to the reduction in [7], and shows that the non-cyclic models there agree with the present definition. When our category is the category of finitely generated projective modules over a ring the result is not new. Together with Morita equivalence for THH this gives a direct proof of the main result in [12].

Even though the reduction does not preserve the cyclic structure, it is still of general interest because the map form K-theory factors through the simpler theory. In particular, the result in [7] was obtained using this model, a fact that is important even when one wants to include the cyclic structure into the picture.

2.2.1. The simplicial abelian group $R(\mathfrak{C})$. In Section 1.4.7 we showed that THH of ring functors with a "linear" bimodule may be rewritten as the bisimplicial abelian group:

$$THH(A,P) \simeq \left\{ p \mapsto \underset{\overrightarrow{x \in I^{p+1}}}{\mathrm{holim}}\ \Omega^{\sqcup x} \bigoplus_{(c_0,\dots,c_p)\in\mathfrak{C}^{p+1}} P^{x_0}(c_0,c_p) \otimes \tilde{\mathbf{Z}}[A^{x_1}(c_1,c_0)] \right.$$
$$\left. \otimes \cdots \otimes \tilde{\mathbf{Z}}[A^{x_p}(c_p,c_{p-1})] \right\}.$$

Let $\mathfrak{C}$ be an additive category. Then in particular it is a linear category, and again, we call the ring functor induced by $X \mapsto \mathfrak{C}(-,-) \otimes \tilde{\mathbf{Z}}[X]$ simply $\mathfrak{C}$. By the discussion in 2.0.7 we have an inclusion which is a homotopy equivalence

$$\mathfrak{C}(a,b)\otimes\tilde{Z}[S^k]\to s\mathfrak{C}(a\otimes\tilde{Z}[S^n],\, b\otimes\tilde{Z}[S^{n+k}])$$

compatible with composition. Hence we may rewrite the above formula in this particular case as

$$THH(\mathfrak{C})\simeq R(\mathfrak{C})=\left\{p\mapsto \underset{x\in I^{p+1}}{\underrightarrow{\operatorname{holim}}}\ \Omega^{\sqcup x}\bigoplus_{(c_0,\dots,c_p)\in\mathscr{C}^{p+1}} W(\mathfrak{C})(x)\right\}$$

where $W(\mathfrak{C})(x)$ denotes

$$s\mathfrak{C}(c_0\otimes\tilde{Z}[S^{x_1+\cdots+x_p}],\, c_p\otimes\tilde{Z}[S^{\sqcup x}])\otimes\tilde{Z}[s\mathfrak{C}(c_1\otimes\tilde{Z}[S^{x_2+\cdots+x_p}],$$
$$c_0\otimes\tilde{Z}[S^{x_1+\cdots+x_p}])]\otimes\cdots\otimes\tilde{Z}[s\mathfrak{C}(c_p,c_{p-1}\otimes\tilde{Z}[S^{x_p}])].$$

One nice aspect about this representation is the ability to express elements of $W(\mathfrak{C})(x)$ as linear combinations of sequences of simplicial "maps" of the form

$$a_{-1}\xleftarrow{\alpha_0}a_0\xleftarrow{\alpha_1}a_1\xleftarrow{\alpha_2}\cdots\xleftarrow{\alpha_{p-1}}a_{p-1}\xleftarrow{\alpha_p}a_p$$

where $a_{-1}=c_p\otimes\tilde{Z}[S^{\sqcup x}]$, $a_k=c_k\otimes\tilde{Z}[S^{x_{k+1}+\cdots+x_p}]$ and $\alpha_i\in s\mathfrak{C}(a_i,a_{i-1})$. Note however that sums of maps beyond $a_{-1}\leftarrow a_0$ are *not* sums of maps in $s\mathfrak{C}$, but in $\tilde{Z}s\mathfrak{C}$.

With our new model we get that $\mathfrak{C}\mapsto R(\mathfrak{C})$ is a functor into simplicial abelian groups satisfying $R(0)=0$ and $R(\mathfrak{C}\times\mathfrak{D})\xrightarrow{\cong}R(\mathfrak{C})\times R(\mathfrak{D})$ (see Section 1.6.14), and so by [7, Lemma 2.5] we have that $d_0+d_2\simeq d_1:R(SS_2\mathfrak{C})\to R(S\mathfrak{C})$ where $d_0,d_1,d_2:S_2\mathfrak{C}\to S_1\mathfrak{C}\cong\mathfrak{C}$ are the face maps in the S construction, sending a short exact sequence $0\to C'\to C\to C''\to 0$ onto C'', C and C' respectively. This is not true for each R_q. The same problem is present in the proof of Theorem 2.6 in [7] and we are grateful to Lannes and Oliver for pointing this out. However, if we stabilize in the S direction we get

$$d_0+d_2\simeq d_1:\lim_{k\to\infty}\Omega^kR(S^{(k)}S_2\mathfrak{C})\to\lim_{k\to\infty}\Omega^kR(S^{(k)}\mathfrak{C})$$

which clearly suffice for the applications in [7]. This follows from [7, Lemma 2.5] and

2.2.2. Lemma. *Let Y be a functor from exact categories to simplicial abelian groups such that $Y(0)=0$. Then*

$$d_0+d_2\simeq d_1:\lim_{k\to\infty}\Omega^kY(S^{(k)}S_2\mathfrak{C})\to\lim_{k\to\infty}\Omega^kY(S^{(k)}\mathfrak{C}).$$

Proof. The only thing we need to show is that $\lim_{k\to\infty}\Omega^kYS^{(k)}$ respects products, but this is easy: regarded as a map of $2k$ multisimplicial objects the (split) map

$$Y(S^{(k)}\mathfrak{C}\times S^{(k)}\mathfrak{D})\to Y(S^{(k)}\mathfrak{C})\times Y(S^{(k)}\mathfrak{D})$$

is an isomorphism for total degree less than $2k$, and hence is $2k$ connected. □

2.2.3. Proposition. *Let $\mathfrak{C}$ be an exact category, and let $s: THH_0 \to THH_p$ be the zeroth degeneracy applied p times. Then*

$$s: \lim_{k\to\infty} \Omega^k THH_0(S^{(k)}\mathfrak{C}) \subseteq \lim_{k\to\infty} \Omega^k THH_p(S^{(k)}\mathfrak{C})$$

is a homotopy equivalence.

Proof. We know that $THH(-) \to R(-)$ is a degreewise equivalence, and so it is enough to show the proposition on R. Let $\mathfrak{R} = \lim_{k\to\infty} \Omega^k RS^{(k)}$. Let $d: \mathfrak{R}_p \to \mathfrak{R}_0$ be the zeroth face map applied p times. Then $d \circ s = id_{\mathfrak{R}_0}$. We want to show $s \circ d: \mathfrak{R}_p(\mathfrak{C}) \to \mathfrak{R}_p(\mathfrak{C})$ is homotopic to the identity. Now, if $x \in I^{p+1}$, then

$$s_x \circ d_x : W(\mathfrak{D})(x) \to W(\mathfrak{D})(x)$$

sends

$$a = (a_{-1} \xleftarrow{\alpha_0} a_0 \xleftarrow{\alpha_1} a_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{p-1}} a_{p-1} \xleftarrow{\alpha_p} a_p)$$

to $a_{-1} \xleftarrow{\beta} a_p = a_p = \cdots = a_p$ where $\beta = \alpha_0 \circ \alpha_1 \circ \cdots \circ \alpha_p$.

Proceedings in analogy with [7, Theorem 2.6] we define two natural transformations T_{id}, $T_\beta: \mathfrak{R}_p \to \mathfrak{R}_p S_2$ induced by transformations t_{id}, $t_\beta: W(\mathfrak{D})(x) \to W(S_2\mathfrak{D})(x)$ where $\mathfrak{D}$ is an exact category and $x \in I^{p+1}$. Let $a \in W(\mathfrak{D})(x)$ as above, and let $\beta_k = \alpha_k \circ \cdots \circ \alpha_p$ with $\beta = \beta_0$. Then

$$t_{id}(a) =$$

$$\begin{array}{ccccccccccc}
a_{-1} & \xleftarrow{0} & a_p & = & a_p & = & \cdots & = & a_p & = & a_p \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} & & & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} \\
a_{-1}\oplus a_{-1} & \xleftarrow{\Delta\alpha_0\pi_2} & a_p\oplus a_0 & \xleftarrow{id\oplus\alpha_1} & a_p\oplus a_1 & \xleftarrow{id\oplus\alpha_2} & \cdots & \xleftarrow{id\oplus\alpha_{p-1}} & a_p\oplus a_{p-1} & \xleftarrow{id\oplus\alpha_p} & a_p\oplus a_p \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\
a_{-1} & \xleftarrow{\alpha_0} & a_0 & \xleftarrow{\alpha_1} & a_1 & \xleftarrow{\alpha_2} & \cdots & \xleftarrow{\alpha_{p-1}} & a_{p-1} & \xleftarrow{\alpha_p} & a_p
\end{array}$$

and

$$t_\beta(a) =$$

$$\begin{array}{ccccccccccc}
a_{-1} & \xleftarrow{\beta} & a_p & = & a_p & = & \cdots & = & a_p & = & a_p \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle (1\oplus\beta_1)\Delta} & & \downarrow{\scriptstyle (1\oplus\beta_2)\Delta} & & & & \downarrow{\scriptstyle (1\oplus\alpha_p)\Delta} & & \downarrow{\scriptstyle \Delta} \\
a_{-1}\oplus a_{-1} & \xleftarrow{\Delta\alpha_0\pi_2} & a_p\oplus a_0 & \xleftarrow{id\oplus\alpha_1} & a_p\oplus a_1 & \xleftarrow{id\oplus\alpha_2} & \cdots & \xleftarrow{id\oplus\alpha_{p-1}} & a_p\oplus a_{p-1} & \xleftarrow{id\oplus\alpha_p} & a_p\oplus a_p \\
\downarrow{\scriptstyle 1-1} & & \downarrow{\scriptstyle \beta_1-1} & & \downarrow{\scriptstyle \beta_2-1} & & & & \downarrow{\scriptstyle \alpha_p-1} & & \downarrow{\scriptstyle 1-1} \\
a_{-1} & \xleftarrow{0} & a_0 & \xleftarrow{\alpha_1} & a_1 & \xleftarrow{\alpha_2} & \cdots & \xleftarrow{\alpha_{p-1}} & a_{p-1} & \xleftarrow{\alpha_p} & a_p
\end{array}$$

where i_j is the jth inclusion, π_j the jth projection and Δ the diagonal.

Again we have the relations (with the d_i being the face maps $S_2\mathfrak{D} \to \mathfrak{D}$ defined above).

$$d_0 T_{id} = id, \quad d_2 T_\beta = s_x \circ d_x, \quad d_2 T_{id} = d_0 T_\beta = 0, \text{ and } d_1 T_{id} = d_1 T_\beta,$$

and using the formula $d_0 + d_2 \simeq d_1$ just before Lemma 2.2.2 we get that

$$id = d_0 T_{id} \simeq d_1 T_{id} = d_1 T_\beta \simeq d_2 T_\beta = s_x \circ d_x. \qquad \square$$

By the realization lemma and 2.0.6 this implies

2.2.4. Corollary. $\lim_{k\to\infty} \Omega^k THH_0(S^{(k)}\mathfrak{C}) \subseteq \lim_{k\to\infty} \Omega^k THH(S^{(k)}\mathfrak{C}) \overset{\simeq}{\leftarrow} \Omega THH(S\mathfrak{C})$ *are homotopy equivalences.*

Note that $\mathfrak{R}_0 = \lim_{k\to\infty} \Omega^k R_0 S^{(k)}$ is a particularly simple object, namely

$$\mathfrak{R}_0(\mathfrak{C}) = \lim_{k\to\infty} \Omega^k \underset{\overrightarrow{x\in I}}{\operatorname{holim}}\, \Omega^x \bigoplus_{c\in S^{(k)}\mathfrak{C}} S^{(k)}\mathfrak{C}(c,c) \otimes \tilde{Z}[S^x] \simeq \lim_{k\to\infty} \Omega^k \bigoplus_{c\in S^{(k)}\mathfrak{C}} S^{(k)}\mathfrak{C}(c,c)$$

which, by a closely analogous proof to the above, was shown in [7] to be equivalent to what there was called the topological Hochschild homology of an exact category.

Perhaps the diagrams in the proof above can be understood a bit better if we forget some structure. For T_{id} we basically have the information

$$\begin{array}{ccccccccccc}
a_{-1} & \xleftarrow{\alpha_0\pi_2} & a_p \oplus a_0 & \xleftarrow{id\oplus\alpha_1} & a_p\oplus a_1 & \xleftarrow{id\oplus\alpha_2} \cdots & \xleftarrow{id\oplus\alpha_{p-1}} & a_p\oplus a_{p-1} & \xleftarrow{(id\oplus\alpha_p)\Delta} & a_p \\
\| & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} & & & \downarrow{\scriptstyle \pi_2} & & \| \\
a_{-1} & \xleftarrow{\alpha_0} & a_0 & \xleftarrow{\alpha_1} & a_1 & \xleftarrow{\alpha_2} \cdots & \xleftarrow{\alpha_{p-1}} & a_{p-1} & \xleftarrow{\alpha_p} & a_p
\end{array}$$

where we have supressed a top row

$$0 \leftarrow a_p = a_p = \cdots = a_p \leftarrow 0.$$

If the upper row in the diagram represented a simplicial map (in the THH direction), say $T : THH(\mathfrak{C}) \to THH(\mathfrak{C})$, this would have represented a simplicial homotopy from T to the identity. Similarly, the second diagram carries the information

$$\begin{array}{ccccccccccc}
a_{-1} & \xleftarrow{\beta} & a_p & = & a_p & = \cdots & = & a_p & = & a_p \\
\| & & \downarrow{\scriptstyle (1\oplus\beta_1)\Delta} & & \downarrow{\scriptstyle (1\oplus\beta_2)\Delta} & & & \downarrow{\scriptstyle (1\oplus\alpha_p)\Delta} & & \| \\
a_{-1} & \xleftarrow{\alpha_0\pi_2} & a_p\oplus a_0 & \xleftarrow{id\oplus\alpha_1} & a_p\oplus a_1 & \xleftarrow{id\oplus\alpha_2} \cdots & \xleftarrow{id\oplus\alpha_{p-1}} & a_p\oplus a_{p-1} & \xleftarrow{(id\oplus\alpha_p)\Delta} & a_p\oplus a_p
\end{array}$$

where we have supressed the bottom row

$$0 \leftarrow a_0 \xleftarrow{\alpha_1} a_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{p-1}} a_{p-1} \xleftarrow{\alpha_p} 0.$$

If both the rows ($s_x \circ d_x(a)$ and $T_x(a)$ respectively) represented simplicial maps this would have given us a homotopy from $s \circ d$ to T, and so composed with the homotopy above given us a homotopy from the identity to $s \circ d$ proving that for any additive category $\mathfrak{C}$ topological Hochschild homology was concentrated in degree zero. Of course, this is not the case: none of the proposed maps are simplicial (except the identity), and it is crucial to use the additivity coming from the S constructions so that we can forget the irrelevant rows and get homotopies, not in the THH direction, but in the S direction.

Let $\mathfrak{C}$ be a linear category. Then the simplicial abelian group $F.\mathfrak{C} \cong CN(\mathbf{Z}\mathfrak{C}, \mathfrak{C})$ of [7] (see 1.4.1) calculates the Hochschild–Mitchell homology of $\mathfrak{C}$, with coefficients in the bifunctor $\mathfrak{C}(-,-): \mathfrak{C}^0 \times \mathfrak{C} \to \mathscr{A}b$, that is $\pi_k(F.(\mathfrak{C})) = \pi_k(CN(\mathbf{Z}\mathfrak{C}, \mathfrak{C}) = H_k(\mathfrak{C}, \mathfrak{C})$. In [12] it was shown, by means of universal properties, that $H_k(\mathscr{P}_A, \mathscr{P}_A)$ is isomorphic to $\pi_k(THH(A)$. This may also be viewed as a immediate corollary of the result above.

2.2.5. Corollary. *Let $\mathfrak{C}$ be an additive category. Then*

$$H_k(\mathfrak{C}, \mathfrak{C}) \cong \pi_k THH(\mathfrak{C}).$$

In particular, if A is an associative ring with unit, then

$$H_k(\mathscr{P}_A, \mathscr{P}_A) \cong \pi_k(THH(A)).$$

Proof. The second statement follows from the first by Morita equivalence of topological Hochschild homology.

Consider $\mathfrak{C}$ as an exact category by choosing the split exact sequences. We have an inclusion $F.(-) \subset R(-)$. In particular $F_0(-) \to R_0(-)$ is a homotopy equivalence. Consider the commutative diagram

$$\begin{array}{ccccc}
F(\mathfrak{C}) & \longrightarrow & R(\mathfrak{C}) & \xleftarrow{\simeq} & THH(\mathfrak{C}) \\
\downarrow{\simeq} & & \downarrow & & \downarrow{\simeq} \\
\lim_{k\to\infty} \Omega^k F(S^{(k)}\mathfrak{C}) & \longrightarrow & \lim_{k\to\infty} \Omega^k R(S^{(k)}\mathfrak{C}) & \xleftarrow{\simeq} & \lim_{k\to\infty} \Omega^k THH(S^{(k)}\mathfrak{C}) \\
\uparrow{\simeq} & & \uparrow{\simeq} & & \\
\lim_{k\to\infty} \Omega^k F_0(S^{(k)}\mathfrak{C}) & \xrightarrow{\simeq} & \lim_{k\to\infty} \Omega^k R_0(S^{(k)}\mathfrak{C}) & &
\end{array}$$

The left vertical maps are equivalences by [7] together with Lemma 2.2.2 and the others follow from results in this section. □

It should again be stressed that these results are false if $\mathfrak{C}$ is only a linear category. Although the S construction does not appear in the statement, it is the ability to make sums within the category which provide the isomorphisms (or more constructively: it is the additivity of $THH(S-)$ and $F(S-)$ which make the parallel reductions possible).

2.2.6. Generalizations of the above results. One immediate generalization of Corollary 2.2.4 is the following. If X is some finite pointed simplicial set then tensoring with $\tilde{\mathbf{Z}}[X]$ makes sense, and we may form the simplicial $S^{(k)}\mathfrak{C}$ bimodule $S^{(k)}\mathfrak{C}(-,-\otimes\tilde{\mathbf{Z}}[X])$ sending $a, b \in S^{(k)}\mathfrak{C}$ and $Y \in fs_*\mathscr{E}ns$ to $S^{(k)}\mathfrak{C}(a, b\otimes\tilde{\mathbf{Z}}[X\wedge Y])$. The proof of Lemma 2.2.3 obviously extends to this case proving that

$$\lim_{k\to\infty} \Omega^k THH_0(S^{(k)}\mathfrak{C}, S^{(k)}\mathfrak{C}(-,-\otimes\tilde{\mathbf{Z}}[X]))$$

$$\subseteq \lim_{k\to\infty} \Omega^k THH(S^{(k)}\mathfrak{C}, S^{(k)}\mathfrak{C}(-,-\otimes\tilde{\mathbf{Z}}[X]))$$

is a homotopy equivalence, and the latter object is equivalent to $\Omega THH(S\mathfrak{C}, S\mathfrak{C}(-,-\otimes\tilde{\mathbf{Z}}[X]))$.

Another thing one should remark is that the reduction of 1.4.7 did not require the ring functor itself to be linear. This may be exploited if one wishes to study the effect of various subcategories of weak equivalences on some particular linear bimodule, say of the type just mentioned. Here one should notice that the natural transformations defined in the proof of Proposition 2.2.3 will not take a category of weak equivalences outside itself. In fact, it was mainly as a preparation for such applications that the changes from [7] in the natural transformations were made. These questions will hopefully be treated in a later paper, and may prove crucial in an attempt to understand stable algebraic K-theory of spaces in this context.

2.3. Reduction by resolution

As an example of a translation from K-theory to the topological Hochschild homology using the description of THH in the previous section, we offer the following calculation. One should note that this approach is only desirable if one is satisfied with getting non-cyclic information. This may however be remedied by use of the principle of Goodwillie (see the discussion following 1.5.3), which roughly says that any natural property of the homotopy type of topological Hochschild homology extends to the fixed point sets of the finite actions. Thus, to prove a theorem of this sort about $\Omega THH(S-)$ it is enough to prove it for $\Omega THH_0(S-)$.

2.3.1. The exact category $\mathscr{E}_X(\mathfrak{C})$. For any exact category $\mathfrak{C}$ and finite pointed set X let $\mathscr{E}_X(\mathfrak{C})$ denote the exact category of pairs (c, v) where $c \in \mathfrak{C}$ and $v: c \to c\otimes\tilde{\mathbf{Z}}[X]$ a map in $\mathfrak{C}$. A morphism in $\mathscr{E}_X(\mathfrak{C})$ form (c, v) to (d, w) is a morphism $f: c \to d$ in $\mathfrak{C}$ such that

$$\begin{array}{ccc} c & \xrightarrow{v} & c \otimes \tilde{Z}[X] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes id} \\ d & \xrightarrow{v} & d \otimes \tilde{Z}[X] \end{array}$$

commutes. A sequence in $\mathscr{E}_X(\mathfrak{C})$ is exact if and only if the associated sequence of objects in $\mathfrak{C}$ is exact in $\mathfrak{C}$. $\mathscr{E}_X(\mathfrak{C})$ is clearly functiorial in both X and $\mathfrak{C}$, and set maps resp. exact functors are sent to exact functors. Note in particular that $\mathfrak{C}$ equals $\mathscr{E}_*(\mathfrak{C})$ and $\mathscr{E}_{S^0}(\mathfrak{C}) = End(\mathfrak{C})$. In fact if X has $n+1$ elements, then $\mathscr{E}_X(\mathfrak{C})$ is the category with objects $(c; \{v_i\})$ where $c \in \mathfrak{C}$ and $v_i \in End(c)$ $i = 1, \dots, n$ and where a morphism $(c, \{v_i\}) \to (d, \{w_i\})$ is a morphism $f: c \to d$ such that $f \circ v_i = w_i \circ f$ for all i. For a simplicial set we apply $\mathscr{E}_-(\mathfrak{C})$ degreewise.

2.3.2. Proposition (The equivalence criterion). *Let F be an exact functor such that $S^{(k)}(\mathscr{E}_X(F))$ is a weak equivalence for some $k > 0$ and all finite sets X. Then $THH(SF)$ is a weak equivalence.*

Proof. Consider the S construction on $\mathscr{E}_X(\mathfrak{C})$:

$$S(\mathscr{E}_X(\mathfrak{C})) = \coprod_{c \in S\mathscr{C}} S\mathfrak{C}(c, c) \otimes \tilde{Z}[X].$$

The homotopy colimit (over n) of the cofibre of the maps $S^{(k)}\mathfrak{C} = S^{(k)}\mathscr{E}_*(\mathscr{C}) \subseteq S^{(k)}(\mathscr{E}_{S^n}(\mathfrak{C}))$ is thus exactly $THH_0(S^{(k)}\mathfrak{C})$, and so to prove a statement in $THH(S-)$, by Proposition 2.2.3 it is enough to prove it for each of the $S^{(k)}(\mathscr{E}_{S^n}(\mathfrak{C}))$ for $n \geq -1$. □

2.3.3. Proposition (Resolution theorem for THH). *Let $\mathfrak{C}$ be a full subcategory of an exact category $\mathfrak{D}$ which is closed under extension. Assume that*

(1) *If $0 \to d' \to c \to d'' \to 0$ is an exact sequence in $\mathfrak{D}$ with $c \in \mathfrak{C}$, then $d' \in \mathfrak{C}$.*

(2) *For any $d'' \in \mathfrak{D}$ there exists such a short exact sequence with c projective in $\mathfrak{D}$.*

Then $THH(S\mathfrak{C}) \to THH(S\mathfrak{D})$ is a weak equivalence.

Proof. Note that for the K-theory situation, the projectivity assumption in (2) is unnecessary. We need to show that, for all finite sets X, the categories $\mathscr{E}_X(\mathfrak{C}) \subseteq \mathscr{E}_X(\mathfrak{D})$ fulfill the requirements. It is clear that $\mathscr{E}_X(\mathfrak{C})$ is a full subcategory of $\mathscr{E}_X(\mathfrak{D})$. Furthermore, as a sequence

$$0 \to (d', v') \to (d, v) \to (d'', v'') \to 0$$

in $\mathscr{E}_X(\mathfrak{D})$ is exact iff $0 \to d' \to d \to d'' \to 0$ is exact, and $(d, v) \in \mathscr{E}_X(\mathfrak{C})$ iff $d \in \mathfrak{C}$ ($\mathfrak{C}$ is full) it is clear that $\mathscr{E}_X(\mathfrak{C})$ is closed under extensions and admissible subobjects (requirement 1). As to requirement 2, let $(d'', v'') \in \mathscr{E}_X(\mathfrak{D})$. Choose an exact sequence

$$0 \to d' \to c \to d'' \to 0$$

with $c \in \mathfrak{C}$. As c is projective there exists a lifting $v: c \to c \otimes \tilde{Z}[X]$ in

$$\begin{array}{ccc} c & & c \otimes \tilde{Z}[X] \\ \downarrow & & \downarrow \\ d'' & \xrightarrow{v''} & d'' \otimes \tilde{Z}[X] \end{array}$$

This gives an exact sequence

$$0 \to (d', v \mid d') \to (c, v) \to (d'', v'') \to 0$$

and we are done. □

Hence, for instance, if A is a regular ring, and $\mathfrak{D}$ is the category of finitely presented modules, we have that the map $THH(A) \to \Omega THH(S\mathfrak{D})$ induced by the inclusion $\mathscr{P}_A \to \mathfrak{D}$ is a weak equivalence.

The resolution theorem also provides us with a simple example showing that the splitness assumption in Proposition 2.1.3 was necessary.

2.3.4. An example where $THH(\mathfrak{C}) \not\simeq \Omega THH(S\mathfrak{C})$. Let $\mathfrak{C}$ be the category of finitely generated abelian groups with all exact sequences. By the resolution theorem we have

$$\Omega THH(S\mathfrak{C}) \simeq \Omega THH(S\mathscr{F}_Z) \simeq THH(\mathscr{F}_Z) \simeq THH(Z)$$

where for any ring A $\mathscr{F}_A$ is the split exact category of finitely generated A-modules. The equivalence in the middle is due to Proposition 2.1.3, whereas the last is clear from the proof of Proposition 2.1.5. For any prime p, consider the linear (*not* exact) functor $\mathrm{Tor}_1^Z(-, Z/pZ): \mathfrak{C} \to \mathscr{F}_{Z/pZ}$. This is split by the inclusion $\mathscr{F}_{Z/pZ} \subseteq \mathfrak{C}$, so

$$THH(\mathfrak{C}) \simeq THH(\mathscr{F}_{Z/pZ}) \times ?$$

But by the calculations of Bökstedt, $\mathrm{THH}(\mathscr{F}_{Z/pZ}) \simeq THH(Z/pZ)$ is *not* a factor of $THH(Z)$:

$$\pi_k THH(Z/pZ) = \begin{cases} Z/pZ, & \text{if } k = 2i \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\not\subseteq \pi_k THH(Z) = \begin{cases} Z, & \text{if } k = 0, \\ Z/iZ, & \text{if } k = 2i - 1 \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

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