THE ANDO-HOPKINS-REZK ORIENTATION IS SURJECTIVE

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Abstract. We show that the map $\pi_*\text{MString} \to \pi_*\text{tmf}$ induced by the Ando-Hopkins-Rezk orientation is surjective. This proves an unpublished claim of Hopkins and Mahowald. We do so by constructing an $\mathbf{E}_1$-ring B and a map $B \to \text{MString}$ such that the composite $B \to \text{MString} \to \text{tmf}$ is surjective on homotopy. Applications to differential topology, and in particular to Hirzebruch's prize question, are discussed.

1. Introduction

The goal of this paper is to show the following result.

Theorem 1.1. *The map* $\pi_*\text{MString} \to \pi_*\text{tmf}$ *induced by the Ando-Hopkins-Rezk orientation is surjective.*

This integral result was originally stated as [Hop02, Theorem 6.25], but, to the best of our knowledge, no proof has appeared in the literature. In [HM02], Hopkins and Mahowald give a proof sketch of Theorem 1.1 for elements of $\pi_*\text{tmf}$ of Adams-Novikov filtration 0.

To prove Theorem 1.1, we construct an $\mathbf{E}_1$-ring B with an $\mathbf{E}_1$-map $B \to \text{MString}$. The Ando-Hopkins-Rezk orientation [AHR10] $\text{MString} \to \text{tmf}$ is an $\mathbf{E}_\infty$-map, and so the composite $B \to \text{MString} \to \text{tmf}$ is an $\mathbf{E}_1$-map. We then prove the following stronger statement:

Theorem 1.2. *The map* $\pi_*B \to \pi_*\text{tmf}$ *is surjective.*

The map $B \to \text{tmf}$ factors through MString, so Theorem 1.1 follows. In Section 3, we prove Theorem 1.2 after localizing at 3 (as Theorem 3.1). In Section 4, we prove Theorem 1.2 after localizing at 2 (as Theorem 4.1); together, these yield Theorem 1.1 by Corollary 2.6. Finally, in Section 5, we study some applications of Theorem 1.1. In particular, we discuss Hirzebruch's prize question [HBJ92, Page 86] along the lines of [Hop02, Corollary 6.26].

The analogue of Theorem 1.1 for bo (namely, the statement that the map $\pi_*\text{MSpin} \to \pi_*\text{bo}$ induced by the Atiyah-Bott-Shapiro orientation is surjective) is classical [Mil63]. This surjectivity result for spin cobordism was considerably strengthened by Anderson, Brown, and Peterson in [ABP67]: they showed that the Atiyah-Bott-Shapiro orientation $\text{MSpin} \to \text{bo}$ in fact admits a spectrum-level splitting. It is a folklore conjecture that the same is true of the Ando-Hopkins-Rezk orientation $\text{MString} \to \text{tmf}$, and there have been multiple investigations in this direction (see, for instance, [Lau04, LS19]). In forthcoming work [Dev19], we will show that an old conjecture in unstable homotopy theory related to the Cohen-Moore-Neisendorfer theorem, coupled with a conjecture about the centrality of a certain element of $\pi_{13}(B)$, implies that the Ando-Hopkins-Rezk orientation admits a spectrum level splitting. This provides another proof of Theorem 1.1, assuming the truth of these conjectures.

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2. Defining B

In this section, we will define the $\mathbf{E}_1$-ring B mentioned in the introduction and study some of its elementary properties. It is a Thom spectrum, with mod 2 homology given by $\mathbf{F}_2[\zeta_1^8, \zeta_2^4]$. The spectrum B appeared under the name $\overline{X}$ in [HM02, Section 10]. We will work *integrally* (i.e., without inverting any primes) unless explicitly mentioned otherwise.

Construction 2.1. There is a fiber sequence

$$S^9 = \mathrm{O}(10)/\mathrm{O}(9) \to \mathrm{BO}(9) \to \mathrm{BO}(10).$$

There is an element $f \in \pi_{12}\mathrm{O}(10) \cong \mathbf{Z}/12$, which is sent to $2\nu \in \pi_{12}(S^9) \cong \mathbf{Z}/24$ under the boundary homomorphism in the long exact sequence on homotopy. Define a space BN as the homotopy pullback

$$\begin{array}{ccccc} S^9 & \longrightarrow & BN & \longrightarrow & S^{13} \\ \| & & \downarrow & & \downarrow f \\ S^9 & \longrightarrow & \mathrm{BO}(9) & \longrightarrow & \mathrm{BO}(10). \end{array}$$

Let N be the loop space ΩBN. There is a map $N \to \mathrm{BString}$ given by the map of fiber sequences

$$\begin{array}{ccccc} N & \longrightarrow & \Omega S^{13} & \longrightarrow & S^9 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{BString} & \longrightarrow & * & \longrightarrow & \mathrm{B}^2\mathrm{String}, \end{array}$$

whose Thom spectrum we will denote by B.

Note that B is defined integrally, and that it admits an $\mathbf{E}_1$-map $B \to \mathrm{MString}$ obtained by Thomifying the map $N \to \mathrm{BString}$.

Proposition 2.2. *The BP_*-algebra $\mathrm{BP}_*(B)$ is isomorphic to a polynomial algebra $\mathrm{BP}_*[b_4, y_6]$, where $|b_4| = 8$ and $|y_6| = 12$. There is a map $B_{(p)} \to \mathrm{BP}$. On BP-homology, the elements b_4 and y_6 map to t_1^4 and t_2^2 mod decomposables at $p = 2$, and y_6 maps to t_1^3 mod decomposables at $p = 3$.*

Proof. There is a fiber sequence

$$\Omega S^9 \to N \to \Omega S^{13}. \tag{1}$$

The J-homomorphism $\mathrm{BString} \to B\mathrm{GL}_1(\mathbb{S})$ gives a map $N \to \mathrm{BString} \to B\mathrm{GL}_1(\mathbb{S})$. The composite with the map $\Omega S^9 \to N$ gives a map $\Omega S^9 \to B\mathrm{GL}_1(\mathbb{S})$. This is the extension of the map $S^8 \to B\mathrm{GL}_1(\mathbb{S})$ detecting $\sigma \in \pi_7(\mathbb{S})$ along $S^8 \to \Omega S^9$. By one of the main theorems of [AB19], we find that the Thom spectrum of the map $\Omega S^9 \to B\mathrm{GL}_1(\mathbb{S})$ is the $\mathbf{E}_1$-quotient $\mathbb{S}/\!\!/\sigma$ of the sphere spectrum by σ.

The fiber sequence (1) exhibits B as the Thom spectrum of a map $\Omega S^{13} \to B\mathrm{GL}_1(\mathbb{S}/\!\!/\sigma)$. The induced map $S^{12} \to B\mathrm{GL}_1(\mathbb{S}/\!\!/\sigma)$ detects an element $\widetilde{\nu} \in \pi_{11}(\mathbb{S}/\!\!/\sigma)$. This element may be described as follows. The relation $\sigma\nu = 0$ in $\pi_*\mathbb{S}$ defines a lift of $\nu \in \pi_3(\mathbb{S})$ to π_{11} of the 15-skeleton $C\sigma$ of $\mathbb{S}/\!\!/\sigma$; this is the element $\widetilde{\nu}$. Since BP is concentrated in even degrees, the element σ vanishes in $\pi_*\mathrm{BP}$. Consequently, $\widetilde{\nu}$ is well-defined, and it, too vanishes in $\pi_*\mathrm{BP}$. The universal property of Thom spectra from [AB19] then supplies an $\mathbf{E}_1$-map $B \to \mathrm{BP}$.

In particular, the Thom isomorphism says that the BP-homology of B is abstractly isomorphic as an algebra to the BP-homology of N. This may in turn be computed by the Atiyah-Hirzebruch spectral sequence. However, the fiber sequence (1) implies that the homology of N is concentrated in even degrees. Since $\pi_*\mathrm{BP}$ is also concentrated in even degrees, this implies that the Atiyah-Hirzebruch spectral sequence calculating $\mathrm{BP}_*(B)$ collapses, and we find that $\mathrm{BP}_*(B) \cong \mathrm{BP}_*[b_4, y_6]$, as desired.

The map $B \to \mathrm{BP}$ induces a map $\mathrm{BP}_*(B) \to \mathrm{BP}_*(\mathrm{BP}) \cong \mathrm{BP}_*[t_1, t_2, \cdots]$. The element $\widetilde{\nu}$ is detected by $[t_2^2]$ in the 2-local Adams-Novikov spectral sequence for $\mathbb{S}/\!\!/\sigma$, and by $[t_1^3]$ in the 3-local Adams-Novikov spectral sequence for $\mathbb{S}/\!\!/\sigma$. The element $\sigma \in \pi_7(\mathbb{S})$ is detected by $[t_1^4]$ in the 2-local Adams-Novikov spectral sequence for the sphere. This yields the final sentence of the proposition. $\square$

Remark 2.3. Proposition 2.2 implies that the mod 2 homology of B is isomorphic to $\mathbf{F}_2[\zeta_1^8, \zeta_2^4]$.

Remark 2.4. The composite $B \to \mathrm{MString} \to \mathrm{tmf}$ is an $\mathbf{E}_1$-ring map (since the first map is an $\mathbf{E}_1$-ring map by construction, and the second is an $\mathbf{E}_\infty$-ring map by [AHR10]), and it is an equivalence in dimensions ≤ 12. This follows from Proposition 2.2.

Proposition 2.5. *The map $B \to \mathrm{tmf}$ induces a surjection on homotopy after inverting 6.*

Proof. By [Bau08, Proposition 4.4], $\pi_*\mathrm{tmf}[1/6]$ is a polynomial generator on two generators c_4 and c_6, in degrees 8 and 12, respectively. Since the map $B[1/6] \to \mathrm{tmf}[1/6]$ is an $\mathbf{E}_1$-map, the map $\pi_*B[1/6] \to \pi_*\mathrm{tmf}[1/6]$ is a ring map. It therefore suffices to lift the elements c_4 and c_6 to $\pi_*B[1/6]$. This follows from Remark 2.4. □

As an immediate consequence, we have:

Corollary 2.6. *If the maps $\pi_*B_{(3)} \to \pi_*\mathrm{tmf}_{(3)}$ and $\pi_*B_{(2)} \to \pi_*\mathrm{tmf}_{(2)}$ are surjective, then Theorem 1.2 is true.*

Remark 2.7. In [Dev19], we show that B is in many ways analogous to tmf. For instance, it satisfies an analogue of the 2-local Wood equivalence $\mathrm{tmf}_{(2)} \wedge DA_1 \simeq \mathrm{tmf}_1(3)_{(2)}$ from [Mat16], where DA_1 is a certain 8-cell complex: the spectrum $B_{(2)} \wedge DA_1$ is a summand of Ravenel's Thom spectrum $X(4)_{(2)}$. (More precisely, it is the summand $T(2)$ of $X(4)_{(2)}$ obtained from the Quillen idempotent, as studied in [Rav86, Chapter 6.5].)

3. Theorem 1.2 after localizing at 3

In Corollary 2.6, we reduced Theorem 1.2 to showing that the maps $\pi_*B_{(3)} \to \pi_*\mathrm{tmf}_{(3)}$ and $\pi_*B_{(2)} \to \pi_*\mathrm{tmf}_{(2)}$ are surjective. Our goal in this section is to study the 3-local case. We shall prove:

Theorem 3.1. *The map $\pi_*B_{(3)} \to \pi_*\mathrm{tmf}_{(3)}$ is surjective on homotopy.*

Convention 3.2. We shall localize at the prime 3 for the remainder of this section.

3.1. The Adams-Novikov spectral sequence for tmf.

In this section, we review the Adams-Novikov spectral sequence for tmf at $p = 3$; as mentioned in Convention 3.2, we shall 3-localize everywhere. The following result is well-known, and is proved in [Bau08]:

Theorem 3.3. *The E_2-page of the descent spectral sequence (isomorphic to the Adams-Novikov spectral sequence) for tmf is*

$$\mathrm{H}^*(M_{\mathrm{ell}}; \omega^{\otimes 2*}) \cong \mathbf{Z}_3[\alpha, \beta, c_4, c_6, \Delta^{\pm 1}]/I,$$

where I is the ideal generated by the relations

$$3\alpha = 3\beta = 0,\ \alpha^2 = 0,\ \alpha c_4 = \beta c_4 = \alpha c_6 = \beta c_6 = 0,\ c_4^3 - c_6^3 = 1728\Delta.$$

Moreover, α and β are in the image of the map of spectral sequences from the Adams-Novikov spectral sequence of the sphere to that of tmf, with preimages α_1 and β_1.

The differentials are all deduced from Toda's relation $\alpha_1\beta_1^3 = 0$ in $\pi_*\mathbb{S}$. There is a d_5-differential $d_5(\beta_{3/3}) = \alpha_1\beta_1^3$ (the "Toda differential"), where $\beta_{3/3}$ lives in bidegree $(t-s, s) = (34, 2)$; see, e.g., [Rav86, Theorem 4.4.22]. Under the $\mathbf{E}_\infty$-ring map $\mathbb{S} \to \mathrm{tmf}$, this pushes forward to the same differential in the Adams-Novikov spectral sequence for tmf. Then:

Lemma 3.4. *There is a relation $\beta_{3/3} = \Delta\beta$ in the E_2-page of the Adams-Novikov spectral sequence for tmf.*

Proof. We explain how to deduce this from the literature. Multiplication by α is an isomorphism in the Adams-Novikov spectral sequence for both the sphere and tmf in stem 34, so it suffices to check that $\alpha\beta_{3/3} = \Delta\alpha\beta$. The class $\alpha_1\beta_{3/3}$ (resp. $\Delta\alpha\beta$) is a permanent cycle in the Adams-Novikov spectral sequence of the sphere (resp. tmf) by the discussion on [Rav86, Page 137]. It is known (see [DFHH14, Chapter 13, page 12]) that $\Delta\alpha\beta$ detects $\alpha_1\beta_{3/3}$ in homotopy. To conclude that they are the same on the E_2-page of the Adams-Novikov spectral sequence for tmf, it suffices to note that $\alpha_1\beta_{3/3}$ maps to (a unit multiple of) $\Delta\alpha\beta$, as desired. □

It follows by naturality that there is a d_5-differential $d_5(\Delta\beta) = \alpha\beta^3$, which gives (by β-linearity):

Proposition 3.5. *In the Adams-Novikov spectral sequence for* tmf*, there is a d_5-differential $d_5(\Delta) = \alpha\beta^2$.*

Since $3\alpha = 0$ in the Adams-Novikov spectral sequence of tmf, we must have $d_5(3\Delta) = 3\alpha\beta^2 = 0$. There are no other possibilities for differentials on 3Δ, so it is a permanent cycle. Proposition 3.5 shows that there is a Toda bracket $3\Delta \in \langle 3, \alpha, \beta^2\rangle$ in $\pi_*\mathrm{tmf}$. This can be expressed by the claim that 3Δ can be expressed a composite

$$S^{24} \to \Sigma^{20} C\alpha_1 \xrightarrow{\beta^2} \mathrm{tmf},$$

where the first map is of degree 3 on the top cell.

Let $X_3 = S^0 \cup_{\alpha_1} e^4 \cup_{2\alpha_1} e^8$. This is the 8-skeleton of the free $\mathbf{E}_1$-ring $\mathbb{S}/\!\!/\alpha_1$ with a nullhomotopy of α_1. Using the filtered $\mathbf{E}_1$-structure on $\mathbb{S}/\!\!/\alpha_1$, we obtain a factorization

$$\beta^4 : \Sigma^{40} C\alpha_1 \wedge C\alpha_1 \to \Sigma^{40} X_3 \to \mathrm{tmf}.$$

By Δ-linearity, there is also a d_5-differential $d_5(\Delta^2) = \alpha\beta^2\Delta$, so $3\Delta^2$ lives in the E_6-page. There are no further possibilities for differentials, so $3\Delta^2$ lives in $\pi_*\mathrm{tmf}$. Again, this shows that $3\Delta^2 \in \langle 3, \Delta\alpha, \beta^2\rangle$. Finally, we turn to Δ^3. We have $d_5(\Delta^3) = 3\Delta^2\alpha\beta^2$, so we find that $\Delta^3 \in \langle 3, \Delta^2\alpha, \beta^2\rangle$. We collect our conclusions in the following:

Corollary 3.6. *The following is true in $\pi_*\mathrm{tmf}$:*

- $3\Delta \in \langle 3, \alpha, \beta^2\rangle$;
- $3\Delta^2 \in \langle 3, \Delta\alpha, \beta^2\rangle$
- $\Delta^3 \in \langle 3, \Delta^2\alpha, \beta^2\rangle$.

Remark 3.7. The indeterminacy of the above Toda brackets in $\pi_*\mathrm{tmf}_{(3)}$ are $3\mathbf{Z}_{(3)}\{3\Delta\}$, $3\mathbf{Z}_{(3)}\{3\Delta^2\}$, and $3\mathbf{Z}_{(3)}\{\Delta^3\}$, respectively.

3.2. **The Adams-Novikov spectral sequence for B.** In this section, we analyze the ring map $B \to \mathrm{tmf}$, and show that the generators of $\pi_*\mathrm{tmf}_{(3)}$ lift to $\pi_* B_{(3)}$. By Corollary 2.6, this implies Theorem 3.1. We begin by showing:

Proposition 3.8. *There is an element in the E_2-page of the Adams-Novikov spectral sequence for B which lifts the element Δ in the E_2-page of the Adams-Novikov spectral sequence for* tmf*.*

Proof. To prove the proposition, we begin by recalling the definition of a representative for the element Δ in the cobar complex computing the E_2-page of the Adams-Novikov spectral sequence for tmf. The Hopf algebroid $(\mathrm{BP}_*\mathrm{tmf}, \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{tmf})$ is isomorphic to the elliptic curve Hopf algebroid (A, Γ) presenting the moduli stack of cubic curves by [Mat16, Corollary 5.3]. Recall from [Bau08, Page 16] (or [Sil86, Section III.1]) that for an elliptic curve in Weierstrass form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \tag{2}$$

we can define quantities

$$b_2 = a_1^2 + 4a_2,\ b_4 = 2a_4 + a_1 a_3,\ b_6 = a_3^2 + 4a_6,\ b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2,$$

which allows us to define elements

$$c_4 = b_2^2 - 24 b_4,\ c_6 = -b_2^3 + 36 b_2 b_4 - 216 b_6.$$

The discriminant is

$$\Delta = -b_2^2 b_8 - 8 b_4^3 - 27 b_6^2 + 9 b_2 b_4 b_6.$$

Now, it is known that upon inverting 2, every elliptic curve in Weierstrass form (2) is isomorphic to one of the form

$$y^2 = x^3 + a_2 x^2 + a_4 x. \tag{3}$$

It follows that the elliptic curve Hopf algebroid is isomorphic to a Hopf algebroid of the form $(A', \Gamma') = (\mathbf{Z}[1/2][a_2, a_4], A'[r]/(r^3 + a_2 r^2 + a_4 r))$, where I is some ideal consisting of complicated relations, and

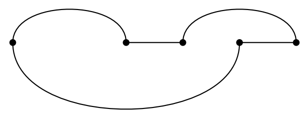

FIGURE 1. Cell structure of the 20-skeleton of B; the bottom cell (in dimension 0) is on the left; straight lines are α_1, and curved lines correspond to α_2 and α_4, in order of increasing length.

where the Hopf algebroid structure can be written down explicitly (as in [Bau08, Section 3]). A straightforward calculation proves that the discriminant is then

$$\Delta = a_2^2 b_4^2 - 16 b_4^3. \tag{4}$$

Turning to B, recall that $\mathrm{BP}_*B \cong \mathrm{BP}_*[b_4, y_6]$. The map $(\mathrm{BP}_*B, \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*B) \to (A', \Gamma')$ of Hopf algebroids induced by the map $B \to \mathrm{tmf}$ sends b_4 to b_4 and y_6 to $a_2 b_4$, and tells us that the discriminant is represented by

$$\Delta = [y_6^2 - 16 b_4^3] \in \mathrm{Ext}(\mathrm{BP}_*B).$$

Using Sage to calculate the 3-series of the formal group law of the elliptic curve (3), one finds that v_1 is a_2 up to a 3-adic unit. We conclude that

$$c_4 = 4v_1^2 - 24b_4, \;\; c_6 = -4v_1^3 - 144y_6.$$

This completes the proof of Proposition 3.8. □

By Remark 2.4, the elements $c_4, c_6 \in \pi_*\mathrm{tmf}$ lift to π_*B. In terms of Toda brackets, they are given by $\langle 3, \alpha_2, 1_B\rangle$ and $\langle 3, \alpha_1, \alpha_2, 1_B\rangle$, respectively. The key to lifting the other elements of $\pi_*\mathrm{tmf}$ is the following:

Theorem 3.9. *There is a differential $d_5(\Delta) = \alpha\beta^2$ in the Adams-Novikov spectral sequence for B. Moreover, $\alpha\beta^2$ vanishes in π_*B, and 3Δ is a permanent cycle.*

Proof. The element $\alpha\beta^2$ is detected in filtration 5 in the Adams-Novikov spectral sequence for the sphere. We first check that there is no class above filtration 5 in stem 23 the Adams-Novikov spectral sequence for B. In Figure 1, we depict the 20-skeleton of B. Now, $\alpha\beta^2$ is the first class in filtration 5 in the Adams-Novikov spectral sequence for the sphere, so there are no classes above filtration 5 in stem 23 in the algebraic Atiyah-Hirzebruch spectral sequence (converging to the Adams-Novikov spectral sequence of B). Consequently, there are no classes above filtration 5 in stem 23 of the Adams-Novikov spectral sequence for B. It follows that $\alpha\beta^2$ must be detected in filtration 5 in the Adams-Novikov spectral sequence for B. Moreover, if the d_5-differential on Δ exists, then it is the longest one (and hence 3Δ is a permanent cycle).

We now prove the d_5-differential. We claim that there is no nonzero target for a d_r-differential on Δ for $2 \le r \le 4$. Indeed, such a class must live in bidegree $(t-s, s) = (23, r)$, so we only need to check that there are no classes in that bidegree. Such a class can only possibly come from those permanent cycles in the algebraic Atiyah-Hirzebruch spectral sequence which are supported on stems $23-8 = 15$, $23-12 = 11$, $23-16 = 7$, or $23-20 = 3$ of the Adams-Novikov spectral sequence of the sphere. The only classes in these stems are in Adams-Novikov filtration 1, so cannot possibly contribute to a class that lives in bidegree $(t-s, s) = (23, r)$ with $2 \le r \le 4$. Therefore, the first possibility for a differential on Δ is the d_5-differential $d_5(\Delta) = \alpha\beta^2$. The existence of this differential is forced by the same differential in the Adams-Novikov spectral sequence for tmf.

Therefore, $\alpha\beta^2$ vanishes in the E_∞-page of the ANSS for B; there may, however, be a multiplicative extension causing $\alpha\beta^2$ to be nonzero in π_*B. But multiplicative extensions have to jump filtration, and we established that there are no classes above filtration 5 in stem 23 of the Adams-Novikov spectral sequence for B. Therefore, $\alpha\beta^2 = 0$ in π_*B, as desired.

□

Corollary 3.10. *The elements* $3\Delta, 3\Delta^2, \Delta^3 \in \pi_*\mathrm{tmf}$ *lift to* $\pi_* B$.

Proof. Theorem 3.9 verifies that 3Δ lifts to $\pi_* B$ and that the brackets in Corollary 3.6 are well-defined in $\pi_* B$. This implies that $3\Delta^2$ and Δ^3 in $\pi_*\mathrm{tmf}$ lift to $\pi_* B$ up to indeterminacy. Remark 3.7 tells us the indeterminacy of the brackets in Corollary 3.6. If $3\Delta^2 + 3n[3\Delta^2] = 3(3n+1)\Delta^2$ (resp. $\Delta^3 + 3n\Delta^3 = (3n+1)\Delta^3$) lifts for some nonzero $n \in \mathbf{Z}_{(3)}$, then so does $3\Delta^2$ (resp. Δ^3) since $3n+1$ is a 3-local unit. □

The elements α, β, c_4, c_6, 3Δ, $3\Delta^2$, Δ^3, and $b = \langle \beta^2, \alpha, \alpha \rangle$ (no indeterminacy) generate the homotopy of tmf. Moreover, $\alpha\beta^2 = 0$ in $\pi_* B$ and $\alpha^2 = 0$ in the sphere, so b admits a lift to $\pi_* B$. Therefore, all generators of $\pi_*\mathrm{tmf}$ admit lifts to $\pi_* B$; this yields Theorem 3.1.

4. Theorem 1.2 after localizing at 2

Our goal in this section is to prove:

Theorem 4.1. *The map* $\pi_* B_{(2)} \to \pi_*\mathrm{tmf}_{(2)}$ *is surjective on homotopy.*

Together with Theorem 3.1 and Corollary 2.6, this proves Theorem 1.2.

Convention 4.2. We shall localize at 2 throughout this section, unless explicitly mentioned otherwise.

4.1. The Adams-Novikov spectral sequence for tmf.

In this section, we review the Adams-Novikov spectral sequence for tmf at $p = 2$. The following result is well-known, and is proved in [Bau08] (see also [Beh20, Proposition 1.4.9]):

Theorem 4.3. *The* E_2*-page of the descent spectral sequence (isomorphic to the Adams-Novikov spectral sequence) for* tmf *is*

$$\mathrm{H}^*(M_{\mathrm{ell}}; \omega^{2*}) \cong \mathbf{Z}_{(2)}[c_4, c_6, \Delta^{\pm 1}, \eta, a_1^2\eta, \nu, \epsilon, \kappa, \overline{\kappa}]/I,$$

where I *is the ideal generated by the relations*

$$2\eta, \eta\nu, 4\nu, 2\nu^2, \nu^3 = \eta\epsilon,$$
$$2\epsilon, \nu\epsilon, \epsilon^2, 2a_1^2\eta, \nu a_1^2\eta, \epsilon a_1^2\eta, (a_1^2\eta)^2 = c_4\eta^2,$$
$$2\kappa, \eta^2\kappa, \nu^2\kappa = 4\overline{\kappa}, \epsilon\kappa, \kappa^2, \kappa a_1^2\eta,$$
$$\nu c_4, \nu c_6, \epsilon c_4, \epsilon c_6, a_1^2\eta c_4 = \eta c_6, a_1^2\eta c_6 = \eta c_4^2,$$
$$\kappa c_4, \kappa c_6, \overline{\kappa} c_4 = \eta^4\Delta, \overline{\kappa} c_6 = \eta^2(a_1^2\eta)\Delta, 1728\Delta = c_4^3 - c_6^2.$$

Remark 4.4. The elements c_4 and $2c_6$ are permanent cycles. There is a map $\mathrm{tmf} \to \mathrm{tmf}_1(3)$, where the target is complex oriented. The elements c_4 and $2c_6$ are nontrivial in $\pi_*\mathrm{tmf}_1(3)$. In fact, the image of the map $\mathrm{tmf} \to \mathrm{tmf}_1(3)$ consists of the elements c_4, $2c_6$, $c_4\Delta^k$, and $2c_6\Delta^k$ for $k \geq 1$, so these elements must be permanent cycles in the Adams-Novikov spectral sequence for tmf.

The ANSS for tmf is essentially determined from Toda's relation $\overline{\kappa}\nu^3 = 0$ in $\pi_{29}\mathbb{S}$. We will explain this statement in the rest of this section. The relation $\overline{\kappa}\nu^3 = 0 \in \pi_{29}\mathbb{S}$ is enforced by the differential $d_5(\beta_{6/2}) = \overline{\kappa}\nu^3$ in the ANSS for the sphere (see [Isa14]). Then:

Lemma 4.5. *There is a relation* $\beta_{6/2} = \Delta\nu^2$ *in the* E_2*-page of the Adams-Novikov spectral sequence for* tmf.

This gives the differential $d_5(\Delta\nu^2) = \overline{\kappa}\nu^3$ in the ANSS for tmf. By ν-linearity, we have $d_5(\Delta) = \overline{\kappa}\nu$. Since $4\nu = 0$ in the E_2-page of the ANSS, the class 4Δ survives. The relation $4\nu = \eta^3$ forces a d_7-differential on 4Δ. In summary:

Theorem 4.6. *There are differentials* $d_5(\Delta) = \overline{\kappa}\nu$ *and* $d_7(4\Delta) = \overline{\kappa}\eta^3$ *in the ANSS for* tmf, *and* $\overline{\kappa}\nu = 0$ *in* $\pi_*\mathrm{tmf}$.

In particular, since $2\eta = 0$ in the ANSS, 8Δ survives to the E_8-page. There are no more differentials, so it is a permanent cycle. Theorem 4.6 then shows that there is a Toda bracket $8\Delta \in \langle 8, \nu, \overline{\kappa}\rangle$ in $\pi_*\mathrm{tmf}$; this bracket is well-defined since $8\nu = 0$ in $\pi_*\mathbb{S}$. This can be expressed by the claim that 8Δ may be expressed as a composite

$$S^{24} \to \Sigma^{20} C\nu \xrightarrow{\overline{\kappa}} \mathrm{tmf},$$

where the first map is degree 8 on the top cell. Similarly, $\Delta\eta \in \langle \eta, \nu, \overline{\kappa}\rangle$ in $\pi_*\mathrm{tmf}$; this bracket is well-defined since $\eta\nu = 0$ in $\pi_*\mathbb{S}$. Arguing in the same way, and using the spherical relations $2\nu^2 = 0$, $\epsilon\nu = 0$, we find that:

Proposition 4.7. *The following Toda brackets exist in* $\pi_*\mathrm{tmf}$*:*

(a) $8\Delta \in \langle 8, \nu, \overline{\kappa}\rangle$;
(b) $\Delta\eta = \langle \eta, \nu, \overline{\kappa}\rangle$;
(c) $2\Delta\nu = \langle 2\nu, \nu, \overline{\kappa}\rangle$;
(d) $\Delta\epsilon = \langle \epsilon, \nu, \overline{\kappa}\rangle$;
(e) $\Delta\eta\kappa = \langle \eta\kappa, \nu, \overline{\kappa}\rangle$;
(f) $\Delta\eta\overline{\kappa} = \langle \eta\overline{\kappa}, \nu, \overline{\kappa}\rangle$.

None of these except the first have any indeterminacy.

To describe the other elements in $\pi_*\mathrm{tmf}$, we adopt a slightly different approach from Section 3.1 — we will not bother writing down all the generators of $\pi_*\mathrm{tmf}$ as Toda brackets of spherical elements unless it is convenient/necessary to do so. This is only to streamline exposition, although one can of course work this out at one's own leisure; see Remark 4.11.

The d_5-differential on Δ forces a differential $d_5(\Delta^k) = k\Delta^{k-1}\overline{\kappa}\nu$. The d_7-differential $d_7(\Delta^4) = \Delta^3\overline{\kappa}\eta^3$ now implies that the classes $\{\Delta^{8k}, 2\Delta^{8k+4}, 4\Delta^{4k+2}, 8\Delta^{2k+1}\}$ survive to the $E_8 = E_9$-page. In fact, these are permanent cycles. A simple induction on k shows:

Proposition 4.8. *Up to units, we have*

(a) $\Delta^{8k} \in \langle 2, \Delta^{8k-1}\eta^3, \overline{\kappa}\rangle$ *with indeterminacy* $2\mathbf{Z}_{(2)}\{\Delta^{8k}\}$;
(b) $2\Delta^{8k+4} \in \langle 2, \Delta^{8k+3}\eta^3, \overline{\kappa}\rangle$ *with indeterminacy* $2\mathbf{Z}_{(2)}\{2\Delta^{8k+4}\}$;
(c) $4\Delta^{4k+2} \in \langle 2, 2\Delta^{4k+1}\nu, \overline{\kappa}\rangle$ *with indeterminacy* $2\mathbf{Z}_{(2)}\{4\Delta^{4k+2}\}$;
(d) $8\Delta^{2k+1} \in \langle 8, \Delta^{2k}\nu, \overline{\kappa}\rangle$ *with indeterminacy* $8\mathbf{Z}_{(2)}\{8\Delta^{2k+1}\}$.

We now turn to the other generators of $\pi_*\mathrm{tmf}$, listed in [Beh20, Figure 1.2].

Proposition 4.9. *We have the following Toda brackets in* $\pi_*\mathrm{tmf}$*, each without any indeterminacy:*

(a) $\Delta^2\nu = \langle \nu, 2\nu\Delta, \overline{\kappa}\rangle$;
(b) $\Delta^4\eta = \langle \eta, \Delta^3\eta^3, \overline{\kappa}\rangle$;
(c) $\Delta^4\nu = \langle \nu, \Delta^3\eta^3, \overline{\kappa}\rangle$;
(d) $\Delta^4\epsilon = \langle \epsilon, \Delta^3\eta^3, \overline{\kappa}\rangle$;
(e) $\Delta^4\kappa = \langle \kappa, 4\nu, 3\nu, 2\nu, \nu, \overline{\kappa}^4\rangle$;
(f) $2\Delta^5\nu = \langle 2\nu, \Delta^4\nu, \overline{\kappa}\rangle$;
(g) $\Delta^5\epsilon = \langle \epsilon, \Delta^4\nu, \overline{\kappa}\rangle$;
(h) $\Delta^6\nu = \langle \nu, 2\Delta^5\nu, \overline{\kappa}\rangle$;

Remark 4.10. We have excluded those elements which can be derived using the multiplicative structure. All other elements (except for $c_4\Delta^k$ and $2c_6\Delta^k$) can be expressed as products of the elements listed in Propositions 4.7, 4.8, and 4.9. Importantly, the proofs of these propositions *only* use $\overline{\kappa}\nu = 0$ in $\pi_*\mathrm{tmf}$ (via Theorem 4.6) and multiplicative relations in the sphere.

Remark 4.11. There are a lot of interesting multiplicative extensions, described in [Bau08, Section 8], but we will not need them. Each of these relations can be derived essentially only using the d_5-differential of Theorem 4.6 and the multiplicative structure in the homotopy of the sphere.

We can recast these extensions from the following perspective. The spectrum $C\nu$ is the Thom spectrum of the Spin-bundle over S^4 determined by the generator of $\pi_4\mathrm{BSpin}$. Since BSpin is an infinite loop space, this bundle extends to one over ΩS^5, and hence over the intermediate James constructions $J_k(S^4)$ for all $k \geq 1$. Let $J_k(S^4)^\mu$ denote the Thom spectrum of this bundle, so $J_1(S^4)^\mu = C\nu$. Since $\{J_k(S^4)\}$ forms a filtered $\mathbf{E}_1$-space, we obtain a map $C\nu^{\wedge k} \to J_k(S^4)^\mu$. Taking the product of $\overline{\kappa} : \Sigma^{20} C\nu \to \mathrm{tmf}$ with itself k times defines a map

$$\overline{\kappa}^k : \Sigma^{20k} J_k(S^4)^\mu \to \Sigma^{20k} J_k(S^4)^\mu \wedge \mathrm{tmf} \to \mathrm{tmf}.$$

If x is a spherical element, a composite $S^{4k+|x|} \to J_k(S^4)^\mu \wedge \mathrm{tmf}$ where the map $S^{4k+|x|} \to J_k(S^4)^\mu$ is given by x on the top ($4k$-dimensional) cell of $J_k(S^4)^\mu$ will define an element of the form $x\Delta^k \in \pi_{24k+|x|}\mathrm{tmf}$. For instance, we have:

(a) $\Delta^2\nu \in \langle \nu, 2\nu, \nu, \overline{\kappa}^2 \rangle$;
(b) $\Delta^4\eta \in \langle \eta, 4\nu, 3\nu, 2\nu, \nu, \overline{\kappa}^4 \rangle$;
(c) $\Delta^4\nu \in \langle \nu, 4\nu, 3\nu, 2\nu, \nu, \overline{\kappa}^4 \rangle$;
(d) $\Delta^4\epsilon \in \langle \epsilon, 4\nu, 3\nu, 2\nu, \nu, \overline{\kappa}^4 \rangle$;
(e) $\Delta^4\kappa \in \langle \kappa, 4\nu, 3\nu, 2\nu, \nu, \overline{\kappa}^4 \rangle$;
(f) $2\Delta^5\nu \in \langle 2\nu, 5\nu, 4\nu, 3\nu, 2\nu, \nu, \overline{\kappa}^5 \rangle$;
(g) $\Delta^5\epsilon \in \langle \epsilon, 5\nu, 4\nu, 3\nu, 2\nu, \nu, \overline{\kappa}^5 \rangle$.

Remark 4.12. Mark Behrens pointed out to us that Mahowald expected $\overline{\kappa}^7 = 0$ in $\pi_*\mathbb{S}_{(2)}$ (it is known that $\overline{\kappa}^6 = 0$ in $\pi_*\mathrm{tmf}_{(2)}$). It would be interesting to know whether this is related to the existence of Δ^8 in $\pi_*\mathrm{tmf}$ via the approach given in Remark 4.11.

Finally, we prove Proposition 4.9.

Proof of Proposition 4.9. We prove this case-by-case.

(a) Since $d_5(\Delta^2) = 2\Delta\overline{\kappa}\nu$ and $2\nu^2 = 0$ in the ANSS for the sphere, we find that $\Delta^2\nu \in \langle \nu, 2\nu\Delta, \overline{\kappa} \rangle$. We provide the argument for indeterminacy in this case, but not for the others since the argument is essentially the same. The indeterminacy lives in $\overline{\kappa}\pi_{31}\mathrm{tmf} + \nu\pi_{48}\mathrm{tmf}$, but $\overline{\kappa}\pi_{31}\mathrm{tmf} \cong \nu\pi_{48}\mathrm{tmf} \cong 0$.
(b) Since $d_7(\Delta^4) = \Delta^3\overline{\kappa}\eta^3$, we have $d_7(\Delta^4\eta) = \Delta^3\overline{\kappa}\eta^4 = 0$. Therefore, $\Delta^4\eta \in \langle \eta, \Delta^3\eta^3, \overline{\kappa} \rangle$. This bracket is well-defined because $\Delta^3\eta^3 = 4\Delta^3\nu$ exists in $\pi_*\mathrm{tmf}$, $\eta\nu = 0$ in the sphere, and $\overline{\kappa}\eta^3 = 0$ in tmf.
(c) Similarly, since $d_7(\Delta^4) = \Delta^3\overline{\kappa}\eta^3$, we have $d_7(\Delta^4\nu) = \Delta^3\overline{\kappa}\eta^3\nu = 0$. Therefore $\Delta^4\nu \in \langle \nu, \Delta^3\eta^3, \overline{\kappa} \rangle$. This bracket is well-defined because $\Delta^3\eta^3$ exists in $\pi_*\mathrm{tmf}$, $\eta\nu = 0$ in the sphere, and $\overline{\kappa}\eta^3$ vanishes in tmf.
(d) Similarly, since $d_7(\Delta^4) = \Delta^3\overline{\kappa}\eta^3$, we have $d_7(\Delta^4\epsilon) = \Delta^3\overline{\kappa}\eta^3\epsilon = 0$, since $2\epsilon = 0$. Therefore, $\Delta^4\epsilon \in \langle \epsilon, \Delta^3\eta^3, \overline{\kappa} \rangle$. This bracket is again well-defined.
(e) This is in [Bau08, Corollary 8.7], where $\Delta^4\kappa$ is denoted $e[110, 2]$.
(f) Since $d_5(\Delta^5) = 5\Delta^4\overline{\kappa}\nu$, we have $d_5(2\Delta^5\nu) = 10\Delta^4\overline{\kappa}\nu^2 = 0$, since $2\nu^2 = 0$. It follows that $2\Delta^5\nu \in 5\langle 2\nu, \Delta^4\nu, \overline{\kappa} \rangle$. This is well-defined because $\Delta^4\nu$ lives in $\pi_*\mathrm{tmf}$, $2\nu^2 = 0$ in the sphere, and $\overline{\kappa}\nu = 0$ in tmf.
(g) Similarly, since $d_5(\Delta^5) = 5\Delta^4\overline{\kappa}\nu$, we have $d_5(\Delta^5\epsilon) = 5\Delta^4\overline{\kappa}\nu\epsilon = 0$, because $\epsilon\nu = 0$. It follows that $\Delta^5\epsilon \in 5\langle \epsilon, \Delta^4\nu, \overline{\kappa} \rangle$, which is well-defined because $\Delta^4\nu$ lives in $\pi_*\mathrm{tmf}$, $\epsilon\nu = 0$ in the sphere, and $\overline{\kappa}\nu = 0$ in tmf.
(h) Since $d_5(\Delta^6) = 6\Delta^5\overline{\kappa}\nu$, we have $d_5(\Delta^6\nu) = 6\Delta^5\overline{\kappa}\nu^2 = 0$. We therefore have $\Delta^6\nu \in 3\langle \nu, 2\Delta^5\nu, \overline{\kappa} \rangle$. This is well-defined because $2\nu\Delta^5$ lives in $\pi_*\mathrm{tmf}$, $2\nu^2 = 0$ in the sphere, and $\overline{\kappa}\nu = 0$ in tmf.

□

4.2. **The Adams-Novikov spectral sequence for B.** In this section, we analyze the ring map $B \to \mathrm{tmf}$, and show that the generators of $\pi_*\mathrm{tmf}_{(2)}$ lift to $\pi_*B_{(2)}$. Again, we will localize at $p = 2$ throughout.

We begin by showing:

Proposition 4.13. *There is an element in the 0-line of the E_2-page of the ANSS for B which lifts the element Δ in the E_2-page of the ANSS for* tmf*.*

Proof. We begin by recalling a representative for Δ in the cobar complex for tmf at $p = 2$. Recall from Proposition 3.8 that the Hopf algebroid $(\mathrm{BP}_*\mathrm{tmf}, \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{tmf})$ is isomorphic to the elliptic curve Hopf algebroid (A, Γ) presenting the moduli stack of cubic curves. As in the 3-complete setting (studied in Proposition 3.8), it is known that upon 2-completion, every elliptic curve in Weierstrass form is isomorphic to one of the form

$$y^2 + a_1xy + a_3y = x^3.$$

Consequently (as in the 3-complete setting), the elliptic curve Hopf algebroid is isomorphic to a Hopf algebroid of the form $(A', \Gamma') = (\mathbf{Z}_2[a_1, a_3], A'[s,t]/I)$, where I is some ideal consisting of complicated relations, and where the Hopf algebroid structure can be written down explicitly (as in [Bau08, Section 3]). A straightforward calculation proves that the discriminant is then

$$\Delta = a_1^3a_3^3 - 27a_3^4 = b_4^3 - 27b_6^2. \tag{5}$$

Turning to B, recall that we may identify BP_*B with $\mathrm{BP}_*[b_4, y_6]$. The map $B \to \mathrm{tmf}$ induces a map $(\mathrm{BP}_*B, \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} \mathrm{BP}_*B) \to (A', \Gamma')$ of Hopf algebroids that sends b_4 to b_4 and y_6 to b_6. It follows from Equation (5) that the element Δ already exists in the 0-line of the Adams-Novikov spectral sequence for B, and is represented by

$$\Delta = [b_4^3 - 27y_6^2] \in \mathrm{Ext}(\mathrm{BP}_*B).$$

This finishes the proof of Proposition 4.13. □

Since the map $B \to \mathrm{tmf}$ is an equivalence in dimensions ≤ 12 (Corollary 2.6), the elements c_4 and $2c_6$ lift to π_*B. We claim that $c_4\Delta^k$ and $2c_6\Delta^k$ live in π_*B; to show this, we argue as in Remark 4.4. There is a map $B \to B \wedge DA_1 \simeq T(2)$ (see also Remark 2.7), and there is a particular complex orientation of $\mathrm{tmf}_1(3)$ exhibiting it as a form of $\mathrm{BP}\langle 2\rangle$, which sits in a commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & T(2) & \longrightarrow & \mathrm{BP} \\ \downarrow & & \downarrow & \swarrow & \\ \mathrm{tmf} & \longrightarrow & \mathrm{tmf}_1(3). & & \end{array}$$

There are choices of indecomposables v_1 and v_2 producing an isomorphism $\pi_*\mathrm{tmf}_1(3) \cong \mathbf{Z}_2[v_1, v_2]$ such that c_4 is sent to v_1^4 and Δ is sent to v_2^4. The map $T(2) \to \mathrm{tmf}_1(3)$ is surjective on homotopy, since v_1 and v_2 live in $\pi_*T(2)$. Since the elements c_4, $2c_6$, $c_4\Delta^k$, and $2c_6\Delta^k$ for $k \geq 1$ therefore already live in the homotopy of $T(2)$, we find by the same argument that these elements already live in the homotopy of B.

We next turn to showing that the other elements of $\pi_*\mathrm{tmf}$ lift to π_*B. The following is the 2-local analogue of Theorem 3.9:

Theorem 4.14. *There are differentials $d_5(\Delta) = \overline{\kappa}\nu$ and $d_7(4\Delta) = \overline{\kappa}\eta^3$ in the ANSS for B. Moreover, $\overline{\kappa}\nu = 0$ in π_*B, and 8Δ is a permanent cycle.*

Proof. To prove the differentials, first note that the d_7-differential follows from the d_5-differential via the spherical relation $4\nu = \eta^3$; it therefore suffices to prove the d_5-differential. The class $\overline{\kappa}\nu$ lives in bidegree $(23, 5)$ in the ANSS for B, since it lives in that bidegree in the ANSS for both the sphere and for tmf. Assume that $\overline{\kappa}\nu$ is the target of a d_r-differential $d_r(x) = \overline{\kappa}\nu$ for $2 \leq r \leq 4$. Then the class x maps to zero in the ANSS for tmf under the unit map $\iota : B \to \mathrm{tmf}$, so since all differentials commute with ι, we find that $0 = d_r(\iota(x)) = \overline{\kappa}\nu$ in the ANSS for tmf. This is obviously a contradiction, so the first possibility for a differential is the d_5-differential $d_5(\Delta) = \overline{\kappa}\nu$. This is forced by the analogous differential in the ANSS of tmf.

Therefore, $\overline{\kappa}\nu$ vanishes in the E_∞-page of the ANSS, but this does not yet imply that $\overline{\kappa}\nu$ vanishes in π_*B, since there may be nontrivial multiplicative extensions. Since multiplicative extensions have to jump filtration by at least one degree, it suffices to show that there are no permanent cycles in the ANSS for B in stem 23 of filtration greater than 5. The class $\overline{\kappa}\nu$ is the first element of filtration 5 in the ANSS for the sphere which does not come from an $\eta = \alpha_1$-tower on the α-family elements, so the only contributions in the ANSS for B in ANSS filtration > 5 come from such elements in the algebraic Atiyah-Hirzebruch spectral sequence (and $\overline{\kappa}\eta^3$). However, all such η-towers are truncated by an ANSS d_3-differential, so they only contribute to elements in filtrations ≤ 3 in the E_4-page. Thus, they contribute no elements of filtration greater than 5 in the E_∞-page of the ANSS for B. Consequently, $\overline{\kappa}\nu$ vanishes in $\pi_* B$, as desired. There also cannot be any longer differentials on Δ, so 8Δ is a permanent cycle. □

Finally:

Proof of Theorem 4.1. Theorem 4.14 implies that 8Δ lifts to $\pi_* B$, and that all the brackets in $\pi_*\mathrm{tmf}$ in Propositions 4.7, 4.8, and 4.9 are well-defined in $\pi_* B$. The elements of $\pi_*\mathrm{tmf}$ in those propositions for which the bracket has no indeterminacy therefore lift to $\pi_* B$. By Remark 4.10, all that remains is to show that the constant multiples of the powers of Δ which live in $\pi_*\mathrm{tmf}$ in fact lift to $\pi_* B$. Theorem 4.14 implies that they lift up to indeterminacy, and this indeterminacy is specified in Proposition 4.8. If $\Delta^{8k} + 2n\Delta^{8k} = (2n+1)\Delta^{8k}$ lifts for some $n \in \mathbf{Z}_{(2)}$, then so does Δ^{8k} since $2n+1$ is a 2-local unit. Similarly, one finds that $2\Delta^{8k+4}$, $4\Delta^{4k+2}$, and $8\Delta^{2k+1}$ also lift to $\pi_* B$, as desired. □

Remark 4.15. We briefly look at the Adams spectral sequence for B. The Steenrod module structure of the 20-skeleton of B is as in Figure 1; since we are at the prime 2, straight lines are Sq^4, and curved lines correspond to Sq^8 and Sq^{16}, in order of increasing length. Using this, we can calculate the Adams spectral sequence in small dimensions. The Adams charts below were created with Hood Chatham's Ext calculator, and the Steenrod module file for B in this range can be found at http://www.mit.edu/~sanathd/input-B-leq-24-prime-2.

The E_2-page for B in the first few dimensions is shown in Figure 2; there are no classes in higher Adams filtration in stem 23. The red class is $g = \overline{\kappa}$, and the purple lines are d_2-differentials. The differential on the class in stem 23 already exists in the Adams spectral sequence for the sphere as $d_2(i) = h_0 P d_0$. The other classes in stem 23 except for the one in filtration 9 are permanent cycles, and there is no multiplicative extension causing any of them to be $\overline{\kappa}\nu$ on homotopy.

As shown in Figure 3, there is also a d_3-differential on the leftmost class $x^{(0)}_{24,1}$ in bidegree $(24, 6)$ (which supports a h_0-tower) to the class in bidegree $(23, 9)$; the class $h_0 x^{(0)}_{24,1}$ is a permanent cycle in the ASS for B which is sent to 8Δ in the ASS for tmf. The class in bidegree $(25, 5)$ is a permanent cycle in the ASS for B which is sent to $\Delta\eta$ in the ASS for tmf.

Remark 4.16. We now compare the approach of this paper with that of [HM02], where the $\mathbf{E}_1$-ring B was constructed under the name $\overline{X}$. The special case of our Theorem 1.2 for elements in $\pi_*\mathrm{tmf}$ of ANSS filtration 0 is stated as [HM02, Theorem 11.1], where a proof is only sketched.

First, their Proposition 11.2 is a combination of our Theorem 3.9 and Theorem 4.14. Secondly, their proof proceeds by calculating the mod 2 Adams spectral sequence of B in dimensions ≤ 24 to show that $\overline{\kappa}\nu$ vanishes in the 2-local homotopy of B. Their argument does not seem to resolve potential multiplicative extensions: as Figure 3 shows, there are two possibilities for multiplicative extensions in the Adams spectral sequence which could make $\overline{\kappa}\nu$ nonzero in $\pi_* B_{(2)}$. (Namely, the classes in bidegrees $(23, 6)$ and $(23, 7)$ could represent $\overline{\kappa}\nu$.) Thirdly, Remark 4.11 essentially gives a proof of their Lemma 11.5, which seems to appear without proof.

5. Applications of Theorem 1.1

In this section, we study some applications of Theorem 1.1. One application of Theorem 1.1 was stated as [Hop02, Corollary 6.26], and provides an answer to Hirzebruch's prize question [HBJ92, Page 86]. See also [HM02].

Corollary 5.1. *There exists a 24-dimensional compact smooth string manifold M with $\hat{A}(M) = 1$ and $\hat{A}(M, \tau_M \otimes \mathbf{C}) = 0$.*

Proof. By the discussion on [HBJ92, Page 86], the conditions on the $\hat{A}$-genus of M are equivalent to the Witten genus of M being $c_4^3 - 744\Delta = \Delta(j - 744)$, where j is the j-function. Let M_0^8 denote the

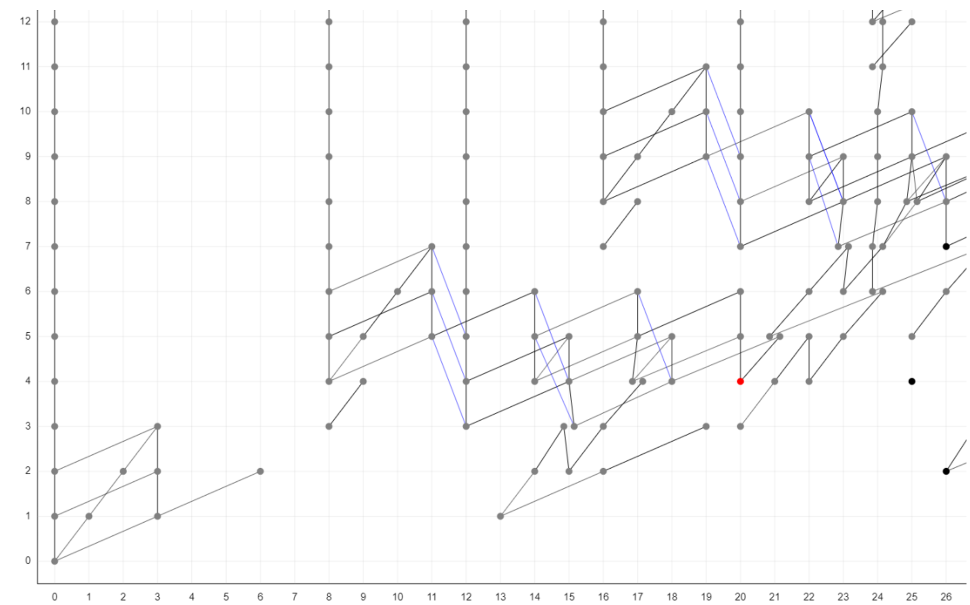

FIGURE 2. E_2-page of the Adams spectral sequence for B. The class highlighted in red is $\overline{\kappa}$.

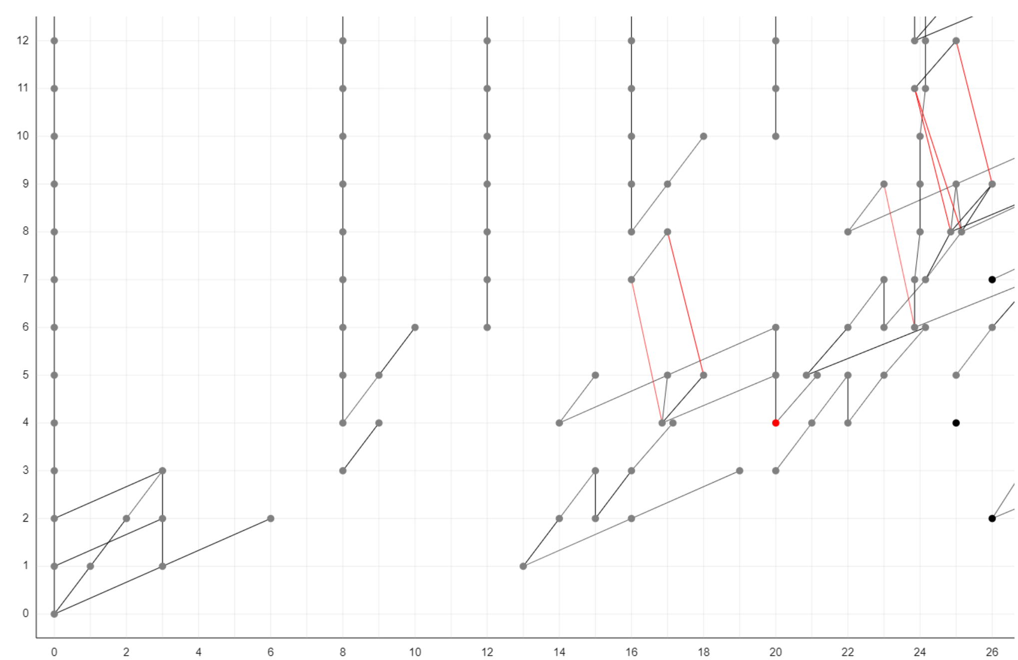

FIGURE 3. E_3-page of the Adams spectral sequence for B. The class highlighted in red is $\overline{\kappa}$. There are no differentials in this range from the E_4-page onwards.

Kervaire-Milnor almost parallelizable 8-manifold; then, the 8-manifold $-M_0^8 - 224\mathbf{H}P^2$ (whose string cobordism class we will denote by $[N_{c_4}]$, where N_{c_4} is the explicit manifold representative above) admits a string structure by [Lau04, Lemma 15]. The map tmf $\to$ bo sends $c_4 \in \pi_8\text{tmf}$ to $v_1^4 \in \pi_8\text{bo}$. By

Lemma 5.2, there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{MString} & \longrightarrow & \mathrm{MSpin} \\ \big\downarrow & & \big\downarrow \\ \mathrm{tmf} & \longrightarrow & \mathrm{bo}, \end{array} \tag{6}$$

where the left vertical map is the Ando-Hopkins-Rezk orientation and the right vertical map is the Atiyah-Bott-Shapiro orientation. Consequently, the Witten genus of $-M_0^8 - 224\mathbf{H}P^2$ is c_4.

By Theorem 1.1, the element $24\Delta \in \pi_{24}\mathrm{tmf}$ lifts to a class $[N_\Delta]$ in $\pi_{24}\mathrm{MString}$, where N_Δ is any manifold representative. Since $744\Delta = 31 \cdot 24\Delta$, we conclude that the string cobordism class of the 24-dimensional compact oriented smooth string manifold $N_{c_4}^3 - 31N_\Delta$ has Witten genus $c_4^3 - 744\Delta$, as desired. □

The proof of Corollary 5.1 utilized the following lemma.

Lemma 5.2. *The diagram* (6) *commutes, where the left vertical map is the Ando-Hopkins-Rezk orientation and the right vertical map is the Atiyah-Bott-Shapiro orientation.*

Proof. We need to show that the composite MString $\to$ tmf $\to$ bo comes from the Atiyah-Bott-Shapiro orientation. By [AHR10, Corollary 7.12], it suffices to show that this composite has the same characteristic series as the restriction of the $\hat{A}$-genus to string manifolds. There is an isomorphism $\pi_*\mathrm{bo} \otimes \mathbf{Q} \cong \mathbf{Z}[\beta^2]$, where β^2 lives in degree 4 and is the square of the Bott element. Moreover, $\pi_*\mathrm{tmf} \otimes \mathbf{Q}$ is isomorphic to the ring of rational modular forms (of weight given by half the degree in $\pi_*\mathrm{tmf} \otimes \mathbf{Q}$) by [Bau08, Proposition 4.4]. The map $\pi_*\mathrm{tmf} \otimes \mathbf{Q} \to \pi_*\mathrm{bo} \otimes \mathbf{Q}$ sends a modular form of weight k with q-expansion $f(q) = \sum a_n q^n$ to the element $a_0(\beta^2)^k \in \pi_{2k}\mathrm{bo} \otimes \mathbf{Q}$. Consequently, the composite $\pi_*\mathrm{MString} \to \pi_*\mathrm{tmf} \otimes \mathbf{Q} \to \pi_*\mathrm{bo} \otimes \mathbf{Q}$ sends a string manifold M to the constant term of the q-expansion of its Witten genus. The lemma will therefore follow if this constant term is the $\hat{A}$-genus of M, but this follows from the discussion on [HBJ92, Page 84]. □

Remark 5.3. The modular form $c_4^3 - 744\Delta$ is $\theta_{\Lambda_{24}} - 24\Delta$, where Λ_{24} is the 24-dimensional Leech lattice and $\theta_{\Lambda_{24}}$ is its theta function.

Remark 5.4. The original motivation for Hirzebruch's prize question was to relate the geometry of the 24-dimensional string manifold M of Corollary 5.1 to representations of the monster group by constructing an action of the monster group on M. The question of constructing this action remains unresolved.

Remark 5.5. The disussion on [HBJ92, Page 86] implies that $\hat{A}(N_\Delta) = 0$ and $\hat{A}(N_\Delta, \tau_{N_\Delta} \otimes \mathbf{C}) = 24$. It follows from [Sto92, Theorem A] that N_Δ (which we may assume is simply-connected by surgery) admits a metric with positive scalar curvature. Since the Witten genus of N_Δ is nonzero, Stolz's conjecture in [Sto96] would imply that it does not admit a metric of positive-definite Ricci curvature. We do not know whether Stolz's conjecture holds in this particular case. Note, however, that there are examples of non-simply-connected manifolds which admit positive scalar curvature metrics but no metrics of positive-definite Ricci curvature: as pointed out to us by Stolz, a connected sum of lens spaces of dimension at least 3 gives such a manifold.

Corollary 5.1 may be generalized in the following manner. Recall the following definition from [Ono04, Section 2.3]. Let $j_1(z) = j(z) - 744$, and define $j_n(z)$ for $n \geq 2$ via $nT_n(j_1(z))$, where T_n is the weight zero Hecke operator, acting on $f(z)$ via

$$T_n f(z) = \sum_{d|n, ad=n} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).$$

By [Ono04, Proposition 2.13], $j_n(z)$ is a monic integral polynomial in $j(z)$ of degree n; for instance,

$$j_2(z) = j(z)^2 - 1488j(z) + 159768, \;\; j_3(z) = j(z)^3 - 2232j(z)^2 + 1069956j(z) - 36866976.$$

The functions $j_n(z)$ for $n \geq 0$ (where $j_0(z) = 1$) form a basis for the complex vector space of weakly holomorphic modular forms of weight 0, and appear in the denominator formula for the monster Lie algebra. They may be defined by Faber polynomials on j. The generalization of Corollary 5.1 is as follows.

Proposition 5.6. *For all $n \geq 0$, there is a $24n$-dimensional compact smooth string manifold M^{24n} whose Witten genus is $\Delta^n j_n(z)$.*

Remark 5.7. By arguing as in [HBJ92, Pages 86-87], we find that the twisted $\hat{A}$-genera of bundles over M^{24n} constructed from the complexified tangent bundle of M are integral linear combinations of dimensions of irreducible representations of the monster group; for instance, $\hat{A}(M^{48}; \mathrm{Sym}^2(\tau_M \otimes \mathbf{C}))$ is the coefficient of q^2 in $\Delta^2 j_2(z)$, which is $2 \times (21296876 + 196883 + 1)$. More generally, $\hat{A}(M^{24n}; \mathrm{Sym}^2(\tau_M \otimes \mathbf{C}))$ is an integral linear combination of the dimensions of the n smallest irreducible representations of the monster group. In light of Hirzebruch's original motivation for his prize question (see Remark 5.4), it seems reasonable to conjecture that the $24n$-dimensional string manifold M^{24n} admits an action of the monster group by diffeomorphisms.

Remark 5.8. It would be interesting to know if there is an analogue of Proposition 5.6 for other McKay-Thompson series.

Before providing the proof, we need the following result.

Theorem 5.9. *A modular form f is in the image of the boundary homomorphism $\pi_*\mathrm{tmf} \to \mathrm{MF}_*$ in the Adams-Novikov spectral sequence if and only if it is expressible as an integral linear combination of monomials of the form $a_{ijk} c_4^i c_6^j \Delta^k$ with $i, k \geq 0$ and $j = 0, 1$, where*

$$a_{ijk} = \begin{cases} 1 & i > 0, j = 0 \\ 2 & j = 1 \\ 24/\gcd(24, k) & i, j = 0. \end{cases}$$

Proof. This is [Hop02, Proposition 4.6], proved in [Bau08]. □

Proof of Proposition 5.6. We have

$$\Delta^n j_n(z) = \sum_{0 \leq k \leq n} \alpha_k j(z)^k \Delta^n = \sum_{0 \leq k \leq n} \alpha_k c_4^{3k} \Delta^{n-k},$$

for some integers α_k (where $\alpha_n = 1$). By Theorem 1.1 and Theorem 5.9, it suffices to show that the constant term α_0 of $j_n(z)$ (when expanded as a monic integral polynomial in $j(z)$) is a multiple of $24/\gcd(24, n)$. The j-function vanishes on a primitive third root of unity, so $\alpha_0 = j_n(\omega)$. Its generating function is

$$\sum_{n \geq 0} j_n(\omega) q^n = -\frac{j'(z)}{j(z)} = \frac{c_6}{c_4},$$

where $q = e^{2\pi i z}$ and ω is a primitive third root of unity.

Let $m \geq 1$; we claim that the coefficients $a_{4,m}$ and $a_{6,m}$ of q^m in the q-expansion for c_4 and c_6 (respectively) are divisible by $24/\gcd(24, m)$. Indeed, the expression for their q-expansion shows that $a_{4,n} = -240\sigma_3(n)$ and $a_{6,n} = 504\sigma_5(n)$, and both 240 and 504 are already divisible by 24. Since the coefficient of q^m in $1/c_4$ can be expressed as an integral linear combination of the $a_{4,k}$, it follows that the coefficient of q^m for $m \geq 1$ in c_6/c_4 (which is $j_m(\omega)$) is divisible by 24, and hence by $24/\gcd(24, m)$, as desired. □

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