Ambidexterity in $T(n)$-Local Stable Homotopy Theory

Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski

Abstract

We extend the theory of ambidexterity and higher semiadditivity developed by M. Hopkins and J. Lurie, and show that the ∞-categories of $T(n)$-local spectra are ∞-semiadditive for all n, where $T(n)$ is the telescope on a v_n-self map of a type n spectrum. This extends and provides a new proof for the analogous result of Hopkins-Lurie on $K(n)$-local spectra. As a consequence, we deduce that $T(n)$-homology of π-finite spaces depends only on the n-th Postnikov truncation. The proof relies on a construction of a certain power operation for commutative ring objects in stable 1-semiadditive ∞-categories. This generalizes some known constructions for Morava E-theory, and is of independent interest. Using this power operation we also give a new proof, and a generalization, of a nilpotence conjecture of J.P. May, which was proved by A. Mathew, N. Naumann, and J. Noel.

Figure 1: A Seaman, holding a telescope with two hands, Louis Peter Boitard [©Trustees of The British Museum]

Contents

1 Introduction

1.1 Main Results

Let X be a spectrum with an action of a finite group G. The spectra of homotopy orbits X_{hG} and homotopy fixed points X^{hG} are related by a canonical norm map $\mathrm{Nm}\colon X_{hG} \to X^{hG}$. This map is usually far from being an equivalence. However, there are certain homology theories, such that when working locally with respect to them, the analogous norm map is always a local equivalence. For a spectrum E, let us denote by Sp_E the ∞-category of E-local spectra. For $X \in \mathrm{Sp}_E$ with a G-action, we denote by X_{hG} and X^{hG} the homotopy orbits and homotopy fixed points respectively, in the ∞-category Sp_E.

Theorem 1.1.1 (Hovey-Sadofsky-Greenlees, [HS96, GS96])**.** *Let $K(n)$ be Morava K-theory of height n. For every $X \in \mathrm{Sp}_{K(n)}$ with an action of a finite group G, the canonical norm map is an equivalence*

$$\mathrm{Nm}\colon X_{hG} \xrightarrow{\sim} X^{hG} \quad \in \mathrm{Sp}_{K(n)}\,.$$

Kuhn proved an analogous result, when one replaces $K(n)$-local spectra with the closely related telescopic localizations.

Theorem 1.1.2 (Kuhn, [Kuh04])**.** *Let $T(n)$ be a telescope on a v_n-self map of a type n-spectrum. For every $X \in \mathrm{Sp}_{T(n)}$ with an action of a finite group G, the canonical norm map is an equivalence*

$$\mathrm{Nm}\colon X_{hG} \xrightarrow{\sim} X^{hG} \quad \in \mathrm{Sp}_{T(n)}\,.$$

Note that since $\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)}$, this in fact generalizes Theorem 1.1.1.

These results have found many applications in chromatic homotopy theory. For example, they were recently used in [Heu18] to generalize Quillen's rational homotopy theory to higher chromatic heights. And they were used to analyze the Balmer spectrum in an equivariant setting [BHN+17].

Lurie and Hopkins generalized Theorem 1.1.1 in a different direction. Considering the classifying space BG as an ∞-groupoid, the data of an E-local spectrum with a G action is equivalent to a functor $F\colon BG \to \mathrm{Sp}_E$. The homotopy orbits and homotopy fixed points of the action, are the colimit and the limit of F respectively (again, in Sp_E).

Definition 1.1.3. Given $m \geq -2$, a space A is called *m-finite* if it is m-truncated, has finitely many connected components and all of its homotopy groups are finite. It is called *π-finite* if it is m-finite for some m.

Theorem 1.1.4 (Hopkins-Lurie, [HL13])**.** *Let A be a π-finite space. For every $F\colon A \to \mathrm{Sp}_{K(n)}$, there is a canonical (and natural) equivalence*

$$\mathrm{Nm}_A\colon \varinjlim F \xrightarrow{\sim} \varprojlim F \quad \in \mathrm{Sp}_{K(n)}\,.$$

When $A = BG$, one recovers Theorem 1.1.1. The main result of this paper is the analogous generalization of Theorem 1.1.2.

Theorem 1.1.5. *Let A be a π-finite space. For every $F\colon A \to \mathrm{Sp}_{T(n)}$, there is a canonical (and natural) equivalence*

$$\mathrm{Nm}_A\colon \varinjlim F \xrightarrow{\sim} \varprojlim F \quad \in \mathrm{Sp}_{T(n)}\,.$$

Again, this implies and therefore also provides an new proof for Theorem 1.1.4.

As a consequence of Theorem 1.1.5 and the general theory that we develop, we obtain a generalization to $T(n)$-homology, of the fact that $K(n)$-homology of a π-finite space depends only on the n-th Postnikov truncation of the space.

Theorem 1.1.6. *Let $n \geq 0$ and let $f\colon A \to B$ be a map with π-finite n-connected homotopy fibers. The induced map $\Sigma^\infty_+ f\colon \Sigma^\infty_+ A \to \Sigma^\infty_+ B$ is a $T(n)$-equivalence.*

The canonical norms of Theorem 1.1.4 (and Theorem 1.1.5) can be set in a broader context developed in [HL13]. Let $\mathcal{C}$ be an ∞-category that admits all (co)limits indexed by π-finite spaces. For every π-finite space A, we have two functors

$$\varinjlim_A, \ \varprojlim_A\colon \operatorname{Fun}(A, \mathcal{C}) \to \mathcal{C}.$$

In [HL13], the authors set up a general process that attempts to construct canonical natural transformations

$$\operatorname{Nm}_A\colon \varinjlim_A \to \varprojlim_A$$

for all m-finite spaces A, by induction on m. The m-th step of this process, requires that all canonical norm maps for $(m-1)$-finite spaces, that were constructed in the previous step, are *isomorphisms*. The property of an ∞-category $\mathcal{C}$, that these canonical norm maps can be constructed and are isomorphisms for all m-finite spaces, is called *m-semiadditivity* (see section 1.2). We can thus restate Theorem 1.1.1 and Theorem 1.1.2 as saying that the ∞-categories $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ are 1-semiadditive, while Theorem 1.1.4 and Theorem 1.1.5 as saying that these ∞-categories are ∞-semiadditive (i.e. m-semiadditive for all m).

A large part of the paper is devoted to establishing some consequences of 1-semiadditivity, especially for stable ∞-categories. The main result in this direction, which is central to the proof of Theorem 1.1.5, but is also of independent interest, is the construction of certain "power operations". In particular, we prove

Theorem 1.1.7. *Let $E \in \mathrm{Sp}$, such that Sp_E is 1-semiadditive (e.g. $E = T(n)$) and let X be an $\mathbb{E}_\infty$-algebra in Sp_E. The commutative ring $R = \pi_0 X$ admits a canonical additive p-derivation $\delta\colon R \to R$ (see Definition 4.1.1). In particular, the operation*

$$\psi(x) = x^p + p\delta(x)$$

is a linear map, which is a canonical lift of the Frobenius endomorphism modulo p. The operation δ (and hence ψ) is functorial with respect to maps of $\mathbb{E}_\infty$-algebra maps.

The lift of Frobenius that we obtain is a generalization of the "canonical lift of Frobenius" constructed in [Sta16] for the Morava E-theory cohomology ring of a space. Employing this power operation, we obtain a straightforward proof of the following result.

Theorem 1.1.8. *Let E be a homotopy commutative ring spectrum, such that Sp_E is 1-semiadditive and let R be an $\mathbb{E}_\infty$-ring spectrum. For every $x \in \pi_* R$, if the image of x in $\pi_*(H\mathbb{Q} \otimes R)$ is nilpotent, then the image of x in $\pi_*(E \otimes R)$ is nilpotent (i.e. the single homology theory $H\mathbb{Q}$ detects nilpotence in all 1-semiadditive multiplicative homology theories).*

In fact, we prove Theorem 1.1.8 for a larger class of spectra E, which we call *sofic* (see Definition 5.2.3). These include for example all spectra whose Bousfield class is contained in a sum of spectra E for which Sp_E is 1-semiadditive. In particular, it can be applied to the Morava E-theories of any height and the finite localizations of the sphere $L_n^f \mathbb{S}$. We also explain how this immediately implies (and generalizes) a conjecture of May, that was recently proved in [MNN15].

1.2 Background on Higher Semiadditivity

The notion of higher semiadditivity deserves a more detailed exposition.

From Norms to Integration

Since the construction of the canonical norm maps is inductive, it will be helpful to begin with describing some *consequences* of having invertible norm maps. This will also clarify their relation to the classical notion of semiadditivity. For an ordinary category $\mathcal{C}$, semiadditivity is a property, whose main feature is the ability to sum a finite family of morphisms between two objects. Similarly, for an ∞-category $\mathcal{C}$, being m-semiadditive is a property, whose main feature is the ability to sum an *m-finite* family of morphisms between two objects. Namely, given an m-finite space A and a map

$$\varphi\colon A \to \mathrm{Map}_{\mathcal{C}}\left(X, Y\right),$$

we define a map

$$\int\limits_A \varphi\colon X \to Y,$$

which we should think of as the sum (or integral) of φ over A, as the composition

$$X \xrightarrow{\Delta} \varprojlim_A X \xrightarrow{\varprojlim \varphi} \varprojlim_A Y \xrightarrow{\mathrm{Nm}_A^{-1}} \varinjlim_A Y \xrightarrow{\nabla} Y.$$

Note, that for an ordinary semiadditive category, summation over a finite set A is indeed obtained in this way using the *canonical* isomorphism

$$\mathrm{Nm}_A\colon \coprod_A X \xrightarrow{\sim} \prod_A X.$$

As a special case, for every object $X \in \mathcal{C}$, integrating the constant A-family on Id_X, produces an endomorphism $[A] \in \mathrm{Map}_{\mathcal{C}}\left(X, X\right)$. This generalizes the “multiplication by k” endomorphism of X for an integer k.

From Integration to Norms

We now turn things around and *construct* norm maps for m-finite spaces by integrating some $(m-1)$-finite families of maps. Quite generally, given any space A and a diagram $F\colon A \to \mathcal{C}$, to specify a morphism

$$\mathrm{Nm}_A\colon \varinjlim_A F \to \varprojlim_A F,$$

roughly amounts to specifying a compatible collection of morphisms

$$a, b \in A\colon \qquad \mathrm{Nm}_A^{a,b}\colon F\left(a\right) \to F\left(b\right).$$

Fixing $a, b \in A$ and denoting by $A_{a,b}$ the space of paths from a to b, the diagram F itself provides a *family* of candidates for $\mathrm{Nm}_A^{a,b}$:

$$F_{a,b}\colon A_{a,b} \to \mathrm{Map}\left(F\left(a\right), F\left(b\right)\right).$$

There is a priori no obvious (compatible) way to choose one of them, but assuming we are able to integrate maps over the spaces $A_{a,b}$, we can just "sum them all"

$$\mathrm{Nm}_A^{a,b} = \int\limits_{A_{a,b}} F_{a,b}.$$

This construction is somewhat easier to grasp when F is constant on an object X. In this special case, a morphism

$$\varinjlim_A X \to \varprojlim_A X,$$

is the same as a map of spaces

$$A \times A \to \mathrm{Map}_{\mathcal{C}}(X, X).$$

That is, an "$A \times A$ matrix" of endomorphism of X, where the $(a, b) \in A \times A$ entry corresponds to $\mathrm{Nm}_A^{a,b}$. The construction sketched above specializes to give $\mathrm{Nm}_A^{a,b} = [A_{a,b}]$. The construction of the norm in the general case can be thought of as a "twisted" version of the one for the constant diagram.

The Inductive Process

To tie things up, we observe that if A is m-finite, then the path spaces $A_{a,b}$ are $(m-1)$-finite. Thus, assuming inductively that we have *invertible* canonical norm maps Nm_A for all $(m-1)$-finite spaces A, we obtain a canonical way to integrate $(m-1)$-finite families of morphisms. As explained above, this allows us to define norm maps for all m-finite spaces. It is now a *property* that all those new norm maps are isomorphisms, which in turn induces an operation of integration over m-finite spaces and so on. We spell out the situation for small values of m.

(-2) We define *every* ∞-category to be (-2)-semiadditive. Indeed, if A is (-2)-finite, then $A \simeq \mathrm{pt}$ and the canonical norm map $\mathrm{Nm}_{\mathrm{pt}}$ is the identity natural transformation of the identity functor. In particular, we get a canonical way to sum a one point family of maps (or anything really), which is just taking the value at the point itself.

(-1) The only non-contractible (-1)-finite space is $A = \varnothing$. The associated norm map is the unique map

$$\mathrm{Nm}_{\varnothing} \colon 0_{\mathcal{C}} \to 1_{\mathcal{C}}$$

from the initial object to the terminal object of $\mathcal{C}$. Requiring this map to be an isomorphism is to require the existence of a zero object. Thus, $\mathcal{C}$ is (-1)-semiadditive if and only if it is *pointed*. This in turn allows us to integrate an empty family of morphisms. Namely, given $X, Y \in \mathcal{C}$, we get a canonical zero map given by the composition

$$X \to 1_{\mathcal{C}} \xrightarrow{\sim} 0_{\mathcal{C}} \to Y.$$

(0) A 0-finite space is one that is equivalent to a finite set A. Given a collection of objects $\{X_a\}_{a \in A}$ in a *pointed* ∞-category $\mathcal{C}$, we get a canonical map

$$\mathrm{Nm}_A \colon \coprod_{a \in A} X_a \to \prod_{a \in A} X_a.$$

This map is given by the "identity matrix" (this uses the zero maps, which in turn use the inverse of $\mathrm{Nm}_{\varnothing}$). Requiring these maps to be isomorphisms is precisely the usual property of being *semiadditive*, which allows to sum a finite family of morphisms.

(1) A connected 1-finite space is of the form $A = BG$ for a finite group G. A diagram $F\colon BG \to \mathcal{C}$ is equivalent to an object $X \in \mathcal{C}$ equipped with an action of G. When $\mathcal{C}$ is *semiadditive*, one can construct the canonical norm map

$$\mathrm{Nm}_{BG}\colon X_{hG} \to X^{hG}$$

and it can be identified with the classical norm of G. If $\mathcal{C}$ is stable, then Nm_{BG} is an isomorphism if and only if its cofiber, the Tate construction X^{tG}, vanishes. It is in this form that Theorem 1.1.1 and Theorem 1.1.2 where originally stated and proved.

Relative and Axiomatic Integration

Just like with ordinary semiadditivity, integration of m-finite families of maps satisfies various compatibilities. These generalize associativity, changing summation order, distributivity with respect to composition etc. To conveniently manage those compatibility relations it is useful to extend the integral operation to the relative case. Given a map of m-finite spaces $q\colon A \to B$, the pullback along q functor

$$q^*\colon \mathrm{Fun}\left(B, \mathcal{C}\right) \to \mathrm{Fun}\left(A, \mathcal{C}\right),$$

admits a left and right adjoint, which we denote by $q_!$ and q_* respectively. If $\mathcal{C}$ is $(m-1)$-semiadditive, one can construct a canonical norm map $\mathrm{Nm}_q\colon q_! \to q_*$ and it is an isomorphism when $\mathcal{C}$ is m-semiadditive. Similarly to the absolute case, given objects $X, Y \in \mathrm{Fun}\left(B, \mathcal{C}\right)$, one can use the inverse of Nm_q to define "integration along the fibers of q",

$$\int_q \colon \mathrm{Map}_{\mathrm{Fun}(A,\mathcal{C})}\left(q^*X, q^*Y\right) \to \mathrm{Map}_{\mathrm{Fun}(B,\mathcal{C})}\left(X, Y\right).$$

The approach we take in this paper, is to further generalize the situation and to put it in an axiomatic framework. We define a *normed functor*, to be a functor

$$q^*\colon \mathcal{C} \to \mathcal{D},$$

that admits a left adjoint $q_!$, a right adjoint q_*, and is equipped with a natural transformation $\mathrm{Nm}_q\colon q_! \to q_*$. If this natural transformation is an isomorphism, we can use the same formulas as above to define an abstract integration operation

$$\int_q \colon \mathrm{Map}_{\mathcal{D}}\left(q^*X, q^*Y\right) \to \mathrm{Map}_{\mathcal{C}}\left(X, Y\right)$$

for all $X, Y \in \mathcal{C}$. We proceed to develop a general calculus of normed functors and integration, which can then be applied to the context of higher semiadditivity. One advantage of this axiomatic approach, is that it separates the formal aspects of this "calculus" from the rather involved inductive construction of the canonical norm maps. Another advantage is that it unifies many seemingly different phenomena as special cases of several general formal statements. This renders the development more economic and streamlined. Finally, we believe that this axiomatic framework might be of use elsewhere.

1.3 Outline of the Proof

We now sketch the proof of the main theorem of the paper, that $\mathrm{Sp}_{T(n)}$ is ∞-semiadditive. The argument is inductive on the level of semiadditivity m. The basis of the induction is $m = 1$, which is given by Theorem 1.1.2. Assume that $\mathrm{Sp}_{T(n)}$ is m-semiadditive. In order to show that $\mathrm{Sp}_{T(n)}$ is $(m+1)$-semiadditive, we need to prove that for every $(m+1)$-finite space B, the natural transformation $\mathrm{Nm}_B \colon \varinjlim_B \to \varprojlim_B$ is an equivalence. We proceed by a sequence of reductions. First, since $\mathrm{Sp}_{T(n)}$ is stable and p-local, by [HL13, Proposition 4.4.16], it suffices to show that

(1) The norm map Nm_B is an equivalence for the single space $B = B^{m+1}C_p$.

Now, consider a fiber sequence of spaces

$$(*) \quad A \to E \to B,$$

where A and E are m-finite, and B is connected and $(m+1)$-finite. We prove that if the natural transformation $[A]$ is *invertible* (we call such A *amenable*), then Nm_B is an equivalence (Proposition 3.1.17). In fact, it suffices to show that the component of $[A]$ at the monoidal unit $\mathbb{S}_{T(n)}$ is invertible (Lemma 3.3.4). By abuse of notation, we denote this component also by $[A]$.

In order to apply the above to $B = B^{m+1}C_p$, we introduce the following class of "candidates" for A. We call a space A, *m-good* if it is connected, m-finite with $\pi_m A \neq 0$, and all homotopy groups of A are p-groups. Since such A is in particular nilpotent, one can always fit it in a fiber sequence $(*)$ with $B = B^{m+1}C_p$. Thus, we are reduced to showing that

(2) There exists an m-good space A, such that $[A] \in \pi_0\mathbb{S}_{T(n)}$ is invertible.

To detect invertibility in the ring $\pi_0\mathbb{S}_{T(n)}$, we transport the problem into a better understood setting. Let E_n be the Morava E-theory $\mathbb{E}_\infty$-ring spectrum of height n, and let $\mathrm{Mod}_{E_n}^{(K(n))}$ be the ∞-category of $K(n)$-local E_n-modules. The functor

$$E_n \hat{\otimes} (-) \colon \mathrm{Sp}_{T(n)} \to \mathrm{Mod}_{E_n}^{(K(n))}$$

is symmetric monoidal, and hence induces a map of commutative rings

$$f \colon \pi_0\mathbb{S}_{T(n)} \to \pi_0 E_n = \mathbb{Z}_p\left[\left[u_1, \dots, u_{n-1}\right]\right].$$

Using the Nilpotence Theorem and standard techniques of chromatic homotopy theory, we show that an element of $\pi_0\mathbb{S}_{T(n)}$ is invertible, if and only if its image under f is invertible (Corollary 5.3.5). Moreover, the functor $E_n \hat{\otimes} (-)$ is colimit preserving. By the general theory of higher semiadditivity that we develop, it follows that $\mathrm{Mod}_{E_n}^{(K(n))}$ is also m-semiadditive (Corollary 3.3.2(2)), and moreover, $f([A])$ coincides with the element $[A]$ of $\pi_0 E_n$ (Corollary 3.2.6). Thus, we can replace $\mathrm{Sp}_{T(n)}$ with the more approachable ∞-category $\mathrm{Mod}_{E_n}^{(K(n))}$. Namely, it suffices to show

(3) There exists an m-good space A, such that $[A] \in \pi_0 E_n$ is *invertible*.

By [BG18, Lemma 1.33], the image of f is contained in the constants $\mathbb{Z}_p$. Hence, $[A]$ is invertible, if and only if its p-adic valuation is zero. On $\mathbb{Z}_p$, we have the Fermat quotient operation

$$\tilde{\delta}(x) = \frac{x - x^p}{p},$$

with the salient property of reducing the p-adic valuation of non-invertible non-zero elements. The heart of the proof comprises of realizing the algebraic operation $\tilde{\delta}$ in a way that acts on the elements $[A]$ in an understood way. It is for this step that it is crucial that our induction base is $m = 1$. Namely, for a presentable, 1-*semiadditive*, stable, p-local, symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$, we construct a "power operation" (Definition 4.1.1 and Theorem 4.3.2)

$$\delta \colon \pi_0 (\mathbb{1}_{\mathcal{C}}) \to \pi_0 (\mathbb{1}_{\mathcal{C}}) \,.$$

This operation shares many of the formal properties of $\tilde{\delta}$. In particular, specializing to the case $\mathcal{C} = \mathrm{Mod}_{E_n}^{(K(n))}$, the operation δ coincides with $\tilde{\delta}$ on $\mathbb{Z}_p \subseteq \pi_0 E_n$. Moreover, for an m-good A, we have

$$\delta ([A]) = [A'] - [A''] \,,$$

where A' and A'' are also m-good (combine Definition 4.3.1 and Theorem 4.2.12). It follows that if $[A]$ is non-zero (and not already invertible), then at least one of $[A']$ and $[A'']$ has *lower* p-adic valuation than $[A]$. The prototypical m-good space is the Eilenberg-MacLane space $B^m C_p$. Hence, it suffices to show that

(4) The element $[B^m C_p] \in \pi_0 E_n$ is *non-zero*.

To get a grip on the elements $[A]$, we reformulate them in terms of the symmetric monoidal dimension (which does not refer at all to higher semiadditivity). Let us denote by $A \otimes E_n$, the colimit of the constant A-shaped diagram on E_n in $\mathrm{Mod}_{E_n}^{(K(n))}$. We show that $A \otimes E_n$ is a dualizable object, and that (Corollary 3.3.10)

$$\dim (A \otimes E_n) = [A^{S^1}] \quad \in \pi_0 E_n.$$

Since

$$[(B^m C_p)^{S^1}] = [B^m C_p \times B^{m-1} C_p] = [B^m C_p][B^{m-1} C_p],$$

it suffices to show that

(5) The element $\dim (B^m C_p \otimes E_n) \in \pi_0 E_n$ is *non-zero*.

Finally, it can be shown that $\dim (A \otimes E_n)$ equals the Euler characteristic of the 2-periodic Morava K-theory (Lemma 5.3.6)[1]

$$\chi_n (A) = \dim_{\mathbb{F}_p} K(n)_0 \, A - \dim_{\mathbb{F}_p} K(n)_1 \, A.$$

Hence, it suffices to prove that

(6) The integer $\chi_n (B^m C_p)$ is *non-zero*.

This is an immediate consequence of the explicit computation of $K(n)_* (B^m C_p)$, carried out in [RW80].

We alert the reader that at several points, this outline diverges from the actual proof we give. Most significantly, we make use of the fact that the steps (1)-(5) are completely formal and the ideas involved can be formalized in a much greater generality. Instead of the functor $E_n \hat{\otimes} (-)$,

[1] We prove this only for $B^m C_p$, as this suffices for our purposes, but this is true in general.

we can consider any colimit preserving symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ between stable, p-local, symmetric monoidal ∞-categories. Given such a functor F, we show how to bootstrap 1-semiadditivity to higher semiadditivity under appropriate conditions (Theorem 4.3.10). This necessitates some technical changes in the argument outlined above[2]. It is only in the final section that we specialize to $\mathcal{C} = \mathrm{Sp}_{T(n)}$, and verify the assumptions of this general criterion.

Remark 1.3.1. Informally, the argument presented above shows that, while 1-semiadditive categories are very special among presentably symmetric monoidal stable p-local ∞-categories, it is quite easy for a 1-semiadditive ∞-category to be ∞-semiadditive. This intuition leads us to ask the following:

Question 1.3.2. *Is every presentable, stable, p-local and 1-semiadditive ∞-category, necessarily ∞-semiadditive?*

1.4 Organization

We now describe the content of each section of the paper.

In section 2, we develop the axiomatic framework of normed functors and integration. We begin by developing some general calculus for this notion and study its functoriality properties. We then study the interaction of integration with symmetric monoidal structure and the notion of duality. We conclude with a discussion of the property of *amenability*.

In section 3, we apply the axiomatic theory of section 2 to the concrete setting of local systems valued in an m-semiadditive ∞-category. We begin by recalling the canonical norm on the pullback functor along an m-finite map (introduced in [HL13, Section 4.1]), and its interaction with various operations. We then consider m-finite colimit preserving functors between m-semiadditive ∞-categories (a.k.a m-semiadditive functors), and their behavior with respect to integration. We continue with studying the interaction of m-semiadditivity with symmetric monoidal structures, duality and dimension. Finally, we study the behavior of equivariant powers in 1-semiadditive ∞-categories, which is used in the sequel in the construction of power operations.

In section 4, we construct the above mentioned power operations for 1-semiadditive stable ∞-categories. First, we introduce the algebraic notion of an additive p-derivation and study some of its properties. We then construct an auxiliary operation α in the presence of 1-semiadditivity. Specializing to the *stable* (p-local) case, we construct from α the additive p-derivation δ and establish its naturality properties. Finally, we formulate and prove the "bootstrap machine" (Theorem 4.3.10), that gives general conditions for a 1-semiadditive ∞-category to be ∞-semiadditive.

In section 5, we apply the abstract theory of sections 2-4 to chromatic homotopy theory. After some generalities, we use the additive p-derivation of section 4 to derive a generalization of a conjecture of May about nilpotence of H_∞-rings. We then apply the "bootstrap machine" to the 1-semiadditive ∞-category $\mathrm{Sp}_{T(n)}$, to show that it is ∞-semiadditive. Finally, we apply the theory of higher semiadditivity to deduce that the $T(n)$-homology of π-finite spaces depends only on the n-th Postnikov truncation (Theorem 5.3.13).

[2] In particular, we bypass [BG18] using a somewhat different and more general argument.

1.5 Acknowledgments

We would like to thank Tobias Barthel, Agnès Beaudry, Gijs Heuts, and Nathaniel Stapleton for useful discussions. We would especially like to thank Michael Hopkins, for suggesting this question and for useful discussions.

The first author is partially supported by the Adams Fellowship of the Israeli Academy of Science. The second author is supported by the Alon Fellowship and ISF1588/18. The third author is supported by the ISF grant 1650/15

The second author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Homotopy Harnessing Higher Structures", where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1.

1.6 Conventions

Throughout the paper we shall use the following terminology and notation:

(1) We use the term *isomorphism* for an invertible morphism of an ∞-category (i.e. an equivalence).

(2) We say that a space A is

  (a) (-2)-finite, if it is contractible.

  (b) m-finite for $m \geq -1$, if $\pi_0 A$ is finite and all the fibers of the diagonal map $\Delta_A \colon A \to A \times A$ are $(m-1)$-finite (for $m \geq 0$, this is equivalent to A having finitely many components, each of them m-truncated with finite homotopy groups).

  (c) π-finite, if it is m-finite for some integer $m \geq -2$.

(3) We say that an π-finite space A is a p-space, if all the homotopy groups of A are p-groups.

(4) Given a map of spaces $q \colon A \to B$, for every $b \in B$ we denote by $q^{-1}(b)$ the homotopy fiber of q over b.

(5) For $m \geq -2$, we say that a map of spaces $q \colon A \to B$ is m-finite (resp. π-finite) if $q^{-1}(b)$ is m-finite (resp. π-finite) for all $b \in B$.

(6) Given an ∞-category $\mathcal{C}$, we say that $\mathcal{C}$ admits all q-limits (resp. q-colimits) if it admits all limits (resp. colimits) of shape $q^{-1}(b)$ for all $b \in B$.

(7) Given a functor $F \colon \mathcal{C} \to \mathcal{D}$ of ∞-categories, we say that F preserves q-colimits (resp. q-limits) if it preserves all colimits (resp. limits) of shape $q^{-1}(b)$ for all $b \in B$.

(8) We use the notation

$$f \colon X \xrightarrow{g} Y \xrightarrow{h} Z$$

to denote that $f \colon X \to Z$ is the composition $h \circ g$ (which is well defined up to a contractible space of choices). We use similar notation for composition of more than two morphisms.

(9) Given functors $F, G\colon \mathcal{C} \to \mathcal{D}$ and $H, K\colon \mathcal{D} \to \mathcal{E}$, and natural transformations $\alpha\colon F \to G$ and $\beta\colon H \to K$, we denote their horizontal composition by $\beta \star \alpha\colon HF \to KG$. The vertical composition of natural transformations is denoted simply by juxtaposition.

(10) For a symmetric monoidal ∞-category $\mathcal{C}$, we denote by $\mathrm{CAlg}(\mathcal{C})$ the ∞-category of $\mathbb{E}_\infty$-algebras in $\mathcal{C}$. We denote $\mathrm{coCAlg}(\mathcal{C}) = \mathrm{CAlg}(\mathcal{C}^{op})^{op}$ the ∞-category of $\mathbb{E}_\infty$-coalgebras in $\mathcal{C}$, where $\mathcal{C}^{op}$ is endowed with the canonical symmetric monoidal structure induced from $\mathcal{C}$.

(11) For an abelian group A and $k \geq 0$, we denote by $B^k A$ the Eilenberg MacLane space with k-th homotopy group equal to A.

2 Norms and Integration

In this section we develop an abstract formal framework of norms on functors between ∞-categories and the operation of integration on maps, that such norms induce. This framework abstracts and axiomatizes the theory of norms and integrals arising from ambidexterity, developed in [HL13, Section 4]. We develop a "calculus" for such integrals and study their functoriality properties and interaction with monoidal structures.

2.1 Normed Functors and Integration

Norms and Iso-Norms

We begin by fixing some terminology regarding adjunctions of ∞-categories.

Definition 2.1.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories.

(1) By a *left adjoint* to F, we mean a pair (L, u), where $L\colon \mathcal{D} \to \mathcal{C}$ is a functor and

$$u\colon \mathrm{Id}_{\mathcal{D}} \to F \circ L$$

a unit natural transformation in the sense of [Lur09, Definition 5.2.2.7].

(2) By a *right adjoint* to F, we mean a pair (R, c), where $R\colon \mathcal{D} \to \mathcal{C}$ is a functor and

$$c\colon F \circ R \to \mathrm{Id}_{\mathcal{D}}$$

a counit natural transformation (i.e. satisfying the dual of [Lur09, Definition 5.2.2.7]).

Given a datum of a left adjoint (L, u), there exists a map $c\colon L \circ F \to \mathrm{Id}_{\mathcal{C}}$, such that u and c satisfy the zig-zag identities up to homotopy. From this also follows that c is a counit map exhibiting (F, c) as a right adjoint to L. This counit map c is unique up to homotopy, and we shall therefore sometimes speak of "the" associated counit map (in fact, the space of such maps together with a homotopy witnessing *one* of the zig-zag identities is contractible [RV16, Proposition 4.4.7]). We shall similarly speak of the unit map $u\colon \mathrm{Id}_{\mathcal{C}} \to R \circ F$ associated with a right adjoint (R, c).

Adjoint functors can be composed in the following (usual) sense.

Definition 2.1.2. Given a pair of composable functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{F'} \mathcal{E}\ ,$$

with left adjoints (L, u) and (L', u') respectively, the composite map

$$u'' \colon \mathrm{Id}_{\mathcal{E}} \xrightarrow{u'} F'L' \xrightarrow{u} F'FLL',$$

which is well defined up to homotopy, is a unit map exhibiting LL' as left adjoint to $F'F$. We define the counit map of the composition of right adjoints in a similar way.

The central notion we are about to study in this section is the following:

Definition 2.1.3. Given ∞-categories $\mathcal{C}$ and $\mathcal{D}$, a *normed functor*

$$q \colon \mathcal{D} \rightarrowtail \mathcal{C},$$

is a functor $q^* \colon \mathcal{C} \to \mathcal{D}$ together with a left adjoint $(q_!, u_!^q)$, a right adjoint (q_*, c_*^q), and a natural transformation

$$\mathrm{Nm}_q \colon q_! \to q_*,$$

which we call a *norm*. We say that q is *iso-normed*, if Nm_q is an isomorphism natural transformation. We also write $X_q = q_! q^* X$, and denote by $c_!^q \colon q_! q^* \to \mathrm{Id}$ and $u_*^q \colon \mathrm{Id} \to q_* q^*$, the associated counit and unit of the respective adjunctions. We drop the superscript q whenever it is clear from the context.

Remark 2.1.4. In subsequent sections, we shall sometimes abuse language and refer to Nm_q as a *norm on* q^* and to q^* itself (with the data of Nm_q) as a normed functor. Since the left and right adjoints of q^* are essentially unique (when they exist), this seems to be a rather harmless convention.

There is a useful criterion for detecting when a normed functor is iso-normed.

Lemma 2.1.5. *A normed functor $q \colon \mathcal{D} \rightarrowtail \mathcal{C}$ is iso-normed if and only if the norm $\mathrm{Nm}_q \colon q_! \to q_*$ is an isomorphism at $q^* X$ for all $X \in \mathcal{C}$.*

Proof. The "only if" part is clear. For the "if" part, consider the two diagrams

$$\begin{tikzcd} & q_! q^* q_* \arrow[r, "c_*"] \arrow[d, "\mathrm{Nm}_q"', "\wr"] & q_! \arrow[d, "\mathrm{Nm}_q"'] \\ q_* \arrow[r, "u_*"] & q_* q^* q_* \arrow[r, "c_*"] & q_*, \end{tikzcd} \qquad \begin{tikzcd} q_! \arrow[r, "u_!"] \arrow[d, "\mathrm{Nm}_q"] & q_! q^* q_! \arrow[r, "c_!"] \arrow[d, "\wr"', "\mathrm{Nm}_q"] & q_! \\ q_* \arrow[r, "u_!"] & q_* q^* q_!, & \end{tikzcd}$$

which commute by naturality of the (co)unit maps. By the zig-zag identities, the composition along the bottom row in the left diagram is the identity. Thus, the left diagram shows that Nm_q has a right inverse. Similarly, the right diagram shows that Nm_q has a left inverse and therefore Nm_q is an isomorphism. $\square$

Given a functor $q^* \colon \mathcal{C} \to \mathcal{D}$ with a left adjoint $(q_!, u_!^q)$ and a right adjoint (q_*, c_*^q), the data of a natural transformation $\mathrm{Nm}_q \colon q_! \to q_*$ is equivalent to the data of its mate $\nu_q \colon q^* q_! \to \mathrm{Id}$. Moreover,

Lemma 2.1.6. *Let $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ be a normed functor. For every $Y \in \mathcal{D}$, the map $\mathrm{Nm}_q\colon q_! \to q_*$ is an isomorphism at $Y \in \mathcal{D}$ if and only if the mate $\nu_q\colon q^*q_! \to \mathrm{Id}$ is a counit map at Y. Namely, for all $X \in \mathcal{C}$, the composition*

$$\mathrm{Map}_{\mathcal{C}}(X, q_!Y) \xrightarrow{q^*} \mathrm{Map}_{\mathcal{D}}(q^*X, q^*q_!Y) \xrightarrow{\nu\circ -} \mathrm{Map}_{\mathcal{D}}(q^*X, Y)$$

is a homotopy equivalence.

Proof. For every $X \in \mathcal{C}$, consider the commutative diagram in the homotopy category of spaces:

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X, q_!Y) \arrow[r, "\mathrm{Nm}_q\circ -"] \arrow[d, "q^*"] & \mathrm{Map}_{\mathcal{C}}(X, q_*Y) \arrow[d, "q^*"] \arrow[dr, "\sim"] & \\
\mathrm{Map}_{\mathcal{D}}(q^*X, q^*q_!Y) \arrow[r, "\mathrm{Nm}_q\circ -"] \arrow[rr, bend right, "\nu_q\circ -"'] & \mathrm{Map}_{\mathcal{D}}(q^*X, q^*q_*Y) \arrow[r, "c_*\circ -"] & \mathrm{Map}_{\mathcal{D}}(q^*X, Y).
\end{tikzcd}$$

By the Yoneda lemma, Nm_q is an isomorphism at Y if and only if the top map in the diagram is an isomorphism for all $X \in \mathcal{C}$. By 2-out-of-3, this is if and only if the composition of the top map and the diagonal map is an isomorphism for all X. Since the diagram commutes, this is if and only if the composition of the left vertical map with the long bottom map is an isomorphism for all X, which is by definition if and only if ν_q is a counit at Y. $\square$

Notation 2.1.7. When Nm_q is an isomorphism at X, and hence ν_q is a counit at X, we denote the associated *unit* by $\mu_{q,X}\colon X \to q_!q^*X = X_q$. If q is iso-normed, we let $\mu_q\colon \mathrm{Id} \to q_!q^*$ be the unit natural transformation associated with ν_q. As usual, we drop the subscript q, whenever the map is understood from the context.

Remark 2.1.8. We will use the two points of view, that of a norm $\mathrm{Nm}_q\colon q_! \to q_*$ and that of a "wrong way counit" $\nu_q\colon q^*q_! \to \mathrm{Id}$ interchangeably. Each point of view has its own advantages. We note that the definition using ν_q seems to be slightly more general as it is available even if q^* does not (a priori) admit a right adjoint. In practice, we are mainly interested in situations where ν_q is indeed a counit map for an adjunction, exhibiting $q_!$ as a right adjoint of q^*. Thus, the gain in generality is rather negligible.

Definition 2.1.9. We define the identity normed functor and composition of normed functors (up to homotopy) as follows.

(1) (Identity) For every ∞-category $\mathcal{C}$, the identity normed functor $\mathrm{Id}\colon \mathcal{C} \rightarrowtail \mathcal{C}$ consists of the identity functor $\mathrm{Id}\colon \mathcal{C} \to \mathcal{C}$ viewed as a left and right adjoint to itself using the identity natural transformation $\mathrm{Id} \to \mathrm{Id}$ as the (co)unit map and with the identity natural transformation $\mathrm{Id} \to \mathrm{Id}$ as the norm.

(2) (Composition) Given a pair of composable normed functors

$$\mathcal{E} \overset{p}{\rightarrowtail} \mathcal{D} \overset{q}{\rightarrowtail} \mathcal{C}\ ,$$

we define their composition $qp\colon \mathcal{E} \rightarrowtail \mathcal{C}$ by composing the adjunctions (Definition 2.1.2)

$$(qp)^* = p^* q^*, \quad (qp)_! = q_! p_!, \quad (qp)_* = q_* p_*$$

and take the norm map to be the horizontal composition of the norms (the order does not matter)

$$q_! p_! \xrightarrow{\mathrm{Nm}_q} q_* p_! \xrightarrow{\mathrm{Nm}_p} q_* p_* .$$

We denote the norm of the composite by Nm_{qp}. If p and q are iso-normed, then so is qp.

Remark 2.1.10. It is possible to define an ∞-category $\widehat{\mathbf{Cat}}_\infty^{\mathrm{Nm}}$, whose objects are ∞-categories and morphisms are normed functors, such that the above constructions give the identity morphisms and composition in the homotopy category. This ∞-category captures the higher coherences manifest in the above definitions. We intend to elaborate on this point in a future work, but for the purposes of this one, which will not use the higher coherences in any way, we shall be content with the above explicit definitions up to homotopy.

Integration

The main feature of iso-normed functors is that they allow us to define a formal notion of "integration" of maps.

Definition 2.1.11. Let $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ be an iso-normed functor. For every $X, Y \in \mathcal{C}$, we define an *integral* map

$$\int\limits_q \colon \mathrm{Map}_{\mathcal{D}}(q^* X, q^* Y) \to \mathrm{Map}_{\mathcal{C}}(X, Y),$$

which is natural in X and Y, as the composition

$$\mathrm{Map}_{\mathcal{D}}(q^* X, q^* Y) \xrightarrow{q_*} \mathrm{Map}_{\mathcal{C}}(q_* q^* X, q_* q^* Y) \xrightarrow{\mathrm{Nm}_q^{-1}} \mathrm{Map}_{\mathcal{C}}(q_* q^* X, q_! q^* Y) \xrightarrow{c_! \circ - \circ u_*} \mathrm{Map}_{\mathcal{C}}(X, Y).$$

Remark 2.1.12. Alternatively, using the *wrong way unit* $\mu_q\colon \mathrm{Id} \to q_! q^*$, one can define the integral as the composition

$$\mathrm{Map}_{\mathcal{D}}(q^* X, q^* Y) \xrightarrow{q_!} \mathrm{Map}_{\mathcal{C}}(q_! q^* X, q_! q^* Y) \xrightarrow{c_! \circ - \circ \mu} \mathrm{Map}_{\mathcal{C}}(X, Y).$$

As a special case we have

Definition 2.1.13. Let $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ be an iso-normed functor. For every $X \in \mathcal{C}$, we define a map

$$[q]_X : X \to X$$

by

$$[q]_X \coloneqq \int\limits_q q^* \mathrm{Id}_X = \int\limits_q \mathrm{Id}_{q^* X}.$$

These are the components of the natural endomorphism $[q] = c_!^q \circ \mu_q$ of $\mathrm{Id}_{\mathcal{C}}$.

Integration satisfies a form of "homogeneity".

Proposition 2.1.14 (Homogeneity)**.** *Let $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ be an iso-normed functor. Given $X, Y, Z \in \mathcal{C}$,*

(1) For all maps $f\colon q^*X \to q^*Y$ and $g\colon Y \to Z$ we have

$$g \circ \left(\int\limits_q f \right) = \int\limits_q (q^*g \circ f) \quad \in \hom_{h\mathcal{C}}(X, Z)\,.$$

(2) For all maps $f\colon X \to Y$ and $g\colon q^*Y \to q^*Z$ we have

$$\left(\int\limits_q g \right) \circ f = \int\limits_q (g \circ q^*f) \quad \in \hom_{h\mathcal{C}}(X, Z)\,.$$

Proof. For (1), consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[d, "\mu"] & q_!q^*X \arrow[r, "f"] \arrow[d, "f"] & q_!q^*Y \arrow[r, "c_!"] \arrow[d, "g"] & Y \arrow[d, "g"] \\
q_!q^*X \arrow[r, "f"] & q_!q^*Y \arrow[r, "g"] & q_!q^*Z \arrow[r, "c_!"] & Z.
\end{tikzcd}$$

The composition along the top and then right path is $g \circ \int\limits_q f$, while the composition along the left and then bottom path is $\int\limits_q (q^*g \circ f)$ (see Remark 2.1.12).

For (2), consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[d, "f"] & q_!q^*X \arrow[r, "f"] \arrow[d, "f"] & q_!q^*Y \arrow[r, "g"] \arrow[d, "g"] & q_!q^*Z \arrow[d, "c_!"] \\
Y \arrow[r, "\mu"] & q_!q^*Y \arrow[r, "g"] & q_!q^*Z \arrow[r, "c_!"] & Z
\end{tikzcd}$$

and apply an analogous argument. ∎

Integration also satisfies a form of "Fubini's Theorem".

Proposition 2.1.15 (Higher Fubini's Theorem)**.** *Given a pair of composable iso-normed functors*

$$\mathcal{E} \overset{p}{\rightarrowtail} \mathcal{D} \overset{q}{\rightarrowtail} \mathcal{C}\ ,$$

for all $X, Y \in \mathcal{C}$, and $f\colon p^*q^*X \to p^*q^*Y$, we have

$$\int\limits_q \left(\int\limits_p f \right) = \int\limits_{qp} f \qquad \in \hom_{h\mathcal{C}}(X, Y)\,.$$

Proof. Since q and p are iso-normed, we can construct the following diagram

$$\begin{tikzcd}
& q_*p_*p^*q^*X \arrow[r, "f"] & q_*p_*p^*q^*Y \arrow[r, "\mathrm{Nm}_{qp}^{-1}"] \arrow[d, "\mathrm{Nm}_p^{-1}"'] & q_!p_!p^*q^*Y \arrow[d, equal] \arrow[dr, "c_!^{qp}"] & \\
X \arrow[ur, "u_*^{qp}"] \arrow[dr, "u_*^q"'] & & q_*p_!p^*q^*Y \arrow[r, "\mathrm{Nm}_q^{-1}"] \arrow[d, "c_!^p"'] & q_!p_!p^*q^*Y \arrow[d, "c_!^p"'] & Y. \\
& q_*q^*X \arrow[uu, "u_*^p"'] \arrow[r, "\int\limits_p f"] & q_*q^*Y \arrow[r, "\mathrm{Nm}_q^{-1}"] & q_!q^*Y \arrow[ur, "c_!^q"'] &
\end{tikzcd}$$

The triangles and the bottom right square commute for formal reasons. The top right square commutes by the way norms are composed (Definition 2.1.9(2)) and the left rectangle commutes by definition of $\int\limits_p f$. Thus, the composition along the top path, which is $\int\limits_{qp} f$, is homotopic to the composition along the bottom path, which is $\int\limits_q \left(\int\limits_p f\right)$. $\square$

2.2 Ambidextrous Squares & Beck-Chevalley Conditions

In this section we study functoriality properties of norms and integrals and develop further the "calculus of integration".

Beck-Chevalley Conditions

We begin by recalling some standard material regarding commuting squares involving adjoint functors (e.g. see beginning of [Lur09, Section 7.3.1]). A commutative square of functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\
\mathcal{D} \arrow[r, "F_{\mathcal{D}}"] & \tilde{\mathcal{D}}
\end{tikzcd} \qquad (\square)$$

is formally a natural isomorphism

$$F_{\mathcal{D}} q^* \xrightarrow{\sim} \tilde{q}^* F_{\mathcal{C}}.$$

If the vertical functors admit left adjoints $q_! \dashv q^*$ and $\tilde{q}_! \dashv \tilde{q}^*$ (suppressing the units), we get a $\mathrm{BC}_!$ (Beck-Chevalley) natural transformation

$$\beta_! \colon \tilde{q}_! F_{\mathcal{D}} \to F_{\mathcal{C}} q_!.$$

Similarly, if the vertical functors admit right adjoints $q^* \dashv q_*$ and $\tilde{q}^* \dashv \tilde{q}_*$, we get a BC_* (Beck-Chevalley) natural transformation

$$\beta_* \colon F_{\mathcal{D}} q_* \to \tilde{q}_* F_{\mathcal{C}}.$$

Definition 2.2.1. We say that the square $\square$ satisfies the $\mathrm{BC}_!$ (resp. BC_*) condition, if q^* and $\tilde{q}^*$ admit left (resp. right) adjoints and the map $\beta_!$ (resp. β_*) is an equivalence.

Remark 2.2.2. It may happen that in $\square$, the horizontal functors $F_{\mathcal{C}}$ and $F_{\mathcal{D}}$ also have left or right adjoints. In this case, there are other BC maps one can write. To avoid confusion, we will always speak about the BC maps with respect to the *vertical* functors.

Given a commutative square $\square$ as above, we denote $u_* = u_*^q$ and $\tilde{u}_* = u_*^{\tilde{q}}$ and similarly for other (co)unit maps (when they exist). It is an easy verification using the zig-zag identities, that the BC-maps are compatible with these units and counits in the following sense.

Lemma 2.2.3. *Given a commutative square of functors $\square$, such that q^* and $\tilde{q}^*$ admit left (resp. right) adjoints, the following four diagrams commute up to homotopy (when they are defined)*

(1)
$$\begin{tikzcd}
& F_{\mathcal{C}} q_* q^* \arrow[d, "\beta_*"] \\
F_{\mathcal{C}} \arrow[ur, "u_*"] \arrow[dr, "\tilde{u}_*"'] & \tilde{q}_* F_{\mathcal{D}} q^* \arrow[d, "\wr"] \\
& \tilde{q}_* \tilde{q}^* F_{\mathcal{C}}
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
F_{\mathcal{D}} q^* q_* \arrow[d, "\wr"] \arrow[dr, "c_*"] & \\
\tilde{q}^* F_{\mathcal{C}} q_* \arrow[d, "\beta_*"] & F_{\mathcal{D}} \\
\tilde{q}^* \tilde{q}_* F_{\mathcal{D}} \arrow[ur, "\tilde{c}_*"'] &
\end{tikzcd}$$

(3)
$$\begin{tikzcd}
& \tilde{q}^* \tilde{q}_! F_{\mathcal{D}} \arrow[d, "\beta_!"] \\
F_{\mathcal{D}} \arrow[ur, "\tilde{u}_!"] \arrow[dr, "u_!"'] & \tilde{q}^* F_{\mathcal{C}} q_! \arrow[d, "\wr"] \\
& F_{\mathcal{D}} q^* q_!
\end{tikzcd}$$

(4)
$$\begin{tikzcd}
\tilde{q}_! \tilde{q}^* F_{\mathcal{C}} \arrow[d, "\wr"] \arrow[dr, "\tilde{c}_!"] & \\
\tilde{q}_! F_{\mathcal{D}} q^* \arrow[d, "\beta_!"] & F_{\mathcal{C}}. \\
F_{\mathcal{C}} q_! q^* \arrow[ur, "c_!"'] &
\end{tikzcd}$$

The BC maps also satisfy some naturality properties with respect to horizontal and vertical pasting, as well as multiplication and exponentiation of squares. We begin with horizontal pasting. Given a commutative diagram of ∞-categories and functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"] & \tilde{\mathcal{C}} \arrow[r, "G_{\mathcal{C}}"] \arrow[d, "\tilde{q}^*"] & \tilde{\tilde{\mathcal{C}}} \arrow[d, "\tilde{\tilde{q}}^*"] \\
\mathcal{D} \arrow[r, "F_{\mathcal{D}}"] & \tilde{\mathcal{D}} \arrow[r, "G_{\mathcal{D}}"] & \tilde{\tilde{\mathcal{D}}},
\end{tikzcd} \qquad (*)$$

we call the big outer square the *horizontal pasting* of the left and right small squares. The following is easy to verify.

Lemma 2.2.4. *Given a horizontal pasting diagram* $(*)$ *as above,*

(1) The $\mathrm{BC}_!$*-map for the outer square is homotopic to the composition of the* $\mathrm{BC}_!$ *maps for the left and right squares*
$$\tilde{\tilde{q}}_! G_{\mathcal{D}} F_{\mathcal{D}} \to G_{\mathcal{C}} \tilde{q}_! F_{\mathcal{D}} \to G_{\mathcal{C}} F_{\mathcal{C}} q_!.$$

(2) The BC_**-map for the outer square is homotopic to the composition of the* BC_* *maps for the left and right squares*
$$G_{\mathcal{C}} F_{\mathcal{C}} q_* \to G_{\mathcal{C}} \tilde{q}_* F_{\mathcal{D}} \to \tilde{\tilde{q}}_* G_{\mathcal{D}} F_{\mathcal{D}}.$$

This immediately implies the following horizontal pasting lemma for BC conditions.

Corollary 2.2.5. *Given a horizontal pasting diagram* $(*)$ *as above, denote by* $\square_L$*,* $\square_R$ *and* $\square$*, the left, right and outer squares respectively.*

(1) If $\square_L$ and $\square_R$ satisfy the $\mathrm{BC}_!$ *(resp.* BC_**) condition, then so does $\square$.*

(2) If $\square_R$ and $\square$ satisfy the $\mathrm{BC}_!$ *(resp.* BC_**) condition and $G_{\mathcal{C}}$ is conservative, the so does $\square_L$.*

We now turn to vertical pasting. Given a commutative diagram of ∞-categories and functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D} \arrow[r, "F_{\mathcal{D}}"] \arrow[d, "p^*"'] & \tilde{\mathcal{D}} \arrow[d, "\tilde{p}^*"] \\ \mathcal{E} \arrow[r, "F_{\mathcal{E}}"] & \tilde{\mathcal{E}}, \end{tikzcd} \qquad (**)$$

we call the big outer square the *vertical pasting* of the top and bottom small squares. The following is easy to verify.

Lemma 2.2.6. *Given a vertical pasting diagram $(**)$ as above,*

(1) The $\mathrm{BC}_!$*-map for the outer square is homotopic to the composition of the* $\mathrm{BC}_!$ *maps for the top and bottom squares*

$$\tilde{q}_!\tilde{p}_!F_{\mathcal{E}} \to \tilde{q}_!F_{\mathcal{D}}p_! \to F_{\mathcal{C}}q_!p_!.$$

(2) The BC_**-map for the outer square is homotopic to the composition of the* BC_* *maps for the top and bottom squares*

$$F_{\mathcal{C}}q_*p_* \to \tilde{q}_*F_{\mathcal{D}}p_* \to \tilde{q}_*\tilde{p}_*F_{\mathcal{E}}.$$

Again, this immediately implies the following vertical pasting lemma for BC conditions.

Corollary 2.2.7. *Given a vertical pasting diagram $(**)$ as above, denote by $\square_T$, $\square_B$ and $\square$, the top, bottom and outer squares respectively. If $\square_T$ and $\square_B$ satisfy the* $\mathrm{BC}_!$ *(resp.* BC_**) condition, then so does $\square$.*

Finally, the BC conditions are also natural with respect to multiplication and exponentiation.

Lemma 2.2.8. *Given a pair of mate squares*

$$\begin{tikzcd} \mathcal{C}\times\mathcal{E} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*\times \mathrm{Id}"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D}\times\mathcal{E} \arrow[r, "F_{\mathcal{D}}"] & \tilde{\mathcal{D}} \end{tikzcd} \quad (\square_1)\,, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\hat{F}_{\mathcal{C}}"] \arrow[d, "q^*"'] & \mathrm{Fun}(\mathcal{E},\tilde{\mathcal{C}}) \arrow[d, "(q^*)^{\mathcal{E}}"] \\ \mathcal{D} \arrow[r, "\hat{F}_{\mathcal{D}}"] & \mathrm{Fun}(\mathcal{E},\tilde{\mathcal{D}}) \end{tikzcd} \quad (\square_2)\,.$$

the square $\square_1$ satisfies the $\mathrm{BC}_!$ *(resp.* BC_**) if and only if $\square_2$ satisfies the* $\mathrm{BC}_!$ *(resp.* BC_**) condition.*

Proof. Under the canonical equivalence of ∞-categories

$$\mathrm{Fun}(\mathcal{D}\times\mathcal{E},\tilde{\mathcal{C}}) \simeq \mathrm{Fun}(\mathcal{D},\mathrm{Fun}(\mathcal{E},\tilde{\mathcal{C}})),$$

the $\mathrm{BC}_!$ (resp. BC_*) map for $\square_1$ corresponds to the $\mathrm{BC}_!$ (resp. BC_*) map of $\square_2$ and isomorphisms correspond to isomorphisms. $\square$

Normed and Ambidextrous Squares

We now consider commuting squares of ∞-categories, where the vertical functors are *normed*.

Definition 2.2.9. We define:

(1) A *normed square* is a pair of normed functors $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ and $\tilde{q}\colon \tilde{\mathcal{D}} \rightarrowtail \tilde{\mathcal{C}}$, together with a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D} \arrow[r, "F_{\mathcal{D}}"] & \tilde{\mathcal{D}}. \end{tikzcd} \qquad (*)$$

It is *iso-normed* if q and $\tilde{q}$ are iso-normed.

(2) Given a normed square as in (1), we have an associated *norm-diagram*:

$$\begin{tikzcd} F_{\mathcal{C}}q_! \arrow[r, "\mathrm{Nm}_q"] & F_{\mathcal{C}}q_* \arrow[d, "\beta_*"] \\ \tilde{q}_!F_{\mathcal{D}} \arrow[u, "\beta_!"] \arrow[r, "\mathrm{Nm}_{\tilde{q}}"] & \tilde{q}_*F_{\mathcal{D}}. \end{tikzcd} \qquad (\square)$$

(3) A *weakly ambidextrous* square is a normed square, such that the associated norm diagram $\square$ commutes up to homotopy. An *ambidextrous square* is a weakly ambidextrous square that is iso-normed (note that an ambidextrous square satisfies the $\mathrm{BC}_!$ condition if and only if it satisfies the BC_* condition).

Remark 2.2.10. We shall often abuse language and say that $(*)$ is a normed (or ambidextrous) square implying by this that we also have normed functors q and $\tilde{q}$ as in the definition.

As with any definition regarding norms, we can recast the definition of an ambidextrous square in terms of wrong way counits. As this will be used in the sequel, we shall spell this out.

Lemma 2.2.11. *Let $(*)$ be a normed square as in Definition 2.2.9(1). Consider the diagrams (where $\lhd$ is defined only when $(*)$ is iso-normed).*

$$\begin{tikzcd} F_Dq^*q_! \arrow[dr, "\nu_q"] & \\ \tilde{q}^*F_{\mathcal{C}}q_! \arrow[u, "\wr"] & F_D \\ \tilde{q}^*\tilde{q}_!F_{\mathcal{D}} \arrow[u, "\beta_!"] \arrow[ur, "\nu_{\tilde{q}}"'] & \end{tikzcd} \quad (\rhd), \qquad \begin{tikzcd} & \tilde{q}_!\tilde{q}^*F_{\mathcal{C}} \arrow[d, "\wr"] \\ F_{\mathcal{C}} \arrow[ur, "\mu_{\tilde{q}}"] \arrow[dr, "\mu_q"'] & \tilde{q}_!F_{\mathcal{D}}q^* \arrow[d, "\beta_!"] \\ & F_{\mathcal{C}}q_!q^*. \end{tikzcd} \quad (\lhd)$$

(1) The norm-diagram $\square$ commutes if and only if the diagram $\rhd$ commutes.

(2) If $(*)$ is iso-normed, satisfies the $\mathrm{BC}_!$ condition and the norm-diagram $\square$ commutes, then the diagram $\lhd$ commutes.

Proof. We begin with (1). The norm-diagram $\square$ commutes if and only if the two maps $\tilde{q}_! F_{\mathcal{D}} \to \tilde{q}_* F_{\mathcal{D}}$ are homotopic. This holds if and only if their mates $\tilde{q}^* \tilde{q}_! F_{\mathcal{D}} \to F_{\mathcal{D}}$ are homotopic. To compute the mate, one applies $\tilde{q}^*$ and post-composes with the counit $\tilde{c}_* \colon \tilde{q}^* \tilde{q}_* \to \mathrm{Id}$ (of the right way adjunction). Now, consider the diagram

$$\begin{tikzcd}
F_D q^* q_! \arrow[r, "\mathrm{Nm}_q"] & F_D q^* q_* \arrow[d, "\wr"] \arrow[dr, "c_*"] & \\
\tilde{q}^* F_{\mathcal{C}} q_! \arrow[u, "\wr"] \arrow[r, "\mathrm{Nm}_q"] & \tilde{q}^* F_{\mathcal{C}} q_* \arrow[d, "\beta_*"] & F_D. \\
\tilde{q}^* \tilde{q}_! F_{\mathcal{D}} \arrow[u, "\beta_!"] \arrow[r, "\mathrm{Nm}_{\tilde{q}}"] & \tilde{q}^* \tilde{q}_* F_{\mathcal{D}} \arrow[ur, "\tilde{c}_*"'] &
\end{tikzcd}$$

The triangle on the right commutes by Lemma 2.2.3(2). The composition of the top maps is $F_{\mathcal{D}} \nu_q$ and of the bottom maps is $\nu_{\tilde{q}} F_{\mathcal{D}}$. Hence, $\square$ commutes, if and only if $\triangleright$ commutes.

We now turn to (2). To check the commutativity of $\triangleleft$, we may replace $\beta_!$ with its inverse. By assumption, all maps in $\square$ are isomorphisms. Thus, the map $\beta_!^{-1}$ in $\triangleleft$ is homotopic to the composition

$$F_{\mathcal{C}} q_! \xrightarrow{\mathrm{Nm}_q} F_{\mathcal{C}} q_* \xrightarrow{\beta_*} \tilde{q}_* F_{\mathcal{D}} \xrightarrow{(\mathrm{Nm}_{\tilde{q}})^{-1}} \tilde{q}_! F_{\mathcal{D}}.$$

This exhibits $\beta_!^{-1}$ as the BC_*-map of the *wrong way* adjunctions $q^* \dashv q_!$ and $\tilde{q}^* \dashv \tilde{q}_!$. The commutativity of $\triangleleft$ now follows from the compatibility of BC-maps with units (Lemma 2.2.3(1)). $\square$

The main feature of ambidextrous squares is that they behave well with respect to the integral operation.

Proposition 2.2.12. *Let*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\
\mathcal{D} \arrow[r, "F_{\mathcal{D}}"] & \tilde{\mathcal{D}}
\end{tikzcd} \qquad (\square)$$

be an ambidextrous square that satisfies the $\mathrm{BC}_!$ *condition (and hence the* BC_* *condition). For all* $X, Y \in \mathcal{C}$ *and* $f \colon q^* X \to q^* Y$, *we have*

$$F_{\mathcal{C}} \left(\int_q f \right) = \int_{\tilde{q}} F_{\mathcal{D}}(f) \quad \in \hom_{h\tilde{\mathcal{C}}}(F_{\mathcal{C}} X, F_{\mathcal{C}} Y).$$

In particular, for all $X \in \mathcal{C}$, *we have*

$$F_{\mathcal{C}}([q]_X) = [\tilde{q}]_{F_{\mathcal{C}}(X)} \quad \in \hom_{h\tilde{\mathcal{C}}}(F_{\mathcal{C}} X, F_{\mathcal{C}} X).$$

Proof. Since $\square$ is iso-normed, we can construct the following diagram:

$$
\begin{tikzcd}
 & F_{\mathcal{C}} q_* q^* X \arrow[r, "f"] \arrow[d, "\beta_*"', "\wr"] & F_{\mathcal{C}} q_* q^* Y \arrow[r, "\mathrm{Nm}_q^{-1}"] \arrow[d, "\beta_*"', "\wr"] & F_{\mathcal{C}} q_! q^* Y \arrow[dr, "c_!"] & \\
F_{\mathcal{C}} X \arrow[ur, "u_*"] \arrow[dr, "\tilde{u}_*"'] & \tilde{q}_* F_{\mathcal{D}} q^* X \arrow[r, "f"] \arrow[d, dash, "\wr"] & \tilde{q}_* F_{\mathcal{D}} q^* Y \arrow[r, "\mathrm{Nm}_{\tilde{q}}^{-1}"] \arrow[d, dash, "\wr"] & \tilde{q}_! F_{\mathcal{D}} q^* Y \arrow[u, "\beta_!", "\wr"'] \arrow[d, dash, "\wr"] & F_{\mathcal{C}} Y. \\
 & \tilde{q}_* \tilde{q}^* F_{\mathcal{C}} X \arrow[r, "f"] & \tilde{q}_* \tilde{q}^* F_{\mathcal{C}} Y \arrow[r, "\mathrm{Nm}_{\tilde{q}}^{-1}"] & \tilde{q}_! \tilde{q}^* F_{\mathcal{C}} Y \arrow[ur, "\tilde{c}_!"'] &
\end{tikzcd}
$$

The left and right triangles commute by compatibility of BC maps with (co)units (Lemma 2.2.3, diagrams (1) and (4) respectively). The top right square commutes by the assumption that the square $\square$ is ambidextrous and satisfies the BC conditions, and the rest of the squares commute for trivial reasons. Hence, the composition along the top path is homotopic to the composition along the bottom path, which proves the first claim. The second claim follows from the first applied to the map $f = q^* \operatorname{Id}_X$. $\square$

Calculus of Normed Squares

As discussed before, squares of functors can be pasted horizontally and vertically. We extend these operations to *normed* squares and consider their compatibility with the notion of ambidexterity. We begin with horizontal pasting. Given normed functors

$$
q \colon \mathcal{D} \rightarrowtail \mathcal{C}, \quad \tilde{q} \colon \tilde{\mathcal{D}} \rightarrowtail \tilde{\mathcal{C}}, \quad \tilde{\tilde{q}} \colon \tilde{\tilde{\mathcal{D}}} \rightarrowtail \tilde{\tilde{\mathcal{C}}},
$$

and a commutative diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"] & \tilde{\mathcal{C}} \arrow[r, "G_{\mathcal{C}}"] \arrow[d, "\tilde{q}^*"] & \tilde{\tilde{\mathcal{C}}} \arrow[d, "\tilde{\tilde{q}}^*"] \\
\mathcal{D} \arrow[r, "F_{\mathcal{D}}"] & \tilde{\mathcal{D}} \arrow[r, "G_{\mathcal{D}}"] & \tilde{\tilde{\mathcal{D}}},
\end{tikzcd} \qquad (*)
$$

we call the big outer *normed* square the *horizontal pasting* of the left and right small *normed* squares. We have the following horizontal pasting lemma for ambidexterity.

Lemma 2.2.13 (Horizontal Pasting)**.** *Let $(*)$ be a horizontal pasting diagram of normed squares as above. We denote by $\square_L$, $\square_R$ and $\square$, the left, right and outer normed squares respectively. If $\square_L$ and $\square_R$ are (weakly) ambidextrous, then so is $\square$.*

Proof. Consider the following diagram composed of whiskerings of the norm diagrams of $\square_L$ and $\square_R$ (with all horizontal maps the respective BC-maps).

$$
\begin{tikzcd}
G_{\mathcal{C}} F_{\mathcal{C}} q_! \arrow[d, "\mathrm{Nm}_q"] & G_{\mathcal{C}} \tilde{q}_! F_{\mathcal{D}} \arrow[l] \arrow[d, "\mathrm{Nm}_{\tilde{q}}"] & \tilde{\tilde{q}}_! G_{\mathcal{D}} F_{\mathcal{D}} \arrow[l] \arrow[d, "\mathrm{Nm}_{\tilde{\tilde{q}}}"] \\
G_{\mathcal{C}} F_{\mathcal{C}} q_* \arrow[r] & G_{\mathcal{C}} \tilde{q}_* F_{\mathcal{D}} \arrow[r] & \tilde{\tilde{q}}_* G_{\mathcal{D}} F_{\mathcal{D}}.
\end{tikzcd}
$$

By Lemma 2.2.4, the outer square is the norm diagram for $\square$, which implies the claim. $\square$

We now turn to vertical pasting. Given normed functors

$$q\colon \mathcal{D} \rightarrowtail \mathcal{C}, \quad \tilde{q}\colon \tilde{\mathcal{D}} \rightarrowtail \tilde{\mathcal{C}}, \quad p\colon \mathcal{E} \rightarrowtail \mathcal{D}, \quad \tilde{p}\colon \tilde{\mathcal{E}} \rightarrowtail \tilde{\mathcal{D}}$$

and a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D} \arrow[r, "F_{\mathcal{D}}"] \arrow[d, "p^*"'] & \tilde{\mathcal{D}} \arrow[d, "\tilde{p}^*"] \\ \mathcal{E} \arrow[r, "F_{\mathcal{E}}"] & \tilde{\mathcal{E}}, \end{tikzcd} \qquad (**)$$

we call the big outer normed square, with respect to the compositions of normed functors qp and $\tilde{q}\tilde{p}$, the *vertical pasting* of the top and bottom small normed squares. We have the following vertical pasting lemma for ambidexterity.

Lemma 2.2.14 (Vertical Pasting)**.** *Let* $(**)$ *be a vertical pasting diagram of normed squares as above. We denote by* $\square_T$*,* $\square_B$ *and* $\square$*, the top, bottom and outer normed squares respectively. If* $\square_T$ *and* $\square_B$ *are (weakly) ambidextrous, then so is* $\square$*.*

Proof. Consider the following diagram composed of whiskerings of the norm diagrams of $\square_T$ and $\square_B$ (with all horizontal maps the respective BC-maps).

$$\begin{tikzcd} F_{\mathcal{C}} q_! p_! \arrow[d, "\mathrm{Nm}_q"] & \tilde{q}_! F_{\mathcal{D}} p_! \arrow[l] \arrow[d, "\mathrm{Nm}_{\tilde{q}}"] & \tilde{q}_! \tilde{p}_! F_{\mathcal{E}} \arrow[l] \arrow[d, "\mathrm{Nm}_{\tilde{q}}"] \\ F_{\mathcal{C}} q_* p_! \arrow[r] \arrow[d, "\mathrm{Nm}_p"] & \tilde{q}_* F_{\mathcal{D}} p_! \arrow[d, "\mathrm{Nm}_p"] & \tilde{q}_* \tilde{p}_! F_{\mathcal{E}} \arrow[l] \arrow[d, "\mathrm{Nm}_{\tilde{p}}"] \\ F_{\mathcal{C}} q_* p_* \arrow[r] & \tilde{q}_* F_{\mathcal{D}} p_* \arrow[r] & \tilde{q}_* \tilde{p}_* F_{\mathcal{E}}. \end{tikzcd}$$

By Lemma 2.2.6, the outer diagram is the norm diagram for $\square$. Thus, it is enough to check that all four small squares commute. The top right and bottom left squares commute for trivial reasons. The top left and bottom right squares are whiskerings of the norm diagrams of $\square_T$ and $\square_B$ respectively and hence commute by assumption. $\square$

2.3 Monoidal Structure and Duality

In this section we study the interaction of norms and integration with (symmetric) monoidal structure on the source and target ∞-categories. Under suitable hypothesis, this interaction allows us to reduce questions about ambidexterity to questions about duality.

Tensor Normed Functors

Definition 2.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal ∞-categories. A $\otimes$-*normed functor* from $\mathcal{D}$ to $\mathcal{C}$, is a normed functor $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$, such that q^* is monoidal (and hence $q_!$ is colax monoidal by the dual of [Lur, Corollary 7.3.2.7]) and for all $Y \in \mathcal{D}$ and $X \in \mathcal{C}$, the compositions of the canonical maps

$$q_!\left(Y \otimes (q^*X)\right) \to (q_!Y) \otimes (q_!q^*X) \xrightarrow{\mathrm{Id} \otimes c_!} (q_!Y) \otimes X$$

and

$$q_!\left((q^*X) \otimes Y\right) \to (q_!q^*X) \otimes (q_!Y) \xrightarrow{c_! \otimes \mathrm{Id}} X \otimes (q_!Y)$$

are isomorphisms.

Remark 2.3.2. The above definition does not depend on the norm and is actually just a property of the functor q^*. However, we shall only be interested in this property in the context of normed functors.

Notation 2.3.3. To make diagrams involving (co)units more readable, we shall employ the following graphical convention. When writing a unit map of an adjunction whiskered by some functors, we enclose in parenthesis the effected terms in the target. Similarly, when writing a counit map of an adjunction whiskered by some functors, we underline the effected terms in the source.

We adopt the definitions and terminology of [HL13] regarding duality in monoidal ∞-categories. In the situation of Definition 2.3.1, substituting $q^*\mathbb{1}_{\mathcal{C}}$ for Y, gives a naturally isomorphism from the functor $q_!q^*$ to the functor $\mathbb{1}_q \otimes -$, where $\mathbb{1}_q = q_!q^*\mathbb{1}_{\mathcal{C}}$. We can therefore consider the map

$$\varepsilon\colon \mathbb{1}_q \otimes \mathbb{1}_q \simeq q_!\underline{q^*q_!}q^*\mathbb{1}_{\mathcal{C}} \xrightarrow{\nu} \underline{q_!q^*}\mathbb{1}_{\mathcal{C}} \xrightarrow{c_!} \mathbb{1}_{\mathcal{C}}.$$

Proposition 2.3.4. *Let $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ be a $\otimes$-normed functor of monoidal ∞-categories. TFAE:*

(1) Nm_q is an isomorphism natural transformation (i.e. q is iso-normed).

(2) Nm_q is an isomorphism at $q^*\mathbb{1}_{\mathcal{C}}$.

(3) The map $\varepsilon\colon \mathbb{1}_q \otimes \mathbb{1}_q \to \mathbb{1}_{\mathcal{C}}$ is a duality datum (exhibiting $\mathbb{1}_q$ as a self dual object in $\mathcal{C}$).

Proof. (1) $\implies$ (2) is obvious. Assume (2). The map $\mathrm{Nm}_q\colon q_! \to q_*$ has a mate $\nu\colon q^*q_! \to \mathrm{Id}$. By Lemma 2.1.6, since Nm_q is an isomorphism at $q^*\mathbb{1}_{\mathcal{C}}$, the map ν is a counit map at $q^*\mathbb{1}_{\mathcal{C}}$ and has an associated unit map $\mu_{\mathbb{1}}\colon \mathbb{1}_{\mathcal{C}} \to q_!q^*\mathbb{1}_{\mathcal{C}}$. Let

$$\eta\colon \mathbb{1}_{\mathcal{C}} \xrightarrow{\mu_{\mathbb{1}}} (q_!q^*)\,\mathbb{1}_{\mathcal{C}} \xrightarrow{u_!} q_!\,(q^*q_!)\,q^*\mathbb{1}_{\mathcal{C}} = \mathbb{1}_q \otimes \mathbb{1}_q.$$

We prove (3) by showing that ε and η satisfy the zig-zag identities. As above, we identify $\mathbb{1}_q$ with

$q_!q^*\mathbb{1}_{\mathcal{C}}$ and $\mathbb{1}_q \otimes \mathbb{1}_q$ with $q_!q^*q_!q^*\mathbb{1}_{\mathcal{C}}$. For the first zig-zag identity, consider the diagram

$$\begin{tikzcd}
q_!q^*\mathbb{1}_{\mathcal{C}} \arrow[r, "\mu_{\mathbb{1}}"] \arrow[dr, equal] \arrow[rr, bend left, "\eta\star\mathrm{Id}"] & (q_!\underline{q^*)q_!}q^*\mathbb{1}_{\mathcal{C}} \arrow[r, "u_!"] \arrow[d, "\nu"] & q_!\left(q^*q_!\right)\underline{q^*q_!}q^*\mathbb{1}_{\mathcal{C}} \arrow[d, "\nu"] \arrow[dd, bend left=60, "\mathrm{Id}\star\varepsilon"] \\
& q_!q^*\mathbb{1}_{\mathcal{C}} \arrow[r, "u_!"] \arrow[dr, equal] & q_!(q^*\underline{q_!)q^*}\mathbb{1}_{\mathcal{C}} \arrow[d, "c_!"] \\
& & q_!q^*\mathbb{1}_{\mathcal{C}}.
\end{tikzcd}$$

The square commutes by the interchange law. The upper triangle by the definition of $\mu_{\mathbb{1}}$ (i.e. the corresponding zig-zag identity at $\mathbb{1}_{\mathcal{C}}$) and the bottom by the zig-zag identities for $u_!$ and $c_!$. For the second zig-zag identity, consider a similar diagram

$$\begin{tikzcd}
q_!q^*\mathbb{1}_{\mathcal{C}} \arrow[r, "\mu_{\mathbb{1}}"] \arrow[dr, equal] \arrow[rr, bend left, "\mathrm{Id}\star\eta"] & q_!\underline{q^*(q_!}q^*)\mathbb{1}_{\mathcal{C}} \arrow[r, "u_!"] \arrow[d, "\nu"] & q_!\underline{q^*q_!}\left(q^*q_!\right)q^*\mathbb{1}_{\mathcal{C}} \arrow[d, "\nu"] \arrow[dd, bend left=60, "\varepsilon\star\mathrm{Id}"] \\
& q_!q^*\mathbb{1}_{\mathcal{C}} \arrow[r, "u_!"] \arrow[dr, equal] & \underline{q_!(q^*}q_!)q^*\mathbb{1}_{\mathcal{C}} \arrow[d, "c_!"] \\
& & q_!q^*\mathbb{1}_{\mathcal{C}}.
\end{tikzcd}$$

Assume (3). By Lemma 2.1.5 and Lemma 2.1.6, it is enough to show that ν is a counit at q^*X for all $X \in \mathcal{C}$. Consider the following diagram

$$\begin{tikzcd}
& \mathrm{Map}\left(Y, \mathbb{1}_q \otimes X\right) \arrow[dl, "\sim"'] \arrow[rr, "\mathbb{1}_q\otimes -"] & & \mathrm{Map}\left(\mathbb{1}_q \otimes Y, \mathbb{1}_q \otimes \mathbb{1}_q \otimes X\right) \arrow[dl, "\sim"'] \arrow[dd, "\varepsilon\circ -"] \\
\mathrm{Map}\left(Y, q_!q^*X\right) \arrow[r, "q^*"] \arrow[dr, dashed] & \mathrm{Map}\left(q^*Y, \underline{q^*q_!}q^*X\right) \arrow[r, "q_!"] \arrow[d, "\nu\circ -"] & \mathrm{Map}\left(q_!q^*Y, q_!\underline{q^*q_!}q^*X\right) \arrow[d, "\nu\circ -"] & \\
& \mathrm{Map}\left(q^*Y, q^*X\right) \arrow[r, "q_!"] \arrow[dr, "\sim"'] & \mathrm{Map}\left(q_!q^*Y, \underline{q_!q^*}X\right) \arrow[d, "c_!\circ -"] & \mathrm{Map}\left(\mathbb{1}_q \otimes Y, X\right) \arrow[dl, "\sim"] \\
& & \mathrm{Map}\left(q_!q^*Y, X\right) &
\end{tikzcd}$$

The triangles commute by definition and the rest by naturality. The composition along the top and then right path is an isomorphism since ε is an evaluation map of a duality datum on $\mathbb{1}_q$. Thus, the dashed arrow is an isomorphism by 2-out-of-3, which proves that ν is a counit at q^*X. $\square$

Remark 2.3.5. A similar result is given in [HL13, Proposition 5.1.8].

Tensor Normed Squares

The following is the analogous notion to a normed square in the monoidal setting.

Definition 2.3.6. A $\otimes$-*normed square* is a pair of $\otimes$-normed functors $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ and $\tilde{q}\colon \tilde{\mathcal{D}} \rightarrowtail \tilde{\mathcal{C}}$ and a commutative square of monoidal ∞-categories and monoidal functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D} \arrow[r, "F_{\mathcal{D}}"] & \tilde{\mathcal{D}}. \end{tikzcd} \qquad (*)$$

For a $\otimes$-normed square $(*)$ as above, we define a colax natural transformation of functors

$$\theta\colon\ (-)_{\tilde{q}}\, F_{\mathcal{C}} = \tilde{q}_!\tilde{q}^* F_{\mathcal{C}} \simeq \tilde{q}_! F_{\mathcal{D}} q^* \xrightarrow{\beta_!} F_{\mathcal{C}} q_! q^* = F_{\mathcal{C}}\,(-)_q\,.$$

Using the isomorphisms from Definition 2.3.1 we define the natural isomorphisms

$$L_q\colon\ (X\otimes Y)_q = q_!q^*\,(X\otimes Y) \simeq q_!\,(q^*X\otimes q^*Y) \xrightarrow{\sim} q_!q^*X\otimes Y = X_q\otimes Y,$$
$$R_q\colon\ (X\otimes Y)_q = q_!q^*\,(X\otimes Y) \simeq q_!\,(q^*X\otimes q^*Y) \xrightarrow{\sim} X\otimes q_!q^*Y = X\otimes Y_q.$$

We shall need a technical lemma regarding the compatibility of the maps L, R, and θ.

Lemma 2.3.7. *Let $(*)$ be a $\otimes$-normed square as above. For all $X, Y \in \mathcal{C}$, the following diagram:*

$$\begin{tikzcd} F_{\mathcal{C}}\,(X\otimes Y)_{\tilde{q}\tilde{q}} \arrow[r, dash, "\sim"] \arrow[d, "\theta_{X\otimes Y}"] & (F_{\mathcal{C}}\,(X)\otimes F_{\mathcal{C}}\,(Y))_{\tilde{q}\tilde{q}} \arrow[r, "R_{\tilde{q}}"] & \left(F_{\mathcal{C}}\,(X)\otimes F_{\mathcal{C}}\,(Y)_{\tilde{q}}\right)_{\tilde{q}} \arrow[r, "L_{\tilde{q}}"] \arrow[d, "\mathrm{Id}\otimes\theta_Y"] & F_{\mathcal{C}}\,(X)_{\tilde{q}}\otimes F_{\mathcal{C}}\,(Y)_{\tilde{q}} \arrow[d, "\mathrm{Id}\otimes\theta_Y"] \\ F_{\mathcal{C}}\left((X\otimes Y)_q\right)_{\tilde{q}} \arrow[r, "R_q"] \arrow[d, "\theta_{(X\otimes Y)_q}"] & F_{\mathcal{C}}\,(X\otimes Y_q)_{\tilde{q}} \arrow[r, dash, "\sim"] \arrow[d, "\theta_{X\otimes Y_q}"] & (F_{\mathcal{C}}\,(X)\otimes F_{\mathcal{C}}\,(Y_q))_{\tilde{q}} \arrow[r, "L_{\tilde{q}}"] & F_{\mathcal{C}}\,(X)_{\tilde{q}}\otimes F_{\mathcal{C}}\,(Y_q) \arrow[d, "\theta_X\otimes\mathrm{Id}"] \\ F_{\mathcal{C}}\left((X\otimes Y)_{qq}\right) \arrow[r, "R_q"] & F_{\mathcal{C}}\left((X\otimes Y_q)_q\right) \arrow[r, "L_q"] & F_{\mathcal{C}}\,(X_q\otimes Y_q) \arrow[r, dash, "\sim"] & F_{\mathcal{C}}\,(X_q)\otimes F_{\mathcal{C}}\,(Y_q) \end{tikzcd}$$

commutes up to homotopy.

Proof. The top right square commutes by naturality of $L_{\tilde{q}}$ and the bottom right square commutes by naturality of θ. We now show the commutativity of the top left rectangle (the commutativity of the bottom right rectangle is completely analogous). By unwinding the definition of R_q, the top left rectangle is obtained by applying $(-)_{\tilde{q}}$ to the following diagram

$$\begin{tikzcd}
\left(F_{\mathcal{C}}(X) \otimes F_{\mathcal{C}}(Y)\right)_{\tilde{q}} \arrow[r] \arrow[d, "\wr"] \arrow[rr, bend left, "R_{\tilde{q}}"] & F_{\mathcal{C}}(X)_{\tilde{q}} \otimes F_{\mathcal{C}}(Y)_{\tilde{q}} \arrow[r, "\tilde{c}_! \otimes \mathrm{Id}"] \arrow[d, "\theta_X \otimes \theta_Y"] & F_{\mathcal{C}}(X) \otimes F_{\mathcal{C}}(Y)_{\tilde{q}} \arrow[d, "\mathrm{Id} \otimes \theta_Y"] \\
F_{\mathcal{C}}(X \otimes Y)_{\tilde{q}} \arrow[d, "\theta_{X \otimes Y}"] & F_{\mathcal{C}}(X_q) \otimes F_{\mathcal{C}}(Y_q) \arrow[r, "c_! \otimes \mathrm{Id}"] \arrow[d, dash, "\wr"] & F_{\mathcal{C}}(X) \otimes F_{\mathcal{C}}(Y_q) \arrow[d, dash, "\wr"] \\
F_{\mathcal{C}}\left((X \otimes Y)_q\right) \arrow[r] \arrow[rr, bend right, "R_q"'] & F_{\mathcal{C}}(X_q \otimes Y_q) \arrow[r, "c_! \otimes \mathrm{Id}"] & F_{\mathcal{C}}(X \otimes Y_q) .
\end{tikzcd}$$

The left rectangle commutes by the monoidality of θ and the bottom right square commutes by naturality. The top right square is a tensor product of two squares

$$\begin{tikzcd}
F_{\mathcal{C}}(X)_{\tilde{q}} \arrow[r, "\tilde{c}_!"] \arrow[d, "\theta_X"] & F_{\mathcal{C}}(X) \arrow[d, "\mathrm{Id}"] \\
F_{\mathcal{C}}(X_q) \arrow[r, "c_!"] & F_{\mathcal{C}}(X)
\end{tikzcd} \quad (\square_1), \qquad
\begin{tikzcd}
F_{\mathcal{C}}(Y)_{\tilde{q}} \arrow[r, "\mathrm{Id}"] \arrow[d, "\theta_Y"] & F_{\mathcal{C}}(Y)_{\tilde{q}} \arrow[d, "\theta_Y"] \\
F_{\mathcal{C}}(Y_q) \arrow[r, "\mathrm{Id}"] & F_{\mathcal{C}}(Y_q) .
\end{tikzcd} \quad (\square_2)$$

The square $\square_2$ commutes for trivial reasons and the square $\square_1$ commutes by the compatibility of BC-maps with counits (Lemma 2.2.3(4)). $\square$

The main fact we shall use about $\otimes$-normed squares is the following.

Proposition 2.3.8. *Let $(*)$ be a $\otimes$-normed square as above. Assume that $(*)$ is weakly ambidextrous and satisfies the* $\mathrm{BC}_!$*-condition. If q is iso-normed, then $\tilde{q}$ is iso-normed and the* BC_* *condition is satisfied as well.*

Proof. By the assumption of the $\mathrm{BC}_!$-condition, the operation θ is an isomorphism. Observe that $\mathbb{1}_{\tilde{q}} \simeq F(\mathbb{1}_{\mathcal{C}})_{\tilde{q}}$ and consider the following diagram

$$\begin{tikzcd}
\mathbb{1}_{\tilde{q}} \otimes \mathbb{1}_{\tilde{q}} \arrow[r, "L_{\tilde{q}}^{-1} R_{\tilde{q}}^{-1}"] \arrow[d, "\theta \otimes \theta"', "\wr"] & \tilde{q}_! \underline{\tilde{q}^* \tilde{q}_!} \tilde{q}^* F_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \arrow[r, "\tilde{\nu}"] \arrow[dd, "\theta \star \theta"] & \underline{\tilde{q}_! \tilde{q}^*} F_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \arrow[dd, "\theta"] \arrow[dr, "\tilde{c}_!"] & \\
F_{\mathcal{C}}(\mathbb{1}_q) \otimes F_{\mathcal{C}}(\mathbb{1}_q) \arrow[d, "\wr"] & & & F_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \simeq \mathbb{1}_{\tilde{\mathcal{C}}}. \\
F_{\mathcal{C}}(\mathbb{1}_q \otimes \mathbb{1}_q) \arrow[r, "L_q^{-1} R_q^{-1}"] & F_{\mathcal{C}}\left(q_! \underline{q^* q_!} q^* \mathbb{1}_{\mathcal{C}}\right) \arrow[r, "\nu"] & F_{\mathcal{C}}\left(\underline{q_! q^*} \mathbb{1}_{\mathcal{C}}\right) \arrow[ur, "c_!"'] &
\end{tikzcd}$$

The middle rectangle and the triangle commute by the compatibility of BC maps with counits (Lemma 2.2.3(4)). The left rectangle commutes by applying Lemma 2.3.7 with $X = Y = \mathbb{1}_{\mathcal{C}}$. By

Proposition 2.3.4, $\varepsilon_q\colon \mathbb{1}_q \otimes \mathbb{1}_q \to \mathbb{1}_{\mathcal{C}}$ is a duality datum and since $F_{\mathcal{C}}$ is monoidal,

$$F_{\mathcal{C}}(\varepsilon_q)\colon F_{\mathcal{C}}(\mathbb{1}_q) \otimes F_{\mathcal{C}}(\mathbb{1}_q) \to F_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \simeq \mathbb{1}_{\tilde{\mathcal{C}}}$$

is a duality datum as well. The commutativity of the above diagram, identifies $F_{\mathcal{C}}(\varepsilon_q)$ with $\varepsilon_{\tilde{q}}$ and hence $\varepsilon_{\tilde{q}}$ is a duality datum for $\mathbb{1}_{\tilde{q}}$. By Proposition 2.3.4 again, $\tilde{q}$ is iso-normed. Finally, the BC_* condition is satisfied by 2-out-of-3 for the norm diagram. □

2.4 Amenability

Definition 2.4.1. An iso-normed functor $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ is called *amenable*, if $[q]$ is an isomorphism natural transformation.

Remark 2.4.2. The name is inspired by the notion of amenability in geometric group theory. Given an object $X \in \mathcal{C}$, the integral operation

$$\int\limits_q \colon\ \mathrm{Map}(q^*X, q^*X) \to \mathrm{Map}(X, X)$$

can be thought of intuitively as "*summation* over the fibers of q". Amenability allows us to "*average* over the fibers of q" by multiplying the integral with $[q]^{-1}$. This is especially suggestive in the prototypical example of local-systems, which we study in the next section.

Lemma 2.4.3. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F_{\mathcal{C}}"] \arrow[d, "q^*"'] & \tilde{\mathcal{C}} \arrow[d, "\tilde{q}^*"] \\ \mathcal{D} \arrow[r, "F_{\mathcal{D}}"] & \tilde{\mathcal{D}} \end{tikzcd}$$

be an ambidextrous square, such that $F_{\mathcal{C}}$ is conservative. If $\tilde{q}$ is amenable, then q is amenable.

Proof. Given $X \in \mathcal{C}$, since the square is ambidextrous, we have by Proposition 2.2.12,

$$F_{\mathcal{C}}([q]_X) = [\tilde{q}]_{F_{\mathcal{C}}(X)}\,.$$

The claim follows from the assumption that $F_{\mathcal{C}}$ is conservative. □

The next result demonstrates how can amenability be profitably used for "averaging".

Theorem 2.4.4 (Higher Maschke's Theorem)**.** *Let $q\colon \mathcal{D} \rightarrowtail \mathcal{C}$ be an iso-normed functor. If q is amenable, then for every $X \in \mathcal{C}$ the counit map $c_!\colon q_!q^*X \to X$ has a section (i.e. left inverse) up to homotopy. In particular, every object of $\mathcal{C}$ is a retract of an object in the essential image of $q_!$.*

Proof. Let $X \in \mathcal{C}$. By the zig-zag identities,

$$q^*X \xrightarrow{u_!} (q^*\underline{q_!)q^*}X \xrightarrow{c_!} q^*X$$

is the identity on q^*X. Integrating along q and using Proposition 2.1.14, we get

$$[q]_X = \int\limits_q \mathrm{Id}_{(q^*X)} = \int\limits_q \Big(q^*(c_!)_X \circ (u_!)_{q^*X}\Big) = (c_!)_X \circ \int\limits_q (u_!)_{q^*X}\,.$$

Hence, if $[q]_X$ is isomorphism, then $(c_!)_X$ has a section up to homotopy. □

Theorem 2.4.5 (Cancellation Theorem)**.** *Let*

$$\mathcal{E} \overset{p}{\rightarrowtail} \mathcal{D} \overset{q}{\rightarrowtail} \mathcal{C}$$

be a pair of composable normed functors. If p is amenable and qp is iso-normed, then q is iso-normed.

Proof. The map Nm_{qp} is given by the composition

$$q_!p_! \xrightarrow{\mathrm{Nm}_q} q_*p_! \xrightarrow{\mathrm{Nm}_p} q_*p_*.$$

Since Nm_{qp} and Nm_p are isomorphisms, so is $q_!p_! \xrightarrow{\mathrm{Nm}_q} q_*p_!$. By Theorem 2.4.4, every $X \in \mathcal{D}$ is a retract of $p_!Y$ for some $Y \in \mathcal{E}$. Isomorphisms are closed under retracts, and so Nm_q is an isomorphism for every $X \in \mathcal{D}$. $\square$

Remark 2.4.6. This is essentially the same argument as the one used in the proof of [HL13, Proposition 4.4.16].

3 Local-Systems and Ambidexterity

The main examples of normed functors that we are interested in, are the ones provided by the theory of semiadditivity developed in [HL13] and further in [Har17]. In what follows, we first briefly recall the relevant definitions and explain how they fit in to the abstract framework developed in the previous section. Then we apply the theory of the previous section to this special case. The theory developed in [HL13] is set up in a rather general framework of Beck-Chevalley fibrations. Even though this framework fits into our theory of normed functors, for concreteness and clarity, we shall confine ourselves to the special case of local systems.

3.1 Local-Systems and Canonical Norms

Let $\mathcal{C}$ be an ∞-category and let A be a space viewed as an ∞-groupoid. We call $\mathrm{Fun}(A, \mathcal{C})$ the ∞-category of $\mathcal{C}$-valued *local systems* on A. Let $q\colon A \to B$ be a map of spaces and assume that $\mathcal{C}$ admits all q-limits and q-colimits. The functor

$$q^*\colon \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$$

admits both a left adjoint $q_!$ and a right adjoint q_* (given by left and right Kan extension respectively). We shall define, after [HL13, Section 4.1], a class of *weakly $\mathcal{C}$-ambidextrous* maps q, to which we associate a canonical norm map $\mathrm{Nm}_q\colon q_! \to q_*$. This norm map gives rise to a normed functor

$$q^{\mathrm{can}}\colon \mathrm{Fun}(A, \mathcal{C}) \rightarrowtail \mathrm{Fun}(B, \mathcal{C}).$$

A map q is called *$\mathcal{C}$-ambidextrous* if it is weakly $\mathcal{C}$-ambidextrous and the associated canonical norm is an isomorphism (i.e. q^{can} is iso-normed).

Base Change & Canonical Norms

We begin with some terminology regarding the operation of base change for local-systems.

Definition 3.1.1. Given an ∞-category $\mathcal{C}$ and a pullback diagram of spaces

$$\begin{tikzcd} \tilde{A} \arrow[r, "s_A"] \arrow[d, "\tilde{q}"'] & A \arrow[d, "q"] \\ \tilde{B} \arrow[r, "s_B"] & B \end{tikzcd} \qquad (*)$$

the associated *base-change square* (of $\mathcal{C}$-valued local-systems) is

$$\begin{tikzcd} \mathrm{Fun}\,(B,\mathcal{C}) \arrow[r, "s_B^*"] \arrow[d, "q^*"'] & \mathrm{Fun}(\tilde{B},\mathcal{C}) \arrow[d, "\tilde{q}^*"] \\ \mathrm{Fun}\,(A,\mathcal{C}) \arrow[r, "s_A^*"] & \mathrm{Fun}(\tilde{A},\mathcal{C}). \end{tikzcd} \qquad (\square)$$

Lemma 3.1.2. *Let $\mathcal{C}$ be an ∞-category and let $(*)$ be a pullback diagram of spaces as in Definition 3.1.1 above. If $\mathcal{C}$ admits all q-colimits (resp. q-limits), then the associated base-change square $\square$ satisfies the* $\mathrm{BC}_!$ *(resp.* BC_* *condition).*

Proof. For $\mathrm{BC}_!$ this is [HL13, Proposition 4.3.3] (note we only need q-colimits). For BC_* a completely analogous argument works. $\square$

The construction of the canonical norm rests on the following more general construction.

Definition 3.1.3. Let $q\colon A \to B$ be a map of spaces and let $\delta\colon A \to A \times_B A$ be the diagonal of q. Let $\mathcal{C}$ be an ∞-category that admits all q-(co)limits and δ-(co)limits. Given an isomorphism natural transformation

$$\mathrm{Nm}_\delta\colon \delta_! \xrightarrow{\sim} \delta_*,$$

we define the *diagonally induced* norm map

$$\mathrm{Nm}_q\colon q_! \to q_*$$

as follows. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[dr, "\delta"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\ & A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & A \arrow[d, "q"] \\ & A \arrow[r, "q"] & B. \end{tikzcd}$$

To the iso-norm Nm_δ, corresponds a wrong way unit map $\mu_\delta\colon \mathrm{Id} \to \delta_! \delta^*$. By Lemma 3.1.2, the base change square associated with $(*)$ satisfies the $\mathrm{BC}_!$ condition, and so we can define the composition

$$\nu_q\colon q^* q_! \xrightarrow{\beta_!^{-1}} (\pi_2)_! \, \pi_1^* \xrightarrow{\mu_\delta} (\pi_2)_! \, \delta_! \delta^* \pi_1^* \xrightarrow{\sim} \mathrm{Id}\,.$$

We define $\mathrm{Nm}_q\colon q_! \to q_*$ to be the mate of ν_q under the adjunction $q^* \dashv q_*$.

Remark 3.1.4. In light of [HL13, Remark 4.1.9], we can informally say that the diagonally induced norm map on q is obtained by integrating the identity map along the diagonal δ. Though we shall not use this perspective, it is helpful to keep it in mind.

Note that if $q\colon A \to B$ is m-truncated for some $m \geq -1$, then δ is $(m-1)$-truncated. This allows us to define canonical norm maps inductively on the level of truncatedness of the map.

Definition 3.1.5. Let $\mathcal{C}$ be an ∞-category and $m \geq -2$ an integer. A map of spaces $q\colon A \to B$ is called

(1) *weakly m-$\mathcal{C}$-ambidextrous*, if q is m-truncated, $\mathcal{C}$ admits q-(co)limits and either of the two holds:

- $m = -2$, in which case the inverse of q^* is both a left and right adjoint of q^*. We define the *canonical norm* map on q^* to be the identity of some inverse of q^*.
- $m \geq -1$, and the diagonal $\delta\colon A \to A \times_B A$ of q is $(m-1)$-*$\mathcal{C}$-ambidextrous*. In this case we define the *canonical norm* on q^* to be the diagonally induced one from the canonical norm of δ.

(2) *m-$\mathcal{C}$-ambidextrous*, if it is weakly m-$\mathcal{C}$-ambidextrous and its canonical norm map is an isomorphism.

A map of spaces $q\colon A \to B$ is called (weakly) $\mathcal{C}$-ambidextrous if it is (weakly) m-$\mathcal{C}$-ambidextrous for some m.

By [HL13, Proposition 4.1.5 (5)], the canonical norm associated with a map $q\colon A \to B$, that is m-truncated for some m, is independent of m.

Definition 3.1.6. In the situation of Definition 3.1.5, given a map $q\colon A \to B$ that is weakly $\mathcal{C}$-ambidextrous, we define the associated *canonical normed functor*

$$q_{\mathcal{C}}^{\mathrm{can}}\colon \mathrm{Fun}\,(A, \mathcal{C}) \rightarrowtail \mathrm{Fun}\,(B, \mathcal{C})\,,$$

By

$$(q_{\mathcal{C}}^{\mathrm{can}})^* = q^*, \quad (q_{\mathcal{C}}^{\mathrm{can}})_! = q_!, \quad (q_{\mathcal{C}}^{\mathrm{can}})_* = q_*,$$

and the norm map $\mathrm{Nm}_q\colon q_! \to q_*$ the canonical norm of Definition 3.1.5.

Note that the normed functor $q_{\mathcal{C}}^{\mathrm{can}}$ is iso-normed if and only if q is $\mathcal{C}$-ambidextrous. We add the following definition.

Definition 3.1.7. Let $\mathcal{C}$ be an ∞-category. A $\mathcal{C}$-ambidextrous map $q\colon A \to B$ is called *$\mathcal{C}$-amenable* if q^{can} is amenable.

Notation 3.1.8. Given a weakly $\mathcal{C}$-ambidextrous map of spaces $q\colon A \to B$, we write q^{can} for $q^{\mathrm{can}}_{\mathcal{C}}$ if $\mathcal{C}$ is understood from the context. We also write $(-)_q$, $\int_q$ and $[q]$ instead of $(-)_{q^{\mathrm{can}}}$, $\int_{q^{\mathrm{can}}}$ and $[q^{\mathrm{can}}]$. For a map $q\colon A \to \mathrm{pt}$, we shall also say that A is (weakly) $\mathcal{C}$-ambidextrous or amenable if q is, and write $(-)_A$, $\int_A$, and $[A]$ instead of $(-)_q$, $\int_q$ and $[q]$.

The next proposition ensures that the canonical norms are preserved under base change, compositions and identity as in Definition 2.1.9.

Proposition 3.1.9. *Let $\mathcal{C}$ be an ∞-category.*

(1) (Identity) Given an isomorphism of spaces $q\colon A \xrightarrow{\sim} B$, the functor q^* is $\mathcal{C}$-ambidextrous and its canonical norm is the identity of the left and right adjoint inverse of q^*.

(2) (Composition) Given (weakly) $\mathcal{C}$-ambidextrous maps $q\colon A \to B$ and $p\colon B \to C$, the composition $pq\colon A \to C$ is (weakly) $\mathcal{C}$-ambidextrous and $(pq)^{\mathrm{can}}$ can be identified with $p^{\mathrm{can}}q^{\mathrm{can}}$.

(3) (Base-change) Let $(*)$ be a pullback diagram of spaces as in Definition 3.1.1. If q is (weakly) $\mathcal{C}$-ambidextrous, then $\tilde{q}$ is (weakly) $\mathcal{C}$-ambidextrous and the associated base-change square

$$\begin{tikzcd}
\mathrm{Fun}(B,\mathcal{C}) \arrow[r, "s_B^*"] \arrow[d, "q^*"'] & \mathrm{Fun}(\tilde{B},\mathcal{C}) \arrow[d, "\tilde{q}^*"] \\
\mathrm{Fun}(A,\mathcal{C}) \arrow[r, "s_A^*"] & \mathrm{Fun}(\tilde{A},\mathcal{C})
\end{tikzcd}$$

is (weakly) ambidextrous.

Proof. (1) follows directly from the definition. (2) is the content of [HL13, Remark 4.2.4]. (3) is a restatement of [HL13, Remark 4.2.3]. □

The following is a central notion for this paper.

Definition 3.1.10. Let $m \geq -2$ be an integer. An ∞-category $\mathcal{C}$ is called m-semiadditive, if it admits all m-finite limits and m-finite colimits and every m-finite map of spaces is $\mathcal{C}$-ambidextrous. It is called ∞-semiadditive if it is m-semiadditive for all m.

Remark 3.1.11. Our definition of m-semiadditivity agrees with [Har17, Definition 3.1] and differs slightly from [HL13, Definition 4.4.2] in that we don't require $\mathcal{C}$ to admit *all* small colimits, but only m-finite ones. Note that using the "wrong way counit" perspective, one could phrase m–semiadditivity without the *assumption* that $\mathcal{C}$ admits m-finite limits, but this would then be a direct *consequence*. Thus, Definition 3.1.10 is somewhat more general then [HL13, Definition 4.4.2].

Base Change and Integration

We can now apply the theory of integration developed in the previous section to the canonically normed functors associated with ambidextrous maps.

Example 3.1.12. (see [HL13, Remark 4.4.11]) Let $\mathcal{C}$ be a 0-semiadditive ∞-category (e.g. $\mathcal{C}$ is stable). For every *finite set* A, the map $q\colon A \to \mathrm{pt}$ is $\mathcal{C}$-ambidextrous. Given $X, Y \in \mathcal{C}$, a map of local systems $f\colon q^*X \to q^*Y$, can be viewed as a collection of maps $\{f_a\colon X \to Y\}_{a\in A}$. We have

$$\int\limits_A f = \sum_{a\in A} f_a \quad \in \hom_{h\mathcal{C}}(X, Y).$$

Applying the general theory of integration to base change squares, we get

Proposition 3.1.13. *Let $\mathcal{C}$ be an ∞-category and let $(*)$ be a pullback diagram of spaces as in Definition 3.1.1, such that q (and hence $\tilde{q}$) is $\mathcal{C}$-ambidextrous. For all $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f\colon q^*X \to q^*Y$, we have*

$$s_B^* \int\limits_q f = \int\limits_{\tilde{q}} s_A^* f \quad \in \hom_{h\,\mathrm{Fun}\left(\tilde{B},\mathcal{C}\right)}(s_B^*X, s_B^*Y).$$

In particular, for all $X \in \mathrm{Fun}(B, \mathcal{C})$ we have

$$s_B^* [q]_X = [\tilde{q}]_{s_B^* X} \quad \in \hom_{h\,\mathrm{Fun}\left(\tilde{B},\mathcal{C}\right)}(s_B^*X, s_B^*X).$$

Proof. Denote by $\square$ the associated base-change square. By Proposition 3.1.9(3), $\square$ is ambidextrous and by Lemma 3.1.2, it satisfies the $\mathrm{BC}_!$ condition. Now, the result follows from Proposition 2.2.12. $\square$

As a consequence, we get a form of “distributivity” for integration.

Corollary 3.1.14. *Let $\mathcal{C}$ be an ∞-category and let $q_1\colon A_1 \to B$ and $q_2\colon A_2 \to B$ be two $\mathcal{C}$-ambidextrous maps of spaces. Consider the pullback square*

$$\begin{tikzcd}
A_2 \times_B A_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, "q_2 \times_B q_1"] & A_1 \arrow[d, "q_1"] \\
A_2 \arrow[r, "q_2"] & B.
\end{tikzcd}$$

The map $q_2 \times_B q_1$ is $\mathcal{C}$-ambidextrous and for all $X, Y, Z \in \mathrm{Fun}(B, \mathcal{C})$ and maps

$$f_1\colon q_1^*X \to q_1^*Y, \quad f_2\colon q_2^*Y \to q_2^*Z,$$

we have

$$\int\limits_{q_2\times_B q_1} (\pi_2^* f_2 \circ \pi_1^* f_1) = \int\limits_{q_2} f_2 \circ \int\limits_{q_1} f_1 \quad \in \hom_{h\,\mathrm{Fun}(B,\mathcal{C})}(X, Z).$$

In particular, for every $X \in \mathrm{Fun}(B, \mathcal{C})$, we have

$$[q_2 \times_B q_1]_X = [q_2]_X \circ [q_1]_X \quad \in \hom_{h\,\mathrm{Fun}(B,\mathcal{C})}(X, X).$$

Proof. The map π_2 is $\mathcal{C}$-ambidextrous by Proposition 3.1.9(3) and therefore $q_2 \times_B q_1 = q_2\pi_2$ is $\mathcal{C}$-ambidextrous by Proposition 3.1.9(2). We now start from the left hand side and use Proposition 2.1.15, Proposition 2.1.14(1), Proposition 3.1.13 and Proposition 2.1.14(2) (in that order).

$$\int\limits_{q_2 \times_B q_1} (\pi_2^* f_2 \circ \pi_1^* f_1) = \int\limits_{q_2\pi_2} (\pi_2^* f_2 \circ \pi_1^* f_1) = \int\limits_{q_2}\int\limits_{\pi_2} (\pi_2^* f_2 \circ \pi_1^* f_1) =$$

$$\int\limits_{q_2}\left(f_2 \circ \int\limits_{\pi_2} \pi_1^* f_1\right) = \int\limits_{q_2}\left(f_2 \circ q_2^* \int\limits_{q_1} f_1\right) = \int\limits_{q_2} f_2 \circ \int\limits_{q_1} f_1.$$

The second claim follows from applying the first to $f_2 = q_2^* \operatorname{Id}_X$ and $f_1 = q_1^* \operatorname{Id}_X$. $\square$

As another consequence, we obtain the additivity property of the integral.

Proposition 3.1.15 (Integral Additivity)**.** *Let $\mathcal{C}$ be a 0-semiadditive ∞-category and let $q_i \colon A_i \to B$ for $i = 1, \dots, k$ be $\mathcal{C}$-ambidextrous maps. Then,*

$$(q_1, \dots, q_k) : A_1 \sqcup \dots \sqcup A_k \to B$$

is $\mathcal{C}$-ambidextrous and for all $X, Y \in \operatorname{Fun}(B, \mathcal{C})$ and maps $f_i \colon q_i^* X \to q_i^* Y$ for $i = 1, \dots, k$, we have

$$\int\limits_{(q_1,\dots,q_k)} (f_1, \dots, f_k) = \sum_{i=1}^{k} \left(\int\limits_{q_i} f_i\right) \quad \in \hom_{h\operatorname{Fun}(B,\mathcal{C})}(X, Y).$$

Proof. By induction, we may assume $k = 2$. Write (q_1, q_2) as a composition

$$A_1 \sqcup A_2 \xrightarrow{q_1 \sqcup q_2} B \sqcup B \xrightarrow{\nabla} B,$$

where ∇ is the fold map. By [HL13, 4.3.5], the map $q_1 \sqcup q_2$ is $\mathcal{C}$-ambidextrous. Consider the pullback square of spaces

$$\begin{tikzcd} A_1 \arrow[r, "j_1"] \arrow[d, "q_1"'] & A_1 \sqcup A_2 \arrow[d, "q_1 \sqcup q_2"] \\ B \arrow[r, "j_1"] & B \sqcup B. \end{tikzcd} \qquad (*)$$

By Proposition 3.1.13 applied to the base-change square of $(*)$, we get that

$$j_1^*\left(\int\limits_{q_1 \sqcup q_2} (f_1, f_2)\right) \simeq \int\limits_{q_1} f_1.$$

Applying the analogous argument to the second component, we get

$$\int\limits_{q_1 \sqcup q_2} (f_1, f_2) = \left(\int\limits_{q_1} f_1, \int\limits_{q_2} f_2\right).$$

Since $\nabla\colon B \sqcup B \to B$ is 0-finite and $\mathcal{C}$ is 0-semiadditive, ∇ is $\mathcal{C}$-ambidextrous and the map (q_1, q_2) is $\mathcal{C}$-ambidextrous as a composition of two such /(Proposition 3.1.9(2)). Using Fubini's Theorem (Proposition 2.1.15), and a direct computation from the definition of the integral over ∇ (identical to Example 3.1.12) we get

$$\int\limits_{(q_1,q_2)} (f_1, f_2) \simeq \int\limits_{\nabla} \int\limits_{q_1 \sqcup q_2} (f_1, f_2) = \int\limits_{\nabla} \left(\int\limits_{q_1} f_1, \int\limits_{q_2} f_2 \right) = \int\limits_{q_1} f_1 + \int\limits_{q_2} f_2.$$

$\square$

Amenable Spaces

Ambidexterity of the base-change square has also a corollary for the notion of amenability.

Corollary 3.1.16. *Let $\mathcal{C}$ be an ∞-category and let $(*)$ be a pullback diagram of spaces as in Definition 3.1.1. If s_B is surjective on connected components and $\tilde{q}$ is $\tilde{\mathcal{C}}$-amenable, then q is $\mathcal{C}$-amenable.*

Proof. Since s_B is surjective on connected components, s_B^* is conservative. Thus, the result follows from Proposition 3.1.9(3) and Lemma 2.4.3. $\square$

The following two propositions give the core properties of amenable spaces.

Proposition 3.1.17. *Let $\mathcal{C}$ be an ∞-category and let $A \to E \xrightarrow{p} B$ be a fiber sequence of weakly $\mathcal{C}$-ambidextrous spaces, where B is connected. If E is $\mathcal{C}$-ambidextrous and A is $\mathcal{C}$-amenable, then B is $\mathcal{C}$-ambidextrous.*

Proof. By assumption, A is $\mathcal{C}$-amenable and B is connected, hence by Corollary 3.1.16, the map p is $\mathcal{C}$-amenable. Denote $q\colon B \to \mathrm{pt}$ and consider the pair of composable canonically normed functors

$$\mathrm{Fun}\,(E, \mathcal{C}) \xrightarrow{p^{\mathrm{can}}} \mathrm{Fun}\,(B, \mathcal{C}) \xrightarrow{q^{\mathrm{can}}} \mathrm{Fun}\,(\mathrm{pt}, \mathcal{C})\,.$$

Since p^{can} is amenable and $(qp)^{\mathrm{can}} = q^{\mathrm{can}} p^{\mathrm{can}}$ is iso-normed, by Theorem 2.4.5, q^{can} is iso-normed. In other words, the map q (namely, the space B) is $\mathcal{C}$-ambidextrous. $\square$

Proposition 3.1.18. *Let $\mathcal{C}$ be an ∞-category and let A be a connected space, such that $\mathcal{C}$ admits all A-(co)limits and ΩA-(co)limits. Denoting $q\colon A \to \mathrm{pt}$, if ΩA is $\mathcal{C}$-amenable, then the counit map*

$$c_!^q\colon q_! q^* \to \mathrm{Id},$$

is an isomorphism.

Proof. Let $e\colon \mathrm{pt} \to A$ be a choice of a base point. The composition

$$\mathrm{Id} = q_! \underline{e_! e^*} q^* \xrightarrow{c_!^e} \underline{q_! q^*} \xrightarrow{c_!^q} \mathrm{Id}$$

is the counit of the adjunction

$$\mathrm{Id} = q_! e_! \dashv e^* q^* = \mathrm{Id},$$

and hence an isomorphism. Thus, the whiskering $q_! c_!^e q^*$ is a left inverse of $c_!^q$ up to isomorphism. It therefore suffices to show that $c_!^e$ has itself a left inverse. Since A is connected and ΩA is $\mathcal{C}$-amenable, the map e is $\mathcal{C}$-amenable by Corollary 3.1.16. Thus, by Theorem 2.4.4, the map $c_!^e$ has a left inverse. $\square$

Higher Semiadditivity & Spans

We conclude with recalling from [Har17] some results regarding the universality of spans of m-finite spaces among m-semiadditive ∞-categories. These results are useful in reducing questions about general m-semiadditive categories to the universal case, in which they are sometimes easier to solve. Let $\mathcal{S}_m \subseteq \mathcal{S}$ be the full subcategory spanned by m-finite spaces and let $\mathcal{S}_m^m$ be the ∞-category of spans of m-finite spaces (see [Bar17]). Roughly,

- The objects of $\mathcal{S}_m^m$ are m-finite spaces.
- A morphism from A to B is a span $A \leftarrow E \rightarrow B$, where E is m-finite as well.
- Composition, up to homotopy, is given by pullback of spans.

By [Har17, Section 2.2], the ∞-category $\mathcal{S}_m^m$ of spans of m-finite spaces inherits a symmetric monoidal structure from the Cartesian symmetric monoidal structure on $\mathcal{S}_m$. While this symmetric monoidal structure is not itself Cartesian, the unit is $\mathrm{pt} \in \mathcal{S}_m^m$ and the tensor of two maps $A_1 \xleftarrow{q_1} E_1 \xrightarrow{r_1} B_1$ and $A_2 \xleftarrow{q_2} E_2 \xrightarrow{r_2} B_2$ is equivalent to

$$A_1 \times A_2 \xleftarrow{q_1 \times q_2} E_1 \times E_2 \xrightarrow{r_1 \times r_2} B_1 \times B_2.$$

One of the main results of [Har17] is that $\mathcal{S}_m^m$ canonically acts on any m-semiadditive ∞-category (and the existence of such an action is in fact equivalent to m-semiadditivity). Formally,

Theorem 3.1.19 (Harpaz, [Har17, Corollary 5.2])**.** *For every m-semiadditive $\mathcal{C}$, there is a unique monoidal m-finite colimit preserving functor $\mathcal{S}_m^m \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$.*

Unwinding the definition of this action, we get that

- The image of an m-finite space $a\colon A \to \mathrm{pt}$ is equivalent to the functor

  $$(-)_A = a_! a^* \colon \mathcal{C} \to \mathcal{C}$$

  (i.e. colimit over the constant A-shaped diagram).

- The image of a "right way" arrow $A \xleftarrow{=} A \xrightarrow{r} B$ is homotopic to the right way counit map

  $$(-)_A = a_! a^* \simeq b_! \underline{r_! r^*} b^* \xrightarrow{c_!^r} b_! b^* = (-)_B\,,$$

  where $a\colon A \to \mathrm{pt}$ and $b\colon B \to \mathrm{pt}$ are the unique maps (i.e. it is the natural map induced on colimits).

- The image of a "wrong way" arrow $B \xleftarrow{q} A \xrightarrow{=} A$ is homotopic to the wrong way unit map

$$(-)_B = b_! b^* \xrightarrow{\mu_q} b_! \left(q_! q^*\right) b^* \simeq a_! a^* = (-)_A$$

(which can informally be thought of as "integration along the fibers of q").

Remark 3.1.20. If one is only interested in this functor on the level of homotopy categories (as we are),

$$h\mathcal{S}_m^m \to h\,\mathrm{Fun}\left(\mathcal{C}, \mathcal{C}\right),$$

one can use the above formulas as a *definition*. The compatibility with composition can be verified using [HL13, Proposition 4.2.1 (2)]

3.2 Higher Semiadditive Functors

In this section we study m-finite colimit preserving functors between m-semiadditive ∞-categories and study their behavior with respect to integration. We call such functors *m-semiadditive.*

Definition 3.2.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and $q\colon A \to B$ a map of spaces. We define the (F, q)*-square* to be the commutative square

$$\begin{tikzcd} \mathrm{Fun}\left(B, \mathcal{C}\right) \arrow[r, "F_*"] \arrow[d, "q^*"'] & \mathrm{Fun}\left(B, \mathcal{D}\right) \arrow[d, "q^*"] \\ \mathrm{Fun}\left(A, \mathcal{C}\right) \arrow[r, "F_*"] & \mathrm{Fun}\left(A, \mathcal{D}\right), \end{tikzcd}$$

where the horizontal functors are post-composition with F and the vertical functors are pre-composition with q. If q is weakly $\mathcal{C}$ and $\mathcal{D}$ ambidextrous, then this square is canonically normed.

Proposition 3.2.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and $q\colon A \to B$ a map of spaces. If $\mathcal{C}$ and $\mathcal{D}$ admit, and F preserves, all q-colimits (resp. q-limits), then the (F, q)-square satisfies the* $\mathrm{BC}_!$ *(resp.* BC_**) condition.*

Proof. This follows from the point-wise formulas for the left and right Kan extensions. □

The following is the main result of this section.

Theorem 3.2.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ a functor of ∞-categories which preserves $(m-1)$-finite colimits. Let $q\colon A \to B$ be an m-finite map of spaces. If q is (weakly) $\mathcal{C}$-ambidextrous and (weakly) $\mathcal{D}$-ambidextrous, then the (F, q)-square is (weakly) ambidextrous.*

Proof. The statement about ambidexterity follows immediately from the ambidexterity of q and the statement about weak ambidexterity. We shall prove the later by induction on m. For $m = -2$, both vertical maps in the (F, q)-square are equivalences, and so the claim follows from Proposition 3.1.9(1). We therefore assume $m \geq -1$. Consider the diagram

$$\begin{tikzcd}
A \arrow[dr, "\delta"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & A \arrow[d, "q"] \\
& A \arrow[r, "q"] & B.
\end{tikzcd} \qquad (\heartsuit)$$

The square in the diagram induces a $\mathrm{BC}_!$ map $\beta\colon (\pi_2)_!\, \pi_1^* \to q^* q_!$, which is an isomorphism by Lemma 3.1.2. By definition, $\nu_q^{\mathcal{C}}$ is the composition of maps

$$q^* q_! \xrightarrow{\beta^{-1}} (\pi_2)_!\, \pi_1^* \xrightarrow{\mu_\delta} (\pi_2)_!\, \delta_! \delta^* \pi_1^* \simeq \mathrm{Id}\,.$$

By Lemma 2.2.11(1), it suffices to show that the wrong way counit diagram of q commutes. This will follow from the commutativity of the (solid) diagram:

$$\begin{tikzcd}
q^* q_! F_* \arrow[r, "\beta^{-1}"] \arrow[dd, "\beta_!"] & (\pi_2)_!\, (\pi_1)^*\, F_* \arrow[r, "\mu_\delta^{\mathcal{D}}"] \arrow[dd, dashed, "\wr"] & (\pi_2)_!\, \delta_! \delta^*\, (\pi_1)^*\, F_* \arrow[d, dashed, "\wr"] \arrow[ddr, bend left, "\sim"] & \\
& & (\pi_2)_!\, \delta_! \delta^* F_*\, (\pi_1)^* \arrow[d, dashed, "\wr"] & \\
q^* F_* q_! \arrow[dd, "\wr"] & (\pi_2)_!\, F_*\, (\pi_1)^* \arrow[ur, dashed, "\mu_\delta^{\mathcal{D}}"] \arrow[dr, dashed, "\mu_\delta^{\mathcal{C}}"'] \arrow[dd, dashed] & (\pi_2)_!\, \delta_! F_* \delta^*\, (\pi_1)^* \arrow[r, dashed, "\simeq"] \arrow[d, dashed] & F_*. \\
& & (\pi_2)_!\, F_* \delta_! \delta^*\, (\pi_1)^* \arrow[d, dashed] & \\
F_* q^* q_! \arrow[r, "\beta^{-1}"] & F_*\, (\pi_2)_!\, (\pi_1)^* \arrow[r, "\mu_\delta^{\mathcal{C}}"] & F_*\, (\pi_2)_!\, \delta_! \delta^*\, (\pi_1)^* \arrow[uur, bend right, "\sim"'] &
\end{tikzcd}$$

The two trapezoids and the upper triangle commute for formal reasons. The bottom triangle commutes by Lemma 2.2.6(1) and the fact that $\pi_2 \circ \delta = \mathrm{Id}$. For the rectangle on the left, it is enough to prove the commutativity of the associated rectangular diagram with β instead of β^{-1} in both horizontal lines, which we denote by $(*)$. We can now consider the commutative cubical diagram

$$
\begin{tikzcd}
\mathrm{Fun}\,(B,\mathcal{C}) \arrow[rr, "F_*"] \arrow[dr, "q^*"] \arrow[dd, "q^*"'] \arrow[drrr, dotted] & & \mathrm{Fun}\,(B,\mathcal{D}) \arrow[dr, "q^*"] \arrow[dd, "q^*" near end] & \\
& \mathrm{Fun}\,(A,\mathcal{C}) \arrow[rr, "F_*" near start] \arrow[dd, "\pi_2^*"' near start] & & \mathrm{Fun}\,(A,\mathcal{D}) \arrow[dd, "\pi_2^*"] \\
\mathrm{Fun}\,(A,\mathcal{C}) \arrow[rr, "F_*" near end] \arrow[dr, "\pi_1^*"] \arrow[drrr, dotted] & & \mathrm{Fun}\,(A,\mathcal{D}) \arrow[dr, "\pi_1^*"] & \\
& \mathrm{Fun}\,(A\times_B A,\mathcal{C}) \arrow[rr, "F_*"] & & \mathrm{Fun}\,(A\times_B A,\mathcal{D})\,.
\end{tikzcd}
\tag{$\spadesuit$}
$$

Applying Lemma 2.2.4(1) once to the back and then right face of ($\spadesuit$) and once to the left and then front face of ($\spadesuit$), we get two presentations of the $\mathrm{BC}_!$ map of the diagram

$$
\begin{tikzcd}
\mathrm{Fun}(B,\mathcal{C}) \arrow[r, dotted] \arrow[d, "q^*"'] & \mathrm{Fun}(A,\mathcal{D}) \arrow[d, "\pi_2^*"] \\
\mathrm{Fun}(A,\mathcal{C}) \arrow[r, dotted] & \mathrm{Fun}(A\times_B A,\mathcal{D}).
\end{tikzcd}
$$

These two presentations correspond precisely to the two paths in $(*)$.

It is left to check the commutativity of the triangle in the middle, which is a whiskering of the diagram

$$
\begin{tikzcd}
& \delta_!\delta^* F_* \arrow[d, "\wr"] \\
F_* \arrow[ur, "\mu_\delta^{\mathcal{D}}"] \arrow[dr, "\mu_\delta^{\mathcal{C}}"'] & \delta_! F_* \delta^* \arrow[d, "\beta_!"] \\
& F_*\delta_!\delta^*.
\end{tikzcd}
\tag{$\lhd$}
$$

The map δ is an $(m-1)$-finite map that is both $\mathcal{C}$-ambidextrous and $\mathcal{D}$-ambidextrous. By assumption, F preserves $(m-1)$-finite colimits and so, by the inductive hypothesis, the norm diagram of the (F,δ)-square commutes. Thus, $\lhd$ commutes by Lemma 2.2.11(2). $\square$

As a corollary, we get a higher analogue of a known fact about 0-semiadditive categories.

Corollary 3.2.4. *Let $F\colon \mathcal{C}\to\mathcal{D}$ be a functor of m-semiadditive ∞-categories. The functor F preserves m-finite colimits if and only if it preserves m-finite limits.*

Proof. We proceed by induction on m. For $m=-2$, there is nothing to prove. For $m\ge -1$, assume by induction the claim holds for $m-1$. Since $\mathcal{C}$ and $\mathcal{D}$ are in particular $(m-1)$-semiadditive and F preserves either $(m-1)$-colimits or $(m-1)$-limits, we deduce that F preserves both. For every m-finite A, consider the map $q\colon A\to \mathrm{pt}$. Since $\mathcal{C}$ and $\mathcal{D}$ are in particular $(m-1)$-semiadditive and F preserves $(m-1)$-colimits, by Theorem 3.2.3, the (F,q)-square is weakly ambidextrous. Since $\mathcal{C}$ and $\mathcal{D}$ are m-semiadditive, the (F,q)-square is in fact ambidextrous. It follows that the (F,q)-square satisfies the $\mathrm{BC}_!$ condition if and only if it satisfies the BC_* condition. Namely, F preserves A-shaped colimits if and only if it preserves A-shaped limits. $\square$

Definition 3.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be m-semiadditive ∞-categories. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *m-semiadditive*, if it preserves m-finite (co)limits.

The fundamental property of m-semiadditive functors, which justifies their name, is

Corollary 3.2.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an m-semiadditive functor and let $q\colon A \to B$ be an m-finite map of spaces. For all $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f\colon q^*X \to q^*Y$, we have*

$$F\left(\int\limits_q f\right) = \int\limits_q F(f) \quad \in \hom_{h\,\mathrm{Fun}(B,\mathcal{D})}(FX, FY).$$

In particular, for all $X \in \mathrm{Fun}(B, \mathcal{C})$ we have

$$F([q]_X) = [q]_{F(X)} \quad \in \hom_{h\,\mathrm{Fun}(B,\mathcal{D})}(FX, FX).$$

Proof. The (F, q)-square is ambidextrous by Theorem 3.2.3 and satisfies the BC conditions by Proposition 3.2.2, and so the claim follows form Proposition 2.2.12. ☐

Remark 3.2.7. In view of Remark 3.1.4, one can reinterpret Theorem 3.2.3 informally, as saying that

$$\int\limits_\delta F(\mathrm{Id}) = F\left(\int\limits_\delta \mathrm{Id}\right),$$

where $\delta\colon A \to A \times_B A$ is the diagonal of $q\colon A \to B$. Since δ is $(m-1)$-finite, this in turn follows inductively from Corollary 3.2.6. Turning this argument into a rigorous proof requires some categorical maneuvers that we preferred to avoid.

Multivariate Functors

We now discuss a multivariate version of higher semiadditive functors.

Definition 3.2.8. Let $\mathcal{C}_1, \dots, \mathcal{C}_k$ and $\mathcal{D}$ be ∞-categories and $F\colon \prod\limits_{i=1}^{k} \mathcal{C}_i \to \mathcal{D}$ a functor. Given a collection of diagrams $X_i\colon A_i \to \mathcal{C}_i$ for $i = 1, \dots, k$, their external product $X_1 \boxtimes \cdots \boxtimes X_k$ is defined to be the composition

$$\prod_{i=1}^{k} A_i \xrightarrow{\prod_{i=1}^{k} X_i} \prod_{i=1}^{k} \mathcal{C}_i \xrightarrow{F} \mathcal{D}.$$

This assembles to give a functor

$$\boxtimes\colon \prod_{i=1}^{k} \mathrm{Fun}(A_i, \mathcal{C}_i) \to \mathrm{Fun}\left(\prod_{i=1}^{k} A_i, \mathcal{D}\right).$$

Given a collection of maps of spaces $q_i\colon A_i \to B_i$ for $i = 1, \dots, k$, we obtain the associated *external product square*:

$$\begin{tikzcd} \prod_{i=1}^{k} \mathrm{Fun}(B_i, \mathcal{C}_i) \arrow[r, "\boxtimes"] \arrow[d, "\prod_{i=1}^{k} q_i^*"] & \mathrm{Fun}\left(\prod_{i=1}^{k} B_i, \mathcal{D}\right) \arrow[d, "\left(\prod_{i=1}^{k} q_i\right)^*"] \\ \prod_{i=1}^{k} \mathrm{Fun}(A_i, \mathcal{C}_i) \arrow[r, "\boxtimes"] & \mathrm{Fun}\left(\prod_{i=1}^{k} A_i, \mathcal{D}\right). \end{tikzcd} \qquad (*)$$

Proposition 3.2.9. *Let $\mathcal{C}_1, \dots, \mathcal{C}_k$ and $\mathcal{D}$ be ∞-categories and $F\colon \prod_{i=1}^{k} \mathcal{C}_i \to \mathcal{D}$ a functor. Additionally, let $q_i\colon A_i \to B_i$ for $i = 1, \dots, k$ be a collection of maps of spaces. If F preserves all q_i-colimits (resp. q_i-limits) in the i-th coordinate, then the external product square $(*)$ satisfies the $BC_!$ (resp. BC_*) condition.*

Proof. We proceed by a sequence reductions. First, by induction on k and horizontal pasting (Corollary 2.2.5), we can reduce to $k = 2$. Write $q_1 \times q_2$ as a composition

$$A_1 \times A_2 \xrightarrow{q_1 \times \mathrm{Id}} B_1 \times A_2 \xrightarrow{\mathrm{Id} \times q_2} B_1 \times B_2.$$

The diagram

$$\begin{tikzcd} \mathrm{Fun}(B_1, \mathcal{C}_1) \times \mathrm{Fun}(B_2, \mathcal{C}_2) \arrow[r, "\boxtimes"] \arrow[d, "\mathrm{Id} \times q_2^*"] & \mathrm{Fun}(B_1 \times B_2, \mathcal{D}) \arrow[d, "(\mathrm{Id} \times q_2)^*"] \\ \mathrm{Fun}(B_1, \mathcal{C}_1) \times \mathrm{Fun}(A_2, \mathcal{C}_2) \arrow[r, "\boxtimes"] \arrow[d, "q_1^* \times \mathrm{Id}"] & \mathrm{Fun}(B_1 \times A_2, \mathcal{D}) \arrow[d, "(q_1 \times \mathrm{Id})^*"] \\ \mathrm{Fun}(A_1, \mathcal{C}_1) \times \mathrm{Fun}(A_2, \mathcal{C}_2) \arrow[r, "\boxtimes"] & \mathrm{Fun}(A_1 \times A_2, \mathcal{D}) \end{tikzcd}$$

exhibits $(*)$ as a vertical pasting of the top and bottom squares. Hence, by Corollary 2.2.7, it is enough to show that each of them satisfies the $\mathrm{BC}_!$ (resp. BC_*) condition. We will focus on the bottom square (the argument for the top square is identical). Since (co)limits in A_2-local systems are computed point-wise, the external product functor

$$F_{A_2}\colon \mathcal{C}_1 \times \mathrm{Fun}(A_2, \mathcal{C}_2) \to \mathrm{Fun}(A_2, \mathcal{D})$$

preserves in each coordinate the (co)limits which are preserved by F. By replacing the ∞-category $\mathcal{C}_2$ with $\mathrm{Fun}(A_2, \mathcal{C}_2)$, the ∞-category $\mathcal{D}$ with $\mathrm{Fun}(A_2, \mathcal{D})$ and the functor F with F_{A_2}, we may assume without loss of generality that $A_2 = \Delta^0$. The bottom square becomes

$$\begin{tikzcd} \mathrm{Fun}(B_1, \mathcal{C}_1) \times \mathcal{C}_2 \arrow[r, "\boxtimes"] \arrow[d, "q_1^* \times \mathrm{Id}"] & \mathrm{Fun}(B_1, \mathcal{D}) \arrow[d, "q_1^*"] \\ \mathrm{Fun}(A_1, \mathcal{C}_1) \times \mathcal{C}_2 \arrow[r, "\boxtimes"] & \mathrm{Fun}(A_1, \mathcal{D}). \end{tikzcd}$$

By the exponential rule (Lemma 2.2.8), it is enough to show that the left square is the following diagram satisfies the $\mathrm{BC}_!$ (resp. BC_*) condition:

$$\begin{tikzcd}
\mathrm{Fun}(B_1,\mathcal{C}_1) \arrow[r, "\boxtimes"] \arrow[d, "q_1^*"] & \mathrm{Fun}(\mathcal{C}_2,\mathrm{Fun}(B_1,\mathcal{D})) \arrow[r, "\sim"] \arrow[d, "(q_1^{\mathcal{C}})^*"] & \mathrm{Fun}(B_1,\mathrm{Fun}(\mathcal{C}_2,\mathcal{D})) \arrow[d, "q_1^*"] \\
\mathrm{Fun}(A_1,\mathcal{C}_1) \arrow[r, "\boxtimes"] & \mathrm{Fun}(\mathcal{C}_2,\mathrm{Fun}(A_1,\mathcal{D})) \arrow[r, "\sim"] & \mathrm{Fun}(A_1,\mathrm{Fun}(\mathcal{C}_2,\mathcal{D}))\,.
\end{tikzcd}$$

Equivalently, it is enough to show that the outer square $\square$ satisfies the $\mathrm{BC}_!$ (resp. BC_*) condition. Observe that $\square$ is the $(F^\vee, q_1)$-square for the functor

$$F^\vee\colon \mathcal{C}_1 \to \mathrm{Fun}(\mathcal{C}_2,\mathcal{D})\,,$$

which is the mate of F. From the assumption on F, the functor $F^\vee$ preserves q_1-colimits (resp. q_1-limits) and therefore $\square$ satisfies the $\mathrm{BC}_!$ (resp. BC_*) condition by the univariate version (Proposition 3.2.2). $\square$

Definition 3.2.10. Let $\mathcal{C}_1,\dots,\mathcal{C}_k$ and $\mathcal{D}$ be m-semiadditive ∞-categories. An *m-semiadditive multi-functor* $F\colon \prod_{i=1}^{k}\mathcal{C}_i \to \mathcal{D}$ is a functor that preserves m-finite colimits in each coordinate separately.

In particular, we get

Corollary 3.2.11. *Let $\mathcal{C}_1,\dots,\mathcal{C}_k$ and $\mathcal{D}$ be m-semiadditive ∞-categories. Let $F\colon \prod_{i=1}^{k}\mathcal{C}_i \to \mathcal{D}$ be an m-semiadditive multi-functor. For every collection of m-finite maps $q_i\colon A_i \to B_i$ for $i = 1,\dots k$, the external product square $(*)$ from Definition 3.2.8 satisfies both* BC*-conditions.*

3.3 Symmetric Monoidal Structure

In this section we study the interaction of higher semiadditivity with (symmetric) monoidal structure.

Monoidal Local Systems

Let $(\mathcal{C},\otimes,\mathbb{1})$ be a (symmetric) monoidal ∞-category. For every space A, the ∞-category $\mathrm{Fun}(A,\mathcal{C})$ acquires a point-wise (symmetric) monoidal structure. Moreover, given a map of spaces $q\colon A \to B$, the functor

$$q^*\colon\ \mathrm{Fun}(B,\mathcal{C}) \to \mathrm{Fun}(A,\mathcal{C})$$

is (symmetric) monoidal in a canonical way ([Lur, Example 3.2.4.4]).

Proposition 3.3.1. *Let $(\mathcal{C},\otimes,\mathbb{1})$ be a monoidal ∞-category. Let $q\colon A \to B$ be a weakly $\mathcal{C}$-ambidextrous map of spaces, such that $\otimes$ distributes over q-colimits. The normed functor*

$$q^{\mathrm{can}}\colon\ \mathrm{Fun}(A,\mathcal{C}) \rightarrowtail \mathrm{Fun}(B,\mathcal{C})$$

is $\otimes$-normed in a canonical way (see Definition 2.3.1).

Proof. Consider the diagram

$$\begin{tikzcd}
q_!(q^*X \otimes Y) \arrow[r, "u_{!,X} \otimes u_{!,Y}"] \arrow[ddr, dashed] & q_!((q^*\underline{q_!)q^*}X \otimes (q^*q_!)Y) \arrow[r, "c_{!,X} \otimes \mathrm{Id}"] \arrow[d, "\wr"] & q_!(q^*X \otimes q^*q_!Y) \arrow[d, "\wr"] \\
& \underline{q_!q^*}(\underline{q_!q^*}X \otimes q_!Y) \arrow[r, "c_{!,X} \otimes \mathrm{Id}"] \arrow[d, "c_{!,(X \otimes q_!Y)}"] & \underline{q_!q^*}(X \otimes q_!Y) \arrow[d, "c_{!,(X \otimes q_!Y)}"] \\
& \underline{q_!q^*}X \otimes q_!Y \arrow[r, "c_{!,X} \otimes \mathrm{Id}"] & X \otimes q_!Y.
\end{tikzcd}$$

The triangle on the left commutes by definition, where the dashed arrow is induced by the colax monoidality of $q_!$. The rest of the diagram commutes for formal reasons. The composition along the bottom path of the diagram is the second map in Definition 2.3.1 and we shall show it is an isomorphism (the proof for the first one follows by symmetry). Since the diagram commutes, it suffices to show that the composition along the top and then right path is an isomorphism. By the zig-zag identities, the later is homotopic to the composition

$$q_!(q^*X \otimes Y) \xrightarrow{\mathrm{Id} \otimes u_{!,Y}} q_!(q^*X \otimes q^*q_!Y) \xrightarrow{\sim} q_!q^*(X \otimes q_!Y) \xrightarrow{c_{!,(X \otimes q!Y)}} X \otimes q_!Y.$$

Finally, this composition is by definition the $\mathrm{BC}_!$ map $\beta_!$ for the square

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathcal{C}) \arrow[r, "X \otimes (-)"] \arrow[d, "q^*"] & \mathrm{Fun}(B, \mathcal{C}) \arrow[d, "q^*"] \\
\mathrm{Fun}(A, \mathcal{C}) \arrow[r, "q^*X \otimes (-)"] & \mathrm{Fun}(A, \mathcal{C}).
\end{tikzcd}$$

To see that $\beta_!$ is an isomorphism, it is enough to check this after pulling back to every point $b \in B$. This in turn follows from the assumption that $\otimes$ distributes over q-colimits. $\square$

This allows us to apply the general results about $\otimes$-normed functors to the setting of local systems.

Corollary 3.3.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be an m-finite colimit preserving monoidal functor between monoidal categories that admit, and the tensor product distributes over, m-finite colimits.*

(1) An m-finite map of spaces $q : A \to B$, that is $\mathcal{C}$-ambidextrous and weakly $\mathcal{D}$-ambidextrous, is $\mathcal{D}$-ambidextrous.

(2) If $\mathcal{C}$ is m-semiadditive, then $\mathcal{D}$ is also m-semiadditive.

Proof. By Proposition 3.3.1, q^{can} is $\otimes$-normed. By Theorem 3.2.3, the (F, q)-square is ambidextrous. Since F preserves m-finite colimits, the (F, q)-square satisfies the $\mathrm{BC}_!$-condition. (1) now follows from Proposition 2.3.8. We prove (2) by induction on m. For $m = -2$, there is nothing to prove, and so we assume $m \geq -1$. By the inductive hypothesis, we may assume $\mathcal{D}$ is $(m-1)$-semiadditive. In this case, every m-finite map $q : A \to B$ is weakly $\mathcal{D}$-ambidextrous and $\mathcal{C}$-ambidextrous, hence by (1), is $\mathcal{D}$-ambidextrous. $\square$

The following definition is the natural notion of (symmetric) monoidal structure in the realm of m-semiadditive ∞-categories.

Definition 3.3.3. An *m-semiadditively* (symmetric) monoidal ∞-category, is an m-semiadditive (symmetric) monoidal ∞-category $\mathcal{C}$, such that the tensor product distributes over m-finite colimits.

Lemma 3.3.4. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an m-semiadditively monoidal ∞-category and A an m-finite space.*

(1) For every $X \in \mathcal{C}$, we have $[A]_X \simeq \mathrm{Id}_X \otimes [A]_{\mathbb{1}}$.

(2) A is $\mathcal{C}$-amenable if and only if $[A]_{\mathbb{1}}$ is an isomorphism.

Proof. We start with (1). Given an object $X \in \mathcal{C}$, the functor $F_X : \mathcal{C} \to \mathcal{C}$, given by

$$F_X(Y) = X \otimes Y,$$

preserves m-finite colimits. Thus, by Corollary 3.2.6 we have:

$$\mathrm{Id}_X \otimes [A]_{\mathbb{1}} = F_X([A]_{\mathbb{1}}) = [A]_{F_X(\mathbb{1})} = [A]_X .$$

(2) is an immediate corollary of (1). $\square$

Notation 3.3.5. For an m-semiadditively symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$ and an m-finite space A, we abuse notation by identifying $[A]_{\mathbb{1}}$ with $[A]$. If we want to emphasize the ∞-category $\mathcal{C}$, we write $[A]_{\mathcal{C}}$. By Lemma 3.3.4, this conflation of terminology is rather harmless.

We also have the following consequence for dualizability.

Proposition 3.3.6. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal ∞-category. For every space A that is $\mathcal{C}$-ambidextrous and such that $\otimes$ distributes over A-colimits, the object $\mathbb{1}_A$ is dualizable. In particular, if $(\mathcal{C}, \otimes, \mathbb{1})$ is m-semiadditively monoidal ∞-category, then $\mathbb{1}_A$ is dualizable for every m-finite space A.*

Proof. By Proposition 3.3.1, the map $q\colon A \to \mathrm{pt}$ corresponds to a $\otimes$-normed functor

$$q^{\mathrm{can}}\colon \mathrm{Fun}(A, \mathcal{C}) \rightarrowtail \mathcal{C}$$

and by definition $\mathbb{1}_A = \mathbb{1}_q = q_! q^* \mathbb{1}$. Thus, the claim follows from Proposition 2.3.4. $\square$

Symmetric Monoidal Dimension

We now specialize to the *symmetric* monoidal case. We begin with recalling the definition of dimension for a dualizable object of a symmetric monoidal ∞-category. A dualizable object X in a symmetric monoidal ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$ has a notion of dimension, which is defined as follows. Let $X^\vee$ be the dual of X and let

$$\varepsilon\colon X^\vee \otimes X \to \mathbb{1}, \quad \eta\colon \mathbb{1} \to X \otimes X^\vee$$

be the evaluation and coevaluation maps respectively.

Definition 3.3.7. We denote by

$$\dim_{\mathcal{C}}(X) \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$$

the composition

$$\mathbb{1} \xrightarrow{\eta} X \otimes X^{\vee} \xrightarrow{\sigma} X^{\vee} \otimes X \xrightarrow{\varepsilon} \mathbb{1},$$

where σ is the swap map of the symmetric monoidal structure. We say that a space A is dualizable in $\mathcal{C}$, if $\mathbb{1}_A$ is dualizable in $\mathcal{C}$ and we denote

$$\dim_{\mathcal{C}}(A) = \dim_{\mathcal{C}}(\mathbb{1}_A).$$

Dualizability of m-finite spaces in $\mathcal{S}_m^m$ assumes a particularly simple form.

Proposition 3.3.8. *Every m-finite space A is self dual in $\mathcal{S}_m^m$ and satisfies*

$$\dim_{\mathcal{S}_m^m}(A) = (\mathrm{pt} \leftarrow A^{S^1} \to \mathrm{pt}) = [A^{S^1}] \in \mathrm{End}_{\mathcal{S}_m^m}(\mathrm{pt}).$$

Proof. It is straightforward to check that the spans

$$\varepsilon\colon (A \times A \xleftarrow{\Delta} A \to \mathrm{pt})$$

$$\eta\colon (\mathrm{pt} \leftarrow A \xrightarrow{\Delta} A \times A),$$

satisfy the zig-zag identities and therefore ε is a duality pairing exhibiting A as self dual. Moreover, since $\varepsilon \circ \sigma$ is homotopic to ε, where $\sigma\colon X \times X \to X \times X$ is the symmetric monoidal swap, we get $\dim(A) = \varepsilon \circ \eta$. Computing the relevant pullback explicitly,

$$\begin{tikzcd}
& & A^{S^1} \arrow[dl] \arrow[dr] & & \\
& A \arrow[dl] \arrow[dr] & & A \arrow[dl] \arrow[dr] & \\
\mathrm{pt} & & A \times A & & \mathrm{pt}
\end{tikzcd}$$

we obtain the desired result. □

As a symmetric monoidal ∞-category, $\mathcal{S}_m^m$ has also the following universal property.

Theorem 3.3.9 (Harpaz, [Har17, Corollary 5.8])**.** *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an m-semiadditively symmetric monoidal ∞-category. There exists a unique m-semiadditive symmetric monoidal functor $\mathcal{S}_m^m \to \mathcal{C}$ and its underlying functor is $\mathbb{1}_{(-)}$.*

From this we immediately get

Corollary 3.3.10. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an m-semiadditively symmetric monoidal ∞-category. Every m-finite space A is dualizable in $\mathcal{C}$ and*

$$\dim_{\mathcal{C}}(A) = [A^{S^1}] \quad \in \hom_{h\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}).$$

In particular, if A is a loop space (e.g. $A = B^k C_p$), we have

$$\dim_{\mathcal{C}}(A) = [A]\,[\Omega A].$$

Proof. By Theorem 3.3.9, there is a canonical m-finite colimit preserving symmetric monoidal functor $F\colon \mathcal{S}_m^m \to \mathcal{C}$. Since $F(A) = \mathbb{1}_A$ and F is symmetric monoidal, we have

$$F\left(\dim_{\mathcal{S}_m^m} A\right) = \dim_{\mathcal{C}}(\mathbb{1}_A).$$

Since F also preserves m-finite colimits, we have by Corollary 3.2.6, that

$$F\left([B]_{\mathrm{pt}}\right) = [B]_{F(\mathrm{pt})} = [B]_{\mathbb{1}_{\mathcal{C}}}$$

for all m-finite B. We are therefore reduced to the universal case $\mathcal{C} = \mathcal{S}_m^m$, which is given by Proposition 3.3.8. The last claim follows from the fact that if A is a loop-space, then $A^{S^1} \simeq A \times \Omega A$ and Corollary 3.1.14. □

3.4 Equivariant Powers

Let $\mathcal{C}$ be a symmetric monoidal ∞-category and p a prime. For every object $X \in \mathcal{C}$, the p-th tensor power $X^{\otimes p}$ carries a natural action of the cyclic group $C_p \subseteq \Sigma_p$. Moreover, given a map $f\colon X \to Y$, we get a C_p-equivariant morphism $f^{\otimes p}\colon X^{\otimes p} \to Y^{\otimes p}$. Namely, there is a functor

$$\Theta^p\colon \mathcal{C} \to \mathrm{Fun}(BC_p, \mathcal{C}),$$

whose composition with $e^*\colon \mathrm{Fun}(BC_p, \mathcal{C}) \to \mathcal{C}$ (where $e\colon \mathrm{pt} \to BC_p$) is homotopic to the p-th power functor $(-)^{\otimes p}: \mathcal{C} \to \mathcal{C}$. In this section we study the functor Θ^p, its naturality and additivity properties.

Functoriality & Integration

We begin by describing Θ^p formally. It will be useful to work in a greater level of generality of $\mathcal{C}$-valued *local-systems* instead of single objects. Given an ∞-category $\mathcal{C}$ and a space A, the C_p-equivariant p-power of ∞-categories induces a functor

$$(-)^p_{hC_p} : \mathrm{Fun}(A, \mathcal{C}) \to \mathrm{Fun}((A^p)_{hC_p}, (\mathcal{C}^p)_{hC_p}).$$

We also write $A \wr C_p = (A^p)_{hC_p}$.

Definition 3.4.1. Given a symmetric monoidal ∞-category $\mathcal{C}$, we define the functor

$$\Theta^p : \mathrm{Fun}(A, \mathcal{C}) \to \mathrm{Fun}(A \wr C_p, \mathcal{C})$$

to be the composition of $(-)^p_{hC_p}$ with

$$(\mathcal{C}^p)_{hC_p} \to (\mathcal{C}^p)_{h\Sigma_p} \xrightarrow{\otimes} \mathcal{C}.$$

The Θ^p operation is functorial in the following sense.

Lemma 3.4.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories. For every space A, the diagram*

$$\begin{tikzcd}
\mathrm{Fun}(A,\mathcal{C}) \arrow[r, "\Theta^p"] \arrow[d, "F_*"'] & \mathrm{Fun}(A \wr C_p,\mathcal{C}) \arrow[d, "F_*"] \\
\mathrm{Fun}(A,\mathcal{D}) \arrow[r, "\Theta^p"] & \mathrm{Fun}(A \wr C_p,\mathcal{D})
\end{tikzcd}$$

commutes up to homotopy.

Proof. The square in question is the outer square of the following diagram

$$\begin{tikzcd}
\mathrm{Fun}(A,\mathcal{C}) \arrow[r] \arrow[d, "F_*"'] & \mathrm{Fun}(A \wr C_p,\mathcal{C}^p_{hC_p}) \arrow[r, "\otimes"] \arrow[d, "F_*"] & \mathrm{Fun}(A \wr C_p,\mathcal{C}) \arrow[d, "F_*"] \\
\mathrm{Fun}(A,\mathcal{D}) \arrow[r] & \mathrm{Fun}(A \wr C_p,\mathcal{D}^p_{hC_p}) \arrow[r, "\otimes"] & \mathrm{Fun}(A \wr C_p,\mathcal{D})\,.
\end{tikzcd}$$

The left square commutes by the functoriality of $\mathcal{C} \mapsto \mathcal{C}^p_{hC_p}$ and the right, since F is symmetric monoidal. $\square$

Definition 3.4.3. For a map of spaces $q\colon A \to B$, the naturality of Definition 3.4.1 gives a commutative square

$$\begin{tikzcd}
\mathrm{Fun}(B,\mathcal{C}) \arrow[r, "\Theta^p"] \arrow[d, "q^*"] & \mathrm{Fun}(B \wr C_p,\mathcal{C}) \arrow[d, "(q\wr C_p)^*"] \\
\mathrm{Fun}(A,\mathcal{C}) \arrow[r, "\Theta^p"] & \mathrm{Fun}(A \wr C_p,\mathcal{C})\,.
\end{tikzcd}$$

We call this the Θ^p-square of q. If q is m-finite, then so is $q \wr C_p$. If additionally $\mathcal{C}$ is $(m-1)$-semiadditive and admits m-finite (co)limits, the Θ^p-square is canonically normed.

Example 3.4.4. For a space A, we have a canonical fiber sequence

$$A^p \to A \wr C_p \xrightarrow{\pi} BC_p.$$

The Θ^p-square of $q\colon A \to \mathrm{pt}$ is

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{pt},\mathcal{C}) \arrow[r, "\Theta^p"] \arrow[d, "q^*"] & \mathrm{Fun}(BC_p,\mathcal{C}) \arrow[d, "\pi^*"] \\
\mathrm{Fun}(A,\mathcal{C}) \arrow[r, "\Theta^p"] & \mathrm{Fun}(A \wr C_p,\mathcal{C})\,.
\end{tikzcd}$$

Lemma 3.4.5. *Let $q\colon A \to B$ be a map of spaces and let $(\mathcal{C},\otimes,\mathbb{1})$ be a symmetric monoidal ∞-category that admits all q-(co)limits. If $\otimes$ distributes over all q-colimits (resp. q-limits), then the Θ^p-square satisfies the $BC_!$ (resp. BC_*) condition.*

Proof. We horizontally paste the Θ^p-square for q with the square induced by the pullback diagram

$$\begin{tikzcd} A^p \arrow[r, "\pi_A"] \arrow[d, "q^p"'] & A \wr C_p \arrow[d, "q \wr C_p"] \\ B^p \arrow[r, "\pi_B"] & B \wr C_p, \end{tikzcd}$$

to obtain

$$\begin{tikzcd} \mathrm{Fun}\,(B, \mathcal{C}) \arrow[r, "\Theta^p_B"] \arrow[d, "q^*"] & \mathrm{Fun}\,(B \wr C_p, \mathcal{C}) \arrow[r, "\pi_B^*"] \arrow[d, "(q \wr C_p)^*"] & \mathrm{Fun}\,(B^p, \mathcal{C}) \arrow[d, "(q^p)^*"] \\ \mathrm{Fun}\,(A, \mathcal{C}) \arrow[r, "\Theta^p_A"] & \mathrm{Fun}\,(A \wr C_p, \mathcal{C}) \arrow[r, "\pi_A^*"] & \mathrm{Fun}\,(A^p, \mathcal{C})\,. \end{tikzcd} \qquad (*)$$

The right square $\square_R$ satisfies both BC-conditions by Lemma 3.1.2. Since π_B^* is conservative (π_B is surjective on connected components), by Corollary 2.2.5(2), it is enough to show that the outer square $\square$ satisfies the $\mathrm{BC}_!$ (resp. BC_*) condition. We can now write $\square$ as a horizontal pasting of two squares $\square'_L$ and $\square'_R$ in a different way:

$$\begin{tikzcd} \mathrm{Fun}\,(B, \mathcal{C}) \arrow[r, "\Delta"] \arrow[d, "q^*"] & \mathrm{Fun}\,(B, \mathcal{C})^p \arrow[r, "\boxtimes^p"] \arrow[d, "(q^*)^p"] & \mathrm{Fun}\,(B^p, \mathcal{C}) \arrow[d, "(q^p)^*"] \\ \mathrm{Fun}\,(A, \mathcal{C}) \arrow[r, "\Delta"] & \mathrm{Fun}\,(A, \mathcal{C})^p \arrow[r, "\boxtimes^p"] & \mathrm{Fun}\,(A^p, \mathcal{C})\,. \end{tikzcd}$$

The square $\square'_L$ satisfies the BC-conditions trivially and $\square'_R$ by Proposition 3.2.9. $\square$

Proposition 3.4.6. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an m-semiadditively symmetric monoidal ∞-category and let $q\colon A \to B$ be an m-finite map of spaces. The corresponding Θ^p-square is ambidextrous.*

Proof. Since $\mathcal{C}$ is m-semiadditive, the Θ^p-square for q is iso-normed and hence it suffices to show that it is weakly ambidextrous. Namely, that the associated norm-diagram commutes. The proof is very similar to the argument given in Theorem 3.2.3, and therefore we shall use similar notation and indicate only the changes that need to be made. We proceed by induction on m using the diagram of spaces

$$\begin{tikzcd} A \arrow[dr, "\delta"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\ & A \times_B A \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & A \arrow[d, "q"] \\ & A \arrow[r, "q"] & B. \end{tikzcd} \qquad (\heartsuit)$$

Denoting $\widetilde{(-)} = (-) \wr C_p$, we consider the diagram of functors from $\mathrm{Fun}(A, \mathcal{C})$ to $\mathrm{Fun}(A \wr C_p, \mathcal{C})$

(where all unnamed arrows are BC-maps).

$$\begin{tikzcd}
\tilde{q}^*\tilde{q}_!\Theta^p_A \arrow[r, "\beta^{-1}"] \arrow[dd, "\beta_!"] & (\tilde{\pi}_2)_!\left(\tilde{\pi}_1\right)^*\Theta^p_A \arrow[r, "\mu_{\tilde{\delta}}"] \arrow[dd, dashed, "\wr"] & (\tilde{\pi}_2)_!\tilde{\delta}_!\tilde{\delta}^*\left(\tilde{\pi}_1\right)^*\Theta^p_A \arrow[d, dashed, "\wr"] \arrow[ddr, bend left, "\sim"] & \\
 & & (\tilde{\pi}_2)_!\tilde{\delta}_!\tilde{\delta}^*\Theta^p_{A\times_B A}\left(\pi_1\right)^* \arrow[d, dashed, "\wr"] & \\
\tilde{q}^*\Theta^p_B q_! \arrow[dd, "\wr"] & (\tilde{\pi}_2)_!\Theta^p_{A\times_B A}\left(\pi_1\right)^* \arrow[ur, dashed, "\mu_{\tilde{\delta}}"] \arrow[dr, dashed, "\mu_\delta"'] \arrow[dd, dashed] & (\tilde{\pi}_2)_!\tilde{\delta}_!\Theta^p_A\delta^*\left(\pi_1\right)^* \arrow[r, dashed, "\sim"] \arrow[d, dashed] & \Theta^p_A. \\
 & & (\tilde{\pi}_2)_!\Theta^p_{A\times_B A}\delta_!\delta^*\left(\pi_1\right)^* \arrow[d, dashed] & \\
\Theta^p_A q^* q_! \arrow[r, "\beta^{-1}"] & \Theta^p_A\left(\pi_2\right)_!\left(\pi_1\right)^* \arrow[r, "\mu_\delta"] & \Theta^p_A\left(\pi_2\right)_!\delta_!\delta^*\left(\pi_1\right)^* \arrow[uur, bend right, "\sim"'] &
\end{tikzcd}$$

By Lemma 2.2.11(1), it suffices to show that the above (solid) diagram commutes. As in the proof of Theorem 3.2.3, all the parts except for the rectangle on the left and the triangle in the middle, commute for formal reasons. Since the operation $(-)\wr C_p\colon \mathcal{S}\to\mathcal{S}$ preserves fiber products, $\tilde{\delta}$ can be identified with the diagonal of $\tilde{q}$. By Lemma 3.4.5, the $\mathrm{BC}_!$ map in the middle triangle is an isomorphism. Thus, the middle triangle commutes by the inductive hypothesis and Lemma 2.2.11(2). As for the rectangle, we apply a similar argument to the one in Theorem 3.2.3, using again that the functor $(-)\wr C_p$ preserves fiber products, and the commutative cubical diagram

$$\begin{tikzcd}
\operatorname{Fun}(B,\mathcal{C}) \arrow[rr, "\Theta^p_B"] \arrow[dr, "q^*"] \arrow[dd, "q^*"'] \arrow[drrr, dotted] & & \operatorname{Fun}(B\wr C_p,\mathcal{C}) \arrow[dr, "\tilde{q}^*"] \arrow[dd, "\tilde{q}^*" near start] & \\
 & \operatorname{Fun}(A,\mathcal{C}) \arrow[rr, "\Theta^p_A" near start] \arrow[dd, "\pi_2^*" near start] & & \operatorname{Fun}(A\wr C_p,\mathcal{C}) \arrow[dd, "\tilde{\pi}_2^*"] \\
\operatorname{Fun}(A,\mathcal{C}) \arrow[rr, "\Theta^p_A" near end] \arrow[dr, "\pi_1^*"'] \arrow[drrr, dotted] & & \operatorname{Fun}(A\wr C_p,\mathcal{C}) \arrow[dr, "\tilde{\pi}_1^*"] & \\
 & \operatorname{Fun}(A\times_B A,\mathcal{C}) \arrow[rr, "\Theta^p_{A\times_B A}"] & & \operatorname{Fun}(A\times_B A\wr C_p,\mathcal{C}).
\end{tikzcd} \qquad (\spadesuit)$$

$\square$

Theorem 3.4.7. *Let $\mathcal{C}$ be an m-semiadditively symmetric monoidal ∞-category and $q\colon A\to B$ an m-finite map of spaces. For every $X,Y\in\operatorname{Fun}(B,\mathcal{C})$ and $f\colon q^*X\to q^*Y$, we have*

$$\Theta^p_B\left(\int_q f\right)=\int_{q\wr C_p}\Theta^p_A(f)\quad\in \hom_{h\operatorname{Fun}(B\wr C_p,\mathcal{C})}\left(\Theta^p X,\Theta^p Y\right).$$

Proof. By Lemma 3.4.5, the Θ^p-square satisfies the BC conditions, and by Proposition 3.4.6, it is ambidextrous. Thus, the claim follows from Proposition 2.2.12. $\square$

Additivity of Theta

We now investigate the interaction of Θ^p with addition of morphisms. Let $\mathcal{C}$ be a 0-semiadditively symmetric monoidal ∞-category. Given two objects $X, Y \in \mathcal{C}$ and two maps $f, g\colon X \to Y$, we can express $f+g$ as an integral of the pair (f,g) over $q\colon \mathrm{pt} \sqcup \mathrm{pt} \to \mathrm{pt}$ (see Example 3.1.12). Applying Theorem 3.4.7 to this special case and analyzing the result, we will derive a formula of the form

$$\Theta^p(f+g) = \Theta^p(f) + \Theta^p(g) + \text{"induced terms"}.$$

The Θ^p-square for $q\colon \mathrm{pt} \sqcup \mathrm{pt} \to \mathrm{pt}$ is

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{pt}, \mathcal{C}) \arrow[r, "\Theta^p_{\mathrm{pt}}"] \arrow[d, "q^*"] & \mathrm{Fun}(BC_p, \mathcal{C}) \arrow[d, "(q \wr C_p)^*"] \\
\mathrm{Fun}(\mathrm{pt} \sqcup \mathrm{pt}, \mathcal{C}) \arrow[r, "\Theta^p_{\mathrm{pt} \sqcup \mathrm{pt}}"] & \mathrm{Fun}((\mathrm{pt} \sqcup \mathrm{pt}) \wr C_p, \mathcal{C}).
\end{tikzcd} \qquad (*)$$

Our first goal is to make this diagram more explicit. First, we can identify q^* with the diagonal

$$\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}.$$

Next, let S be the set

$$S = \{w \in \{x,y\}^p \mid w \neq x^p, y^p\},$$

with x, y formal variables and let $\overline{S}$ is the set of orbits of S under the action of C_p by cyclic shift. We have a homotopy equivalence of spaces

$$(\mathrm{pt} \sqcup \mathrm{pt}) \wr C_p \simeq BC_p \sqcup BC_p \sqcup \overline{S},$$

and therefore an equivalence of ∞-categories

$$\mathrm{Fun}((\mathrm{pt} \sqcup \mathrm{pt}) \wr C_p, \mathcal{C}) \simeq \mathcal{C}^{BC_p} \times \mathcal{C}^{BC_p} \times \prod_{\overline{w} \in \overline{S}} \mathcal{C}.$$

Choosing a base point map $e\colon \mathrm{pt} \to BC_p$, we see that up to homotopy, we have

$$q \wr C_p = (\mathrm{Id}, \mathrm{Id}, e, \dots, e).$$

Similarly, the bottom arrow of $(*)$ can be identified with a functor

$$\Phi\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}^{BC_p} \times \mathcal{C}^{BC_p} \times \prod_{\overline{w} \in \overline{S}} \mathcal{C},$$

which we now describe. For each $\overline{w} \in \overline{S}$, let

$$e_{\overline{w}}\colon \mathrm{pt} \to (\mathrm{pt} \sqcup \mathrm{pt}) \wr C_p$$

be the map choosing the point $\overline{w} \in \overline{S}$ and let $e_w\colon \mathrm{pt} \to \overline{w}$ be the map choosing the point $w \in \overline{w}$. Given an element $w \in \{x,y\}^p$ we define a functor $w(-,-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as the composition

$$\mathrm{Fun}(\mathrm{pt} \sqcup \mathrm{pt}, \mathcal{C}) \xrightarrow{\Delta} \mathrm{Fun}(\mathrm{pt} \sqcup \mathrm{pt}, \mathcal{C})^p \xrightarrow{\boxtimes} \mathrm{Fun}((\mathrm{pt} \sqcup \mathrm{pt})^p, \mathcal{C}) \xrightarrow{w^*} \mathrm{Fun}(\mathrm{pt}, \mathcal{C}).$$

Informally, for objects $X, Y \in \mathcal{C}$, we have

$$w(X,Y) = Z_1 \otimes Z_2 \otimes \cdots \otimes Z_p, \qquad Z_i = \begin{cases} X & w_i = x \\ Y & w_i = y \end{cases}.$$

Lemma 3.4.8. *There is a natural isomorphism of functors*

$$\Phi \simeq \left(\Theta^p \circ p_1, \Theta^p \circ p_2, \{w(-,-)\}_{\overline{w} \in S}\right),$$

where $p_i \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the projection to the i-th component (it does not matter which representative w we take for each $\overline{w} \in S$).

Proof. The claim about the first two components follows from the commutativity of the Θ^p-square applied to the two inclusion maps $\mathrm{pt} \hookrightarrow \mathrm{pt} \sqcup \mathrm{pt}$. The pullback square

$$\begin{tikzcd}
\overline{w} \arrow[r] \arrow[d] & (\mathrm{pt} \sqcup \mathrm{pt})^p \arrow[d, "\pi"] \\
\mathrm{pt} \arrow[r, "e_{\overline{w}}"] & (\mathrm{pt} \sqcup \mathrm{pt}) \wr C_p
\end{tikzcd}$$

induces the commutative square in the following diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{pt} \sqcup \mathrm{pt}, \mathcal{C}) \arrow[r, "\Theta^p_{\mathrm{pt} \sqcup \mathrm{pt}}"] & \mathrm{Fun}((\mathrm{pt} \sqcup \mathrm{pt}) \wr C_p, \mathcal{C}) \arrow[r, "\pi^*"] \arrow[d, "e^*_{\overline{w}}"] & \mathrm{Fun}((\mathrm{pt} \sqcup \mathrm{pt})^p, \mathcal{C}) \arrow[d] \arrow[dr, dashed, "w^*"] & \\
 & \mathrm{Fun}(\mathrm{pt}, \mathcal{C}) \arrow[r, "\Delta"] & \mathrm{Fun}(\overline{w}, \mathcal{C}) \arrow[r, "e^*_w"] & \mathrm{Fun}(\mathrm{pt}, \mathcal{C}).
\end{tikzcd}$$

Observe that the composition of the leftmost horizontal functor and the left vertical functor is the $\overline{w}$ component of Φ. Since the composition of the two bottom horizontal functors is the identity, it suffices to show that the resulting functor

$$\mathrm{Fun}(\mathrm{pt} \sqcup \mathrm{pt}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{pt}, \mathcal{C}),$$

obtained from the composition along the entire bottom path of the diagram, is naturally isomorphic to $w(-,-)$. Since the diagram commutes, this is isomorphic to the composition along the top path of the diagram, which is $w(-,-)$ by definition. ☐

Summing up, we have identified the Θ^p-square $(*)$ with the following square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Theta^p"] \arrow[d, "\Delta"] & \mathcal{C}^{BC_p} \arrow[d, "(\mathrm{Id}, \mathrm{Id}, e, \dots, e)^*"] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "\left(\Theta^p \circ p_1, \Theta^p \circ p_2, \{w(-,-)\}_{\overline{w} \in \overline{S}}\right)"] & \mathcal{C}^{BC_p} \times \mathcal{C}^{BC_p} \times \prod_{\overline{w} \in \overline{S}} \mathcal{C}.
\end{tikzcd} \qquad (**)$$

Using this we can compute the effect of Θ^p on the sum of two maps.

Proposition 3.4.9. *Let $\mathcal{C}$ be a 0-semiadditively symmetric monoidal ∞-category, Given $X, Y \in \mathcal{C}$ and a pair of maps $f, g \colon X \to Y$, we have*

$$\Theta^p (f+g) = \Theta^p (f) + \Theta^p (g) + \sum_{\overline{w} \in \overline{S}} \left(\int_e w(f,g) \right).$$

Proof. The pair (f,g) can be considered as a map $(f,g) : q^* X \to q^* Y$. By Theorem 3.4.7 and the additivity of the integral (Proposition **??**) we have

$$\Theta^p (f+g) = \Theta^p \left(\int_q (f,g) \right) = \int_{(\mathrm{Id}, \mathrm{Id}, e, \ldots, e)} \left(\Theta^p (f), \Theta^p (g), \{ w(f,g) \}_{\overline{w} \in \overline{S}} \right)$$

$$= \Theta^p (f) + \Theta^p (g) + \sum_{\overline{w} \in \overline{S}} \left(\int_e w(f,g) \right).$$

□

4 Higher Semiadditivity and Additive Derivations

Let $\mathcal{C}$ be a *stable* symmetric monoidal ∞-category and p a fixed prime. For every pair

$$X \in \mathrm{coCAlg}(\mathcal{C}), \quad Y \in \mathrm{CAlg}(\mathcal{C}),$$

the set

$$\hom_{h\mathcal{C}} (X, Y) = \pi_0 \operatorname{Map}_{\mathcal{C}} (X, Y)$$

has a structure of a commutative ring. Assuming further that $\mathcal{C}$ is 1-semiadditively symmetric monoidal, we will construct in this section an operation (which depends on p)

$$\delta \colon \hom_{h\mathcal{C}} (X, Y) \to \hom_{h\mathcal{C}} (X, Y),$$

and show that it is an "additive p-derivation". We begin with a general discussion of the algebraic notion of an additive p-derivation. We proceed to construct an auxiliary operation α (which does not require stability) and study its properties. We then specialize to the stable case, construct the operation δ above and study it's behavior on elements of the form $[A]$. Finally, we shall use the properties of the operation δ to provide a general criterion that allows to deduce ∞-semiadditivity of a presentably symmetric monoidal, 1-semiadditive, stable, p-local ∞-category.

4.1 Additive p-derivations

This section is devoted to the algebraic notion of an additive p-derivation. We recall the definition and establish some of its basic properties.

Definition & Properties

The following is a variant on the notion of a p-derivation (e.g. see [Bui05, Definition 2.1]), in which we do not require the multiplicative property.

Definition 4.1.1. Let R be a commutative ring. An *additive p-derivation* on R, is a function of sets

$$\delta\colon R \to R,$$

that satisfies:

(1) (additivity) $\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$ for all $x, y \in R$.

(2) (normalization) $\delta(0) = \delta(1) = 0$.

The pair (R, δ) is called a *semi-δ-ring*. A *semi-δ-ring homomorphism* from (R, δ) to (R', δ'), is a ring homomorphism $f\colon R \to R'$, that satisfies $f \circ \delta = \delta' \circ f$.

Remark 4.1.2. The expression

$$\frac{x^p + y^p - (x+y)^p}{p}$$

is actually a polynomial with integer coefficients in the variables x and y and does not involve division by p. In particular, this is well defined for all $x, y \in R$, even when R has p-torsion.

Remark 4.1.3. In fact, the condition $\delta(0) = 0$ is superfluous, as it follows from the additivity property, and we include it in the definition only for emphasis.

The following follows immediately from the definitions.

Lemma 4.1.4. *Let $\delta\colon R \to R$ be an additive p-derivation on a commutative ring R. The function $\psi\colon R \to R$ given by*

$$\psi(x) = x^p + p\delta(x)$$

is an additive lift of Frobenius. i.e. it is a homomorphism of abelian groups and agrees with the Frobenius modulo p.

Example 4.1.5. The following are some examples of additive p-derivations.

(1) For R a subring of $\mathbb{Q}$, the *Fermat quotient*

$$\tilde{\delta}(x) = \frac{x - x^p}{p}$$

is an additive p-derivation (we shall soon show that it is the *unique* additive p-derivation on any such R).

(2) The same formula as for the Fermat quotient defines the unique additive p-derivation on the ring of p-adic integers $\mathbb{Z}_p$.

(3) Fix $m \geq 1$. Let $\mathcal{R}_m^{\square}$ be the commutative ring freely generated by formal elements $[A]$, where A is an m-finite space, subject to the relations

$$[A \sqcup B] = [A] + [B]\,, \quad [A \times B] = [A]\,[B]\,.$$

It is easy to verify that the operation

$$\delta\left([A]\right) = [BC_p \times A] - [A \wr C_p]$$

is well defined and is an additive p-derivation on $\mathcal{R}_m^{\square}$.

Definition 4.1.6. For every $x \in \mathbb{Q}$, we denote by $v_p\left(x\right) \in \mathbb{Z} \cup \{\infty\}$ the p-adic valuation of x.

The fundamental property of the Fermat quotient is

Lemma 4.1.7. *For every $x \in \mathbb{Q}$, if $0 < v_p\left(x\right) < \infty$, then*

$$v_p\left(\tilde{\delta}\left(x\right)\right) = v_p\left(x\right) - 1.$$

Proof. Since $v_p\left(x\right) > 0$, we have

$$v_p\left(x^p\right) = p v_p\left(x\right) > v_p\left(x\right).$$

Thus,

$$v_p\left(\frac{x - x^p}{p}\right) = v_p\left(x - x^p\right) - 1 = v_p\left(x\right) - 1.$$

$\square$

Definition 4.1.8. Let R be a commutative ring. Let $\phi_0\colon \mathbb{Z} \to R$ be the unique ring homomorphism and let S_R be the set of primes p, such that $\phi_0\left(p\right) \in R^{\times}$. We denote

$$\mathbb{Q}_R = \mathbb{Z}\left[S_R^{-1}\right] \subseteq \mathbb{Q}$$

and $\phi\colon \mathbb{Q}_R \to R$, the unique extension of ϕ_0. We call an element $x \in R$ *rational* if it is in the image of ϕ. By Example 4.1.5, $\left(\mathbb{Q}_R, \tilde{\delta}\right)$ is a semi-δ-ring.

The following elementary lemma will have several useful consequences.

Lemma 4.1.9. *Let (R, δ) be a semi-δ-ring and let $\tilde{\delta}$ denote the Fermat quotient on $\mathbb{Q}_R$. For all $t \in \mathbb{Q}_R$ and $x \in R$, we have*

$$\delta\left(tx\right) = t\delta\left(x\right) + \tilde{\delta}\left(t\right) x^p.$$

Proof. Fix $x \in R$ and consider the function $\varphi\colon \mathbb{Q}_R \to R$ given by

$$\varphi\left(t\right) = \delta\left(tx\right) - \tilde{\delta}\left(t\right) x^p.$$

Since

$$\delta\left(tx + sx\right) = \delta\left(tx\right) + \delta\left(sx\right) + \frac{\left(tx\right)^p + \left(sx\right)^p - \left(tx + sx\right)^p}{p} =$$
$$\delta\left(tx\right) + \delta\left(sx\right) + \left(\tilde{\delta}\left(t + s\right) - \tilde{\delta}\left(t\right) - \tilde{\delta}\left(s\right)\right) x^p = \varphi\left(t\right) + \varphi\left(s\right) + \tilde{\delta}\left(t + s\right) x^p.$$

we get

$$\varphi\left(t + s\right) = \delta\left(tx + sx\right) - \tilde{\delta}\left(t + s\right) x^p = \varphi\left(t\right) + \varphi\left(s\right).$$

Hence, φ is additive and $\varphi\left(1\right) = \delta\left(x\right)$. Since $\mathbb{Q}_R$ is a localization of $\mathbb{Z}$, φ is a map of $\mathbb{Q}_R$-modules and we deduce that $\varphi\left(t\right) = t\delta\left(x\right)$ for all $t \in \mathbb{Q}_R$. $\square$

p-Local Rings

In the case where R is a *p-local* commutative ring, which is the case we are mainly interested in, the existence of an additive p-derivation on R has several interesting implications.

Proposition 4.1.10. *Let (R, δ) be a p-local semi-δ-ring. If $x \in R$ is torsion, then x is nilpotent.*

Proof. Since R is p-local, if x is torsion, then there is $d \in \mathbb{N}$, such that $p^d x = 0$. By Lemma 4.1.9, we have

$$0 = \delta(0) = \delta\left(p^d x\right) = p^d \delta(x) + \tilde{\delta}\left(p^d\right) x^p.$$

Multiplying by x, we obtain $\tilde{\delta}\left(p^d\right) x^{p+1} = 0$. By Lemma 4.1.7, $v_p\left(\tilde{\delta}\left(p^d\right)\right) = d - 1$, and since R is p-local, we get $p^{d-1} x^{p+1} = 0$. Iterating this d times we get $x^{(p+1)^d} = 0$. $\square$

Proposition 4.1.11. *Let (R, δ) be a non-zero p-local semi-δ-ring. The map $\phi\colon \mathbb{Q}_R \to R$ is an injective semi-δ-ring homomorphism. In particular $\tilde{\delta}$ is the unique additive p-derivation on $\mathbb{Q}_R$.*

Proof. Applying Lemma 4.1.9 to $x = 1$, we see that $\phi \circ \tilde{\delta} = \delta \circ \phi$. If ϕ is non-injective, then so is $\phi_0\colon \mathbb{Z} \to R$ and hence $1 \in R$ is torsion. By Proposition 4.1.10, 1 is nilpotent and hence $R = 0$. $\square$

Remark 4.1.12. For a non-zero p-local semi-δ-ring (R, δ), we abuse notation by identifying $\mathbb{Q}_R$ with the subset of rational elements of R. There are two options:

(1) If $p \in R^\times$, then $\mathbb{Q}_R = \mathbb{Q} \subseteq R$ and all non-zero rational elements are invertible.

(2) If $p \notin R^\times$, then $\mathbb{Q}_R = \mathbb{Z}_{(p)} \subseteq R$, and $x \in \mathbb{Q}_R$ is invertible if and only if $v_p(x) = 0$.

Proposition 4.1.13. *Let (R, δ) be a p-local semi-δ-ring. The ideal $I_{\mathrm{tor}} \subseteq R$ of torsion elements is closed under δ.*

Proof. For $x \in I_{\mathrm{tor}}$, there is $d \in \mathbb{N}$, such that $p^d x = 0$. By Lemma 4.1.9,

$$0 = \delta\left(p^{d+1} x\right) = p^{d+1} \delta(x) + \tilde{\delta}\left(p^{d+1}\right) x^p.$$

By Lemma 4.1.7, $v_p\left(\tilde{\delta}\left(p^{d+1}\right)\right) = d$ and therefore $\tilde{\delta}\left(p^{d+1}\right) x^p = 0$. We get $p^{d+1}\delta(x) = 0$ and hence $\delta(x) \in I_{\mathrm{tor}}$. $\square$

Definition 4.1.14. For every commutative ring R, we define $I_{\mathrm{tor}} \subseteq R$ to be the ideal of torsion elements, and $R^{\mathrm{tf}} = R/I_{\mathrm{tor}}$ to be the torsion free ring obtained from R.

The following proposition will allow us to "ignore torsion" when dealing with questions of invertibility in p-local semi-δ-ring. First,

Definition 4.1.15. Given a ring homomorphism $f\colon R \to S$, we say that f *detects invertibility* if for every $x \in R$, if $f(x)$ is invertible, then x is invertible.

Proposition 4.1.16. *Let (R, δ) be a p-local semi-δ-ring. There is a unique additive p-derivation δ on R^{tf}, such that the quotient map $g\colon R \twoheadrightarrow R^{\mathrm{tf}}$ is a homomorphism of semi-δ-rings. In addition, g detects invertibility.*

Proof. Let $x \in R$ and $y \in I_{\mathrm{tor}}$. We have

$$\delta(x+y) - \delta(x) = \delta(y) + \left(\frac{x^p + y^p - (x+y)^p}{p}\right) \in I_{\mathrm{tor}}$$

since $\delta(y) \in I_{\mathrm{tor}}$ by Proposition 4.1.13 and the expression in parenthesis is a multiple of y. Thus,

$$\delta(x + I_{\mathrm{tor}}) := \delta(x) + I_{\mathrm{tor}}$$

is a well defined function on R^{tf}. The operation δ is an additive p-derivation and makes g a homomorphism of semi-δ-rings. The operation δ is unique by the surjectivity of g. For the second claim, the kernel of g consists of nilpotent elements by Proposition 4.1.10 and hence g detects invertibility. $\square$

4.2 The Alpha Operation

Let $\mathcal{C}$ be a 0-semiadditively symmetric monoidal ∞-category and let

$$X \in \mathrm{coCAlg}(\mathcal{C}), \qquad Y \in \mathrm{CAlg}(\mathcal{C}).$$

Fix a prime p. The set

$$\hom_{h\mathcal{C}}(X,Y) = \pi_0 \operatorname{Map}_{\mathcal{C}}(X,Y)$$

has a structure of a *commutative rig* (i.e. like a ring, but without additive inverses). Assuming further that $\mathcal{C}$ is 1-semiadditively symmetric monoidal, we construct an operation α (which depends on p) on $\hom_{h\mathcal{C}}(X,Y)$ and study its properties and interaction with the rig structure.

Throughout the section we denote

$$\mathrm{pt} \xrightarrow{e} BC_p \xrightarrow{r} \mathrm{pt}.$$

Definition and Naturality

The $\mathbb{E}_\infty$-coalgebra and $\mathbb{E}_\infty$-algebra structures, on X and Y respectively, provide symmetric comultiplication and multiplication maps:

$$\bar{t}_X \colon X \to \left(X^{\otimes p}\right)^{hC_p} = r_* \Theta^p(X)$$

$$\overline{m}_Y \colon r_! \Theta^p(Y) = \left(Y^{\otimes p}\right)_{hC_p} \to Y.$$

These maps have mates

$$t_X \colon r^* X \to \Theta^p(X), \qquad m_Y \colon \Theta^p(Y) \to r^* Y,$$

such that

$$e^* t_X \colon X = e^* r^* X \to e^* \Theta^p(X) = X^{\otimes p}$$

$$e^* m_Y \colon Y^{\otimes p} = e^* \Theta^p(Y) \to e^* r^* Y = Y,$$

are the ordinary comultiplication and multiplication maps.

Definition 4.2.1. Let $\mathcal{C}$ be a 1-semiadditively symmetric monoidal ∞-category and let

$$X \in \mathrm{coCAlg}(\mathcal{C}), \qquad Y \in \mathrm{CAlg}(\mathcal{C}).$$

(1) Given $g\colon \Theta^p(X) \to \Theta^p(Y)$, we define $\overline{\alpha}(g) : X \to Y$ to be either of the compositions in the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\bar{t}_X"] & r_*\Theta^p(X) \arrow[r, "\mathrm{Nm}_r^{-1}"] \arrow[d, "g"] & r_!\Theta^p(X) \arrow[d, "g"] & \\
& r_*\Theta^p(Y) \arrow[r, "\mathrm{Nm}_r^{-1}"] & r_!\Theta^p(Y) \arrow[r, "\overline{m}_Y"] & Y.
\end{tikzcd}$$

(2) Given $f\colon X \to Y$, we define $\alpha(f) = \overline{\alpha}(\Theta^p(f))$.

Remark 4.2.2. In fact, the definition of α uses only the H_∞-algebra structure of Y and the H_∞-coalgebra structure of X. Moreover, everything we state and prove in this section about the properties of α holds when we replace $\mathbb{E}_\infty$ with H_∞.

Remark 4.2.3. The operation $\overline{\alpha}$ is additive, as it is composed of applying additive functors and pre/post-composition with fixed maps in a 0-semiadditive ∞-category.

The operation α is natural with respect to (co)algebra homomorphisms in the following sense.

Lemma 4.2.4. *Let $\mathcal{C}$ be a 1-semiadditively symmetric monoidal ∞-category and let*

$$X, X' \in \mathrm{coCAlg}(\mathcal{C}), \qquad Y, Y' \in \mathrm{CAlg}(\mathcal{C}).$$

Given maps $g\colon Y \to Y'$ and $h\colon X' \to X$ of commutative algebras and coalgebras respectively, for every map $f\colon X \to Y$, we have

$$\alpha(g \circ f \circ h) = g \circ \alpha(f) \circ h \quad \in \hom_{h\mathcal{C}}(X', Y').$$

Proof. Consider the diagram

$$\begin{tikzcd}
X' \arrow[r, dotted, "\bar{t}_{X'}"] \arrow[d, dotted, "h"] & r_*\Theta^p X' \arrow[r, "\mathrm{Nm}_r^{-1}"] \arrow[d, "h"] \arrow[dddr, dotted] & r_!\Theta^p X' \arrow[d, "h"] & \\
X \arrow[r, "\bar{t}_X"] \arrow[drrr, dotted] & r_*\Theta^p X \arrow[r, "\mathrm{Nm}_r^{-1}"] \arrow[d, "f"] & r_!\Theta^p X \arrow[d, "f"] & \\
& r_*\Theta^p Y \arrow[r, "\mathrm{Nm}_r^{-1}"] \arrow[d, "g"] & r_!\Theta^p Y \arrow[r, "\overline{m}_Y"] \arrow[d, "g"] & Y \arrow[d, dotted, "g"] \\
& r_*\Theta^p Y' \arrow[r, "\mathrm{Nm}_r^{-1}"] & r_!\Theta^p Y' \arrow[r, dotted, "\overline{m}_{Y'}"] & Y'.
\end{tikzcd}$$

The squares in the middle column commute by the naturality of the norm map. The homotopy rendering the bottom right square commutative is provided by the data that makes g into a morphism of commutative algebras and similarly for the upper left square and h. The composition along one of the dotted paths is $\alpha(g \circ f \circ h)$, while composition along the other dotted path is $g \circ \alpha(f) \circ h$, which completes the proof. $\square$

The operation α is also functorial in the following sense.

Lemma 4.2.5. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a 1-semiadditive symmetric monoidal functor between 1-semiadditively symmetric monoidal ∞-categories, and let $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$. The induced map of commutative rings*

$$F\colon\ \hom_{h\mathcal{C}}(X, Y) \to \hom_{h\mathcal{D}}(FX, FY),$$

commutes with the operation α.

Proof. Given a map $g\colon X \to Y$, consider the following diagram:

$$\begin{tikzcd}
 & F\left(r_*\Theta^p(X)\right) \arrow[r, "\mathrm{Nm}_r^{-1}"] \arrow[d, "\beta_*", "\wr"'] & F\left(r_!\Theta^p(X)\right) \arrow[r, "g"] & F\left(r_!\Theta^p(Y)\right) \arrow[dr, "\overline{m}_Y"] & \\
FX \arrow[ur, "\bar{t}_X"] \arrow[dr, "\bar{t}_{FX}"'] & r_*F\left(\Theta^p(X)\right) \arrow[r, "\mathrm{Nm}_r^{-1}"] \arrow[d, no head, "\wr"] & r_!F\left(\Theta^p(X)\right) \arrow[u, "\beta_!"', "\wr"] \arrow[r, "g"] \arrow[d, no head, "\wr"] & r_!F\left(\Theta^p(Y)\right) \arrow[u, "\beta_!"', "\wr"] \arrow[d, no head, "\wr"] & FY. \\
 & r_*\Theta^p(FX) \arrow[r, "\mathrm{Nm}_r^{-1}"] & r_!\Theta^p(FX) \arrow[r, "g"] & r_!\Theta^p(FY) \arrow[ur, "\overline{m}_{FY}"'] &
\end{tikzcd}$$

The bottom squares are defined by Lemma 3.4.2 and commute by the interchange law. The top left square commutes by the ambidexterity of the (F, r)-square (Theorem 3.2.3) and the top right square by naturality of the $\mathrm{BC}_!$ map. The triangles commute by the definition of the commutative coalgebra (resp. algebra) structure on $F(X)$ (resp. $F(Y)$). Thus, the composition along the top path, which is $F(\alpha(g))$, is homotopic to the composition along the bottom path, which is $\alpha(F(g))$. $\square$

Additivity of Alpha

Our next goal is to understand the interaction of α with sums. For this we first need to describe the effect of $\overline{\alpha}$ on "induced maps". Recall the notation

$$\mathrm{pt} \xrightarrow{e} BC_p \xrightarrow{r} \mathrm{pt}\,.$$

Lemma 4.2.6. *Let $\mathcal{C}$ be a 1-semiadditively symmetric monoidal ∞-category and let $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$. For every map*

$$h\colon X^{\otimes p} = e^*\Theta^p(X) \to e^*\Theta^p(Y) = Y^{\otimes p},$$

the map $\overline{\alpha}\left(\int\limits_e h\right)$ is homotopic to the composition

$$X \xrightarrow{e^*t_X} X^{\otimes p} \xrightarrow{h} Y^{\otimes p} \xrightarrow{e^*m_Y} Y.$$

Proof. Unwinding the definition of the integral, the map $\int\limits_e h$ is homotopic to the composition of the following maps

$$\Theta^p(X) \xrightarrow{u_*^e} e_*e^*\Theta^p(X) \xrightarrow{h} e_*e^*\Theta^p(X) \xrightarrow{\mathrm{Nm}_e^{-1}} e_!e^*\Theta^p(X) \xrightarrow{c_!^e} \Theta^p(X)\,.$$

Plugging this into the definition of $\overline{\alpha}$, we get that $\overline{\alpha}\left(\int\limits_e h\right)$ equals the composition along the top and then right path in the following diagram

$$\begin{tikzcd}
X \arrow[r, "\bar{t}_X"] \arrow[drr, "e^* t_X"'] & r_*\Theta^p(X) \arrow[r, "u_*^e"] & r_*e_*e^*\Theta^p(X) \arrow[r, "h"] \arrow[d, equal] & r_*e_*e^*\Theta^p(Y) \arrow[r, "\mathrm{Nm}_e^{-1}"] \arrow[d, equal] & r_*e_!e^*\Theta^p(Y) \arrow[r, "c_!^e"] \arrow[d, "\mathrm{Nm}_r^{-1}"] & r_*\Theta^p(Y) \arrow[d, "\mathrm{Nm}_r^{-1}"] \\
 & & e^*\Theta^p(X) \arrow[r, "h"] & e^*\Theta^p(Y) \arrow[r, equal] \arrow[drr, "e^* m_Y"'] & r_!e_!e^*\Theta^p(Y) \arrow[r, "c_!^e"] & r_!\Theta^p(Y) \arrow[d, "\overline{m}_Y"] \\
 & & & & & Y.
\end{tikzcd}$$

We denote this diagram by $(*)$. The left square commutes for trivial reasons, the right square by the interchange law and the middle by

$$\mathrm{Nm}_r^{-1} \circ \mathrm{Nm}_e^{-1} = (\mathrm{Nm}_e \circ \mathrm{Nm}_r)^{-1} = (\mathrm{Nm}_{re})^{-1} = \mathrm{Id}\,.$$

To see that the left triangle commutes, consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\bar{t}_X"] \arrow[dr, "u_*^r"'] & r_*\Theta^p(X) \arrow[r, "u_*^e"] & r_*e_*e^*\Theta^p(X) & \\
 & r_*r^*X \arrow[u, "t_X"] \arrow[r, "u_*^e"] & r_*e_*e^*r^*X \arrow[u, "t_X"] \arrow[r, equal] & X. \arrow[ul, "e^* t_X"']
\end{tikzcd}$$

The square commutes by naturality, and the left triangle by the definition of mates. Note that the composition along the bottom path is the unit of the composed adjunction

$$\mathrm{Id} = e^*r^* \dashv r_*e_* = \mathrm{Id},$$

and hence is the identity map. It follows that the left triangle in $(*)$ is commutative. The proof that the right triangle in $(*)$ commutes is completely analogous. Thus, $(*)$ is commutative and $\overline{\alpha}\left(\int\limits_e h\right)$ equals the composition along the bottom diagonal path in $(*)$, which completes the proof. $\square$

The main property of α is that it satisfies the following "addition formula".

Proposition 4.2.7. *Let $\mathcal{C}$ be a 1-semiadditively symmetric monoidal ∞-category and let*

$$X \in \mathrm{coCAlg}(\mathcal{C}), \quad Y \in \mathrm{CAlg}(\mathcal{C})\,.$$

For every $f, g\colon X \to Y$, we have

$$\alpha(f+g) = \alpha(f) + \alpha(g) + \frac{(f+g)^p - f^p - g^p}{p} \quad \in \hom_{h\mathcal{C}}(X, Y)$$

(as in Remark 4.1.2, this expression does not actually involve division by p).

Proof. Since $\overline{\alpha}$ is additive (see Remark 4.2.3), we get by Proposition 3.4.9,

$$\alpha(f+g)=\overline{\alpha}(\Theta^p(f+g))=\overline{\alpha}\left(\Theta^p(f)+\Theta^p(g)+\sum_{\overline{w}\in\overline{S}}\left(\int_e w(f,g)\right)\right)$$

$$=\alpha(f)+\alpha(g)+\sum_{\overline{w}\in\overline{S}}\overline{\alpha}\left(\int_e w(f,g)\right).$$

Now, by Lemma 4.2.6, the map $\overline{\alpha}\left(\int_e w(f,g)\right)$ is homotopic to the composition

$$X\xrightarrow{e^*t_X}X^{\otimes p}\xrightarrow{w(f,g)}Y^{\otimes p}\xrightarrow{e^*m_Y}Y.$$

This is by definition $f^{w_x}g^{w_y}$, where w_x and w_y are the number of x-s and y-s in w respectively and this completes the proof. □

Alpha and The Unit

We shall now apply the above discussion of the operation α to the special case where $X=Y=\mathbb{1}$ is the unit of a symmetric monoidal ∞-category $\mathcal{C}$. The unit $\mathbb{1}\in\mathcal{C}$ has a unique $\mathbb{E}_\infty$-algebra structure and this structure makes it initial in $\mathrm{CAlg}(\mathcal{C})$. The same argument applied to $\mathcal{C}^{op}$ shows that $\mathbb{1}$ has also a unique $\mathbb{E}_\infty$-coalgebra structure and it is terminal with respect to it.

Definition 4.2.8. Let $(\mathcal{C},\otimes,\mathbb{1})$ be a symmetric monoidal ∞-category. We denote

$$R_{\mathcal{C}}=\hom_{h\mathcal{C}}(\mathbb{1},\mathbb{1})$$

as a commutative monoid. If $\mathcal{C}$ is 0-semiadditive, then R is naturally a commutative *rig* and if $\mathcal{C}$ is stable, then it is a commutative *ring*. Given a symmetric monoidal functor $F\colon\mathcal{C}\to\mathcal{D}$, the induced map $\varphi: R_{\mathcal{C}}\to R_{\mathcal{D}}$ is a monoid homomorphism. It is also a rig (resp. ring) homomorphism, when $\mathcal{C}$ and $\mathcal{D}$ are 0-semiadditive (resp. stable) and F is a 0-semiadditive functor.

The goal of this section is to study the operation α on $R_{\mathcal{C}}$. We begin with a few preliminaries. Recall the notation

$$\mathrm{pt}\xrightarrow{e}BC_p\xrightarrow{r}\mathrm{pt}\,.$$

Lemma 4.2.9. *Let $(\mathcal{C},\otimes,\mathbb{1})$ be a symmetric monoidal ∞-category. The action of C_p on $\mathbb{1}^{\otimes p}\simeq\mathbb{1}$ is trivial. Namely, $\Theta^p(\mathbb{1})=r^*\mathbb{1}$.*

Proof. We shall show more generally that the action of Σ_k on $\mathbb{1}^{\otimes k}\simeq\mathbb{1}$ is trivial. The forgetful functor $U\colon\mathrm{CAlg}(\mathcal{C})\to\mathcal{C}$ is symmetric monoidal with respect to the coproduct on $\mathrm{CAlg}(\mathcal{C})$ and $\otimes$ on $\mathcal{C}$ (by [Lur, Example 3.2.4.4] and [Lur, Proposition 3.2.4.7]). In particular, for every algebra A, the action of Σ_k on $U(A)^{\otimes k}$ is induced by the action of Σ_k on $A^{\sqcup k}$. Since $\mathbb{1}$ has a canonical commutative algebra structure, and as an object $\underline{\mathbb{1}}\in\mathrm{CAlg}(\mathcal{C})$ it is *initial* ([Lur, Corollary 3.2.1.9]), any Σ_k action on it as a commutative algebra is trivial. □

It follows by the above that $r_!\Theta^p(\mathbb{1}) \simeq r_! r^* \mathbb{1}$ and $r_*\Theta^p(\mathbb{1}) \simeq r_* r^* \mathbb{1}$.

Lemma 4.2.10. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category. The maps*

$$\bar{t}_{\mathbb{1}} \colon \mathbb{1} \to r_*\Theta^p(\mathbb{1}), \quad \overline{m}_{\mathbb{1}} \colon r_!\Theta^p(\mathbb{1}) \to \mathbb{1},$$

induced from the commutative algebra and coalgebra structures, are equivalent to the unit and counit maps (respectively)

$$u_* \colon \mathbb{1} \to r_* r^* \mathbb{1}, \qquad c_! \colon r_! r^* \mathbb{1} \to \mathbb{1}.$$

Proof. This is equivalent to showing that the mate (in both cases) is the identity map $r^*\mathbb{1} \to r^*\mathbb{1}$. The algebra structure on $\mathbb{1} \in \mathcal{C}$ is induced from the algebra structure on $\underline{\mathbb{1}} \in \mathrm{CAlg}(\mathcal{C})$, where $\mathrm{CAlg}(\mathcal{C})$ is endowed with the coCartesian symmetric monoidal structure and in which $\underline{\mathbb{1}}$ is initial ([Lur, Corollary 3.2.1.9]). Now, the object $r^*\underline{\mathbb{1}}$ is initial in $\mathrm{Fun}(BC_p, \mathrm{CAlg}(\mathcal{C}))$, and therefore the only map $r^*\underline{\mathbb{1}} \to r^*\underline{\mathbb{1}}$ is the identity. A similar argument applies for the comultiplication map. $\square$

As a consequence, we can describe the effect of α on any element of $R_{\mathcal{C}}$ using the integral operation.

Proposition 4.2.11. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a 1-semiadditively symmetric monoidal ∞-category. For every $f \in R_{\mathcal{C}}$, we have*

$$\alpha(f) = \int\limits_{BC_p} \Theta^p(f) \quad \in \mathcal{R}_{\mathcal{C}}.$$

Proof. Unwinding the definition of $\overline{\alpha}$ (Definition 4.2.1) in this case and using Lemma 4.2.10, we get $\overline{\alpha}(-) = \int\limits_{BC_p} (-)$. Hence,

$$\alpha(f) = \overline{\alpha}(\Theta^p(f)) = \int\limits_{r} \Theta^p(f) = \int\limits_{BC_p} \Theta^p(f).$$

$\square$

In particular, we get an explicit formula for the operation α on elements of the form $[A] \in \mathcal{R}_{\mathcal{C}}$.

Theorem 4.2.12. *Let $\mathcal{C}$ be an m-semiadditively symmetric monoidal ∞-category for $m \geq 1$. For every m-finite space A, we have*

$$\alpha([A]) = [A \wr C_p] \quad \in \mathcal{R}_{\mathcal{C}}.$$

Proof. Consider the maps

$$q \colon A \to \mathrm{pt}, \quad \pi = q \wr C_p \colon A \wr C_p \to BC_p, \quad r \colon BC_p \to \mathrm{pt}.$$

By definition of α, Proposition 4.2.11, the definition of $[A]$, the ambidexterity of the Θ^p-square (Theorem 3.4.7) and Fubini's Theorem (Proposition 2.1.15) (in that order) we have

$$\alpha([A]) = \overline{\alpha}(\Theta^p([A])) = \int\limits_{r} \Theta^p([A]) = \int\limits_{r} \Theta^p\left(\int\limits_{q} \mathrm{Id}_{\mathbb{1}}\right) = \int\limits_{r}\int\limits_{\pi} \Theta^p(\mathrm{Id}_{\mathbb{1}}) = \int\limits_{r\pi} \mathrm{Id}_{\mathbb{1}} = [A \wr C_p].$$

$\square$

As a consequence, we can identify the action of α on the identity element of the rig $\hom_{h\mathcal{C}}(X,Y)$, for any $X \in \mathrm{coCAlg}(\mathcal{C})$ and $Y \in \mathrm{CAlg}(\mathcal{C})$.

Lemma 4.2.13. *Let $\mathcal{C}$ be a 1-semiadditively symmetric monoidal ∞-category and let*

$$X \in \mathrm{coCAlg}(\mathcal{C}), \quad Y \in \mathrm{CAlg}(\mathcal{C}).$$

Denoting $\mathcal{R} = \hom_{h\mathcal{C}}(X,Y)$, we have

$$\alpha(1_R) = [BC_p] \circ 1_R \quad \in \mathcal{R},$$

where $1_R \in \mathcal{R}$ is the multiplicative unit element.

Proof. The map $1_R \colon X \to Y$ is the composition of the canonical maps $X \xrightarrow{x} \mathbb{1} \xrightarrow{y} Y$, encoding the counit and unit of the coalgebra and algebra structures of X and Y respectively. The maps x and y are naturally maps of commutative coalgebras and commutative algebras respectively. By Lemma 4.2.4, we have

$$\alpha(1_R) = \alpha(y \circ 1 \circ x) = y \circ \alpha(1) \circ x,$$

where $1 \in \mathcal{R}_{\mathcal{C}}$ is the multiplicative unit element. Observing that $1 = [\mathrm{pt}]$ and using Theorem 4.2.12, we get

$$y \circ \alpha(1) \circ x = y \circ \alpha([\mathrm{pt}]) \circ x = y \circ [BC_p] \circ x = [BC_p] \circ y \circ x = [BC_p] \circ 1_R.$$

$\square$

4.3 Higher Semiadditivity and Stability

In this section we specialize to the *stable* case. Using the operation α and stability, we construct additive p-derivations and use their properties to formulate a general detection principle for higher semiadditivity.

Stability and Additive p-Derivations

Definition 4.3.1. Let $\mathcal{C}$ be a *stable* 1-semiadditively symmetric monoidal ∞-category with

$$X \in \mathrm{coCAlg}(\mathcal{C}), \quad Y \in \mathrm{CAlg}(\mathcal{C}),$$

and so $R = \hom_{h\mathcal{C}}(X,Y)$ is a commutative *ring*. We define an operation $\delta \colon R \to R$ by

$$\delta(f) = [BC_p]\, f - \alpha(f),$$

for every $f \in R$. In particular, this applies to $R_{\mathcal{C}} = \hom_{h\mathcal{C}}(\mathbb{1},\mathbb{1})$.

Theorem 4.3.2. *Let $\mathcal{C}$ be a stable 1-semiadditively symmetric monoidal ∞-category with*

$$X \in \mathrm{coCAlg}(\mathcal{C}), \quad Y \in \mathrm{CAlg}(\mathcal{C}).$$

The operation δ from Definition 4.3.1 is an additive p-derivation on $R = \hom_{h\mathcal{C}}(X,Y)$.

Proof. The additivity condition follows from Proposition 4.2.7 and the normalization follows from Lemma 4.2.13. $\square$

The additive p-derivation of Theorem 4.3.2 is natural in the following sense.

Proposition 4.3.3. *Let $\mathcal{C}$ be a stable 1-semiadditively symmetric monoidal ∞-category with*

$$X, X' \in \mathrm{coCAlg}(\mathcal{C}), \qquad Y, Y' \in \mathrm{CAlg}(\mathcal{C}).$$

Given maps $g\colon Y \to Y'$ and $h\colon X' \to X$ of commutative algebras and coalgebras respectively, the function

$$g \circ - \circ h\colon \hom_{h\mathcal{C}}(X, Y) \to \hom_{h\mathcal{C}}(X', Y')$$

is a homomorphism of semi-δ-rings.

Proof. This follows from Lemma 4.2.4 and naturality of $[BC_p]$. □

The additive p-derivation of Theorem 4.3.2 is also functorial in the following sense.

Proposition 4.3.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal 1-semiadditive functor between* stable *1-semiadditively symmetric monoidal ∞-categories. Given*

$$X \in \mathrm{coCAlg}(\mathcal{C}), \quad Y \in \mathrm{CAlg}(\mathcal{C}),$$

the map

$$F\colon \hom_{h\mathcal{C}}(X, Y) \to \hom_{h\mathcal{D}}(FX, FY),$$

is a homomorphism of semi-δ-rings.

Proof. By Lemma 4.2.5, F preserves α, and by Corollary 3.2.6, F preserves multiplication by $[BC_p]$. Combined with ordinary additivity, it follows that F preserves δ. □

The theory of p-local semi-δ-rings has the following consequence for stable, p-local, 1-semiadditive ∞-categories.

Corollary 4.3.5. *Let $\mathcal{C}$ be a stable, p-local, 1-semiadditively symmetric monoidal ∞-category with*

$$X \in \mathrm{coCAlg}(\mathcal{C}), \qquad Y \in \mathrm{CAlg}(\mathcal{C}),$$

and consider the commutative ring $R = \hom_{h\mathcal{C}}(X, Y)$. Every torsion element of R is nilpotent. In particular, if $\mathbb{Q} \otimes R = 0$, then $R = 0$.

Proof. The commutative ring R is p-local and admits an additive p-derivation by Theorem 4.3.2, and so the result follows by Proposition 4.1.10. The last claim follows by considering the element $1 \in R$. □

Detection Principle for Higher Semiadditivity

We now formulate the main detection principle for m-semiadditivity for symmetric monoidal, stable, p-local ∞-categories. For convenience, we formulate these results for presentable ∞-categories and colimit preserving functors, though what we actually use is only the existence and preservation of certain limits and colimits.

Lemma 4.3.6. *Let $m \geq 1$ and let $\mathcal{C}$ be an m-semiadditive presentably symmetric monoidal, stable, p-local ∞-category. If there exists a connected m-finite p-space A, such that $\pi_m(A) \neq 0$ and $[A]_{\mathbb{1}}$ is an isomorphism, then $\mathcal{C}$ is $(m+1)$-semiadditive.*

Proof. Since $m \geq 1$, the space $B^{m+1}C_p$ is connected. Since the ∞-category $\mathcal{C}$ is m-semiadditive, the map $q\colon B^{m+1}C_p \to \mathrm{pt}$ is weakly $\mathcal{C}$-ambidextrous. By [HL13, Corollary 4.4.23], it suffices to show that q is $\mathcal{C}$-ambidextrous. Since m-finite p-spaces are nilpotent, there is a fiber sequence $A \to B \xrightarrow{\pi} B^{m+1}C_p$ with B an m-finite space. Since $[A]_{\mathbb{1}}$ is invertible, by Lemma 3.3.4(2), A is $\mathcal{C}$-amenable. Hence, by Proposition 3.1.17, the space $B^{m+1}C_p$ is $\mathcal{C}$-ambidextrous. □

We can exploit the extra structure given by the additive p-derivation on $\mathcal{R}_{\mathcal{C}}$ to find a space A as in Lemma 4.3.6.

Proposition 4.3.7. *Let $m \geq 1$ and let $\mathcal{C}$ be an m-semiadditive presentably symmetric monoidal, stable, p-local ∞-category. Let $h\colon \mathcal{R}_{\mathcal{C}} \to S$ be a semi-δ-ring homomorphism that detects invertibility, and such that $h([BC_p]), h([B^mC_p]) \in S$ are rational and non-zero. Then $\mathcal{C}$ is $(m+1)$-semiadditive.*

Proof. A space A will be called *h-good* if

(a) A is a connected m-finite p-space, such that $\pi_m(A) \neq 0$.

(b) $h([A])$ is rational.

By Lemma 4.3.6, it is enough to show that there exists an h-good space A, such that $[A]$ is invertible in $R_{\mathcal{C}}$. Since h detects invertibility, it suffices to find such A with $h([A])$ invertible in S. By assumption, $h([B^mC_p])$ is rational and therefore B^mC_p is h-good. If $p \in S^\times$, then all non-zero rational elements in S are invertible and we are done by the assumption that $h([B^mC_p]) \neq 0$. Hence, we assume that $p \notin S^\times$. In this case, a rational element $x \in S$ is invertible if and only if $v_p(x) = 0$. Denoting $v(A) = v_p(h([A]))$, it is enough to show that there exists an h-good space A with $v(A) = 0$.

Since $h([B^mC_p])$ is non-zero and p is not invertible, we get $0 \leq v(B^mC_p) < \infty$. It therefore suffices to show that given an h-good space A with $0 < v(A) < \infty$, there exists an h-good space A' with $v(A') = v(A) - 1$. For this we exploit the operation δ. We compute using Corollary 3.1.14 and Theorem 4.2.12:

$$\delta([A]) = [BC_p][A] - \alpha([A]) = [BC_p][A] - [A \wr C_p] = [BC_p \times A] - [A \wr C_p].$$

Thus,

$$\delta(h([A])) = h(\delta([A])) = h([BC_p \times A]) - h([A \wr C_p]).$$

Since by assumption $h([BC_p])$ is rational, then by Corollary 3.1.14 we get that

$$h([BC_p \times A]) = h([BC_p])\, h([A])$$

is also rational, and moreover, as $p \notin S^\times$, we obtain $v(A) \leq v(BC_p \times A)$. Furthermore, since $h([A])$ is rational, by Lemma 4.1.11, the same is true for $\delta(h([A]))$. Therefore,

$$h([A \wr C_p]) = h([BC_p \times A]) - h(\delta([A]))$$

is also rational. It is clear that $A \wr C_p$ satisfis (a), and so is h-good. Since $0 < v(A) < \infty$, by Lemma 4.1.7, we get $v_p(\delta(h([A]))) = v(A) - 1$. Thus, $v(A \wr C_p) = v(A) - 1$ and this completes the proof. □

Remark 4.3.8. The proof did not actually use anything specific to the space $B^m C_p$. It would have sufficed to have some good space A with $h([A])$ rational and non-zero. The space $B^m C_p$ is just the “simplest” one.

In practice, the situation of Proposition 4.3.7 arises as follows.

Proposition 4.3.9. *Let $m \geq 1$, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit preserving symmetric monoidal functor between presentably symmetric monoidal, stable, p-local, m-semiadditive ∞-categories. Assume that the map $\varphi\colon R_{\mathcal{C}} \to R_{\mathcal{D}}$, induced by F, detects invertibility and that the images of $[BC_p]_{\mathcal{D}}, [B^m C_p]_{\mathcal{D}} \in \mathcal{R}_{\mathcal{D}}$ in the ring $\mathcal{R}^{\mathrm{tf}}_{\mathcal{D}}$ are rational and non-zero. Then $\mathcal{C}$ and $\mathcal{D}$ are $(m+1)$-semiadditive.*

Proof. It is enough to prove that $\mathcal{C}$ is $(m+1)$-semiadditive, since by Corollary 3.3.2, this implies that $\mathcal{D}$ is $(m+1)$-semiadditive. We shall apply Proposition 4.3.7 to the composition

$$\mathcal{R}_{\mathcal{C}} \xrightarrow{\varphi} \mathcal{R}_{\mathcal{D}} \xrightarrow{g} \mathcal{R}^{\mathrm{tf}}_{\mathcal{D}},$$

where g is the canonical projection. By Proposition 4.3.4, φ is a semi-δ-ring homomorphism and it detects invertibility by assumption. On the other hand, g is a semi-δ-ring homomorphism and it detects invertibility by Proposition 4.1.16. It is only left to observe that $\varphi([A]_{\mathcal{C}}) = [A]_{\mathcal{D}}$, which follows from Corollary 3.2.6. $\square$

We conclude with a variant of Proposition 4.3.9, in which the condition on the elements $[B^m C_p]_{\mathcal{D}}$, is replaced by a condition on the closely related elements $\dim_{\mathcal{D}}(B^m C_p)$, and which assembles together the individual statements for different $m \in \mathbb{N}$.

Theorem 4.3.10. *(Bootstrap Machine) Let $1 \leq m \leq \infty$ and let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit preserving symmetric monoidal functor between presentably symmetric monoidal, stable, p-local ∞-categories. Assume that*

(1) $\mathcal{C}$ is 1-semiadditive.

(2) The map $\varphi\colon R_{\mathcal{C}} \to R_{\mathcal{D}}$, induced by F, detects invertibility.

(3) For every $0 \leq k < m$, if the space $B^k C_p$ is dualizable in $\mathcal{D}$, then the image of $\dim_{\mathcal{D}}\left(B^k C_p\right)$ in $\mathcal{R}^{\mathrm{tf}}_{\mathcal{D}}$ is rational and non-zero.

Then $\mathcal{C}$ and $\mathcal{D}$ are m-semiadditive.

Proof. It suffices to show that $\mathcal{C}$ is m-semiadditive, since by Corollary 3.3.2, $\mathcal{D}$ is then also m-semiadditive. We prove by induction on k, that the images of the elements $\left[B^i C_p\right]_{\mathcal{D}}$ in $\mathcal{R}^{\mathrm{tf}}_{\mathcal{D}}$ are rational and non-zero for all $0 \leq i < k$, and that $\mathcal{C}$ is k-semiadditive. The base case $k = 1$ holds by assumption (1) and the fact that $[C_p]_{\mathcal{D}} = p$ is rational and nonzero in $\mathcal{R}^{\mathrm{tf}}_{\mathcal{D}}$, since the unique ring homomorphism $\mathbb{Z} \to \mathcal{R}^{\mathrm{tf}}_{\mathcal{D}}$ is injective by Lemma 4.1.11. Assuming the inductive hypothesis for some $k < m$, we first prove that $\left[B^k C_p\right]_{\mathcal{D}}$ in $\mathcal{R}^{\mathrm{tf}}_{\mathcal{D}}$ are rational and non-zero. By Corollary 3.3.10, $B^k C_p$ is dualizable in $\mathcal{D}$, and we have

$$\dim_{\mathcal{D}}\left(B^k C_p\right) = \left[B^k C_p\right]_{\mathcal{D}} \left[B^{k-1} C_p\right]_{\mathcal{D}} \in \mathcal{R}^{\mathrm{tf}}_{\mathcal{D}}.$$

By assumption (3), $\dim_{\mathcal{D}}\left(B^k C_p\right)$ is rational and non-zero and by the inductive hypothesis, the image of $\left[B^{k-1} C_p\right]_{\mathcal{D}}$ in $\mathcal{R}_{\mathcal{D}}^{\mathrm{tf}}$ is rational and non-zero as well. Consequently, the image of $\left[B^k C_p\right]_{\mathcal{D}}$ in $\mathcal{R}_{\mathcal{D}}^{\mathrm{tf}}$ must also be rational and non-zero since $\mathcal{R}_{\mathcal{D}}^{\mathrm{tf}}$ is torsion-free. We shall now deduce that $\mathcal{C}$ is $(k+1)$-semiadditive by applying Proposition 4.3.9 to the functor F. Since $\left[B^k C_p\right]_{\mathcal{D}}$ is rational and non-zero, it suffices to show that $[BC_p]_{\mathcal{D}}$ is rational and non-zero. For $k=1$ there is nothing to prove and for $k \geq 2$ this follows by the inductive hypothesis. $\square$

Remark 4.3.11. The proof shows that the assumptions of the theorem above *imply* that the spaces $B^k C_p$ are dualizable in $\mathcal{D}$. Thus, in retrospect, the "if" in assumption (3) is superfluous.

5 Applications to Chromatic Homotopy Theory

Throughout, we fix a prime p which will be implicit in all definitions that depend on it. In this final section we apply the general theory developed in the previous sections to chromatic homotopy theory. After fixing some notation and terminology,[3] we study the consequences of 1-semiadditivity of $\mathrm{Sp}_{T(n)}$ to nilpotence in $\mathbb{E}_\infty$ (and H_∞)-ring spectra. We then use the general theory of the previous sections to establish the ∞-semiadditivity of $\mathrm{Sp}_{T(n)}$ and some consequences.

5.1 Generalities of Chromatic homotopy Theory

We begin with some generalities, mainly to fix terminology and notation. Let $(\mathrm{Sp}, \otimes, \mathbb{S})$ be the symmetric monoidal ∞-category of spectra (see [Lur, Corollary 4.8.2.19]).

Localizations, Rings and Modules

For every spectrum $E \in \mathrm{Sp}$, we denote by $\mathrm{Sp}_E \subseteq \mathrm{Sp}$, the full subcategory spanned by E-local spectra.

Proposition 5.1.1. *For every spectrum $E \in \mathrm{Sp}$, the ∞-category Sp_E is stable, presentable and admits a canonical structure of a presentably symmetric monoidal ∞-category $\left(\mathrm{Sp}_E, \hat{\otimes}, L_E\mathbb{S}\right)$. Moreover, the inclusion $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$ is lax symmetric monoidal and admits a symmetric monoidal left adjoint $L_E\colon \mathrm{Sp} \to \mathrm{Sp}_E$. Finally, for all $X, Y \in \mathrm{Sp}_E$ we have*

$$X \hat{\otimes} Y \simeq L_E\left(X \otimes Y\right).$$

Proof. Applying [Lur09, Proposition 5.5.4.15] to the collection of E-local morphisms in Sp_E, we deduce that Sp_E is presentable and the inclusion $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$ admits a left adjoint $L_E\colon \mathrm{Sp} \to \mathrm{Sp}_E$. By [Lur, Lemma 1.1.3.3], Sp_E is a stable subcategory of Sp. Since E-local morphisms are compatible with $\otimes$, we get the rest of the claim by [Lur, Proposition 2.2.1.9]. $\square$

Proposition 5.1.2. *Let $E \in \mathrm{Sp}$ and R an E-local $\mathbb{E}_\infty$-ring. The ∞-category $\mathrm{Mod}_R^{(E)}$ of left modules over R in the symmetric monoidal ∞-category Sp_E, is presentable and admits a structure of a presentably symmetric monoidal ∞-category. Moreover, we have a free-forgetful adjunction*

$$F_R\colon \mathrm{Sp}_E \leftrightarrows \mathrm{Mod}_R^{(E)} \colon U_R,$$

[3] We refer the reader to [Rav16] for a comprehensive treatment of the fundamentals of chromatic homotopy theory.

in which F_R is symmetric monoidal.

Proof. [Lur, Corollary 4.5.1.5] identifies modules over R as an $\mathbb{E}_\infty$-ring with left modules over R as an $\mathbb{E}_1$-ring. By [Lur, Theorem 4.5.3.1] and [Lur, Corollary 4.2.3.7] this ∞-category is equipped with a presentably symmetric monoidal structure. By [Lur, Remark 4.2.3.8] and [Lur, Remark 4.5.3.2] applied to the map of algebras $\mathbb{S}_E \to R$, we have the adjunction $F_R \dashv U_R$, such that F_R is symmetric monoidal. □

Morava Theories

Given an integer $n \geq 0$, let E_n be a 2-periodic Morava E-theory of height n with coefficients (for $n \geq 1$)

$$\pi_* E_n \simeq \mathbb{Z}_p\left[[u_1, \dots, u_n]\right]\left[u^{\pm 1}\right], \quad |u_i| = 0,\ |u| = 2,$$

and let $K(n)$ be a 2-periodic Morava K-theory of height n with coefficients ($n \geq 1$)

$$\pi_* K(n) = \mathbb{F}_p\left[u^{\pm 1}\right], \quad |u| = 2.$$

The spectrum E_n admits an $\mathbb{E}_\infty$-ring structure in Sp (by [GH04]) and the spectrum $K(n)$ admits an $\mathbb{E}_1$-ring structure with an $\mathbb{E}_1$-ring map $E_n \to K(n)$. Since E_n is $K(n)$-local, we can also view it as an $\mathbb{E}_\infty$-ring in the ∞-category $\mathrm{Sp}_{K(n)}$.

Telescopic Localizations

We now recall some terminology and basic properties regarding type n spectra.

Definition 5.1.3. A finite p-local spectrum X, i.e. a compact object in the ∞-category $\mathrm{Sp}_{(p)}$, is said to be of *type* n, if $K(n) \otimes X \neq 0$ and $K(j) \otimes X = 0$ for $j = 0, \dots, n-1$.

Let $X(n)$ be a finite p-local spectrum of type n. Let $\mathbb{D}X(n) = F\left(X(n), \mathbb{S}_{(p)}\right)$ be the Spanier-Whitehead dual of $X(n)$. The finite p-local spectrum

$$R = \mathbb{D}X(n) \otimes X(n) = F\left(X(n), X(n)\right),$$

is also of type n by the Kunneth isomorphism. By replacing $X(n)$ with R, we may assume that $X(n)$ is an $\mathbb{E}_1$-ring (see [Lur, Section 4.7.1]). Every type n spectrum $X(n)$ admits a v_n-self map, which is a map

$$v \colon \Sigma^k X(n) \to X(n),$$

that is an isomorphism on $K(n)_* X$ and zero on $K(j)_*(X)$ for $j \neq n$. We let

$$T(n) = v^{-1} X(n) = \varinjlim_k \left(X(n) \xrightarrow{v} \Sigma^{-k} X(n) \xrightarrow{v} \Sigma^{-2k} X(n) \xrightarrow{v} \dots \right),$$

be the telescope on v. The canonical map $X(n) \to T(n)$ exhibits $T(n)$ as the $T(n)$-localization of $X(n)$ (e.g.[MS95, Proposition 3.2]). Since the functor $L_{T(n)}$ is symmetric monoidal, we can consider $T(n) = L_{T(n)} X(n)$ as an $\mathbb{E}_1$-ring in $\mathrm{Sp}_{T(n)}$. It is known (e.g. [MS95, Section 6 (4)]) that

$$\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)} \subseteq \mathrm{Sp}.$$

Thus, both E_n and $K(n)$ are also $T(n)$-local, and so we can consider them as an $\mathbb{E}_\infty$-ring and an $\mathbb{E}_1$-ring in $\mathrm{Sp}_{T(n)}$ respectively.

5.2 Consequences of 1-Semiadditivity

In this section we discuss some applications of the theory of 1-semiadditivity in stable ∞-categories.

Lifts of Frobenius

Definition 5.2.1. Given a commutative ring R, a *lift of Frobenius* for R is a homomorphism of abelian groups $\psi\colon R \to R$, such that

$$\psi(x) \equiv x^p \quad \in R/pR.$$

We define the category FrobRing, whose objects are pairs (R, ψ) with R a commutative ring and ψ a lift of Frobenius for R, and a morphism $(R, \psi) \to (R', \psi')$ is a ring homomorphism $f\colon R \to R'$ satisfying

$$f \circ \psi = \psi' \circ f.$$

There is an obvious forgetful functor FrobRing $\to$ CommRing, where the later is the category of commutative rings.

The p-th Adams operation ψ^p on the complex K-theory ring of a space is a lift of Frobenius. In [Sta16], a canonical lift of Frobenius is constructed for the Morava E-theory ring of a space in every height. We generalize these facts as follows.

Theorem 5.2.2. *The functor*

$$\pi_0\colon \operatorname{CAlg}(\operatorname{Sp}_{T(n)}) \to \operatorname{CommRing}$$

has a lift to a functor

$$\operatorname{CAlg}(\operatorname{Sp}_{T(n)}) \to \operatorname{FrobRing}.$$

Proof. The ∞-category $\operatorname{Sp}_{T(n)}$ is 1-semiadditive by [Kuh04] and therefore satisfies the conditions of Theorem 4.3.2. Thus, for every $R \in \operatorname{CAlg}(\operatorname{Sp}_{T(n)})$, the commutative ring

$$\pi_0 R = \hom_{h\operatorname{Sp}_{T(n)}}(\mathbb{S}_{T(n)}, R)$$

admits an additive p-derivation δ. By Lemma 4.1.4, the operation $\psi(x) = x^p + p\delta(x)$ is a lift of Frobenius. The functoriality follows from Proposition 4.3.3. $\square$

On May's Conjecture

Definition 5.2.3. Let $\mathcal{C}$ be a stable, presentably symmetric monoidal ∞-category. We say that $\mathcal{C}$ is *sofic*,[4] if there exists a stable, 1-semiadditive, presentably symmetric monoidal ∞-category $\mathcal{D}$ and a colimit preserving, conservative, lax symmetric monoidal functor $\mathcal{C} \to \mathcal{D}$. We call a spectrum $E \in \operatorname{Sp}$ sofic, if Sp_E is sofic.

Every stable, 1-semiadditive, presentably symmetric monoidal ∞-category is of course sofic, but the later condition is considerably weaker.

[4] The term *sofic* is derived from the Hebrew word "sofi" for $finite$ (See [Wei73]).

Example 5.2.4. The spectrum $H\mathbb{Q}$ is sofic and more generally, the spectra $K(n)$ and $T(n)$ for all n. Any sum of sofic spectra is sofic, and since being sofic depends only on the Bousfield class of the spectrum, so are the Morava theories E_n for all n and the telescopic localizations of the sphere spectrum $L_n^f\mathbb{S}$.

Theorem 5.2.5. *Let $E \in \mathrm{Sp}$ be a sofic homotopy commutative ring spectrum and let R be an $\mathbb{E}_\infty$-ring. For every $x \in \pi_* R$, if the image of x in $\pi_*(H\mathbb{Q} \otimes R)$ is nilpotent, then the image of x in $\pi_*(E \otimes R)$ is nilpotent.*

Namely, the single homology theory $H\mathbb{Q}$ detects nilpotence in all sofic homology theories.

Proof. First, observe that

$$\pi_*(H\mathbb{Q} \otimes R) \simeq \mathbb{Q} \otimes \pi_* R.$$

Replacing x with a suitable power, we can assume that x is torsion in $\pi_* R$. Since the homogeneous components of a torsion element are torsion, we may assume without loss of generality that $x \in \pi_k R$ for some k (i.e. x is homogeneous). Consider the corresponding map $[x] : R \to \Sigma^{-k} R$ given by multiplication by x. The telescope

$$x^{-1}R = \varinjlim \left(R \xrightarrow{[x]} \Sigma^{-k} R \xrightarrow{[x]} \Sigma^{-2k} R \xrightarrow{[x]} \dots \right)$$

carries a structure of an $\mathbb{E}_\infty$-ring and the map $R \to x^{-1}R$ induces the localization by x map on π_*. In particular, the unit map $\mathbb{S} \to x^{-1}R$ is torsion. Let $F\colon \mathrm{Sp}_E \to \mathcal{C}$ be a functor as in Definition 5.2.3. Since F and L_E are both lax symmetric monoidal and exact, $F\left(L_E\left(x^{-1}R\right)\right)$ is an $\mathbb{E}_\infty$-algebra and its unit is also torsion. By Corollary 4.3.5, $F\left(L_E\left(x^{-1}R\right)\right) = 0$ and since F is conservative, $L_E\left(x^{-1}R\right) = 0$. It follows that $E \otimes x^{-1}R = 0$. Let $\tilde{x}$ be the image of x in $\pi_*(E \otimes R)$. Since,

$$\pi_*\left(E \otimes x^{-1}R\right) \simeq \pi_*\left(\tilde{x}^{-1}(E \otimes R)\right) \simeq \tilde{x}^{-1}\pi_*(E \otimes R),$$

we get that $\tilde{x}$ is nilpotent in $\pi_*(E \otimes R)$. $\square$

Remark 5.2.6. We could have replaced $\mathbb{E}_\infty$ by H_∞ (see Remark 4.2.2). Applying the theorem in this form for $E = K(n)$, and using the Nilpotence Theorem, one can deduce the conjecture of May, that was proved in [MNN15]. We also note that the above theorem can be extended to a general stable presentably symmetric monoidal ∞-category $\mathcal{C}$ with a compact unit (instead of Sp) and $x\colon I \to R$ any map from an invertible object I (i.e. an object of the Picard group of $\mathcal{C}$).

5.3 Higher Semiadditivity of $T(n)$-Local Spectra

In this section we prove the main theorem of the paper. Namely, that the ∞-category $\mathrm{Sp}_{T(n)}$ is ∞-semiadditive for all $n \geq 0$. Our strategy is to apply Theorem 4.3.10 to the functor given by the following definition.

Definition 5.3.1. Let $\widehat{E}_n[-]$ be the composition

$$\mathrm{Sp}_{T(n)} \xrightarrow{L_{K(n)}} \mathrm{Sp}_{K(n)} \xrightarrow{F_{E_n}} \mathrm{Mod}_{E_n}^{(K(n))},$$

where we abuse notation and write $L_{K(n)}$ also for the left adjoint of the inclusion $\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)}$. The functor $\widehat{E}_n[-]$ is a colimit preserving symmetric monoidal functor as a composition of two such.

Realizing this strategy consists of verifying assumptions (1)-(3) of Theorem 4.3.10. Namely,

(1) Showing that the ∞-categories $\mathrm{Sp}_{T(n)}$ are 1-semiadditive.

(2) Showing that $\widehat{E}_n[-]$ detects invertibility.

(3) Computing the symmetric monoidal dimension of the spaces $B^k C_p$ in $\mathrm{Mod}_{E_n}^{(K(n))}$.

Part (1) is a theorem of Kuhn (see [Kuh04] and [CM17] for a shorter proof). Part (2) will follow from the Nilpotence Theorem using standard techniques of chromatic homotopy. Part (3) will follow from the computations of Ravenel-Wilson of the Morava K-theory of $B^k C_p$.

Detecting Invertibility

Definition 5.3.2. We say that a map of homotopy ring spectra $f\colon R \to S$ *detects invertibility*, if the induced map of ordinary rings $\pi_0 f\colon \pi_0 R \to \pi_0 S$ detects invertibility.

We shall use the notation $\mathbb{S}_E$ for $L_E\mathbb{S}$. Every $\mathbb{E}_1$-ring R is R-local and hence we have a unit map $h\colon \mathbb{S}_R \to R$. We have the following general fact.

Lemma 5.3.3. *Let R be an $\mathbb{E}_1$-ring. The unit map $h\colon \mathbb{S}_R \to R$ detects invertibility.*

Proof. Consider both $\mathbb{S}_R$ and R as $\mathbb{E}_1$-rings in Sp_R. An element $x \in \pi_0\mathbb{S}_R$ is represented by a map $[x] : \mathbb{S}_R \to \mathbb{S}_R$, and x is invertible if and only if $[x]$ is an isomorphism. Since $\mathbb{S}_R$ is R-local, $[x]$ is an isomorphism if and only if the induced map

$$[x] \otimes R\colon \mathbb{S}_R \otimes R \to \mathbb{S}_R \otimes R$$

is an isomorphism. Since $\mathbb{S}_R \otimes R$ is R-local (being an R-module), we have

$$\mathbb{S}_R \otimes R \simeq L_R\left(\mathbb{S}_R \otimes R\right) \simeq \mathbb{S}_R \hat{\otimes} R \simeq R.$$

Via this isomorphism, the map $[x] \otimes R$ can be identified with $[h(x)] : R \to R$, which is the multiplication from the right by $h(x) \in \pi_0 R$. Thus, $x \in \pi_0\mathbb{S}_R$ is invertible if and only if $[h(x)]$ is an isomorphism if and only if $h(x) \in \pi_0 R$ is invertible and we are done. $\square$

Proposition 5.3.4. *The unit map $g\colon \mathbb{S}_{T(n)} \to K(n)$ detects invertibility.*

Proof. Consider the unit maps

$$g\colon \mathbb{S}_{T(n)} \to K(n)\,, \quad h\colon \mathbb{S}_{T(n)} \to T(n)\,.$$

Tensoring g and h in $\mathrm{Sp}_{T(n)}$, we obtain a commutative square of $T(n)$-local $\mathbb{E}_1$-rings

$$\begin{tikzcd} \mathbb{S}_{T(n)} \arrow[r, "h"] \arrow[d, "g"'] & T(n) \arrow[d, "\tilde{g}"] \\ K(n) \arrow[r, "\tilde{h}"] & K(n)\,\hat{\otimes} T(n)\,. \end{tikzcd} \qquad (*)$$

Since $K(n) \otimes T(n)$ is $T(n)$-local (being a $T(n)$-module), we have

$$K(n) \hat{\otimes} T(n) \simeq L_{T(n)} (K(n) \otimes T(n)) \simeq K(n) \otimes T(n) .$$

Let $x \in \pi_0 \mathbb{S}_{T(n)}$, such that $g(x)$ is invertible in $\pi_0 K(n) = \mathbb{F}_p$. Without loss of generality, we may assume that $g(x) = 1$, and therefore

$$\tilde{g}(h(x)) = \tilde{h}(g(x)) = 1.$$

It follows that $\tilde{g}(h(x) - 1) = 0$. For $j \neq n$, we have $K(j) \otimes T(n) = 0$ (e.g. [Rav87, Proposition A.2.13]), hence by the Nilpotence Theorem, $h(x) - 1$ is nilpotent in $\pi_0 T(n)$. Thus, $h(x)$ is invertible. Finally, by Lemma 5.3.3, the map h detects invertibility and therefore x is invertible. ☐

Corollary 5.3.5. *The unit map $f \colon \mathbb{S}_{T(n)} \to E_n$ detects invertibility.*

Proof. The unit map $g \colon \mathbb{S}_{T(n)} \to K(n)$ is homotopic to the composition

$$\mathbb{S}_{T(n)} \xrightarrow{f} E_n \to K(n) ,$$

and so the result follows from Proposition 5.3.4. ☐

Dimension of Eilenberg-MacLane Spaces

Next, we need to compute the $\mathrm{Mod}_{E_n}^{(K(n))}$-dimension of Eilenberg-MacLane spaces.

Lemma 5.3.6. *Let $n \geq 0$ and let X be a space. If*

$$\dim_{\mathbb{F}_p} (K(n)_0 (X)) = d < \infty \qquad \text{and} \qquad K(n)_1 (X) = 0,$$

then X is dualizable in $\mathrm{Mod}_{E_n}^{(K(n))}$ and

$$\dim_{\mathrm{Mod}_{E_n}^{(K(n))}} (X) = d.$$

Proof. By [HL13, Proposition 3.4.3], there is an isomorphism of E_n-modules

$$L_{K(n)} (E_n \otimes \Sigma^\infty X_+) \simeq E_n^d,$$

from which the claim follows immediately. ☐

Remark 5.3.7. Using [HL13, Proposition 3.4.3] together with [Mat16, Proposition 10.11], one can deduce that for every dualizable object $M \in \mathrm{Mod}_{E_n}^{(K(n))}$, we have

$$\dim_{\mathrm{Mod}_{E_n}^{(K(n))}} (M) = \dim_{\mathbb{F}_p} (\pi_0 (K(n) \otimes_{E_n} M)) - \dim_{\mathbb{F}_p} (\pi_1 (K(n) \otimes_{E_n} M)) .$$

But we shall not need this fact.

Using the classical computations of Ravenel and Wilson we deduce the following.

Corollary 5.3.8. *For all $k \in \mathbb{N}$, we have*

$$\dim_{\mathrm{Mod}_{E_n}^{(K(n))}}\left(B^k C_p\right) = p^{\binom{n}{k}} \quad \in \pi_0\left(E_n\right).$$

In particular, these are all rational and non-zero.

Proof. By [RW80, Theorem 9.2], we have

$$\dim_{\mathbb{F}_p} K(n)_0\left(B^k C_p\right) = p^{\binom{n}{k}} \qquad \text{and} \qquad K(n)_1\left(B^k C_p\right) = 0.$$

Hence, the result follows from Lemma 5.3.6. □

Main Result and Consequences

We are now ready to prove the main theorem of the paper.

Theorem 5.3.9. *For all $n \geq 0$, the ∞-categories $\mathrm{Sp}_{T(n)}$ and $\mathrm{Mod}_{E_n}^{(K(n))}$ are ∞-semiadditive.*

Proof. We verify the assumptions (1)-(3) of Theorem 4.3.10 for the colimit preserving symmetric monoidal functor

$$\widehat{E}_n\left[-\right] \colon \mathrm{Sp}_{T(n)} \to \mathrm{Mod}_{E_n}^{(K(n))}.$$

(1) is proved in [Kuh04], (2) follows from Corollary 5.3.5 and (3) is Corollary 5.3.8. □

This readily implies the original result of [HL13].

Corollary 5.3.10. *For all $n \geq 0$, the ∞-category $\mathrm{Sp}_{K(n)}$ is ∞-semiadditive.*

Proof. Apply Corollary 3.3.2 to the localization functor $L_{T(n)} \colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(n)}$. Alternatively, one could just use the same argument as in Theorem 5.3.9. □

By Theorem 5.3.9, both ∞-categories $\mathrm{Sp}_{T(n)}$ and $\mathrm{Mod}_{E_n}^{(K(n))}$ are ∞-semiadditive. Hence, for every π-finite space A, we have an element $[A] \in \pi_0\mathbb{S}_{T(n)}$, which maps to the corresponding element $[A] \in \pi_0 E_n$ (since the map is induced by a colimit preserving functor). We shall make some computations regarding these elements and use them to deduce some new facts about $\mathrm{Sp}_{T(n)}$.

Lemma 5.3.11. *For every $k, n \geq 0$ we have*

$$\left[B^k C_p\right]_{\mathrm{Mod}_{E_n}^{(K(n))}} = p^{\binom{n-1}{k}} \quad \in \pi_0 E_n.$$

Proof. By Corollary 3.3.10 and Corollary 5.3.8, we have

$$p^{\binom{n}{k}} = \dim_{\mathrm{Mod}_{E_n}^{(K(n))}}\left(B^k C_p\right) = [B^k C_p][B^{k-1} C_p].$$

The result now follows by induction on k, using the identity

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

and the fact that the ring $\pi_0 E_n$ is torsion free. □

Lemma 5.3.12. *For every $k \geq n \geq 0$ the element $\left[B^k C_p\right]_{\mathrm{Sp}_{T(n)}} \in \pi_0 \mathbb{S}_{T(n)}$ is invertible.*

Proof. By Corollary 5.3.5, the map

$$f \colon \pi_0 \mathbb{S}_{T(n)} \to \pi_0 E_n$$

detects invertibility and by Lemma 5.3.11,

$$f\left(\left[B^k C_p\right]\right) = p^{\binom{n-1}{k}} = 1.$$

□

Theorem 5.3.13. *Let $n \geq 0$ and let $f \colon A \to B$ be a map with π-finite n-connected homotopy fibers. The induced map $\Sigma^\infty_+ f \colon \Sigma^\infty_+ A \to \Sigma^\infty_+ B$ is a $T(n)$-equivalence.*

Proof. We begin with a standard general argument that reduces the statement to the case $B = \mathrm{pt}$, by passing to the fibers. Consider the equivalence of ∞-categories

$$\mathcal{S}_{/B} \xrightarrow{\sim} \mathrm{Fun}\left(B, \mathcal{S}\right),$$

given by the Grothedieck construction. Let $X \in \mathrm{Fun}\left(B, \mathcal{S}\right)$ be the local system of spaces on B, that corresponds to f and let $Y \in \mathrm{Fun}\left(B, \mathcal{S}\right)$ be the constant local system on $\mathrm{pt} \in \mathcal{S}$. As Y is terminal, there is an essentially unique map $X \to Y$, which at each point $b \in B$, is the essentially unique map from X_b, the homotopy fiber of f at b, to $Y_b = \mathrm{pt}$. We recover f, up to homotopy, as the induced map on colimits

$$A \simeq \varinjlim X \to \varinjlim Y \simeq B.$$

For each $E \in \mathrm{Sp}$, the functor

$$E \otimes \Sigma^\infty_+ \left(-\right) : \mathcal{S} \to \mathrm{Sp}$$

preserves colimits. Therefore, if the induced map for each homotopy fiber

$$E \otimes \Sigma^\infty_+ X_b \to E \otimes \Sigma^\infty_+ \mathrm{pt},$$

is an isomorphism, then the induced map on colimits is also an isomorphism

$$E \otimes \Sigma^\infty_+ A \xrightarrow{\sim} E \otimes \Sigma^\infty_+ B.$$

Now, if $B = \mathrm{pt}$, we have that A is a π-finite n-connected space. For $n = 0$, the claim is obvious, and so we may assume that $n \geq 1$. Therefore, A is simply connected and in particular nilpotent. Thus, we can refine the Postnikov tower of A to a finite tower

$$A = A_0 \to A_1 \to \cdots \to A_d = \mathrm{pt},$$

such that the homotopy fiber of each $A_i \to A_{i+1}$ is of the form $B^k C_q$, for q a prime and $k \geq n+1$. It thus suffices to show that the map

$$\Sigma^\infty_+ B^k C_q \to \Sigma^\infty_+ \mathrm{pt} \simeq \mathbb{S},$$

induced by

$$g \colon B^k C_q \to \mathrm{pt},$$

is a $T(n)$-equivalence. For $q \neq p$ this is clear. For $q = p$ we apply Proposition 3.1.18 to the map g. For this, we need to check that

$$[\Omega B^k C_p] = [B^{k-1} C_p]$$

is invertible in $\pi_0 \mathbb{S}_{T(n)}$, which follows from Lemma 5.3.12. $\square$

Remark 5.3.14. The analogous result for $K(n)$ instead of $T(n)$ is a consequence of the [RW80] computation of the $K(n)$-homology of Eilenberg-MacLane spaces. A weaker result for $T(n)$, namely that the conclusion holds if the homotopy fibers of f are π-finite and k-connected for $k \gg 0$, can be deduced from [Bou82, Theorem 3.1].

References

[Bar17] Clark Barwick. Spectral Mackey functors and equivariant algebraic K-theory I. *Advances in Mathematics*, 304:646–727, 2017.

[BG18] Irina Bobkova and Paul G. Goerss. Topological resolutions in $K(2)$-local homotopy theory at the prime 2. *Journal of Topology*, 11(4):917–956, 2018.

[BHN+17] Tobias Barthel, Markus Hausmann, Niko Naumann, Thomas Nikolaus, Justin Noel, and Nathaniel Stapleton. The Balmer spectrum of the equivariant homotopy category of a finite abelian group. *arXiv preprint arXiv:1709.04828*, 2017.

[Bou82] A.K. Bousfield. On homology equivalences and homological localizations of spaces. *American Journal of Mathematics*, 104(5):1025–1042, 1982.

[Bui05] Alexandru Buium. *Arithmetic differential equations*. Number 118. American Mathematical Soc., 2005.

[CM17] Dustin Clausen and Akhil Mathew. A short proof of telescopic Tate vanishing. *Proceedings of the American Mathematical Society*, 145(12):5413–5417, 2017.

[GH04] Paul G. Goerss and Michael J. Hopkins. Moduli spaces of commutative ring spectra. *Structured ring spectra*, 315(151-200):22, 2004.

[GS96] John P.C. Greenlees and Hal Sadofsky. The Tate spectrum of v_n-periodic complex oriented theories. *Mathematische Zeitschrift*, 222(3):391–405, 1996.

[Har17] Yonatan Harpaz. Ambidexterity and the universality of finite spans. arXiv preprint arXiv:1703.09764, 2017.

[Heu18] Gijs Heuts. Lie algebras and v_n-periodic spaces. 2018. arXiv preprint arXiv:1803.06325.

[HL13] Michael Hopkins and Jacob Lurie. Ambidexterity in $K(n)$-local stable homotopy theory. 2013. http://www.math.harvard.edu/ lurie/.

[HS96] Mark Hovey and Hal Sadofsky. Tate cohomology lowers chromatic Bousfield classes. *Proceedings of the American Mathematical Society*, 124(11):3579–3585, 1996.

[Kuh04] Nicholas J. Kuhn. Tate cohomology and periodic localization of polynomial functors. *Inventiones mathematicae*, 157(2):345–370, 2004.

[Lur] Jacob Lurie. Higher algebra. http://www.math.harvard.edu/ lurie/.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Mat16] Akhil Mathew. The Galois group of a stable homotopy theory. *Adv. Math.*, 291:403–541, 2016.

[MNN15] Akhil Mathew, Niko Naumann, and Justin Noel. On a nilpotence conjecture of J.P. May. *Journal of Topology*, 8(4):917–932, 2015.

[MS95] Mark Mahowald and Hal Sadofsky. v_n telescopes and the Adams spectral sequence. *Duke Math. J.*, 78(1):101–129, 1995.

[Rav87] Douglas C. Ravenel. Localization and periodicity in homotopy theory. In *Homotopy theory (Durham, 1985)*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 175–194. Cambridge Univ. Press, Cambridge, 1987.

[Rav16] Douglas C. Ravenel. *Nilpotence and Periodicity in Stable Homotopy Theory.(AM-128)*, volume 128. Princeton University Press, 2016.

[RV16] Emily Riehl and Dominic Verity. Homotopy coherent adjunctions and the formal theory of monads. *Advances in Mathematics*, 286:802–888, 2016.

[RW80] Douglas C. Ravenel and W. Stephen Wilson. The Morava K-theories of Eilenberg-MacLane spaces and the Conner-Floyd conjecture. *American Journal of Mathematics*, 102(4):691–748, 1980.

[Sta16] Nathaniel Stapleton. A canonical lift of Frobenius in Morava E-theory. 2016. arXiv preprint arXiv:1603.04811.

[Wei73] Benjamin Weiss. Subshifts of finite type and sofic systems. *Monatshefte für Mathematik*, 77(5):462–474, 1973.