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THE LOCALIZATION OF SPECTRA WITH RESPECT TO HOMOLOGY

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IN [8] WE studied localizations of spaces with respect to homology, and we now develop the analogous stable theory. Let Ho^s denote the stable homotopy category of CW-spectra. We show that each spectrum $E \in Ho^s$ gives rise to a natural E_*-localization functor $(\ )_E: Ho^s \to Ho^s$ and $\eta: 1 \to (\ )_E$. For $A \in Ho^s$, $\eta: A \to A_E$ is the terminal example of an E_*-equivalence going out of A in Ho^s. After proving the existence of E_*-localizations, we develop their basic properties and discuss in detail the cases where E is connective and $E = K$. Using E_*-localization theory, we obtain results on the convergence of the E_*-Adams spectral sequence, and on the class $\underset{\sim}{A}(Ho^s)$ of acyclicity types of spectra, see [9].

The paper is sectioned as follows: §1. E_*-localizations and E_*-acyclizations of spectra; §2. Localizations with respect to Moore spectra; §3. General examples of E_*-localizations; §4. K-theoretic localizations of spectra; §5. The E_*-Adams spectral sequence and the E-nilpotent completion; §6. Convergence theorems for the E_*-Adams spectral sequence; §7. Duality in $\underset{\sim}{A}(Ho^s)$.

E_*-localizations of spectra were previously studied by Adams, who sketched a useful outline of the subject in [4]. However, the first complete existence proof for such localizations was obtained by the author as an immediate corrollary of work on E_*-localizations of spaces, see [8]. The relevant proofs in [8] may be stabilized by using Kan's semi-simplicial spectra in place of simplicial sets. That approach also shows the existence of "E_*-factorizations" for maps of spectra and leads to a Quillen model category framework for "stable homotopy theory with respect to E_*." To make the present exposition more understandable we omit these topics and work in the homotopy category of CW-spectra.

This paper grew out of an attempt to understand Doug Ravenel's work on localizations of spectra with respect to certain periodic homology theories [17], and we are indebted to him for explaining his ideas. We are also indebted to Mark Mahowald and Haynes Miller for supplying the homotopy theoretic theorem underlying our approach to K_*-localizations, and we have been significantly influenced by Guido Mislin's work on K_*-localizations of spaces [16]. We understand that various results in this paper, particularly in §4, were obtained by Frank Adams and David Baird in earlier unpublished work, and that some have been obtained independently by W. Dwyer (unpublished).

We essentially use the notation and terminology of [4]. However, we let Ho^s be the category of CW-spectra and homotopy classes of maps of degree 0, see [4], p. 144. Thus Ho^s is an additive category equipped with an equivalence $\Sigma: Ho^s \to Ho^s$ induced by the "shift" suspension for CW-spectra. We let $[X, Y]$ be the group of morphisms

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$X \to Y$ in Ho^s, and write $[X, Y]_n = [\Sigma^n X, Y]$ for $n \in Z$. By a *triangle* in Ho^s we mean a sequence $X \to Y \to Z \to \Sigma X$ in Ho^s such that there exists a commutative diagram

$$\begin{array}{ccccccc} V & \xrightarrow{f} & W & \longrightarrow & WU_fCV & \longrightarrow & \text{Susp } V \overset{\alpha_1}{\simeq} \Sigma V \\ \downarrow i & & \downarrow j & & \downarrow k & & \downarrow \Sigma i \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

in Ho^s where i, j, k are equivalences, f is a cellular may between CW-spectra, and the upper row is as in pp. 155 and 186 of [4]. Recall that Ho^s has coproducts $\vee_\alpha X_\alpha$, products $\Pi_\alpha X_\alpha$, and smash products $X \wedge Y$ with the usual properties. Also, there are (by Brown representability) natural function spectra $F(Y, Z)$ in Ho^s such that $F(Y, -): Ho^s \to Ho^s$ is right adjoint to $- \wedge Y: Ho^s \to Ho^s$. By a *ring spectrum* we mean a spectrum E with an associative (but not necessarily commutative) multiplication $E \wedge E \to E \in Ho^s$ and a two sided unit $S \to E \in Ho^s$ where S is the sphere spectrum. As usual we let $X_*(Y) = \pi_*(X \wedge Y) = [S, X \wedge Y]_*$ for X, $Y \in Ho^s$.

§1. E-LOCALIZATIONS AND E_*-ACYCLIZATIONS OF SPECTRA

For $E \in Ho^s$ we now construct E_*-localizations and E_*-acyclizations in Ho^s and outline some elementary properties of E_*-localizations. (We omit most of the corresponding properties of E_*-acyclizations). A map $f: A \to B \in Ho^s$ is called an *E_*-equivalence* if $f_*: E_*A \approx E_*B$, and a spectrum $A \in Ho^s$ is called *E_*-acyclic* if $E_*A = 0$. We define a spectrum $C \in Ho^s$ to be *E_*-local* if each E_*-equivalence $f: A \to B$ in Ho^s induces a bijection $f^*: [B, C]_* \approx [A, C]_*$, or equivalently if $[A, C]_* = 0$ for each E_*-acyclic spectrum A. Each spectrum A can be decomposed into E_*-local and E_*-acyclic spectra as follows.

THEOREM 1.1. *Given E, $A \in Ho^s$ there is a natural* (*in A*) *triangle*

$$_EA \xrightarrow{\theta} A \xrightarrow{\eta} A_E \longrightarrow \Sigma(_EA)$$

in Ho^s such that $_EA$ is E_*-acyclic and A_E is E_*-local.

The proof is in 1.12–1.15 at the end of this section. We call $\eta: A \to A_E$ the *E_*-localization* of A and call $\theta: {}_EA \to A$ the *E_*-acyclization* of A. Clearly η is the initial example of a map from A to an E_*-local spectrum in Ho^s, and θ is the terminal example of a map from an E_*-acyclic spectrum to A. Also, η is the terminal example of an E_*-equivalence going out of A in Ho^s. By a formal argument, the E_*-localization is idempotent, i.e. for each $A \in Ho^s$ the maps η, $\eta_E: A_E \to (A_E)_E$ are equivalences with $\eta = \eta_E$. Note that if $f: A \to B$ is any E_*-equivalence in Ho^s with B E_*-local, then f is canonically equivalent to $\eta: A \to A_E$. Such a map $f: A \to B$ will be called an *E_*-localization* of A. We now list some elementary results on E_*-local spectra.

LEMMA 1.2. (*The E_*-Whitehead Theorem*) *If X, $Y \in Ho^s$ are E_*-local and $f: X \to Y$ is an E_*-equivalence, then $f: X \simeq Y$.*

LEMMA 1.3. *If X is a module spectrum over a ring spectrum E, then X is E_*-local* (*see e.g.* [4] *Lemma* 13.1, *p.* 283).

LEMMA 1.4. *If $W \to X \to Y \to \Sigma W$ is a triangle in Ho^s and any two of W, X, T are E_*-local, then so is the third.*

LEMMA 1.5. *The product of a set of E_*-local spectra is E_*-local.*

LEMMA 1.6. *A retract of an E_*-local spectrum is E_*-local.*

LEMMA 1.7. *We call a communative square*

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "h"] & X \arrow[d, "j"] \\ Y \arrow[r, "k"] & Z \end{tikzcd}$$

in Ho^s a homotopy fibre square if there is a given map $Z \to \Sigma W \in Ho^s$ such that

$$W \xrightarrow{(i,h)} X \vee Y \xrightarrow{(j,-k)} Z \longrightarrow \Sigma W$$

is a triangle in Ho^s. In this case, if any two of W, $X \vee Y$, Z are E_*-local, then so is the third.

LEMMA 1.8. *We call X_∞ a homotopy inverse limit of a sequence*

$$X_0 \xleftarrow{j_1} X_1 \xleftarrow{j_2} X_2 \longleftarrow \cdots$$

in Ho^s if there is a given triangle

$$X_\infty \longrightarrow \prod_{n\ge 0} X_n \xrightarrow{1-g} \prod_{n\ge 0} X_n \longrightarrow \Sigma X_\infty$$

in Ho^s where g is induced by the maps $j_1, j_2, \ldots$. In this case, if $X_0, X_1, X_2, \ldots$ are E_*-local, then so is X_∞.

Actually, a "homotopy inverse limit" of an arbitrary diagram of E_*-local spectra is E_*-local. However, an adequate discussion of such homotopy limits would take us too far afield since it would involve "homotopy categories of diagrams of spectra" rather than "diagrams in the homotopy category of spectra".

It is now easy to show that the E_*-localization functor commutes with various "finite" constructions.

LEMMA 1.9. *For $X, Y \in Ho^s$ there are canonical equivalences $(X \vee Y)_E \simeq X_E \vee Y_E$, $(\Sigma X)_E \simeq \Sigma(X_E)$, $(\Sigma^{-1}X)_E \simeq \Sigma^{-1}(X_E)$.*

LEMMA 1.10. *If $W \to X \to Y \to \Sigma W$ is a triangle in Ho^s, then so is $W_E \to X_E \to Y_E \to (\Sigma W)_E \simeq \Sigma W_E$.*

To prove 1.10, first prolong $W_E \to X_E$ to a triangle $W_E \to X_F \to C \to \Sigma W_E$ and then construct a map of triangles

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\eta"] & X \arrow[r] \arrow[d, "\eta"] & Y \arrow[r] \arrow[d, "j"] & \Sigma W \arrow[d, "\Sigma\eta"] \\ W_E \arrow[r] & X_E \arrow[r] & C \arrow[r] & \Sigma W_E. \end{tikzcd}$$

It is not hard to show that j is an E_*-localization, and that the sequence $W_E \to X_E \to Y_E \to \Sigma W_E$ is equivalent to the lower triangle.

Although the E_*-localization functor need not preserve smash products, there is a canonical map $X_E \wedge Y_E \to (X \wedge Y)_E$ with the obvious properties for $X, Y \in Ho^s$. Thus if G is a commutative ring spectrum, then so is G_E. In particular, S_E is a commutative ring spectrum, and each E_*-local spectrum is canonically a module spectrum over S_E. Moreover, for each $X \in Ho^s$ there is a canonical map $S_E \wedge X \to X_E$ of S_E-module spectra, which is an equivalence whenever X is a finite CW-spectrum.

To compare the E_*-localizations for various $E \in Ho^s$, we use the notation of [9]. Thus $\underset{\sim}{A}(Ho^s)$ is the class of all $\langle E\rangle$ for $E \in Ho^s$, where $\langle E\rangle$ is the equivalence class consisting of all $G \in Ho^s$ such that the G_*-acyclic spectra are the same as the E_*-acyclic spectra. Clearly, the E_*-localization and G_*-localization functors are equivalent iff $\langle E\rangle = \langle G\rangle$, and a determination of $\underset{\sim}{A}(Ho^s)$ would provide an inventory of all possible homological localization functors. As in [9], 2.1, we partially order $\underset{\sim}{A}(Ho^s)$ by defining $\langle E\rangle \leq \langle G\rangle$ if each G_*-acyclic spectrum is E_*-acyclic. Note that $\langle 0\rangle$ is the smallest member of $\underset{\sim}{A}(Ho^s)$, and $\langle S\rangle$ is the largest.

PROPOSITION 1.11. *(Ravenel) If $\langle E\rangle \leq \langle G\rangle$ in $\underset{\sim}{A}(Ho^s)$ then: (i) Each E_*-local spectrum is G_*-local; (ii) For each $X \in Ho^s$ there are natural equivalences $(X_G)_E \simeq X_E \simeq (X_E)_G$, ${}_G(X_E) \simeq 0 \simeq ({}_GX)_E$, ${}_E({}_GX) \simeq {}_GX \simeq {}_G({}_EX)$, and ${}_E(X_G) \simeq ({}_EX)_G$; (iii) For each $X \in Ho^s$ there are natural triangles ${}_EX_G \to X_G \to X_E \to \Sigma({}_EX_G)$ and ${}_GX \to {}_EX \to {}_EX_G \to \Sigma({}_GX)$ in Ho^s where ${}_EX_G$ denotes ${}_E(X_G) \simeq ({}_EX)_G$.*

Proof. These are straightforward in the given order. For instance, ${}_E(X_G) \simeq ({}_EX)_G$ follows by showing that the natural map ${}_EX \to {}_E(X_G)$ is a G_*-localization. It is a G_*-equivalence because there is a triangle ${}_E({}_GX) \to {}_EX \to {}_E(X_G) \to \Sigma({}_E({}_GX))$ obtained by applying ${}_E(\ )$ to the triangle ${}_GX \to X \to X_G \to \Sigma({}_GX)$, and because ${}_E({}_GX) \simeq {}_GX$ is G_*-acyclic. Furthermore, ${}_E(X_G)$ is G_*-local because there is a triangle ${}_E(X_G) \to X_G \to (X_G)_E \to \Sigma({}_E(X_G))$ where X_G and $(X_G)_E$ are G_*-local.

We now proceed to prove 1.1. Let σ be a fixed infinite cardinal number which is at least equal to the cardinality of π_*E. For a CW-spectrum X, let $\#X$ denote the number of stable cells in X (see e.g. [4], p. 143), and note that $E_*(X)$ has at most σ elements if $\#X \leq \sigma$. Recall that a subspectrum B of a CW-spectrum X is *closed* if "whenever a cell of X gets into B later, it is in B to start with", or equivalently if X/B is a CW-spectrum (see e.g. [4], p. 155).

LEMMA 1.12. *Let X be a CW-spectrum and let $B \subset X$ be a proper closed subspectrum with $E_*(X/B) = 0$. Then there exists a closed subspectrum $W \subset X$ such that $\#W \leq \sigma$, $W \not\subset B$, and $E_*(W/W \cap B) = 0$.*

Proof. The desired W is given by the union $W = \bigcup_{n\geq 1} W_n$ where $W_1 \subset \ldots \subset W_n \subset W_{n+1} \subset \ldots$ is a sequence of closed subspectra of X such that $\#W_n \leq \sigma$, $W_n \not\subset B$, and the map $E_*(W_n/W_n \cap B) \to E_*(W_{n+1}/W_{n+1} \cap B)$ is zero for each $n \geq 1$. To inductively construct $\{W_n\}$, first choose a closed subspectrum $W_1 \subset X$ such that $\#W_1 \leq \sigma$ and $W_1 \not\subset B$. Then, given W_n, choose for each element $x \in E_*(W_n/W_n \cap B)$ a closed finite subspectrum $F_x \subset X$ such that x goes to zero in $E_*(W_n \cup F_x/(W_n \cup F_x) \cap B)$. This is possible because $E_*(X/B) = 0$. Finally, let W_{n+1} be the union of W_n with all F_x for $x \in E_*(W_n/W_n \cap B)$.

For $Y \in Ho^s$, let Class-Y denote the smallest class of spectra in Ho^s such that: (i) $Y \in$ Class-Y; (ii) if $\{X_\alpha\}$ is a set of spectra in Class-Y, then $\bigvee_\alpha X_\alpha \in$ Class-Y; (iii) if $V \to W \to X \to \Sigma V$ is a triangle in Ho^s and any two of V, W, X are in Class-Y, then so is the third. In ([9] 1.6) we showed that Class-Y equals the class of "$[Y,\ ]_*$-colocal" spectra in Ho^s. Thus, for instance, if $\bigvee_\alpha X_\alpha \in$ Class-Y then each X_α is in Class-Y.

LEMMA 1.13. *There exists an E_*-acyclic spectrum $aE \in Ho^s$ such that Class-aE equals the class of all E_*-acyclic spectra in Ho^s.*

Proof. Let $\{K_\alpha\}$ be a set of E_*-acyclic CW-spectra such that $\{K_\alpha\}$ contains, up to equivalence, each E_*-acyclic $W \in Ho^s$ with $\#W \leq \sigma$. Now define $aE = \bigvee_\alpha K_\alpha$ and note that Class-aE is clearly contained in the class of E_*-acyclic spectra. It remains to show that each E_*-acyclic CW-spectrum X is in Class-aE. Using 1.12 construct a transfinite sequence

$$0 = B_0 \subset B_1 \subset \ldots \subset B_s \subset B_{s+1} \subset \ldots$$

of E_*-acyclic closed subspectra of X such that: (i) if λ is a limit ordinal then $B_\lambda = \bigcup_{s<\lambda} B_s$, (ii) if $B_s = X$ then $B_{s+1} = X$, and (iii) if $B_s \neq X$ then $B_{s+1} = B_s \cup W$ for some closed subspectrum $W \subset X$ with $\#W \leq \sigma$, $W \not\subset B$, and $E_*(W/W \cap B_s) = 0$. Then $X = B_\gamma$ for some γ, and it suffices to show $B_s \in$ Class-aE for all s. Clearly $B_0 \in$ Class-aE. If $B_s \in$ Class-aE, then $B_{s+1}/B_s \simeq K_\alpha$ for some α, and $K_\alpha \in$ Class-aE since $aE = \bigvee_\alpha K_\alpha$. If λ is a limit ordinal and $B_s \in$ Class-aE for all $s < \lambda$, then $B_\lambda \in$ Class-aE since there is a triangle

$$\bigvee_{s<\lambda} B_s \xrightarrow{1-g} \bigvee_{s<\lambda} B_s \longrightarrow B_\lambda \longrightarrow \Sigma(\bigvee_{s<\lambda} B_s) \in Ho^s$$

where g is induced by the inclusion maps $B_s \to B_{s+1}$. Thus $B_s \in$ Class-aE for all s.

LEMMA 1.14. *Let aE be as in 1.13. Then a spectrum $X \in Ho^s$ is E_*-local $\Leftrightarrow$ $[aE, X]_* = 0$.*

The proof is straightforward.

1.15 *Proof of* 1.1. Let $aE \in Ho^s$ be as in 1.13. Then by ([9], 1.6 and 1.7) there is a natural (in A) triangle

$${}^{aE}A \longrightarrow A \longrightarrow A^{aE} \longrightarrow \Sigma({}^{aE}A) \in Ho^s$$

such that ${}^{aE}A \in$ Class-aE and $[aE, A^{aE}]_* = 0$. Thus ${}^{aE}A$ is E_*-acyclic and A^{aE} is E_*-local, so we let ${}_EA \xrightarrow{\theta} A \xrightarrow{\eta} A_E \to \Sigma({}_EA)$ denote the above triangle.

Our proof of 1.1 actually shows somewhat more: it leads to the duality theorem for $\underline{A}(Ho^s)$ in §7; and it leads to constructions (which we omit) of E_*-localization functors on underlying categories of Ho^s, e.g. the category of CW-spectra and cellular functions of degree zero.

§2. LOCALIZATIONS WITH RESPECT TO MOORE SPECTRA

We will determine the SG_*-localizations of spectra where SG is a Moore spectrum. These localizations are essentially well-known, and are very similar to the usual localizations and completions of simply connected spaces, see e.g. [7, 8]. We also compare EG_*-localizations with E_*-localizations, where EG is a spectrum E with "coefficients" G, i.e. $EG \simeq E \wedge SG$. These results will be used in §3 to determine localizations of connective spectra with respect to arbitrary connective homology theories.

For an abelian group G, let $SG \in Ho^s$ be a Moore spectrum of type $(G, 0)$, i.e. $\pi_i SG = 0$ for $i < 0$, $H_0 SG = G$, and $H_i SG = 0$ for $i \neq 0$. Note that SG is determined up to (non-canonical) equivalence by G, and for a spectrum X there are short exact sequences

$$0 \to G \otimes \pi_* X \to \pi_* SG \wedge X \to \operatorname{Tor}(G, \pi_{*-1} X) \to 0 \tag{2.1}$$

$$0 \to \operatorname{Ext}(G, \pi_{*+1} X) \to [SG, X]_* \to \operatorname{Hom}(G, {''}_* X) \to 0. \tag{2.2}$$

We say that two abelian groups G_1 and G_2 have the same *type of acyclicity* if:

(i) G_1 is a torsion group iff G_2 is a torsion group.

(ii) For each prime p, G_1 is uniquely p-divisible iff G_2 is uniquely p-divisible.

By ([9], 2.13) and its proof, we have

PROPOSITION 2.3. Two abelian groups G_1 and G_2 have the same type of acyclicity $\Leftrightarrow$ $\langle SG_1 \rangle = \langle SG_2 \rangle$ in $\underline{A}(Ho^s)$ $\Leftrightarrow$ SG_1 and SG_2 give equivalent localization functors on Ho^s.

Thus to determine all SG_*-localization functors, it suffices to assume $G = Z_{(J)}$ or $G = \bigoplus_{p \in J} Z/p$, where J is a set of primes and $Z_{(J)} \subset Q$ denotes the integers localized at J.

PROPOSITION 2.4. *Let $G = Z_{(J)}$ for a set J of primes. Then $X_{SG} \simeq SG \wedge X$ and $\pi_* X_{SG} \simeq G \otimes \pi_* X$ for each $X \in Ho^s$. Moreover, a spectrum $X \in Ho^s$ is SG_*-local $\Leftrightarrow$ the groups $\pi_* X$ are uniquely p-divisible for all primes $p \notin J$.*

Proof. Since SG is a ring spectrum, $SG \wedge X$ is SG_*-local by 1.3. Thus the map $X \simeq S \wedge X \xrightarrow{i \wedge 1} SG \wedge X$ is an SG_*-localization where $i : S \to SG$ represents $1 \in Z_{(J)} \approx \pi_0 SG$, and the proposition follows easily.

To obtain a corresponding result for $G = Z/p$ with p prime, let

$$SZ/p \xleftarrow{q_1} SZ/p^2 \xleftarrow{q_2} SZ/p^3 \xleftarrow{q_3} \cdots$$

be a sequence in Ho^s which is carried by H_0 to the sequence of quotient maps $Z/p \leftarrow Z/p^2 \leftarrow Z/p^3 \leftarrow \ldots$. For $X \in Ho^s$ the function spectrum $F(S^{-1}Z/p^\infty, X)$ is a homotopy inverse limit (see 1.8) of the sequence

$$SZ/p \wedge X \xleftarrow{q_1 \wedge 1} SZ/p^2 \wedge X \xleftarrow{q_2 \wedge 1} SZ/p^3 \wedge X \longleftarrow \cdots$$

where $S^{-1}Z/p^\infty \in Ho^s$ is the desuspension of the Moore spectrum SZ/p^∞ with $Z/p^\infty = Z[1/p]/Z$. This follows since $S^{-1}Z/p^\infty$ is a "homotopy direct limit" of the sequence $DSZ/p \xrightarrow{Dq_1} DSZ/p^2 \xrightarrow{Dq_2} \ldots$ where D is the Spanier-Whitehead duality functor.

PROPOSITION 2.5. *Let $G = Z/p$ for p prime. Then $X_{SG} \simeq F(S^{-1}Z/p^\infty, X)$ and there is a splittable short exact sequence*

$$0 \to \operatorname{Ext}(Z/p^\infty, \pi_* X) \longrightarrow \pi_* X_{SG} \longrightarrow \operatorname{Hom}(Z/p^\infty, \pi_{*-1} X) \to 0$$

for each $X \in Ho^s$. If the groups $\pi_* X$ are finitely generated, then $\pi_* X_{SG} \approx Z_p^\wedge \otimes \pi_* X$

where $Z_p^\wedge$ denotes the p-adic integers. Moreover, a spectrum $X \in Ho^s$ is SG_*-local $\Leftrightarrow$ the groups $\pi_* X$ are Ext-*p-complete in the sense of* ([7], *p.* 172).

Proof. The obvious triangle

$$S^{-1}Z/p^\infty \longrightarrow S \longrightarrow SZ[1/p] \longrightarrow \Sigma S^{-1}Z/p^\infty$$

induces a triangle

$$F(SZ[1/p], X) \to F(S, X) \to F(S^{-1}Z/p^\infty, X) \to \Sigma F(SZ[1/p], X)$$

in Ho^s, with $F(S, X) \simeq X$. The spectrum $F(S^{-1}Z/p^\infty, X)$ is SZ/p_*-local since any SZ/p_*-acyclic spectrum is also $S^{-1}Z/p^\infty{}_*$-acyclic by 2.3. The spectrum $F(SZ[1/p], X)$ is SZ/p_*-acyclic since its homotopy groups are uniquely p-divisible. Thus $X \to F(S^{-1}Z/p^\infty, X)$ is a SZ/p_*-localization. Now the results on $\pi_* X_{SZ/p}$ follow from (2.2) and ([7], p. 183). Finally, a spectrum $X \in Ho^s$ is SZ/p_*-local $\Leftrightarrow$ $F(SZ[1/p], X) \simeq 0$ $\Leftrightarrow$ $\mathrm{Ext}(Z[1/p], \pi_* X) = 0 = \mathrm{Hom}(Z[1/p], \pi_* X)$ $\Leftrightarrow$ the groups $\pi_* X$ are Ext-p-complete.

The above proof easily generalizes to show

PROPOSITION 2.6. *Let $G = \bigoplus_{p \in J} Z/p$ for a set J of primes. Then $X_{SG} \simeq \prod_{p \in J} X_{SZ/p}$ for each $X \in Ho^s$. Moreover, a spectrum $X \in Ho^s$ is SG_*-local $\Leftrightarrow$* $\mathrm{Ext}(Z[J^{-1}], \pi_* X) = 0 = \mathrm{Hom}(Z[J^{-1}], \pi_* X)$.

This completes our determination of SG_*-localizations, and we next discuss EG_*-localizations where EG is a spectrum E with coefficients in G, i.e. $EG = E \wedge SG$. The results are similar to those in the unstable case, see e.g. [14, 16]. First, 2.3 implies

PROPOSITION 2.7. *Let G_1 and G_2 be abelian groups with the same type of acyclicity, and let $E \in Ho^s$. Then $\langle EG_1 \rangle = \langle EG_2 \rangle$, and EG_1 and EG_2 give equivalent localization functors on Ho^s.*

Thus it suffices to consider EG_*-localizations where $G = Z_{(J)}$ or $G = \bigoplus_{p \in J} Z/p$ for a set J of primes. It is easy to prove

PROPOSITION 2.8. *If $EQ \neq 0$ then $X_{EQ} \simeq X_{SQ}$; and if $EQ \simeq 0$ then $X_{EQ} \simeq 0$ for $X \in Ho^s$.*

We next establish a useful "arithmetic square" result.

PROPOSITION 2.9. *For $E, X \in Ho^s$ there is a homotopy fibre square (see* 1.7)

$$\begin{array}{ccc} X_E & \xrightarrow{i} & \prod_{p \text{ prime}} X_{EZ/p} \\ \downarrow h & & \downarrow k \\ X_{EQ} & \xrightarrow{j} & \left(\prod_{p \text{ prime}} X_{EZ/p}\right)_{EQ} \end{array}$$

in Ho^s where h, i, j, k are the obvious maps.

Proof. Construct a homotopy fibre square

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"] & \prod_{p \text{ prime}} X_{EZ/p} \arrow[d, "k"] \\ X_{EQ} \arrow[r, "j"] & (\prod_{p \text{ prime}} X_{EZ/p})_{EQ} \end{tikzcd}$$

in Ho^s, and note that P is E_*-local. Let $u: X \to P$ be a map which induces the obvious maps $\eta: X \to X_{EQ}$ and $w: X \to \prod_{p \text{ prime}} X_{EZ/p}$. It suffices to show u is an E_*-equivalence. By a Mayer–Vietoris argument, g is an EQ_*-equivalence and f is an EZ/p_*-equivalence for each p. Clearly, η is an EQ_*-equivalence, and w is an EZ/p_*-equivalence for each p because $\prod_{q \neq p} X_{EZ/q}$ is SZ/p_*-acyclic since $X_{EZ/q}$ is SZ/q_*-local. Thus $u: X \to P$ is an EQ_*-equivalence and an EZ/p_*-equivalence for each prime p, and therefore u is an E_*-equivalence.

Using similar techniques one also proves

PROPOSITION 2.10. *Let $E, X \in Ho^s$, let J be a set of primes, and let K be the complement of J. Then there is a homotopy fibre square*

$$\begin{tikzcd} X_E \arrow[r, "i"] \arrow[d, "h"] & X_{EZ_{(J)}} \arrow[d, "k"] \\ X_{EZ_{(K)}} \arrow[r, "j"] & X_{EQ} \end{tikzcd}$$

in Ho^s where h, i, j, k are the obvious maps.

In general, the spectra $(X_E)_{SG}$ and X_{EG} need not be equivalent; for instance $(H_{SZ/p})_{SQ} \neq 0$ for p prime where H is the spectrum of integral homology. However, we have

PROPOSITION 2.11. *Let $E, X \in Ho^s$ and let G be an abelian group. If either $EQ \neq 0$ or G is a torsion group, then $X_{EG} \simeq (X_E)_{SG}$.*

Proof. First suppose $G = Z/q$ for q prime. Then by 2.9, the map $X_E \to \prod_{p \text{ prime}} X_{EZ/p}$ is an SZ/q_*-equivalence, and thus $X_E \to X_{EZ/p}$ is also. Hence $X_{EZ/q} \simeq (X_E)_{SZ/q}$ because $X_{EZ/q}$ is clearly SZ/q_*-local. Next suppose $G = \bigoplus_{p \in J} Z/p$ for a set J of primes. Then clearly $X_{EG} \simeq \prod_{p \in J} X_{EZ/p}$ and $(X_E)_{SG} \simeq \prod_{p \in J} (X_E)_{SZ/p}$ by 2.6. Thus $X_{EG} \simeq (X_E)_{SG}$. Now suppose $EQ \neq 0$ and $G = Z_{(J)}$ for a set J of primes. Then by 2.10 the map $X_E \to X_{EZ_{(J)}}$ is an SZ/p_*-equivalence for each prime $p \in J$, and is an SQ_*-equivalence since $EQ \neq 0$. Hence $X_{EZ_{(J)}} \simeq (X_E)_{SZ_{(J)}}$ because $X_{EZ_{(J)}}$ is clearly $SZ_{(J)*}$-local. Finally, the proposition follows by combining the above cases with 2.3 and 2.7.

§3. GENERAL EXAMPLES OF E_*-LOCALIZATIONS

We first compute the E_*-localizations of connective spectra where E is an arbitrary connective spectrum. Then we determine, in principle, the E_*-localizations of arbitrary spectra where E is either a wedge of finite CW-spectra or is the "complement" of such a wedge. Finally, we introduce "E-nilpotent" spectra and discuss their applications to E_*-localization theory. This will be used later when we study K_*-localizations and the E_*-Adams spectra sequence.

Let E be a connective spectrum (i.e. $\pi_n E = 0$ for sufficiently small n) and let G be an abelian group having the same type of acyclicity as $\bigoplus_n \pi_n E$. We can assume G is of the form $Z_{(J)}$ or $\bigoplus_{p \in J} Z/p$ for a set J of primes. Then

THEOREM 3.1. *$X_E \simeq X_{SG}$ for each connective spectrum $X \in Ho^s$. Of course X_{SG} can be computed as in* 2.4, 2.5, *or* 2.6. *Our proof depends on the following lemmas.*

LEMMA 3.2. *$\langle HG \rangle \le \langle E \rangle \le \langle SG \rangle$ in $\underline{A}(Ho^s)$ for E and G as above.*

Proof. It suffices to show $\langle HG \rangle = \langle H \wedge E \rangle$ and $\langle E \rangle = \langle SG \wedge E \rangle$ in $\underline{A}(Ho^s)$. Since E is connective, the groups $\bigoplus_n \pi_n E$ and $\bigoplus_n H_n E$ have the same type of acyclicity, and thus

$$\langle HG \rangle = \langle H(\bigoplus_n H_n E) \rangle = \langle \bigvee_n \Sigma^n H(H_n E) \rangle = \langle H \wedge E \rangle.$$

Now let G_1 be a sum of members of $\{Q, Z/2, Z/3, Z/5, \ldots\}$ such that G_1 has the same type of acyclicity as G, and let G_2 be the sum of the complementary members. Then $SG_2 \wedge E \simeq 0$ and thus

$$\langle E \rangle = \langle (SG_1 \vee SG_2) \wedge E \rangle = < SG_1 \wedge E \rangle = \langle SG \wedge E \rangle.$$

LEMMA 3.3. *If $X \in Ho^s$ is connective, then X is H_*-local.*

Proof. Each Eilenberg–MacLane spectrum $H(\pi_n X)$ is H_*-local by (1.3). Thus the Postnikov sections of X are H_*-local by (1.4), and X is H_*-local by (1.8).

Proof of 3.1. The SG_*-localization $X \to X_{SG}$ is an E_*-equivalence by 3.2, and we must show that X_{SG} is E_*-local. But $X_{SG} \simeq (X_H)_{SG} \simeq X_{HG}$ by 3.3 and 2.11, and thus X_{SG} is E_*-local by 3.2.

We remark that Theorem 3.1 can easily be generalized to show $X_E \simeq X_{SG}$ whenever X is H_*-local and the groups $\bigoplus_n \pi_n E$ and $\bigoplus_n H_n E$ have the same type of acyclicity.

We now turn to a different sort of generalization of our results on SG_*-localizations. Let B be a (possibly infinite) wedge of finite CW-spectra in Ho^s. As in ([9] 1.8), let $\nu : S \to S^B$ denote the $[B, \ ]_*$-trivialization of the sphere spectrum, i.e. ν is the initial example of a map $S \to X$ in Ho^s with $[B, X]_* = 0$. For example, if $B = \bigvee_{p \in J} SZ/p$ where J is a set of primes, then $S^B \simeq SZ[J^{-1}]$. We remark that $\langle S^B \rangle$ is the "complement" of $\langle B \rangle$ in $\underline{A}(Ho^s)$ by ([9], 2.9). Furthermore, S^B is a commutative ring spectrum with unit ν, and has the unusual property that the multiplication map $S^B \wedge S^B \to S^B$ is an equivalence, see ([9], 2.9ff.). There is some reason to think

CONJECTURE 3.4. *If E is a ring spectrum in Ho^s such that the multiplication map $E \wedge E \to E$ is an equivalence, then $E \simeq S^B$ for some wedge B of finite CW-spectra.*

Now the following two propositions generalize our results on SG_*-localizations, see 2.4, 2.5 and 2.6.

PROPOSITION 3.5. *If $\langle E\rangle = \langle S^B\rangle$ where B is a (possibly infinite) wedge of finite CW-spectra and $X \in Ho^s$, then $X_E \simeq S^B \wedge X$. Moreover, a spectrum X is E_*-local $\Leftrightarrow$ $[B, X]_* = 0$.*

Proof. Since S^B is a ring spectrum with unit $\nu : S \to S^B$ and multiplication $S^B \wedge S^B \xrightarrow{\simeq} S^B$, it is easy to show that the map $X \simeq S \wedge X \xrightarrow{\nu \wedge 1} S^B \wedge X$ is an S^B_*-localization. The "$\Leftrightarrow$" statement follows since the map $X \to S^B \wedge X$ is a $[B, \;]_*$-trivialization by ([9], Proof 2.9].

As in ([9], 1.7), let $\varphi : {}^BS \to S$ denote the $[B, \;]_*$-colocalization of S in Ho^s. Recall that ${}^BS$ is the "fibre" of $\nu : S \to S^B$.

PROPOSITION 3.6. *If $\langle E\rangle = \langle B\rangle$ where B is a (possibly infinite) wedge of finite CW-spectra and $X \in Ho^s$, then $X_E \simeq F({}^BS, X)$. Moreover, a spectrum X is E_*-local $\Leftrightarrow$ $[S^B, X]_* = 0$.*

Proof. The triangle ${}^BS \xrightarrow{\varphi} S \xrightarrow{\nu} S^B \longrightarrow \Sigma({}^BS)$ of ([9], 1.7) induces a triangle of function spectra

$$F(S^B, X) \longrightarrow F(S, X) \longrightarrow F({}^BS, X) \longrightarrow \Sigma F(S^B, X)$$

with $F(S, X) \cong X$. Thus it suffices to show $F({}^BS, X)$ is B_*-local, and $F(S^B, X)$ is B_*-acyclic. The former follows since $\langle B\rangle = \langle {}^BS\rangle$ by ([9], 2.9). For the latter, we must show $B_\alpha \wedge F(S^B, X) \simeq 0$ where B_α is a finite CW-summand of B. This follows since $B_\alpha \wedge F(S^B, X) \simeq F(F(B_\alpha, S^B), X)$ and $F(B_\alpha, S^B) \simeq 0$.

Finally we introduce E-nilpotent spectra and describe E_*-localizations when S is "E-prenilpotent."

Definition 3.7. For a ring spectrum E, the *E-nilpotent* spectra form the smallest class $\underset{\sim}{C}$ of spectra in Ho^s such that: (i) $E \in \underset{\sim}{C}$; (ii) If $N \in \underset{\sim}{C}$ and $X \in Ho^s$, then $N \wedge X \in \underset{\sim}{C}$; (iii) If $X \to Y \to Z \to \Sigma X$ is a triangle in Ho^s and in Ho^s and two of X, Y, Z are in $\underset{\sim}{C}$, then so the third; (iv) If $N \in \underset{\sim}{C}$ and M is a retract of N in Ho^s, then $M \in \underset{\sim}{C}$.

Note that a module spectrum X over E is E-nilpotent since X is a retract of $E \wedge X$.

LEMMA 3.8. *If $X \in Ho^s$ is E-nilpotent, then X is E_*-local.*

Proof. We filter the class $\underset{\sim}{C}$ of E-nilpotent spectra as follows. Let $\underset{\sim}{C}_0$ consist of all spectra equivalent to $E \wedge X$ for some $X \in Ho^s$; and given $\underset{\sim}{C}_m$ with $m \geq 0$, let $\underset{\sim}{C}_{m+1}$ consist of all spectra $N \in Ho^s$ such that either N is a retract of a member of $\underset{\sim}{C}_m$ or there is a triangle $X \to N \to Z \to \Sigma X$ in Ho^s with $X, Z \in \underset{\sim}{C}_m$. Induction shows that if $N \in \underset{\sim}{C}_m$ for some m, then N is E_*-local and $N \wedge X \in \underset{\sim}{C}_m$ for each $X \in Ho^s$. Thus $\underset{\sim}{C} = \bigcup_{m \geq 0} \underset{\sim}{C}_m$ and the E-nilpotent spectra are E_*-local.

We call $Y \in Ho^s$ *E-prenilpotent* if there exists an E_*-equivalence $Y \to N$ such that N is E-nilpotent. This is equivalent to saying that Y_E is E-nilpotent. Since the smash product of an E-nilpotent spectrum with an arbitrary spectrum is E-nilpotent, it is not hard to show

PROPOSITION 3.9. *If the sphere spectrum S is E-prenilpotent for a ring spectrum E, then*: (i) $S_E \wedge Y \xrightarrow{\simeq} Y_E$ *for each $Y \in Ho^s$*; (ii) *Each $Y \in Ho^s$ is E-prenilpotent, and* *the E-nilpotent spectra are the same as the E_*-local spectra*; (iii) *The ring spectrum S_E has multiplication map $S_E \wedge S_E \xrightarrow{\simeq} S_E$*; (iv) $\langle E\rangle = \langle S_E\rangle$ *in* $\underset{\sim}{A}(Ho^s)$.

The hypotheses of 3.9 clearly hold when E has multiplication $E \wedge E \xrightarrow{\simeq} E$, in which case $S_E \simeq E$. They also hold when $E = K$ (see 4.7) and when the E_*-Adams spectral sequence satisfies appropriate vanishing conditions (see 6.12).

§4. *K*-THEORETIC LOCALIZATIONS OF SPECTRA

Let K be the spectrum of non-connective complex K-theory. We now show that K_*-localizations are actually examples of the general localizations given in 3.5 and 3.9. In more detail, S_K is a K-nilpotent spectrum closely related to the ImJ spectrum, and $X_K \simeq S_K \wedge X$ for all X. Furthermore, $\langle K\rangle = \langle S^B\rangle$ for a certain wedge B of finite CW-spectra. Thus $\langle K\rangle$ is the complement of $\langle B\rangle$ in the Boolean algebra of spectra (see [9]), and a spectrum Y is K_*-local iff $[B, Y]_* = 0$. Our B is the wedge of the cofibres of Adams' K_*-equivalences $A_2 : \Sigma^8 M_2 \to M_2$ and $A_P : \Sigma^{2(p-1)} M_p \to M_p$ for odd primes p, where M_p is the Moore spectrum $SZ/p*S^0U_pe^1$, (see §12 of [2]). Consequently, a spectrum Y is K_*-local iff its mod-p homotopy groups are periodic via A_p for each prime p. We remark that K_*-localizations are the same as KO_*-localizations since $\langle K\rangle = \langle KO\rangle$ by [14], [9], or [17]. The author learned some of the main results in this section (Theorem 4.3 and its corollaries) from Doug Ravenel and understands that these results were also obtained by Frank Adams and David Baird in earlier unpublished work. The present results lead to a description of K_*-localizations for infinite loop spaces which we hope to give in a future paper.

For p prime we form the sequence

$$M_p \xrightarrow{\Sigma^{-d}A_p} \Sigma^{-d}M_p \xrightarrow{\Sigma^{-2d}A_p} \Sigma^{-2d}M_p \xrightarrow{\Sigma^{-3d}A_p} \cdots$$

in Ho^s where $d = 2(p-1)$ for p odd and $d = 8$ for $p = 2$, and we let $M_p^\infty \in Ho^s$ denote a homotopy direct limit (mapping telescope) of this sequence. Our proofs in this section are based on

4.1. THEOREM (*Mahowald and Miller*). *The groups $\pi_i M_2^\infty$ have orders*: 4 *for $i \equiv 0$ mod 8*; 8 *for $i \equiv 1$ mod 8*; 8 *for $i \equiv 2$ mod 8*; 4 *for $i \equiv 3$ mod 8*; 2 *for $i \equiv 4$ mod 8*; 2 *for $i \equiv 7$ mod 8*; *and* 1 *otherwise. For an odd prime p, the groups $\pi_i M_p^\infty$ have orders*: p *for $i \equiv 0$ mod $2(p-1)$*; p *for $i \equiv -1$ mod $2(p-1)$*; *and* 1 *otherwise.*

The case $p = 2$ was proved by Mark Mahowald in his work on the J-homomorphism [10–12]. The case p odd was recently proved by Haynes Miller in [15]. Actually the K_*-equivalences A_p for p prime were not explicitly chosen in ([2] §12), but we can (and do) assume they are now chosen so that 4.1 holds.

The canonical map $i : M_p \to M_p^\infty$ is clearly a K_*-equivalence and we will show that it is actually a K_*-localization for each prime p. If $p = 2$ let $r = 3$, and if p is odd let r be a positive integer which reduces to a generator of the group of units of Z/p^2. Form a map

$$\begin{array}{ccccccc}
j_{(p)} & \longrightarrow & KOZ_{(p)}\langle 0,\ldots,\infty\rangle & \xrightarrow{\psi^r-1} & KOZ_{(p)}\langle 4,\ldots,\infty\rangle & \longrightarrow & \Sigma j_{(p)} \\
\downarrow a & & \downarrow b & & \downarrow c & & \downarrow \Sigma a \\
\mathcal{J}_{(p)} & \longrightarrow & KOZ_{(p)} & \xrightarrow{\psi^r-1} & KOZ_{(p)} & \longrightarrow & \Sigma\mathcal{J}_{(p)}
\end{array}$$

of triangles in Ho^s, where ψ^r is the stable Adams operation and b, c are the connective covers. Note that the spectra $KOZ_{(p)}$, $\mathcal{J}_{(p)}$, and $M_p \wedge \mathcal{J}_{(p)}$ are K_*-local since they are KO_*-local by 1.3–1.4 and since $\langle K\rangle = \langle KO\rangle$. Now let $e: S \to j_{(p)}$ be the map in Ho^s which is carried to the unit of $KOZ_{(p)}\langle 0, \dots, \infty\rangle$. Because $i: M_p \to M_p^\infty$ is a K_*-equivalence and $M_p \wedge \mathcal{J}_{(p)}$ is K_*-local, there is a unique map $\theta: M_p^\infty \to M_p \wedge \mathcal{J}_{(p)}$ in Ho^s making the diagram

$$\begin{array}{ccc} M_p & \xrightarrow{\simeq} & M_p \wedge S \\ \downarrow i & & \downarrow 1 \wedge ac \\ M_p^\infty & \overset{\theta}{\dashrightarrow} & M_p \wedge \mathcal{J}_{(p)} \end{array}$$

commute.

PROPOSITION 4.2. *For each prime p, the map* $\theta: M_p^\infty \to M_p \wedge \mathcal{J}_{(p)}$ *is an equivalence in* Ho^s *and* $i: M_p \to M_p^\infty$ *is a* K_**-localization.*

Proof. To show θ is an equivalence it suffices to prove $\theta_*: \pi_* M_p^\infty \to \pi_*(M_p \wedge \mathcal{J}_{(p)})$ is onto, since the groups $\pi_*(M_p \wedge \mathcal{J}_{(p)})$ have the same finite orders as $\pi_* M_p^\infty$ (given in 4.1). The map $e: S \to j_{(p)}$ induces a splittable epimorphism $e_*: \pi_m S \to \pi_m j_{(p)}$ for $m > 0$ by J-theory as outlined in ([13] pp. 476–480). Specifically, the map $e: S_{(p)} \to j_{(p)}$ of spectra restricts (by taking the 1-components of the 0^{th} spaces of the associated Ω-spectra) to a map $e: SF_{(p)} \to J_{\oplus(p)}$; there is a map $\alpha: J_{(p)} \to SF_{(p)}$ such that the composite $e\alpha: J_{(p)} \to J_{\otimes(p)}$ is an equivalence; and thus $e_*: \pi_m S \to \pi_m j_{(p)}$ is a splittable epimorphism for $m > 0$. It follows that

$$(1 \wedge ae)_*: \pi_m(M_p \wedge S) \to \pi_m(M_p \wedge \mathcal{J}_{(p)})$$

is onto for $m > 2$. Since $A_p: \Sigma^d M_p \to M_p$ is a K_*-equivalence and $\mathcal{J}_{(p)}$ is K_*-local, $A_p^*: [M_p, \mathcal{J}_{(p)}]_* \approx [\Sigma^d M_p, \mathcal{J}_{(p)}]_*$ and thus $A_p \wedge 1: \Sigma^d M_p \wedge \mathcal{J}_{(p)} \simeq M_p \wedge \mathcal{J}_{(p)}$ by ([9] 2.10). The desired surjectively of $\theta_*: \pi_* M_p^\infty \to \pi_*(M_p \wedge \mathcal{J}_{(p)})$ now follows using the commutative ladder

$$\begin{array}{ccccccccc} M_p \wedge S & \longrightarrow & \Sigma^{-d} M_p \wedge S & \longrightarrow & \Sigma^{-2d} M_p \wedge S & \to & \dots \to & & M_p^\infty \wedge S \\ \downarrow 1 \wedge ae & & \downarrow \Sigma^{-d}(1 \wedge ae) & & \downarrow \Sigma^{-2d}(1 \wedge ae) & & & & \downarrow \theta \\ M_p \wedge \mathcal{J}_{(p)} & \xrightarrow{\simeq} & \Sigma^{-d} M_p \wedge \mathcal{J}_{(p)} & \xrightarrow{\simeq} & \Sigma^{-2d} M_p \wedge \mathcal{J}_{(p)} & \xrightarrow{\simeq} & \dots & \xrightarrow{\simeq} & M_p \wedge \mathcal{J}_{(p)} \end{array}$$

where the horizontal maps are induced by A_p. Now $i: M_p \to M_p^\infty$ is a K_*-localization, since it is a K_*-equivalence and $M_p^\infty \simeq M_p \wedge \mathcal{J}_{(p)}$ is K_*-local.

It is now easy to obtain the $KZ_{(p)*}$-localization of a spectrum X for a prime p. Construct a triangle

$$\tilde{\mathcal{J}}_{(p)} \longrightarrow \mathcal{J}_{(p)} \xrightarrow{k} S^{-1}Q \longrightarrow \Sigma \tilde{\mathcal{J}}_{(p)}$$

in Ho^s such that $k_*: Q \otimes \pi_{-1} \mathcal{J}_{(p)} \approx Q \otimes \pi^{-1} S^{-1} Q$, and let $\lambda: S \to \tilde{\mathcal{J}}_{(p)}$ be the lifting of $ae: S \to \mathcal{J}_{(p)}$. The following theorem and corollaries have essentially been obtained by Adams–Baird (unpublished) and Ravenel[17] using a different approach.

THEOREM 4.3. (*Adams–Baird, Ravenel*) *For* $X \in Ho^s$ *the map*

$$X \simeq X \wedge S \xrightarrow{1 \wedge \lambda} X \wedge \tilde{\mathcal{J}}_{(p)}$$

is a $KZ_{(p)*}$*-localization for p prime.*

Proof. By 4.2 the map $ae: S \to \mathcal{J}_{(p)}$ is a KZ/p_*-equivalence, and hence $\lambda: S \to \tilde{\mathcal{J}}_{(p)}$ is also. Since $\lambda: S \to \tilde{\mathcal{J}}_{(p)}$ is an SQ_*-equivalence it is a KQ_*-equivalence, and thus $1 \wedge \lambda$ is a $KZ_{(p)*}$- equivalence. $X \wedge KOZ_{(p)}$ and $X \wedge S^{-1}Q$ are $KZ_{(p)*}$-local, since they are module spectra over $KOZ_{(p)}$ and SQ respectively. Thus $X \wedge \tilde{\mathcal{J}}_{(p)}$ is $KZ_{(p)*}$-local, and the theorem follows.

By 2.4 and 2.11 there is an isomorphism $\pi_* X_{KZ_{(p)}} \approx Z_{(p)} \otimes \pi_* X_K$. To compute $Z_{(p)} \otimes \pi_* X_K$, especially when $Q \otimes \pi_* X = 0$, one can use

COROLLARY 4.4. (*Adams–Baird, Ravenel*). *For $X \in Ho^s$ and p prime there are long exact sequences*

$$\ldots \to \pi_* X \wedge \mathcal{J}_{(p)} \to Z_{(p)} \otimes KO_* X \xrightarrow{\psi^r - 1} Z_{(p)} \otimes KO_* X \to \pi_{*-1} X \wedge \mathcal{J}_{(p)} \to \ldots$$

$$\ldots \to Z_{(p)} \otimes \pi_* X_K \to \pi_* X \wedge \mathcal{J}_{(p)} \to Q \otimes \pi_{*+1} X \to Z_{(p)} \otimes \pi_{*-1} X_K \to \ldots$$

For p odd the first sequence can be replaced by $\ldots \to \pi_* X \wedge \mathcal{J}_{(p)} \to$
$$Z_{(p)} \otimes K_* X \xrightarrow{\psi^r - 1} Z_{(p)} \otimes K_* X \to \pi_{*-1} X \wedge \mathcal{J}_{(p)} \to \ldots$$

because $\mathcal{J}_{(p)}$ also arises (for p odd) as the fibre of

$$KZ_{(p)} \xrightarrow{\psi^r - 1} KZ_{(p)}.$$

COROLLARY 4.5. (*Adams–Baird, Ravenel*). *The groups $Z_{(2)} \otimes \pi_i S_K$ are: $Z_{(2)} \oplus Z/2$ for $i = 0$; $Q/Z_{(2)} = Z/2^\infty$ for $i = -2$; $Z/2$ for $i = 0 \bmod 8$ with $i \neq 0$; $(Z/2)^2$ for $i \equiv 1 \bmod 8$; $Z/2$ for $i \equiv 2 \bmod 8$; $Z/8$ for $i \equiv 3 \bmod 8$; $Z/2^v$ for $i \equiv 7 \bmod 8$ with $i \neq -1$ where $i + 1 = q2^{v-1}$ with q odd; and 0 otherwise. For an odd prime p, the groups $Z_{(p)} \otimes \pi_i S_K$ are: $Z_{(p)}$ for $i = 0$; $Q/Z_{(p)} = Z/p^\infty$ for $i = -2$; Z/p^v for $i \equiv -1 \bmod 2(p-1)$ with $i \neq -1$ where $i + 1 = 2(p-1)qp^{v-1}$ with q not divisible by p; and 0 otherwise.*

The proof is reasonably straightforward using ([1], Lemma 2.12).

COROLLARY 4.6. (*Adams–Baird, Ravenel*). *The groups $\pi_i S_K$ are: $Z \oplus Z/2$ for $i = 0$; Q/Z for $i = -2$; and $\bigoplus_{p \text{ prime}} Z_{(p)} \otimes \pi_i S_K$ for $i \neq 0$.*

This follows immediately from 4.5.

COROLLARY 4.7. (*Adams–Baird, Ravenel*). *The spectrum S_K is K-nilpotent. Hence $S_K \wedge Y \xrightarrow{\simeq} Y_K$ for each $Y \in Ho^s$, and the ring spectrum S_K has multiplication $S_K \wedge S_K \xrightarrow{\simeq} S_K$ by 3.9.*

Proof. The product $\prod_{p \text{ prime}} KO_{SZ/p}$ is K-nilpotent since it is a KO-module spectrum and since KO is K-nilpotent by a straightforward argument using $K \simeq KO \wedge (S^0 \cup_\eta e^2)$ and $\eta^4 = 0$. Now $\prod_{p \text{ prime}} S_{KZ/p}$ is K-nilpotent since $S_{KZ/p} \simeq (S_{KZ_{(p)}})_{SZ/p}$ by 2.11 and since $(S_{KZ_{(p)}})_{SZ/p}$ is the fibre of a map $KO_{SZ/p} \to KO_{SZ/p}$ by 4.3. In the arithmetic fibre square (see 2.9)

$$\begin{array}{ccc} S_K & \longrightarrow & \prod_{p \text{ prime}} S_{KZ/p} \\ \downarrow & & \downarrow \\ S_{KQ} & \longrightarrow & (\prod_{p \text{ prime}} S_{KZ/p})_{KQ} \end{array}$$

the rational terms are also K-nilpotent since SQ is, and thus S_K is K-nilpotent.

We next show that the K_*-local spectra may be detected by a periodicity property of their mod-p homotopy groups.

THEOREM 4.8. *For a spectrum $X \in Ho^s$ the following are equivalent:*(i) *X is K_*-local.* (ii) $A_p^*: [M_p, X]_* \approx [M_p, X]_{*+d}$ for each prime p. (iii) $A_{p*}: \pi_* M_p \wedge X \approx \pi_{*+d} M_p \wedge X$ for each prime p.

Proof. Clearly (i) $\Rightarrow$ (ii), and by ([9] 2.10) (ii) $\Rightarrow$ (iii). To show (iii) $\Rightarrow$ (i) we suppose A_{p*} is iso for each p. Then the maps

$$M_p \wedge X \xrightarrow{\Sigma^{-d}A_p \wedge 1} \Sigma^{-d} M_p \wedge X \xrightarrow{\Sigma^{-2d}A_p \wedge 1} \Sigma^{-2d} M_p \wedge X \longrightarrow \dots$$

are equivalences so $M_p \wedge X \simeq M_p^\infty \wedge X$, and thus $M_p \wedge X$ is K_*-local by 4.2 and 4.7. By induction, $SZ/p^n \wedge X$ is K_*-local for $n \geq 1$, and hence the homotopy inverse limit $X_{SZ/p}$ is K_*-local. In the arithmetic fibre square (see 2.9)

$$\begin{array}{ccc} X & \longrightarrow & \prod_{p \text{ prime}} X_{SZ/p} \\ \downarrow & & \downarrow \\ X_{SQ} & \longrightarrow & (\prod_{p \text{ prime}} X_{SZ/p})_{SQ} \end{array}$$

the rational terms are also K_*-local, so X is K_*-local.

For each prime p, we form a triangle

$$\Sigma^d M_P \xrightarrow{A_p} M_p \longrightarrow Cof A_p \longrightarrow \Sigma^{d+1} M_p$$

in Ho^s. (Of course, $Cof A_p$ is equivalent to Toda's $V(1)$ for p odd). Let $B = \bigvee_{p \text{ prime}} Cof A_p$ and let S^B be as in 3.5.

COROLLARY 4.9. *$\langle K \rangle = \langle S^B \rangle$ in $\underline{A}(Ho^s)$, and thus $\langle K \rangle$ is the complement of $\langle B \rangle$ by* ([9], 2.9).

Proof. By 3.5 and 4.8, a spectrum $X \in Ho^s$ is S_*^B-local $\Leftrightarrow [B, X]_* = 0 \Leftrightarrow X$ is K_*-local. For any $Y, E \in Ho^s$, a formal argument using 1.1 shows that Y is E_*-acyclic $\Leftrightarrow [Y, W]_* = 0$ for each E_*-local spectrum W. Thus a spectrum $Y \in Ho^s$ is S_*^B-acyclic $\Leftrightarrow Y$ is K_*-acyclic. Hence $\langle K \rangle = \langle S^B \rangle$.

All K_*-acyclic spectra can be built from the $Cof A_p$ in an elementary way.

COROLLARY 4.10. If $X \in Ho^s$ is K_*-acyclic, then X is equivalent to some $Y \in Ho^s$ which is the union of closed subspectra $0 = F_0 Y \subset F_1 Y \subset F_2 Y \subset \dots$ such that each $F_n Y / F_{n-1} Y$ is equivalent to a possibly infinite wedge of the $\Sigma^i Cof A_p$.

Proof. As in ([9], Proof 1.5), form a sequence of CW-spectra $X = W_0 \subset W_1 \subset W_2 \subset \dots$ such that each W_n / W_{n-1} is equivalent to a wedge of the $\Sigma^i Cof A_p$

and such that $[Cof A_p, \underset{n}{\cup} W_n]_* = 0$ for each p. Then $X \to \underset{n}{\cup} W_n$ is a K_*-localization by 4.8, and thus $\underset{n}{\cup} W_n \simeq 0$. Now $Y = \Sigma^{-1}(\underset{n}{\cup} W_n/X)$ has the desired properties.

For $X \in Ho^s$ the sequence

$$\pi_* M_p \wedge X \xrightarrow{A_{p*}} \pi_{*+d} M_p \wedge X \xrightarrow{A_{p*}} \pi_{*+2d} M_p \wedge X \longrightarrow \dots$$

of mod-p homotopy groups has direct limit $\pi_* M_p^\infty \wedge X$. Using these periodic mod-p homotopy groups, we have

THEOREM 4.11. (i) *A spectrum $X \in Ho^s$ is K_*-local* $\Leftrightarrow i_* : \pi_* M_p \wedge X \approx \pi_* M_p^\infty \wedge X$ *for each prime p.* (ii) *A map $f : X \to Y$ in Ho^s is a K_*-equivalence* $\Leftrightarrow f_* : Q \otimes \pi_* X \approx Q \otimes \pi_* Y$ *and* $f_* : \pi_* M_p^\infty \wedge X \approx \pi_* M_p^\infty \wedge Y$ for each prime p.

Proof. Part (i) is immediate from 4.8. Clearly $f_* : Q \otimes \pi_* X \approx Q \otimes \pi_* Y \Leftrightarrow f_* : KQ_* X \approx KQ_* Y$. By 4.2 and 4.7, $f_* : \pi_* M_p^\infty \wedge X \approx \pi_* M_p^\infty \wedge Y \Leftrightarrow f_* : \pi_*(M_p \wedge X)_K \approx \pi_*(M_p \wedge Y)_K \Leftrightarrow f_* : K_*(M_p \wedge X) \approx K_*(M_p \wedge Y) \Leftrightarrow f_* : (KZ/p)_* X \approx (KZ/p)_* Y$. This implies (ii).

§5. THE E_*-ADAMS SPECTRAL SEQUENCE AND THE E-NILPOTENT COMPLETION

In ([4], §15) Adams presented the E_*-Adams spectral sequence and observed that it may be used to approach E_*-localizations. We believe this approach gives considerable insight into the convergence problem for the spectral sequence, and provides a promising method for computing certain E_*-localizations. We devote §5 and §6 to developing these ideas. Some of the results have been used by Ravenel in his recent study of localizations[17]. In §5 we first construct the E_*-Adams spectral sequence $\{E_r^{s,t}(X, Y)\}$ for $X, Y \in Ho^s$, and then introduce the "E-nilpotent completion" $E^\wedge Y$. In general $[X, E^\wedge Y]_*$ seems to be the appropriate target of $\{E_r^{s,t}(X, Y)\}$, and we conclude §5 by studying $E^\wedge Y$. In §6 we prove convergence theorems which guarantee that "usually" $\{E_r^{s,t}(X, Y)\}$ converges to $[X, E^\wedge Y]_*$ and that "often" $E^\wedge Y \simeq Y_E$.

For a ring spectrum E, form a triangle $\bar{E} \xrightarrow{i} S \xrightarrow{\eta} E \xrightarrow{k} \Sigma \bar{E}$ where η is the unit of E, and let $\bar{E}^s = \bar{E} \wedge \dots \wedge \bar{E}$ with s factors. Smashing the triangle with $\bar{E}^s \wedge Y$ for $Y \in Ho^s$, we obtain triangles

$$\bar{E}^{s+1} \wedge Y \xrightarrow{i} \bar{E}^s \wedge Y \xrightarrow{\eta} E \wedge \bar{E}^s \wedge Y \xrightarrow{k} \Sigma(\bar{E}^{s+1} \wedge Y)$$

for $s \geq 0$ where $\bar{E}^0 = S$. For $X \in Ho^s$ the $[X, \ ]_*$-exact sequences of these triangles give rise to the *E_*-Adams spectral sequence* $\{E_r^{s,t}(X, Y)\}$ where

$$E_1^{s,t}(X, Y) = [X, E \wedge \bar{E}^s \wedge Y]_{t-s} \qquad \text{for } s \geq 0$$

$$d_r : E_r^{s,t}(X, Y) \longrightarrow E_r^{s+r, t+r-1}(X, Y).$$

Note that $\{E_r^{s,t}(X, Y)\}$ is natural in $X, Y \in Ho^s$ and that E_*-equivalences $X' \to X$ and $Y \to Y'$ induce $E_r^{s,t}(X, Y) \approx E_r^{s,t}(X', Y')$ for $r \geq 1$. Under various assumptions, Adams has shown

$$E_2^{s,t}(X, Y) \approx \mathrm{Ext}_{E_*E}^{s,t}(E_*X, E_*Y)$$

and the reader should consult ([4], §15) for some details.

We have derived the E_*-Adams spectral sequence from a tower $\{\bar{E}^s \wedge Y\}$ over Y; however, we need to derive it instead from a tower $\{\bar{E}_s \wedge Y\}$ under Y with "$\bar{E}_{s-1} = S/\bar{E}^s$". More precisely, for each $s \geq 0$ form a triangle $\bar{E}^s \xrightarrow{i^s} S \xrightarrow{b} \bar{E}_{s-1} \xrightarrow{c} \Sigma\bar{E}^s$ where $i^s : \bar{E}^s \longrightarrow S = \bar{E}^0$ is the composite of the obvious tower maps and where $\bar{E}_{-1} = 0$. Using Verdier's axiom (see e.g. pp. 212–213 of [4]) for the composition $\bar{E}^{s+1} \xrightarrow{i} \bar{E}^s \xrightarrow{i^s} S$, construct maps $u : E \wedge \bar{E}^s \longrightarrow \bar{E}_s$ and $v : \bar{E}_s \longrightarrow \bar{E}_{s-1}$ such that the sequence

$$E \wedge \bar{E}^s \xrightarrow{u} \bar{E}_s \xrightarrow{v} \bar{E}_{s-1} \xrightarrow{(\Sigma\eta)c} \Sigma(E \wedge \bar{E}^s) \tag{5.1}$$

is a triangle in Ho^s and such that the diagrams of triangles

$$\begin{array}{ccccccc}
\bar{E}^{s+1} & \xrightarrow{i^{s+1}} & S & \xrightarrow{b} & \bar{E}_s & \xrightarrow{c} & \Sigma\bar{E}^{s+1} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \Sigma i} \\
\bar{E}^s & \xrightarrow{i^s} & S & \xrightarrow{b} & \bar{E}_{s-1} & \xrightarrow{c} & \Sigma\bar{E}^s
\end{array} \tag{5.2}$$

$$\begin{array}{ccccccc}
E \wedge \bar{E}^s & \xrightarrow{u} & \bar{E}_s & \xrightarrow{v} & \bar{E}_{s-1} & \xrightarrow{(\Sigma\eta)c} & \Sigma(E \wedge \bar{E}^s) \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Sigma 1} \\
E \wedge \bar{E}^s & \xrightarrow{k} & \Sigma\bar{E}^{s+1} & \xrightarrow{\Sigma i} & \Sigma\bar{E}^s & \xrightarrow{\Sigma\eta} & \Sigma(E \wedge \bar{E}^s)
\end{array} \tag{5.3}$$

commute. Now smashing (5.1) with $Y \in Ho^s$, we obtain triangles

$$E \wedge \bar{E}^s \wedge Y \to \bar{E}_s \wedge Y \to \bar{E}_{s-1} \wedge Y \to \Sigma(E \wedge \bar{E}^s \wedge Y) \tag{5.4}$$

for $s \geq 0$. Using (5.3) we see that the E_*-Adams spectral sequence can be derived from the $[X, \ ]_*$-exact sequences of (5.4).

PROPOSITION 5.5. For $Y \in Ho^s$, the tower $\{\bar{E}_s \wedge Y\}$ has a homotopy inverse limit (see 1.8) $E^\wedge Y \in Ho^s$ together with a map $\alpha : Y \to E^\wedge Y$ in Ho^s inducing the maps $b : Y \to \bar{E}_s \wedge Y$, and such that the above structures are all natural in Y.

Proof. To achieve the required naturality we work in the underlying category *Spec* of spectra and functions of degree 0 (see pp. 140, 141 of [4]). We can assume each $\bar{E}_s$ is an Ω-*CW*-spectrum, and then construct maps $\bar{E}_s \to \bar{E}_{s-1}$ in *Spec* representing the given homotopy classes. Then there is an induced tower $\{\bar{E}_s \wedge Y\}$ in *Spec*. Let $Q(\bar{E}_s \wedge Y)$ be the associated Ω-spectrum of $\bar{E}_s \wedge Y$, and form the tower $\{Q(\bar{E}_s \wedge Y)\}$ in *Spec*. Now let $E^\wedge Y$ be the homotopy equalizer of the maps

$$1, g : \prod_{s \geq 0} Q(\bar{E}_s \wedge Y) \to \prod_{s \geq 0} Q(\bar{E}_s \wedge Y)$$

in *Spec* where g is induced by the maps $Q(\bar{E}_s \wedge Y) \to Q(\bar{E}_{s-1} \wedge Y)$. Since Ho^s can be obtained from *Spec* by inverting the weak equivalences, we regard $E^\wedge Y$ as an object of Ho^s. The remaining constructions and verifications are straightforward.

We call $\alpha : Y \to E^\wedge Y$ the *E-nilpotent completion* of Y. It is actually a stable analogue of the R-completion functor studied in [7]. In §6 we show that $[X, E^\wedge Y]_*$ may be considered the target of $\{E_r^{s,t}(X, Y)\}$ and we now discuss other properties of $E^\wedge Y$.

For $Y \in Ho^s$ and $s \geq 0$, $\bar{E}_s \wedge Y$ is E-nilpotent by induction. Thus $E^\wedge Y$ is E_*-local by 1.8 and 3.8. Consequently, there is a unique map $\beta : Y_E \to E^\wedge Y$ in Ho^s such that the composite $Y \xrightarrow{\eta} Y_E \xrightarrow{\beta} E^\wedge Y$ is α. The map β is often, but not always, an equivalence, see 6.7. Note that $\alpha_* : E_*(Y) \to E_*(E^\wedge Y)$ is an injection with a natural left inverse constructed using the map $E^\wedge Y \longrightarrow \bar{E}_0 \wedge Y \simeq E \wedge Y$. Thus a map $X \to Y \in Ho^s$ is an E_*-equivalence $\Leftrightarrow E^\wedge X \to E^\wedge Y \in Ho^s$ is an equivalence. For $Y \in Ho^s$ it follows easily that $\beta : Y_E \simeq E^\wedge Y \Leftrightarrow \alpha_* : E_* Y \approx E_* E^\wedge Y \Leftrightarrow E^\wedge(\alpha) : E^\wedge Y \simeq E^\wedge E^\wedge Y$. Thus the E-nilpotent completion of Y agrees with the E_*-localization of Y in exactly those cases where the completion is idempotent. It is also straightforward to show that $E^\wedge$ preserves finite wedges, (de)suspensions, and triangles in Ho^s.

To permit a more flexible construction of $E^\wedge Y$, we give

Definition 5.6. For a ring spectrum E, an *E-nilpotent resolution* of a spectrum Y is a tower $\{W_s\}_{s \geq 0}$ under Y in Ho^s such that:

(i) W_s is E-nilpotent for each $s \geq 0$.

(ii) For each E-nilpotent spectrum N, the map $\operatorname{colim}_s [W_s, N]_* \to [Y, N]_*$ is iso.

LEMMA 5.7. *For a ring spectrum E and $Y \in Ho^s$, the tower $\{\bar{E}_s \wedge Y\}$ is an E-nilpotent resolution of Y.*

Proof. Since $\bar{E}_s \wedge Y$ is E-nilpotent, we must check 5.6 (ii). By (5.2) there is a long exact sequence

$$\ldots \to \operatorname{colim} [\bar{E}^{s+1} \wedge Y, N]_{*+1} \to \operatorname{colim} [\bar{E}_s \wedge Y, N]_* \to [Y, N]_* \to \operatorname{colim} [\bar{E}^{s+1} \wedge Y, N]_* \to \ldots$$

so it suffices to show $\operatorname{colim} [\bar{E}^{s+1} \wedge Y, N]_* = 0$ when N is E-nilpotent. This follows when $N \simeq E \wedge X$ because $[\bar{E}^s \wedge Y, N]_* \to [\bar{E}^{s+1} \wedge Y, N]_*$ is zero. It follows in general by induction using the filtration $\{C_m\}$ given in the proof of 3.8.

Any E-nilpotent resolution of Y can be used to compute $E^\wedge Y$.

PROPOSITION 5.8. For a ring spectrum E and $Y \in Ho^s$, let $\{W_s\}$ be an E-nilpotent resolution of Y with homotopy inverse limit W_∞. Then $W_\infty \simeq E^\wedge Y$.

We prove 5.8 in 5.12 after developing some pro-category notions needed in §6.

5.9. *The category Tow-A (see e.g.* [5], *Appendix).* By a tower $\{V_s\}$ in a category $\underset{\sim}{A}$ we mean a sequence of objects $V_s \in \underset{\sim}{A}$ for $s \geq 0$ together with maps $V_{s+1} \to V_s$ for $s \geq 0$. These form a category Tow-$\underset{\sim}{A}$ where

$$\operatorname{Hom}(\{V_s\}, \{W_s\}) = \lim_t \operatorname{colim}_s \operatorname{Hom}(V_s, W_t)$$

or equivalently where $\operatorname{Hom}(\{V_s\}, \{W_s\})$ is the set of natural transformations from $\operatorname{colim}_s \operatorname{Hom}(V_s, \ ) : \underset{\sim}{A} \to Sets$ to $\operatorname{colim}_s \operatorname{Hom}(W_s, \ ) : \underset{\sim}{A} \to Sets$. Thus if $\{V_{i(s)}\}$ is a confinal subtower of $\{V_s\}$ then $\{V_{i(s)}\} \approx \{V_s\}$ in Tow-A. Note that a tower $\{W_s\}$ under Y in $\underset{\sim}{A}$ corresponds to a map $\{Y\} \to \{W_s\}$ in Tow-$\underset{\sim}{A}$ where $\{Y\}$ is the constant tower of Y.

Using 5.9 it is easy to show

LEMMA 5.10. *If $\{W_s\}$ is an E-nilpotent resolution of Y in Ho^s, then there exists a unique isomorphism $e:\{\bar{E}_s \wedge Y\}\to\{W_s\}$ in Tow-Ho^s such that*

$$\begin{array}{ccc} \{Y\} & \xrightarrow{1} & \{Y\} \\ \downarrow & & \downarrow \\ \{\bar{E}_s \wedge Y\} & \xrightarrow{e} & \{W_s\} \end{array}$$

commutes.

A map $f:\{V_s\}\to\{W_s\}$ in *Tow-Ho^s* is called a *weak equivalence* if it induces an isomorphism $f_*:\{\pi_i V_s\}\to\{\pi_i W_s\}$ in *Tow-Ab* for each *i*, where *Ab* is the category of abelian groups.

LEMMA 5.11. *Let $f:\{V_s\}\to\{W_s\}$ be a weak equivalence in Tow-Ho^s (e.g. let f be an isomorphism in Tow-Ho^s). If V_∞, $W_\infty \in Ho^s$ are homotopy inverse limits of $\{V_s\}$, $\{W_s\}$ respectively, then there exists an equivalence $u: V_\infty \simeq W_\infty$ such that*

$$\begin{array}{ccc} \{V_\infty\} & \xrightarrow{\{u\}} & \{W_\infty\} \\ \downarrow & & \downarrow \\ \{V_s\} & \xrightarrow{f} & \{W_s\} \end{array}$$

commutes in Tow-Ho^s.

Proof. First suppose f is represented by a strict tower map $\{f_s\}$, where $f_s: V_s \to W_s \in Ho^s$ for $s \geq 0$. Then choose $u: V_\infty \to W_\infty$ so that

$$\begin{array}{ccccccc} V_\infty & \longrightarrow & \Pi V_s & \longrightarrow & \Pi V_s & \longrightarrow & \Sigma V_\infty \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \Pi f_s} & & \downarrow{\scriptstyle \Pi f_s} & & \downarrow{\scriptstyle \Sigma u} \\ W_\infty & \longrightarrow & \Pi W_s & \longrightarrow & \Pi W_s & \longrightarrow & \Sigma W_\infty \end{array}$$

commutes (see 1.8). In the induced diagram

$$\begin{array}{ccccccccc} 0 \to & \lim^1 \pi_{*+1} V_s & \to & \pi_* V_\infty & \to & \lim \pi_* V_s & \to 0 \\ & \downarrow & & \downarrow{\scriptstyle u_*} & & \downarrow & \\ 0 \to & \lim^1 \pi_{*+1} W_s & \to & \pi_* W_\infty & \to & \lim \pi_* W_s & \to 0 \end{array}$$

the outer vertical maps are iso by e.g. of p. 75 of [7]. Thus $u_*: \pi_* V_\infty \approx \pi_* W_\infty$ and $u: V_\infty \simeq W_\infty$ as desired. In the general case f can be factored as $\{V_s\} \xrightarrow{\{t_s\}^{-1}} \{V_{i(s)}\} \xrightarrow{\{\varphi_s\}} \{W_s\}$ where $\{t_s\}:\{V_{i(s)}\}\to\{V_s\}$ is the canonical map from some cofinal subtower to $\{v_s\}$, and where $\{\varphi_s\}$ is a strict tower map. Since $\{t_s\}$ and $\{\varphi_s\}$ are "strict" weak equivalences, we can construct equivalences $v: V_{i(\infty)} \simeq V_\infty$ and $w: V_{i(\infty)} \simeq W_\infty$ as above where $V_{i(\infty)}$ is a homotopy inverse limit of $\{V_{i(s)}\}$. Now let $u = wv^{-1}$.

5.12 *Proof of* 5.8. By 5.10 there is an isomorphism $e:\{\bar{E}_s \wedge Y\}\to\{W_s\}$ in *Tow-Ho^s*, and thus by 5.11 there is an equivalence $E^\wedge Y \simeq W_\infty$ in Ho^s.

§6. CONVERGENCE THEOREMS FOR THE E_*-ADAMS SPECTRAL SEQUENCE

Consider the E_*-Adams spectral sequence $\{E_r^{s,t}(X, Y)\}$ constructed using the tower $\{\bar{E}_s \wedge Y\}$ under Y, where E is a ring spectrum and $X, Y \in Ho^s$. We will show that "usually" $\{E_r^{s,t}(X, Y)\}$ converges to $[X, E^\wedge Y]_*$ and that "often" $E^\wedge Y \simeq Y_E$.

Since $E^\wedge Y$ is a homotopy inverse limit of $\{\bar{E}_s \wedge Y\}$, there is a short exact sequence

$$0 \to \lim_s{}^1 [X, \bar{E}_s \wedge Y]_{*+1} \to [X, E^\wedge Y]_* \to \lim_s [X, \bar{E}_s \wedge Y]_* \to 0.$$

Filter $[X, E^\wedge Y]_*$ by letting

$$F^s[X, E^\wedge Y]_* = \ker([X, E^\wedge Y]_* \longrightarrow [X, \bar{E}_{s-1} \wedge Y]_*)$$

and note that the map

$$[X, E^\wedge Y]_* \to \lim [X, E^\wedge Y]_*/F^s[X, E^\wedge Y]_* \tag{6.1}$$

is always surjective. Since $E_{r+1}^{s,t}(X, Y) \subset E_r^{s,t}(X, Y)$ for $r > s$, let $E_\infty^{s,t}(X, Y) = \bigcap_{r>s} E_r^{s,t}(X, Y)$ and note that the map

$$F^s[X, E^\wedge Y]_*/F^{s+1}[X, E^\wedge Y]_* \to E_\infty^{s,s+*}(X, Y) \tag{6.2}$$

is always injective. We say $\{E_r^{s,t}(X, Y)\}$ *converges completely* to $[X, E^\wedge Y]_*$ if (6.1) and (6.2) are both isomorphisms. A standard argument (see e.g. p. 263 of [7]) shows

PROPOSITION 6.3. *The E_*-Adams spectral sequence $\{E_r^{s,t}(X, Y)\}$ converges completely to $[X, E^\wedge Y]_*$ if and only if $\lim_r{}^1 E_r^{s,t}(X, Y) = 0$ for each s, t.*

Thus $\{E_r^{s,t}(X, Y)\}$ converges completely to $[X, E^\wedge Y]_*$ if $\{E_r^{s,t}(X, Y)\}$ is *Mittag-Leffler* (i.e. for each s, t there exists $r_0 < \infty$ such that $E_r^{s,t}(X, Y) = E_\infty^{s,t}(X, Y)$ whenever $r > r_0$). This condition is automatic when the terms $E_r^{s,t}(X, Y)$ are finite. We will call $\{E_r^{s,t}(X, Y)\}$ *strongly Mittag-Leffler* if it is Mittag-Leffler and for each m there exists $s_0 < \infty$ such that $E_\infty^{s,s+m}(X, Y) = 0$ whenever $s > s_0$. This is automatic when $\{E_2^{s,t}(X, Y)\}$ has an appropriate "vanishing line", and it implies that $F_s[X, E^\wedge Y]_m = 0$ for sufficiently large s (depending on m).

Our main convergence theorems will guarantee that $E^\wedge Y \simeq Y_E$ and that the above Mittag-Leffler properties hold under suitable hypotheses. We first suppose E is *connective* (i.e. $\pi_i E = 0$ for sufficiently small i). Then the convergence behavior of the E_*-Adams spectral sequence depends largely on the core of the ring $\pi_0 E$, and we recall from [6] and [18]:

6.4 *The core of a ring.* For a (possibly non-commutative) ring R with identity 1, the *core* of R is the subring

$$cR = \{r \in R \mid r \otimes 1 = 1 \otimes r \text{ in } R \otimes_Z R\}.$$

Thus cR is the equalizer of the two obvious ring homomorphisms $R \to R \otimes_Z R$. By 1.1, 1.3, and 5.1 of [18] (or by 2.1 of [6]) when R is commutative), cR is a *solid* ring, i.e. cR is a commutative ring such that the multiplication map $cR \otimes_Z cR \to cR$ is an isomorphism. In 3.5 of [6] we determined all solid rings, and they are:

(I) The rings $Z[J^{-1}] \subset Q$ where J is a set of primes.

(II) The cyclic rings Z/n for $n \geq 2$.

(III) The product rings $Z[J^{-1}] \times Z/n$ where J is a set of primes, $n \geq 2$, and each prime factor of n is in J.

(IV) The rings

$$R = c(Z[J^{-1}] \times \prod_{p \in K} Z/p^{e(p)})$$

where J and K are infinite sets of primes with $K \subset J$ and where each $e(p)$ is a positive integer. (One shows that $R^t = \bigoplus_{p \in K} Z/p^{e(p)}$ and $R/R^t = Z[J^{-1}]$ where R^t is the torsion subgroup of R.)

The following three theorems generalize the convergence results of pp. 316–318 of [4].

THEOREM 6.5. *Let E be a connective ring spectrum with $c\pi_0 E \approx Z[J^{-1}]$, and let Y be a connective spectrum. Then $\beta: Y_E \simeq E \wedge Y$ and $E \wedge Y \simeq Y_{SZ[J^{-1}]}$. If X is a finite CW-spectrum, then $\{E_r^{s,t}(X, Y)\}$ is strongly Mittag–Leffler and converges completely to $[X, E \wedge Y]_* \approx Z[J^{-1}] \otimes_Z [X, Y]_*$.*

For $n \geq 2$, we say that an abelian group G has "n-torsion of bounded order" if for each prime factor p of n, the p-torsion subgroup of G is annihilated by some power of p. Of course, this is automatic when G is finitely generated.

THEOREM 6.6. *Let E be a connective ring spectrum with $c\pi_0 E = Z/n$ for $n \geq 2$, and let Y be a connective spectrum. Then $\beta: Y_E \simeq E \wedge Y$ and $E \wedge Y \simeq Y_{SZ/n}$. If X is a finite CW-spectrum and if each $[X, Y]_m$ has n-torsion of bounded order, then $\{E_r^{s,t}(X, Y)\}$ is Mittag–Leffler and converges completely to $[X, E \wedge Y]_* \approx \lim_s [X, Y]_*/n^s[X, Y]_*$.*

THEOREM 6.7. *Let E be a connective ring spectrum with $c\pi_0 E$ of type (III) or (IV). Then $\beta: H_E \to E \wedge H$ is not an equivalence, where H is the spectrum of integral homology.*

To prove 6.5–6.7, let R denote the solid ring $c\pi_0 E$, and call an abelian group G *R-nilpotent* if it has a finite filtration

$$G = G_0 \supset \dots \supset G_j \supset \dots \supset G_n = 0$$

by subgroups such that each quotient G_j/G_{j+1} admits an R-module structure (which by 2.5 of [6] is unique). When $R = Z[J^{-1}]$ this means that G is uniquely j-divisible, and when $R = Z/n$ it means that $n^i G = 0$ for sufficiently large i. In general, if $A \to B \to G \to C \to D$ is an exact sequence of abelian groups with A, B, C, D R-nilpotent, then G is R-nilpotent (see e.g. p. 85 of [7]). Now let $\underset{\sim}{N}(R)$ be the class of all spectra $W \in Ho^s$ such that each $\pi_i W$ is R-nilpotent and $\pi_i W \neq 0$ for only finitely many i. An *R-resolution* of a spectrum Y is a tower $\{W_s\}$ under Y in Ho^s such that:

(i) $W_s \in \underset{\sim}{N}(R)$ for each $s \geq 0$.

(ii) For each $W \in \underset{\sim}{N}(R)$ the map $\operatorname{colim}_s [W_s, W]_* \to [Y, W]_*$ is iso.

Any two R-resolutions of Y are canonically isomorphic in *Tow-Ho*^{s} (see 5.9), and

LEMMA 6.8. *Let E be a connective ring spectrum and $R = c\pi_0 E$. Then for each connective $Y \in Ho^s$, the tower $\{P_s(\bar{E}_s \wedge Y)\}$ is an R-resolution of Y, where P_s is the Postnikov functor $P_s X = X\langle -\infty, \dots, s\rangle$.*

Proof. Since each $\pi_i(E \wedge \bar{E}^s \wedge Y)$ is an R-module, each $\pi_i(\bar{E}_s \wedge Y)$ is R-nilpotent by induction using (5.4). Thus $P_s(\bar{E}_s \wedge Y) \in \underset{\sim}{N}(R)$ for $s \geq 0$, and it suffices to show that each $W \in \underset{\sim}{N}(R)$ is E-nilpotent, since this implies

$$\operatorname*{colim}_s [\bar{E}_s \wedge Y, W]_* \approx [Y, W]_*$$

by 5.7. If V is a connective E-nilpotent spectrum, then the Eilenberg–MacLane spectra $H\pi_i V$ are all E-nilpotent by an argument using 3.7 (ii) with $X = H$. One now shows successively that $H\pi_0 E$, $H(\pi_0 E \otimes_Z \pi_0 E)$, HR, and $H(R\text{-module})$ are E-nilpotent. Thus each $W \in \underset{\sim}{N}(R)$ is E-nilpotent.

Proof of 6.5. The towers $\{P_s(\bar{E}_s \wedge Y\}$ and $\{P_s(SZ[J^{-1}) \wedge Y)\}$ are both R-resolutions of Y, and have homotopy inverse limits $E^\wedge Y$ and $SZ[J^{-1}] \wedge Y$ respectively. By 5.11, there is a diagram

$$\begin{array}{ccc} \{E^\wedge Y\} & \xrightarrow{\{u\}} & \{SZ[J^{-1}] \wedge Y\} \\ \downarrow & & \downarrow \\ \{P_s(\bar{E}_s \wedge Y)\} & \xrightarrow{\approx} & \{P_s(SZ[J^{-1}] \wedge Y)\} \end{array} \tag{6.9}$$

in Tow-Ho^s where $u: E^\wedge Y \simeq SZ[J^{-1}] \wedge Y$. Since $SZ[J^{-1}] \wedge Y \simeq Y_{SZ[J^{-1}]} \simeq Y_E$ by 2.4 and 3.1, we see that $Y_E \simeq E^\wedge Y$. However, more work is needed to show $\beta: Y_E \simeq E^\wedge Y$. Applying E_i to (6.9), we deduce that $\{E_i(E^\wedge Y)\} \xrightarrow{\approx} \{E_i P_s(\bar{E}_s \wedge Y)\}$ in Tow-Ab, and clearly $\{E_i(\bar{E}_s \wedge Y)\} \xrightarrow{\approx} \{E_i P_s(E_s \wedge Y)\}$. Since $\{E_i(\bar{E}^s \wedge Y)\}$ is trivial in Tow-Ab, we have $\{E_i Y\} \xrightarrow{\approx} \{E_i(\bar{E}_s \wedge Y)\}$. Now applying E_i to

$$\begin{array}{ccc} \{Y\} & \longrightarrow & \{E^\wedge Y\} \\ \downarrow & & \downarrow \\ \{\bar{E}_s \wedge Y\} & \longrightarrow & \{P_s(\bar{E}_s \wedge Y)\} \end{array}$$

we deduce that $E_i Y \xrightarrow{\approx} E_i(E^\wedge Y)$, and thus $\beta: Y_E \simeq E^\wedge Y$. Let X be a finite CW-spectrum. Since $\{P_s(\bar{E}_s \wedge Y)\} \approx \{P_s(SZ[J^{-1}] \wedge Y)\}$ in Tow-Ho^s, $\{[X, \bar{E}_s \wedge Y]_m\} \approx \{X, SZ[J^{-1}] \wedge Y]_m\}$ in Tow-Ab for each $m \in Z$. Hence each $\{[X, \bar{E}_s \wedge Y]_m\}$ is isomorphic to a constant tower in Tow-Ab, and thus $\{E_r^{s,t}(X, Y)\}$ is strongly Mittag–Leffler by (pp. 200, 264 of Ref. [7]). Finally since X is finite and $E^\wedge Y \simeq SZ[J^{-1}] \wedge Y$, we have $[X, E^\wedge Y]_* \approx Z[J^{-1}] \otimes [X, Y]_*$.

Proof of 6.6. Using the diagram

$$\begin{array}{ccc} \{E^\wedge Y\} & \xrightarrow{\{u\}} & \{Y_{SZ/n}\} \\ \downarrow & & \downarrow \\ \{P_s(\bar{E}_s \wedge Y)\} & \xrightarrow{\approx} & \{P_s(SZ/n^s \wedge Y)\} \end{array}$$

in place of (6.9), one proves $\beta: Y_E \simeq E^\wedge Y$ and $Y_E \simeq Y_{SZ/n}$ as above. Let X be a finite

CW-spectrum such that each $[X, Y]_m$ has n-torsion of bounded order. One obtains isomorphisms

$$\{[X, \bar{E}_s \wedge Y]_m\} \overset{1}{\approx} \{[X, SZ/n^s \wedge Y]_m\} \overset{2}{\approx} \{Z/n^s \otimes [X, Y]_m\}$$

in Tow-Ab, where $\overset{1}{\approx}$ comes from the above diagram, and $\overset{2}{\approx}$ is deduced using

$$0 \to Z/n^s \otimes [X, Y]_m \to [X, SZ/n^s \wedge Y]_m \to \mathrm{Tor}\,(Z/n^s, [X, Y]_{m-1}) \to 0$$

and the triviality of $\{\mathrm{Tor}\,(Z/n^s, [X, Y]_{m-1})\}$ in Tow-Ab. Hence each $\{[X, \bar{E}_s \wedge Y]_m\}$ is isomorphic to a surjective tower in Tow-Ab, and thus $\{E_r^{s,t}(X, Y)\}$ is Mittag–Leffler by ([7], pp. 200, 264). Finally, one has

$$[X, E^\wedge Y]_* \approx [X, Y_{SZ/n}]_* \approx \lim_s [X, SZ/n^s \wedge Y]_* \approx \lim_s (Z/n^s \otimes [X, Y]_*).$$

Proof of 6.7. Let $R = c\pi_0 E$ and suppose it is of type (IV). (The type (III) case is similar). For $j \geq 1$ let R_j be the solid ring

$$R_j = c(Z[J^{-1}] \times \prod_{p \in K} Z/p^{je(p)})$$

and note that $R_1 = R$. Using the obvious ring homomorphisms $Z \to R_j$ and $R_j \to R_{j-1}$, we obtain an associated tower of Eilenberg–MacLane spectra $\{HR_j\}$ under $H = HZ$ in Ho^s, and we claim that this is an R-resolution of H. Clearly, $HR_j \in \underset{\sim}{N}(R)$ for all j since there is an exact sequence $0 \to R^t \to R_{j+1} \to R_j \to 0$ for $j \geq 1$, and we must show

$$\operatorname*{colim}_j H^*(H(R_j/Z); M) = 0$$

for each R-module M. The map $R_{j+1}/Z \to R_j/Z$ is isomorphic to the direct sum of maps $p^{e(p)}: Z/p^\infty \to Z/p^\infty$ for $p \in K$ and maps $1: Z/p^\infty \to Z/p^\infty$ for $p \in J - K$. Using the isomorphism $R \otimes_Z M \approx M$ and the exact sequence

$$0 \to R^t \otimes_Z M \to R \otimes_Z M \to Z[J^{-1}] \otimes_Z M \to 0$$

one shows that the map

$$H^*(H(R_j/Z); M) \to H^*(H(R_{j+1}/Z); M)$$

is zero, and thus $\{HR_j\}$ is an R-resolution of H. Now $E^\wedge H \simeq H(\lim_j R_j)$ by 5.11 and 6.8, while $H_E \simeq HZ(L^{-1}]$ by 3.1 where $L = J - K$. Thus $E^\wedge H \not\simeq H_E$.

When E is not connective, we must usually impose strong conditions on Y to prove $\beta: Y_E \simeq E^\wedge Y$. Recall from §3 that Y is called E-prenilpotent if there exists an E_*-equivalence $Y \to N$ with N E-nilpotent.

THEOREM 6.10. *Let $Y \in Ho^s$ be E-prenilpotent for a ring spectrum E. Then $\beta: Y_E \simeq E^\wedge Y$; and for each $X \in Ho^s$, $\{E_r^{s,t}(X, Y)\}$ is strongly Mittag–Leffler and* *converges completely to* $[X, E^\wedge Y]_*$. *Furthermore, there exists* $s_0 \in Z^+ = \{n \in Z | n \geq 0\}$ *and* $\varphi : Z^+ \to Z^+$ *such that, for each* $X \in Ho^s$, $E_\infty^{s,*}(X, Y) = 0$ *when* $s > s_0$ *and* $E_r^{s,*}(X, Y) = E_\infty^{s,*}(X, Y)$ *when* $r > \varphi(s)$.

Proof. Since Y is E-prenilpotent, the constant tower $\{Y_E\}$ is an E-nilpotent resolution of Y. Thus $\{Y_E\} \approx \{\bar{E}_s \wedge Y\}$ in Tow-Ho^s and there is an equivalence $u : Y_E \simeq E^\wedge Y$ such that

$$\begin{array}{ccc} \{Y_E\} & \xrightarrow{\{u\}} & \{E^\wedge Y\} \\ \downarrow{\scriptstyle \{1\}} & & \downarrow \\ \{Y_E\} & \xrightarrow{\approx} & \{\bar{E}_s \wedge Y\} \end{array}$$

commutes in *Tow-Ho*^{s}. It follows that $E_*Y \xrightarrow{\approx} E_*(E^\wedge Y)$ and $\beta : Y_E \simeq E^\wedge Y$, and the proof is completed using pp. 200 and 264 of Ref. [7].

For some choices of E, all spectra are E-prenilpotent and thus the E_*-Adams spectral sequence always converges as in 6.10. For instance, this happens when $E = SZ[J^{-1}]$, $E = K$, or $E = S^B$ where B is a wedge of finite CW-spectra (see 3.5, 3.9, and 4.7). Other interesting examples have recently been suggested by Ravenel [17], and the following theorem seems useful for verifying such examples. We use the condition:

(6.11) There exist $s_0 \in Z^+$ and $\varphi : Z^+ \to Z^+$ such that, for each finite CW-spectrum W, $E_\infty^{s,*}(S, W) = 0$ when $s > s_0$ and $E_r^{s,*}(S, W) = E_\infty^{s,*}(S, W)$ when $r > \varphi(s)$.

Of course, this holds if there exist r_0, $s_0 < \infty$ such that $E_{r_0}^{s,*}(S, W) = 0$ for each $s > s_0$ and each finite CW-spectrum W.

THEOREM 6.12. *Let E be a ring spectrum such that π_*E is countable. Then all spectra are E-prenilpotent $\Leftrightarrow$ the E_*-Adams spectral sequence satisfies* (6.11).

Proof. By 3.9, all spectra are E-prenilpotent $\Leftrightarrow S$ is E-prenilpotent. For each finite CW-spectrum W, the spectral sequence $\{E_r^{s,t}(S, W)\}$ is isomorphic to $\{E_r^{s,t}(DW, S)\}$ where DW is the Spanier–Whitehead dual of W. Thus "$\Rightarrow$" follows from 6.10. We now suppose (6.11), and arrange that $\varphi(s) \geq s$ and $\varphi(s) \geq \varphi(s-1)$ for all s. For $r > s$ there is an exact sequence

$$0 \to E_r^{s,t}(S, W) \to [DW, \bar{E}_s]_{t-s}^{(r-1)} \to [DW, \bar{E}_{s-1}]_{t-s}^{(r)} \to 0$$

where $[DW, \bar{E}_s]_*^{(m)}$ denotes the image of the map $[DW, \bar{E}_{s+m}]_* \to [DW, \bar{E}_s]_*$. For each finite CW-spectrum W, it follows that $[DW, \bar{E}_s]_*^{(r-1)} = [DW, \bar{E}_s]_*^{(\infty)}$ for $r > \varphi(s)$ and $[DW, \bar{E}_s]_*^{(\infty)} \xrightarrow{\approx} [DW, \bar{E}_{s-1}]_*^{(\infty)}$ for $s > s_0$ where $[DW, \bar{E}_s]_*^{(\infty)} = \bigcap_m [DW, \bar{E}_s]_*^{(m)}$. Using the canonical maps $E^\wedge S \to \bar{E}_s \to \bar{E}_{s-1}$, construct a tower of triangles in Ho^s where the towers maps are

$$\begin{array}{ccccccc} E^\wedge S & \longrightarrow & \bar{E}_s & \longrightarrow & \bar{E}_s/E^\wedge S & \longrightarrow & \Sigma E^\wedge S \\ \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ E^\wedge S & \longrightarrow & \bar{E}_{s-1} & \longrightarrow & \bar{E}_{s-1}/E^\wedge S & \longrightarrow & \Sigma E^\wedge S. \end{array}$$

Now for $s \geq s_0$ there is an exact sequence

$$0 \to [DW, E^\wedge S]_* \to [DW, \bar{E}_s]_* \to [DW, \bar{E}_s/E^\wedge S]_* \to 0$$

where the image of the left map is $[DW, \bar{E}_s]_*^{(\infty)}$. Thus for each finite CW-spectrum W, the map

$$[DW, \bar{E}_{s+r}/E^\wedge S]_* \to [DW, \bar{E}_s/E^\wedge S]_*$$

is zero when $s \geq s_0$ and $r \geq \varphi(s)$. For any countable CW-spectrum $X \in Ho^s$, we claim that the map

$$[X, \bar{E}_{s+r}/E^\wedge S]_* \to [X, \bar{E}_s/E^\wedge S]_*$$

is zero when $s \geq s_0$ and $r \geq s + \varphi(s) + \varphi(s + \varphi(s))$. For this, note that X is a union of finite CW-spectra $B_0 \subset B_1 \subset B_2 \subset \cdots$, so there is a triangle

$$\bigvee_{n\geq 0} B_n \longrightarrow \bigvee_{n\geq 0} B_n \longrightarrow X \longrightarrow \Sigma(\bigvee_{n\geq 0} B_n)$$

in Ho^s. The claim now follows since the maps

$$[\bigvee_{n\geq 0} B_n, \bar{E}_{s+r}/E^\wedge S]_* \longrightarrow [\bigvee_{n\geq 0} B_n, \bar{E}_s/E^\wedge S]_*$$

are zero when $s \geq s_0$ and $r \geq \varphi(s)$. Since $\pi_* E$ is countable, it is not hard to show that each $\bar{E}_{s+r}/E^\wedge S$ is equivalent to a countable CW-spectrum, and thus the map $\bar{E}_{s+r}/E^\wedge S \to \bar{E}_s/E^\wedge S$ is zero when $s \geq s_0$ and $r \geq s + \varphi(s) + \varphi(s + \varphi(s))$. It follows easily that $E_* S \to E_*(E^\wedge S)$ is iso and $E^\wedge S$ is a retract of some $\bar{E}_s$. Thus S is E-prenilpotent.

Combining 6.12 with 3.9 and 6.10 we obtain

COROLLARY 6.13. Let E be a ring spectrum such that $\pi_* E$ is countable and (6.11) holds. Then for $X, Y \in Ho^s$:

(i) $S_E \wedge Y \simeq Y_E \simeq E^\wedge Y$

(ii) The spectral sequence $\{E_r^{s,t}(X, Y)\}$ is strongly Mittag-Leffler and converges completely to $[X, E^\wedge Y]_*$.

Condition (6.11) holds for $E = KZ[1/2]$ by the vanishing theorem of [3], p. 3, and thus 6.13 may be used to derive the odd prime cases of some results in §4 on K_*-localizations. More importantly, as shown by Ravenel [17], these methods provide a promising approach to localizations with respect to other interesting homology theories.

§7. DUALITY IN $\underset{\sim}{A}(Ho^s)$

In 1.13 we constructed, for each $E \in Ho^s$, a spectrum $aE \in Ho^s$ which "generates" the E_*-acyclic spectra. Using aE we now obtain a duality on the class $\underset{\sim}{A}(Ho^s)$ of "acyclicity types" of spectra, see 1.11 and [9]. Recall that $\underset{\sim}{A}(Ho^s)$ has a partial order relation $\leq$, and has operations $\vee$ and $\wedge$ induced by the wedge and smash product in Ho^s.

By the following proposition, there is a well-defined function $a(\ ): \underset{\sim}{A}(Ho^s) \to \underset{\sim}{A}(Ho^s)$ with $a\langle E\rangle = \langle aE\rangle$.

PROPOSITION 7.1. *For $E \in Ho^s$, let aE be as in* 1.13. *Then $\langle aE\rangle$ is the greatest member of $\underset{\sim}{A}(Ho^s)$ with $\langle E\rangle \wedge \langle aE\rangle = \langle 0\rangle$.*

Proof. Suppose $\langle E\rangle \wedge \langle X\rangle = \langle 0\rangle$. Then $E \wedge X \simeq 0$, so $X \in$ Class-aE by 1.13, and thus $\langle X\rangle \leq \langle aE\rangle$.

We call $a(\ ): \underline{A}(Ho^s) \to \underline{A}(Ho^s)$ a duality because of

Proposition 7.2. For $\langle E\rangle, \langle G\rangle \in \underline{A}(Ho^s)$: (i) $aa\langle E\rangle = \langle E\rangle$; (ii) $\langle E\rangle \le \langle G\rangle \Leftrightarrow a\langle G\rangle \le a\langle E\rangle$.

Proof. For (i) it suffices by 7.1 to show that $\langle E\rangle$ is the greatest solution to $a\langle E\rangle \wedge \langle X\rangle = \langle 0\rangle$. If $a\langle E\rangle \wedge \langle X\rangle = \langle 0\rangle$, then $aE \wedge X \simeq 0$. Thus by 1.13 $Y \wedge X \simeq 0$ whenever Y is E_*-acyclic, and thus $\langle X\rangle \le \langle E\rangle$ as desired. The "$\Rightarrow$" part of (ii) is straightforward, and the "$\Leftarrow$" part follows from "$\Rightarrow$" and (i).

Unfortunately, the DeMorgan law $a(\langle X\rangle \vee \langle Y\rangle) = a\langle X\rangle \wedge a\langle Y\rangle$ does not always hold in $\underline{A}(Ho^s)$. If it did, we would have

$$\langle E\rangle \wedge \langle E\rangle = aa\langle E\rangle \wedge aa\langle E\rangle = a(a\langle E\rangle \vee a\langle E\rangle) = aa\langle E\rangle = \langle E\rangle$$

for all $\langle E\rangle \in Ho^s$, but this contradicts 2.5 of [9]. In [9] we obtained a Boolean algebra $\underline{BA}(Ho^s)$ and a distributive lattice $\underline{DL}(Ho^s)$ with $\underline{BA}(Ho^s) \subset \underline{DL}(Ho^s) \subset \underline{A}(Ho^s)$.

PROPOSITION 7.3. *If $\langle E\rangle \in \underline{BA}(Ho^s)$. then $a\langle E\rangle$ is the complement of $\langle E\rangle$ in $\underline{BA}(Ho^s)$. However, $\underline{DL}(Ho^s)$ is not closed under $a(\ )$.*

Proof. The complement $\langle E\rangle^c$ of $\langle E\rangle$ in $\underline{BA}(Ho^s)$ satisfies $\langle E\rangle \wedge \langle E\rangle^c = \langle 0\rangle$ and $\langle E\rangle \vee \langle E\rangle^c = \langle S\rangle$. If $\langle E\rangle \wedge \langle X\rangle = \langle 0\rangle$, then $\langle X\rangle \le \langle E\rangle^c$ because

$$\langle X\rangle = \langle S\rangle \wedge \langle X\rangle = (\langle E\rangle \vee \langle E\rangle^c) \wedge \langle X\rangle = \langle E\rangle^c \wedge \langle X\rangle.$$

Thus $\langle E\rangle^c$ is the greatest solution to $\langle E\rangle \wedge \langle X\rangle = \langle 0\rangle$, and $\langle E\rangle^c = a\langle E\rangle$. To show $\underline{DL}(Ho^s)$ is not closed under $a(\ )$, it suffices to show $a(\langle H\rangle \vee a\langle H\rangle) \notin \underline{DL}(Ho^s)$ where H is the spectrum of integral homology. Let $\langle L\rangle = \langle H\rangle \vee a\langle H\rangle$ and note that $a\langle L\rangle \ne \langle 0\rangle$ because $\langle L\rangle \ne \langle S\rangle$ by 2.7 of [9]. Clearly $a\langle L\rangle \le a\langle H\rangle \le \langle L\rangle$ since $\langle H\rangle \le \langle L\rangle$, and thus $a\langle L\rangle \wedge a\langle L\rangle = \langle 0\rangle$ because $a\langle L\rangle \wedge A\langle L\rangle \le \langle L\rangle \wedge a\langle L\rangle = \langle 0\rangle$. Consequently $a\langle L\rangle \notin \underline{DL}(Ho^s)$.

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