

ON THE COMPARISON OF STABLE AND UNSTABLE p -COMPLETION

TOBIAS BARTHEL AND A. K. BOUSFIELD

(Communicated by Michael A. Mandell)

ABSTRACT. In this note we show that a p -complete nilpotent space X has a p -complete suspension spectrum if and only if its homotopy groups π_*X are bounded p -torsion. In contrast, if π_*X is not all bounded p -torsion, we locate uncountable rational vector spaces in the integral homology and in the stable homotopy groups of X . To prove this, we establish a homological criterion for p -completeness of connective spectra. Moreover, we illustrate our results by studying the stable homotopy groups of $K(\mathbb{Z}_p, n)$ via Goodwillie calculus.

1. INTRODUCTION

The notion of p -completion plays a fundamental role in algebra and topology, for it provides effective means to isolate and study p -primary properties. Applied to homotopy theory by Bousfield and Kan [4] as well as Sullivan [13] and developed further in [2, 3], it has since become one of the standard tools in the hands of algebraic topologists. However, there appears to be no general account of the comparison between unstable and stable p -completion in the literature, which is the question we address in the present note.

Our main goal is to characterize p -complete spaces which have p -complete suspension spectra:

Theorem 4.7. *If X is a p -complete nilpotent space, then $\Sigma^\infty X$ is p -complete if and only if $\pi_n X$ is bounded p -torsion for each n .*

In fact, we exhibit a sharp dichotomy of p -complete nilpotent spaces: if X is a p -complete nilpotent space whose homotopy groups are not all bounded p -torsion, then the integral homology groups and stable homotopy groups of X both contain an uncountable rational vector space. As a consequence, we deduce that a nilpotent space X with derived p -complete integral homology and unstable homotopy must have both $H_n(X; \mathbb{Z})$ and $\pi_n X$ of bounded p -torsion for all n .

In a first step towards the proof of the theorem, we complement the second author's characterization of p -complete spectra in terms of homotopy groups with an integral homological criterion, using a mild generalization of Serre classes appropriate for stable homotopy theory. This is in sharp contrast to the aforementioned fact that the integral homology of p -complete spaces is not well-behaved, and thus cannot be used to characterize p -completeness of spaces.

Received by the editors December 22, 2017, and, in revised form, May 28, 2018.

2010 *Mathematics Subject Classification.* Primary 55P60, 55P42.

The first author has been partially supported by the DNR92.

Corollary 3.3. *A bounded below spectrum X is p -complete if and only if $H_*(X; \mathbb{Z})$ is derived p -complete in each degree.*

In order to use this result to prove the theorem, we need to detect rational classes in the homology of p -complete spaces whose homotopy is not bounded p -torsion. This rests on the study of the integral homology of p -complete spheres. We end this note with a sample computation, illustrating how Goodwillie calculus allows us to detect rational classes in the stable homotopy groups of the Eilenberg–MacLane space $K(\mathbb{Z}_p, n)$.

Proposition 5.3. *For $n \geq 1$ and $k > 1$, the stable homotopy group $\pi_{nk} \Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not p -complete.*

In fact, we also give a short alternative argument based on the integral homology of $K(\mathbb{Z}_p, n)$.

Conventions. Throughout this paper, p will be a fixed prime number and \mathbb{Z}_p denotes the p -adic integers. We say that a nilpotent group N is bounded p -torsion if there exists an m such that for all $x \in N$, we have $x^{p^m} = 1$. A graded nilpotent group N_* is said to be of bounded p -torsion if N_k is bounded p -torsion for each k ; however, we do not require a uniform bound. Whenever we are in a graded context, we indicate the degree of an abelian group A by square brackets, i.e., $A[n]$ refers to A placed in degree n . If X is a topological space, then $H_*(X; A)$ is the reduced homology of X with coefficients in A . For a space or spectrum X , we write $\tau_{\leq n} X = \tau_{< n+1} X$ for the n th Postnikov section of X and $\tau_{\geq n+1} X = \tau_{> n} X$ for the fiber of the canonical map $X \rightarrow \tau_{\leq n} X$.

2. PRELIMINARIES ON p -COMPLETION

We briefly recall the basic properties of p -completion for nilpotent groups, topological spaces, and spectra. With the exceptions of Lemma 2.2 and Proposition 2.4, this material is mostly taken from [2–4], and we refer to these sources as well as [8, 10] for further references.

2.1. Algebraic p -completion for abelian groups. In general, the p -completion functor $M \mapsto \lim_i M/p^i M$ on the category of abelian groups is neither left nor right exact, so one studies its zeroth and first left derived functors L_0 and L_1 , which may be expressed as $L_0 M = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p^\infty, M)$ and $L_1 M = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ by [4, Ch. VI, 2.1]. An abelian group M is called derived p -complete (or Ext- p -complete or L -complete) if the natural completion map $M \rightarrow L_0 M$ is an isomorphism. For each abelian group M , the map $M \rightarrow L_0 M$ will then be the universal homomorphism from M to a derived p -complete abelian group by [4, Ch. VI, 3.2]. We will denote the full subcategory of derived p -complete abelian groups by \mathcal{C}_p .

Proposition 2.1. *The category \mathcal{C}_p is a full abelian subcategory of $\text{Mod}_{\mathbb{Z}}$ closed under extensions and limits. Furthermore, for any $M \in \text{Mod}_{\mathbb{Z}}$ there is a short exact sequence*

$$0 \longrightarrow \lim_i^1 \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^i, M) \longrightarrow L_0 M \longrightarrow \lim_i M/p^i M \longrightarrow 0$$

relating derived p -completion to ordinary p -completion.

Proof. This is essentially proven in [4, Ch. VI, 2.1], but can also be deduced as a special case of [8, Thms. A.2 and A.6]. \square

We will later make use of the following observation.

Lemma 2.2. *If $A \in \mathcal{C}_p$ is torsion, then A is bounded p -torsion.*

Proof. We give two proofs, a conceptual one and an elementary argument. First, any derived p -complete group A has $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], A) = 0 = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}[1/p], A)$ by [4, Ch. VI, 3.4], and hence A satisfies $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, A) = 0 = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, A)$ since \mathbb{Q} is a quotient of free $\mathbb{Z}[1/p]$ -modules. Thus, A is a cotorsion group with no nontrivial divisible subgroups, so the Baer–Fomin theorem [1] implies that A is a bounded p -torsion group.

Second, suppose that the conclusion of the lemma is false, i.e., that there exists a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of A such that the order of a_i is p^i . Set $x_j = \sum_{i=0}^{j-1} a_{2i+1} p^i$; then the element $x = (x_1, x_2, x_3, \dots) \in \prod_{j \in \mathbb{N}} A$ lies in $\lim_j A/p^j$. By construction, x is not p -torsion, which contradicts the fact that $A \rightarrow \lim_j A/p^j$ is surjective, forcing $\lim_j A/p^j$ to be p -torsion. \square

Remark 2.3. By a theorem of Prüfer, the conclusion of the lemma implies that A must in fact be a direct sum of cyclic p -groups.

2.2. Algebraic p -completion for nilpotent groups. Recall from [4, Ch. VI, §2] that the notion of derived p -completion can be extended to nilpotent groups, as follows: If X_p^\wedge denotes the Bousfield–Kan p -completion of a nilpotent space X as recalled in the next subsection, then we define the derived p -completion of the nilpotent group N as $L_0 N = \pi_1(K(N, 1)_p^\wedge)$ and $L_1 N = \pi_2(K(N, 1)_p^\wedge)$. A nilpotent group N is called derived p -complete if the completion map $N \rightarrow L_0 N$ is an isomorphism; for each nilpotent group N , the map $N \rightarrow L_0 N$ will then be the universal homomorphism from N to a derived p -complete nilpotent group by [4, Ch. VI, 3.2]. We denote the category of derived p -complete nilpotent groups by \mathcal{N}_p .

The inclusion functor $\mathcal{C}_p \rightarrow \mathcal{N}_p$ has a left adjoint given by taking a derived p -complete nilpotent group N to the derived p -completion of its abelianization $L_0(N/[N, N])$. Note that the unit of this adjunction is surjective, i.e., for any derived p -complete nilpotent group N , the canonical map $N \rightarrow L_0(N/[N, N])$ is surjective. Indeed, since L_0 preserves epimorphisms of nilpotent groups, all maps in the following commutative diagram are surjective:

$$\begin{array}{ccc}
 N & \longrightarrow & N/[N, N] \\
 \cong \downarrow & & \downarrow \\
 L_0 N & \longrightarrow & L_0(N/[N, N]).
 \end{array}$$

We obtain the following generalization of Lemma 2.2.

Proposition 2.4. *The following conditions are equivalent for $N \in \mathcal{N}_p$:*

- (1) N is torsion.
- (2) $L_0(N/[N, N])$ is torsion.
- (3) N is bounded p -torsion.

Proof. The surjectivity of the map $N \rightarrow L_0(N/[N, N])$ observed above immediately gives the implication (1) \Rightarrow (2), while (3) \Rightarrow (1) is trivial.

Assume that $L_0(N/[N, N])$ is torsion and thus bounded p -torsion by Lemma 2.2. Consider the lower central series of N ,

$$N = \gamma_1 N \supseteq \gamma_2 N \supseteq \dots \supseteq \gamma_m N = 1,$$

with successive abelian quotients $Q_i(N) = \gamma_i N / \gamma_{i+1} N$. We claim that, for each $i \geq 1$, $Q_i(N)$ is a direct sum of a p -divisible group and a bounded p -torsion group. Indeed, we start with the abelianization $Q_1(N) = N/[N, N]$ of N . Lemma 3.7 in [4, Ch. VI] implies that the kernel of the completion map $Q_1(N) \rightarrow L_0 Q_1(N)$ is p -divisible, so the claim holds for $Q_1(N)$. The general case follows from this, because $\bigoplus_{i \geq 1} Q_i(N)$ is generated as a Lie algebra by $Q_1(N)$. By [4, Ch. VI, 2.5], there is an exact sequence

$$L_0 Q_i(N) \longrightarrow L_0(N/\gamma_{i+1} N) \longrightarrow L_0(N/\gamma_i N) \longrightarrow 1$$

for any $i \geq 1$. Using the previous claim, $L_0 Q_i(N)$ is bounded p -torsion, so we see inductively that $L_0(N/\gamma_i N)$ is bounded p -torsion for all $i \geq 1$, hence (3) holds. \square

Remark 2.5. The implication (1) \Rightarrow (3) in the previous proposition could also be proven more directly via the upper central series of N , whose quotients are known to be derived p -complete by [4, VI. 3.4(ii)], but this result would be insufficient for our later use.

2.3. Topological p -completion. In [4], Bousfield and Kan introduced the notion of p -completion for topological spaces, lifting the algebraic notion defined above to topology. In general, the p -completion of a space is difficult to describe, but the theory simplifies significantly for nilpotent spaces; in particular, in this case p -completion coincides with $H\mathbb{F}_p$ -localization [2]. Furthermore, for nilpotent spaces with \mathbb{F}_p -homology of finite-type, p -completion can be identified with p -profinite completion due to Sullivan [13]. Similarly, the category of spectra admits (at least) two notions of p -completion, given either by $H\mathbb{F}_p$ -localization or, the one we will use here, localization at the mod p Moore spectrum S^0/p ; see [3]. The next result summarizes the relation between these constructions and lists their basic properties.

Theorem 2.6 (Bousfield, Kan).

- (1) A nilpotent space X is p -complete if and only if $\pi_n X$ is derived p -complete for all $n \in \mathbb{N}$. Moreover, the notions of p -completion and $H\mathbb{F}_p$ -localization coincide for nilpotent spaces.
- (2) A spectrum X is p -complete if and only if $\pi_n X$ is derived p -complete for all $n \in \mathbb{Z}$. If X is bounded below, then X is p -complete if and only if X is $H\mathbb{F}_p$ -local.

Moreover, if X is a nilpotent space or spectrum, then there exists a splittable short exact sequence computing the unstable or stable homotopy groups of its p -completion, respectively:

$$0 \longrightarrow L_0 \pi_n X \longrightarrow \pi_n(X_p^\wedge) \longrightarrow L_1 \pi_{n-1} X \longrightarrow 0$$

for any n , where $L_i(-) \cong \text{Ext}_{\mathbb{Z}}^{1-i}(\mathbb{Z}/p^\infty, -)$ are the derived functors of p -completion.

3. GENERALIZED SERRE THEORY

The full subcategory \mathcal{C}_p of $\text{Mod}_{\mathbb{Z}}$ is not closed under subobjects or quotients, and thus does not form a Serre class in the usual sense. This necessitates a mild generalization of Serre’s mod \mathcal{C} theory which we develop in this section.

Definition 3.1. A weak Serre class is a full subcategory $\mathcal{C} \subseteq \text{Mod}_{\mathbb{Z}}$ such that if

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5$$

is an exact sequence in $\text{Mod}_{\mathbb{Z}}$ with $A_1, A_2, A_4, A_5 \in \mathcal{C}$, then also $A_3 \in \mathcal{C}$.

More explicitly, this means that $\mathcal{C} \subseteq \text{Mod}_{\mathbb{Z}}$ is a full additive subcategory closed under kernels, cokernels, and extensions. It follows that \mathcal{C} is also closed under tensoring and $\text{Tor}_1^{\mathbb{Z}}$ with respect to finitely generated abelian groups. For instance, any Serre subcategory of $\text{Mod}_{\mathbb{Z}}$ is a weak Serre class, but the converse does not hold. The main example of interest to us here is the category \mathcal{C}_p of derived p -complete abelian groups; see Proposition 2.1.

Proposition 3.2. *Suppose \mathcal{C} is a weak Serre class. If X is a bounded below spectrum, then the following two conditions are equivalent:*

- (1) $\pi_n X \in \mathcal{C}$ for all $n \in \mathbb{Z}$.
- (2) $H_n(X; \mathbb{Z}) \in \mathcal{C}$ for all $n \in \mathbb{Z}$.

Proof. Assume the first condition holds; we will argue via the Postnikov tower $(\tau_{\leq n} X)$ of X . For simplicity, we will write $H_*(Y)$ for the integral homology of a spectrum Y throughout this proof.

To start with, we need to show that $H_*(HA) \in \mathcal{C}$ for $A \in \mathcal{C}$. Using the isomorphisms $H_*(HA) \cong H_*(H\mathbb{Z}; A)$, the universal coefficient theorem gives a short exact sequence

$$0 \longrightarrow H_*(H\mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow H_*(HA) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(H_{*-1}(H\mathbb{Z}), A) \longrightarrow 0.$$

In each degree, the integral Steenrod algebra $H_*(H\mathbb{Z})$ is finitely generated over \mathbb{Z} , as follows from Serre theory for the class of finitely generated abelian groups. Therefore, the outer terms of this sequence are in \mathcal{C} . This shows $H_*(HA) \in \mathcal{C}$ as well.

Given $n \in \mathbb{Z}$, we will now prove that $H_n(X) \in \mathcal{C}$. Since $H_n(\tau_{>n} X) = 0 = H_{n-1}(\tau_{>n} X)$ by connectivity, we see that $H_n(X) \cong H_n(\tau_{\leq n} X)$. This reduces the claim to proving that $H_*(\tau_{\leq n} X) \in \mathcal{C}$. This follows inductively, using the exact sequence

$$\begin{aligned} H_{*+1}(\tau_{\leq n-1} X) &\longrightarrow H_*(\Sigma^n H\pi_n X) \longrightarrow H_*(\tau_{\leq n} X) \longrightarrow H_*(\tau_{\leq n-1} X) \\ &\hspace{15em} \longrightarrow H_{*-1}(\Sigma^n H\pi_n X) \end{aligned}$$

associated to the fiber sequence $\Sigma^n H(\pi_n X) \rightarrow \tau_{\leq n} X \rightarrow \tau_{\leq n-1} X$. Since $H_k(H\pi_n X) \in \mathcal{C}$ for all $k \in \mathbb{Z}$, this gives the implication (1) \Rightarrow (2).

For the converse, consider the convergent Atiyah–Hirzebruch spectral sequence

$$E_{s,t}^2 \cong H_s(X; \pi_t S^0) \implies \pi_{s+t} X.$$

Since $\pi_t S^0$ is finitely generated over \mathbb{Z} for each $t \in \mathbb{Z}$, $H_s(X; \pi_t S^0) \in \mathcal{C}$ for each bidegree (s, t) , hence $\pi_n X$ is also in \mathcal{C} for all $n \in \mathbb{Z}$. □

When applied to the weak Serre class \mathcal{C}_p , we obtain a homological characterization of p -completeness for bounded below spectra.

Corollary 3.3. *For a bounded below spectrum X , the following conditions are equivalent:*

- (1) X is p -complete.
- (2) $\pi_n X$ is derived p -complete for all n .
- (3) $H_n(X; \mathbb{Z})$ is derived p -complete for all n .

Proof. The equivalence of (1) and (2) is the content of Theorem 2.6(2), while (2) is equivalent to (3) by Proposition 3.2. □

We deduce that the integral homology of nilpotent p -complete spaces is well-behaved in the stable range.

Corollary 3.4. *Suppose X is a nilpotent p -complete space. If X is n -connected, then $H_k(X; \mathbb{Z})$ is derived p -complete for all $k \leq 2n$.*

Proof. Since $\pi_k \Sigma^\infty X \cong \pi_k X$ for $k \leq 2n$ by the Freudenthal suspension theorem, Theorem 2.6 implies that $\pi_* \tau_{\leq 2n} \Sigma^\infty X$ is derived p -complete in each degree, hence so is $H_*(\tau_{\leq 2n} \Sigma^\infty X; \mathbb{Z})$ by Corollary 3.3. We thus get that $H_k(X; \mathbb{Z}) \cong H_k(\Sigma^\infty X; \mathbb{Z}) \cong H_k(\tau_{\leq 2n} \Sigma^\infty X; \mathbb{Z})$ is derived p -complete for $k \leq 2n$. □

Corollary 3.5. *For a bounded below spectrum X , there exists a splittable short exact sequence computing the integral homology groups of its p -completion:*

$$0 \longrightarrow L_0 H_n(X; \mathbb{Z}) \longrightarrow H_n(X_p^\wedge; \mathbb{Z}) \longrightarrow L_1 H_{n-1}(X; \mathbb{Z}) \longrightarrow 0$$

for any n .

Proof. Since the spectrum $H\mathbb{Z} \wedge X_p^\wedge$ is p -complete by Corollary 3.3, there is a canonical map $(H\mathbb{Z} \wedge X)_p^\wedge \rightarrow H\mathbb{Z} \wedge X_p^\wedge$, and this map must be an equivalence because it is an $H\mathbb{F}_p$ -equivalence of p -complete bounded below spectra. Hence, the claim follows from Theorem 2.6. □

From Corollary 3.5, we obtain the following description of the p -complete sphere spectrum as a Moore spectrum.

Example 3.6. There is a canonical equivalence $S_p^0 \xrightarrow{\sim} M\mathbb{Z}_p$.

4. THE COMPARISON

In this section, we first study the relation between p -completion for spectra and spaces under the infinite loop space functor Ω^∞ , and then prove our main theorem.

4.1. Infinite loop spaces. It is easy to deduce from Theorem 2.6 the following relation between unstable and stable p -completion under Ω^∞ .

Proposition 4.1. *For 0-connected spectra X and Y , we have:*

- (1) X is p -complete if and only if $\Omega^\infty X$ is p -complete.
- (2) A map $f: X \rightarrow Y$ is an $H\mathbb{F}_p$ -equivalence if and only if $\Omega^\infty f$ is an $H\mathbb{F}_p$ -equivalence.
- (3) The canonical comparison map $(\Omega^\infty X)_p^\wedge \rightarrow \Omega^\infty(X_p^\wedge)$ is an equivalence.

Proof. Since $\pi_* \Omega^\infty X \cong \pi_* X$ and $\Omega^\infty X$ is nilpotent, the first claim is a direct consequence of Theorem 2.6. In order to prove (2), note that f is an $H\mathbb{F}_p$ -equivalence if and only if the homotopy groups $\pi_* \operatorname{cof}(f)$ of the cofiber of f are uniquely p -divisible.

This is equivalent to the statement that the \mathbb{F}_p -homology $H_*(\Omega^\infty \operatorname{cof}(f); \mathbb{F}_p)$ is trivial. The Serre spectral sequence associated to the fiber sequence

$$\Omega^\infty X \xrightarrow{\Omega^\infty f} \Omega^\infty Y \longrightarrow \Omega^\infty \operatorname{cof}(f)$$

thus shows that this happens if and only if $\Omega^\infty f$ is an $H\mathbb{F}_p$ -equivalence.

Statement (1) implies that $\Omega^\infty(X_p^\wedge)$ is p -complete, so the map $\Omega^\infty(X) \rightarrow \Omega^\infty(X_p^\wedge)$ factors canonically through $\phi: (\Omega^\infty X)_p^\wedge \rightarrow \Omega^\infty(X_p^\wedge)$, making the following diagram commute:

$$\begin{array}{ccc} \Omega^\infty X & \longrightarrow & (\Omega^\infty X)_p^\wedge \\ & \searrow & \downarrow \\ & & \Omega^\infty(X_p^\wedge). \end{array}$$

By Statement (2), both the horizontal and the diagonal map are $H\mathbb{F}_p$ -equivalences, hence so is the vertical comparison map. \square

Remark 4.2. Let Ω_0^∞ be the 0-component of Ω^∞ . The last part of the proposition can be strengthened to an equivalence $(\Omega_0^\infty X)_p^\wedge \rightarrow \Omega_0^\infty(X_p^\wedge)$ for any connective spectrum X such that $\pi_0 X$ does not contain any copies of \mathbb{Z}/p^∞ . To prove this directly, one may use the short exact sequences displayed at the end of Theorem 2.6.

4.2. Suspension spectra. We now turn to the comparison under Σ^∞ . In odd dimensions, the next result has also been observed in [4, Rem. VI.5.7]; see also [10, Rem. 11.1.5].

Lemma 4.3. *Let $n \geq 1$ and write S_p^n for the p -completion of S^n . There exists an uncountable rational vector space in $H_{2n}(S_p^n; \mathbb{Z})$ which injects into $H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z})$ under the map $S_p^n \rightarrow \tau_{\leq n} S_p^n \simeq K(\mathbb{Z}_p, n)$.*

Proof. Consider the following segment of the Serre long exact sequence for the fibration $F \rightarrow S_p^n \rightarrow K(\mathbb{Z}_p, n)$:

$$H_{2n}(F; \mathbb{Z}) \longrightarrow H_{2n}(S_p^n; \mathbb{Z}) \longrightarrow H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z}) \longrightarrow H_{2n-1}(F; \mathbb{Z}) \longrightarrow \dots$$

Corollary 3.4 implies that $H_{2n}(F; \mathbb{Z})$ and $H_{2n-1}(F; \mathbb{Z})$ are derived p -complete. Recalling that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, A) = 0 = \operatorname{Ext}_{\mathbb{Z}}^1(\mathbb{Q}, A)$ whenever A is derived p -complete, we see that the natural map $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, H_{2n}(S_p^n; \mathbb{Z})) \rightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z}))$ is surjective. Thus, it will suffice to show that $H_{2n}(K(\mathbb{Z}_p, n); \mathbb{Z})$ contains an uncountable rational vector space, which will be verified in the homological proof of Proposition 5.3 below. \square

Note that, because $H_*(S_p^n; \mathbb{F}_p) \cong H_*(S^n; \mathbb{F}_p) \cong \mathbb{F}_p[n]$, an application of the universal coefficient theorem shows that $H_k(S_p^n; \mathbb{Z})$ is rational for all $k > n$.

Lemma 4.4. *Suppose N is a derived p -complete nilpotent (abelian) group and $n = 1$ ($n \geq 1$) If N is not bounded p -torsion, then there exists an element $x \in N$ of infinite order inducing a monomorphism $H_*(K(\mathbb{Z}_p, n); \mathbb{Q}) \rightarrow H_*(K(N, n); \mathbb{Q})$.*

Proof. By assumption on N and Proposition 2.4, $L_0(N/[N, N])$ contains elements of infinite order. Let \bar{x} be such an element and let $x \in N$ be a lift of \bar{x} . For the

remainder of the proof we assume $n = 1$; the (easier) case $n \geq 2$ and N abelian is proven similarly. The element x induces a map

$$K(\mathbb{Z}_p, 1) \longrightarrow K(N, 1) \longrightarrow K(L_0(N/[N, N]), 1)$$

such that the composite is injective on π_1 . It follows that the rationalization $K(\mathbb{Z}_p, 1)_{\mathbb{Q}} \rightarrow K(L_0(N/[N, N]), 1)_{\mathbb{Q}}$ of this map is split, hence the composite

$$H_*(K(\mathbb{Z}_p, 1); \mathbb{Q}) \longrightarrow H_*(K(N, 1); \mathbb{Q}) \longrightarrow H_*(K(L_0(N/[N, N]), 1); \mathbb{Q})$$

is a split monomorphism, which implies the claim. □

Proposition 4.5. *If X is a p -complete nilpotent space whose homotopy groups are not all bounded p -torsion, then the integral homology groups $H_*(X; \mathbb{Z})$ and the stable homotopy groups $\pi_*\Sigma^\infty X$ both contain an uncountable rational vector space.*

Proof. Assume that π_*X is not all bounded p -torsion, and let $\pi_n X$ be the lowest such group. It then follows from Lemma 4.4 that $\pi_n X$ contains a class x of infinite order inducing a monomorphism $H_*(K(\mathbb{Z}_p, n); \mathbb{Q}) \rightarrow H_*(K(\pi_n X, n); \mathbb{Q})$. Since the map $\tau_{\geq n} X \rightarrow X$ is a rational homology equivalence, any rational subgroup of $H_*(\tau_{\geq n} X; \mathbb{Z})$ must map monomorphically to $H_*(X; \mathbb{Z})$, so it suffices to prove the homological claim for $\tau_{\geq n} X$. The element x yields a map $S_p^n \rightarrow \tau_{\geq n} X$ such that the composite $S_p^n \rightarrow \tau_{\geq n} X \rightarrow K(\pi_n X, n)$ factors as

$$\begin{array}{ccc} \tau_{\geq n} X & \longrightarrow & \tau_{\leq n} \tau_{\geq n} X \simeq K(\pi_n X, n) \\ \uparrow & & \uparrow \\ S_p^n & \longrightarrow & \tau_{\leq n} S_p^n \simeq K(\mathbb{Z}_p, n). \end{array}$$

It follows from Lemma 4.3 and the choice of x that the induced homomorphism in homology

$$H_{2n}(S_p^n; \mathbb{Z}) \longrightarrow H_{2n}(\tau_{\geq n} X; \mathbb{Z}) \longrightarrow H_{2n}(K(\pi_n X, n); \mathbb{Z})$$

maps an uncountable rational vector space monomorphically to $H_{2n}(K(\pi_n X, n); \mathbb{Z})$, hence so does the map $H_{2n}(S_p^n; \mathbb{Z}) \rightarrow H_{2n}(\tau_{\geq n} X; \mathbb{Z})$. This verifies the claim about the integral homology of X .

Recall that, for any connective spectrum Y , the Hurewicz map $\pi_* Y \rightarrow H_*(Y; \mathbb{Z})$ has kernel and cokernel of bounded torsion in each degree. Indeed, the fiber sequence $Y \wedge \tau_{>0} S^0 \rightarrow Y \rightarrow Y \wedge H\mathbb{Z}$ reduces this claim to showing that $\pi_*(Y \wedge \tau_{>0} S^0)$ is bounded torsion in each degree. This follows from the convergent Atiyah–Hirzebruch spectral sequence

$$H_s(Y; \pi_t \tau_{>0} S^0) \implies \pi_{s+t}(Y \wedge \tau_{>0} S^0),$$

because $H_s(Y; \pi_t \tau_{>0} S^0)$ is bounded torsion for all s and t . Therefore, any rational vector space in $H_*(Y; \mathbb{Z})$ may be lifted back to $\pi_* Y$. In particular, an uncountable rational vector space in $H_{2n}(X; \mathbb{Z})$ may be lifted back to $\pi_{2n}(\Sigma^\infty X)$ after suspension. □

Remark 4.6. Suppose X is a p -complete nilpotent space such that $\pi_n X$ is the lowest homotopy group not of bounded p -torsion. The above argument shows that $H_{2n}(X; \mathbb{Z})$ contains an uncountable rational vector space. With more work, we can also show that $H_k(X; \mathbb{Z})$ is derived p -complete for $k \leq 2n - 2$ and thus cannot

contain any rational classes. Note that when X is $(n - 1)$ -connected, this follows immediately from Corollary 3.4 since $H_k(X; \mathbb{Z})$ is in the stable range.

We can now prove our main theorem.

Theorem 4.7. *If X is a p -complete nilpotent space, then $\Sigma^\infty X$ is p -complete if and only if $\pi_n X$ is bounded p -torsion for each n .*

Note that the torsion exponent of $\pi_n X$ may vary with n and does not need to be bounded uniformly for all n .

Proof. First assume that X is a p -complete nilpotent space with $\pi_n X$ of bounded p -torsion for each n ; we can apply [4, Ch. II, 4.7] to see that the Postnikov tower of X can be refined to a tower of principal fibrations whose fibers are Eilenberg–MacLane spaces for bounded p -torsion abelian groups. The category of bounded p -torsion abelian groups forms a Serre class, so Serre theory implies that $H_*(X; \mathbb{Z}) \cong H_*(\Sigma^\infty X; \mathbb{Z})$ is degreewise bounded p -torsion. Hence, $\Sigma^\infty X$ is p -complete as a spectrum by Corollary 3.3.

The converse is a consequence of Proposition 4.5: if $\pi_* X$ is not all bounded torsion, then $H_*(\Sigma^\infty X; \mathbb{Z})$ contains rational classes and thus cannot be derived p -complete, hence $\Sigma^\infty X$ is not p -complete by Corollary 3.3. \square

The next result generalizes [11, Prop. 2.4].

Corollary 4.8. *If X is a pointed connected space with degreewise finite homotopy groups, then the canonical map $(\Sigma^\infty X)_p^\wedge \rightarrow \Sigma^\infty X_p^\wedge$ is an equivalence.*

Proof. By [4, Ch. VII, 4.3], X is a \mathbb{Z}/p -good space and X_p^\wedge is a p -complete nilpotent space whose homotopy groups are all finite p -groups. Hence $\Sigma^\infty X_p^\wedge$ is p -complete by Theorem 4.7. It follows that the natural map $(\Sigma^\infty X)_p^\wedge \rightarrow \Sigma^\infty X_p^\wedge$ is an $H\mathbb{F}_p$ -equivalence between $H\mathbb{F}_p$ -local spectra, which implies the claim. \square

Corollary 4.9. *If X is a nilpotent space with $H_n(X; \mathbb{Z})$ and $\pi_n X$ derived p -complete for all n , then $H_n(X; \mathbb{Z})$ and $\pi_n X$ are bounded p -torsion for all n .*

Proof. The assumption on $\pi_* X$ implies that X is p -complete by Theorem 2.6, while the assumption on $H_*(X; \mathbb{Z})$ shows that $\Sigma^\infty X$ is p -complete, using Corollary 3.3. It thus follows from Theorem 4.7 that $\pi_* X$ is degreewise bounded p -torsion, hence so is $H_*(X; \mathbb{Z})$ by the proof of Theorem 4.7. \square

The analogue of this corollary does not hold stably, as the following example demonstrates.

Example 4.10. Let $M(\mathbb{Z}_p, n)$ be the Moore space for \mathbb{Z}_p in degree $n \geq 2$. As $H_*(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z})$ is isomorphic to $\mathbb{Z}_p[n]$, we see that $\Sigma^\infty M(\mathbb{Z}_p, n)$ is p -complete and consequently has derived p -complete stable homotopy groups and integral homology groups. However, $H_n(\Sigma^\infty M(\mathbb{Z}_p, n); \mathbb{Z}) \cong \mathbb{Z}_p$ is clearly not bounded p -torsion. In particular, $M(\mathbb{Z}_p, n)$ is not p -complete, so this also shows that the assumption that X be p -complete cannot be dropped in Theorem 4.7.

5. RATIONAL CLASSES IN THE STABLE HOMOTOPY GROUPS OF $K(\mathbb{Z}_p, n)$

In this section, we present an example that illustrates how the rational classes in the stable homotopy groups of p -complete spaces arise. In fact, we present two different approaches: one using the integral homology of $K(\mathbb{Z}_p, n)$, and one

using Goodwillie calculus. The latter derivation is entirely stable and might be of independent interest.

First, we need a well-known auxiliary result; we outline a proof because we were unable to find a published reference for it. For an abelian group A and any $k \geq 0$, let $\text{Sym}_{\mathbb{Z}}^k(A)$ and $\Lambda_{\mathbb{Z}}^k(A)$ be the k th symmetric power and the k th exterior power on A , respectively.

Lemma 5.1. *If $k > 1$, then $\Lambda_{\mathbb{Z}}^k(\mathbb{Z}_p)$ and the kernel of the multiplication map $\text{Sym}_{\mathbb{Z}}^k(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p$ are uncountable rational vector spaces.*

Proof. Since both symmetric and exterior power commute with base-change along $\mathbb{Z} \rightarrow \mathbb{Z}/l$ for any prime l , the indicated maps are isomorphisms mod l . Moreover, $\text{Sym}_{\mathbb{Z}}^k(A)$ and $\Lambda_{\mathbb{Z}}^k(A)$ are torsion-free whenever A is, so both $\ker(\text{Sym}_{\mathbb{Z}}^k(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p)$ and $\Lambda_{\mathbb{Z}}^k(\mathbb{Z}_p)$ are rational vector spaces. We may therefore base-change to \mathbb{Q} , where it is easy to verify that the \mathbb{Q} -dimension of the groups under consideration is that of \mathbb{Q}_p . □

Remark 5.2. A similar argument also shows that $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ is a rational vector space with the same \mathbb{Q} -dimension as \mathbb{Q}_p .

Proposition 5.3. *For $n \geq 1$ and all $k > 1$, the stable homotopy group $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not p -complete.*

First proof. Let A be an abelian group and recall that $H_*(K(A, n); \mathbb{Z})$ equipped with the Pontryagin product is a graded commutative algebra such that squares of odd dimensional elements are zero; in fact, it has the structure of a graded divided power algebra; see [5, 6] or more recently [12]. With notation as in the previous lemma, the canonical isomorphism $A \rightarrow H_n(K(A, n); \mathbb{Z})$ thus extends to a natural homomorphism

$$\begin{cases} \phi^k(A, n): \Lambda_{\mathbb{Z}}^k(A) \longrightarrow H_{kn}(K(A, n); \mathbb{Z}) & \text{if } n \text{ odd,} \\ \phi^k(A, n): \text{Sym}_{\mathbb{Z}}^k(A) \longrightarrow H_{kn}(K(A, n); \mathbb{Z}) & \text{if } n \text{ even,} \end{cases}$$

for any $n, k > 0$. Moreover, we know that $\phi^k(A, n) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational isomorphism. It then follows from Lemma 5.1 that, for $k > 1$, there exists an uncountable rational vector space which is mapped monomorphically to $H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z})$ via $\phi^k(\mathbb{Z}_p, n)$. We thus obtain an uncountable rational vector space in $H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z})$ that may be lifted back to give the desired uncountable rational vector space in $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ for $k > 1$, as in the proof of Proposition 4.5. □

Second proof. We will compute the homotopy groups of

$$\Sigma^\infty K(\mathbb{Z}_p, n) \simeq \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p$$

using Goodwillie calculus [7]. To this end, recall that the Goodwillie tower $(P_k)_{k \geq 1}$ associated to the functor $\Sigma^\infty \Omega^\infty: \text{Sp} \rightarrow \text{Sp}$ is assembled from fiber sequences of functors

$$(5.1) \quad D_k \longrightarrow P_k \longrightarrow P_{k-1}$$

with layers $D_k X \simeq X_{h\Sigma_k}^{\wedge k}$, where the homotopy orbits are formed with respect to the permutation action of Σ_k (see for example [9] and the references given therein).

Moreover, the Goodwillie tower $(P_k)_{k \geq 0}$ converges for connective spectra, i.e., there is a canonical equivalence

$$\Sigma^\infty \Omega^\infty X \xrightarrow{\sim} \lim_k P_k X$$

for any connective $X \in \text{Sp}$. We will apply this in the case $X = \Sigma^n H\mathbb{Z}_p$.

In order to understand the layers, we start by analyzing $\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge k}$ via the universal coefficient theorem. We claim that, for all $k \geq 1$, the homotopy groups have the following form:

$$(5.2) \quad \pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge k} \cong \begin{cases} 0 & * < nk, \\ \mathbb{Z}_p^{\otimes_{\mathbb{Z}} k} & * = nk, \\ \text{finite} & * > nk. \end{cases}$$

By the universal coefficient theorem, we have an isomorphism

$$\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge k} \cong (\pi_*(\Sigma^n H\mathbb{Z})^{\wedge k}) \otimes_{\mathbb{Z}} \mathbb{Z}_p^{\otimes k}.$$

In degrees $* > nk$, the groups $\pi_*(\Sigma^n H\mathbb{Z})^{\wedge k}$ are torsion, so the only torsion-free summand appears in degree nk . Since $\pi_*(\Sigma^n H\mathbb{Z})^{\wedge k}$ is finitely generated over \mathbb{Z} in each degree, the claim follows.

We now plug the formula (5.2) into the convergent homotopy orbit spectral sequence

$$H_s(\Sigma_k, \pi_t(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \implies \pi_{s+t} D_k(\Sigma^n H\mathbb{Z}_p).$$

There are two cases: If $t > nk$ or $t < nk$, then the groups $H_s(\Sigma_k, \pi_t(\Sigma^n H\mathbb{Z}_p)^{\wedge k})$ are finite or trivial for all s , respectively. Let $t = nk$. By Lemma 5.1 and (5.2), there is an isomorphism $H_s(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \cong H_s(\Sigma_k, \mathbb{Z}_p)$ for $s > 0$ and $H_0(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k})$ contains an uncountable rational vector space V_k if $k > 1$. To see the last statement, it suffices to compute the coinvariants on the rational submodule of $\mathbb{Z}_p^{\otimes_{\mathbb{Z}} k}$ by choosing a \mathbb{Q} -bases, as in the proof of Lemma 5.1. Furthermore, since the integral homology of Σ_k is finitely generated over \mathbb{Z} in each degree and rationally trivial in positive degrees, $H_s(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k})$ is finite for all $s > 0$. Combining all this information, we obtain $D_1 \Sigma^n H\mathbb{Z}_p \simeq \Sigma^n H\mathbb{Z}_p$ and for $k > 1$:

$$(5.3) \quad \pi_* D_k(\Sigma^n H\mathbb{Z}_p) \cong \begin{cases} 0 & * < nk, \\ V_k \oplus W_k & * = nk, \\ \text{finite} & * > nk, \end{cases}$$

where V_k is an uncountable rational vector space and W_k is some abelian group.

This allows us to derive a structural formula for $\pi_* P_k \Sigma^n H\mathbb{Z}_p$. Consider the following segment of the long exact sequence of homotopy groups associated to the fiber sequence (5.1):

$$\dots \longrightarrow \pi_{nk+1} P_{k-1} \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} D_k \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} P_k \Sigma^n H\mathbb{Z}_p \longrightarrow \dots$$

Because $n \geq 1$, it follows inductively from (5.3) that the term on the left is finite, hence V_k must be a summand in $\pi_{nk} P_k \Sigma^n H\mathbb{Z}_p$. This yields for all $k \geq 1$:

$$(5.4) \quad \pi_* P_k \Sigma^n H\mathbb{Z}_p \cong \begin{cases} 0 & * < n, \\ V_l \oplus W'_l & * = nl \text{ with } 1 \leq l \leq k, \\ \text{finite} & \text{otherwise,} \end{cases}$$

where V_l is as above for $l \geq 2$, and V_1 and W'_l are some abelian groups.

Finally, since $D_k \Sigma^n H\mathbb{Z}_p$ is nk -connective for all k , the tower $(\pi_* P_k \Sigma^n H\mathbb{Z}_p)_{k \geq 0}$ stabilizes after finally many steps in each degree and hence is Mittag-Leffler. The corresponding Milnor sequence thus degenerates to an isomorphism

$$\pi_* \Sigma^\infty K(\mathbb{Z}_p, n) \cong \pi_* \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p \cong \lim_k \pi_* P_k \Sigma^n H\mathbb{Z}_p.$$

Therefore, the claim follows from (5.4). \square

ACKNOWLEDGMENTS

The authors are grateful to Peter May for suggesting the authors get in touch over this problem. Furthermore, the first author would like to thank Bjørn Dundas, Frank Gounelas, Jesper Grodal, and Thomas Nikolaus for helpful conversations about p -completion.

REFERENCES

- [1] Reinhold Baer, *The subgroup of the elements of finite order of an abelian group*, Ann. of Math. (2) **37** (1936), no. 4, 766–781, DOI 10.2307/1968617. MR1503308
- [2] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150, DOI 10.1016/0040-9383(75)90023-3. MR0380779
- [3] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281, DOI 10.1016/0040-9383(79)90018-1. MR551009
- [4] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972. MR0365573
- [5] *Séminaire Henri Cartan de l'École Normale Supérieure, 1954/1955. Algèbres d'Eilenberg-MacLane et homotopie* (French), Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955. MR0087934
- [6] Samuel Eilenberg and Saunders Mac Lane, *On the groups $H(\Pi, n)$. II. Methods of computation*, Ann. of Math. (2) **60** (1954), 49–139, DOI 10.2307/1969702. MR0065162
- [7] Thomas G. Goodwillie, *Calculus. III. Taylor series*, Geom. Topol. **7** (2003), 645–711, DOI 10.2140/gt.2003.7.645. MR2026544
- [8] Mark Hovey and Neil P. Strickland, *Morava K -theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100, DOI 10.1090/memo/0666. MR1601906
- [9] Nicholas Kuhn and Jason McCarty, *The mod 2 homology of infinite loopspaces*, Algebr. Geom. Topol. **13** (2013), no. 2, 687–745, DOI 10.2140/agt.2013.13.687. MR3044591
- [10] J. P. May and K. Ponto, *More concise algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories. MR2884233
- [11] S. Precht Reeh, T. M. Schlank, and N. Stapleton, *A formula for p -completion by way of the Segal conjecture*, ArXiv e-prints (2017).
- [12] Birgit Richter, *Divided power structures and chain complexes*, Alpine perspectives on algebraic topology, Contemp. Math., vol. 504, Amer. Math. Soc., Providence, RI, 2009, pp. 237–254, DOI 10.1090/conm/504/09882. MR2581911
- [13] Dennis Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) **100** (1974), 1–79, DOI 10.2307/1970841. MR0442930

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, 2100 KØBENHAVN Ø, DENMARK

Email address: tbarthel@math.ku.dk

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCES, UNIVERSITY OF ILLINOIS AT CHICAGO, 851 S. MORGAN STREET (M/C 249), CHICAGO, ILLINOIS 60607-7045

Email address: bous@uic.edu