Tangent scheme + derivations II

R = comm ring

B = R-algebra

M = B-module

$$\begin{tikzcd} B \arrow[r, "d_M"] \arrow[d, "d_U"'] & M \\ \Omega_{B/R} \arrow[ur, dashed] & \end{tikzcd}$$

$$B \otimes_R B \xrightarrow{\mu} B$$

$$b_1 \otimes b_2 \longmapsto b_1 b_2$$

$$\ker \mu =: J$$

$$d_J : B \longrightarrow J/J^2$$

$$b \longmapsto b \otimes 1 - 1 \otimes b$$

$$\langle \Omega_{B/R}, d_U \rangle \cong \langle J/J^2, d_J \rangle$$

$$\left\{ \begin{array}{c} \text{closed} \\ \text{subschemes} \\ \text{of } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{quasi-coherent} \\ \text{ideal sheaves} \end{array} \right\}$$

$\mathcal{F}$ is quasi-coh if

① $\exists$ cover $\{U_i\}$ of X such that

$$\mathcal{O}_{U_i}^{(I)} \to \mathcal{O}_{U_i}^{(J)} \to \mathcal{F}|_{U_i} \to 0$$

is exact

② For each diagram

$$\begin{tikzcd} \operatorname{Spec}(B) \arrow[d, "g"'] \arrow[dr, "b"] & \\ \operatorname{Spec}(A) \arrow[r, "a"'] & X \end{tikzcd}$$

Note $\mathcal{O}_X(a) = A$, $\mathcal{O}_X(b) = B$

$$g^* \mathcal{F}(a) = B \otimes_A \mathcal{F}(a) \xrightarrow{\simeq} \mathcal{F}(b)$$

If $\mathcal{F}$ is the sheaf of ideals corresponding to $\Delta(X)$. ($f: X \to S$ is separated if $X \xrightarrow{\Delta} X \times_S X$ is closed

$$\Omega_{X/S} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

Def:

R comm ring

M an R-module

$R \ltimes M$ is the <u>square zero extension</u> of R by M

$R \ltimes M$ as an R-module is $R \times M$

There is a mult

$$(a,x)(b,y) = (ab,\ ay + bx)$$

<u>Example</u> Suppose that $f : B \to B \ltimes M$ is a ring map

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, equal] & B \ltimes M \arrow[d, "p_1"] \\ & B \end{tikzcd}$$

Then the composite

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[rr, bend right, "d_M"'] & B \ltimes M \arrow[r, "p_2"] & M \end{tikzcd}$$

$$f(m_1 m_2) = f(m_1) f(m_2)$$
$$= (m_1 m_2,\ m_2\, dm_1 + m_1\, dm_2)$$

Def Let $\mathcal{F}$ be a quasi-coh sheaf over X. Define the $\mathcal{O}_X$-algebraic sheaf. $\mathcal{O}_X \ltimes \mathcal{F}$ on X to be the square-zero extension of $\mathcal{O}_X$ by $\mathcal{F}$

$$\mathcal{O}_X \ltimes \mathcal{F}(U) = \mathcal{O}_X(U) \ltimes \mathcal{F}(U).$$

A <u>derivation of X with coeffs in $\mathcal{F}$</u> is a diagram of comm rings

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr, "1"] \arrow[dr, equal] & & \mathcal{O}_X \ltimes \mathcal{F} \arrow[dl, "p"] \\
& \mathcal{O}_X &
\end{tikzcd}$$

$q: X \to S$ (if X is a scheme over S) then a <u>S-derivation</u> of X with coeffs in $\mathcal{F}$

is a deriv. of X with coeff. in $\mathcal{J}$

$$q_* f : q_* \mathcal{O}_X \longrightarrow q_* \mathcal{O}_X \oplus q_* \mathcal{J}$$

is a morphism of $\mathcal{O}_S$-algebras.

Write $\mathrm{Der}_S(X, \mathcal{J})$ for the set of such derivs.

<u>Example</u> $X \to S$ is a separated morphism of schemes ($\Delta(X)$ is closed in $X \times_S X$).

Let $\mathcal{J}$ be the corresponding ideal sheaf.

There is an ideal sheaf corr. to the vanishing of $\mathcal{J}^2$.

Let $(X \times_S X)$ be the closed subscheme that corr to this

$\Delta^{\vee} \mathcal{O}_{(X \times_S X)} = \mathcal{O}_X \ltimes \Omega_{X/S}$ is a sheaf over X and

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "fn"] \arrow[rr, bend right, "dn"] & \mathcal{O}_X \ltimes \Omega_{X/S} \arrow[r] & \Omega_{X/S} \end{tikzcd}$$

is the universal derivation.

The module of S-derivations are the global sections of

$$Der_S(X, \mathcal{F})(U \xrightarrow{open} X) = Der_S(U, \mathcal{F}|_U)$$

is an $\mathcal{O}_X$-module sheaf not going to be quasi-coherent in general.

Prop There is a natural isomorphism of $\mathcal{O}_X$-module sheaves

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \xrightarrow{\cong} \mathcal{D}er_S(X, \mathcal{F})$$

given by composing with the universal derivation.

Example

$$\begin{tikzcd} X \arrow[d, phantom, "\shortparallel"] & S \arrow[d, phantom, "\shortparallel"] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[rr, "d_M"] \arrow[dr, "d_M"'] & & M \\ & \Omega_{B/R} \arrow[ur, "f_M"'] & \end{tikzcd}$$

R-derivation

$$\mathrm{Mod}_B(\Omega_{B/R}, M) \overset{?}{\cong} \mathrm{Der}_R(B, M)$$

$$f_M \mapsto f_M \circ d_M$$

Corollary

$$\mathrm{Mod}_X(\Omega_{X/S}, \mathcal{F}) \cong \mathrm{Der}_S(X, \mathcal{F})$$

category of $\mathcal{O}_X$-module sheaves in Zariski Topology

Note If $\mathcal{F}$ is a quasi-coh sheaf of $\mathcal{O}_X$-modules then we can "ringify" it by

$$\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}) = \mathcal{O}_{\mathbb{V}(\mathcal{F})} \quad \text{where}$$

$$\mathbb{V}(\mathcal{F})(A) = \coprod_{\mathrm{Spec}(A) \to X} \mathrm{Mod}_A(\mathcal{F}(A), A)$$

Note: iso of $\mathcal{O}_X \cong \mathcal{O}_Y$ gives $X \cong Y$.

First let $\mathcal{F} = \mathcal{O}_X$ in the prop. We get

$$\mathrm{hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X) \xrightarrow{\cong} \mathrm{Der}_S(X, \mathcal{O}_X)$$

$\mathbb{V}(\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X)) = \mathbb{V}(\mathcal{D}er_S(X, \mathcal{O}_X))$

$\qquad \| $

$\mathbb{V}(\Omega^*_{X/S})$

has structure sheaf

$\mathcal{O}_X \times \mathcal{O}_X = \mathcal{O}_X(\varepsilon)$

where $\mathcal{O}_X(\varepsilon)(A) = X(A(\varepsilon))$ and $A(\varepsilon) = A[x]/(x^2)$

$= \mathcal{O}_{\mathrm{Tan}\, X}$

As an A-module $A(\varepsilon) = A \times A$

$(a+bx)(c+dx) = ac + (ad+bc)x + bdx^2$

$\equiv ac + (ad+bc)x \mod x^2$

As an A-alg, $A(\varepsilon) = A \rtimes A$.