

# BAILEY DERIVATIONS II

DL, TW, MW, JP

Note Title

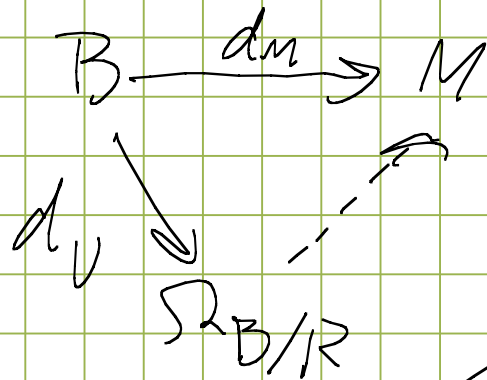
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## Tangent scheme + derivations II

$R =$  comm ring

$B = R$ -algebra

$M = B$ -module



$$B \otimes_R B \xrightarrow{\mu} B$$

$$b_1 \otimes b_2 \mapsto b_1 b_2$$

$$\ker \mu =: J$$

$$d_J: B \rightarrow J/J^2$$

$$b \mapsto b \otimes 1 - 1 \otimes b$$

$$\langle \Omega_{B/R}, d_U \rangle \cong \langle J/J^2, d_J \rangle$$

{ closed subschemes of  $X$  }

{ quasi-coherent ideal sheaves }

$\longleftrightarrow$

$\mathcal{F}$  is quasi-coh if

(1)  $\mathcal{F}$  covers  $\{U_i\}$  of  $X$  such that

$$\mathcal{O}_{U_i}^{(i)} \rightarrow \mathcal{O}_{U_i}^{(j)} \rightarrow \mathcal{F}|_{U_i} \rightarrow 0$$

is exact

(2) For each diagram

$$\begin{array}{ccc} \text{Spec}(B) & \xrightarrow{f} & X \\ \downarrow g & & \downarrow \alpha \\ \text{Spec}(A) & \xrightarrow{a} & X \end{array}$$

with  $\mathcal{O}_x(A) = A, \mathcal{O}_x(B) = B$

$$f^* \mathcal{F}(a) = B \otimes_A \mathcal{F}(a) \cong \mathcal{F}(a)$$

If  $\mathcal{F}$  is the sheaf of ideals corresponding to  $\Delta(X)$ . ( $f: X \rightarrow S$  is separated if  $X \xrightarrow{\Delta} X \times_S X$  is closed

$$\Omega_{X/S} = \Delta^* (\Omega / \Omega^2)$$

Def:

$R$  comm ring

$M$  an  $R$ -module

$R \times M$  is the square  
zero extension of  $R$  by  $M$

$R \times M$  as an  $R$ -module is  $R \times M$

There is a mult

$$(a, x) (b, y) = (ab, ay + bx)$$

Example Suppose that

$f: B \rightarrow B \times M$  is a  
ring map

$$\begin{array}{ccc} B & \xrightarrow{f} & B \times M \\ & \searrow & \swarrow \\ & B & \leftarrow p_1 \end{array}$$

Then the composite

$$\begin{array}{ccccc} B & \xrightarrow{f} & B \times M & \xrightarrow{p_2} & M \\ & & \downarrow & & \downarrow \\ & & & & dM \end{array}$$

$$f(m_1, m_2) = f(m_1) f(m_2)$$

$$= (m_1, m_2, m_2 dm_1 + m_1 dm_2)$$

Def Let  $\mathcal{F}$  be a quasi-coh sheaf over  $X$ . Define the  $\mathcal{O}_X$ -algebraic sheaf,  $\mathcal{O}_X \rtimes \mathcal{F}$  on  $X$  to be the square root extension of  $\mathcal{O}_X$  by  $\mathcal{F}$

$$\mathcal{O}_X \rtimes \mathcal{F}(U) = \mathcal{O}_X(U) \rtimes \mathcal{F}(U).$$

A derivation of  $X$  with coeffs in  $\mathcal{F}$  is a diagram of comm rings

$$\begin{array}{ccc}
 \mathcal{O}_X & \xrightarrow{\quad} & \mathcal{O}_X \rtimes \mathcal{F} \\
 & \searrow & \swarrow \text{pr} \\
 & & \mathcal{O}_X
 \end{array}$$

$g: X \rightarrow S$  (if  $X$  is a scheme over  $S$  then  
 a  $S$ -derivation of  $X$  with coeffs in  $\mathcal{F}$ )

is a deriv. of  $X$  with coefficient  $\mathcal{F}$   
 $g_x f: g_x \mathcal{O}_x \longrightarrow g_x \mathcal{O}_x \otimes g_x \mathcal{F}$   
is a morphism of  $\mathcal{O}_x$ -algebras

Write  $\text{Der}_S(X, \mathcal{F})$  for the set of such derivs.

Example  $X \rightarrow S$  is a separated morphism of schemes ( $\Delta(X)$  is closed in  $X \times_S X$ )  
Let  $\mathcal{I}$  be the corresponding ideal sheaf.  
There is an ideal sheaf corr. to the vanishing of  $\mathcal{I}^2$ .

Let  $(X \times_S X)$  be the closed subscheme that corresponds to this

$$\Delta^* \mathcal{O}_{(X \times_S X)} = \mathcal{O}_X \otimes \Omega_{X/S} \text{ is a sheaf over } X \text{ and } \mathcal{O}_X \xrightarrow{d\pi} \mathcal{O}_X \otimes \Omega_{X/S} \rightarrow \Omega_{X/S}$$

is the universal derivation.

The module of  $S$ -derivations are the global sections of

$$\mathcal{D}_{X/S}(U, \mathcal{F}) (U \xrightarrow{\text{open}} X) = \text{Der}_S(U, \mathcal{F}|_U)$$

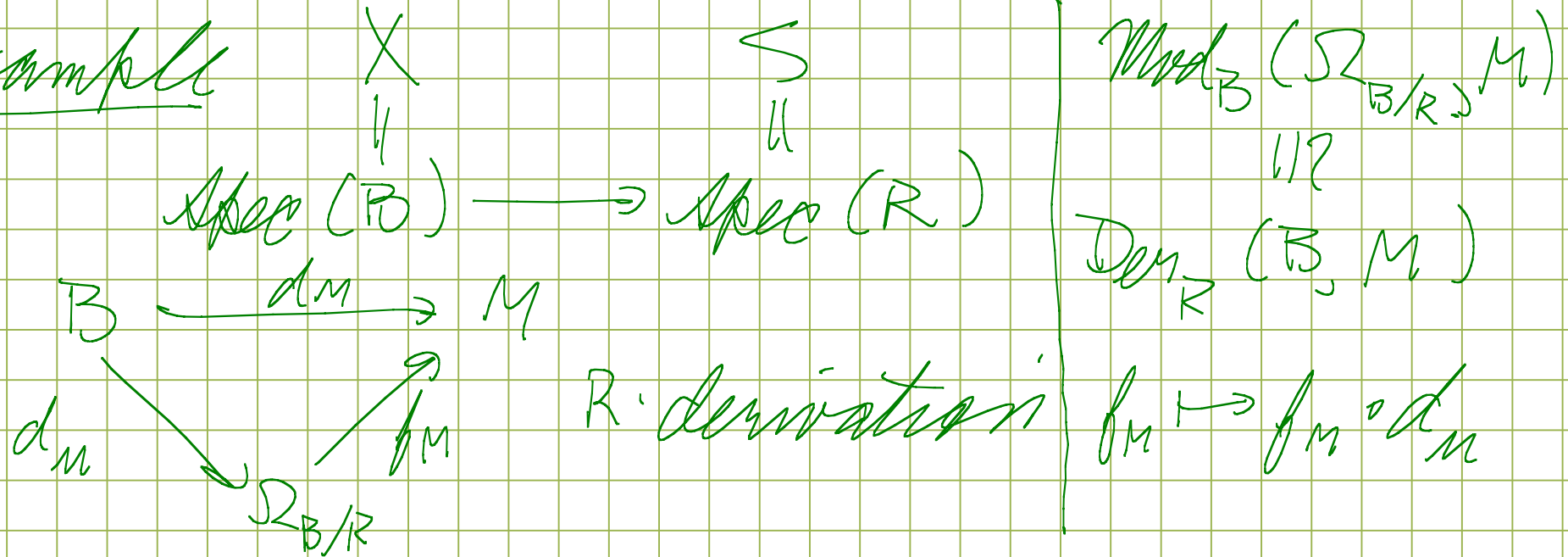
is an  $\mathcal{O}_X$ -module sheaf not going to be quasi-coherent in general.

Prop There is a natural isomorphism of  $\mathcal{O}_X$ -module sheaves

$$\text{hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \xrightarrow{\cong} \text{Der}_S(X, \mathcal{F})$$

given by composing with the universal derivation.

Example



Corollary

$$\text{Mod}_X(\Omega_{X/k}, \mathcal{F}) \cong \text{Der}_S(X, \mathcal{F})$$

category  
of  $\mathcal{O}_X$ -module  
sheaves in  
Zariski topology



Note If  $\mathcal{F}$  is a quasi-coh sheaf of  $\mathcal{O}_X$ -modules  
then we can "ringify" it by

$$\text{Sym}_{\mathcal{O}_X}(\mathcal{F}) = \mathcal{O}_{W(\mathcal{F})} \quad \text{where}$$

$$W(\mathcal{F})(A) = \coprod_{\text{Spec}(A) \rightarrow X} \text{Mod}_A(\mathcal{F}(A), A)$$

Note : ism of  $\mathcal{O}_X \cong \mathcal{O}_Y$  gives  $X \cong Y$ .

First let  $\mathcal{F} = \mathcal{O}_X$  in the prop. We get  
 $\text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X) \xrightarrow{\cong} \text{Der}_S(X, \mathcal{O}_X)$

$$\mathbb{V}(\text{hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X)) = \mathbb{V}(\text{Der}_S(X, \mathcal{O}_X))$$

$$\Downarrow$$

$$\mathbb{V}(\Omega_{X/S}^*)$$

has structure sheaf

$$\mathcal{O}_X \times \mathcal{O}_X = \mathcal{O}_X(\varepsilon)$$

where  $\mathcal{O}_X(\varepsilon)(A) = X(A(\varepsilon))$  and  $A(\varepsilon) = A[x]/(x^2)$   
 $= \mathcal{O}_{\text{Tan } X}$

As an  $A$ -module  $A(\varepsilon) = A \times A$

$$(a + bx)(c + dx) = ac + (ad + bc)x + bdx^2$$

$$= ac + (ad + bc)x \pmod{x^2}$$

As an  $A$ -alg,  $A(\varepsilon) = A \rtimes A$ .

