BROWDER HISTORY OF THE PROBLEM

Note Title 2/11/2010

If M^{4k} is a closed oriented mfd, we have a cup product $H^{2k}(M) \otimes H^{2k}(M) \xrightarrow{(\,,\,)} \mathbb{Z}$ which is symm, bilinear + nonsing. It has a signature $\operatorname{sgn}((\,,\,))$, the # of positive − # negative eigenvalues.

If $M = \partial W$ (W oriented) then $\operatorname{sgn}(M) = 0$, so it is a cobordism invariant.

For M^{4k+2}, $H^{2k+1} \otimes H^{2k+1} \longrightarrow \mathbb{Z}$ is nonsing, skew symm, bilin and therefore has a canonical form. So there is no such invariant

Homotopy theory began with the Hopf map

$$S^3 \xrightarrow{h} S^2 = \mathbb{C} \cup \{\infty\}$$

$$(z_1, z_2) \longmapsto \begin{cases} z_1/z_2 & z_2 \neq 0 \\ \infty & z_2 = 0 \end{cases}$$

$$S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1 \}$$

$$S^3_+ = \{ (z_1, z_2) : |z_1| \geq |z_2| \}$$

$$S^3_- = \{ \quad : \quad \leq$$

If h extends to D^4 then

$h^{-1}(O_+) = S^1_+$, $h^{-1}(O_-) = S^1_-$

linking # 1

If h extends this linking # would be 0.

Pontryagin (193?)

$$S^{n+k} \xrightarrow{f} S^k$$

k = codim

n = dim

For a regular value $\lambda \in S^k$, $f^{-1}(\lambda) = N^n$ with a framing; i.e. an embedding

$$N^n \times \mathbb{R}^k \hookrightarrow S^{n+k}$$

$$\pi_{n+k} S^k = \Omega^{fr}_n \quad \text{for } k >> 0$$

Pontryagin used this method to show $\pi_1 = \mathbb{Z}/2$ and $\pi_2 = 0$ (WRONG) via an interesting mistake, assuming a certain map is linear rather than quadratic

Let V be an $\mathbb{F}_2$-vector space with quad. form φ, i.e. $\varphi(x+y) - \varphi(x) - \varphi(y) = (x, y)$ bilinear

"democratic invariant"

$\varphi : V \longrightarrow \{\pm 1\}$ $\varphi(x)$ is the vote of x

majority rules. The radical is

$\{ y \in V : (x, y) = 0 \ \forall x \}$

Lemma The election is not a tie provided if no radical element votes negatively.

$d(\psi) = \{\pm 1, 0\}$ (0 for a tie)

Kervaire[1960] Let M^{10} be 4-connected. Defined $\psi : H^5 M \to \mathbb{F}_2$ and showed $d(\psi) = 1$ (trivial Arf invariant) if M is C^∞. Constructed a PL M, smooth outside a single pt with $d(\psi) = -1$. Hence M is not the htpy type of a smooth mfd.

Let M be 4-connected + smooth with $\dim = 10$, $p \in M$.

For each $x \in H^5 M$ we can find an embedding $S^5 \to M$ rep'ing x (Hard Whitney embedding thm)

Let $\varphi(x) = \begin{cases} 0 & \text{if normal bundle } \nu \text{ of } S^5 \hookrightarrow M \text{ is trivial} \\ 1 & \text{if not} \end{cases}$

Any such bundle is stably trivial, so

$\nu = T(S^5)$ (tangent bundle)

or trivial

This φ is quadratic. Consider $S^5 \times S^5$

Kervaire - Milnor paper "Groups of homotopy spheres"

Milnor introduced plumbing + surgery.

Kervaire's M^{10} is a plumbing construction

Let E = tangent disk bundle of S^5

$E_1 \xrightarrow{f_1} S^5_1$ $\quad f_1^{-1}(D^5) \simeq D^5 \times D^5 \simeq f_2^{-1}(D^5)$

$E_2 \xrightarrow{f_2} S^5_2$

Glue E_1 and E_2 along common $D^5 \times D^5$

and get M^{10}_0 with $\partial(M^{10}_0) = \Sigma^9$ homeo

to S^9. $\quad M^{10} = M^{10}_0 \cup \text{cone}(\Sigma^9)$

Hence Σ^9 is exotic S^9

Kervaire - Milnor (KM) Let W^n be a framed mfd with $\partial W = N$, meaning

$$W^n \times D^k \hookrightarrow D^{n+k}$$

$$\partial W \times D^k \hookrightarrow \partial D^{n+k}$$

Use framed surgery to simplify homotopy type of W. Suppose W is $(l-1)$ - connected and $S^l \subset W$ framed embedding.

$$\overset{\shortparallel}{\partial D^{l+1}} \subset D^{l+1} \subset D^{n+k}$$

S^l has a k-frame $\subset \nu(S^l \subset D^{n+k})$

If this k-frame extends to $D^{l+1} \subset D^{n+k}$, then we can use this embedding to change W to W' via framed surgery.

There are no difficulties below the middle dimension.

KM showed that $\partial W = N$ $\left[\frac{n-1}{2}\right]$ - connected

W $\left[\frac{n}{2}\right]$ - connected

Can go from $N^{2l} \times \mathbb{R}^k \subset \mathbb{R}^{2l+k}$ to N' $(l-1)$-conn.

The obstruction to making it $(2l-1)$ connected is

$sgn(W)$ if $\partial W \neq 0$, $l = 2q$, $n = 4$

$KI(W) = d(\psi)$, ψ defined by normal bundle normal bundle of embedded spheres or by obstruction to extending the k-frame over a disk.

How can we determine this obstruction without doing surgery first?

Question: When are there framed $2n$-mflds N $(n = 2q+1)$ in $\mathbb{R}^{2n+k}$ with $KI(M) \neq 0$?

$S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ YES

dim 10 NO

For many years it was thought answer was no unless $n = 2, 6, 14$

1963 EH Brown showed KI could be defined for spin mfd for $n = 8k + 2$

1965 Brown + Peterson showed it vanishes for $k > 0$.

<u>1968 Browder</u> Gave different definition.

For q odd let M^{2q} be framed in some mfd W^{2q+k}

Thom's work implied

a) Given $x \in H_q(M; \mathbb{F}_2)$, $\exists$ mfd $N^q \overset{i}{\hookrightarrow} M^{2q}$

such that $i_*[N^q] = x$

b) If $x = \partial Y$ for $Y \in H_{q+1}(W, M)$,

$\exists\ U^{q+1} \subset W \times [0,1]$ with $\partial U = N$

Over N in $M \subset W$, given k-frame over M

hence over $N \subset W$

Obstruction to extending over U is $v_{q+1}|U$

v_{q+1} = relative characteristic class

(Kervaire)

= Wu class

W is roughly
$$\begin{array}{ccccc} K(\mathbb{Z}/2, q) & \longrightarrow & W & \longrightarrow & BO \\ & & & & \downarrow \\ & & & & K(\mathbb{Z}/2, q+1) \end{array}$$

This leads to reduction of framing problem to easier homotopy theory???