CHAPTER VII

H_∞ RING SPECTRA VIA SPACE-LEVEL HOMOTOPY THEORY

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Our main goal in this chapter is to show that the spectrum KU representing periodic complex K-theory has an H_∞ structure. The existence of such a structure is important since it will allow us to develop a complete theory of Dyer-Lashof operations in K-theory, including the computation of $K_*(QX)$; this program is carried out in chapter IX. Of course, we already know that the connective spectrum kU has an H_∞ structure since it has an E_∞ structure by [71, VIII. 2.1]. However, it is not known whether KU has an E_∞ structure, and the distinction between kU and KU is crucial for our work in chapter IX. We therefore require a new method for constructing H_∞ ring spectra.

As usual, the case of ordinary ring spectra provides a useful analogy. The easiest way to give KU a ring structure is to use Whitehead's original theory of spectra [108]. We use the term "prespectrum" for a spectrum in the sense of Whitehead [108, p. 240], reserving the term "spectrum" for the stricter definition of I§1. The Bott periodicity theorem for BU gives rise at once to a prespectrum ([108, p. 241]; more work is needed in order to get a spectrum), and the tensor product of vector bundles gives this prespectrum a ring structure in the sense of [108, p. 270]. Now the Whitehead category is not equivalent to the stable category $\bar{h}\mathscr{S}$, but it is a quotient of it, and one can lift structures in this category to $\bar{h}\mathscr{S}$ as long as certain $\lim^1$ terms vanish. These $\lim^1$ terms do vanish for KU and we obtain the desired ring structure.

In order to carry this through for H_∞ structures we must give the Bott prespectrum a "Whitehead" H_∞ structure (which is fairly easy) and show how to lift it to $\bar{h}\mathscr{S}$ (which is considerably more difficult). Our main concern in this chapter is with the lifting process, which is called the cylinder construction and denoted by Z. We begin in Sections 1 and 2 by giving a careful development of the cases already mentioned, namely the passage from prespectra to spectra and from ring prespectra to ring spectra. Our account is based on that in [67] and [71, II §3] but is adapted to allow generalization to the H_∞ case to which we turn next. In section 3 we give a general result allowing construction of maps $D_\pi E \to F$ in $\bar{h}\mathscr{S}$ from prespectrum-level data. Although the basic idea is similar to that of section 2 this situation requires new hypotheses and methods. Section 4 is a digression which gives a convenient sufficient condition for the vanishing of the $\lim^1$ terms encountered in sections 1, 2, and 3. In section 5 we define H_∞ structures on prespectra (for technical reasons, these are called H_∞^d structures) and show that they lift to H_∞ structures in $\bar{h}\mathscr{S}$ when the relevant $\lim^1$ terms vanish. In section 6 we observe that spectra obtained in this way actually have H_∞^d structures as defined in I.4.3 and that there is in fact an "approximate equivalence" between H_∞^d structures on spectra and prespectra. Section 7 gives the application to K-theory. The necessary H_∞^d structure on the Bott prespectrum is obtained from the E_∞ structure on kU; a more elementary construction not depending on E_∞ theory (but still using the results of this chapter) will be given in VIII §4. Section 8 gives a technical result which is used in section 3. Except for section 8 and one place in section 1 we use only the formal properties of $\overline{h\mathcal{A}}$ and D_π given in I§1 and I§2.

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§1. The Whitehead category and the stable category

In this section we describe the relation between the Whitehead category, denoted $\overline{w\mathcal{P}}$, and the stable category $\overline{h\mathcal{S}}$. The results are well-known, but we give them in some detail in order to fix notation and because we need particularly precise statements for our later work.

We begin by defining $\overline{w\mathcal{P}}$. An object T, called a prespectrum, is a sequence of spaces T_i (for $i \geq 0$) and maps $\sigma_i:\Sigma T_i \to T_{i+1}$ in $\overline{h\mathcal{J}}$ (see I§1; the use of $\overline{h\mathcal{J}}$ here is technically convenient but could be avoided by systematic use of CW-approximations). If the adjoints $\tilde{\sigma}_i:T_i \to \Omega T_{i+1}$ are weak equivalences we call T an Ω-prespectrum. A morphism $f:T \to U$ is a sequence of maps $f_i:T_i \to U_i$ such that $f_{i+1}\circ\sigma_i \simeq \sigma_i \circ \Sigma f_i$ in $\overline{h\mathcal{J}}$. This should be compared with the much stricter definition of morphism in $\overline{h\mathcal{S}}$ given in I§1; it is precisely because morphisms in $\overline{w\mathcal{P}}$ are defined in terms of homotopy that this category is a useful intermediate step between space-level and spectrum-level homotopy theory. The set of maps in $\overline{w\mathcal{P}}$ from T to U is denoted $[T,U]_w$. If U is an Ω-prespectrum then this set is an abelian group and is equal to the inverse limit $\lim_i [T_i,U_i]$ with respect to the maps

$$[T_{i+1},U_{i+1}] \xrightarrow{\Omega} [\Omega T_{i+1},\Omega U_{i+1}] \xrightarrow{\tilde{\sigma}_i^*} [T_i,\Omega U_{i+1}] \xrightarrow{(\tilde{\sigma}_i)_*^{-1}} [T_i,U_i].$$

There is an evident forgetful functor $z: \overline{h\mathcal{S}} \to \overline{w\mathcal{P}}$. Although there is no useful functor in the other direction, there is an "approximately functorial" construction Z, called the cylinder construction. This can be defined in several essentially equivalent ways (see I§6 of the sequel). For our purposes it is easiest to define

$$ZT = \operatorname*{Tel}_{i} \Sigma^{-i}\Sigma^{\infty}T_i,$$

where the telescope is taken with respect to the maps

$$\Sigma^{-i}\Sigma^{\infty}T_i \simeq \Sigma^{-i}\Sigma^{-1}\Sigma\Sigma^{\infty}T_i \simeq \Sigma^{-i-1}\Sigma^{\infty}\Sigma T_i \longrightarrow \Sigma^{-i-1}\Sigma^{\infty}T_{i+1} .$$

We write θ_i for the inclusion $\Sigma^{\infty}T_i \to \Sigma^{i}ZT$. If $f:T \to U$ is any map in $\bar{w}\mathcal{P}$ there exists a map $F:ZT \to ZU$ induced by f in the sense that the diagram

$$\begin{tikzcd}
\Sigma^{-i}\Sigma^{\infty}T_i \arrow[r, "\Sigma^{-i}\Sigma^{\infty}f_i"] \arrow[d, "\Sigma^{-i}\theta_i"] & \Sigma^{-i}\Sigma^{\infty}U_i \arrow[d, "\Sigma^{-i}\theta_i"] \\
ZT \arrow[r, "F"] & ZU
\end{tikzcd}$$

commutes for all $i \geq 0$. Unfortunately, this map is not in general unique. To clarify the situation consider the Milnor $\lim^1$ sequence

$$0 \longrightarrow \lim{}^1\,[\Sigma^{1-i}\Sigma^{\infty}T_i, ZU] \longrightarrow [ZT,ZU] \longrightarrow \lim[\Sigma^{-i}\Sigma^{\infty}T_i, ZU] \longrightarrow 0.$$

Clearly, the map induced by f is unique if and only if the $\lim^1$ term vanishes. We shall use the notation Zf for this map when this condition is satisfied (and not otherwise). We have $Z(f \circ g) = Zf \circ Zg$ whenever all three are defined.

The $\lim^1$ term just mentioned is only the first of many which will arise in our work. For applications we wish to know when they vanish. This question will be considered in detail in §4; for the moment we simply remark that for the cases of interest to us (namely Bott spectra and certain bordism spectra) all relevant $\lim^1$ terms do in fact vanish.

Although Z is not a functor, it has several useful properties. In fact, one may think of the pair (z,Z) as an "approximate adjoint equivalence" between $\bar{h}\mathcal{S}$ and the full subcategory of Ω-prespectra in $\bar{w}\mathcal{P}$. The following result makes this precise.

Theorem 1.1. For each $T \in \bar{w}\mathcal{P}$ and $E \in \bar{h}\mathcal{S}$ there exists maps $\kappa:T \to zZT$ and $\lambda:ZzE \to E$ with the following properties.

(i) κ is natural in the sense that $zZf \circ \kappa = \kappa \circ f$ whenever Zf is defined.

(ii) κ is an equivalence whenever T is an Ω-prespectrum.

(iii) λ is natural in the sense that $f \circ \lambda = \lambda \circ Zzf$ whenever Zzf is defined.

(iv) λ is an equivalence for all $E \in \overline{h}\mathcal{S}$.

(v) $z\lambda \circ \kappa$ is the identity map of zE.

(vi) The map $\tau:[ZT,E] \to [T,zE]_w$ defined by $\tau f = zf \circ \kappa$ is an isomorphism whenever $\lim^1 E^{i-1}T_i = 0$.

(vii) The map Zf, whenever it is defined, is uniquely determined by the equation $\tau(Zf) = \kappa \circ f$.

The rest of this section gives the proof of 1.1. In order to construct κ and λ we need an alternative description of the i-th space functor from $\overline{h}\mathcal{S}$ to $\overline{h}\mathcal{J}$.

Lemma 1.2. There is a natural equivalence $E_i \simeq \Omega^\infty\Sigma^i E$. If $\theta_i^!$ denotes the adjoint map $\Sigma^\infty E_i \to \Sigma^i E$ then the following diagrams commute.

$$\begin{tikzcd}
\Sigma E_i \arrow[r, phantom, "\simeq"] \arrow[d, "\sigma_i"] & \Sigma\Omega^\infty\Sigma^i E \arrow[d] \\
E_{i+1} \arrow[r, phantom, "\simeq"] & \Omega^\infty\Sigma^{i+1}E
\end{tikzcd} \qquad (1)$$

$$\begin{tikzcd}
\Sigma^\infty\Sigma E_i \arrow[r, phantom, "\cong"] \arrow[d, "\Sigma^\infty\sigma_i"] & \Sigma\Sigma^\infty E_i \arrow[d, "\Sigma\theta_i^!"] \\
\Sigma^\infty E_{i+1} \arrow[r, "\theta_{i+1}^!"] & \Sigma^{i+1}E
\end{tikzcd} \qquad (2)$$

For the proof see I§7 of the sequel. The fact that such an equivalence exists should not be surprising since it is well-known that the reduced E-cohomology groups E^iX of a based space X can be defined either as $[\Sigma^\infty X, \Sigma^i E]$ or as $[X,E_i]$. The diagrams of Lemma 1.2 (which are adjoints of each other) simply say that one obtains the same suspension isomorphism with either of these two definitions.

Given $T \in \overline{w}\mathcal{P}$ we can now define $\kappa:T \to zZT$ by letting the i-th component $\kappa_i:T_i \to (ZT)_i$ be the composite

$$T_i \longrightarrow \Omega^\infty\Sigma^\infty T_i \xrightarrow{\Omega^\infty\theta_i} \Omega^\infty\Sigma^i ZT \simeq (ZT)_i .$$

We note for later use that the following diagram commutes.

$$\begin{tikzcd}
\Sigma^\infty T_i \arrow[rr, "\Sigma^\infty\kappa_i"] \arrow[dr, "\theta_i"'] & & \Sigma^\infty(ZT)_i \arrow[dl, "\theta_i^!"] \\
& \Sigma^i ZT &
\end{tikzcd} \qquad (3)$$

The verification that κ is in fact a $\overline{w}\mathcal{P}$ -map is a routine diagram chase using diagram (1) above. It is clear that κ satisfies 1.1(i); in fact it has the stronger property that $zF \circ \kappa = \kappa \circ f$ whenever $F:ZT \to ZU$ is induced by f. For part (ii) we first compute

$$\pi_k(ZT)_i = \pi_{k-i}ZT = \operatorname*{colim}_j \pi_{k-i+j}\Sigma^\infty T_j$$
$$= \operatorname*{colim}_j \operatorname*{colim}_\ell \pi_{k-i+j+\ell}\, \Sigma^\ell T_j \ .$$

A cofinality argument shows that the inclusion of $\operatorname*{colim}_j \pi_{k-i+j}\ T_j$ in the last group is an isomorphism. If T is an Ω-prespectrum, then the inclusion

$$\pi_i T_k \to \operatorname*{colim}_j \pi_{k-i+j}\ T_j$$

is an isomorphism and the result follows.

Next we define $\lambda : ZzE \to E$ to be any map obtained by passage to the telescope from the maps

$$\Sigma^{-i}\theta_i' : \Sigma^{-i}\Sigma^\infty E_i \to E.$$

Part (v) is immediate, and (iv) follows from (ii) and (v). For (iii) it suffices, by the definition of Zzf, to show that $\lambda^{-1} \circ f \circ \lambda : ZzE \to ZzE'$ is induced by zf, i.e., that the diagram

$$\begin{tikzcd}
\Sigma^{-i}\Sigma^\infty E_i \arrow[rrr, "\Sigma^{-i}\Sigma^\infty f_i"] \arrow[d, "\Sigma^{-i}\theta_i"] & & & \Sigma^{-i}\Sigma^\infty E_i' \arrow[d, "\Sigma^{-i}\theta_i"] \\
ZzE \arrow[r, "\lambda"] & E \arrow[r, "f"] & E' & ZzE' \arrow[l, "\lambda"']
\end{tikzcd}$$

commutes for all $i \geq 0$. This in turn follows from the definition of λ and the naturality of θ_i'.

For part (vi) consider the $\lim^1$ sequence

$$0 \longrightarrow \lim{}^1[\Sigma\Sigma^{-i}\Sigma^\infty T_i, E] \longrightarrow [ZT,E] \xrightarrow{\overline{\tau}} \lim[\Sigma^{-i}\Sigma^\infty T_i, E] \longrightarrow 0.$$

The map $\overline{\tau}$ agrees with τ under the isomorphism

$$\lim[\Sigma^{-i}\Sigma^\infty T_i, E] \cong \lim[T_i, E_i] = [T, zE]_W$$

and the result follows.

Finally for (vii) we calculate

$$\tau(Zf) = zZf \circ \kappa = k \circ f.$$

The uniqueness follows from (vi).

§2. Pairings of spectra and prespectra.

In this section we give a multiplicative version of the results of §1 which in particular will allow us to produce a ring spectrum in $\bar{h}\mathcal{S}$ from suitable input in $\bar{w}\mathcal{P}$. Again the results are well-known.

For the rest of the chapter we fix an integer $d > 0$ and consider prespectra indexed on nonnegative multiples of d. This is convenient in the present section (for dealing with Bott spectra) and will be crucial in §3.

Let $E, E', F \in \bar{h}\mathcal{S}$. By a pairing of E and E' into F we mean simply a map $\phi: E \wedge E' \to F$. Although the category $\bar{w}\mathcal{P}$ has no smash product, a suitable prespectrum-level notion of pairing has been given by Whitehead [108, p. 255]; we recall it here.

Definition 2.1. Let $T, T', U \in \bar{w}\mathcal{P}$. A pairing $\psi: (T,T') \to U$ consists of a collection of maps

$$\psi_{i,j}: T_{di} \wedge T'_{dj} \to U_{d(i+j)}$$

such that the following diagram commutes in $\bar{h}\mathcal{T}$ for all $i,j \geq 0$.

$$\begin{tikzcd}
\Sigma^d T_{di} \wedge T'_{dj} \arrow[r, "\sigma_i \wedge 1"] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "(-1)^{dj}"] & T_{d(i+1)} \wedge T'_{dj} \arrow[dr, "\psi_{i+1,j}"] & \\
\Sigma^d (T_{di} \wedge T'_{dj}) \arrow[r, "\Sigma^d \psi_{i,j}"] \arrow[d, phantom, "\wr"] & \Sigma^d U_{d(i+j)} \arrow[r, "\sigma_{i+j}"] & U_{d(i+j+1)} \\
T_{di} \wedge \Sigma^d T'_{dj} \arrow[r, "1 \wedge \sigma_j"] & T_{di} \wedge T'_{d(j+1)} \arrow[ur, "\psi_{i,j+1}"'] &
\end{tikzcd}$$

If $\phi: E \wedge E' \to F$ is a pairing in $\bar{h}\mathcal{S}$ and $f: \hat{E} \to E$, $f': \hat{E}' \to E'$, and $g: F \to \hat{F}$ are maps in $\bar{h}\mathcal{S}$ there is an evident pairing

$$g \circ \phi \circ (f \wedge f'): \hat{E} \wedge \hat{E}' \to \hat{F}.$$

Similarly, if $\psi: (T,T') \to U$ is a pairing in $\bar{w}\mathcal{P}$ and $f: \hat{T} \to T$, $f': \hat{T}' \to T'$, and $g: U \to \hat{U}$ are maps in $\bar{w}\mathcal{P}$ there is a composite pairing

$$g \circ \psi \circ (f, f'): (\hat{T}, \hat{T}') \to \hat{U}.$$

Next we show how to lift pairings from $\bar{w}\mathcal{P}$ to $\bar{h}\mathcal{S}$. If $\psi: (T,T') \to U$ is a pairing then $ZT \wedge ZT'$ is equivalent to

$$\operatorname{Tel} \Sigma^{-2di}\Sigma^{\infty}(T_{di} \wedge T'_{di})$$

and we can obtain an induced pairing $ZT \wedge ZT' \to ZU$ by passage to telescopes from the maps $\Sigma^{-2di}\Sigma^{\infty}\psi_{i,i}$. The induced pairing is unique if the group

$$\lim{}^{1}(ZU)^{2di-1}(T_{di} \quad T'_{di})$$

vanishes, and we denote it by $Z\psi$ when this condition is satisfied. Note that we now have two distinct, but analogous, meanings for the symbol Z, and we shall give another in section 3. There is no risk of confusion since the context will always indicate whether Z is being applied to a map in $\overline{w\mathcal{P}}$, a pairing, or an extended pairing as defined in section 3. Clearly we have

$$Zg \circ Z\psi \circ (Zf \wedge Zf') = Z(g \circ \psi \circ (f,f'))$$

whenever both sides are defined.

Next, given a pairing $\phi: E \wedge E' \to F$ in $\overline{h\mathcal{S}}$ we wish to define a pairing $z\phi: (zE, zE') \to zF$ (again, this use of the notation z is distinct from that in section 1). In contrast to section 1, it is inconvenient to do this directly from the definitions since the definition of $E \wedge E'$ is too complicated. Instead, we use the maps provided by Lemma 1.2. First let

$$\phi_{i,j}: \Sigma^{\infty}(E_{di} \wedge E'_{dj}) \to \Sigma^{d(i+j)}F$$

be the composite

$$\Sigma^{\infty}(E_{di} \wedge E'_{dj}) \simeq \Sigma^{\infty}E_{di} \wedge \Sigma^{\infty}E'_{dj} \xrightarrow{\theta'_i \wedge \theta'_j} \Sigma^{di}E \wedge \Sigma^{dj}E' \simeq \Sigma^{d(i+j)}E \wedge E' \longrightarrow \Sigma^{d(i+j)}F$$

Then the diagram

$$\begin{tikzcd}
\Sigma^{\infty}(\Sigma^{d}E_{di} \wedge E'_{dj}) \arrow[r] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "(-1)^{dj}"] & \Sigma^{\infty}(E_{d(i+1)} \wedge E'_{dj}) \arrow[dr, "\phi_{i+1,j}"] & \\
\Sigma^{\infty}(\Sigma^{d}(E_{di} \wedge E'_{dj})) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & \Sigma^{d}\Sigma^{\infty}(E_{di} \wedge E'_{dj}) \arrow[r, "\Sigma^{d}\phi_{i,j}"] & \Sigma^{d(i+j+1)}F \\
\Sigma^{\infty}(E_{di} \wedge \Sigma^{d}E'_{dj}) \arrow[r] & \Sigma^{\infty}(E_{di} \wedge E'_{d(j+1)}) \arrow[ur, "\phi_{i,j+1}"'] &
\end{tikzcd}$$

commutes by Lemma 1.2. We now define

$$(z\phi)_{i,j}: E_{di} \wedge E'_{dj} \to F_{d(i+j)}$$

to be the composite

$$E_{di} \wedge E'_{dj} \longrightarrow \Omega^\infty\Sigma^\infty(E_{di} \wedge E'_{dj}) \xrightarrow{\Omega^\infty \phi_{i,j}} \Omega^\infty\Sigma^{d(i+j)}F \simeq F_{d(i+j)} \, .$$

The fact that $z\phi$ is a pairing follows from the diagram above and another application of Lemma 1.2. We clearly have

$$z(g \circ \phi \circ (f \wedge f')) = zg \circ z\phi \circ (zf, zf').$$

Finally, given a pairing $\phi: ZT \wedge ZT' \to F$ we can define a pairing $\tau(\phi):(T,T') \to zF$ by $\tau(\phi) = z\phi \circ (\kappa,\kappa)$. In analogy with Theorem 1.1 we have

Proposition 2.2 (i) If ψ is a pairing in $\overline{w}\mathcal{P}$ then $zZ\psi \circ (\kappa,\kappa) = \kappa \circ \psi$ whenever $Z\psi$ is defined.

(ii) If ϕ is a pairing in $\overline{h}\mathcal{S}$ then $\lambda \circ Zz\phi = \phi \circ (\lambda \;\; \lambda)$ whenever $Zz\phi$ is defined.

(iii) If $\lim^1 F^{2di-1}(T_{di} \wedge T'_{di}) = 0$ then τ is a one-to-one correspondence between pairings $ZT \wedge ZT' \to F$ and pairings $(T,T') \to zF$.

(iv) The pairing $Z\psi$, whenever it is defined, is uniquely determined by the equation $\tau(Z\psi) = \kappa \circ \psi$.

The proof is completely parallel to that of 1.1 and will be omitted.

As a special case we consider ring spectra and prespectra. Let S be the zero-sphere in $\overline{h}\mathcal{S}$ and let $\underline{S}$ be the prespectrum whose di-th term is S^{di} (with the evident structural maps). A ring spectrum is a spectrum E with maps $\phi: E \wedge E \to E$ and $e: S \to E$ satisfying the usual associativity, commutativity and unit axioms. Similarly, a ring prespectrum is a prespectrum T with a pairing $\psi:(T,T) \to T$ and a map $e: \underline{S} \to T$ satisfying associativity, commutativity and unit axioms. The unit axiom in this case is the commutativity of the following diagram in $\overline{h}\mathcal{J}$.

$$\begin{tikzcd}
S^{di} \wedge T_{dj} \arrow[r, "e_{di} \wedge 1"] \arrow[dr, "\sigma"'] & T_{di} \wedge T_{dj} \arrow[d, "\psi_{i,j}"] & T_{di} \wedge S^{dj} \arrow[l, "1 \wedge e_{dj}"'] \arrow[dl, "\sigma"] \\
 & T_{d(i+j)} &
\end{tikzcd}$$

There are also evident notions of morphism for these structures. As a consequence of Proposition 2.2 we have the following.

Corollary 2.3. (i) If E is a ring spectrum then zE is a ring prespectrum. If f is a ring map in $\overline{h}\mathcal{S}$ then zf is a ring map in $\overline{w}\mathcal{P}$.

(ii) If T is a ring prespectrum with $\lim^1 (ZT)^{2di-1}(T_{di} \wedge T_{di}) = 0$ then ZT is a ring spectrum and $\kappa: T \to zZT$ is a ring map. If in addition $f: T \to T'$ is a ring map and

$$\lim^1 (ZT')^{2di-1}(T_{di} \wedge T_{di}) = \lim^1 (ZT')^{2di-1}(T'_{di} \wedge T'_{di}) = 0$$

then Zf is a ring map. If E is a ring spectrum and $\lim^1 E^{2di-1}(E_{di} \wedge E_{di}) = 0$ then $\lambda : ZzE \to E$ is a ring map.

§3. Extended pairings of spectra and prespectra

Let π be a fixed subgroup of Σ_j. In this section we generalize the results of section 2 by relating maps of the form $f : D_\pi E \to F$ in $\bar{h}\mathscr{S}$ to certain structures in $\bar{w}\mathscr{P}$ called extended pairings. This is our basic technical result, which will be applied in this chapter and the next to various problems in the theory of H_∞ ring spectra.

First we need a generalization of Definition 2.1. The difficulty is that, unlike the smash product, D_π does not commute with suspension. The situation becomes clearer when one realizes that $D_\pi \Sigma^d X$ is a relative Thom complex. For if p is the bundle

$$E\pi \times_\pi (R^d)^j \to B\pi$$

and p_X is the pullback of this bundle along the map

$$E\pi \times_\pi X^j \to B\pi,$$

then $D_\pi \Sigma^d X$ is the quotient $T(p_X)/T(p_*)$, where $*$ denotes the basepoint of X. The failure of D_π to commute with suspension arises from the fact that the bundle p is nontrivial. This suggests that we consider theories for which this bundle is at least orientable and replace the suspension isomorphisms which were implicitly present in section 2 with Thom isomorphisms. Note that the orientability of p with respect to a certain theory may well depend on the positive integer d.

Definition 3.1. Let F be a ring spectrum. A π-orientation for F is a map

$$\mu : D_\pi S^d \to \Sigma^{dj} F$$

such that the diagram

$$\begin{tikzcd}
(S^d)^{(j)} \arrow[r, "\iota"] \arrow[d, "\cong"'] & D_j S^d \arrow[d, "\mu"] \\
S^{dj} \arrow[r, "\Sigma^{dj} e"] & \Sigma^{dj} F
\end{tikzcd}$$

commutes in $\bar{h}\mathscr{S}$. If U is a ring prespectrum, a π-orientation for U is a map

$$\nu : D_\pi S^d \to U_{dj}$$

such that the diagram

$$\begin{tikzcd}
(S^d)^{(j)} \arrow[r, "\iota"] \arrow[d, phantom, "\wr"] & D_\pi S^d \arrow[d, "\nu"] \\
S^{dj} \arrow[r, "e_{dj}"] & U_{dj}
\end{tikzcd}$$

commutes in $\overline{h\mathcal{J}}$. A ring spectrum F or a ring prespectrum U with a fixed choice of π-orientation is called π-oriented. A ring map of π-oriented spectra or prespectra is π-oriented if it preserves the orientation.

It is now easy to give an analog for Definition 2.1. Recall the natural map δ defined in I§2.

Definition 3.2. Let T be a prespectrum and let (U,ν) be a π-oriented ring prespectrum. An extended pairing

$$\zeta:(\pi,T) \to (U,\nu)$$

is a sequence of maps

$$\zeta_i:D_\pi T_{di} \to U_{dij}$$

such that the following diagram commutes in $\overline{h\mathcal{J}}$ for all $i \geq 0$.

$$\begin{tikzcd}
D_\pi(T_{di}\wedge S^d) \arrow[rr, "\delta"] \arrow[d, "D_\pi \sigma"] & & D_\pi T_{di}\wedge D_\pi S^d \arrow[d, "\zeta_i \wedge \nu"] \\
D_\pi T_{d(i+1)} \arrow[dr, "\zeta_{i+1}"'] & & U_{dij}\wedge U_{dj} \arrow[dl, "\psi_{ij,j}"] \\
& U_{d(i+1)j} &
\end{tikzcd}$$

We shall usually suppress the orientation ν from the notation.

Definition 3.1 is general enough for our purposes, but it could be made more general by allowing U to be a module prespectrum over some π-oriented ring prespectrum. Everything which follows would work in this generality.

If $g:U \to U'$ is a π-oriented ring map and $f:T' \to T$ is any map in $\overline{w\mathcal{P}}$ we define the composite

$$g \circ \zeta \circ (\pi,f):(\pi,T') \to U'$$

by letting $(g \circ \zeta \circ (\pi,f))_i = g_{dji} \circ \zeta_i \circ D_\pi(f_{di})$. We also have composites in the π-variable: if ρ is a subgroup of π and U has a ρ-orientation consistent with its π-orientation then the maps

$$\zeta_i \circ \iota : D_\rho T_{di} \to U_{dij}$$

form an extended pairing denoted $\zeta \circ (\iota, 1)$.

There is an evident stable version of 3.2: if F is a π-oriented ring spectrum we define an extended pairing from E to F to be a map $\xi : D_\pi E \to F$. We do not assume any relation between ξ and the orientation μ, but the presence of μ is necessary for the comparison with the prespectrum level. We can define composites $g \circ \xi \circ D_\pi f$ and $\xi \circ \iota$ as in the prespectrum case.

To complete the program of section 2 must show how to define $z\xi$ and $Z\zeta$ with suitable properties. Both of these will be defined by using a spectrum-level variant of the Thom homomorphism to which we turn next. If F is a π-oriented ring spectrum and $f : D_\pi E \to \Sigma^n F$ is any map we write $\Phi(f)$ for the composite

$$D_\pi \Sigma^d E \xrightarrow{\delta} D_\pi E \wedge D_\pi S^d \xrightarrow{f \wedge \mu} \Sigma^n F \wedge \Sigma^{dj} F \xrightarrow{\phi} \Sigma^{n+dj} F.$$

Since each class in $F^n(D_\pi E)$ is represented by some f we obtain a homomorphism

$$\Phi : F^n(D_\pi E) \to F^{n+dj}(D_\pi \Sigma^d E)$$

called the Thom homomorphism. We write $\Phi^{(i)}$ for the iterate $F^n(D_\pi E) \to F^{n+dij}(D_\pi \Sigma^{di} E)$. If $E = \Sigma^\infty X$ for some space X then it is easy to see that Φ is the relative Thom homomorphism for the bundle p_X and is therefore an isomorphism. Thus the following result should not be surprising.

Theorem 3.3. Φ is an isomorphism for every $E \in \overline{h\mathscr{S}}$.

The proof of this result, while not difficult, involves the definition of D_π and not just its formal properties and is deferred until section 8.

We can now define $z\xi$ for an extended pairing $\xi : D_\pi E \to F$. Give zF the orientation

$$z(\mu) : D_\pi S^d \longrightarrow \Omega^\infty \Sigma^\infty D_\pi S^d \simeq \Omega^\infty D_\pi S^d \longrightarrow \Omega^\infty \Sigma^{dj} F \simeq F_{dj} .$$

For each $i \geq 0$ let $(z\xi)_i$ be the composite

$$D_\pi E_{di} \longrightarrow \Omega^\infty D_\pi \Sigma^\infty E_{di} \xrightarrow{\Omega^\infty D_\pi \theta_{di}} \Omega^\infty D_\pi \Sigma^{di} E \xrightarrow{\Omega^\infty \Phi^{(i)} \xi} \Omega^\infty \Sigma^{dij} F \simeq F_{dij} .$$

The verification that $z\xi$ is in fact an extended pairing is completely similar to the analogous verification in section 2. Further, z is natural in the sense that $z(g \circ \xi \circ D_\pi f) = zg \circ z\xi \circ (\pi, zf)$ and $z(\xi \circ \iota) = z\xi \circ (\iota, 1)$. Note that $z\xi$ depends not just on the map ξ but also on the orientation μ.

Unfortunately, $Z\zeta$ cannot be constructed directly as in sections 1 and 2. Instead we observe that we could have used 1.1(vi) and 2.2(iv) to define Zf and $Z\psi$ by means of the equations $\tau(Zf) = \kappa \circ f$ and $\tau(Z\psi) = \kappa \circ \psi$. If ξ is an extended pairing from ZT to F let $\tau(\xi)$ be the extended pairing

$$z\xi \circ (\pi,\kappa) \colon (\pi,T) \to zF.$$

At the end of this section we shall prove

Theorem 3.4. If $\lim^1 F^{-1}(D_\pi \Sigma^{-di}\Sigma^\infty T_{di}) = 0$ then τ is a bijection between extended pairings $D_\pi ZT \to F$ and extended pairings $(\pi,T) \to zF$.

We can now define $Z\zeta$ for an extended pairing $\zeta\colon(\pi,T) \to U$ when the relevant $\lim^1$ terms vanish. Give ZU the π-orientation

$$Z(\nu)\colon D_\pi S^d \simeq \Sigma^\infty D_\pi S^d \to \Sigma^\infty U_{dj} \to \Sigma^{dj} ZU.$$

and let $Z(\zeta)$ be $\tau^{-1}(\kappa \circ \zeta)$.

Corollary 3.5. (i) $zZ\zeta \circ (\pi,\kappa) = \kappa \circ \zeta$ whenever $Z\zeta$ is defined.

(ii) $Z(g \circ \zeta \circ (\pi,f)) = Zg \circ Z\zeta \circ D_\pi Zf$ and $Z(\zeta \circ (\iota,1)) = Z\zeta \circ \iota$ whenever both sides are defined.

(iii) $\lambda \circ Zz\xi = \xi \circ D_\pi \lambda$ whenever $Zz\xi$ is defined.

Proof of 3.5. (i) is the definition of $Z\zeta$. For the first equation in (ii) we calculate

$$\begin{aligned}\tau(Zg \circ Z\zeta \circ D_\pi Zf) &= zZg \circ zZ\zeta \circ (\pi,zZf) \circ (\pi,\kappa)\\ &= zZg \circ zZ\zeta \circ (\pi,\kappa) \circ (\pi,f)\\ &= zZg \circ \kappa \circ \zeta \circ (\pi,f)\\ &= \kappa \circ g \circ \zeta \circ (\pi,f)\\ &= \tau(Z(g \circ \zeta \circ (\pi,f)));\end{aligned}$$

the result follows by 3.4. The verification of the other equation in (ii) is similar. For part (iii) we have

$$\begin{aligned}\tau(\lambda^{-1} \circ \xi \circ D_\pi \lambda) &= z\lambda^{-1} \circ z\xi \circ (\pi,z\lambda) \circ (\pi,\kappa)\\ &= \kappa \circ z\xi = \tau(Z\xi)\end{aligned}$$

with the second equality following from 1.1(v); the result follows by 3.4.

Next we make some observations that will be important in sections 5 and 6. Part (iii) of our next result gives an alternate description of $Z\zeta$ which is similar to the definitions of Zf and $Z\psi$ in sections 1 and 2.

Corollary 3.6. Let $\xi : D_\pi ZT \to F$ be an extended pairing.

(i) $\tau(\xi)_i$ is the composite

$$D_\pi T_{di} \longrightarrow \Omega^\infty D_\pi \Sigma^\infty T_{di} \xrightarrow{\Omega^\infty D_\pi \theta_i} \Omega^\infty D_\pi \Sigma^{di} ZT \xrightarrow{\Omega^\infty \Phi^{(i)}\xi} \Omega^\infty \Sigma^{dij} F \simeq F_{dij}$$

(ii) If $\xi' : D_\pi ZT \to F$ is another extended pairing and τ is a bijection then $\xi = \xi'$ if and only if

$$\Phi^{(i)}\xi \circ D_\pi \theta_i = \Phi^{(i)}\xi' \circ D_\pi \theta_i$$

for all $i \geq 0$.

(iii) If $\zeta : (\pi, T) \to U$ is an extended pairing and $Z\zeta$ is defined then $Z\zeta$ is the unique map for which the following diagram commutes for all $i \geq 0$.

$$\begin{tikzcd}
D_\pi \Sigma^\infty T_{di} \arrow[r, "D_\pi \theta_{di}"] \arrow[d, phantom, "\wr"] & D_\pi \Sigma^{di} ZT \arrow[dd, "\Phi^{(i)} Z\zeta"] \\
\Sigma^\infty D_\pi T_{di} \arrow[d, "\Sigma^\infty \zeta_i"] & \\
\Sigma^\infty U_{dij} \arrow[r, "\theta_{dij}"] & \Sigma^{dij} ZU
\end{tikzcd}$$

Proof of 3.6. Part (i) is immediate from the definition of τ and diagram (3) of section 1. Part (ii) follows at once from part (i). In part (iii) the commutativity follows from part (i) and the definition of $Z\zeta$, while the fact that $Z\zeta$ is the only such map follows from (ii).

Remark 3.7. Let D be a functor which is naturally equivalent to D_π for some π. More precisely, we assume that there are space and spectrum level functors , both called D and compatible with Σ^∞, and space and spectrum level equivalences $D \simeq D_\pi$ which are also compatible under Σ^∞; the cases of interest are $D_j \wedge D_k$ and $D_j D_k$. We can clearly carry through everything in this section with D_π replaced everywhere by D. The necessary maps

$$\delta : D(X \wedge Y) \to DX \wedge DY$$

and

$$\iota : X^{(j)} \to DX$$

may be obtained from those for D_π by means of the given natural equivalence. Of course, D may already possess transformations δ and ι compatible with those for D_π; this is the case for $D = D_j \wedge D_k$ and $D = D_j D_k$. If π is a subgroup of $\rho \subset \Sigma_j$ and ι' denotes the composite

$$D \simeq D_\pi \xrightarrow{\iota} D_\rho$$

then (provided that ι' preserves the orientations) we can compose an extended pairing $\xi : D_\rho E \to F$ with ι' to get an extended pairing in the new sense from DE to F. Clearly z and Z will preserve such composites. The examples of interest for ι' are the maps $\alpha_{j,k}$ and $\beta_{j,k}$ defined in I§2.

We conclude this section with the proof of 3.4. If $\xi : D_\pi ZT \to F$ is an extended pairing we write $[\xi]$ for the element of $F^0 D_\pi ZT$ represented by ξ. Now D_π preserves telescopes by I.1.2(iii) so

$$D_\pi ZT \simeq \operatorname{Tel} D_\pi \Sigma^{-di} \Sigma^\infty T_{di} \ .$$

Hence the $\lim^1$ hypothesis implies

$$F^0 D_\pi ZT \cong \lim F^0 D_\pi \Sigma^{-di} \Sigma^\infty T_{di} \ .$$

The image of $[\xi]$ in the i-th term of the limit is $(D_\pi \Sigma^{-di} \theta_i)^*[\xi]$.

On the other hand if $\zeta : (\pi, T) \to zF$ is an extended pairing then each ζ_i represents an element $[\zeta_i] \in F^{dij} D_\pi T_{di}$, and Definition 3.2 says precisely that

$$\Phi[\zeta_i] = (D_\pi \sigma)^* [\zeta_{i+1}].$$

Hence the extended pairings $(\pi, T) \to zF$ are in one-to-one correspondence with the elements of

$$\lim F^{dij} D_\pi T_{di},$$

where the maps of the inverse system are the composites

$$F^{d(i+1)j} D_\pi T_{d(i+1)} \xrightarrow{(D_\pi \sigma)^*} F^{d(i+1)j} D_\pi \Sigma^d T_{di} \xrightarrow{\Phi^{-1}} F^{dij} D_\pi T_{di} \ .$$

Thus τ gives a map

$$\lim F^0 (D_\pi \Sigma^{-di} \Sigma^\infty T_{di}) \longrightarrow \lim F^{dij} (D_\pi T_{di}).$$

We claim this map is $\lim \Phi^{(i)}$, from which the result follows by 3.3. For by 3.6(i) and the naturality of Φ we have

$$[(\tau\xi)_i] = (D_\pi \theta_{di})^* \Phi^{(i)} [\xi] = \Phi^{(i)} ((D_\pi \theta_{di})^* [\xi]).$$

§4. A vanishing condition for $\lim^1$ terms

In order to apply the results of sections 1,2, and 3, one must have some way of showing that the relevant $\lim^1$ terms vanish. In this section, which is based on a paper of D. W. Anderson [10], we give a simple sufficient condition which is satisfied in our applications.

If F is a spectrum and X is a space we denote the F-cohomology Atiyah-Hirzebruch spectral sequence of X by $E_r(X;F)$. We say that the pair (X,F) is *Mittag-Leffler* (abbreviated M-L) if for each p and q there is an r with $E_r^{p,q}(X;F) = E_\infty^{p,q}(X;F)$; in particular this is true if the spectral sequence collapses.

Definition 4.1. A pair (T,F) with $T \in \overline{w}\mathcal{P}$ and $F \in \overline{h}\mathcal{S}$ is *$\lim^1$-free* if

(i) F and each T_{di} have finite type.

(ii) The pair (T_{di},F) is M-L for each $i \geq 0$.

(iii) If d is odd then $H^n(T_{di})$ and $\pi_n F$ are finite for all n. If d is even they are finite for odd n.

We say that $T \in \overline{w}\mathcal{P}$ is *$\lim^1$-free* if the pair (T,ZT) is.

The integer d in part (iii) is the one which was fixed at the beginning of section 2.

In practice it is easy to see whether a particular pair satisfies (i) and (iii). It is sometimes easier to deal with condition (ii) in the following equivalent form ([10, p. 291]).

Proposition 4.2. Suppose $E_2(X;F)$ has finite type. Then the pair (X,F) is M-L if and only if for each p and q the infinite cycles $Z_\infty^{p,q}(X;F)$ have finite index in $E_2^{p,q}(X;F)$.

Proof. Fix p and q. Let $C_r^{p,q}$ be the quotient of $E_r^{p,q}$ by its infinite cycles. If $Z_\infty^{p,q}$ has finite index in $E_r^{p,q}$ then $C_2^{p,q}$ is finite. Since $C_{r+1}^{p,q}$ is a subquotient of $C_r^{p,q}$ there must be an r_0 with $C_r^{p,q} = C_{r_0}^{p,q}$ for all $r \geq r_0$. But then clearly $C_{r_0}^{p,q} = 0$, hence $E_{r_0}^{p,q} = E_\infty^{p,q}$.

For the converse we recall that the rationalization $F \to F_{\mathbf{Q}}$ induces a rational isomorphism of E_2 terms. Since $F_{\mathbf{Q}}$ splits as a wedge of rational Eilenberg-Mac Lane spectra the spectral sequence $E_r(X;F_{\mathbf{Q}})$ collapses. Hence an element of infinite order in $E_r^{p,q}(X;F)$ cannot have as boundary another element of infinite order. It follows that $Z_r^{p,q}$ has finite index in $E_r^{p,q}$ and that the projection $Z_r^{p,q} \to E_{r+1}^{p,q}$ has finite kernel. But if $E_{r_0}^{p,q} = E_r^{p,q}$ then $C_{r_0}^{p,q} = 0$ and hence $C_2^{p,q}$ is finite as required.

Corollary 4.3. Suppose $E_r(X;F)$ and $E_r(X';F')$ have finite type. If $f:E_r(X;F) \to E_r(X';F')$ is a map of spectral sequences which induces a rational epimorphism in each bidegree of the E_2-terms, and if the pair (X,F) is M-L, then so is the pair (X',F').

As a consequence we get a way of generating new $\lim^1$-free pairs from known ones.

Corollary 4.4. Let (T,F) be a $\lim^1$-free pair and let $f:F \to F'$ and $g:T' \to T$ be maps inducing rational epimorphisms onto π_*F' and $H^*T'_{di}$ for each i. If F' and each T'_{di} have finite type then the pair (T',F') is $\lim^1$-free.

Proof. The pair (T',F') clearly satisfies 4.1(iii), and it also satisfies 4.1(ii) since

$$f_*g^*_{di}:E_2(T_{di};F) \to E_2(T'_{di};F')$$

is a rational epimorphism in each bidegree.

In the remainder of this section we show that $\lim^1$ terms arising in previous sections do in fact vanish for $\lim^1$-free pairs. The reader willing to believe this can proceed to section 5.

By a filtered group we mean an abelian group A with a descending filtration

$$A = A^0 \supset A^1 \supset A^2 \supset \cdots .$$

A is complete if the map $A \to \lim A/A^n$ is an isomorphism (this includes the Hausdorff property), or equivalently if $\lim A^n = \lim^1 A^n = 0$. Filtered groups form a category whose morphisms are the filtration preserving maps.

Let $\{A_i\}_{i\geq 0}$ be an inverse system of filtered groups, and let A_i^n be the n-th filtration of A_i. Let $G^nA_i = A_i^n/A_i^{n+1}$. We need an algebraic fact ([10, Lemma 1.13]).

Proposition 4.5. Suppose that $\lim\limits_i{}^1 G^nA_i = 0$ for each n and that A_i is complete for each i. Then $\lim^1 A_i = 0$.

Using this we can prove the standard result about convergence of the Atiyah-Hirzebruch spectral sequence ([10, Theorem 2.1]). Recall that the skeletal filtration of F^mX has as its n-th filtration the kernel of the restriction to the (n-1)-skeleton X(n-1). The associated graded groups of this filtration are the E_∞-term of the Atiyah-Hirzebruch spectral sequence.

Corollary 4.6. If the pair (X,F) is M-L then

(i) $\lim_n F^m X(n) = 0$ for each m,

(ii) The map $F^m X \to \lim_n F^m X(n)$ is an isomorphism, and

(iii) The skeletal filtration of $F^m X$ is complete.

Proof. Clearly (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) so we need only prove (i). Let $A_i = F^m X(i)$ with its skeletal filtration. This filtration is discrete, hence certainly complete, so by 4.5 it suffices to show $\lim_i^1 E_\infty^{p,q}(X(i);F) = 0$ for each p and q. Now the restriction

$$E_1^{p,q}(X;F) \to E_1^{p,q}(X(i);F)$$

is an isomorphism for $p \leq i$, hence the map

$$E_r^{p,q}(X;F) \to E_r^{p,q}(X(i);F)$$

is an isomorphism for $p \leq i-r+1$. Thus, if r_0 is such that $E_\infty^{p,q}(X;F) = E_{r_0}^{p,q}(X;F)$ we see that $E_\infty^{p,q}(X;F) \to E_\infty^{p,q}(X(i);F)$ is an isomorphism for $i \geq p+r_0-1$, so that $\lim_i^1 E_\infty^{p,q}(X(i);F) = 0$.

Now we can deal with the $\lim^1$ term of section 1.

Corollary 4.7. If the pair (T,F) is $\lim^1$-free then $\lim^1 F^{di-1}T_{di} = 0$.

Proof. Give $F^{di-1}T_{di}$ the skeletal filtration, which is complete by 4.6. Then each group of the associated graded is finite by 4.1(iii), hence the hypothesis of 4.5 is satisfied and we conclude that $\lim^1 F^{di-1}T_{di} = 0$.

Next we consider the relation with multiplicative structures.

Proposition 4.8. [10, p. 291] Suppose that F is a spectrum of finite type having the form ZU for a ring prespectrum U (in particular F may be a ring spectrum). If X and Y are spaces of finite type and the pairs (X,F) and (Y,F) are M-L, then so is $(X \wedge Y,F)$.

Proof. The hypothesis on F makes F-cohomology a ring-valued theory on spaces (but not necessarily on spectra). For each p and q the resulting product map

$$\bigoplus_{p'+p''=p} (E_2^{p',0}(X;F) \times E_2^{p'',q}(Y;F)) \to E_2^{p,q}(X \wedge Y;F)$$

is a rational epimorphism. Now $Z_\infty^{p',0}(X;F)$ and $Z_\infty^{p'',q}(Y;F)$ have finite index in $E_2^{p',0}(X;F)$ and $E_2^{p'',q}(Y;F)$ by 4.2, and the image of $Z_\infty^{p',0} \otimes Z_\infty^{p'',q}$ is contained in $Z_\infty^{p,q}(X \wedge Y;F)$. Hence $Z_\infty^{p,q}(X \wedge Y;F)$ has finite index in $E_2^{p,q}(X \wedge Y;F)$ and the result follows by 4.2.

This allows us to handle the $\lim^1$ term in section 2.

Corollary 4.9. If (T,F) and (T',F) are $\lim^1$-free and F has the form ZU for a ring prespectrum U then $\lim^1 F^{2di-1}(T_{di} \wedge T'_{di}) = 0$.

Proof. The skeletal filtration of $F^{2di-1}(T_{di} \wedge T'_{di})$ is complete by 4.6 and 4.8, and each group of the associated graded is finite by 4.1(iii). The result follows by 4.5.

We now consider extended powers.

Corollary 4.10. If X and F have finite type, F has the form ZU for a ring prespectrum U, and the pair (X,F) is M-L, then so is $(D_\pi X,F)$ for any $\pi \subset \Sigma_j$.

Proof. The transfer, which is a stable map from $D_\pi X$ to $X^{(j)}$, gives a rational epimorphism

$$E_2^{p,q}(X^{(j)};F) \to E_2^{p,q}(D_\pi X;F).$$

The result follows by 4.2 and 4.8.

Next we dispose of the $\lim^1$ term of section 3.

Corollary 4.11. If (T,F) is $\lim^1$-free and F is a π-oriented ring spectrum then $\lim^1 F^{-1} D_\pi \Sigma^{-di} \Sigma^\infty T_{di} = 0$.

Proof. The proof of 3.4 shows that the given inverse system is isomorphic to the inverse system $F^{dij-1} D_\pi T_{di}$ with structural maps $\Phi^{-1} \circ (D_\pi \sigma)^*$. Now the Thom isomorphism Φ preserves the skeletal filtration so we have a filtered inverse system of groups which are complete by 4.10. The associated graded groups are finite by 4.1(iii) and the proof of 4.10. The result follows by 4.5.

Finally, we record a result of Anderson which generalizes 4.6.

Proposition 4.12 [10, Corollary 2.4]. Suppose that X and F have finite type and (X,F) is M-L. If X is a countable CW-complex then the map

$$F^n X \to \lim_{\alpha} F^n X_\alpha ,$$

where $\{X_\alpha\}$ is the set of finite subcomplexes of X, is an isomorphism for each n.

§5. H_∞ ring spectra and prespectra

In this section we show that H_∞ ring spectra can be obtained by lifting the following structures in $\overline{w\mathcal{P}}$.

Definition 5.1. An H_∞^d ring prespectrum is a ring prespectrum U with maps

$$\zeta_{j,i}: D_j U_{di} \to U_{dij}$$

for all i,j $\geq$ 0 such that each $\zeta_{1,i}$ is the identity map and the following diagrams commute in $\overline{h\mathcal{J}}$ for all i,j,k $\geq$ 0.

$$\begin{tikzcd}
D_j U_{di} \wedge D_k U_{di} \arrow[r, "\alpha"] \arrow[d, "\zeta_{j,i} \quad \zeta_{k,i}"] & D_{j+k} U_{di} \arrow[d, "\zeta_{j+k,i}"] \\
U_{dij} \wedge U_{dik} \arrow[r, "\psi"] & U_{di(j+k)}
\end{tikzcd}
\qquad
\begin{tikzcd}
D_j D_k U_{di} \arrow[r, "\beta"] \arrow[d, "D_j \zeta_{k,i}"] & D_{jk} U_{di} \arrow[d, "\zeta_{jk,i}"] \\
D_j U_{dik} \arrow[r, "\zeta_{j,ik}"] & U_{dijk}
\end{tikzcd}$$

$$\begin{tikzcd}
D_k(U_{di} \wedge U_{dj}) \arrow[rr, "\delta"] \arrow[d, "D_k\psi"] & & D_k U_{di} \wedge D_k U_{dj} \arrow[d, "\zeta_{k,i} \wedge \zeta_{k,j}"] \\
D_k U_{d(i+j)} \arrow[dr, "\zeta_{k,i+j}"] & & U_{dik} \wedge U_{djk} \arrow[dl, "\psi"] \\
& U_{d(i+j)k} &
\end{tikzcd}$$

A ring map $f: U \to U'$ between H_∞^d ring prespectra is an H_∞^d ring map if $\zeta_{j,i} \circ D_j f_{di} = f_{dij} \circ \zeta_{j,i}$ for all i,j $\geq$ 0.

The significance of the positive integer d in this definition is that a prespectrum may have an H_∞^d structure but not an $H_\infty^{d'}$ structure for d' < d. (Some examples of this phenomenon are given in the next section.) The third diagram in Definition 5.1 has no analog in the definition of H_∞ ring spectrum since in that situation the analog of the third diagram follows from the other two by (ii) and (iii) of I.3.4.

Definition 5.1 has several consequences. The first diagram implies the commutativity of

$$\begin{tikzcd}
(U_{di})^{(j)} \arrow[rr, "\iota"] \arrow[dr, "\psi"'] & & D_j U_{di} \arrow[dl, "\zeta_{j,i}"] \\
& U_{dij} &
\end{tikzcd}$$

for all i and j. In particular the composite

$$\nu_j : D_j S^d \xrightarrow{D_j e_d} D_j U_d \xrightarrow{\zeta_{j,1}} U_{dj}$$

is a Σ_j-orientation for U. These orientations are consistent in the sense that the diagrams

$$\begin{tikzcd}
\text{(1)} & D_j S^d \wedge D_k S^d \arrow[r, "\alpha"] \arrow[d, "\nu_j \wedge \nu_k"] & D_{j+k} S^d \arrow[d, "\nu_{j+k}"] & \text{(2)} & D_j D_k S^d \arrow[r, "\beta"] \arrow[d, "D_j \nu_k"] & D_{jk} S^d \arrow[d, "\nu_{jk}"] \\
& U_{dj} \wedge U_{dk} \arrow[r, "\psi"] & U_{d(j+k)} & & D_j U_{dk} \arrow[r, "\zeta_{j,k}"] & U_{djk}
\end{tikzcd}$$

commute for all j and k. Now the unit diagram in the definition of a ring prespectrum and the third diagram in Definition 5.1 imply that for each fixed j the maps $\zeta_{j,i}$ give an extended pairing

$$\zeta_j : (\Sigma_j, U) \to (U, \nu_j).$$

Theorem 5.2. If U is a $\lim^1$-free H^d_∞ ring prespectrum then the maps

$$Z(\zeta_j) : D_j ZU \to ZU$$

give ZU an H_∞ ring structure. If $f : U \to U'$ is an H^d_∞ ring map and U,U' and the pair (U,ZU') are $\lim^1$-free then Zf is an H_∞ ring map.

The proof will occupy the rest of this section. We write F for ZU, ξ_j for $Z(\zeta_j)$ and ϕ for the multiplication $Z\psi$. Let μ_j be the orientation

$$Z(\nu_j) : D_j S^d \to \Sigma^{dj} ZU = \Sigma^{dj} F,$$

as defined after Theorem 3.4. First we claim that the μ_j are consistent in the following sense.

Lemma 5.3. The diagrams

$$\text{(3)}\quad \begin{tikzcd} D_jS^d \wedge D_kS^d \arrow[r, "\alpha"] \arrow[d, "\mu_j \wedge \mu_k"] & D_{j+k}S^d \arrow[d, "\mu_{j+k}"] \\ \Sigma^{dj}F \wedge \Sigma^{dk}F \arrow[r, "\Sigma^{d(j+k)}\phi"] & \Sigma^{d(j+k)}F \end{tikzcd} \qquad \text{(4)}\quad \begin{tikzcd} D_jD_kS^d \arrow[r, "\beta"] \arrow[d, "D_j\mu_k"] & D_{jk}S^d \arrow[d, "\mu_{jk}"] \\ D_j\Sigma^{dk}F \arrow[r, "\Phi^{(k)}(\xi_j)"] & \Sigma^{djk}F \end{tikzcd}$$

commute for all $j,k \geq 0$.

Proof. For diagram (4) recall that μ_i is the composite $\theta_{di} \circ \Sigma^\infty \nu_i$, where θ_{di} is the natural map $\Sigma^\infty U_{di} \to \Sigma^{di}F$. Hence

$$\begin{aligned} \mu_{jk} \circ \beta &= \theta_{djk} \circ \Sigma^\infty(\nu_{jk} \circ \beta) \\ &= \theta_{djk} \circ \Sigma^\infty(\zeta_{j,k}) \circ \Sigma^\infty D_j \nu_k && \text{by diagram (2)} \\ &= \Phi^{(k)}(\xi_j) \circ D_j\theta_{dk} \circ D_j\Sigma^\infty\nu_k && \text{by Corollary 3.6(iii)} \\ &= \Phi^{(k)}(\xi_j) \circ D_j\mu_k . \end{aligned}$$

The proof for diagram (3) is similar.

Next we need another preliminary result.

Lemma 5.4. The diagram

$$\begin{tikzcd} D_k(F \wedge F) \arrow[rr, "\delta"] \arrow[d, "D_k\phi"] & & D_kF \wedge D_kF \arrow[d, "\xi_k \wedge \xi_k"] \\ D_kF \arrow[dr, "\xi_k"'] & & F \wedge F \arrow[dl, "\phi"] \\ & F & \end{tikzcd}$$

commutes for all $k \geq 0$.

In order to prove 5.4 we need the following variant of 3.6(ii).

Lemma 5.5. Let η_1 and η_2 be two maps

$$D_\pi(ZT \wedge ZT') \to F,$$

where F is a π-oriented ring spectrum and the pairs (T,F) and (T',F) are $\lim^1$-free. Then $\eta_1 = \eta_2$ if and only if the equation

$$(5)\qquad \Phi^{(2i)}(\eta_1) \circ D_\pi(\theta_i \wedge \theta_i) = \Phi^{(2i)}(\eta_2) \circ D_\pi(\theta_i \wedge \theta_i)$$

holds for all $i \geq 0$.

Proof of 5.5. The composite isomorphism

$$F^0(D_\pi(ZT \wedge ZT')) \xrightarrow{\cong} \operatorname{Lim} F^0 D_\pi \Sigma^{-2di}(T_{di} \wedge T'_{di}) \xrightarrow{\lim \Phi^{(2i)}} \lim F^{2dij} D_\pi(T_{di} \wedge T'_{di})$$

takes η_1 to $\Phi^{(2i)}(\eta_1) \circ D_\pi(\theta_i \wedge \theta_i)$, and similarly for η_2.

Proof of 5.4. Let η_1 be the counterclockwise composite in the diagram and η_2 the clockwise composite. Consider the following diagram of spectra, where we have suppressed Σ^∞ to simplify the notation and the unlabeled arrows are all induced by maps θ_{di}.

$$\begin{tikzcd}
D_k(U_{di} \wedge U_{di}) \arrow[rrrr, "\delta"] \arrow[dr] \arrow[dddd, "D_k\psi"'] & & & & D_kU_{di} \wedge D_kU_{di} \arrow[dl] \arrow[dddd, "\zeta_{k,i} \wedge \zeta_{k,i}"] \\
& D_k(\Sigma^{di}F \wedge \Sigma^{di}F) \arrow[rr, "\delta"] \arrow[d, "D_k\Sigma^{2di}\phi"] & \textcircled{A} & D_k\Sigma^{di}F \wedge D_k\Sigma^{di}F \arrow[d, "\phi^{(i)}(\xi_k) \wedge \phi^{(i)}(\xi_k)"] & \\
\textcircled{E} & D_k\Sigma^{2di}F \arrow[dr, "\phi^{(2i)}(\xi_k)"'] & & \Sigma^{dik}F \wedge \Sigma^{dik}F \arrow[dl, "\Sigma^{2dik}\phi"] \arrow[ddr] & \textcircled{B} \\
& & \Sigma^{2dik}F \arrow[dd, no head] & & \\
D_kU_{2dik} \arrow[uur] \arrow[drr, "\zeta_{k,2i}"'] & \textcircled{D} & & \textcircled{C} & U_{dik} \wedge U_{dik} \arrow[dll, "\psi"] \\
& & U_{2dik} & &
\end{tikzcd}$$

It is easy to see that the counterclockwise and clockwise composites in the inner pentagon are $\Phi^{(2i)}(\eta_1)$ and $\Phi^{(2i)}(\eta_2)$. To verify equation (5) it suffices to show that the outer pentagon and parts A, B, C, D and E commute. But the outer pentagon is the third diagram of Definition 5.1. Part A commutes by naturality of δ, parts C and E by definition of $\phi = Z\psi$, and parts B and D by 3.6(iii).

We now turn to the main part of the proof of 5.2. We shall show that the following diagram commutes; the other is similar.

(6)

$$\begin{tikzcd}
D_jD_kF \arrow[r, "\beta"] \arrow[d, "D_j\xi_k"] & D_{jk}F \arrow[d, "\xi_{jk}"] \\
D_jF \arrow[r, "\xi_j"] & F
\end{tikzcd}$$

We shall apply Remark 3.7 with $D = D_jD_k$. First orient $D_jD_kS^d$ using either of the two equal composites in diagram (4) of Lemma 5.3, and denote the associated Thom isomorphism by $\overline{\Phi}$. We write η_1 and η_2 for the counterclockwise and clockwise composites in diagram (6); these are extended pairings in the sense of Remark 3.7. By 3.6(ii) it suffices to show

$$\overline{\Phi}^{(i)}\eta_1 \circ D_jD_k\theta_i = \overline{\Phi}^{(i)}\eta_2 \circ D_jD_k\theta_i \tag{7}$$

for each $i \geq 0$. Consider the following diagram, where we have again suppressed Σ^∞ and the unlabeled arrows are all induced by maps θ_{di}.

$$\begin{tikzcd}
D_jD_kU_{di} \arrow[rrrrr, "\beta"] \arrow[ddrr] \arrow[ddddd, "D_j\zeta_{k,i}"] & & & & & D_{jk}U_{di} \arrow[ddll] \arrow[ddddd, "\zeta_{jk,i}"] \\
 & & \arrow[r, phantom, "\textcircled{A}"] & & & \\
 & & D_jD_k\Sigma^{di}F \arrow[r, "\beta"] \arrow[d, "D_j\Phi^{(i)}(\xi_k)"] & D_{jk}\Sigma^{di}F \arrow[d, "\Phi^{(i)}(\xi_{jk})"] & & \\
 & \textcircled{D} & D_j\Sigma^{dik}F \arrow[r, "\Phi^{(ik)}(\xi_j)"] & \Sigma^{dijk}F & \textcircled{B} & \\
 & & & \textcircled{C} & & \\
D_jU_{dik} \arrow[uurr] \arrow[rrrrr, "\zeta_{j,ik}"] & & & & & U_{dijk} \arrow[uull]
\end{tikzcd}$$

In the inner square the clockwise composite is clearly $\overline{\Phi}^{(i)}(\eta_2)$. Using Lemma 5.4 one can show that the counterclockwise composite is $\overline{\Phi}^{(i)}(\eta_1)$. To verify equation (7) we must show that the outer square and parts A, B, C and D commute. The outer square is the second diagram of Definition 5.1. Part A commutes by naturality of β and parts B,C, and D by 3.6(iii). This completes the proof.

§6. H_∞^d ring spectra.

Theorem 5.2 gives a useful relation between H_∞ structures in $\bar{h}\mathscr{S}$ and H_∞^d structures in $\overline{w}\mathscr{P}$. However, it does not provide a satisfactory analog for Corollary 2.3 since an arbitrary H_∞ ring spectrum F need not possess the Σ_j-orientations necessary to give an H_∞^d structure for zF. For example, if F = S then zF is not an H_∞^d prespectrum for <u>any</u> d > 0 (cf. Proposition 6.1). What is needed is a notion of H_∞ ring spectrum with built-in orientations. It turns out that the right objects to look at are H_∞^d ring spectra as defined in I.4.3.

If F is an H_∞ ring spectrum we say that a sequence of Σ_j-orientations is <u>consistent</u> if the diagrams of Lemma 5.3 commute. If F has an H_∞^d structure let μ_j be the composite

$$\begin{tikzcd} D_j S^d \arrow[r, "D_j \Sigma^d e"] & D_j \Sigma^d F \arrow[r, "\xi_{j,1}"] & \Sigma^{dj} F. \end{tikzcd}$$

Then each μ_j is a Σ_j-orientation by I.4.4(iii) and an easy diagram chase shows that the μ_j are consistent. On the other hand, some H_∞ ring spectra do not even have Σ_2-orientations, and thus are certainly not H_∞^d. This is illustrated by our next result.

<u>Proposition 6.1</u>. (i) The sphere spectrum S is not an H_∞^d ring spectrum for any d > 0.

(ii) If F is an H_∞^d ring spectrum for d odd, then $\pi_* F$ has characteristic 2. If, in addition, F is connective and $\pi_0 F$ is augmented over Z_2 then F splits as a wedge of suspensions of HZ_2.

<u>Proof</u>. Let p^d be the bundle

$$E\Sigma_2 \times_{\Sigma_2} (R^d)^2 \to B\Sigma_2.$$

Then p^d is the d-fold Whitney sum of p^1 with itself, and p^1 is the sum of the Hopf bundle with a trivial bundle. The Thom complex of p^d is $D_2 S^d$, and so p^d is F-orientable if and only if F has a Σ_2-orientation (for the given value of d).

For (i) we recall (e.g. from [71, III.2.7]) that a bundle is S-orientable if and only if it is stably fibre-homotopy trivial. But p^d clearly has nontrivial Stiefel-Whitney classes for every $d \geq 1$.

(ii) Let $R = \pi_0 F$ and observe that F-orientability implies HR-orientability by virtue of the canonical map F → HR. Consider the spectral sequence with

$$E_2^{p,q} = H^p(Z_2; H^q(S^d \quad S^d; R))$$

converging to $H^*(D_2 S^d; R)$. There is only one nonzero row and so $H^{2d}(D_2 S^d; R)$ is isomorphic to $H^0(Z_2; H^2(S^d \wedge S^d; R))$, which is the Z_2-fixed subgroup of

$H^{2d}(S^d \wedge S^d;R) \cong R$. But Z_2 acts on R as multiplication by -1, so we conclude that $H^{2d}(D_2S^d;R)$ is isomorphic to the 2-torsion subgroup of R. If on the other hand p^1 has an HR-orientation then $H^{2d}(D_2S^d;R) \cong R$, so that R must have characteristic 2. If in addition F is connective and R is augmented over Z_2 then the proof of Steinberger's splitting theorem III.4.1 gives the splitting of F.

Now let F be an H_∞^d ring spectrum. An easy diagram chase shows that the equation

$$\xi_{j,i} = \Phi^{(i)}(\xi_{j,0}):D_j\Sigma^{di}F \to \Sigma^{dij}F$$

holds for each i and j, where $\Phi^{(i)}$ is the Thom isomorphism determiend by the induced Σ_j-orientation of F. Thus the H_∞^d structure on F is uniquely determined by its underlying H_∞ structure and the set of induced Σ_j-orientations. Conversely, we have

Proposition 6.2. If F is an H_∞ ring spectrum with consistent Σ_j-orientations then the maps $\xi_{j,i}$ defined by $\xi_{j,i} = \Phi^{(i)}(\xi_j)$ give F an H_∞^d structure.

Using this, we can give a precise analog of 2.3.

Corollary 6.3 (i) If F is an H_∞^d ring spectrum then zF is an H_∞^d ring prespectrum. If f is an H_∞^d ring map in $\bar{h}\mathscr{S}$ then zf is an H_∞^d ring map in $\bar{w}\mathscr{P}$.

(ii) If U is a $\lim^1$-free H_∞^d ring prespectrum then ZU is an H_∞^d ring spectrum and $\kappa:U \to zZU$ is an H_∞^d ring map. If in addition $f:U \to U'$ is an H_∞^d ring map and U' and (U,ZU') are $\lim^1$-free then Zf is an H_∞^d ring map. If F is an H_∞^d ring spectrum and zF is $\lim^1$-free then $\lambda:ZzF \to F$ is an H_∞^d ring map.

Proof of 6.3. For part (i), the adjoint of the composite

$$\Sigma^\infty D_jF_{di} \cong D_j\Sigma^\infty F_{di} \xrightarrow{D_j\theta'_{di}} D_j\Sigma^{di}F \xrightarrow{\xi_{j,i}} \Sigma^{dij}F$$

is a map $\zeta_{j,i}:D_jF_{di} \to F_{dij}$. An easy diagram chase shows that the $\zeta_{j,i}$ satisfy Definition 5.1. Part (ii) is immediate from 5.2, 5.3 and 6.2.

The rest of this section gives the proof of 6.2. Let ω_j denote the composite

$$D_jS \xrightarrow{D_je} D_jF \xrightarrow{\xi_j} F$$

and let $\mu_j^{(i)} = \Phi^{(i)}\omega_j:D_jS^{di} \to \Sigma^{dij}F$; in particular $\mu_j^{(1)} = \mu_j$. Then $\xi_{j,i}$ is the composite

$$D_j\Sigma^{di}F \xrightarrow{\delta} D_jF \wedge D_jS^{di} \xrightarrow{\xi_j \wedge \mu_j^{(i)}} F \wedge \Sigma^{dij}F \xrightarrow{\Sigma^{dij}\phi} \Sigma^{dij}F.$$

It clearly suffices to show the commutativity of the following diagrams for all i,j,k.

(1)
$$\begin{tikzcd}
D_j S^{di} \wedge D_k S^{di} \arrow[r, "\alpha"] \arrow[d, "\mu_j^{(i)} \wedge \mu_k^{(i)}"] & D_{j+k} S^{di} \arrow[d, "\mu_{j+k}^{(i)}"] \\
\Sigma^{dij} F \wedge \Sigma^{dik} F \arrow[r, "\Sigma^{di(j+k)}\phi"] & \Sigma^{di(j+k)} F
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
D_j D_k S^{di} \arrow[r, "\beta"] \arrow[d, "D_j \mu_k^{(i)}"] & D_{jk} S^{di} \arrow[d, "\mu_{jk}^{(i)}"] \\
D_j \Sigma^{dik} F \arrow[r, "\Phi^{(ik)}(\xi_j)"] & \Sigma^{dijk} F
\end{tikzcd}$$

(3)
$$\begin{tikzcd}
D_k(S^{di} \wedge S^{dj}) \arrow[rr, "\delta"] \arrow[d, equal] & & D_k S^{di} \wedge D_k S^{dj} \arrow[d, "\mu_k^{(i)} \wedge \mu_k^{(j)}"] \\
D_k S^{d(i+j)} \arrow[dr, "\mu_k^{(i+j)}"] & & \Sigma^{dik} F \wedge \Sigma^{djk} F \arrow[dl, "\Sigma^{d(i+j)k}\phi"] \\
& \Sigma^{d(i+j)k} F &
\end{tikzcd}$$

In diagram (3) the clockwise composite is $\Phi^{(j)}\mu_k^{(i)} = \Phi^{(j)}\Phi^{(i)}\omega_k = \Phi^{(i+j)}\omega_k$. Hence the diagram commutes. Diagrams (1) and (2) commute when $i = 0$ since $e:S \to F$ is an H_∞ ring map. They commute when $i = 1$ by the consistency of the μ_j, and for $i \geq 1$ by induction. A similar induction shows that they will commute for all negative i if they do for $i = -1$. We prove commutativity of (2) when $i = -1$; the proof for (1) is similar. We apply Remark 3.7 with $D = D_j D_k$. Give $D_j D_k S^d$ either of the two equal orientations indicated in the second diagram of Lemma 5.3 and let Φ denote the associated Thom isomorphism. Let η_1 be the counterclockwise composite in diagram (2) and let η_2 be the clockwise composite. Clearly, we have $\Phi(\eta_2) = \omega_{jk} \circ \beta$, and since $\omega_{jk} \circ \beta = \xi_j \circ D_j \omega_k$ (this is the case $i = 0$ of diagram (2)) it suffices to show

$$\overline{\Phi}(\eta_1) = \xi_j \circ D_j \omega_k.$$

This is demonstrated by the following commutative diagram.

$$\begin{tikzcd}
D_jD_k(S^{-d}\wedge S^d) \arrow[rr, "D_j\delta"] \arrow[d, "\cong"] & & D_j(D_kS^{-d}\wedge D_kS^d) \arrow[r, "\delta"] \arrow[d, "D_j(\mu_k^{(-1)}\wedge\mu_k)"] \arrow[dr, phantom, "\textcircled{B}"] & D_jD_kS^{-d}\wedge D_jD_kS^d \arrow[d, "D_j\mu_k^{(-1)}\wedge D_j\mu_k"] \\
D_jD_kS^0 \arrow[dr, "D_j\omega_k"] & \textcircled{A} & D_j(\Sigma^{-dk}F\wedge\Sigma^{dk}F) \arrow[r, "\delta"] \arrow[dl, "D_j\phi"'] \arrow[dr, phantom, "\textcircled{C}"] & D_j\Sigma^{-dk}F\wedge D_j\Sigma^{dk}F \arrow[d, "\Phi^{(-k)}(\xi_j)\wedge\Phi^{(k)}(\xi_j)"] \\
& D_jF \arrow[r, "\xi_j"] & F & \Sigma^{-djk}F\wedge\Sigma^{djk}F \arrow[l, "\phi"']
\end{tikzcd}$$

Here part Ⓐ is D_j applied to one case of diagram (3), part Ⓑ commutes by naturality of δ, and part Ⓒ follows from diagram (3) and the fact that ϕ is an H_∞ ring map (see parts (ii) and (iii) of I.3.4). This completes the proof.

§7. K-theory spectra

For our work in chapter IX with Dyer-Lashof operations in K-theory it will be essential to know that the spectrum KU representing periodic complex K-theory is an H_∞ ring spectrum. This is immediate from Corollary 6.3 once one has the necessary space-level input. We begin this section with a quick proof using as input the fact that the connective spectrum kU has an E_∞ ring structure. This in turn raises a consistency question which is settled in the remainder of the section. In VIII §4 we shall use Atiyah's power operations as input to give a more leisurely and elementary proof that KU is an H_∞ ring spectrum. Although we concentrate on the complex case in this section, everything goes through in the orthogonal case with the usual changes.

First recall from [71, VIII §2] that the spectrum kU representing connective complex K-theory is an E_∞ ring spectrum. Hence (as explained in I§4) it is an H_∞ ring spectrum. Throughout this section we will write ξ_j for the structural maps $D_j kU \to kU$. Now by I.3.9 the zero-th space of kU, which we denote by X, is an $H_{\infty 0}$ space with structural maps $D_jX \to X$ which will be denoted by ζ_j. The space X is of course equivalent to $BU \times Z$, and by Bott periodicity we can define an Ω-prespectrum $\mathcal{K}U$ with $\mathcal{K}U_{2i} = X$. We give $\mathcal{K}U$ an H_∞^2 structure by letting each map $D_j\mathcal{K}U_{2i} \to \mathcal{K}U_{2ij}$ be $\zeta_j : D_jX \to X$. We define KU to be $Z\mathcal{K}U$. At this point we need to know something about $\lim^1$ terms.

Proposition 7.1. $\mathcal{K}U$ and $\mathcal{K}O$ are $\lim^1$-free.

Proof. The pair $(\mathcal{K}U, KU)$ clearly satisfies 4.1(i) and (iii). Since $E_r(BU \times Z;KU)$ collapses for dimensional reasons it also satisfies 4.1(ii) and hence is $\lim^1$-

free. The result for $\mathcal{K}O$ follows from 4.4 by letting $f:KU \to KO$ be realification and $g:\mathcal{K}O \to \mathcal{K}U$ be complexification.

Now we can apply 6.3 to get

Theorem 7.2. KU is an H_∞^2 ring spectrum and KO is an H_∞^8 ring spectrum.

Remark 7.3. (i) We shall see in VIII§6 that the H_∞^8 structure of KO extends to an H_∞^4 structure.
(ii) It is shown in [71, VIII. 2.6 and VIII. 2.9] that the Adams operation ψ^k induces an E_∞ ring map of kU when completed away from k. We shall see in VIII§7 that ψ^k also induces an H_∞ ring map of $KU_{(p)}$ for p prime to k but that this is not an H_∞^2 ring map. Since the methods of the present section can only give H_∞^2 ring maps they cannot be applied directly to this question.

Next we wish to show that the H_∞ structure on KU is consistent with the original structure on kU. The point is that (as we shall see in a moment) kU inherits an H_∞ structure from that just given for KU, and we would like to know that the inherited structure is its original one. The proof will occupy the rest of this section.

First recall the n-connected-cover functors in $\bar{h}\mathcal{S}$ ([71, II.2.11]). We write c for the connective (i.e., -1-connected) cover functor. These functors have the usual property that any map from an n-connected spectrum lifts uniquely to the n-connected cover of its target ([71, II.2.10]). In particular, we have

Proposition 7.4. If F is an H_∞ ring spectrum then cF has a unique H_∞ structure for which the map $cF \to F$ is H_∞.

We shall prove

Proposition 7.5. There is an H_∞ ring map from kU (with its E_∞ structure) to cKU (with the H_∞ structure given by 7.2 and 7.4) which is an equivalence.

The analogous comparison of ring structures was given in]71, II§3].

First we observe that the iterated Bott map

$$B:\Sigma^{2i}kU \to kU$$

is equivalent to the (2i-1)-connected cover of kU. We can therefore define

$$\mu_j : D_j S^2 \to \Sigma^{2j} kU$$

to be the unique lift of the composite

$$D_jS^2 \xrightarrow{D_j\Sigma^2 e} D_j\Sigma^2 kU \xrightarrow{D_jB} D_j kU \xrightarrow{\xi_j} kU.$$

The μ_j are consistent Σ_j-orientations in the sense of 6.2 and hence kU is an H_∞^2 ring spectrum. It follows that zkU is an H_∞^2 ring prespectrum. We write

$$\eta_{j,i}: D_j(kU)_{2i} \to (kU)_{2ij}$$

for its structural maps.

Now define a map

$$\gamma: zkU \to \mathcal{K}U$$

by letting γ_{2i} be the composite

$$(zkU)_{2i} \simeq \Omega^\infty\Sigma^{2i}kU \xrightarrow{\Omega^\infty B} \Omega^\infty kU = X = (\mathcal{K}U)_{2i}.$$

We claim that γ is an H_∞^2 ring map. This is demonstrated by the commutativity of the following diagram.

$$\begin{tikzcd}
D_j(kU)_{2i} \arrow[rrr, "D_j\gamma_i"] \arrow[dr] \arrow[ddddd, "\eta_{j,i}"'] & & & D_jX \arrow[dl] \arrow[ddddd, "\zeta_j"] \\
& \Omega^\infty\Sigma^\infty D_j(kU)_{2i} \arrow[r, "\Omega^\infty\Sigma^\infty D_j\gamma_i"] \arrow[d, "\wr"] \arrow[ur, phantom, "\textcircled{A}"] & \Omega^\infty\Sigma^\infty D_j(kU)_0 \arrow[d, "\wr"] & \\
\arrow[r, phantom, "\textcircled{F}"] & \Omega^\infty D_j\Sigma^\infty(kU)_{2i} \arrow[r, "\Omega^\infty D_j\Sigma^\infty\gamma_i"] \arrow[d, "\Omega^\infty D_j\theta'_{2i}"] \arrow[ur, phantom, "\textcircled{B}"] & \Omega^\infty D_j\Sigma^\infty(kU)_0 \arrow[d, "\Omega^\infty D_j\theta'_0"] & \\
& \Omega^\infty D_j\Sigma^{2i}kU \arrow[r, "\Omega^\infty D_jB"] \arrow[d, "\Omega^\infty\xi_{j,i}"] \arrow[ur, phantom, "\textcircled{C}"] & \Omega^\infty D_j kU \arrow[d, "\Omega^\infty\xi_j"] \arrow[r, phantom, "\textcircled{G}"] & \\
& \Omega^\infty\Sigma^{2i}kU \arrow[r, "\Omega^\infty B"] \arrow[ur, phantom, "\textcircled{D}"] & \Omega^\infty kU & \\
(kU)_{2ij} \arrow[ur, "\simeq"] \arrow[rrr, "\gamma_{ij}"] \arrow[urr, phantom, "\textcircled{E}"] & & & X \arrow[ul, "="']
\end{tikzcd}$$

Parts F and G commute by definition of $\eta_{j,i}$ and ζ_j. Parts A and B commute by naturality, parts C and E by the definition of γ. Commutativity of part D follows from the definitiion of μ_i.

Next we need more $\lim^1$ information.

Proposition 7.6. zkU, zkO and the pairs (zkU,KU) and (zkO,KO) are $\lim^1$ free.

Proof. The Serre spectral sequence shows that the pairs (zkU,kU) and (zkU,KU) satisfy the finiteness requirement of 4.1(i) and (iii). Now by [10,4.3] and the proof of [10,3.13] (specifically the fifth line on p.301) we see that the pair $((kU)_{2i},kU)$ is M-L for each i and hence zkU is $\lim^1$-free. Since

$$E_2^{p,q}((kU)_{2i};KU) = E_2^{p,q}((kU)_{2i};kU)$$

for $q \leq 0$ it follows that $Z_\infty^{p,q}((kU)_{2i});KU)$ has finite index in $E_2^{p,q}((kU)_{2i};KU)$ for $q \leq 0$, hence for all q by Bott periodicity. Thus the pair (kU,KU) is $\lim^1$-free. The orthogonal case follows as in the proof of 7.1.

We can now define

$$\Gamma : kU \to KU$$

to be $Z_\gamma \circ \lambda^{-1}$, where Z and λ are as in §1. Then Γ is an H_∞^2 ring map by 6.3 and is clearly an equivalence of zeroth spaces. Hence the unique lift of Γ to cKU is an H_∞^2 ring map and an equivalence. This completes the proof of 7.5.

The fact that Γ is an H_∞^2 ring map, and thus preserves the orientations, has the following additional consequence which will be used in VIII §4.

Corollary 7.7. $\mu_j : D_j S^2 \to \Sigma^{2j} KU$ is the composite

$$D_j S^2 \xrightarrow{D_j \Sigma^2 e} D_j \Sigma^2 KU \xrightarrow{D_j B} D_j KU \xrightarrow{\xi_j} KU \xrightarrow{B^{-2j}} \Sigma^{2j} KU.$$

§8. A Thom isomorphism for spectra

In this section we prove Theorem 3.3. This is the only place in our work where we need the actual definition of D_π, instead of just its formal properties. We accordingly begin by giving a form of the definition; for a general discussion see the sequel.

Let $\mathcal{L}(j)$ be the space of linear isometries from $(R^\infty)^j$ to R^∞. Then $\mathcal{L}(j)$ is a free contractible π-space and hence there is a π-map $\chi : E\pi \to \mathcal{L}(j)$. Choose an increasing sequence W_i of finite π-subcomplexes of $E\pi$ with $\bigcup_i W_i = E\pi$. If $V \subset (R^\infty)^j$ is a finite-dimensional subspace then (since W_i is compact) the union

$$\bigcup_{w \in W_i} \chi(w)(V) \subset R^\infty$$

is contained in a finite-dimensional subspace. In particular, if we let A_i be the standard copy of R^{di} in R^∞ then there is a finite-dimensional subspace A_i' of R^∞ with

$$\chi(w)(A_i \oplus \cdots \oplus A_i) \subset A_i'$$

for every $w \in W_i$. Let a_i be the dimension of A_i'. We may assume that the A_i' form an increasing sequence, and we write B_i and B_i' for the orthogonal complements of A_i in A_{i+1} and of A_i' in A_{i+1}'.

Now consider the map from $W_i \times (A_i)^j$ to $W_i \times A_i'$ which takes $(w,x_1,\ldots,x_j)$ to $(w,\chi(w)(x_1 \oplus \cdots \oplus x_j))$. This gives an embedding of the trivial bundle

(1) $\quad W_i \times (A_i)^j \to W_i$

in the trivial bundle

(2) $\quad W_i \times A_i' \to W_i$.

The orthogonal complement is a nontrivial vector bundle over W_i. We let η_i be the associated sphere bundle (obtained by fibrewise one-point compactification). We write $S(\eta_i)$ and $T(\eta_i)$ for the total space and the Thom complex of η_i. If we let π act through permutations on $(A_i)^j$ and trivially on A_i' we obtain diagonal actions on the bundles (1) and (2) and hence on $S(\eta_i)$ and $T(\eta_i)$.

Next observe that the diagram of embeddings

$$\begin{tikzcd}
W_i \times (A_i)^j \arrow[r] \arrow[d] & W_{i+1} \times (A_i)^j \arrow[d] \\
W_i \times A_i' \arrow[d] & W_{i+1} \times (A_{i+1})^j \arrow[d] \\
W_i \times A'_{i+1} \arrow[r] & W_{i+1} \times A_{i+1}'
\end{tikzcd}$$

commutes. Hence there is a bundle map

$$\eta_i \oplus B_i' \to \eta_{i+1} \oplus (B_i)^j$$

covering the inclusion $W_i \to W_{i+1}$. The induced map

$$T(\eta_i) \wedge S^{B_i'} \longrightarrow T(\eta_{i+1}) \wedge (S^{B_i})^{(j)}$$

of Thom complexes is a π-map if we give each side the diagonal π-action; here S^{B_i} is the one-point compactification of B_i, etc.

Now let U be a prespectrum (indexed on multiples of d as usual). We define a new prespectrum U^X indexed on the set $\{a_i\}$ as follows (we haven't previously considered prespectra indexed on sets like $\{a_i\}$, but everything in section 1 goes through with the obvious modifications). Let $(U^{\chi})_{a_i}$ be the space

$$T(\eta_i) \wedge_{\pi} (U_{di})^{(j)}$$

with the structural maps σ indicated in the following diagram.

$$\begin{tikzcd}
\Sigma^{a_{i+1}-a_i} T(\eta_i)\wedge_{\pi}(U_{di})^{(j)} \arrow[r, phantom, "\cong"] \arrow[d, "\sigma"] & (T(\eta_i)\wedge S^{B'_i})\wedge_{\pi}(U_{di})^{(j)} \arrow[r] & (T(\eta_{i+1})\wedge(S^{B_i})^{(j)}) \wedge_{\pi}(U_{di})^{(j)} \arrow[d, phantom, sloped, "\cong"] \\
T(\eta_{i+1})\wedge_{\pi}(U_{d(i+1)})^{(j)} & T(\eta_{i+1})\wedge_{\pi}(\Sigma^{d}U_{di})^{(j)} \arrow[l] \arrow[r, phantom, "\cong"] & T(\eta_{i+1} \;{}_{\pi}(S^{B_i}\wedge U_{di})^{(j)}
\end{tikzcd}$$

Finally, given $E \in \overline{h\mathscr{S}}$ we choose a prespectrum U with $ZU \simeq E$ (for example, we could let $U = zE$) and define

$$D_{\pi}E = Z(U^{\chi}) = \underset{i}{\mathrm{Tel}}\ \Sigma^{-a_i}\Sigma^{\infty}[T(\eta_i) \wedge_{\pi} (U_{di})^{(j)}].$$

This agrees up to weak equivalence with the more sophisticated definition given in the sequel, and in particular it does not depend on the choice of χ or U.

Now we can give the proof of 3.3. First we observe that the Thom isomorphism theorem holds in F-cohomology of <u>spaces</u> for any F-orientable bundle. This is well-known when the base space is finite-dimensional (see e.g. [71,III. 1.4]) and the general case follows since the Thom homomorphism induces a map of Milnor $\lim^1$ sequences. Similarly, the relative Thom isomorphism theorem holds for any F-oriented bundle over a pair (X,Y). For example, let U be a prespectrum, let

$$X = S(\eta_i) \times_{\pi} (U_{di})^{j}$$

and let Y be the subspace in which at least one coordinate is a point at ∞ or the basepoint of U_{di}. Note that X/Y is $(U^{\chi})_{a_i}$. Let q be the pullback of the bundle

$$p{:}E\pi \times_{\pi} (R^{d})^{j} \to B\pi$$

along the map

$$X = S(\eta_i) \times_{\pi} (U_{di})^{j} \to E\pi \times_{\pi} * = B\pi.$$

Then the relative Thom complex $T(q)/T(q|Y)$ is

$$T(\eta_i) \;{}_{\pi} (\Sigma^{d}U_{di})^{(j)} = (\Sigma^{d}U)^{\chi}_{a_i} .$$

Let δ_i denote the composite indicated in the following diagram.

$$\begin{tikzcd}
(\Sigma^d U)^{\chi}_{a_i} = T(\eta_i) \wedge_{\pi} (\Sigma^d U_{di})^{(j)} \arrow[r, "\Delta \wedge 1"] \arrow[d, "\delta_i"] & (T(\eta_i) \wedge E\pi^{+}) \wedge_{\pi} (\Sigma^d U_{di})^{(j)} \arrow[dl] \\
U^{\chi}_{a_i} \wedge D_{\pi} S^d = [T(\eta_i) \wedge_{\pi} (U_{di})^{(j)}] \wedge [E\pi^{+} \wedge_{\pi} (S^d)^{(j)}] &
\end{tikzcd}$$

If F is a π-oriented ring spectrum then the relative Thom isomorphism for q is the composite

$$F^n(U^{\chi}_{a_i}) \longrightarrow F^{n+dj}(U^{\chi}_{a_i} \;\; D_{\pi} S^d) \xrightarrow{\delta_i^*} F^{n+dj}((\Sigma^d U)^{\chi}_{a_i}),$$

where the first map is multiplication by the π-orientation μ. We denote this composite by Φ_i.

Next, we note that if $E \simeq ZU$ then $\Sigma^d E \simeq Z(\Sigma^d U)$. It is shown in the sequel that the map

$$\delta : D_{\pi} \Sigma^d E \to D_{\pi} E \wedge D_{\pi} S^d$$

is obtained by passage to telescopes from the δ_i. We therefore have a map of Milnor $\lim^1$ sequences

$$\begin{tikzcd}
0 \arrow[r] & \lim\limits_i{}^1 F^{n+a_i-1}(U^{\chi}_{a_i}) \arrow[r] \arrow[d, "\lim^1 \Phi_i"] & F^n D_{\pi} E \arrow[r] \arrow[d, "\Phi"] & \lim\limits_i F^{n+a_i}(U^{\chi}_{a_i}) \arrow[r] \arrow[d, "\lim \Phi_i"] & 0 \\
0 \arrow[r] & \lim\limits_i{}^1 F^{n+dj+a_i-1}((\Sigma^d U)^{\chi}_{a_i}) \arrow[r] & F^{n+dj} D_{\pi} \Sigma^d E \arrow[r] & \lim\limits_i F^{n+dj+a_i}((\Sigma^d U)^{\chi}_{a_i}) \arrow[r] & 0
\end{tikzcd}$$

The result follows by the five lemma.

We conclude this section with a technical fact which will be needed in VIII §6. Let $\zeta : (\pi, T) \to U$ be an extended pairing and suppose that the pair (T, ZU) is $\lim^1$-free. Then $Z\zeta$ exists and is clearly determined by the composites

$$T(\eta_i) \wedge_{\pi} (T_{di})^{(j)} = T^{\chi}_{a_i} \xrightarrow{\kappa} (D_{\pi} ZT)_{a_i} \xrightarrow{(Z\zeta)_{a_i}} (ZU)_{a_i}$$

for $i \geq 0$. It is natural to ask for an explicit description of the elements

$$z_i \in (ZU)^{a_i}(T(\eta_i) \wedge_{\pi} (T_{di})^{(j)})$$

represented by these composites. We shall give such a description by calculating the image of z_i under the relative Thom isomorphism

$$\Psi : (ZU)^{a_i}(T(\eta_i) \wedge_{\pi} (T_{di})^{(j)}) \longrightarrow (ZU)^{a_i+dij}(T(\eta_i) \wedge_{\pi} (\Sigma^{di} T_{di})^{(j)}).$$

Let $y_i \in (ZU)^{dij}(W_i^+ \wedge_\pi (T_{di})^{(j)})$ be represented by the composite

$$W_i^+ \wedge_\pi (T_{di})^{(j)} \hookrightarrow D_\pi T_{di} \xrightarrow{\zeta_i} U_{dij} \xrightarrow{\kappa} (ZU)_{dij}$$

and recall the homeomorphism

$$T(n_i) \wedge_\pi (\Sigma^{di} T_{di})^{(j)} \cong \Sigma^{a_i} W_i^+ \wedge_\pi (T_{di})^{(j)}.$$

Proposition 8.1. $\Psi z_i = \Sigma^{a_i} y$.

Proof. Write a for a_i. It will be shown in the sequel that the following diagram commutes for any space X.

$$\begin{tikzcd}
T(n_i) \wedge_\pi (\Sigma^{di} X)^{(j)} \arrow[rr, "\kappa_a"] \arrow[d, phantom, "\wr"] & & (D_\pi \Sigma^\infty X)_a \arrow[d, phantom, "\wr"] \\
\Sigma^a (W_i^+ \wedge_\pi (T_{di})^{(j)}) \arrow[r, hook] & \Sigma^a D_\pi T_{di} \arrow[r, "\kappa_a"] & (\Sigma^\infty D_\pi X)_a
\end{tikzcd}$$

Letting $X = T_{di}$ gives the commutativity of the left square in the next diagram.

$$\begin{tikzcd}
\Sigma^\infty (T(n_i) \wedge_\pi (\Sigma^{di} T_{di})^{(j)}) \arrow[r, "\theta_a"] \arrow[d, phantom, "\wr"] & \Sigma^a D_\pi \Sigma^\infty T_{di} \arrow[rr, "\Sigma^a D_\pi \theta_{di}"] \arrow[dd, "\simeq"] & & \Sigma^a D_\pi \Sigma^{di} ZT \arrow[dd, "\Sigma^a \phi(i)_{Z\zeta}"] \\
\Sigma^\infty \Sigma^a (W_i^+ \wedge_\pi (T_{di})^{(j)}) \arrow[d] & & & \\
\Sigma^\infty \Sigma^a D_\pi T_{di} \arrow[r, "\simeq"] & \Sigma^a \Sigma^\infty D_\pi T_{di} \arrow[r, "\Sigma^a \Sigma^\infty \zeta_i"] & \Sigma^a \Sigma^\infty U_{dij} \arrow[r, "\Sigma^a \theta_{dij}"] & \Sigma^{a+dij} ZU
\end{tikzcd}$$

The right square commutes by Corollary 3.6(iii), and we therefore have equality of the two composites around the outside. But the counterclockwise composite is clearly $\Sigma^a y_i$, and the proof of Theorem 3.3 given in this section shows that the clockwise composite is Ψz_i. This completes the proof.