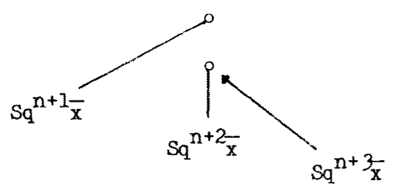
CHAPTER VI

THE ADAMS SPECTRAL SEQUENCE of H_∞ RING SPECTRA

by Robert R. Bruner

In this chapter we show how to use an H_∞ ring structure on a spectrum Y to produce formulas for differentials in the Adams spectral sequence of π_*Y. We shall confine attention to the Adams spectral sequence based on mod p homology, although it is clear that similar results will hold in generalized Adams spectral sequences as well.

The differentials have two parts. The first is the reflection in the Adams spectral sequence of relations in homotopy like those in Chapter V. For example, when $x \in \pi_n Y$ and $n \equiv 1$ (4), there is no homotopy operation $P^{n+1}x$ since the n+1 cell of P_n^∞ is attached to the n cell by a degree 2 map. In the Adams spectral sequence there is a Steenrod operation $Sq^{n+1}\,\bar{x}$ and a differential $d_2 Sq^{n+1}\,\bar{x} = h_0 Sq^n\,\bar{x}$ $= h_0\bar{x}^2$. Therefore $h_0\bar{x}^2 = 0$ in E_{∞}. This in itself only implies that $2x^2$ has filtration greater than that of $h_0\bar{x}^2$ in the Adams spectral sequence, but by examining its origin as a homotopy operation we see that $2x^2 = 0$. Thus, the formulas we produce for differentials are most effective when combined with the results about homotopy operations in Chapter V. The differential $d_2 Sq^{n+3}\,\bar{x} =$ $h_0 Sq^{n+2}\,\bar{x}$, still assuming $n \equiv 1$ (4), is a perfect illustration of this. The corresponding relation in homotopy is $2P^{n+2}x = h_1 P^{n+1}x$ where $h_1 P^{n+1}$ is an indecomposable homotopy operation detected by $h_1 Sq^{n+1}$ in the Adams spectral sequence. The differential on $Sq^{n+3}\bar{x}$ represented geometrically is the sum of maps representing $h_0 Sq^{n+2}\bar{x}$ and $h_1 Sq^{n+1}\bar{x}$, but since $h_1 Sq^{n+1}\bar{x}$ has filtration one greater



than does $h_0 Sq^{n+2}\bar{x}$, it does not appear in the differential. This reflects a hidden extension in the Adams spectral sequence: $2P^{n+2}x$ appears to be 0 in the Adams spectral sequence (i.e. $h_0 Sq^{n+2}\bar{x} = 0$ in E_∞) only because of the filtration shift. In fact, $2P^{n+2}x = h_1 P^{n+1}x$. The moral of this is just the obvious fact mentioned above: the differentials should not be considered in isolation but should be combined with the homotopy operations of Chapter V. Further examples will be given in section 1.

The second part of the differentials arises when we consider Steenrod operations on elements that are not permanent cycles. If x in filtration s survives until E_r we can make x into a permanent cycle by truncating the spectral sequence at filtration s+r. Thus the differentials of the type just discussed apply to x until we get to E_r. However, by analyzing the contribution of $d_r x$ we can show that it will not affect the differentials on $\beta^{\varepsilon}P^j x$ until E_{pr-p+1} where it contributes $\beta^{\varepsilon}P^j d_r x$. Thus the differentials of the first type apply far beyond the range in which we are justified in pretending that x is a permanent cycle. (To be precise we should note that $d_r x$ can occasionally affect differentials on $\beta^{\varepsilon}P^j x$ through a term containing $x^{p-1}d_r x$ in E_{r+1}.)

The first results of this type were established by D. S. Kahn [45] who showed that the H_∞ ring map $\xi_2 : W \ltimes_{Z_2} S^{(2)} \to S$ (obtained through coreductions of stunted projective spaces) could be filtered to obtain maps representing the results of Steenrod operations in $\mathrm{Ext}_A(Z_2, Z_2)$ and that some differentials were implied by this. Milgram [81] extended Kahn's work to the odd primary case and introduced the spectral sequence of IV.6 which is by far the most effective tool for computing the first part of the differential. His work was confined to the range in which it is possible to act as if one is operating on a permanent cycle. Nonetheless he was able to use the resulting formulas for differentials to substantially shorten Mahowald and Tangora's calculation [61] of the first 45 stems at the prime 2 and to catch a mistake in their calculation. The next step was taken by Makinen [62], who showed how to incorporate the contribution of $d_r x$ in the differentials on $Sq^j x$ for p = 2. Unfortunately, he apparently did not apply his formulas to the known calculations of the stable stems, for one of his most interesting formulas (published in 1973),

$$d_3 Sq^j x = h_1 Sq^{j-2} x + Sq^j d_2 x \quad \text{if } n \equiv 1\ (4),$$

combined with Milgram's calculation of Steenrod operations [81], implies that $d_3 e_1 = h_1 t$, contradicting Theorem 8.6.6 of Mahowald and Tangora [61]. This application was left for the author to discover in 1983. Note that the differential is out of Milgram's range since a nonzero $d_2 x$ prevents us from calculating $d_3 Sq^j x$ unless we incorporate terms involving $d_2 x$. The argument in [61] that e_1 is a permanent cycle is an intricate one, involving the existence of various Toda brackets, while the proof that $d_3 Sq^j x = h_1 Sq^{j-2} x + Sq^j d_2 x$ if $n \equiv 1$ (4) is relatively straightforward. This appears to be convincing evidence that the H_∞ structure in the form of Steenrod operations in Ext is a powerful computational tool.

One other piece of related work is the thesis of Clifford Cooley [30]. He obtains formulas similar to Milgram's [61] by using the spectral sequence connecting homomorphism for a cofiber sequence of stunted projective spaces to reduce them to d_1's which he gets from a lambda algebra resolution of the cohomology of the appropriate stunted projective space. Calculating differentials this way or by the spectral sequence of IV.6 is probably a matter of indifference. The most interesting aspect of Cooley's thesis is that he works unstably, examining the interaction of the Steenrod operations and the EHP sequence. As in all other earlier work on this subject he views the H_∞ ring structure in terms of coreductions of stunted projective spaces. The interaction of the Steenrod operations and the EHP sequence had been discovered by William Singer [97] using the algebraic EHP sequence obtained from the lambda algebra.

In the work at hand, we extend the ideas of Makinen to the odd primary case to obtain comprehensive formulas for the first nontrivial differential on $\beta^\varepsilon P^j x$, which we state in §1. These apply to the mod p Adams spectral sequence of any H_∞ ring spectrum. The remainder of §1 consists of calculations using these formulas in the Adams spectral sequence of a sphere, including the differential discussed above. These are intended to illustrate especially the interaction between the homotopy operations and the differentials, specifically to obtain better formulas in particular cases than hold in general. One of these is $d_3 r = h_1 d_0^2$, which forces h_4^2 to be a permanent cycle. This is the shortest proof we know of this fact.

In §§2 and 3 we describe the natural Σ_p equivariant cell decomposition of $(\Sigma X)^{(p)}$ and use it to relate extended powers of X and of ΣX.

In §4 we start the proof of the formulas in §1, using the results of §§2 and 3. We also prove that the geometry splits naturally into three cases, which we deal with one at a time in the remaining §§5-7.

1. Differentials in the Adams spectral sequence

In this section we state our theorems concerning differentials, explain some of the subtleties involved in understanding what they are really saying, and calculate some examples in order to illustrate their use and demonstrate their power.

Localize everything at p. Let Y be an H_∞ ring spectrum. Let $E_r^{s,n+s}(S,Y) \Rightarrow \pi_n Y$ be the Adams spectral sequence based on ordinary mod p homology. We shall adopt the following shorthand notation for differentials. If A is in filtration s and B_1 and B_2 are in filtrations $s+r_1$ and $s+r_2$ respectively, then

$$d_* A = B_1 \dotplus B_2$$

means that $d_i A = 0$ for $i < \min(r_1, r_2)$ and

$$d_{r_1} A = B_1 \qquad \text{if } r_1 < r_2$$

$$d_r A = B_1 + B_2 \qquad \text{if } r_1 = r = r_2, \text{ and}$$

$$d_{r_2} A = B_2 \qquad \text{if } r_1 > r_2$$

Note. This does not mean that this differential is necessarily nonzero. Nor does it mean that if B_1 happens to be 0, then $d_{r_2}A = B_2$ regardless of whether $r_2 > r_1$ or not. More likely, B_1 is zero because it comes from a map which lifts to filtration $s+r_1+1$ or more and, hence, B_1 could conceivably lead to a nonzero $d_{r_1+1}A$. The point is that you can't tell what B_1 is contributing to the differential if all you know is that it is zero in filtration $s+r_1$. However, when we explicitly state that $T_p = 0$ in Theorem 1.2 we mean that it is to be treated as having filtration ∞.

The geometry behind the formula $d_*A = B_1 \dotplus B_2$ will make it clear exactly what the formula can and cannot tell you. The formula means that for some $r_0 > \max(r_1, r_2)$, A is represented by a map whose boundary splits into a sum $\overline{B}_1 + \overline{B}_2 + \overline{B}_0$, where each $\overline{B}_i$ lifts to filtration $s+r_i$, and where $\overline{B}_1$ and $\overline{B}_2$ represent B_1 and B_2 respectively. It is irrelevant what $\overline{B}_0$ represents because $\overline{B}_1 + \overline{B}_2$ lies in a lower filtration. This is fortunate, since in general $\overline{B}_0$ is very complicated. In particular cases however, we can often analyze $\overline{B}_0$ in order to get more complete information about d_*A. For examples of this, see Proposition 1.17(ii) (the formula $d_3 r_0 = h_1 d_0^2$) and Proposition 1.6.

Two remaining points about the formula are best made using examples. The formulas we will shortly prove say that, under appropriate circumstances,

$$d_* Sq^j x = Sq^j d_r x \dotplus \overline{a} x d_r x$$

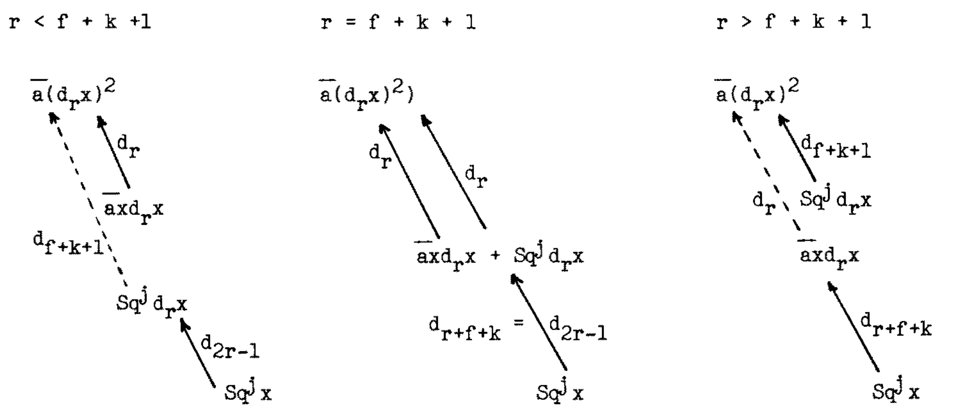
and

$$d_* Sq^j d_r x = \overline{a}(d_r x)^2$$

where $\overline{a} \in E_\infty(S,S)$. The algebra structure also implies that

$$d_r(\overline{a} x d_r x) = \overline{a}(d_r x)^2.$$

If the filtration of $Sq^j x$ is s, then the filtration of $Sq^j d_r x$ is $s+2r-1$, while that of $\overline{a} x d_r x$ is $s+r+f+k$ (f is the filtration of $\overline{a}$ and k will be defined shortly). The three ways these differentials can combine are illustrated below



Taken individually, the terms $Sq^j d_r x$ and $\bar{a}xd_r x$ do not always appear to survive long enough for $Sq^j x$ to be able to hit them. For example, when $r > f+k+1$, the differential $d_{r+f+k}Sq^j x = \bar{a}xd_r x$ is preceded by the differential $d_r(\bar{a}xd_r x) = \bar{a}(d_r x)^2$, which would have prevented $\bar{a}xd_r x$ from surviving until E_{r+k+f}, had it not happened that a still earlier differential $(d_{f+k+1}Sq^j d_r x = \bar{a}(d_r x)^2)$ had already hit $\bar{a}(d_r x)^2$. This is completely typical. The formula $d_*A = B_1 \dot{+} B_2$, as used here, carries with it the claim that the right-hand side will survive long enough for this differential to occur, and even shows the "coconspirator" which will make this possible when it seems superficially false.

The other point illustrated by this example occurs when $Sq^j d_r x$ and $xd_r x$ are permanent cycles and $r > f+k+1$. Then the differential $d_{r+k+f}Sq^j x = \bar{a}xd_r x$ reflects a hidden extension: $\bar{a}(xd_r x)$ is zero in E_∞ because of a filtration shift. It is actually detected by $Sq^j d_r x$. Relations among homotopy operations typically cause such phenomena. Note that the cell which carries $Sq^j x$ is also the cell which produces the relation in homotopy. In a suitably relative sense this is the meaning of all differentials in the Adams spectral sequence ("relative" because the terms in a relation corresponding to a differential will typically be relative homotopy classes which do not survive to E_∞ to become absolute homotopy classes).

We can now state our main theorems. Assume given $x \in E_r^{s,n+s}$ and consider the element $\beta^\varepsilon P^j x$ (as usual, $\varepsilon = 0$ and $P^j = Sq^j$ if $p = 2$). Let

$$k = \begin{cases} j-n & p = 2 \\ (2j-n)(p-1)-\varepsilon & p > 2 \,, \end{cases}$$

so that $\beta^\varepsilon P^j x \in E_2^{ps-k,p(n+s)}$, which lies in the $k+np$ stem. Using the functions v_p and a_p of V.2.15, V.2.16 and V.2.17 we define $v = v_p(k+n(p-1))$ and $a = a_p(k+n(p-1)) \in \pi_{v-1}S$. Recall that a is the top component of an attaching map of a stunted lens space after the attaching map has been compressed into the lowest possible skeleton. Let

$$\bar{a} \in E_\infty^{f,f+v-1}(S,S)$$

detect a (this defines f as well). Recall that $a_0 \in E_\infty^{1,1}$ detects the map of degree p when $p > 2$.

<u>Theorem 1.1</u>. There exists an element $T_p \in E_2^{**}(S,Y)$ such that

(i) if $p = 2$ then $d_*Sq^j x = Sq^j d_r x \dot{+} T_2$,

(ii) if $p > 2$ then

$$d_{r+1}P^j x = d_{r+1}x^p = a_0 x^{p-1} d_r x \quad \text{if } 2j = n\,,$$

$$d_2 P^j x = a_0 \beta P^j x \quad \text{if } 2j > n, \text{ and}$$

$$d_*\beta P^j x = -\beta P^j d_r x \dot{+} T_p\,.$$

Theorem 1.2.

$$T_2 = \begin{cases} 0 & v > k+1 \text{ or } 2r-2 < v < k \\ \bar{a} x d_r x & v = k+1 \\ \bar{a} Sq^{j-v} x & v = k \text{ or } (v < k \text{ and } v \leq 10) \end{cases}$$

If $p > 2$ then

$$T_p = \begin{cases} 0 & v > k+1 \text{ or } pr-p < v < k \\ (-1)^e \, \bar{a} x^{p-1} d_r x & v = k+1 \\ (-1)^{e-1} \, \bar{a} \beta P^{j-e-1} x & v < k \text{ and } v \leq pq. \end{cases}$$

where e is the exponent of p in the prime factorization of j.

Note. When $p > 2$, k and v have opposite parity so that $v = k$ never occurs.

Theorems 1.1 and 1.2 give complete information on the first possible nonzero differential except when

$$pq < v < \min(k, pr-p+1) \quad \text{if } p > 2,$$

or

$$10 < v < \min(k, 2r-1) \quad \text{if } p = 2.$$

The sketch of the proof given in Section 4 should make it clear what the obstruction is in these cases. We do have some partial information which we collect in the following theorem.

Theorem 1.3. If $p > 2$ and $v > q$ then $d_i \beta P^j x = 0$ if $i < v+2 \leq pr-p+1$, while $d_{pr-p+1} \beta P^j x = -\beta P^j d_r x$ if $v + 2 > pr-p+1$. If $p = 2$ and $v > 8$ then $d_i Sq^j x = 0$ if $i < v+2 \leq 2r-1$, while $d_{2r-1} Sq^j x = Sq^j d_r x$ if $v+2 > 2r-1$.

To apply these results we must know the values of the Steenrod operations in $E_2 = \mathrm{Ext}_{\mathcal{A}}(Z_p, H_* Y)$. For our examples we will concentrate primarily on $p = 2$ and $Y = S^0$, since this is a case in which there are many nontrivial examples. We cannot resist also showing how useful the Steenrod operations are in the purely algebraic task of determining the products in Ext.

We begin with the elements $h_n \in E_2^{1,2^n-1}$ dual to the Sq^{2^n}. Parts (i) and (iii) of the following propositon may also be found in [88].

Proposition 1.4. (i) (Adams [3]) $Sq^{2^n} h_n = h_{n+1}$ and $Sq^{2^n-1} h_n = h_n^2$.

(ii) (Adams [2]) $h_n h_{n+1} = 0$, $h_{n+1}^3 = h_n^2 h_{n+1}$ and $h_n h_{n+2}^2 = 0$.

(iii) (Novikov [91]) $h_n^2 h_{n+3}^2 = 0$, $h_0^{2^n} h_{n+2}^2 = 0$ and, if $n > 0$, $h_0^{2^n} h_n = 0$.

Proof $Sq^{2^n-1}h_n = h_n^2$ because the first operation is always the square. If we let $S:Ext^{s,*} \to Ext^{s,*}$ be Sq^{n+s} on $Ext^{s,n+s}$, then Proposition 11.10 of [68] shows that in the cobar construction $S[x_1|\cdots|x_j] = [x_1^2|\cdots|x_j^2]$. Since h_n is represented by $[\xi_1^{2^n}]$, it follows that $Sq^{2^n}h_n = S(h_n) = h_{n+1}$. For dimensional reasons, the Cartan formula reduces to $S(xy) = S(x)S(y)$. Thus, to show (ii) we need only show $h_0h_1 = 0$, $h_1^3 = h_0^2h_2$, and $h_0h_2^2 = 0$. These occur in such low dimensions that they may be checked "by hand". In fact, only the first and third must be done this way since $Sq^2(h_0h_1) = h_0^2h_2 + h_1^3$. The relation $h_n^2h_{n+3}^2 = 0$ follows similarly from $h_0^2h_3^2 = Sq^8(h_0h_2^2) = 0$. The only nonzero operation on h_{n+2}^2 is $Sq^{2^{n+3}}h_{n+2}^2 = h_{n+3}^2$ since (ii) implies that $h_{n+2}^4 = h_{n+2}(h_{n+1}^2h_{n+3}) = 0$. The relation $h_0^{2^n}h_{n+2}^2 = 0$ then follows by induction from $h_0^2h_3^2 = 0$. Finally, $h_0^{2^n}h_n = 0$ follows by induction from $h_0^2h_1 = 0$ since

$$Sq^{2^n}(h_0^{2^n}h_n) = h_0^{2^{n+1}}h_{n+1} .$$

As is well known, the preceding proposition implies the Hopf invariant one differentials.

Corollary 1.5. $d_2h_{n+1} = h_0h_n^2$ for all $n > 0$.

Proof. By Theorems 1.1 and 1.2 we find that

$$d_*h_{n+1} = d_*Sq^{2^n}h_n = Sq^{2^n}d_2h_n \dot{+} h_0h_n^2$$

so that $$d_2h_{n+1} = h_0h_n^2$$

since $Sq^{2^n}d_2h_n$ is in filtration 4. (It follows, of course, that $Sq^{2^n}d_2h_n = Sq^{2^n}h_0h_{n-1}^2 = h_0^2h_n^2$.)

The next result shows how we may use the relation with homotopy operations to get stronger results than the differentials themselves give.

Proposition 1.6. h_1h_4 and h_2h_4 are permanent cycles.

Proof. Since $h_1h_4 = Sq^9(h_0h_3)$, it is carried by the 9-cell of P_7^9. The attaching map is η, to the 7-cell, and hence its boundary is $\eta(2\sigma)^2 = 0$. Similarly, $h_2h_4 = Sq^{10}(h_1h_3)$, so h_2h_4 is carried by the 10-cell of $P_8^{10} \simeq S^8 \vee (S^9 \cup_2 e^{10})$. The 9-cell carries $P^9(\eta\sigma)$, which has order 2 by the Cartan formula in Theorem V.1.10. Thus, the boundary of the 10-cell maps to 0 and h_2h_4 is a permanent cycle.

Before turning to other families of elements we should note that the Hopf invariant one differentials of Corollary 1.5 account for only a few of the nontrivial differentials on the $h_0^i h_{n+1}$. In fact, Proposition 1.4 implies $d_2 h_0^i h_{n+1} = h_0^{i+1} h_n^2$ is 0 if $i+1 \geq 2^{n-2}$. On the other hand, $h_0^i h_{n+1} \neq 0$ for $i < 2^{n+1}$, and from the known order of Im J, there must be higher differentials on many of the $h_0^i h_{n+1}$ which survive to E_3. It seems difficult to determine these higher differentials in terms of the Steenrod operations, though Milgram [81] has indicated that it may be possible with a sufficiently good hold on the chain level operations. More disappointing is the fact that it doesn't seem possible to propagate these higher differentials. That is, even if we accept as given a differential like $d_3 h_0 h_4 = h_0 d_0$, we don't seem to get any information on $d_3 h_0^3 h_5$.

The operation we call S in Proposition 1.4 will be very useful so we collect its properties before proceeding.

Proposition 1.7. If $S = Sq^{n+s}: Ext^{s,n+s} \to Ext^{s,2(n+s)}$ then

(i) $S[x_1|\cdots|x_k] = [x_1^2|\cdots|x_k^2]$ in the cobar construction

(ii) $S(xy) = S(x)S(y)$

(iii) $Sq^j Sx = SSq^{j-n-s}x$

(iv) $S\langle x_0, x_1, \ldots, x_n\rangle \subset \langle Sx_0, Sx_1, \ldots, Sx_n\rangle$

Proof. (i) is Proposition 11.10 of [68], while (ii) and (iii) are immediate from the Cartan and Adem relations since all the other terms must be 0 for dimensional reasons. Part (iv) is proved in [78].

For our remaining sample calculations we will explore the consequences of the squaring operations on the elements c_0, d_0, e_0 and f_0. The key elements we will be concerned with are collected in Table 1.1 along with Massey product representations. With the exception of f_0 and y_0, the Massey products have no indeterminacy.

s	n = t-s	Name	Massey product
3	8	c_0	$\langle h_1,h_0,h_2^2\rangle$
4	14	d_0	$\langle h_0,h_2^2,h_0,h_2^2\rangle$
4	17	e_0	$\langle h_0^2,h_3^2,h_1,h_0\rangle$
4	18	f_0	$\langle h_0^2,h_3^2,h_2\rangle$
4	20	g_1	-------
6	30	r_0	$\langle h_0^2,h_3^2,h_3^2,h_0^2\rangle$
7	35	m_0	$\langle h_2,h_1,r_0\rangle$
6	36	t_0	------
5	37	x_0	$\langle h_3,h_4,d_0\rangle$
6	38	y_0	$\langle h_0^4,h_4^2,h_3\rangle$

TABLE 1.1

Also, note that the elements Mahowald and Tangora call r,m,t,x and y, we are calling r_0, m_0, t_0, x_0 and y_0. The reason for the subscript will be apparent from the following definition.

Definition 1.8. If $i \geq 0$ and $a \in \{c,d,e,f,g,r,m,t,x,y\}$, let $a_0 = a$ and

$$a_{i+1} = Sa_i.$$

Applying Proposition 1.7(iv) we find immediately that

$$c_i \in \langle h_{i+1},h_i,h_{i+2}^2\rangle$$

$$d_i \in \langle h_i,h_{i+2}^2,h_i,h_{i+2}^2\rangle$$

$$e_i \in \langle h_i^2,h_{i+3}^2,h_{i+1},h_i\rangle$$

$$f_i \in \langle h_i^2,h_{i+3}^2,h_{i+2}\rangle$$

$$r_i \in \langle h_i^2,h_{i+3}^2,h_{i+3}^2,h_i^2\rangle$$

$$m_i \in \langle h_{i+2},h_{i+1},r_i\rangle$$

$$x_i \in \langle h_{i+3},h_{i+4},d_i\rangle$$

and

$$y_i \in \langle h_i^4,h_{i+4}^2,h_{i+3}\rangle \ .$$

However, we shall not make any use of these Massey product representations here.

From the calculations of Mukohda [88] or Milgram [81] we collect the values of the Steenrod operations on c_0, d_0, e_0 and f_0. The following abbreviation will be very convenient: if $x \in Ext^{s,n+s}$ let $Sq^*(x) = (Sq^n x, Sq^{n+1}, \ldots, Sq^{n+s}x) = (x^2, \ldots, Sx)$

<u>Theorem 1.9.</u>

$$Sq^* c_0 = (c_0^2, h_0e_0, f_0, c_1)$$

$$Sq^* d_0 = (d_0^2, 0, r_0, 0, d_1)$$

$$Sq^* e_0 = (e_0^2, m_0, t_0, x_0, e_1)$$

$$Sq^* f_0 = (0, h_3r_0, y_0, 0, f_1)$$

The indeterminacy in the Massey product representations of f_0 and y_0 suggests that we should define them by the squaring operations above:

$$f_0 = Sq^{10}c_0 \quad \text{and} \quad y_0 = Sq^{20}f_0.$$

Applying Proposition 1.7.(iii) we immediately obtain the following corollary.

<u>Corollary 1.10.</u>

$$Sq^* c_i = (c_i^2, h_ie_i, f_i, c_{i+1})$$
$$Sq^* d_i = (d_i^2, 0, r_i, 0, d_{i+1})$$
$$Sq^* e_i = (e_i^2, m_i, t_i, x_i, e_{i+1})$$
$$Sq^* f_i = (0, h_{i+3}r_i, y_i, 0, f_{i+1}).$$

Before computing the differentials that this corollary implies, it will be useful to obtain a number of relations in Ext. This also gives us an opportunity to illustrate how powerful the Steenrod operations are in propagating relations. The relations we will assume known are all calculated by Tangora [103] by means of the May spectral sequence. In general, this technique only yields relations modulo terms of lower weight. However, the particular relations we need do not suffer from this ambiguity, since there are no terms of lower weight in their bidegree.

<u>Proposition 1.11</u> (i) $h_0c_0 = 0$, $h_2c_0 = 0$, $h_3c_0 = 0$, $h_0c_1 = 0$, $h_1f_0 = 0$, $h_1r_0 = 0$, $h_1m_0 = 0$.

(ii) $c_0^2 = h_1^2d_0$, $h_2d_0 = h_0e_0$, $h_1e_0 = h_0f_0$, $h_2e_0 = h_0g_1$, $h_0^2d_0 = P^1h_2^2$, $h_2t_0 = c_1g_1$.

(iii) $h_0^6r_0 = 0$, $h_4f_0 = 0$, $h_3d_0^2 = 0$, $h_2d_1 = h_4g_1$, $h_0^6x_0 = 0$, $h_2m_0 = h_0^2y_0$, $h_0^2f_1 = h_1^2e_1$.

These relations are grouped as follows: (i) holds because the relevant bidegree is 0 or is not annihilated by h_0, as multiples of h_1 must be; (ii) follows from [103] since, again by [103], there are no elements of lower weight in the given bidegrees; (iii) now follows either by applying Steenrod operations to relations in (i) and (ii) or by the same argument as (ii). (The point is that the relations in (iii) are dependent on those in (i) and (ii) under the action of the Steenrod algebra.)

<u>Corollary 1.12</u>. (i) $h_i c_i = 0$, $h_{i+2}c_i = 0$, $h_{i+3}c_i = 0$, $h_{i-1}c_i = 0$, $h_{i+1}f_i = 0$, $h_{i+1}r_i = 0$, $h_{i+1}m_i = 0$.

(ii) $c_i^2 = h_{i+1}^2 d_i$, $h_{i+2}d_i = h_i e_i$, $h_{i+1}e_i = h_i f_i$, $h_{i+2}e_i = h_i g_{i+1}$, $h_{i+2}t_i = c_{i+1}g_{i+1}$.

(iii) $h_{i+4}f_i = 0$, $h_{i+3}d_i^2 = 0$, $h_{i+1}d_i = h_{i+3}g_i$, $h_{i+2}m_i = h_i^2 y_i$, $h_{i-1}^2 f_i = h_i^2 e_i$.

<u>Proof</u> These are immediate from Proposition 1.11 since S is a ring homomorphism by Proposition 1.7(ii).

A comparison of the preceding proposition and corollary will show that if we view the periodicity operator as a Massey product

$$P^r x = \langle h_{r+2}, h_0^{2^{r+1}}, x\rangle ,$$

then we have only Milgram's theorem (Proposition 1.7.(iv)) to use in calculating $S(P^r x)$, and this generally leaves us with too much indeterminacy. For example, $P^1 h_1 h_3 = c_0^2$ so $S(P^1 h_1 h_3) = Sc_0^2 = c_1^2$. On the other hand, $S(P^1 h_1 h_3) = S\langle h_3, h_0^4, h_1 h_3\rangle \in \langle h_4, 0, h_2 h_4\rangle = 0$ modulo indeterminacy which is divisible by h_4. Of course, since $c_1^2 \neq 0$, it follows that $h_2 h_4 g = c_1^2$ since $h_4(h_2 g)$ is the only possible nonzero element divisible by h_4. This example shows that to calculate $S(P^r x)$, we need another representation of $P^r x$. It also shows that the Massey product representation can lead to useful information (although in this case the product $h_2 h_4 g = c_1^2$ was already true in the associated graded). Accordingly, we provide the following formula for the interaction of the Sq^i and the periodicity homomorphisms P^r.

<u>Proposition 1.13</u>. Let $Sq_i = Sq^{t-i}: Ext^{s,t} \to Ext^{s+i,2t}$. Modulo the ideal generated by $\{h_{r+1}^2, h_{r+2}, Sq_0 x, \ldots, Sq_i x\}$ we have

$$Sq_i P^{r-1} x = \begin{cases} 0 & i < 2^r \\ P^r Sq_{i-2^r} x + \langle h_{r+1}^2, h_0^{2^{r+1}}, Sq_{i-2^r-1} x\rangle & i \geq 2^r. \end{cases}$$

If $i = 0$, the indeterminacy (of $Sq_0 = S$) is generated by h_{r+2} and $Sq_0 x$.

Proof. This is a special case of Milgram's general result [78], which, for three-fold Massey products says

$$Sq_i \langle a,b,c\rangle \subset \left\langle (Sq_i a, \ldots, Sq_0 a), \begin{pmatrix} Sq_0 b & & \\ \vdots & \ddots & \\ Sq_i b & \cdots & Sq_0 b \end{pmatrix}, \begin{pmatrix} Sq_0 c \\ \vdots \\ Sq_i c \end{pmatrix} \right\rangle,$$

since $Sq_0 h_0^n = h_1^n = 0$ for $n \geq 4$, $Sq_n h_0^n = h_0^{2n}$, and $Sq_i h_0^n = 0$ otherwise.

Corollary 1.14. $\langle h_4, h_0^8, h_3^2\rangle = P^2 h_3^2 = h_0^4 r_0$ with no indeterminacy.

Proof. By Proposition 1.11, $P^1 h_2^2 = h_0^2 d_0$. By Theorem 1.9 we have $Sq^{16} h_0^2 d_0 = h_0^4 r_0 + h_1^2 d_0^2 = h_0^4 r_0$, since $h_1 d_0^2$ must be divisible by h_0 so $h_1^2 d_0^2 = 0$. By Proposition 1.13, $Sq^{16} P^1 h_2^2 = Sq_4 P^1 h_2^2 = P^2 h_3^2$ with indeterminacy generated by h_3^2 and h_4. For dimensional reasons the indeterminacy is 0.

Combining Proposition 1.11 with Theorem 1.9 we can produce a number of relations in Ext which do not hold in the associated graded calculated by Tangora.

Proposition 1.15.

(i)	$h_0 r_0 = s_0$	and hence	$h_i r_i = s_i$
(ii)	$h_3 r_0 = h_1 t_0 + h_0^2 x_0$	and hence	$h_{i+3} r_i = h_{i+1} t_i + h_i^2 x_i$
(iii)	$h_2 e_0^2 = h_0^4 x_0$	and hence	$h_{i+2} e_i^2 = 0$ if $i > 0$
(iv)	$h_2^2 d_1 = h_1 x_0$	and hence	$h_{i+1}^2 d_i = h_i x_{i-1}$
(v)	$h_1 y_0 = h_2 t_0$	and hence	$h_{i+1} y_i = h_{i+2} t_i$
(vi)	$h_2 x_0 = 0$	and hence	$h_{i+2} x_i = 0$
(vii)	$h_1 f_1 = h_0^2 c_2$	and hence	$h_i f_i = h_{i-1}^2 c_{i+1}$
(viii)	$h_2 y_0 = 0$	and hence	$h_{i+2} y_i = 0$
(ix)	$h_3 x_0 = h_0^2 g_2$	and hence	$h_{i+3} x_i = h_i^2 g_{i+2}$

Note. Mahowald and Tangora [61] found (i)-(iii) by other techniques. Barratt, Mahowald and Tangora [20] also found (iv), (vii), and (ix) by other techniques. Milgram [81] found (i) and (ii) by using the Steenrod operations. Mukohda [88] found (iv)-(vi) and (ix), partly by using the Steenrod operations and the cobar construction, and partly by means of a minimal resolution.

Proof. Given (ii), (i) follows because $h_0h_3r_0 = h_0^3x_0 \neq 0$, from which it follows that $h_0r_0 \neq 0$. The only possibility is $h_0r_0 = s_0$. To prove (ii), apply Sq^{20} to the relation $h_2d_0 = h_0e_0$. To prove (iii), apply Sq^{19} to the relation $h_1e_0 = h_0f_0$ and use the fact that $h_1m_0 = 0$. To prove (iv), apply Sq^{21} to the relation $h_2d_0 = h_0e_0$ and use the fact that $h_0^2e_1 = 0$. To prove (v), apply Sq^{21} to the relation $h_1e_0 = h_0f_0$ and use (iv) to show that $h_1^2x_0 = h_1(h_2^2d_1) = 0$. To prove (vi), apply Sq^{22} to the relation $h_1e_0 = h_0f_0$ to show that $h_2x_0 = h_1^2e_1 + h_0^2f_1$, and apply Proposition 1.11.(iii) to show that this is 0. For (vii), we apply Sq^{22} to $h_0c_1 = 0$. Similarly, Sq^{21} applied to $h_1f_0 = 0$ yields (viii). Finally, (ix) follows by applying Sq^{24} to the relation $h_2e_0 = h_0g_1$ to get $h_0^2g_2 = h_3x_0 + h_2^2e_1$, and noting that $h_2^2e_1 = h_2(h_1f_1) = 0$. The calcultion of $Sq^{24}(h_0g_1)$ is possible because $Sq^{24}g_1 = g_2$ by definition, while $Sq^{23}g_1 = 0$ for dimensional reasons.

Now we examine the differentials implied by the squaring operations in the c_i, d_i, e_i and f_i families. The results we obtain for $t-s \geq 45$ are all new. In the range $t-s \leq 45$ they are due to May [66], Maunder [65], Mahowald and Tangora [61], Milgram [81] and Barratt, Mahowald and Tangora [20] with the exception of $d_3e_1 = h_1t$, which is new and corrects a mistake in [20]. As noted by Milgram [81] the proofs using Steenrod operations are usually far simpler and more direct than the original proofs. In addition, when they replace proofs which relied on prior knowledge of the relevant homotopy groups we obtain independent verification of the calculation of those homotopy groups.

If $x \in E_r^{s,n+s}$, let us write $x \in (s,n)$ or $x \in (s,n)_r$ for convenience. Theorems 1.1, 1.2 and 1.3 imply that

$$d_*Sq^jx = Sq^jd_rx \dot{+} \begin{cases} 0 & v > k+1 \text{ or } 2r-2 < v < k \\ \bar{a}xd_rx & v = k+1 \\ \bar{a}Sq^{j-v}x & v = k \text{ or } (v < k \text{ and } v \leq 10) \end{cases}$$

where $k = j-n$, $v = 8a + 2^b$ if $j+1 = 2^{4a+b}(\text{odd})$, and $\bar{a}$ detects a generator of Im J in $\pi_{v-1}S^0$.

We start with a general observation about families $\{a_i\}$ with $a_{i+1} = S(a_i)$. If $a_i \in (s,n_i)$ then

$$n_i + s = 2(n_{i-1} + s) = 2^i(n_0 + s).$$

If N is the integer such that $2^{N-1} < s+2 \leq 2^N$ then the differentials on the elements $Sq^j a_i$ depend on the congruence class of n_i modulo 2^N. Clearly, $n_i \equiv -s$ modulo 2^N if $i \geq N$. Thus, the differentials on all but the first N members of such a family follow a pattern which depends only on the filtration in which the family lives.

Consider the c_i family. We have $c_0 \in (3,8)_\infty$, so in general $c_i \in (3, 2^i \cdot 11-3)$.

<u>Proposition 1.16</u>. (i) $c_1 \in E_\infty$ while $d_2 c_i = h_0 f_{i-1}$ for $i \geq 2$

(ii) $d_2 f_0 = h_0^2 e_0$, $f_1 \in E_5$, and $d_3 f_i = h_1 y_{i-1}$ for $i \geq 2$

(iii) $d_3 c_i^2 = h_0^2 h_{i+2} r_{i-1}$ for $i \geq 2$

<u>Note</u>. We will show shortly that $d_2 h_0 y_{i-1} = h_0^2 h_{i+2} r_{i-1}$. This, together with (iii) implies that $d_3 c_i^2 = 0$.

<u>Corollary 1.17</u>. $d_2 e_0 = c_0^2$ and $\nu\theta_4 \neq 0$, where θ_4 is the Arf invariant one element detected by h_4^2.

<u>Proof</u>. Since $c_0 \in (3,8)_\infty$, $Sq^* c_0 = (c_0^2, h_0 e_0, f_0, c_1)$ is carried by $\Sigma^8 P_8^{11} = S^{16} \vee (S^{17} \cup_2 e^{18}) \vee S^{19}$. Therefore $c_1 \in E_\infty$ and $d_2 f_0 = h_0^2 e_0$. Applying Proposition 1.11 we find that $d_2 h_1 e_0 = d_2 h_0 f_0 = h_0^3 e_0 = h_1^3 d_0 = h_1 c_0^2$, from which it follows that $d_2 e_0 = c_0^2$.

Since $c_1 \in (3,19)_\infty$, $Sq^* c_1 = (c_1^2, h_1 e_1, f_1, c_2)$ is carried by $\Sigma^{19} P_{19}^{23} = (S^{38} \cup_2 e^{39} \cup_\eta e^{40}) \cup_2 e^{41}$. Therefore $d_2 c_2 = h_0 f_1$ and $d_3 f_1 = h_1 c_1^2 = h_1 h_2^2 d_1 = 0$, so that $f_1 \in E_5$ for dimensional reasons. Since $c_2 = \langle h_3, h_2, h_4^2 \rangle$ and $c_2 \notin E_\infty$, the Toda bracket $\langle \sigma, \nu, \theta_4 \rangle$ does not exist. We shall show in the next proposition that $h_4^2 \in E_\infty$ so that θ_4 exists. Since $\sigma\nu = 0$, it follows that $\nu\theta_4 \neq 0$.

Now assume for induction that $d_2 c_i = h_0 f_{i-1}^2$ and that $i \geq 2$. We can arrange the relevant information in the following table.

j (mod 4)	$Sq^j c_i$	$Sq^j(h_0 f_{i-1}^2)$	k	v	$\bar{a}$
1	c_i^2	$h_0^2 h_{i+2} r_{i-1}$	0	2	h_1
2	$h_i e_i$	$h_0^2 y_{i-1} + h_1 h_{i+2} r_{i-1}$	1	1	h_0
3	f_i	$h_1 y_{i-1}$	2	≥ 4	-
4	c_{i+1}	$h_0^2 f_i$	3	1	h_0

It follows that $d_3c_i^2 = h_0^2h_{i+2}r_{i-1}$, $d_2h_ie_i = h_0c_i^2$, $d_3f_i = h_1y_{i-1}$ and $d_2c_{i+1} = h_0f_i$. This completes the inductive step and finishes the proof of Propositon 1.16 and Corollary 1.17. Note that we have omitted $d_2h_ie_i$ from the statement of the proposition because it will follow from our calculation of d_2e_i below.

<u>Proposition 1.18</u>. (i) $d_2k = h_0d_0^2$

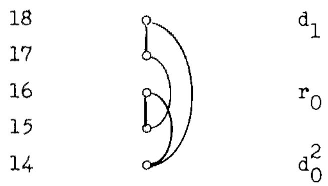
(ii) $d_3r_0 = h_1d_0^2$ and $h_4^2 \in E_\infty$

(iii) $r_i \in E_3$ for $i \geq 1$

(iv) $d_i \in E_3$ for $i \geq 1$

<u>Note</u>. Mahowald and Tangora show [61] that d_1 is actually in E_∞, not just E_3. Also, the proof given here that $h_4^2 \in E_\infty$ is much simpler than the proof in [61].

<u>Proof</u>. Since $d_0 \in (4,14)_\infty$, $Sq^*d_0 = (d_0^2, 0, r_0, 0, d_1)$ is carried by $\Sigma^{14}P_{14}^{18}$, which has attaching maps as shown



Now $d_3h_0h_4 = h_0d_0$ implies $h_0d_0^2 = 0$ in E_4. The only possibility is that $d_2k = h_0d_0^2$. This implies that $2\pi_{29} = 0$. Since the boundary of the 16 cell carries $h_1d_0^2$ plus twice something, we get $d_3r_0 = h_1d_0^2$. Nothing is left for h_4^2 to hit, so $h_4^2 \in E_\infty$. Finally, $d_2(d_1) = h_0 \cdot 0 = 0$ so $d_1 \in E_3$. Now assume for induction that $i \geq 1$ and $d_i \in E_3$. The terms $Sq^jd_3d_i$ in the differentials on Sq^jd_i will not contribute until E_5, so will not affect the proof of (iii) and (iv). Since $Sq^*d_i = (d_i^2, 0, r_i, 0, d_{i+1})$ we find that $d_2r_i = h_0 \cdot 0 = 0$ and $d_2(d_{i+1}) = h_0 \cdot 0 = 0$, proving (iii) and (iv) and completing the induction.

<u>Proposition 1.19</u>. (i) $d_2m_0 = h_0e_0^2$, $t_0 \in E_{11}$ and $d_3e_1 = h_1t_0$

(ii) $e_1^2 \in E_5$, $d_5m_1 = Sq^{39}h_1t_0$, $d_2t_1 = h_0m_1$, $d_3x_1 = h_1m_1$ and $d_2e_2 = h_0x_1$.

(iii) If $i \geq 2$ and $n = 2^i \cdot 21 - 4$ then $d_3e_i^2 = h_0^2e_ix_{i-1} + Sq^nh_0x_{i-1}$, $d_3m_i = Sq^{n+1}h_0x_{i-1}$, $d_2t_i = h_0m_i$, $d_3x_i = Sq^{n+3}h_0x_{i-1}$, and $d_2e_i = h_0x_{i-1}$.

<u>Proof</u>. By Corollary 1.17, $d_2e_0 = c_0^2$. The information needed to calculate the differentials on the Sq^je_0 is most conveniently presented in a table.

j	$Sq^j e_0$	k	v	$\bar{a}$	$Sq^j c_0^2$	conclusion
17	e_0^2	0	2	h_1	0	$d_3 e_0^2 = 0$
18	m_0	1	1	h_0	$h_0^2 e_0^2$	$d_2 m_0 = h_0 e_0^2$
19	t_0	2	4	h_2	0	$d_3 t_0 = 0$
20	x_0	3	1	h_0	0	$d_2 x_0 = h_0 t_0 = 0$
21	e_1	4	2	h_1	0	$d_3 e_1 = h_1 t_0$

We omit $d_3 e_0{}^2$ and $d_2 x_0 = 0$ from the proposition because they also follow simply for dimensional reasons. Similarly, since t_0 is in E_4 it must be in E_{11} for dimensional reasons. Thus (i) is proved.

Since $d_3 e_1 = h_1 t_0$, the term $Sq^j h_1 t_0$ will contribute to $d_5 Sq^j e_1$ if $Sq^j e_1$ lives that long. Again, the information is most conveniently organized into a table.

j	$Sq^j e_1$	k	v	$\bar{a}$	conclusion
38	e_1^2	0	1	h_0	$d_4 e_1^2 = h_0 e_1 h_1 t_0 = 0$
39	m_1	1	8	h_3	$d_5 m_1 = Sq^{39} h_1 t_0$
40	t_1	2	1	h_0	$d_2 t_1 = h_0 m_1$
41	x_1	3	2	h_1	$d_3 x_1 = h_1 m_1$
42	e_2	4	1	h_0	$d_2 e_2 = h_0 x_1$

All of (ii) follows immediately . Now assume for induction that $d_2 e_i = h_0 x_{i-1}$ and $i \geq 2$. Again we organize the information in tabular form. Let $n = 2^i \cdot 21 - 4$ so that $e_i \in (4,n)_2$.

j	$Sq^j e_i$	k	v	$\bar{a}$	conclusion
n	e_i^2	0	1	h_0	$d_3 e_i^2 = h_0^2 e_i x_{i-1} + Sq^n h_0 x_{i-1}$
$n+1$	m_i	1	2	h_1	$d_3 m_i = Sq^{n+1} h_0 x_{i-1}$
$n+2$	t_i	2	1	h_0	$d_2 t_i = h_0 m_i$
$n+3$	x_i	3	4	h_2	$d_3 x_i = Sq^{n+3} h_0 x_{i-1}$
$n+4$	e_{i+1}	4	1	h_0	$d_2 e_{i+1} = h_0 x_i$

This establishes (iii) and completes the induction.

Note that three of the 5 entries in the above table satisfy $v = k+1$. The corresponding differentials therefore contain terms of the form $\overline{a}xd_rx$, specifically $\overline{a}h_0e_ix_{i-1}$ in this instance.

Only one of the differentials on the Sq^jf_i is interesting.

Proposition 1.20. For all $i \geq 0$, $d_2y_i = h_0h_{i+3}r_i$.

Proof. The terms in d_*Sq^jx involving d_rx do not contribute to d_2Sq^jx. If $n = 2^i \cdot 22 - 4$ so that $f_i \in (4,n)$ then $Sq^{n+1}f_i = h_{i+3}r_i$ and $Sq^{n+2}f_i = y_i$. Since $n+2$ is even the proposition follows immediately.

This completes our sampler. We have calculated only about one fourth of the differentials found by Mahowald and Tangora, but they include some of the most difficult. The remaining differentials follow more or less directly from those calculated here just as in Mahowald and Tangora's original paper [61].

2. Extended Powers of Cells

In order to study Steenrod operations on elements of the Adams spectral sequence which are not permanent cycles, we need a relative version of the extended power construction. The extended power functor $E\pi \ltimes_\pi X^{(p)}$, for $\pi \subset \Sigma_p$, factors as the composite of the functors

$$X \longmapsto X^{(p)}$$

and

$$Y \longmapsto E\pi \ltimes_\pi Y$$

If we replace X by a pair (X,A) then $X^{(p)}$ is replaced by a length p+1 filtration $X^{(p)} \supset \cdots \supset A^{(p)}$ of π spectra and we may apply $E\pi \ltimes_\pi (?)$ to this termwise. The resulting diagram is the relativization which we need. While the formalism applies to any pair (X,A), we will confine attention to pairs (CX,X), where CX is the cone on X, both for notational simplicity and because the p^{th} power of such a pair has special properties which we shall exploit. In particular, note that Lemma 2.4 is the geometric analog of the fact that a trivial one-dimensional representation splits off the permutation representation of $\pi \subset \Sigma_p$ on $\mathbb{R}^p$. Most of this section is devoted to this fact and its consequences.

An element $x \in E_r^{s,n+s}(X,Y)$ can be represented by a map of pairs

$$(CX,X) \longrightarrow (Y_s,Y_{s+r}).$$

Extended powers of (CX,X) can be used to construct a map representing $\beta^\epsilon P^jx$. The final bit of the section establishes the facts about extended powers which will enable us to construct and analyze such a map.

We shall work first in the category of based π-spaces and based π-maps and the homotopy category of based π-spaces and π-homotopy classes of based π-maps with weak equivalences inverted. The results are then transferred to the category of π-spectra by small smash products, desuspensions, and colimits.

Let I be the unit interval. We choose 0 as the basepoint, justifying our choice by the resulting simplicity of the formulas in the proof of Lemma 2.4. For a space or spectrum X, let $CX = X \wedge I$. The isomorphism $X \cong X \wedge \{0,1\}$ and the cofibration $\{0,1\} \subset I$ induce a cofibration $X \to CX$ with cofiber ΣX.

Definition 2.1. For a space X, define a Σ_p-space $\Gamma_i(X)$ by

$$\Gamma_i(X) = \{c_1 \wedge \cdots \wedge c_p \in (CX)^{(p)} \mid \text{at least } i \text{ of the } c_j \text{ lie in } X\}.$$

If X is a spectrum, define a Σ_p spectrum $\Gamma_i(X) = X^{(p)} \wedge \Gamma_i(S^0)$.

Lemma 2.2. (i) For a space X, $\Gamma_i(X)$ is naturally and Σ_p equivariantly homeomorphic to $X^{(p)} \wedge \Gamma_i(S^0)$.

(ii) $\Gamma_i(\Sigma^\infty X) \cong \Sigma^\infty \Gamma_i(X)$ if X is a space.

(iii) $\Gamma_{i+1}(X) \to \Gamma_i(X)$ is a Σ_p-cofibration.

(iv) $\Gamma_i(X)/\Gamma_{i+1}(X)$ is equivalent to the wedge of all (i,p-i) permutations of $X^{(i)} \wedge (\Sigma X)^{(p-i)}$. In particular, if (p) is the permutation representation of Σ_p on $\mathbb{R}^p$ then $\Gamma_0(X)/\Gamma_1(X) \cong (\Sigma X)^{(p)} \cong \Sigma^{(p)} X^{(p)}$ and $\Gamma_p(X) \cong X^{(p)}$.

(v) $\Gamma_1(X) \cong \Sigma^{p-1} X^{(p)}$ as Σ_p spaces or spectra, where S^{p-1} has the Σ_p action inherited from the p-cell $\Gamma_0(S^0) = I^{(p)}$.

Proof. (i) follows immediately from the shuffle map

$$(x_1 \wedge t_1) \wedge \cdots \wedge (x_p \wedge t_p) \longmapsto (x_1 \wedge \cdots \wedge x_p) \wedge (t_1 \wedge \cdots \wedge t_p).$$

(ii) is a consequence of the commutation of Σ^∞ and smash products.

(iii) follows for spectra if it holds for spaces. By (i) it holds for spaces if it holds for S^0. For S^0, it follows because $\Gamma_i(S^0)$ is the (p-i) skeleton of a CW decomposition of $\Gamma_0(S^0) = I^{(p)}$.

Similarly, (iv) holds in general if it holds for S^0, for which it is immediate.

(v) follows from the fact that $\Gamma_1(S^0)$ is the boundary of the p-cell $\Gamma_0(S^0)$.

Remark 2.3: We will complete what we have begun in (iv) and (v) above in Lemma 3.5, which shows that

$$\Gamma_i(X) \simeq \bigvee_{(p-i,i-1)} \Sigma^{np-i}X^{(p)}.$$

The next lemma is the key result of this section. Let I and S^1 have trivial Σ_p actions so that if X is a Σ_p space or spectrum then $CX = X \wedge I$ and $\Sigma X = X \wedge S^1$ are also.

Lemma 2.4. There are natural equivariant equivalences $\Gamma_0(X) \cong C\Gamma_1(X)$ and $\Sigma\Gamma_1(X) \cong (\Sigma X)^{(p)}$ such that the triangle

$$\begin{array}{ccc} & \subset & C\Gamma_1(X) \\ \Gamma_1(X) & & \wr\| \\ & \subset & \Gamma_0(X) \end{array}$$

commutes.

Proof. By definition and by 2.2(i) we may assume $X = S^0$. We define a Σ_p homeomorphism $\Gamma_0(S^0) \to C\Gamma_1(S^0)$ by

$$t_1 \wedge \cdots \wedge t_p \longrightarrow \left(\frac{t_1}{t} \wedge \cdots \wedge \frac{t_p}{t}\right) \wedge t$$

where $t = \max\{t_i\}$. The inverse homeomorphism is given by

$$(t_1 \wedge \cdots \wedge t_p) \wedge t \longmapsto tt_1 \wedge tt_2 \wedge \cdots \wedge tt_p .$$

Commutativity of the triangle is immediate. The equivalence $\Sigma\Gamma_1(X) \cong (\Sigma X)^{(p)}$ follows since $\Sigma\Gamma_1(X) \cong C\Gamma_1(X)/\Gamma_1(X) \cong \Gamma_0(X)/\Gamma_1(X) \cong (\Sigma X)^{(p)}$, the latter equivalence by 2.2(iv).

Lemma 2.5. For any $\pi \subset \Sigma_p$ and any π-free π space W, there are natural equivalences

$$W \ltimes_\pi \Gamma_0(X) \cong C(W \ltimes_\pi \Gamma_1(X))$$

and

$$\Sigma(W \ltimes_\pi \Gamma_1(X)) \cong W \ltimes_\pi (\Sigma X)^{(p)}$$

such that the following triangle commutes.

$$\begin{array}{ccc} & \nearrow & W \ltimes_\pi \Gamma_0(X) \\ W \ltimes_\pi \Gamma_1(X) & & \wr\| \\ & \searrow & C(W \ltimes_\pi \Gamma_1(X)) \end{array}$$

Proof. By Lemma 2.4, $W \ltimes_\pi \Gamma_0(X) \cong W \ltimes_\pi(\Gamma_1(X) \wedge I)$ and by I.1.2.(ii) $W \ltimes_\pi(\Gamma_1(X) \wedge I) \simeq (W \ltimes_\pi \Gamma_1(X)) \wedge I = C(W \ltimes_\pi \Gamma_1(X))$. The second equivalence follows similarly. Commutativity of the triangle follows from naturality with respect to $\{0,1\} \subset I$.

In the remainder of this section we shall restrict attention to the special case of interest in section 4. The general case presents no additional difficulties but is notationally more cumbersome.

Let $\pi \subset \Sigma_p$ be cyclic of order p and let $W = S^\infty$ with the cell structure which makes $C_* W \cong \mathscr{W}$, the usual $Z[\pi]$ resolution of Z. Let W^k be the k-skeleton of W. As in V.2, W^k/π is the lens space $\tilde{L}^k$, and, by I.1.3.(ii), if $\Gamma_i = \Gamma_i(S^{n-1})$ then $W^k \ltimes_\pi \Gamma_i / W^{k-1} \ltimes_\pi \Gamma_i \simeq \Sigma^k \Gamma_i$. By Lemmas 2.2 and 2.5 we then have the following corollary of Theorems V.2.6 and V.2.14.

<u>Corollary 2.6</u>: $$W^k \ltimes_\pi \Gamma_p \simeq \Sigma^{n-1} \tilde{L}^{(n-1)(p-1)+k}_{(n-1)(p-1)}$$

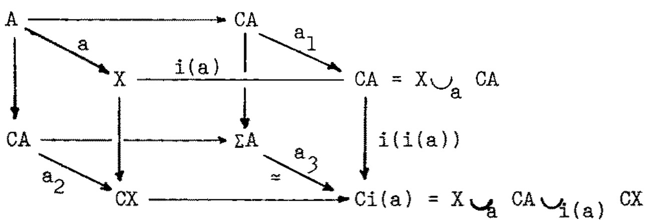
and $$W^k \ltimes_\pi \Gamma_1 \simeq \Sigma^{n-1} \tilde{L}^{n(p-1)+k}_{n(p-1)} .$$

Now note that Lemma 2.5 also implies that $W^k \ltimes_\pi \Gamma_1 \cup W^{k-1} \ltimes_\pi \Gamma_0$ is the cofiber of the inclusion $W^{k-1} \ltimes_\pi \Gamma_1 \to W^k \ltimes_\pi \Gamma_1$. By Corollary 2.6 or by Lemma 2.2 and I.1.3.(ii) it follows that

$$W^k \ltimes_\pi \Gamma_1 \cup W^{k-1} \ltimes_\pi \Gamma_0 \simeq S^{np+k-1}.$$

To get this equivalence in a maximally useful form, first consider a more general situation. In order to analyze the Barratt-Puppe sequence of a map $a:A \to X$ one constructs the diagram below.

(2.1)



In diagram (2.1) the front and back squares are pushouts, a_3 is an equivalence, $a_2 = Ca = a \wedge 1$, a_1 is the obvious natural inclusion, and the maps a, i(a), and $a_3^{-1} i(i(a))$ are the beginning of the cofiber sequence of a. The following obvious fact about such diagrams will be used repeatedly.

<u>Lemma 2.7</u>. Let $B \to Y$ be a cofibration and let $\pi: Y \to Y/B$ be the natural map. For any map

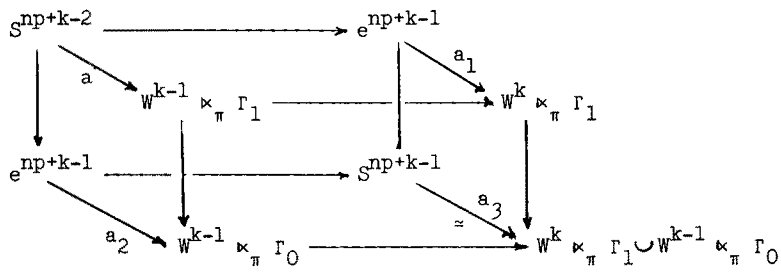
$$f:(Ci(a),X) \to (Y,B),$$

we have $\pi f a_3 = \overline{fa_1} - \overline{fa_2}$ in $[\Sigma A, Y/B]$, where $\overline{fa_i}$ is the map $\Sigma A \to Y/B$ induced by $(fa_i, fa):(CA,A) \to (Y,B)$.

<u>Proof</u>. The only question is whether we should get $\overline{fa_1} - \overline{fa_2}$ or its negative. We choose $\overline{fa_1} - \overline{fa_2}$ for consistency with the Barratt-Puppe sequence signs. The point is that a_3 is a homotopy inverse to the map from Ci(a) to ΣA which collapses CX, and the orientations on the two cones are determined by this fact.

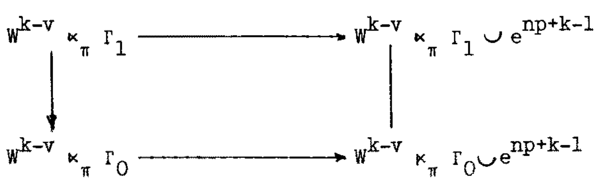
Returning to the special case which prompted these generalities, let $a: S^{np+k-2} \to W^{k-1} \ltimes_\pi \Gamma_1$ be the attaching map of the top cell of $W^k \ltimes_\pi \Gamma_1$. Then diagram (2.1) becomes diagram (2.2) below.

(2.2)



<u>Corollary 2.8</u>. Let $B \to Y$ and $\pi: Y \to Y/B$ be as in Lemma 2.7. For any map $f:(W^k \ltimes_\pi \Gamma_1 \cup W^{k-1} \ltimes_\pi \Gamma_0, W^{k-1} \ltimes_\pi \Gamma_1) \to (Y,B)$ we have $\pi f a_3 = \overline{fa_1} - \overline{fa_2}$ in $\pi_{np+k-1}(Y/B)$.

Let $v = v_p(n(p-1)+k)$ in the notation of Definition V.2.15, so that $a \in \pi_{np+k-2} W^{k-1} \ltimes_\pi \Gamma_1$ factors through $W^{k-v} \ltimes_\pi \Gamma_1$. Then we may replace the front face of diagram (2.2) by



in which the np+k-1 cell is attached by a lift of a. This gives us a version of Corollary 2.8 in which f need only map $W^{k-v} \ltimes_\pi \Gamma_1$ into B and the map $\overline{fa_2}$ factors through $W^{k-v} \ltimes_\pi \Gamma_0$.

§3. Chain Level Calculations

In this section we define and study certain elements in the cellular chains of $W \ltimes_\pi \Gamma_0(S^{n-1})$. In sections 5-7 they will be used to investigate the homotopy groups of various pairs of subspaces of $W \ltimes_\pi \Gamma_0(S^{n-1})$. Here we use them to determine the effect in homology of a compression (lift) of the natural map $W^k \ltimes_\pi \Gamma_p(S^{n-1}) \to W^k \ltimes_\pi \Gamma_1(S^{n-1})$.

Let $\Gamma_i = \Gamma_i(S^{n-1})$. Give $e^n = C(S^{n-1})$ the cell structure with one n-cell x and one (n-1)-cell dx. Let $C_*(?)$ denote cellular chains and $C_*(?;R) = C_*(?) \otimes R$. Then $C_*\Gamma_0 = \langle x,dx\rangle^p$, the p-fold tensor product of copies of $C_*(e^n) = \langle x,dx\rangle$, and

$$C_i\Gamma_j = \begin{cases} C_i\Gamma_0 & i \leq np-j \\ 0 & i > np-j \end{cases} .$$

We shall find it convenient to omit the tensor product sign in writing elements of $C_*\Gamma_j$, so that, for example, $x^{p-1}dx$ denotes $x \otimes x \otimes \cdots \otimes x \otimes dx$. Let $W = S^\infty$ with the usual π-equivariant cell structure. Then C_*W is the minimal resolution $\mathscr{W}$ of Z over $Z[\pi]$. Let

$$\mathscr{W}(k)_j = \begin{cases} \mathscr{W}_j & j \leq k \\ 0 & j > k \end{cases}$$

so that $\mathscr{W}(k) = C_*(W^k)$, where W^k is the k-skeleton of W. Then by I.2.1, $C_*(W^k \ltimes_\pi \Gamma_i) \cong \mathscr{W}(k) \otimes_\pi C_*\Gamma_i$.

Let α be the p-cycle $(1\ 2 \cdots p)$ in $\pi \subset \Sigma_p$, and let π and Σ_p act on $C_*\Gamma_i$ by permuting factors. Following [68, Theorem 3.1] we define elements $t_i \in C_*\Gamma_0$ as follows. Define a contracting homotopy for $C_*\Gamma_0$ by $s(ax) = 0$ and $s(adx) = (-1)^{|a|}ax$.

<u>Definition 3.1</u>. If p = 2, let $t_0 = dx^2$, $t_1 = xdx$, and $t_2 = x^2$. If $p > 2$, let $N = 1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}$. Let

$t_0 = dx^p$, $t_1 = dx^{p-1}x$,
$t_{2i} = s((\alpha^{-1} - 1)t_{2i-1})$, and
$t_{2i+1} = s(Nt_{2i})$.

<u>Lemma 3.2</u>. (i) If p = 2 then $d(t_2) = (\alpha + (-1)^n)t_1$ and $d(t_1) = t_0$.

(ii) If $p > 2$ then $d(t_1) = t_0$,
$d(t_{2i}) = (\alpha^{-1} - 1)t_{2i-1}$
and $d(t_{2i+1}) = Nt_{2i}$ if $i > 0$.

(iii) If $p > 2$ then $t_p = (-1)^{mn} m! x^p$ and

$$t_{p-1} = m! x^{p-1} dx + (m-1)!(\alpha^{-1} - 1) Q x^{p-1} dx$$

where $\quad m = (p-1)/2$ and $Q = (\alpha+1) \sum_{i=1}^{m} i\alpha^{2i}$.

Proof. (i) and (ii) are easy calculations, by induction on i for $d(t_{2i})$ and $d(t_{2i+1})$ using $(\alpha^{-1}-1)N = 0 = N(\alpha^{-1}-1)$ and $ds + sd = 1$.

In [68,Theorem 3.1] it is shown that $t_p = (-1)^{mn} m! x^p$ and that $t_{p-1} = (m-1)! P x^{p-1} dx$, where $P = \alpha + \alpha^3 + \cdots \alpha^{p-2}$. Since $P = m + (\alpha^{-1} - 1)Q$, (iii) follows.

Lemma 3.3. If $p = 2$, then in $C_*(W^{i+1} \ltimes_\pi \Gamma_1)$

$$e_{i+1} \otimes dx^2 \sim \begin{cases} (-1)^i e_i \otimes d(x^2) & n \not\equiv i \ (2) \\ (-1)^i e_i \otimes d(x^2) - 2e_i \otimes xdx & n \equiv i \ (2) \end{cases}$$

Proof. We have $d(e_i) = (\alpha + (-1)^i)e_{i-1}$ and $d(x^2) = dx\, x + (-1)^n x\, dx$. Therefore

$$\begin{aligned} d(e_{i+1} \otimes xdx) &= (\alpha + (-1)^{i+1})e_i \otimes xdx + (-1)^{i+1} e_{i+1} \otimes dx^2 \\ &= e_i \otimes dx\, x + (-1)^{i+1}\, e_i \otimes xdx + (-1)^{i+1} e_{i+1} \otimes dx^2 , \end{aligned}$$

from which we obtain

$$\begin{aligned} e_{i+1} \otimes dx^2 &\sim (-1)^i e_i \otimes dx\, x - e_i \otimes xdx \\ &= (-1)^i e_i \otimes d(x^2) - (1 + (-1)^{i+n}) e_i \otimes xdx . \end{aligned}$$

Lemma 3.4. Let $p > 2$. If i is odd then, in $C_*(W^{i+p-1} \ltimes_\pi \Gamma_i)$,

$$e_{i+p-1} \otimes dx^p \sim (-1)^{mn+m}\, m! e_i \otimes d(x^p).$$

If i is even then, in $C_*(W^{i+p-1} \ltimes_\pi \Gamma_1)$,

$$e_{i+p-1} \otimes dx^p \sim (-1)^{mn+m} m!\ e_i \otimes d(x^p) - p \sum_{j=1}^{p-1} (-1)^{[j/2]} e_{i+p-j-1} \otimes t_j .$$

Hence, for any i,

$$e_{i+p-1} \otimes dx^p \sim (-1)^{mn+m}\ m!\ e_i \otimes d(x^p)$$

in $C_*(W^{i+p-1} \ltimes_\pi \Gamma_1, Z_p)$.

Proof. By Lemma 3.1 and the definition of $\mathcal{N}$ we find that if i is even then

$$d(e_{i+p-j} \otimes t_j) = \begin{cases} N(e_{i+p-j-1} \otimes t_j + e_{i+p-j} \otimes t_{j-1}) & j \text{ odd}, j \neq 1 \\ T(e_{i+p-j-1} \otimes t_j - e_{i+p-j} \otimes t_{j-1}) & j \text{ even} \\ Ne_{i+p-2} \otimes t_1 + e_{i+p-1} \otimes t_0 & j = 1 \end{cases}$$

and if i is odd then

$$d(e_{i+p-j} \otimes t_j) = \begin{cases} Te_{i+p-j-1} \otimes t_j - Ne_{i+p-j} \otimes t_{j-1} & j \text{ odd}, j \neq 1 \\ Ne_{i+p-j-1} \otimes t_j + Te_{i+p-j} \otimes t_{j-1} & j \text{ even} \\ Te_{i+p-2} \otimes t_1 - e_{i+p-1} \otimes t_0 & j = 1\,, \end{cases}$$

where $N = 1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}$ and $T = \alpha - 1$.

Suppose i is odd. We define

$$c = \sum_{j=1}^{m} (-1)^{j-1}(e_{i+p-2j+1} \otimes t_{2j-1} - e_{i+p-2j} \otimes t_{2j}).$$

A routine calculation then shows that

$$d(c) = -e_{i+p-1} \otimes t_0 + (-1)^m e_i \otimes Nt_{p-1}\,,$$

and hence, by Lemma 3.2.(ii) and (iii)

$$e_{i+p-1} \otimes t_0 \sim (-1)^m e_i \otimes Nt_{p-1} = (-1)^m e_i \otimes d(t_p) = (-1)^{mn+m}\, m!\; e_i \otimes d(x^p).$$

This establishes the result for odd i.

Now suppose i is even. We define

$$c = \sum_{j=1}^{m} (-1)^{j-1}(Me_{i+p-2j} \otimes t_{2j} + e_{i+p-2j+1} \otimes t_{2j-1})$$

where $M = \alpha^{p-2} + 2\alpha^{p-3} + \cdots + (p-2)\alpha + (p-1)$. One easily checks that $N = TM + p = MT + p$. A routine calculation then shows that

$$d(c) = e_{i+p-1} \otimes t_0 + p\sum_{j=1}^{m} (-1)^{j-1}(e_{i+p-2j} \otimes t_{2j-1} - e_{i+p-2j-1} \otimes t_{2j})$$
$$- (-1)^m e_i \otimes Nt_{p-1}\,,$$

from which the result follows for even i by Lemma 3.2.(ii) and (iii) just as for odd i.

In order to prove the compression result (Lemma 3.6) we need to show that, ignoring the Σ_p action, $\Gamma_i(x)$ is just a wedge of suspensions of $X^{(p)}$.

Lemma 3.5. In $\overline{h}\mathcal{J}$ or $\overline{h}\mathcal{S}$, $\Gamma_i(X) \simeq \bigvee_{(p-i,i-1)} \Sigma^{np-i}X^{(p)}$.

Proof. By Definition 2.1 and Lemma 2.2.(i) we may assume $X = S^0$. Again let $\Gamma_i = \Gamma_i(S^0)$. Since $\Gamma_0 = e^{np}$ is contractible, $C_*\Gamma_0$ is exact. It follows that $C_*\Gamma_i$ is exact except in dimension np-i and that

$$H_k\Gamma_i = \begin{cases} 0 & k \neq np-i \\ \ker(C_{np-i}\Gamma_0 \to C_{np-i-1}\Gamma_0) & k = np-i \end{cases}$$

Thus $H_{np-i}\Gamma_i$ is free abelian, being a subgroup of the free abelian group $C_{np-i}\Gamma_0$. By the Hurewicz and Whitehead theorems Γ_i is a wedge of np-i spheres. Splitting $C_*\Gamma_0$ into short exact sequences shows that

$$\text{rank } H_{np-i}\Gamma_i + \text{rank } H_{np-i-1}\Gamma_{i+1} = \text{rank } C_{np-i}\Gamma_0 = (p-i,i).$$

(Recall $(a,b) = (a+b)!/a!b!$). Since $H_{np-1}\Gamma_1$ has rank 1 by Lemma 2.2(v), we see by induction on i that

$$\text{rank } H_{np-i}\Gamma_i = (p-i,i-1).$$

We are now prepared to prove the key result.

Lemma 3.6. The natural inclusion $W^{i+1} \ltimes_\pi \Gamma_{j+1} \to W^{i+1} \ltimes_\pi \Gamma_j$ is homotopic to a map $e:W^{i+1} \ltimes_\pi \Gamma_{j+1} \to W^i \ltimes_\pi \Gamma_j$. In integral homology $e = ee \cdots e:W^{i+p-1} \ltimes_\pi \Gamma_p \to W^i \ltimes_\pi \Gamma_1$ satisfies

(i) $\quad e_*(e_{i+p-1} \otimes (dx)^p) = (-1)^{mn+m} m! e_i \otimes d(x^p)$ if $p > 2$ and i is odd,

(ii) $\quad e_*(e_{i+1} \otimes (dx)^2) = (-1)^i e_i \otimes d(x^2)$ if $p = 2$ and $n \not\equiv i$ (2),

where we denote homology classes by representative cycles. In mod p homology, (i) and (ii) hold for all i and n. In integral homology $e:W^{p-1} \ltimes_\pi \Gamma_p \to W^0 \ltimes_\pi \Gamma_2 \simeq \Gamma_2$ satisfies

(iii) $\quad e_*(e_{p-2} \otimes (dx)^p) = (-1)^{m-1} Te_0 \otimes t_{p-2}$ if $p > 2$.

Proof. The map compresses because $W^{i+1} \ltimes_\pi \Gamma_{j+1}$ is np+i-j dimensional while $W^{i+1} \ltimes_\pi \Gamma_j / W^i \ltimes_\pi \Gamma_j \simeq \bigvee S^{np+i-j+1}$ by the preceding lemma. In order to evaluate e_*, first assume $p > 2$ and consider the commutative triangle,

$$\begin{array}{ccc} W^{i+p-1} \ltimes_{\pi} \Gamma_p & \longrightarrow & W^{i+p-1} \ltimes_{\pi} \Gamma_1 \\ & \searrow^{e} & \uparrow \\ & & W^{i} \ltimes_{\pi} \Gamma_1 \end{array}$$

in which the unlabelled maps are the natural inclusions. In mod p homology the vertical map is an isomorphism, so it suffices to note that $e_{i+p-1} \otimes dx^p \sim (-1)^{mn+m} m! e_i \otimes d(x^p)$ by 3.4. Now assume i is odd. The vertical map is the quotient map $Z \to Z_p$, and the mod p case implies e_* is correct up to a multiple of p. The indeterminacy of the lift from $W^{i+1} \ltimes_{\pi} \Gamma_1$ to $W^{i} \ltimes_{\pi} \Gamma_1$ consists of maps

$$W^{i+p-1} \ltimes_{\pi} \Gamma_p \xrightarrow{c} S^{np+i-1} \xrightarrow{b} S^{np+i-1} \xrightarrow{a} W^{i} \ltimes_{\pi} \Gamma_1$$

in which c is projection onto the top cell, b is arbitrary, and a is the attaching map of the np+i cell. On integral homology c_* is the identity and a_* is multiplication by p. Thus it is possible to choose the lift e such that e_* is as stated in integral homology. (This is a general fact about maps obtained by cellular approximation, but we only need it here so do not bother with the general statement.)

The argument for p = 2 is exactly analogous to that just given.

§4. Reduction to three cases

In this section we start with an overview of the proof, then establish notations which we shall use in the remainder of this chapter, and finally start the proof of Theorems 1.1, 1.2 and 1.3 by showing that it splits into three parts and by proving some results which will be used in all three.

If $\Gamma_j = \Gamma_j(S^{n-1})$ as in Section 2, we would like to prove Theorems 1.1, 1.2 and 1.3 by doing appropriate calculations in a spectral sequence $E_r(S, \mathcal{D})$ where $\mathcal{D}$ is an inverse sequence constructed from the $W^i \ltimes_{\Sigma_p} \Gamma_j$'s. However, there are technical difficulties which have prevented this. If a proof can be constructed along these lines, it should immediately imply that T_p (see Theorem 1.2) is a linear combination of $\beta^{\delta} P^{j-i} x$ and $x^{p-k}(d_r x)^k$ for various δ, i and k, with coefficients in $E_2(S,S)$. The coefficient of the lowest filtration term would be $\bar{a}$, and the determination of the other coefficients would give complete information on the first possible nonzero differential on $\beta^{\varepsilon} P^j x$.

The proof we give runs as follows. The spectrum $W \ltimes_{\Sigma_p} \Gamma_j$ is a wedge summand of $W \ltimes_{\pi} \Gamma_j$, $\pi \subset \Sigma_p$ cyclic of order p. In a very convenient abuse of notation, we will write $D^i \Gamma_j$ for the np + i-j skeleton of this summand. There is a homotopy equivalence of (e^{k+np}, S^{k+np-1}) with $(D^k \Gamma_0, D^{k-1}\Gamma_0 \cup D^k \Gamma_1)$. The element $\beta^{\varepsilon} P^j x$ is represented by a map of $(D^k\Gamma_0, D^{k-1}\Gamma_0 \cup D^k\Gamma_1)$ into the Adams resoluton of our H_∞ ring spectrum Y. Thus, we must study lifts of the boundary $D^{k-1}\Gamma_0 \cup D^k\Gamma_1$ in order to compute $d_*\beta^\varepsilon P^j x$. Since $D^k\Gamma_1$ is homotopy equivalent to the stunted lens space $\Sigma^n L^{n(p-1)+k}_{n(p-1)}$ and $D^k\Gamma_0$ is the cone on $D^k\Gamma_1$, $D^{k-1}\Gamma_0 \cup D^k\Gamma_1 \simeq D^k\Gamma_1/D^{k-1}\Gamma_1 \simeq S^{k+np-1}$. Now $D^{k+p-1}\Gamma_p$ is also a stunted lens space and the natural inclusion $D^{k+p-1}\Gamma_p \to D^{k+p-1}\Gamma_1$ factors through $D^k\Gamma_1$ (Lemma 3.6). The resulting map $D^{k+p-1}\Gamma_p \to D^k\Gamma_1$ is equivalent to the cofiber of the inclusion of the bottom cell of $D^{k+p-1}\Gamma_p$. Thus $D^k\Gamma_1/D^{k-1}\Gamma_1 \simeq D^{k+p-1}\Gamma_p/D^{k+p-2}\Gamma_p$. The top cell of $D^{k+p-1}\Gamma_p$ carries the element $\beta^\varepsilon P^j d_r x$ and this is where this term comes from. The other term comes in because we are given a map of $D^{k-1}\Gamma_0 \cup D^k\Gamma_1$, not $D^k\Gamma_1/D^{k-1}\Gamma_1$, into the Adams resolution. Thus we must find another cell whose boundary is the same as the boundary of the top cell of $D^k\Gamma_1$ or $D^{k+p-1}\Gamma_p$, and we must lift it until it detects an element in homotopy or until it has filtration higher than that of $\beta^\varepsilon P^j d_r x$. Since $D^i\Gamma_0 \cong CD^i\Gamma_1$, we can simply cone off the attaching map of the top cell of $D^k\Gamma_1$ as long as this cell is nontrivially attached. This produces the terms $\overline{a}P^{j-v}x$, $\overline{a}\beta P^{j-e-1}x$ and $a_0\beta P^j x$. If the top cell of $D^k\Gamma_1$ is unattached, the top cell of $D^{k+p-1}\Gamma_p$ may still be attached to the cell $D^{p-2}\Gamma_p$. There is a nullhomotopy of this cell in Γ_1 which carries $x^{p-1}d_r x$. This is the source of the terms $\overline{a}x^{p-1}d_r x$. Finally, when the top cell of $D^{k+p-1}\Gamma_p$ is unattached, it carries the entire boundary.

There are two complications to the above picture. First, the map $D^{k+p-1}\Gamma_p \to D^k\Gamma_1$ is a lift of the natural inclusion $D^{k+p-1}\Gamma_p \to D^{k+p-1}\Gamma_1$ and does not commute with the maps into the Adams resolution until we pass to a lower filtration. This necessitates extra work at some points. Second, the attaching map ataches the top cell to the whole lens space, not just to the cell carrying $P^{j-v}x$ or $\beta P^{j-e-1}x$. As the filtration of $\overline{a}$ increases, the possibility arises that a piece of the attaching map which attaches to a lower cell will show up in a lower filtration than the term $\overline{a}P^{j-v}x$ or $\overline{a}\beta P^{j-e-1}x$. This possibility accounts for the cases in which we do not have complete information.

Now let us establish notation to be used in this and the remaining sections. As in section 1 we assume given a p-local H_∞ ring spectrum Y and an element $x \in E_r^{s,n+s}(S,Y)$, the E_r term of the ordinary Adams spectral sequence converging to π_*Y. We wish to describe the first nontrivial differential on $\beta^\varepsilon P^j x$ in terms of x and $d_r x$. (Here $\varepsilon = 0$ if $p = 2$.) Recall from §1 the definition

$$k = \begin{cases} j-n & p = 2 \\ (2j-n)(p-1) - \varepsilon & p > 2 \end{cases}$$

Let

$$Y \simeq Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \cdots$$

be an Adams resolution of Y and let

$$Y^{(p)} \simeq Y_0^{(p)} = F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$$

be its p^{th} power as in IV.4. Represent x by a map $(e^n,S^{n-1}) \to (Y_s,Y_{s+r})$ and let $\Gamma_i = \Gamma_i(S^{n-1})$ be the i^{th} filtration of $\Gamma_0 = e^{np}$ as in Definition 2.1. Recall that the spectrum $W \ltimes_{\Sigma_p} \Gamma_i$ is a wedge summand of $W \ltimes_\pi \Gamma_i$ where $\pi \subset \Sigma_p$ is cyclic of order p. In the remainder of this chapter, $D^k\Gamma_i$ will denote the np+k-i skeleton of this summand. Let us use ξ generically to denote the composites

$$\xi_{k,ps+ir}(1 \ltimes x^p):D^k\Gamma_i \to W^k \ltimes_\pi \Gamma_i \to W^k \ltimes_\pi F_{ps+ir} \to Y_{ps+ir-k},$$

the maps of pairs and unions constructed from them, and their composites with the maps $Y_{j+t} \to Y_j$. We will use the following consequence of Lemma 3.6 repeatedly. Recall that e is defined in Lemma 3.6.

<u>Lemma 4.1</u>. The following diagram commutes.

$$\begin{array}{ccc}
D^{k+p-1}\Gamma_p & \xrightarrow{\ e\ } & D^k\Gamma_1 \\
 & & \downarrow \xi \\
\xi \downarrow & & Y_{ps+r-k} \\
 & & \downarrow \\
Y_{ps+pr-k-p+1} & \longrightarrow & Y_{ps+r-k-1}
\end{array}$$

<u>Proof</u>. In the diagram below, the triangle commutes because $r \geq 1$ and the quadrilateral commutes by Lemma 3.6.

$$\begin{array}{ccc}
D^{k+1}\Gamma_{j+1} & \longrightarrow & D^k\Gamma_j \\
\downarrow & & \downarrow \\
Y_{ps+(j+1)r-k-1} & & Y_{ps+jr-k} \\
\downarrow & & \downarrow \\
Y_{ps+(j+1)r-k-2} & \longrightarrow & Y_{ps+jr-k-1}
\end{array}$$

The lemma follows by composing the diagrams for $j = 1,2,\ldots,p-1$.

In IV.2 we constructed a chain homomorphism $\Phi: \mathcal{N} \otimes \mathcal{C}^p \to \mathcal{C}$, where is the cobar construction, which we used to construct Steenrod operations, and in IV.5 we showed that ξ induces such a homomorphism. In particular, Definition IV.2.4 says

$$\beta^{\varepsilon}P^j x = (-1)^j \nu(n)\Phi_*(e_k \otimes x^p) \qquad p > 2$$

and
$$Sq^j x = \Phi_*(e_k \otimes x^2) \qquad p = 2.$$

The following relative version of Corollary IV.5.4 gives us maps which represent these elements. In it we let $\mathcal{C}$ be the cobar construction $C(Z_p, \mathcal{A}_p, H_*Y)$ so that $\mathcal{C}_{s,n+s} \cong \pi_n(Y_s/Y_{s+1}) \cong \pi_n(Y_s, Y_{s+1})$ and let $\mathcal{W} = C_*(W)$ so that $\mathcal{W}_k = C_k(W) \cong \pi_k(W^k/W^{k-1}) \cong \pi_k(W^k, W^{k-1})$.

<u>Lemma 4.2</u>. If $e \in \mathcal{W}_k$ is represented by $e \in \pi_k(W^k, W^{k-1})$ then $\Phi_*(e \otimes x^p)$ is represented by the composite

$$\begin{array}{ccc}
(e^{np+k}, S^{np+k-1}) & \xrightarrow{\Phi_*(e\otimes x^p)} & (Y_{ps-k}, Y_{ps-k+1}) \\
\wr\!\!\mid & & \\
(e^k \ltimes \Gamma_0, e^k \ltimes \Gamma_1 \cup S^{k-1} \ltimes \Gamma_0) & & \\
\downarrow e \ltimes 1 & & \uparrow \xi \\
(W^k \ltimes \Gamma_0, W^k \ltimes \Gamma_1 \cup W^{k-1} \ltimes \Gamma_0) & & \\
\downarrow u & & \\
(W^k \ltimes_\pi \Gamma_0, W^k \ltimes_\pi \Gamma_1 \cup W^{k-1} \ltimes_\pi \Gamma_0) & \xrightarrow[1 \ltimes_\pi x^p]{} & (W^k \ltimes_\pi F_{ps}, W^k \ltimes_\pi F_{ps+r} \cup W^{k-1} \ltimes_\pi F_{ps})
\end{array}$$

where u is the passage to orbits map.

<u>Note</u>: If $e \in \mathcal{W}_k$ is a $Z[\pi]$ generator (e.g. $e = \alpha^i e_k$ for some i) then the vertical composite in the diagram is an equivalence by the same argument which was used to construct diagrams (2.1) and (2.2).

<u>Proof</u>. This is simply the relative version of Corollary IV.5.4. The natural isomorphism $\pi_*(X,A) \cong \pi_*(X/A)$ for cofibrations $A \to X$ enable one to pass freely between this version and the absolute version of IV.5.4.

We shall refer to the boundary of the map in Lemma 4.2 so frequently that we give it a name.

<u>Definition 4.3</u>. Let $\partial\Phi \in \pi_{np+k-1}Y_{ps-k+1}$ be the restriction to S^{np+k-1} of the map $\Phi_*(e_k \otimes x^p)$ of Lemma 4.2. Let $\iota \in \pi_{np+k-1}(D^k\Gamma_1 \cup D^{k-1}\Gamma_0)$ be the map with Hurewicz image

$$(-1)^k e_k \otimes d(x^p) + \begin{cases} 0 & k=0 \text{ or } k \text{ odd}, \ p > 2 \\ 0 & k+n \text{ odd}, \ p = 2 \\ pe_{k-1} \otimes x^p & 0 \neq k \text{ even}, \ p > 2 \\ (-1)^k 2e_{k-1} \otimes x^2 & k+n \text{ even}, \ p = 2 \end{cases}$$

Lemma 4.4. (i) $\partial\Phi = \xi_*(\iota)$

(ii) ι is an equivalence

(iii) Orienting the top cell of $D^k\Gamma_1$ correctly, the homotopy class ι contains the map a_3 of diagram (2.2).

Proof (i) holds because we are in the Hurewicz dimension of $D^k\Gamma_1 \cup D^{k-1}\Gamma_0 \simeq S^{np+k-1}$ so the Hurewicz image of ι is sufficient to determine ι, and its Hurewicz image is the boundary of the cell $e_k \otimes x^p$. Statement (ii) is immediate from the Hurewicz isomorphism, and statement (iii) is immediate from the fact that a_3 is an equivalence.

The differentials on $\beta^\varepsilon P^j x$ are given by the successive lifts of $(-1)^j \nu(n)\partial\Phi$ when $p > 2$, and of $\partial\Phi$ when $p = 2$. Corollary 2.8 and the discussion following it show that the attaching maps of lens spaces, and hence elements of Im J, enter into the question of lifting this boundary. In the remainder of this section we establish various facts about the numerical relations between the filtrations and dimensions involved, the last of which will enable us to split our proof into three very natural special cases.

Lemma 4.5. If $p > 2$, the generator of Im J in dimension $jq-1$ has filtration $\leq j$. If $p = 2$ the generator of Im J in dimension $8a+\varepsilon$ $(\varepsilon = 0,1,3,7)$ has filtration $\leq 4a+\varepsilon$ if $\varepsilon \neq 7$, and $\leq 4a+4$ if $\varepsilon = 7$.

Proof. The vanishing theorem for $\mathrm{Ext}_{\mathcal{A}_p}(Z_p,Z_p)$ says that $\mathrm{Ext}^{st} = 0$ if $0 < t-s < U(s)$, where $U(s) = qs-2$ if $p > 2$ and

$$U(4a+\varepsilon) = \begin{cases} 8a-1 & \varepsilon = 0 \\ 8a+1 & \varepsilon = 1 \\ 8a+2 & \varepsilon = 2 \\ 8a+3 & \varepsilon = 3 \end{cases}$$

if $p = 2$ by [4] and [56]. First suppose $p > 2$. The Im J generator in dimension $jq-1$ is detected by an element of $\mathrm{Ext}^{s,t}$ where $t-s = jq-1$. Hence $jq-1 \geq U(s) = sq-2$, which implies $j \geq s$. Now, suppose $p = 2$. A trivial calculation shows that if

$s > 4a + \varepsilon$, $\varepsilon = 0,1,3,4$, then $U(s) > 8a + \varepsilon$ if $\varepsilon \neq 4$, $8a + 7$ if $\varepsilon = 4$. This immediately implies the lemma.

We apply this to prove the following three lemmas. As in §1 let v be $v_p(k + n(p-1))$, and let f be the Adams filtration of the generator of Im J in $\pi_{v-1}S^0$.

__Lemma 4.6.__ Assume $p > 2$. If $v = k+1$ and $f \geq r-1$ then $pr-p-k+1 < 2r-1$.

__Proof.__ Equivalently, we must show $k > (p-2)(r-1)$. By Lemma 4.5

$$f \leq \frac{v}{q} = \frac{k+1}{q} .$$

Thus $k+1 \geq qf \geq q(r-1)$ and hence it is sufficient to show that $q(r-1) - 1 > (p-2)(r-1)$. This is immediate since $r > 1$.

__Lemma 4.7.__ Either $\min\{pr-p+1, v+f\} < v+r-1$ or $r = p = 2$ and $v = 1$ or 2.

__Proof.__ Suppose $p > 2$. Then $f \leq v/q$. If $pr-p+1 \geq v+r-1$ then $v \leq (p-1)(r-1) + 1$ and hence

$$f \leq \frac{r-1}{2} + \frac{1}{q} < r-1.$$

Now suppose $p = 2$. We must show that if $r \geq v$ then $f < r-1$. It suffices to show $f < v-1$. This follows from Lemma 4.5 except when $v = 1,2$, or 4. In these cases $f = 1$ so the lemma holds when $v = 4$. If $v = 1$ or 2 then $f < r-1$ unless $r = 2$. This completes the lemma.

__Lemma 4.8.__ Exactly one of the following holds:

(a) $v > k + p-1$,

(b) $v = k+1$ and if $p > 2$ then n is even,

(c) $v \leq k$.

__Proof.__ There is nothing to prove if $p = 2$, so assume $p > 2$. We must show that if $k < v \leq k+p-1$ then $v = k+1$ and n is even. Recall that $k = (2j-n)(p-1)-\varepsilon$ and $v = v_p(k+n(p-1)) = v_p(2j(p-1)-\varepsilon)$. If $\varepsilon = 0$ then $v = 1$. Hence $k = 0$ and $n = 2j$ so that (b) holds as required. If $\varepsilon = 1$ then $v = q(1 + \varepsilon_p(j))$. Dividing the inequalities $k < v \leq k+p-1$ by $p-1$ yields

$$2j-n-\frac{1}{p-1} < 2(1+\varepsilon_p(j)) \leq 2j-n - \frac{1}{p-1} + 1$$

which has only one solution: $2(1 + \varepsilon_p(j)) = 2j-n$. Hence n is even and $v = q(1+\varepsilon_p(j)) = (2j-n)(p-1) = k+1$.

Lemma 4.8 is a consequence of the splitting of the mod p lens space into wedge summands, the summand of interest to us being the Σ_p extended power of a sphere. To see the relation, recall that v tells us how far we can compress the attaching map of the top cell of $W^k \ltimes_\pi \Gamma_1 \simeq \Sigma^{n-1} \tilde{L}^{n(p-1)+k}_{n(p-1)}$. When $v \leq k$, it compresses to $W^{k-v} \ltimes_\pi \Gamma_1$ and no further. When $v > k$ it is not attached to $W^k \ltimes_\pi \Gamma_1$. However, recall that there are equivalences

$$\begin{array}{ccc} W^{k+p-1} \ltimes_\pi \Gamma_p & \simeq & \Sigma^{n-1} \tilde{L}^{n(p-1)+k}_{(n-1)(p-1)} \\ \big\downarrow & & \big\downarrow \\ W^k \ltimes_\pi \Gamma_1 & \simeq & \Sigma^{n-1} \tilde{L}^{n(p-1)+k}_{n(p-1)} \end{array}$$

by Corollary 2.6, and that the top cell of $W^k \ltimes_\pi \Gamma_1$ is the image of the top cell of $W^{k+p-1} \ltimes_\pi \Gamma_p$ by Lemma 3.6. When $v > k$ this cell compresses to $W^{p-2} \ltimes_\pi \Gamma_p$. The first possibility is that it goes no further, and in this case the wedge summand of the lens space we are interested in has cells in dimensions $n(p-1)$ and $n(p-1)-1$ so that n must be even. By the splitting of the lens space into wedge summands, the next possibility is $v = k+p-1$, which would have the top cell of $W^{k+p-1} \ltimes_p \Gamma_p$ attached to the bottom cell. In fact this cannot happen because the attaching map is in Im J and thus is not in an even stem. So $v > k+p-1$ is the only possibility if $v > k+1$, and this says that top cells of $W^{k+p-1} \ltimes_\pi \Gamma_p$ and $W^k \ltimes_\pi \Gamma_1$ are unattached. This "geometry" explains why the differentials on $\beta^\varepsilon P^j x$ are so different in these three cases. We shall start with the simplest of the three cases, and proceed to the most complicated.

§5. Case (a): $v > k+p-1$

Since $v > k+p-1 \geq 1$, it follows that $\varepsilon = 1$ if $p > 2$. Thus Theorems 1.1 and 1.2 say that

$$d_{2r-1} P^j x = P^j d_r x \qquad \text{if } p = 2$$

and

$$d_{pr-p+1} \beta P^j x = -\beta P^j d_r x \quad \text{if } p > 2.$$

Theorem 1.3 follows automatically from these facts, so these are what we shall establish.

By Lemma 4.1, the following diagram commutes.

$$\begin{array}{ccccc}
D^{k+p-1}\Gamma_p & \xrightarrow{e} & D^k\Gamma_1 & \longrightarrow & D^k\Gamma_1 \cup D^{k-1}\Gamma_0 \\
 & & \downarrow \xi & & \\
\downarrow \xi & & E_{ps+r-k} & & \downarrow \xi \\
 & & \downarrow & & \\
E_{ps+pr-k-p+1} & \longrightarrow & E_{ps+r-k+1} & \longrightarrow & E_{ps-k+1}
\end{array}$$

Because $v > k+p-1$, the top cell of $D^{k+p-1}\Gamma_p$ is not attached (Corollary 2.6 and Definition V.2.15). Thus there exists a reduction $\rho \in \pi_{np+k-1}(D^{k+p-1}\Gamma_p)$ whose Hurewicz image is $e_{k+p-1} \otimes dx^p$ (it is easy to check that $e_{k+p-1} \otimes dx^p$ generates H_{np+k-1}). Also, $v > k+p-1 \geq 1$ immplies that k is odd if $p > 2$ and that k+n is odd if $p = 2$ by Proposition V.2.16. Combining Lemmas 3.6 and 4.4 we find that $\xi_*(\rho)$ is a lift of $\partial\Phi$ when $p = 2$, and of $(-1)^{mn+m-1}m!\partial\Phi$ when $p > 2$. Applying Lemma 4.2 or Corollary IV.5.4 we see that $\xi_*(\rho)$ represents $\Phi_*(e_{k+p-1} \otimes dx^p)$. Thus, if $p = 2$ we have

$$d_{2r-1}P^j x = \xi_*(\rho) = \Phi_*(e_{k+1} \times dx)^2 = P\, d^j x_r.$$

If $p > 2$, we have

$$d_{pr-p+1}\beta P^j x = (-1)^j \nu(n)(-1)^{mn+m-1} \frac{1}{m!} \xi_*(\rho)$$

$$= (-1)^{mn+m-1}(\nu(n)/m!\,\nu(n-1))\beta P^j d_r x.$$

It is easy to check that $\nu(n)/m!\,\nu(n-1) \equiv (-1)^{mn+m} \bmod p$ so that $d_{pr-p+1}\beta P^j x = -\beta P^j d_r x$.

§6. Case (b): v = k+1

We will begin by considering $p = 2$. Theorems 1.1 and 1.2 say that

$$\begin{array}{ll}
d_{2r-1}P^j x = P^j d_r x & \text{if } 2r-1 < r+f+k, \\
d_{2r-1}P^j x = P^j d_r x + \overline{a}x d_r x & \text{if } 2r-1 = r+f+k, \text{ and} \\
d_{r+f+k}P^j x = \overline{a}x d_r x & \text{if } 2r-1 > r+f+k.
\end{array}$$

Since the filtration f of $\overline{a}$ is positive and $r \geq 2$, Theorem 1.3 follows from Theorems 1.1 and 1.2.

Let $N = k+2n-1$ and let $C_2 \in \pi_N(D^{k+1}\Gamma_2, \Gamma_2)$ be the top cell of $D^{k+1}\Gamma_2$ with its boundary compressed as far as it will go. Then the Hurewicz image

$h(C_2) = e_{k+1} \otimes dx^2$ and $\partial C_2 = a = a_2(k+n) \in \pi_{N-1}\Gamma_2 \cong \pi_k S^0$. Since $\Gamma_2 \simeq S^{2n-2}$ and $\Gamma_1/\Gamma_2 \simeq S^{2n-1} \vee S^{2n-1}$ by Lemma 2.2, the Hurewicz homomorphisms in

$$\begin{array}{ccc} \pi_{2n-1}(\Gamma_1,\Gamma_2) & \xrightarrow{\ h\ } & H_{2n-1}(\Gamma_1,\Gamma_2) \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ \pi_{2n-2}\Gamma_2 & \xrightarrow{\ h\ } & H_{2n-2}\Gamma_2 \end{array}$$

are isomorphisms. Let $R \in \pi_{2n-1}(\Gamma_1,\Gamma_2)$ satisfy $h(R) = x\,dx = e_0 \otimes x\,dx$ in the notation of §3. Then $\partial R \in \pi_{2n-2}\Gamma_2$ is an equivalence since $h(\partial R) = dx^2 = e_0 \otimes dx^2$. Let a also denote $(Ca,a) \in \pi_N(e^{2n-1},S^{2n-2})$. Let i be the natural inclusion $i:(\Gamma_1,\Gamma_2) \to (D^{k-1}\Gamma_0,\Gamma_2)$ if $k > 0$ and let $i = 1:(\Gamma_1,\Gamma_2) \to (\Gamma_1,\Gamma_2)$ if $k = 0$. Let eC_2 denote $(e,1)_*(C_2) \in \pi_N(D^k\Gamma_1,\Gamma_2)$.

Lemma 6.1: $\partial\Phi = \xi_*(eC_2 \cup iRa)$ in $\pi_N Y_{2s-k+1}$.

Proof. First note that $eC_2 \cup iRa$ is defined since $\partial C_2 = \partial(iRa) = a \in \pi_{N-1}\Gamma_2$. By Lemma 4.4, $\partial\Phi = \xi_*(eC_2 \cup iRa)$ will follow if $eC_2 \cup iRa \in \pi_N(D^k\Gamma_1 \cup D^{k-1}\Gamma_0)$ has Hurewicz image $(-1)^k e_k \otimes d(x^2)$, since $\nu_2(k+n) = k+1$ implies that either $k+n$ is odd or $k = 0$. If $k \neq 0$ then $\pi: D^k\Gamma_1 \cup D^{k-1}\Gamma_0 \to D^k\Gamma_1/D^{k-1}\Gamma_1$ is an equivalence and Lemma 2.7 says that $\pi(eC_2 \cup iRa) = \overline{eC_2} \in \pi_N D^k\Gamma_1/D^{k-1}\Gamma_1$ since iRa factors through $D^{k-1}\Gamma_1$. Then $h(\overline{eC_2}) = e_* h(C_2) = (-1)^k e_k \otimes d(x^2)$ by Lemma 3.6 (since $k+n$ is odd) and we are done. If $k = 0$ then n is even, since $\nu_2(n) = 1$, and $eC_2 \cup Ra \in \pi_{2n-1}\Gamma_1$. Also, $a = -2 \in \pi_{2n-2}S^{2n-2}$ since $h(\partial C_2) = d(e_1 \otimes dx^2) = (\alpha-1)e_0 \otimes dx^2 = -2e_0 \otimes dx^2$. To compute $h(eC_2 \cup Ra)$, project to Γ_1/Γ_2 since $H_{2n-1}\Gamma_1 \to H_{2n-1}\Gamma_1/\Gamma_2$ is the monomorphism which sends $e_0 \otimes d(x^2)$ to $e_0 \otimes x dx + e_0 \otimes dx\, x$. By Lemma 2.7, $\pi(eC_2 \cup Ra): S^{2n-1} \to \Gamma_1 \to \Gamma_1/\Gamma_2$ equals $\overline{eC_2} - \overline{Ra}$ so

$$\begin{aligned} h(\pi(eC_2 \cup Ra)) &= h(\overline{eC_2}) - h(\overline{Ra}) \\ &= e_*(e_1 \otimes dx^2) + 2e_0 \otimes xdx \\ &= e_0 \otimes (dx)x - e_0 \otimes xdx + 2e_0 \otimes xdx \\ &= e_0 \otimes (dx)x + e_0 \otimes xdx. \end{aligned}$$

Therefore $h(eC_2 \cup Ra) = e_0 \otimes d(x^2)$ and we're done, proving Lemma 6.1.

Since $\xi_*\partial C_2 \in \pi_*Y_{2s+2r}$, $\xi_*(eC_2 \cup iRa) = \xi_*(eC_2) - \xi_*(iRa)$ in $\pi_*(Y_{2s-k+1},Y_{2s+2r})$. By Lemma 4.1 (or 3.6), $\xi_*(eC_2)$ and ξ_*C_2 have the same image in $\pi_*(Y_{2s-k+1},Y_{2s+2r})$. Since $h(C_2) = e_{k+1} \otimes dx^2$, $\xi_*C_2 \in \pi_*(Y_{2s-k+2r-1},Y_{2s+2r})$ represents $P^j d_r x$ by Lemma 4.2. Similarly, $h(R) = e_0 \otimes x\, dx$ implies that $\xi_*R \in \pi_*(Y_{2s+r},Y_{2s+2r})$ represents xd_rx, and hence $\xi_*(Ra) \in \pi_*(Y_{2s+r+f},Y_{2s+2r})$ represents $\bar{a}xd_rx$. This completes case (b) when $p = 2$.

When $p > 2$ (and $v = k+1$) we will treat $k = 0$ and $k > 0$ separately. First suppose $k = 0$. Then $v = 1$, $n = 2j$ and $\varepsilon = 0$. Also, $f = 1$, $\bar{a} = a_0 \in E_\infty^{1,1}(S,S)$ and $a \in \pi_0 S$ is the map of degree p. Thus, we must show

$$d_{r+1}x^p = a_0x^{p-1}d_rx.$$

Heuristically this is exactly what one would expect from the fact that $d_rx^p = p(x^{p-1}d_rx)$. That this is too casual is shown by the fact that we have just proved (for $p = 2$) that

$$d_3x^2 = h_0xd_2x + P^nd_2x.$$

The extra term arises because when we lift the map representing $2xd_2x$ to the next filtration, we find also the map representing P^nd_2x which we added in order to replace $xd_2x + (d_2x)x$ by $2xd_2x$. Thus, our task for $p > 2$ is to show the analogous elements can always be lifted to a higher filtration than that in which $a_0x^{p-1}d_rx$ lies. The following lemma will do this for us.

Lemma 6.2. There exists elements

$$C_1 \in \pi_{np-1}\Gamma_1 \qquad Y \in \pi_{np-1}(D^1\Gamma_2,\Gamma_2 \cup D^1\Gamma_3)$$
$$X \in \pi_{np-1}(\Gamma_1,\Gamma_2) \qquad Z \in \pi_{np-1}(D^2\Gamma_3,D^1\Gamma_3 \cup D^2\Gamma_4)$$

such that

$$C_1 = pX + pY + Z \text{ in } \pi_{np-1}(D^1\Gamma_1 \cup D^2\Gamma_2,\Gamma^2 \cup D^1\Gamma_3 \cup D^2\Gamma_4),$$
$$h(C_1) = e_0 \otimes d(x^p), \text{ and}$$
$$h(X) = e_0 \otimes x^{p-1}dx.$$

Proof. Since np-1 is the Hurewicz dimension of all the spectra or pairs of spectra involved, we may define C_1,X,Y and Z by their Hurewicz images. Thus C_1 and X are given, and we let

$$h(Y) = \frac{1}{m}\, e_1 \otimes Qd(x^{p-1})dx - \frac{1}{m!}\, e_1 \otimes t_{p-2}, \text{ and}$$
$$h(Z) = -\frac{1}{m!}\, e_2 \otimes Nt_{p-3}.$$

As in section 3, $N = \sum \alpha^i$ and $Q = (\alpha+1)\sum_{i=1}^{m} i\alpha^{2i}$. We also let $M = \sum i\alpha^{p-i-1}$ and note that $M(\alpha-1) = N-p$. Define

$$C = \frac{1}{m!}(Me_1 \otimes t_{p-1} + e_2 \otimes t_{p-2}) + \frac{p}{m} e_1 \otimes Qx^{p-1}\, dx$$

in $C_*(D^1\Gamma_1 \cup D^2\Gamma_2, \Gamma_1 \cup D^1\Gamma_2 \cup D^2\Gamma_3)$. By Lemma 3.2 it follows that

$$d(C) = h(C_1) - ph(X) - ph(Y) - h(Z)$$

which shows that $C_1 = pX + pY + Z$.

By Lemmas 4.4 and 6.2, $\partial\Phi \in \pi_* Y_{ps+1}$ is the image of $\xi_* C_1 \in \pi_* Y_{ps+r}$. Lemma 6.2 also implies that

$$\xi_* C_1 = p\xi_* X + p\xi_* Y + \xi_* Z$$

in $\pi_*(Y_{ps+r-1}, Y_{ps+2r})$. Since $\xi_* Y \in \pi_*(Y_{ps+2r-1}, Y_{ps+2r})$ and $\xi_* Z \in \pi_*(Y_{ps+3r-2}, Y_{ps+3r-1})$ it follows that $\xi_* C_1 = p\xi_* X$ in $\pi_*(Y_{ps+r-1}, Y_{ps+2r})$ and that $\partial\Phi = p\xi_* X$ in $\pi_*(Y_{ps+1}, Y_{ps+2r})$. Lemma 4.2 implies that $\xi_* X \in \pi_*(Y_{ps+r}, Y_{ps+2r})$ represents $x^{p-1}d_r x$ and hence $p\xi_* X$ lifts to $\pi_*(Y_{ps+r+1}, Y_{ps+2r})$ where it represents $a_0 x^{p-1} d_r x$. Finally, IV.3.1 implies

$$d_{r+1} p^j x = d_{r+1} x^p = a_0 x^{p-1} d_r x.$$

Now suppose that $k > 0$. Then $v = k+1$ is greater than 1 and hence congruent to 0 mod $2(p-1)$ by V.2.16. Also by V.2.16, $\varepsilon = 1$ and $k = (2j-n)(p-1)-\varepsilon$ is therefore odd. Lemma 4.4 then implies $\partial\Phi = \xi_*(\iota)$ with $h(\iota) = -e_k \otimes d(x^p)$. The next three lemmas describe the pieces into which we will decompose $\partial\Phi$. In the first we define an element of π_{np-1} of the cofiber of $e: D^{p-2}\Gamma_p \to \Gamma_1$, which we think of as an element of a relative group $\pi_{np-1}(\Gamma_1, D^{p-2}\Gamma_p)$. In order to specify the image of such an element under the Hurewicz homomorphism, we use the cellular chains of the cofiber in the guise of the mapping cone of $e_*: C_* D^{p-2}\Gamma_p \to C_*\Gamma_1$. That is, we let

$$C_i(\Gamma_1, D^{p-2}\Gamma_p) = C_i\Gamma_1 \oplus C_{i-1}D^{p-2}\Gamma_p$$

with $d(a,b) = (d(a) - e_*(b), -d(b))$.

<u>Lemma 6.3</u>. There exists $R \in \pi_{np-1}(\Gamma_1, D^{p-2}\Gamma_p)$ such that

(i) $h(R) = ((-1)^{m-1}e_0 \otimes t_{p-1},\ e_{p-2} \otimes t_0) \in H_*(\Gamma_1, D^{p-2}\Gamma_p)$

(ii) $h(\partial R) = e_{p-2} \otimes t_0 = e_{p-2} \otimes (dx)^p$, and

(iii) $\partial R \in \pi_{np-2} D^{p-2}\Gamma_p$ is an equivalence.

Proof. Since $d(e_0 \otimes t_{p-1}) = Te_0 \otimes t_{p-2}$ by Lemma 3.2 and $e_*(e_{p-2} \otimes t_0) = (-1)^{m-1}Te_0 \otimes t_{p-2}$ by Lemma 3.6.(iii), and since $d(e_{p-2} \otimes t_0) = 0$, it follows that $((-1)^m e_0 \otimes t_{p-1}, e_{p-2} \otimes t_0)$ is a cycle of $(\Gamma_1, D^{p-2}\Gamma_p)$. Since $\Gamma_1 \simeq S^{np-1}$ and $D^{p-2}\Gamma_p \simeq S^{np-2}$, the Hurewicz homomorphism is onto and R satisfying (i) exists. Now (ii) is obvious since the boundary homomorphism simply projects onto the second factor. Part (iii) is immediate from the fact that $e_{p-2} \otimes t_0$ generates $H_{np-2}D^{p-2}\Gamma_p$.

Now we split R into a piece we want and another piece modulo Γ_2.

Lemma 6.4. There exist $X \in \pi_{np-1}(\Gamma_1,\Gamma_2)$ and $Y \in \pi_{np-1}(D^1\Gamma_2,\Gamma_2)$ such that

(i) $h(X) = (-1)^{m-1}m!e_0 \otimes x^{p-1}dx$, and

(ii) $(i,e)_*(R) = i_*X + j_*Y$ in $\pi_*(D^1\Gamma_1,\Gamma_2)$ where $i:\Gamma_1 \to D^1\Gamma_1$, $j:D^1\Gamma_2 \to D^1\Gamma_1$ and $e:D^{p-2}\Gamma_p \to \Gamma_2$.

Proof. We are working in the Hurewicz dimension of all the pairs involved so it suffices to work in homology. We define X by (i) and define Y by

$$h(Y) = (-1)^{m-1}(m-1)!e_1 \otimes Qd(x^{p-1})dx.$$

On cellular chains, the map $(i,e):(\Gamma_1,D^{p-2}\Gamma_p) \to (D^1\Gamma_1,\Gamma_2)$ induces the homomorphism

$$C_k\Gamma_1 \oplus C_{k-1}D^{p-2}\Gamma_p \longrightarrow C_k\Gamma_1 \xrightarrow{i_*} C_kD^1\Gamma_1 \longrightarrow C_kD^1\Gamma_1/C_k\Gamma_2$$

in which the unlabelled maps are the obvious quotient maps. Thus, denoting equivalence classes by representative elements,

$$\begin{aligned} h((i,e)_*R) &= (-1)^{m-1}e_0 \otimes t_{p-1} \\ &= (-1)^{m-1}m!e_0 \otimes x^{p-1}dx + (-1)^{m-1}(m-1)!Te_0 \otimes Qx^{p-1}dx \end{aligned}$$

by Lemma 3.2. Since

$$d(e_1 \otimes Qx^{p-1}dx) = Te_0 \otimes Qx^{p-1}dx - e_1 \otimes Qd(x^{p-1})dx,$$

it follows that $h((i,e)_*R) = h(i_*X + j_*Y)$.

In our last lemma we split $\partial\Phi$ into two pieces modulo $D^{p-2}\Gamma_p$. Let $N = k+np-1$.

Lemma 6.5. If $v = k+1$ and $k > 0$, and if $C_p \in \pi_N(D^{k+p-1}\Gamma_p, D^{p-2}\Gamma_p)$ is the top cell $(h(C_p) = e_{k+p-1} \otimes dx^p)$ with its boundary compressed as far as possible, then $\partial C_p =$

∂Ra in $\pi_{N-1}D^{p-2}\Gamma_p$ and

$$\partial\Phi = (-1)^{m-1}\frac{1}{m!}\,\xi_*(eC_p \cup iRa) \quad \text{in} \quad \pi_*Y_{ps-k+1}\;.$$

Proof. Since $v = k+1$, the attaching map of the top cell factors through $D^{p-2}\Gamma_p$. Since ∂R is an equivalence by Lemma 6.3.(iii), the definition of $a = a_p(k+n(p-1))$ ensures that $\partial C_p = (\partial R)a = \partial Ra$. Now $D^k\Gamma_1 \cup D^{k-1}\Gamma_0 \simeq D^k\Gamma_1/D^{k-1}\Gamma_1$ and, since $k > 0$, Ra factors through $\Gamma_1 \subset D^{k-1}\Gamma_1$. Hence, in $H_*(D^k\Gamma_1 \cup D^{k-1}\Gamma_0)$,

$$\begin{aligned} h(eC_p \cup iRa) &= h(eC_p) \\ &= e_*(e_{k+p-1} \otimes dx^p) \\ &= (-1)^m m!\,e_k \otimes d(x^p) \end{aligned}$$

by Lemma 3.6 (since k is odd and n is even). By Lemma 4.4, it follows that $\partial\Phi = (-1)^{m-1}\frac{1}{m!}\,\xi_*(eC_p \cup iRa)$.

We are now ready to prove Theorems 1.1, 1.2, and 1.3 in this remaining case ($p > 2$, $v = k+1$, and $k > 0$). We must show that

$$d_*\beta P^j x = -\beta P^j d_r x \dotplus (-1)^e\,\bar{a}\,x^{p-1}d_r x.$$

By Lemma 6.5, $d_*\beta P^j x$ is obtained by lifting

$$(-1)^j\nu(n)\partial\Phi = (-1)^{j+m-1}\nu(n)\,\frac{1}{m!}\,\xi_*(eC_p \cup iRa)$$

from $\pi_*(Y_{ps-k+1})$ to the highest filtration possible. Since $\xi_*(eC_p)$ and $\xi_*(iRa)$ have common boundary in $Y_{ps+pr-p+2}$, $\xi_*(eC_p \cup iRa) = \xi_*(eC_p) - \xi_*(iRa)$ in $\pi_*(Y_{ps-k+1}, Y_{ps+pr-p+2})$. By naturality of ξ, $\xi_*(iRa)$ is the image of

$$\xi_*Ra \in \pi_*(Y_{ps+r}, Y_{ps+pr-p+2})$$

and by Lemma 4.1, $\xi_*(eC_p)$ is the image of

$$\xi_*C_p \in \pi_*(Y_{ps+pr-k-p+1}, Y_{ps+pr-p+2}).$$

Lemma 6.4 implies that $\xi_*R = \xi_*X$ in $\pi_*(Y_{ps+r-1}, Y_{ps+2r-1})$ since ξ_*Y is in filtration $2r-1$ or higher. (Note that since ∂R is mapped into Γ_2 by e in 6.4.(ii), Lemma 4.1 forces us to work modulo filtration $2r-1$, the filtration into which ξ maps $D^1\Gamma_2$.) Thus

$$\xi_*(eC_p \cup iRa) = \xi_*C_p - \xi_*Xa \quad \text{in} \quad \pi_*(Y_{ps-k+1}, Y_{ps+2r-1}),$$

and, since $\bar{a}$ has filtration f, ξ_*Xa comes from $\pi_*(Y_{ps+r+f}, Y_{ps+2r})$. By Lemma 4.6, either $r+f$ or $pr-k-p+1$ is less than $2r-1$, so that at least one of ξ_*C_p and ξ_*Xa is nontrivial in $\pi_*(Y_{ps-k+1}, Y_{ps+2r-1})$ in general. Since $h(C_p) = e_{k+p-1} \otimes dx^p$ and $h(X) = (-1)^{m-1}\,m!e_0 \otimes x^{p-1}dx$, Lemma 4.2 implies that

$$\xi_* C_p \text{ represents } (-1)^j \frac{1}{\nu(n-1)} \beta P^j d_r x\ , \quad \text{and}$$

$$\xi_* Xa \text{ represents } (-1)^{m-1} m!\ \bar{a} x^{p-1} d_r x.$$

It then follows that

$$\begin{aligned} d_* \beta P^j x &= (-1)^j \nu(n) \partial\Phi \\ &= (-1)^{j+m-1} \nu(n) \frac{1}{m!} (\xi_* C_p - \xi_* Xa) \\ &= (-1)^{m-1} \frac{\nu(n)}{\nu(n-1)} \frac{1}{m!} \beta P^j d_r x - (-1)^j \nu(n)\ \bar{a}\ x^{p-1} d_r x \\ &= -\ \beta P^j d_r x \dot{+} (-1)^e\ \bar{a}\ x^{p-1} d_r x \end{aligned}$$

since $\nu(n)/\nu(n-1) \equiv (-1)^m\ m!\ (\text{mod } p)$ and since $v = k+1$ implies $2(e+1)(p-1) = (2j-n)(p-1)$ so that $n = 2(j-e-1)$ and hence

$$-(-1)^j \nu(n) = (-1)^{j+1} (-1)^{j-e-1} = (-1)^e.$$

This completes case (b).

§7. Case (c): $v \leq k$.

In this case the boundary $\partial\Phi$ splits into a piece which represents the same operation (P^j or $\beta^\varepsilon P^j$) on $d_r x$ and another piece which is an operation of lower degree applied to x times an attaching map of a stunted lens space. We begin with the lemma needed to identify this latter piece exactly. Recall the spectral sequence of IV.6, and recall the notations established in §1.

Lemma 7.1. Let $\alpha \in \pi_{k+np-1} D^{k-v} S^{n(p)}$ be the attaching map of the top cell of $D^k S^{n(p)}$ and let f be the filtration of $\rho_*(\alpha) = a_p(k+n(p-1))$, where $\rho: D^{k-v} S^{n(p)} \to S^{k+np-v}$ is projection onto the top cell. Let $\mathcal{D}$ be the sequence

$$D^{k-v} S^{n(p)} \leftarrow D^{k-v-1} S^{n(p)} \leftarrow \cdots \leftarrow S^{n(p)}.$$

In the spectral sequence $E_r(S, \mathcal{D})$ the following hold:

(a) $1 \leq \text{filt}(\alpha) \leq f$,

(b) if $\text{filt}(\alpha) = f$ then α is detected by

$$\bar{a} e_{k-v} + \sum_{i=0}^{k-v-1} c_i e_i$$

for some $c_i \in E_2(S,S)$,

(c) if $p = 2$ and $v \leq 10$ or $p > 2$ and $v \leq pq$ then $\operatorname{filt}(\alpha) = f$ and α is detected by $\overline{a}e_{k-v}$.

Proof. (a) Since $\alpha_* = 0$ in mod p homology, $\operatorname{filt}(\alpha) > 0$. Note that this fact (applied to all the attaching maps of $D^{k-v}S^{n(p)}$) ensures that the spectral sequence can be constructed. Since ρ induces a homomorphism from $E_r(S, \mathcal{B})$ to $E_r(S,S)$, and $\rho_*(\alpha)$ has filtration f, α must have filtration $\leq f$.

(b) By IV.6.1(i), every element has the form

$$\sum_{i=0}^{k-v} c_i e_i$$

for some c_i. If $\operatorname{filt}(\alpha) = f$ then the element detecting α projects to $\overline{a}$ in the Adams spectral sequence of the top cell. Hence $c_{k-v} = \overline{a}$. (In fact this argument shows that if $c_{k-v} \neq 0$ then $\operatorname{filt}(\alpha) = f$ and $c_{k-v} = \overline{a}$.)

(c) Under the stated hypothesis, $\overline{a}e_{k-v}$ is the only element of filtration $\leq f$ in degree $k+np-1$.

To prove Theorems 1.1, 1.2 and 1.3, let us first assume that $v = 1$. Then k is even and $\varepsilon = 0$ if $p > 2$, and $k+n$ is even if $p = 2$. Theorems 1.1 and 1.2 say that

$$d_2 P^j x = h_0 P^{j-1} x \qquad \text{if } p = 2, \text{ and}$$

$$d_2 P^j x = a_0 \beta P^j x \qquad \text{if } p > 2.$$

Theorem 1.3 follows from Theorems 1.1 and 1.2 in this case. The first step is to split the element ι of Definition 4.3 into two pieces. Recall that

$$h(\iota) = (-1)^k (e_k \otimes d(x^p) + pe_{k-1} \times x^p).$$

Lemma 7.2: If $k \geq v = 1$ and $C_1 \in \pi_{k+np-1}(D^k\Gamma_1, D^{k-1}\Gamma_1)$ is the top cell, oriented so that $h(C_1) = (-1)^k e_k \otimes d(x^p)$, there exists $A \in \pi_{k+np-1}(D^{k-1}\Gamma_0, D^{k-1}\Gamma_1)$ such that

$$h(A) = (-1)^{k-1} pe_{k-1} \otimes x^p$$

and $$\iota = C_1 \cup A \in \pi_{k+np-1}(D^k\Gamma_1 \cup D^{k-1}\Gamma_0).$$

Proof. Let $N = k+np-1$. To see that A exists, consider the boundary maps and Hurewicz homomorphisms

$$\begin{array}{ccccc}
\pi_N(D^{k-1}\Gamma_0, D^{k-1}\Gamma_1) & \xrightarrow[\partial]{\cong} & \pi_{N-1}D^{k-1}\Gamma_1 & \xleftarrow[\partial]{} & \pi_N(D^k\Gamma_1, D^{k-1}\Gamma_1) \\
\Big\downarrow h & & \Big\downarrow h & & \Big\downarrow h \ \cong \\
H_N(D^{k-1}\Gamma_0, D^{k-1}\Gamma_1) & \xrightarrow[\partial]{\cong} & H_{N-1}D^{k-1}\Gamma_1 & \xleftarrow[\partial]{} & H_N(D^k\Gamma_1, D^{k-1}\Gamma_1)
\end{array}$$

The isomorphisms are isomorphisms because $D^{k-1}\Gamma_0 \simeq *$ by Lemma 2.4 and because $D^k\Gamma_1/D^{k-1}\Gamma_1 \simeq S^{k+np-1}$. Certainly A exists satisfying $\partial A = \partial C_1$. It follows that

$$\partial(h(A)) = \partial(h(C_1)) = \partial((-1)^{k-1} pe_{k-1} \otimes x^p),$$

showing that $h(A) = (-1)^{k-1} pe_{k-1} \otimes x^p$.

To show that $\iota = C_1 \cup A$, it is enough to show $h(\iota) = h(C_1 \cup A)$, since $D^k\Gamma_1 \cup D^{k-1}\Gamma_0 \simeq S^{k+np-1}$. With $N = k+np-1$, note that $H_N D^{k-1}\Gamma_1 = 0$. This implies that the homomorphism

$$H_N D^k\Gamma_1 \cup D^{k-1}\Gamma_0 \xrightarrow{i_*} H_N(D^k\Gamma_1 \cup D^{k-1}\Gamma_0, D^{k-1}\Gamma_1)$$

is injective, so that we need only show $i_*h(\iota) = i_*h(C_1 \cup A)$. By Lemma 2.7, $i_*h(C_1 \cup A) = h(C_1) - h(A)$ and the result follows.

We now have $\partial\Phi = \xi_*\iota = \xi_*(C_1 \cup A) = \xi_*C_1 - \xi_*A$ modulo $Y_{ps+r-k+1}$ since $\xi_*(D^{k-1}\Gamma_1) \subsetneq Y_{ps+r-k+1}$. Applying Lemma 7.1 we find that ξ_*A represents $(-1)^{k-1}a_0\Phi_*(e_{k-1} \otimes x^p)$ in $\pi_*(Y_{ps-k+2}, Y_{ps+r-k+1})$ (with $a_0 = h_0$ if $p = 2$). Sorting out the constants, we find using Definition IV.2.4 that $-\xi_*A$ contributes $a_0\beta P^j x$, if $p > 2$, and $h_0 P^{j-1}x$, if $p = 2$, to the differential on $P^j x$. Thus, it remains only to show that ξ_*C_1 is in a higher filtration than ξ_*A.

<u>Lemma 7.3</u>. If i_1 and i_2 are the maps

$$\begin{array}{ccc} (D^k\Gamma_1, D^{k-1}\Gamma_1) & \xrightarrow{\ i_1\ } & \\ \downarrow & & (D^{k+1}\Gamma_1, D^{k-1}\Gamma_1 \cup D^{k+1}\Gamma_2) \\ (D^k\Gamma_1, D^{k-1}\Gamma_1 \cup D^k\Gamma_2) & \xrightarrow{\ i_2\ } & \end{array}$$

then there exists X such that $i_{1*}C_1 = p(i_{2*}X)$.

<u>Proof</u>. Since $k+np-1$ is the Hurewicz dimension of the domain and codomain of i_2, it suffices to work in homology. First suppose $p > 2$. We let $h(X) = e_k \otimes x^{p-1}dx$, which is obviously a cycle modulo $D^{k-1}\Gamma_1 \cup D^k\Gamma_2$. Then, in the codomain of i_1 and i_2 we have

$$\begin{aligned} e_k \otimes d(x^p) &= e_k \otimes Nx^{p-1}dx \\ &= Te_k \otimes Mx^{p-1}dx + pe_k \otimes x^{p-1}dx \\ &\sim e_{k+1} \otimes M^{-1}d(x^{p-1})dx + pe_k \otimes x^{p-1}dx \\ &\equiv p\ e_k \otimes x^{p-1}dx, \end{aligned}$$

where $N = \sum \alpha^i$, $T = \alpha - 1$, and $M = \sum_{1}^{p-1} i\alpha^{p-i-1}$. The homology is due to $d(e_{k+1} \otimes Mx^{p-1}dx)$ and the congruence holds modulo $D^{k+1}\Gamma_2 \cup D^{k-1}\Gamma_1$. This implies that $i_{1*}C_1 = pi_{2*}X$.

Now suppose $p = 2$. We again let $h(X) = e_k \otimes xdx$ and again this is obviously a cycle. By Lemma 3.3 we have

$$\begin{aligned}(-1)^k e_k \otimes d(x^2) &\sim e_{k+1} \otimes dx^2 + 2e_k \otimes xdx \\ &\equiv 2e_k \otimes xdx,\end{aligned}$$

where the congruence holds modulo $D^{k+1}\Gamma_2 \cup D^{k-1}\Gamma_1$. This implies that $i_{1*}C_1 = 2i_{2*}X$.

We can now finish the proof of Theorems 1.1-1.3 for $v = 1$. By Lemma 7.3, the image of ξ_*C_1 in $\pi_*(Y_{ps-k+1}, Y_{ps-k+r+1})$ is zero, since it is the image of ξ_*pX, with $\xi_*X \in \pi_*(Y_{ps-k+r}, Y_{ps-k+r+1})$ so that $\xi_*pX \in \pi_*(Y_{ps-k+r+1}, Y_{ps-k+r+1}) = 0$. Thus the entire differential is given by $-\xi_*A$ and we are done.

Now suppose $1 < v \leq k$. Then, since $v = v_p(k+n(p-1))$, Lemma V.2.16 implies that $k+n$ is odd if $p = 2$ and that k is odd and $\varepsilon = 1$ if $p > 2$. Also, by Definition 4.3, $h(\iota) = (-1)^k e_k \otimes d(x^p)$. Let $N = k+np-1$.

<u>Lemma 7.4</u>. If $C_p \in \pi_N(D^{k+p-1}\Gamma_p, D^{k+p-1-v}\Gamma_p)$ is the top cell, oriented so that $h(C_p) = e_{k+p-1} \otimes dx^p$, then there exists $A \in \pi_N(D^{k-v}\Gamma_0, D^{k-v}\Gamma_1)$ such that $\partial A = e_*\partial C_p$ and $\iota \in \pi_N(D^k\Gamma_1 \cup D^{k-1}\Gamma_0)$ is the image of

$$\begin{Bmatrix} (-1)^{k+mn+m} \frac{1}{m!} (eC_p \cup A) & p > 2 \\ \\ eC_2 \cup A & p = 2 \end{Bmatrix} \in \pi_N(D^k\Gamma_1 \cup D^{k-v}\Gamma_0)$$

<u>Proof</u>. To see that A exists consider the following diagram, whose upper square commutes and whose lower square anticommutes.

$$\begin{array}{ccc}
\pi_{N-1}D^{k+p-1-v}\Gamma_p & \xleftarrow{\ \partial\ } & \pi_N(D^{k+p-1}\Gamma_p, D^{k+p-1-v}\Gamma_p) \\
\downarrow e_* & & \cong \downarrow (e,e)_* \\
\pi_{N-1}D^{k-v}\Gamma_1 & \xleftarrow{\ \partial\ } & \pi_N(D^k\Gamma_1, D^{k-v}\Gamma_1) \\
\cong \uparrow \partial & (-1) & \cong \uparrow \partial \\
\pi_N D^{k-v}\Gamma_0/\Gamma_1 & \xleftarrow{\ \partial\ } & \pi_{N+1}(D^k\Gamma_0/\Gamma_1, D^{k-v}\Gamma_0/\Gamma_1)
\end{array}$$

The isomorphisms are isomorphisms because $D^k\Gamma_0 \simeq * \simeq D^{k-v}\Gamma_0$ by Lemma 2.4 and (e,e) is an equivalence by Lemma 3.6. Thus, we may define $A = \partial^{-1}e_*\partial C_p$. To see that ι is the image of the claimed elements, it suffices to work in homology, as in Lemma 7.2. Here, $h(eC_p \cup A) = e_*h(C_p) - h(A) = e_*h(C_p)$ since $H_{N-1}D^{k-v}\Gamma_1 = 0$ for dimensional reasons. By hypothesis, $h(C_p) = e_{k+p-1} \otimes dx^p$, so

$$h(eC_p \cup A) = \begin{cases} (-1)^{mn+m} m!\, e_k \otimes d(x^p) & p > 2 \\ (-1)^k e_k \otimes d(x^2) & p = 2 \end{cases}$$

by Lemma 3.6. Comparing this with $h(\iota) = (-1)^k e_k \otimes d(x^p)$ finishes the proof.

Now,

$$d_*\beta^\varepsilon P^j x = \begin{cases} (-1)^j \nu(n)\xi_*\iota & p > 2 \\ \xi_*\iota & p = 2 \end{cases}$$

so, up to a scalar multiple, our differential is $\xi_*(eC_p \cup A) \in \pi_N Y_{ps-k+1}$. By Corollary 2.8 and Lemma 4.1 we find that

$$\begin{aligned} \xi_*(eC_p \cup A) &= \xi_* eC_p - \xi_* A \quad \text{in } \pi_N(Y_{ps-k+1}, Y_{ps-k+r+v}) \\ &= \xi_* C_p - \xi_* A \quad \text{in } \pi_N(Y_{ps-k+1}, Y_{ps-k+r+v-1}). \end{aligned}$$

It follows from the definition of C_p that ξ_*C_p lifts to $\pi_*(Y_{ps-k+pr-p+1}, Y_{ps-k+r+v})$. By Lemma 4.2, ξ_*C_p represents $\Phi_*(e_{k+p-1} \otimes dx^p)$, which equals $\beta^\varepsilon P^j d_r x$ up to a scalar multiple. When p = 2 this shows that ξ_*C_2 contributes $P^j d_r x$ to $d_* P^j x$. When $p > 2$, the coefficient of $\beta P^j d_r x$ is

$$(-1)^{2j+k+mn+m} \frac{\nu(n)}{\nu(n-1)} \frac{1}{m!} \equiv -1 \pmod p.$$

The congruence follows from the definition of ν, $\nu(2a+b) = (-1)^a (m!)^b$ if b = 0 or 1, and the congruence $(m!)^2 \equiv (-1)^{m-1} \pmod p$. This almost proves Theorem 1.1, with T_p consisting of $-\xi_*A \in \pi_N(Y_{ps-k+1}, Y_{ps-k+r+v})$ plus a possible "error term" in $\pi_N(Y_{ps-k+r+v-1}, Y_{ps-k+r+v})$ coming from the use of Lemma 4.1 above. "Almost" because this decomposition is only valid modulo filtration ps-k+r+v and we must still show that either $\beta^\varepsilon P^j d_r x$ or T_p will be a filtration lower than this in order to finish the proof of Theorem 1.1. To do this, we must identify ξ_*A. Referring to the diagram in the proof of Lemma 7.4, the element C_p in the upper right corner goes to A in the lower left corner if we follow the top and left arrows, while it goes to

$$\begin{cases} (-1)^{k+mn+m} m!\alpha & p > 2 \\ \alpha & p = 2, \end{cases}$$

where α is the attaching map of the cell $e_k \otimes x^p$, if we follow the bottom and right arrows. Since the lower square anticommutes and since k is odd if $p > 2$, it follows that

$$A = \begin{cases} (-1)^{mn+m} m!\alpha & p > 2 \\ -\alpha & p = 2. \end{cases}$$

Applying Lemma 7.1(a) we see that $\xi_* A$ has filtration less than or equal to ps-k+v+f. Lemma 4.7 implies that, unless $r = p = 2$ and $v = 1$ or 2, one of $\xi_* C_p$ and $\xi_* A$ will occur in a filtration less than ps-k+v+r-1. Thus Theorem 1.1 is proved unless $r = p = v = 2$ (since $v = 1$ has already been dealt with). Applying the rest of Lemma 7.1 we find that

$$\xi_* A = \begin{cases} (-1)^{mn+m} m!\ \bar{a}\ \Phi_*(e_{k-v} \otimes x^p) & p > 2 \\ -\bar{a}\ \Phi_*(e_{k-v} \otimes x^2) & p = 2 \end{cases}$$

if $v = k$ (since $D^{k-v}\Gamma_0/\Gamma_1 \simeq S^{n(p)}$ has only one cell in this case) or if $p = 2$ and $v \leq 10$ or if $p > 2$ and $v \leq pq$. Combining constants, we find that $T_2 = \bar{a}P^{j-v}x$ and that $T_p = (-1)^{e-1}\bar{a}\beta P^{j-e-1}x$ if $p > 2$ (recall that $e = \varepsilon_p(j)$). The constant in the odd primary case comes from the fact that $v = v_p(k+n(p-1)) = v_p(2j(p-1) - 1) = 2(p-1)(1+e)$ by V.2.16, so $k-v = (2(j-e-1) - n)(p-1) - 1$. This completes the proof of Theorem 1.2 except when $r = p = v = 2$ (as noted above) or when $pr-p < v < k$. In the latter case, Lemma 7.1.(a) still ensures us that

$$\begin{aligned} \text{filt}(\xi_* A) &\geq ps-k + v+1 \\ &> ps-k + pr - p+1 \\ &= \text{filt}(\xi_* C_p). \end{aligned}$$

Hence the term contributed to $d_*\beta^{\varepsilon}P^j x$ by $\xi_* C_p$ appears alone in this case. This completes the proof of Theorem 1.2 except when $r = p = v = 2$. Deferring the latter case until the end, we shall now prove Theorem 1.3. If $p = 2$ we may assume $v > 8$, while if $p > 2$ we may assume $v > q$. The attaching map α of Lemma 7.1 must then have filtration 2 or more. This is so because

(i) all but the top two cells are in filtration 2 or more,

(ii) the next to top cell component is the product of a positive dimensional element of $E_2(S,S)$ (since $v > 0$) and a cell in filtration 1, so has filtration at least 2,

(iii) the top cell component is a permanent cycle (being the image of the permanent cycle α), hence has filtration at least 2 by the nonexistence of Hopf invariant one elements in dimension v-1.

This implies that ξ_*A has filtration ps-k + v+2 or more. Since ξ_*C_p has filtration ps-k + pr - p+1 and $\partial\Phi$ splits into these pieces modulo filtration ps-k + r +v-1, we have $d_i\beta^\varepsilon P^j x = 0$ if

$$i \le \min\{v+1, pr-p, v+r-2\}$$
$$= \min\{v+1, pr-p\} ,$$

the equality holding because $v+r-2 < v+1$ implies $r = 2$, so that $pr-p = p < v = v+r-2$ by our assumption on v. This proves Theorem 1.3.

It remains only to prove Theorems 1.1 and 1.2 when $r = p = v = 2$. Together, they say $d_3P^jx = P^jd_2x + h_1P^{j-2}x$. Let $N = k+2n-1$ and let $C_1 \in \pi_N(D^k\Gamma_1, D^{k-2}\Gamma_1)$ and $C_2 \in \pi_N(D^{k+1}\Gamma_2, D^{k-1}\Gamma_2)$ be the top cells, oriented so that $h(C_1) = (-1)^k e_k \otimes d(x^2)$ and $h(C_2) = e_{k+1} \otimes dx^2$.

<u>Lemma 7.5</u>. There exists $A \in \pi_N(D^{k-2}\Gamma_0, D^{k-2}\Gamma_1)$ such that $\partial A = \partial C_1$ and $\iota = C_1 \cup A$ in $\pi_N(D^k\Gamma_1 \cup D^{k-1}\Gamma_0)$.

<u>Proof</u>. Since $D^{k-2}\Gamma_0 \simeq *$ we may define $A = \partial^{-1}\partial C_1$

$$\pi_N(D^k\Gamma_1, D^{k-2}\Gamma_1) \xrightarrow{\partial} \pi_{N-1}D^{k-2}\Gamma_1 \xleftarrow[\cong]{\partial} \pi_N(D^{k-2}\Gamma_0, D^{k-2}\Gamma_1).$$

Clearly, $h(A) = 0$, so $h(C_1 \cup A) = h(C_1) = h(\iota)$. Thus $\iota = C_1 \cup A$.

It follows that

$$\partial\Phi = \xi_*\iota = \xi_*(C_1 \cup A) = \xi_*C_1 - \xi_*A \in \pi_N(Y_{2s-k+1}, Y_{2s-k+4}).$$

As before, we wish to replace ξ_*C_1 by ξ_*C_2 plus an error term which we can ignore. The following lemma is what we need in order to do this.

<u>Lemma 7.6</u>. Let $i_1 : D^{k-2}\Gamma_1 \to D^{k-1}\Gamma_2 \cup D^{k-2}\Gamma_1$,

$i_2 : D^{k-1}\Gamma_2 \to D^{k-1}\Gamma_2 \cup D^{k-2}\Gamma_1$,

and $j : D^{k-1}\Gamma_1 \to D^k\Gamma_1$

be the natural inclusions. Then there exists $X \in \pi_N(D^{k-1}\Gamma_1, D^{k-1}\Gamma_2 \cup D^{k-2}\Gamma_1)$ with positive filtration in the Adams spectral sequence, such that in $\pi_N(D^k\Gamma_1, D^{k-1}\Gamma_2 \cup D^{k-2}\Gamma_1)$

$$(1,i_1)_*C_1 = (e,i_2)_*C_2 + (j,1)_*X$$

Proof. Since $\rho:(D^k\Gamma_1, D^{k-2}\Gamma_1 \cup D^{k-1}\Gamma_2) \to (D^k\Gamma_1, D^{k-1}\Gamma_1)$ is the cofiber of $(j,1)$, we need only show $\rho_*(1,i_1)_*C_1 = \rho_*(e,i_2)_*C_2$ in order to establish the existence of X satisfying

$$(1,i_1)_*C_1 = (e,i_2)_*C_2 + (j,1)_*X.$$

The filtration of X is necessarily positive because

$$D^{k-1}\Gamma_1/D^{k-1}\Gamma_2 \cup D^{k-2}\Gamma_1 \simeq \bigvee S^{N-1}$$

by I.1.3 and Lemma 2.2. Since N is the Hurewicz dimension of $(D^k\Gamma_1, D^{k-1}\Gamma_1)$ it suffices to show $h(\rho_*(e,i_2)_*C_2) = h(\rho_*(1,i_1)_*C_1)$. This is immediate from Lemma 3.6.

With Lemma 7.6 we can now finish the proof of Theorems 1.1 and 1.2. The element ξ_*X is in $\pi_N(Y_{2s-k+3}, Y_{2s-k+4})$, but since X has filtration greater than 0, $\xi_*X = 0$ in $\pi_N(Y_{2s-k+3}, Y_{2s-k+4})$. Thus $\xi_*C_1 = \xi_*(1,i_1)_*C_1 = \xi_*(e,i_2)_*C_2$ in $\pi_N(Y_{2s-k+2}, Y_{2s-k+4})$. By Lemma 4.1, $\xi_*(e,i_2)_*C_2 = \xi_*C_2$ in $\pi_N(Y_{2s-k+1}, Y_{2s-k+4})$, and ξ_*C_2 lifts to $\pi_N(Y_{2s-k+3}, Y_{2s-k+4})$ where it represents $P^j d_2 x$ by Lemma 4.2. Finally, ξ_*A also lifts to $\pi_N(Y_{2s-k+3}, Y_{2s-k+4})$ where it represents $h_1 P^{j-2}x$ by Lemma 7.1. Thus

$$d_3 P^j x = P^j d_2 x + h_1 P^{j-2} x.$$