CHAPTER III.

HOMOLOGY OPERATIONS FOR H_∞ AND H_n RING SPECTRA

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Since H_∞ ring spectra are analogs of H_∞ spaces and H_n ring spectra are analogs up to homotopy of n-fold loop spaces, it is to be expected that their homologies admit operations analogous to those introduced by Araki and Kudo [12], Browder [22], Dyer and Lashof [33] and Cohen [28]. We define such operations in section 1 for H_∞ ring spectra and in section 3 for H_n ring spectra.

As an amusing example, we end section 1 with the observation, due independently to Haynes Miller and Jim McClure, that our homology operations in $H_*F(X^+,S) = H^*X$ coincide with the Steenrod operations when X is a finite complex.

For connective H_∞ ring spectra, we show that the resulting ring of operations is precisely the Dyer-Lashof algebra. Moreover, if X is an H_∞ space with zero (as in II.1.7), then the new operations for the H_∞ ring spectrum $\Sigma^\infty X$ coincide with the space level operations of $\tilde{H}_*X$.

As will be shown by Lewis in the sequel, the Thom spectrum Mf of an n-fold or infinite loop map $f:X \to BF$ is an H_n or H_∞ ring spectrum and the Thom isomorphism carries the space level operations to the new operations in H_*Mf. This applies in particular to the Thom spectra of the classical groups (although a simpler argument could be used here).

In section 2 we present calculations of the new operations in less obvious cases (with the proofs deferred until sections 5 and 6). Our central calculations concern Eilenberg-MacLane spectra, where , in contrast to the additive homology operations for Eilenberg-MacLane spaces, these operations are highly nontrivial. In fact, they provide a conceptual framework for the splittings of various cobordism spectra into wedges of Eilenberg-MacLane spectra or Brown-Peterson spectra. The proofs of these splittings in the literature are based on computations of the Steenrod operations on the Thom class. We show in section 4 that the presence of an H_n ring structure, $n \geq 2$ ($n \geq 3$ for the BP splittings), reduces these computations to a check of at most one low dimensional operation, depending on the type of splitting. In addition, we have placed these splitting theorems in a more general context which, as explained in the previous chapter, leads to a reproof of Nishida's bound on the order of nilpotency of an element of order p in the stable stems. All of our splittings are deduced directly from our computation of the new operations in the homology of Eilenberg-MacLane spectra.

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§1. Construction and properties of the operations

Just as the space level operations of Araki and Kudo, Browder, and Dyer and Lashof are based on maps

$$E\Sigma_j \times_{\Sigma_j} X^j \to X,$$

so our new spectrum level operations are based on the structural maps

$$\xi_j : D_j E \to E$$

of H_∞ ring spectra (see I.3.1). We consider homology with mod p coefficients for a prime p. The following omnibus theorem describes our operations. Properties of the operations at the prime 2 which are distinct from the properties at odd primes are indicated in square brackets. As usual, β denotes the homology Bockstein operation, and P^r_* denotes the dual of the Steenrod operations P^r, with $P^r = Sq^r$ if $p = 2$.

Theorem 1.1. For integers s there exist operations Q^s in the homology of H_∞ ring spectra E. They enjoy the following properties.

(1) The Q^s are natural homomorphisms.

(2) Q^s raises degree by $2s(p-1)$ [by s].

(3) $Q^s x = 0$ if $2s < \text{degree}(x)$ [if $s < \text{degree}(x)$].

(4) $Q^s x = x^p$ if $2s = \text{degree}(x)$ [if $s = \text{degree}(x)$].

(5) $Q^s 1 = 0$ for $s \neq 0$, where $1 \in H_0X$ is the algebraic unit element of H_*X.

(6) The external and internal Cartan formulas hold:

$$Q^s(x \times y) = \sum_{i+j=s} Q^i x \times Q^j y \quad \text{for } x \times y \in H_*(E \wedge F);$$

$$Q^s(xy) = \sum_{i+j=s} (Q^i x)(Q^j y) \qquad \text{for } x, y \in H_*E.$$

(7) The Adem relations hold: if $p \geq 2$ and $r > ps$, then

$$Q^r Q^s = \sum_i (-1)^{r+i}(pi - r,\ r - (p-1)s - i - 1) Q^{r+s-i} Q^i;$$

if $p > 2$ and $r \geq ps$, then

$$Q^r\beta Q^s = \sum_i (-1)^{r+i}(pi - r, r - (p - 1)s - i)\beta Q^{r+s-i}Q^i$$
$$- \sum_i (-1)^{r+i}(pi - r - 1), r - (p - 1)s - i)Q^{r+s-i}\beta Q^i.$$

(8) The Nishida relations hold: For $p \geq 2$ and n sufficiently large,

$$P^r_* Q^s = \sum_i (-1)^{r+i}(r - pi, p^n + s(p - 1) - pr + pi)Q^{s-r+i}P^i_* .$$

In particular, for $p = 2$, $\beta Q^s = (s - 1)Q^{s-1}$. For $p > 2$ and n sufficiently large,

$$P^r_* \beta Q^s = \sum_i (-1)^{r+i}(r - pi, p^n + s(p - 1) - pr + pi - 1)\beta Q^{s-r+i}P^i_*$$
$$- \sum_i (-1)^{r+i}(r - pi-1,\ p^n + s(p - 1) - pr + pi)Q^{s-r+i}P^i_*\beta.$$

(9) The homology suspension $\sigma:\tilde{H}_*E_0 \to H_*E$ carries the operations given by the multiplicative H_∞ space structure of E_0 to the operations in the homology of E.

(10) If $E = \Sigma^\infty X$ for an $H_{\infty 0}$-space X, then the operations in H_*E agree with the space level operations in $\tilde{H}_*X$.

The statement here is identical to that for the space level operations except that operations of negative degree can act on homology classes of negative degree and that a high power of p is added to the right entry in the binomial coefficients appearing in the Nishida relations. For spaces, the same answer is obtained with or without the power of p because of the restrictions on the degrees of dual Steenrod operations acting nontrivially on a given homology class. Our conventions are that (a,b) is zero if either $a < 0$ or $b < 0$ and is the binomial coefficient $(a + b)!/a!b!$ otherwise. The Nishida relations become cleaner when written in terms of classical binomial coefficients since

$$(a,p^n + b) = \binom{p^n+a+b}{a} = \binom{a+b}{a} \quad \text{for } a < p^n \text{ and } b \geq 0.$$

The Q^s and βQ^s generate an algebra of operations. If we restrict attention to the operations on connective H_∞ ring spectra, then the resulting algebra is precisely the Dyer-Lashof algebra in view of relations (3) and (8) and application of (10) to the $H_{\infty 0}$ space obtained by adjoining a disjoint basepoint to the additive H_∞ space structure on QS^0.

We sketch the proof of the theorem in the rest of this section. With the exception of the proof of the Nishida relations, the argument is precisely parallel to the treatment of the space level homology operations in [28] and is based on the general algebraic approach to Steenrod type operations developed in [68] and summarized by Bruner in IV§2.

Let π be the cyclic group of order p embedded as usual in Σ_p and let W be the standard π-free resolution of Z_p (see IV.2.2). Let $C_*(E\Sigma_p)$ be the cellular chains of the standard Σ_p-free contractible space $E\Sigma_p$ and choose a morphism $j:W \to C_*(E\Sigma_p)$ of π-complexes over Z_p. We may assume that our H_∞ ring spectrum E is a CW-spectrum with cellular structure maps $\xi_j:D_jE \to E$. By I.2.1, D_jE is a CW-spectrum with cellular chains isomorphic to $C_*(E\Sigma_j) \otimes_{\Sigma_j} (C_*E)^j$. Thus we have a composite chain map

$$W \otimes_\pi (C_*E)^p \xrightarrow{j \otimes 1} C_*(E\Sigma_p) \otimes_{\Sigma_p} (C_*E)^p \cong C_*(D_pE) \xrightarrow{\xi_*} C_*E.$$

The homology of the domain has typical elements $e_i \otimes x^p$ (and $e_0 \otimes x_1 \otimes \cdots \otimes x_p$), where $x \in H_*E$, and we let $Q_i(x) \in H_*E$ be the image of $e_i \otimes x^p$. Let x have degree q. If p = 2 define

$$Q^s(x) = 0 \text{ if } s < q \quad \text{and} \quad Q^s(x) = Q_{s-q}(x) \text{ if } s \geq q.$$

for p > 2, define

$$Q^s(x) = 0 \text{ if } 2s < q \quad \text{and} \quad Q^s(x) = (-1)^s\nu(q)Q_{(2s-q)(p-1)}(x) \text{ if } 2s \geq q$$

where $\nu(q) = (-1)^{q(q-1)m/2}(m!)^q$, with $m = \frac{1}{2}(p-1)$. By [68] the Q^s and βQ^s account for all non-trivial Q_i when p > 2. Since ξ_p restricts on $E^{(p)}$ to the p-fold product of E and since the unit $e:S \to E$ is an H_∞-map, parts (1)-(5) of the theorem are immediate from [68].

It is proven in the sequel [Equiv, VIII.2.9] that the maps ι_j, $\alpha_{j,k}$, $\beta_{j,k}$, and δ_j discussed in I§2 have the expected effect on cellular chains. For example, δ_{j*} can be identified with the homomorphism

$$C_*(E\Sigma_j) \otimes (C_*E \otimes C_*E)^j \xrightarrow{(1 \otimes t \otimes 1)(\Delta'_* \otimes u)} C_*(E\Sigma_j) \otimes (C_*E)^j \otimes C_*(E\Sigma_j) \otimes (C_*E)^j$$

where Δ' is a cellular approximation to the diagonal of $E\Sigma_j$ and u and t are shuffle and twist isomorphisms (with the usual signs). The Cartan formula and Adem relations follow. For the former, the smash product of H_∞ ring spectra E and F is an H_∞ ring spectrum with structural maps the composites

$$D_j(E \wedge F) \xrightarrow{\delta_j} D_jE \wedge D_jF \xrightarrow{\xi_j \wedge \xi_j} E \wedge F,$$

and the product $E \wedge E \to E$ of an H_∞ ring spectrum is an H_∞ map; see I.3.4. For the latter, we use the case j = k = p of the second diagram in the definition, I.3.1, of an H_∞ ring spectrum. The requisite algebra is done once and for all in [68].

The Steenrod operations in $H_*(D_\pi E)$ are computed in [Equiv. VIII §3], and the Nishida relations follow by naturality. (See also II.5.5 and VIII §3 here.)

Since $\sigma_*:\tilde{H}_*(E_0) \to H_*E$ is the composite of the identification $\tilde{H}_*(E_0) \cong H_*(\Sigma^\infty E_0)$ and the natural map $\varepsilon_*:H_*(\Sigma^\infty E_0) \to H_*E$ and since $\varepsilon:\Sigma^\infty E_0 \to E$ is an H_∞ map when E is an H_∞ ring spectrum, by I.3.10, part (9) of the theorem is a consequence of part (10). In turn, part (10) is an immediate comparison of definitions in view of I.2.2 and I.3.8. The essential point is that the isomorphism $D_\pi\Sigma^\infty X \cong \Sigma^\infty D_\pi X$ induces the obvious identification on passage to cellular chains, by [Equiv. VIII.2.9].

As promised, we have the following observation of Miller and McClure.

Remark 1.2. Let X be a finite CW complex. By II.3.2, the dual $F(X^+,S)$ of $\Sigma^\infty X^+$ is an H_∞ ring spectrum with p^{th} structural map the adjoint of the composite

$$D_pF(X^+,S)\wedge X^+ \xrightarrow{\Delta} D_p(F(X^+,S)\wedge X^+) \xrightarrow{D_p\varepsilon} D_pS \xrightarrow{\xi_p} S.$$

Here Δ_* is computed in II.5.8, ε_* is the Kronecker product $H^*X \otimes H_*X \to Z_p$, and ξ_{p*} is the identity in degree zero and is zero in positive degrees. For $y \in H_{-q}F(X^+,S) = H^qX$, we find by a simple direct calculation that $Q^{-s}y = P^sy$ for all $s \geq 0$. A more conceptual proof by direct comparison of McClure's abstract definitions of homology and cohomology operations is also possible; see VIII §3.

§2. Some calculations of the homology operations

For R a commutative ring, let HR be the spectrum representing ordinary cohomology with coefficients in R. We wish to compute the operations on the homology of HZ_p and some related spectra. We shall state our results here, but shall present proofs of the computations for HZ_p in sections 5 and 6. Recall that the mod p homology of HZ_p is A_*, the dual of the Steenrod algebra.

Notations 2.1. We shall adopt the notations of Milnor in our analysis of A_* [86]. Thus, at the prime 2, A_* has algebra generators ξ_i of degree 2^i-1 for $i \geq 1$. At odd primes, A_* has generators ξ_i of degree $2p^i-2$ for $i \geq 1$ and generators τ_i of degree $2p^i-1$ for $i \geq 0$. We shall denote the conjugation in A_* by χ.

We have the following theorems.

Theorem 2.2. For $p = 2$, A_* is generated by ξ_1 as an algebra over the Dyer-Lashof algebra. In fact, for $i > 1$,

$$Q^{2^i-2}\xi_1 = \chi\xi_i \ .$$

Moreover, $Q^s\xi_1$ is nonzero for each $s > 0$ and, for $i > 1$,

$$Q^s\chi\xi_i = \begin{cases} Q^{s+2^i-2}\xi_1 & \text{if } s \equiv 0 \text{ or } -1 \bmod 2^i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $Q^{2^i}\chi\xi_i = \chi\xi_{i+1}$ for $i > 0$.

Theorem 2.3. For $p > 2$, A_* is generated by τ_0 as an algebra over the Dyer-Lashof algebra. In fact, for $i > 0$

$$Q^{\rho(i)}\tau_0 = (-1)^i\chi\tau_i \quad \text{and}$$

$$\beta Q^{\rho(i)}\tau_0 = (-1)^i\chi\xi_i \ ,$$

where $\rho(i) = (p^i-1)/(p-1)$. Moreover, $\beta Q^s\tau_0$ is nonzero for each $s > 0$ and, for $i > 0$,

$$Q^s\chi\xi_i = \begin{cases} (-1)^i\beta Q^{s+\rho(i)}\tau_0 & \text{if } s \equiv -1 \bmod p^i \\ (-1)^{i+1}\beta Q^{s+\rho(i)}\tau_0 & \text{if } s \equiv 0 \bmod p^i \\ 0 & \text{otherwise,} \end{cases}$$

while

$$Q^s\chi\tau_i = \begin{cases} (-1)^{i+1}Q^{s+\rho(i)}\tau_0 & \text{if } s \equiv 0 \bmod p^i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $Q^{p^i}\chi\xi_i = \chi\xi_{i+1}$ for $i > 0$ and $Q^{p^i}\chi\tau_i = \chi\tau_{i+1}$ for $i \geq 0$.

Thus, for $p \geq 2$, the operations on the higher degree generators are determined by the operations on the generator of degree one. A complete determination of the operations on this degree one generator does not seem feasible. However, we do have a conceptual determination of these classes. For $p \geq 2$, let ξ be the total ξ class

$$\xi = 1 + \xi_1 + \xi_2 + \cdots$$

For $p > 2$, let τ be the total τ class

$$\tau = 1 + \tau_0 + \tau_1 + \cdots$$

Since the component of these classes in degree zero is one, we may take arbitrary powers of these classes.

Theorem 2.4. For $p = 2$ and $s > 0$,

$$Q^s\xi_1 = (\xi^{-1})_{s+1};$$

that is, $Q^s\xi_1$ is the (s+1)-st coordinate of the inverse of the total ξ class. For $p > 2$ and $s > 0$,

$$Q^s\tau_0 = (-1)^s(\xi^{-1}\tau)_{2s(p-1)+1}, \quad \text{and}$$

$$\beta Q^s\tau_0 = (-1)^s(\xi^{-1})_{2s(p-1)},$$

that is, $Q^s\tau_0$ is $(-1)^s$ times the (2s(p-1)+1)-st coordinate of the product of the total τ class and the inverse of the total ξ class, and $\beta Q^s\tau_0$ is $(-1)^s$ times the (2s(p-1))th coordinate of the inverse of the total ξ class.

Here we are using the H_∞ ring structure on HZ_p derived in I.3.6. In the following corollaries, we consider connective ring spectra E together with morphisms of ring spectra $i:E \to HZ_p$ which induce monomorphisms on mod p homology. When E is an H_∞ ring spectrum, i is an H_∞ ring map by I.3.6.

For $p > 2$, the homology of HZ or $HZ_{(p)}$ embeds as the subalgebra of A_* generated by $\chi\xi_i$ and $\chi\tau_i$ for $i \geq 1$. For $p = 2$, the homology of HZ or $HZ_{(2)}$ embeds as the subalgebra of A_* generated by ξ_1^2 and $\chi\xi_i$ for $i > 1$.

Corollary 2.5. For $p > 2$, the homology of HZ or $HZ_{(p)}$ is generated by $\chi\xi_1$ and $\chi\tau_1$ as an algebra over the Dyer-Lashof algebra. For $p = 2$, the homology of HZ or $HZ_{(2)}$ is generated by ξ_1^2 and $\chi\xi_2$ as an algebra over the Dyer-Lashof algebra.

Similarly, at the prime 2, the homology of kO, the spectrum representing real connective K-theory, embeds as the subalgebra of A_* generated by ξ_1^4, $\chi\xi_2^2$ and $\chi\xi_i$ for $i > 2$. The homology of kU embeds as the subalgebra of A_* generated by ξ_1^2, $\chi\xi_2^2$ and $\chi\xi_i$ for $i > 2$.

Corollary 2.6. At the prime 2, the homology of kO is generated by ξ_1^4, $\chi\xi_2^2$ and $\chi\xi_3$ as an algebra over the Dyer-Lashof algebra, while the homology of kU is generated by ξ_1^2 and $\chi\xi_3$ as an algebra over the Dyer-Lashof algebra.

Proof. By the Cartan formula,

$$Q^4\xi_1^2 = (Q^2\xi_1)^2 = \chi\xi_2^2 .$$

We have analogous results for the p-local Brown-Peterson spectrum BP. Let $i:BP \to HZ_p$ be the unique map of ring spectra. By the Cartan formula, if $p = 2$, or by Theorem 2.4, if $p > 2$, i_* embeds H_*BP as a subalgebra of A_* which is closed under the action of the Dyer-Lashof algebra.

Corollary 2.7. For $p > 2$, H_*BP is generated by $\chi\xi_1$ as an algebra over the Dyer-Lashof algebra. For $p = 2$, H_*BP is generated by ξ_1^2 as an algebra over the Dyer-Lashof algebra.

It is not known whether or not BP is an H_∞ ring spectrum. However, suppose that E is a connective H_∞ ring spectrum and that $f:E \to BP$ has the property that $if:H \to HZ_p$ induces a ring homomorphism on π_0. Then if is an H_∞ ring map, so that $(if)_*$ commutes with the operations. Since i_* is a monomorphism, so does f_*.

We shall also examine the operations on the homology of HZ_{p^n} for $n > 1$. Let B_* be the homology of HZ and let $x \in H_1HZ_{p^n}$ be the element dual to the n-th Bockstein operation on the fundamental cohomology class (so that $\beta_n x = -1$). Then $H_*HZ_{p^n}$ is the truncated polynomial algebra

$$H_*HZ_{p^n} = B_*[x]/(x^2),$$

as an algebra over the dual Steenrod operations. Here the inclusion of B_* in $H_*HZ_{p^n}$ is induced by the natural map $HZ \to HZ_{p^n}$, x maps to zero in the homology of HZ_p, and x is annihilated by the dual Steenrod operations.

Corollary 2.8. For $p > 2$, $H_*HZ_{p^n}$ is generated by x and the elements $\chi\xi_1$ and $\chi\tau_1$ of B_* as an algebra over the Dyer-Lashof algebra. For $p = 2$, $H_*HZ_{p^n}$ is generated by x and the elements ξ_1^2 and $\chi\xi_2$ of B_* as an algebara over the Dyer-Lashof algebra. For $p \geq 2$, the element x is annihilated by all of the operations Q^s.

Proof. For the last assertion, note that $Q^s x$ is an element of B_*x for all s since $Q^s x$ maps to zero in A_*. Since x is annihilated by the dual Steenrod operations, the Nishida relations reduce to

$$P_*^r Q^s x = (-1)^r (r, p^m + s(p - 1) - pr) Q^{s-r} x,$$

and

$$P_*^r \beta Q^s x = (-1)^r (r, p^m + s(p - 1) - pr - 1) \beta Q^{s-r} x$$

for $p > 2$. Since B_*x is isomorphic to B_* as a module over the dual Steenrod operations, and since no nontrivial element of B_* is annihilated by P_*^r for $r > 0$, and β if $p > 2$, $Q^s x = 0$ by induction.

§3. Homology operations for H_n ring spectra, $n < \infty$

Cohen, [28], by computing the equivariant homology of the space $\mathcal{C}_n(j)$ of j little n-cubes, completed the theory of homology operations for n-fold loop spaces begun by Araki and Kudo, Browder and Dyer and Lashof. Since an H_n ring spectrum (cf. [I,§4]) E is defined by structure maps $\mathcal{C}_n(j) \ltimes_{\Sigma_j} E^{(j)} \to E$, we can use Cohen's calculations to obtain analogous theorems for H_n ring spectra.

Theorem 3.1. For integers s there are operations Q^s in the homology of H_n ring spectra. $Q^s x$ is defined when $2s - \text{degree}(x) < n-1$ [$s - \text{degree}(x) < n-1$] and the operations satisfy properties (1)-(8) of Theorem 1.1 and the analogues of (9) and (10) for $n < \infty$. Moreover, these operations are compatible as n increases.

The Browder operation, λ_{n-1}, is also defined for H_n ring spectra.

Theorem 3.2. There is a natural homomorphism $\lambda_{n-1}: H_qE \otimes H_rE \to H_{q+r+n-1}E$, which satisfies the following properties.

(1) If E is an H_{n+1} ring spectrum, λ_{n-1} is the zero homomorphism,

(2) $\lambda_0(x,y) = xy - (-1)^{qr}yx$,

(3) $\lambda_{n-1}(x,y) = (-1)^{qr+1+(n-1)(q+r+1)}\lambda_{n-1}(y,x)$; $\lambda_{n-1}(x,x) = 0$ if $p = 2$,

(4) $\lambda_{n-1}(1,x) = 0 = \lambda_{n-1}(x,1)$, where $1 \in H_*E$ is the algebraic unit,

(5) The analog of the external and internal Cartan formulas hold:

$$\lambda_{n-1}(x \otimes y, x' \otimes y') = (-1)^{|x'|(|y|+n-1)} xx' \otimes \lambda_{n-1}(y,y') + (-1)^{|y|(|x'|+|y'|+n-1)}\lambda_{n-1}(x,x') \otimes yy',$$

where $|z|$ denotes the degree of z,

$$\begin{aligned}\lambda_{n-1}(xy,x'y') &= x\lambda_{n-1}(y,x')y' \\ &+ (-1)^{|y|(n-1+|x'|)}\lambda_{n-1}(x,x')yy' \\ &+ (-1)^{|x'|(n-1+|x|+|y|)}x'x\lambda_{n-1}(y,y') \\ &+ (-1)^{|y|(n-1+|y'|)+|x'||y'|}\lambda_{n-1}(x,y')yx'\end{aligned}$$

(6) The Jacobi identity holds:

$$(-1)^{(q+n-1)(s+n-1)}\lambda_{n-1}(x,\lambda_{n-1}(y,z)) + (-1)^{(r+n-1)(q+n-1)}\lambda_{n-1}(y,\lambda_{n-1}(z,x)) + (-1)^{(s+n-1)(r+n-1)}\lambda_{n-1}(z,\lambda_{n-1}(x,y)) = 0$$

for $x \in H_qE$, $y \in H_rE$, $z \in H_sE$; $\lambda_{n-1}(x,\lambda_{n-1}(x,x)) = 0$ for all x if $p = 3$.

(7) $P^s_*\lambda_{n-1}(x,y) = \sum_{i+j=s} \lambda_{n-1}(P^i_*x \otimes P^j_*y)$,

and

$$\beta\lambda_{n-1}(x,y) = \lambda_{n-1}(\beta x,y) + (-1)^{|x|+n-1}\lambda_{n-1}(x,\beta y)$$

(8) $\lambda_{n-1}(x,Q^sy) = 0$.

There is also a "top" operation, ξ_{n-1}.

Theorem 3.3. There is a function $\xi_{n-1}:H_qE \to H_{q+(n-1+q)(p-1)}E$ $[H_qE \to H_{2q+n-1}]$ defined when $q+n-1$ is even [for all q], which is natural with respect to maps of H_n ring spectra and satisfies the following properties. Here $\mathrm{ad}(x)(y) = \lambda_{n-1}(y,x)$, $\mathrm{ad}^i(x)(y) = \mathrm{ad}(x)(\mathrm{ad}^{i-1}(x)(y))$, and $\zeta_{n-1}x$ is defined, for $p > 2$, by the formula $\zeta_{n-1}x = \beta\xi_{n-1}x - \mathrm{ad}^{p-1}(x)(\beta x)$.

(1) If E is an H_{n+1} ring spectrum, $\xi_{n-1}x = Q^{(n-1+q)/2}x$ $[\xi_{n-1}x = Q^{n-1+q}x]$, hence $\zeta_n x = \beta Q^{(n-1+q)/2}x$ for $x \in H_qE$.

(2) If we let $Q^{(n-1+q)/2}x$ $[Q^{n-1+q}x]$ denote $\xi_{n-1}x$, then $\xi_{n-1}x$ satisfies formulas (3)-(5) of Theorem 1.1, the external Cartan formula, the Adem relations, and the following analogue of the internal Cartan formula:

$$\xi_{n-1}(xy) = \sum_{i+j=s} Q^ixQ^jy + \sum_{\substack{0<i+j<p \\ 0<i,j}} x^iy^j\Gamma_{ij} \qquad \text{for } n > 1,$$

where $s = \frac{n-1+q}{2}$ $[n-1+q]$, $q = \mathrm{degree}(xy)$, and Γ_{ij} is a function of x and y specified in [28, III.1.3(2)]. In particular, if $p = 2$,

$$\xi_{n-1}(xy) = \sum_{i+j=s} Q^ixQ^jy + x\lambda_{n-1}(x,y)y.$$

Moreover, the Nishida relations for ξ_{n-1} are the usual ones plus an unstable error term given by sums of Pontrjagin products which contain nontrivial iterated Browder operations.

(3) $\lambda_{n-1}(x,\xi_{n-1}y) = \mathrm{ad}^p(y)(x)$ and $\lambda_{n-1}(x,\zeta_{n-1}y) = 0$.

(4) $\xi_{n-1}(x+y) = \xi_{n-1}x + \xi_{n-1}y$ + a sum of iterated Browder operations specified in [28, III.1.3(5)].

In the remainder of this section we sketch the proofs of these theorems.

After replacing E by a CW spectrum and replacing $\mathcal{C}_n(j)$ by the geometric realization of its total singular complex, we have that $\mathcal{C}_n(j) \ltimes_\pi E^{(j)}$, is a CW spectrum, for any $\pi \subset \Sigma_j$, with cellular chains naturally isomorphic to

$C_* \mathcal{C}_n(j) \otimes_\pi (C_*E)^j$ (cf. [Equiv., VIII. 2.9]). With field coefficients, $(C_*E)^j$ is equivariantly chain homotopy equivalent to $(H_*E)^j$, so we can apply Cohen's calculations. We define $Q_i x$ to be the image under the structure map of $e_i \otimes x^p$, where $e_i \in H_i \mathcal{C}_n(p)/\pi_p$ is Cohen's class, $\pi_p \subset \Sigma_p$ the cyclic group of order p. Define $Q^s x$ and $\xi_{n-1}x$ by the formula in §1. Since $\mathcal{C}_n(2)$ is homotopy equivalent to S^{n-1}, we can define $\lambda_{n-1}(x,y)$ to be the image under the structure map of $(-1)^{(n-1)q+1}\iota \otimes x \otimes y$, where $\iota \in H_{n-1} \mathcal{C}_n(2)$ is the fundamental class and $x \in H_q E$.

As noted by Cohen, Theorem 3.1 is a consequence of Theorem 3.3, with 3.3(1) immediate from the definition. With the exception of those statements involving Steenrod operations, all of the statements in Theorems 3.2 and 3.3 follow from equalities between the images under the structure map γ of the operad $\mathcal{C}_n$ of the classes in the equivariant homology of the $\mathcal{C}_n(j)$ which induce the stipulated operations. These equalities follow from Cohen's work. This leaves Theorem 3.2(7), the Nishida relations, and the verification that $\zeta_{n-1}x$ is the image under the structure map of the appropriate multiple of $e_{(n-1)(p-1)} \otimes x^p$, this last giving the definition of $\zeta_{n-1}x$ which Cohen uses in deriving his formulas.

Since the Browder operation is defined nonequivariantly, Theorem 3.2(7) follows from the Cartan formula for Steenrod operations. The Nishida relations follow from the computation of the Steenrod operations in $H_*D_{\pi_p}E$ [Equiv, VIII §3], together with the fact that the kernel of $H_*(\mathcal{C}_n(p) \ltimes_{\pi_p} E) \to H_*D_{\pi_p}E$ consists of classes which are carried to sums of Pontrjagin products of the type stated [28, III §5 and 12.3].

For the last statement, we calculate $\beta(e_{(n-1)(p-1)} \otimes x^p)$. Let ε be a chain in $C_* \mathcal{C}_n(p)$ which projects to a cycle in $C_* \mathcal{C}_n(p)/\pi_p$ representing $e_{(n-1)(p-1)}$ and let a be a chain in the integral cellular chains of E, representing x mod p. Let $da = pb$. Let $N = 1 + \alpha + \cdots + \alpha^{p-1}$ in $Z[\pi_p]$, where α is a generator of π_p. Then

$$d(a^p) = pNba^{p-1},$$

so that

$$d(\varepsilon \otimes a^p) = p\varepsilon N \otimes ba^{p-1} + (d\varepsilon) \otimes a^p.$$

Since ε projects to a cycle mod p in $C_* \mathcal{C}_n(p)/\pi_p$, the transfer homomorphism shows that εN is a cycle mod p in $C_* \mathcal{C}_n(p)$. Thus, $\varepsilon N \otimes ba^{p-1}$ gives rise to a sum of Pontrjagin products of Browder operations in βx and x [28, III. 12.3], which, by the space level calculation, must be the appropriate multiple of $ad^{p-1}(x)(\beta x)$. Since $d\varepsilon$ projects to zero in the mod p chains of $\mathcal{C}_n(p)/\pi_p$, and since a^p is fixed under the action of π_p, we can find a chain δ such that

$$(d\varepsilon) \otimes a^p = \delta N \otimes a^p = \delta \otimes Na^p = p\delta \otimes a^p$$

for all a. By naturality and the space level result, δ must project to a cycle representing $e_{(n-1)(p-1)-1}$ in $H_*(\mathcal{C}_n(p)/\pi_p)$, so that $\delta \otimes a^p$ reduces mod p to a representative of $e_{(n-1)(p-1)} \otimes x^p$.

§4. The Splitting Theorems

We present simple necessary and sufficient conditions for a more general class of spectra than previously mentioned to split as wedges of p-local Eilenberg-MacLane spectra or as wedges of suspensions of BP. The spectra we consider are pseudo H_n ring spectra, defined as in Definition II.6.6, but with $D_j\Sigma^{dq}E_q$ replaced by $\mathcal{C}_n(j) \ltimes_{\Sigma_j} (\Sigma^{dq}E_q)^{(j)}$, with $n \geq 2$.

Fix a pseudo H_n ring spectrum E = Tel E_q, and assume that π_*E is of finite type over $\pi_0 E$ and that $\pi_0 E = \pi_0 E_q$ for q sufficiently large. Let $i:E \to HZ_p$ be such that $ie:S^0 \to HZ_p$ is the unit of HZ_p and regard i as an element of $H^0(E;Z_p)$; under our hypotheses i will be unique. Let $Z_{(p)}$ be the integers localized at p.

Theorem 4.1. If $\pi_0 E = Z_p$, then E splits as a wedge of suspensions of HZ_p.

Theorem 4.2. If $\pi_0 E = Z_{p^r}$, $r > 1$, or $\pi_0 E = Z_{(p)}$ and if p = 2 and $Sq^3 i \neq 0$ or p > 2 and $\beta P^1 i \neq 0$, then E splits as a wedge of suspensions of HZ_{p^s}, $s \geq 1$, and $HZ_{(p)}$.

Theorem 4.3. Let $n \geq 3$. If $\pi_0 E = Z_{(p)}$ and $H_*(E;Z_{(p)})$ is torsion free and if p = 2 and $Sq^2 i \neq 0$ or p > 2 and $P^1 i \neq 0$, then E splits as a wedge of suspensions of the p-local Brown-Peterson spectrum BP.

Remarks 4.4. The various known splittings of Thom spectra are direct consequences of these theorems. Obviously the splitting of MO and the other Thom spectra of unoriented cobordism theories follow from Theorem 4.1. When $\pi_0 MG = Z_{(p)}$, the mod p Thom isomorphism commutes with the Bockstein. At 2, the splittings of MSO and of the Thom spectra into which MSO maps follow from Theorem 4.2 and the facts that $Sq^2 i$ is the image of w_2 under the Thom isomorphism and that $Sq^1 w_2 = w_3$ in H^*BSO. The BP splittings of MU at all primes and of MSO and MSU at odd primes follow from Theorem 4.3 and similar trivial calculations. Most strikingly perhaps, the splitting of MSF at odd primes follows trivially from Theorem 4.2. Indeed, $P^1 i$ is nonzero by consideration of the first Wu class in MSO. Since the p-component of $\pi_q^s = \pi_q SF = \pi_{q+1} BSF$ is Z_p for q = 2p-3 and zero for 0 < q < 2p-3,

$$H_q(BSF;Z_{(p)}) = \begin{cases} Z_p & \text{for } q = 2p-2 \\ 0 & \text{for } 0 < q < 2p-2. \end{cases}$$

Thus, $H_{2p-2}(BSF;Z_p) = Z_p$, and the Bockstein

$$\beta : H_{2p-1}(BSF;Z_p) \to H_{2p-2}(BSF;Z_p)$$

is an epimorphism. Thus, the dual cohomology Bockstein is a monomorphism.

We turn to the proof of the splitting theorems. Define

$$HZ_p[x,x^{-1}] = \bigvee_{q\varepsilon Z} \Sigma^{dq} HZ_p ,$$

where $d = 1$ if $p = 2$ and $d = 2$ if $p > 2$. As pointed out in I.4.5 and II.1.3, $HZ_p[x,x^{-1}]$ is an H_∞ ring spectrum. We think of it as the Laurent series spectrum on HZ_p.

Let $A_* \subset H_*(HZ_p[x,x^{-1}])$ be the homology of the zero-th wedge summand HZ_p. Since HZ_p is a sub-H_∞ ring spectrum of $HZ_p[x,x^{-1}]$, we know the operations on A_*. Moreover, if $x \varepsilon H_d HZ_p[x,x^{-1}]$ comes from the canonical generator of $H_d\Sigma^d HZ_p$, then the homology of $HZ_p[x,x^{-1}]$ is isomorphic as an algebra over the dual Steenrod operations to $A_*[x,x^{-1}]$, the ring of Laurent polynomials in x over A_*. We could easily calculate the operations on the powers, x^n, of x by use of the techniques of the next section. However, remarkably, we shall only need the p-th power operation on x. We should remark that multiplication by x,

$$H_*\Sigma^{dq}HZ_p \to H_*\Sigma^{d(q+1)}HZ_p,$$

is the homology suspension.

<u>Lemma 4.7</u>. In $A_*[x,x^{-1}]$, for $p \geq 2$, $i > 0$ and q an integer

$$Q^{pq+p^i}(\chi\xi_i \cdot x^{pq}) = \chi\xi_{i+1} \cdot x^{p^2 q} ,$$

hence

$$Q^{p^2q+p^{i+1}}(\chi\xi_i{}^p \cdot x^{p^2q}) = \chi\xi_{i+1}^p \cdot x^{p^3q} .$$

For $p > 2$, $i \geq 0$ and q an integer,

$$Q^{pq+p^i}(\chi\tau_i \cdot x^{pq}) = \chi\tau_{i+1} \cdot x^{p^2q}.$$

<u>Proof</u>. The internal Cartan formula, together with the degree of $\chi\xi_i$ and of x^{pq} gives

$$Q^{pq+p^i}(\chi\xi_i \cdot x^{pq}) = (Q^{p^i}\chi\xi_i)(Q^{pq}x^{pq}) + Q^{p^i-1}\chi\xi_i)(Q^{pq+1}x^{pq}) .$$

By the Cartan formula, $Q^{pq+1}x^{pq} = 0$. Of course, $Q^{pq}x^{pq} = x^{p^2q}$ (Theorem 1.2.(4)). The first statement follows from Theorem 2.2 or Theorem 2.3 and the fact $A_* \subset A_*[x,x^{-1}]$ is a subalgebra over the Dyer-Lashof algebra. Since $\chi\xi_i{}^p \cdot x^{p^2q} =$

$(\chi\xi_i \cdot x^{pq})^p$, the second statement now follows by the Cartan formula. The proof of the third statement is almost identical to the proof of the first.

It should be noted that the full strength of Theorems 2.2 and 2.3 is quite unnecessary for the computations above. They could be derived quite simply and directly. We shall apply these computations to the proofs of the splitting theorems by means of the following commutative diagram, analogous to that of II.6.8.

$$\begin{array}{ccc} \zeta_n(j) \ltimes_{\Sigma_j} (\Sigma^{dq}E_q)^{(j)} & \xrightarrow{\;1 \ltimes \Sigma^{dq} i_q^{(j)}\;} & \zeta_n(j) \ltimes_{\Sigma_j} (\Sigma^{dq}HZ_p)^{(j)} \\ \downarrow{\scriptstyle \xi_j} & & \downarrow{\scriptstyle \xi_j} \\ \Sigma^{djq}E_{jq} & \xrightarrow{\;\Sigma^{djq} i_{jq}\;} & \Sigma^{djq}HZ_p \end{array}$$

Here, i_s is the restriction of $i:E \to HZ_p$ to E_s, the right-hand map ξ_j is the induced H_n ring structure of $HZ_p[x,x^{-1}]$ restricted to the (dq)-th wedge summand. The commutativity of the diagram is an easy cohomology calculation provided tht $E_q \to E_s$ induces an isomorphism of π_0 for $s > q$.

The key step in the proofs of Theorems 4.1, 4.2 and 4.3 is the following result.

Proposition 4.8. Let $E = \text{Tel}\, E_q$ satisfy the hypotheses of Theorem 4.1, 4.2 or 4.3. For the first two cases, let $j:E \to H\pi_0E$ be such that $je:S \to H\pi_0E$ is the unit. In the third case, let $j:E \to BP$ be a lift of j above to BP. Then j induces a monomorphism of p-primary cohomology.

Proof. We shall show that j induces an epimorphism of p-primary homology. Recall that i is the projection of j above into HZ_p. In the second case, if $\pi_0E = Z_{p^r}$ for $r > 1$, the nontriviality of the r-th Bockstein operation on i shows that the generator $x \in H_*HZ_{p^r} = B_*[x]/(x^2)$ is in the image of j_*. (Here $B_* = H_*HZ_{(p)}$.) Thus, for the second case as a whole, it suffices to show that $B_* \subset A_*$ is in the image of i_*. Similarly, for the third case, it suffices to show that $H_*BP \subset A_*$ is in the image of i_*. The hypotheses of the theorems give us the following conclusions. In Theorem 4.1, the nontriviality of the Bockstein operation on i_q, for q sufficiently large, shows that τ_0, if $p > 2$, or ξ_1, if $p = 2$, is in the image of i_{q*}. In Theorem 4.2, the nontriviality of P^1i and $\beta P^1 i$, for $p > 2$, or of $Sq^2 i$ and $Sq^3 i$, for $p = 2$, shows that for q sufficiently large, $\chi\xi_1$ and $\chi\tau_1$, for $p > 2$, or ξ_1^2 and $\chi\xi_2$ for $p = 2$, are in the image of i_{q*}. In Theorem 4.3, the nontriviality of $P^1 i$, for $p > 2$, or of $Sq^2 i$, for $p = 2$, shows that for q sufficiently large, $\chi\xi_1$, for $p > 2$ or ξ_1^2, for $p = 2$, is in the image of i_{q*}. Thus, the following consequences of Lemma 4.7 and the diagram preceding the statement will suffice.

(1) If $p = 2$ or if $p > 2$ and $n \geq 3$ and if $\chi\xi_i$ is in the image of i_{dpq*}, then $\chi\xi_{i+1}$ is in the image of i_{dp^2q*}.

(2) If $p > 2$ and $\chi\tau_i$ is in the image of i_{dpq*}, then $\chi\tau_{i+1}$ is in the image of i_{dp^2q*}.

(3) If $p = 2$, $n \geq 3$, and $\chi\xi_i^2$ is in the imge of i_{4q*}, then $\chi\xi_{i+1}^2$ is in the imge of i_{8q*}.

The conditions on n are just enough to ensure that $H_*(\mathcal{C}_n(p) \ltimes_{\Sigma_p} \Sigma^{dq}E_q)$ contains preimages of the operations needed to carry out the argument.

The passage from the proposition above to the splitting theorems is well known and has been exploited in the literature to prove the splittings of the cobordism theories. Theorems 4.1 and 4.3 follow from the algebraic splitting theorem of Milnor and Moore [87] together with standard properties of HZ_p and BP. For Theorem 4.2, H^*E splits as a direct sum of suspensions of $A/A\beta$ and of A as a module over the Steenrod algbra A. However, the E_2 term of the Bockstein spectral sequence of H^*E is spanned by the A-module generators of the summands isomorphic to $A/A\beta$. By pairing up these generators with respect to their higher order Bocksteins, we may construct a map of E into a wedge of p-local cyclic Eilenberg-MacLane spectra which induces an isomorphism on mod p cohomology. In all cases, the hypothesis on $\pi_0 E$ ensures that E is p-local, and the cohomology isomorphisms yield equivalences.

§5. Proof of Theorem 2.4; Some low-dimensional calculations

We shall exploit the following observation of Liulevicius.

Proposition 5.1. Let $C = Z_2[x,x^{-1}]$ be the algebra over the Steenrod algebra A which is obtained by inverting the polynomial generator of H^*RP^∞. Let C_* be the dual of C, with a generator e_t in degree t. Let $f_t : C_* \to A_*$ be the unique nontrivial morphism of A_* comodules of degree $-t$ (i.e., $f_t e_t = 1$). Then $f_t e_n$ is the component of the t-th power of the total ξ class in degree $n-t$:

$$f_t e_n = (\xi^t)_{n-t}.$$

Proof. Let $\lambda : C \to C \hat{\otimes} A_*$ be the dual of the module structure of C_* over the dual operations. Recall that for $c \in C$ and $a \in A$, if $\lambda c = \sum c_i \otimes \alpha_i$, then $ac = \sum \langle a,\alpha_i\rangle c_i$. Here $\langle\ ,\ \rangle : A \otimes A_* \to Z_2$ is the Kronecker product. In particular, if $\lambda x^t = \sum x^i \otimes \alpha_i$, then $f_t e_n = \alpha_n$: for $a \in A$,

$$\begin{aligned} \langle a, f_t e_n \rangle &= \langle f_t^* a, e_n \rangle \\ &= \langle a x^t, e_n \rangle \\ &= \langle \langle a, \alpha_n \rangle x^n, e_n \rangle \\ &= \langle a, \alpha_n \rangle, \end{aligned}$$

since $\langle x^n, e_n \rangle = 1$. However, λ is an algebra map, and Milnor has shown that

$$\lambda x = \sum_{i \geq 0} x^{2^i} \otimes \xi_i = \sum_{i \geq 1} x^i \otimes (\xi)_{i-1}.$$

Thus

$$\lambda x^t = \sum_{i \geq t} x^i \otimes (\xi^t)_{i-t}.$$

We also have an odd primary analogue.

<u>Proposition 5.2</u>. For $p > 2$, let C be the A-algebra obtained by inverting the polynomial generator in the cohomology of the lens space L^∞. Thus, C is the tensor product of an exterior algebra on a generator x of degree one and an inverted polynomial algebra on $y = \beta x$. Let C_* be the dual of C and let $e_{2n} \in C_*$ be dual to y^n and let $e_{2n+1} \in C_*$ be dual to xy^n. Let $f_t : C_* \to A_*$ be the A_* comodule map such that $f_t e_t = 1$.

(1) If $t = 2s$, then $f_t e_n$ is $(-1)^n$ times the $(n-t)$-th component of the s-th power of the total ξ class:

$$f_t e_n = (-1)^n (\xi^s)_{n-t}.$$

(2) If $t = 2s+1$, then $f_t e_n$ is the $(n-t)$-th component of the product of the total τ class with the s-th power of the total ξ class:

$$f_t e_n = (\xi^s \tau)_{n-t}.$$

<u>Proof</u>. Let $z_i \in C$ be the dual of e_i. Suppose that $\lambda z_t = \sum z_i \otimes \alpha_i$. The sign convention here is that for $a \in A$,

$$a z_t = \sum (-1)^{i(i-t)} \langle a, \alpha_i \rangle z_i \ .$$

A similar argument to that when p = 2 shows that $f_t e_n = (1)^{n(n-t)} \alpha_n$. Here, Milnor's calculations are that

$$\lambda x = x \otimes 1 + \sum_{i \geq 1} y^i \otimes (\tau)_{2i-1} \quad \text{and}$$

$$\lambda y = \sum_{i \geq 1} y^i \otimes (\xi)_{2i-2} .$$

Thus

$$\lambda y^s = \sum_{i \geq s} y^i \otimes (\xi^s)_{2i-2s} \quad \text{and}$$

$$\lambda (xy^s) = \sum_{i \geq 2s+1} z_i \otimes (\xi^s \tau)_{i-2s-1} .$$

In the remainder of this section and in the next, we shall need to evaluate binomial coefficients mod p. The standard technique is the following.

Lemma 5.3. Let $a = \sum a_i p^i$ and $b = \sum b_i p^i$ be the p-adic expansions of a and b. Then $(a,b) \equiv 0$ mod p unless $a_i + b_i < p$ for all i, when

$$(a,b) \equiv \prod_i (a_i, b_i) \bmod p.$$

Moreover, for $a \leq p^n - 1$,

$$(a, p^n - 1 - a) \equiv (-1)^a \bmod p.$$

We shall not bother to quote the first statment, but shall use it implicitly.

The following proposition is the key step in proving Theorem 2.4.

Proposition 5.4. For p = 2, the map $f: C_* \to A_*$ given by

$$fe_n = \begin{cases} Q^n \xi_1 & \text{for } n > 0 \\ \xi_1 & \text{for } n = 0 \\ 1 & \text{for } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

is a map of A_* coalgebras. For p > 2, the map $f: C_* \to A_*$ given by

$$fe_n = \begin{cases} (-1)^s Q^s \tau_0 & \text{if } n = 2s(p-1) \\ (-1)^s \beta Q^s \tau_0 & \text{if } n = 2s(p-1)-1 \\ -\tau_0 & \text{for } n = 0 \\ 1 & \text{for } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

is a map of A_* coalgebras. Thus, in either case, the map f coincides with the map f_{-1} described above.

Proof. Of course $f:C_* \to A_*$ is a map of A_* comodules if and only if $f^*:A \to C$ is a map of A-modules. But this latter condition is equivalent to the statement that f_* commutes with the action of the dual Steenrod operations $P_*^{p^k}$ for $k \geq 0$ and also commutes with the Bockstein β when $p > 2$

For $p > 2$, $\beta e_{2s} = e_{2s-1}$ and $\beta\tau_0 = -1$. (We have adopted the covention that for $y \in H^qX$ and $x \in H_{q+1}X$, $\langle x,\beta x\rangle = (-1)^{q+1}\langle\beta y,x\rangle$.) Moreover, the subspace of C_* spanned by $e_{2s(p-1)}$ and $e_{2s(p-1)-1}$ for s an integer is a direct summand of C_* as a module over the dual Steenrod operations. We have specified that f = 0 on the complementary summand. Thus, for $p \geq 2$, it will suffice to show that the dual Steenrod operations in C_* agree under f with the Nishida relations on the pertinent homology operations on ξ_1 or τ_0.

For symmetry, we shall write y for the polynomial generator of C when p = 2. For $p \geq 2$, the computation is divided into three cases. First, those e_i which are carried by $P_*^{p^k}$ to an element of positive degree, second, those which have image in degree zero, and third, those which have image in degree -1.

In the first case, we show that for p = 2 and $2^k < s$,

$$P_*^{2^k} e_s = (2^k, s-2^{k+1})e_{s-2^k},$$

and that for p > 2 and $p^k < s$,

$$P_*^{p^k} e_{2s(p-1)} = (p^k, s(p-1) - p^{k+1})e_{2(s-p^k)(p-1)}.$$

Let d = 1 when p = 2 and let d = 2 when p > 2. Then the statements above reduce to

$$P_*^{p^k} e_{ds(p-1)} = (p^k, s(p-1) - p^{k+1})e_{d(s-p^k)(p-1)}$$

for $p \geq 2$. However, since C was obtained from the cohomology of RP^∞ or L^∞,

$$P^r y = \begin{cases} y & \text{for } r = 0 \\ y^p & \text{for } r = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, for n > 0, $P^r y^n = (r,n-r)y^{n+r(p-1)}$ by the Cartan formula. Our claim follows from the calculation

$$\langle y^{d(s-p^k)(p-1)}, P_*^{p^k} e_{ds(p-1)} \rangle = \langle P^{p^k} y^{d(s-p^k)(p-1)}, e_{ds(p-1)} \rangle$$

$$= (p^k, s(p-1) - p^{k+1}).$$

For $p > 2$ and $s > p^k$, we have similarly that

$$P_*^{p^k} e_{2s(p-1)-1} = (p^k, s(p-1) - p^{k+1} - 1)e_{2(s-p^k)(p-1) - 1}.$$

Here, $P^r x = 0$ for $r > 0$, so that

$$\langle xy^{s(p-1)-p^k(p-1)-1}, P_*^{p^k} e_{2s(p-1)-1} \rangle = (p^k, s(p-1) - p^{k+1} - 1).$$

On the other hand, the Nishida relations give us, for $s > p^k$,

$$P_*^{2^k} Q^s \xi_1 = (2^k, 2^m + s - 2^{k+1}) Q^{s-2^k} \xi_1$$

for $p = 2$, and, for $p > 2$,

$$P_*^{p^k} Q^s \tau_0 = -(p^k, p^m + s(p-1) - p^{k+1}) Q^{s-p^k} \tau_0 ,$$

and

$$P_*^{p^k} \beta Q^s \tau_0 = -(p^k, p^m + s(p - 1) - p^{k+1} - 1) \beta Q^{s-p^k} \tau_0 .$$

Here, the initial -1 is cancelled by the conventions in the definition of f, and the additional high power of p in the right-hand side does not alter the binomial coefficients unless the right-hand side would otherwise be negative. Thus, we must check that for $s > p^k$, if $s(p-1) < p^{k+1}$, then $(p^k, p^m + s(p-1) - p^{k+1})$ and $(p^k, p^m + s(p-1) - p^{k+1} - 1)$ are zero. Since $s(p-1) \leq p^{k+1} - 1$, we have $s \leq \rho(k+1) = 1 + p + \cdots + p^k$. But since $p^k < s$, we have $s = p^k + t$ with $0 < t \leq \rho(k)$. Thus, $s(p-1) = p^k(p-1) + t_1$, with $0 < t_1 < p^k$. Thus, the specified coefficients are zero.

It remains to check those operations $P_*^{p^k}$ whose images have degree 0 or -1 in C_*. However, e_0 may not be in the image of any $P_*^{p^k}$, as $P^r 1 = 0$ for $r > 0$. $P_*^r Q^r \xi_1$ and $P_*^r Q^r \tau_0$ are zero by the Nishida relations. (Q_0 kills ξ_1 or τ_1.) For the remaining case, we shall show that for $p = 2$,

$$P_*^{2^k} e_{2^k-1} = e_{-1} ,$$

and for $p > 2$,

$$P_*^{p^k} e_{2p^k(p-1)-1} = -e_{-1}.$$

To do this, we must compute the Steenrod operations on y^{-1} when $p = 2$ and on xy^{-1}

when $p > 2$. For $p \geq 2$ and $r > 0$,

$$0 = P^r(yy^{-1}) = (P^0y)(P^ry^{-1}) + (P^1y)(P^{r-1}y^{-1})$$

$$= yP^ry^{-1} + y^pP^{r-1}y^{-1}$$

by the Cartan formula. Thus, $P^ry^{-1} = -y^{p-1}P^{r-1}y^{-1}$, so that

$$P^ry^{-1} = (-1)^r y^{r(p-1)-1} \ ,$$

by induction. For $p > 2$, since $P^rx = 0$ for $r > 0$,

$$P^r(xy^{-1}) = (-1)^r xy^{r(p-1)-1}.$$

Thus, for $p = 2$,

$$\langle y^{-1}, P_*^{2^k} e_{2^k-1} \rangle = \langle y^{2^k-1}, e_{2^k-1} \rangle = 1$$

and for $p > 2$

$$\langle xy^{-1}, P_*^{p^k} e_{2p^k(p-1)-1} \rangle = (-1)^{p^k} \langle xy^{p^k(p-1)-1}, e_{2p^k(p-1)-1} \rangle = -1.$$

The following lemma will complete the proof.

Lemma 5.5. For $p = 2$,

$$P_*^{s+1}Q^s\xi_1 = 1.$$

For $p > 2$,

$$P_*^s\beta Q^s\tau_0 = (-1)^{s-1}.$$

Proof. For $p = 2$, the Nishida relations reduce to

$$P_*^{s+1}Q^s\xi_1 = (s-1, 2^n-s)Q^0P_*^1\xi_1 = 1,$$

by Lemma 4.3. For $p > 2$, the Nishida relations reduce to

$$P_*^s\beta Q^s\tau_0 = -(s-1, p^n-s)Q^0P_*^0\beta\tau_0$$

$$= (-1)^{s-1}$$

by Lemma 4.3, since $\beta\tau_0 = -1$.

Proof of Theorem 2.4. For p = 2 and s > 0, the fact that

$$Q^s\xi_1 = (\xi^{-1})_{s+1}$$

follows immediately from Propositions 5.1 and 5.4. For p > 2 and s > 0, the fact that

$$Q^s\tau_0 = (-1)^s(\xi^{-1}\tau)_{2s(p-1)+1} \quad \text{and}$$

$$\beta Q^s\tau_0 = (-1)^s(\xi^{-1}\tau)_{2s(p-1)}$$

follows immediately from Proposition 5.2 and 5.4. However, all of the even degree coordinates of $\xi^{-1}\tau$ come from ξ^{-1}. Thus,

$$\beta Q^s\tau_0 = (-1)^s(\xi^{-1})_{2s(p-1)} .$$

One can identify certain algorithms such as the following curiosity when p = 2:

$$Q^{2^i}\xi_1 = \sum_{j=1}^{2^{i-1}-1}(Q^j\xi_1)(Q^{2^i-j-1}\xi_1)$$

Thus, the actual computations can get quite ugly. We have the following low-dimensional computations of $Q^s\xi_1$ for p = 2. In the next section we shall show that $Q^{2t-1}\xi_1 = (Q^{t-1}\xi_1)^2$. Thus, we shall only list $Q^{2t}\xi_1$. We shall write $\chi\xi_i = \beta_i$ for $i \geq 1$.

$Q^{2t}\xi_1$ for $0 < t \leq 15$, where $p = 2$:

s	$Q^s\xi_1$
2	β_2
4	$\beta_1^2\beta_2$
6	β_3
8	$\beta_1^6\beta_2 + \beta_1^2\beta_3 + \beta_2^3$
10	$\beta_1^4\beta_3$
12	$\beta_2^2\beta_3$
14	β_4
16	$\beta_1^2\beta_4 + \beta_2\beta_3^2 + \beta_1^4\beta_2^2\beta_3 + \beta_1^2\beta_2^5 + \beta_1^8(\beta_1^6\beta_2 + \beta_1^2\beta_3 + \beta_2^3)$
18	$\beta_1^4\beta_4 + \beta_1^{12}\beta_3 + \beta_2^4\beta_3$
20	$\beta_1^8\beta_2^2\beta_3 + \beta_2^2\beta_4 + \beta_3^3$
22	$\beta_1^8\beta_4$
24	$\beta_1^4\beta_2^2\beta_4 + \beta_1^4\beta_3^3 + \beta_2^6\beta_3$
26	$\beta_2^4\beta_4$
28	$\beta_3^2\beta_4$
30	β_5

§6. Proofs of Theorems 2.2 and 2.3

We shall compute the operations on $H_*HZ_p = A_*$. The elements of A_* are completely determined by the effect of the dual Steenrod operations $P_*^{p^k}$ for $k \geq 0$, along with the Bockstein operation if $p > 2$. Thus, our computations will be based on induction arguments using the Nishida relations.

Theorems 2.2 is the composite of Lemma 5.5 and Propsitions 6.4 and 6.7. Theorem 2.3 is the composite of Lemma 5.5, Propositions 6.4, 6.7 and 6.9, and Corollary 6.5.

We begin by recalling some basic facts about the dual Steenrod operations in A_*.

Lemma 6.1. The following equalities hold in A_*. For $p \geq 2$ and $i > 0$,

$$P_*^r \chi\xi_1 = \begin{cases} -\chi\xi_{i-k}^{p^k} & \text{if } r = \rho(k) \\ 0 & \text{otherwise} \end{cases}$$

(Recall that $\rho(k) = \frac{p^k - 1}{p - 1}$.) For $p > 2$ and $i \geq 0$,

$$P_*^r \chi\tau_i = 0 \quad \text{for } r > 0,$$

and

$$\beta\chi\tau_i = \chi\xi_i .$$

Here, ξ_0 is identified with the unit, 1, of A_*.

Remarks 6.2. Notice that the added high power of p in the right-hand side of the binomial coefficients in the Nishida relations allows us to make the following simplification. For $p \geq 2$,

$$P_*^{p^k} Q^s = \sum_i (-1)^{i+1}(p^k - pi, s(p-1) - pi)Q^{s-p^k+i} P_*^i .$$

For $p > 2$,

$$P_*^{p^k} \beta Q^s = \sum_i (-1)^{i+1}(p^k - pi, s(p-1) + pi - 1)\beta Q^{s-p^k+i} P_*^i + \sum_i (-1)^{i+1}(p^k - pi - 1, s(p-1) + pi)Q^{s-p^k+i} P_*^i \beta .$$

One of the key observations in our calculations is the following.

Lemma 6.3. (The p-th power lemma). For $p = 2$ and $s > 1$,

$$Q^{2s-1}\xi_1 = (Q^{s-1}\xi_1)^2.$$

For $p > 2$ and $s > 0$,

$$\beta Q^{ps}\tau_0 = (\beta Q^s\tau_0)^p.$$

Proof. We argue by induction on s. We shall show that both sides of the proposed equalities agree under $P_*^{p^k}$ for $k \geq 0$ and under β when $p > 2$. Of course, β is no problem, and both sides of both equations vanish under P_*^1. For the right hand side, this follows from the Cartan formula. For the left-hand side, the Nishida relations give

$$P_*^1 Q^s = (s-1)Q^{s-1}\text{ , and for } p > 2$$

$$P_*^1 \beta Q^s = s\beta Q^{s-1} - Q^{s-1}\beta\ .$$

Thus, we may restrict attention to $P_*^{p^k}$ for $k > 0$. If $s = p^{k-1}$, Lemma 5.5 and the Cartan formula show that both sides of the equations are carried to 1 by $P_*^{p^k}$. Thus, the lemma is true for $p = 2$ and $s = 2$, and for $p > 2$ and $s = 1$. In the remaining cases, $k > 0$ and $s > p^{k-1}$. Here for $p = 2$,

$$P_*^{2^k} Q^{2s-1}\xi_1 = (2^k, 2s-1)Q^{2s-2^k-1}\xi_1\ ,$$

while

$$\begin{aligned} P_*^{2^k}(Q^{s-1}\xi_1)^2 &= (P_*^{2^{k-1}} Q^{s-1}\xi_1)^2 \\ &= (2^{k-1}, s-1)(Q^{s-2^{k-1}-1}\xi_1)^2 \\ &= (2^{k-1}, s-1)Q^{2s-2^k-1}\xi_1\ , \end{aligned}$$

by the Cartan formula, the Nishida relations and induction. For $p > 2$,

$$\begin{aligned} P_*^{p^k}(\beta Q^s\tau_0)^p &= (P_*^{p^{k-1}} \beta Q^s\tau_0)^p \\ &= -(p^{k-1}, s(p-1) - 1)(\beta Q^{s-p^{k-1}}\tau_0)^p \\ &= -(p^{k-1}, s(p-1) - 1)\beta Q^{ps-p^k}\tau_0\ , \end{aligned}$$

by the Cartan formula, the Nishida relations and induction. The conclusion follows easily from Lemma 5.3.

We can now evaluate certain of the operations.

Proposition 6.4. For $p = 2$ and $i > 1$,

$$Q^{2^i-2}\xi_1 = \chi\xi_i .$$

For $p > 2$ and $i > 0$,

$$\beta Q^{\rho(i)}\tau_0 = (-1)^i\chi\xi_i .$$

(Again $\rho(i) = \frac{p^i-1}{p-1}$.)

Proof. We argue by induction on i. Again it will be sufficient to show that both sides of the equations agree under $P_*^{p^k}$ for $k \geq 0$. For $p = 2$,

$$P_*^{2^k}Q^{2^i-2}\xi_1 = (2^k,2^i-2)Q^{2^i-2-2^k}\xi_1 .$$

For $0 < k < i$, the binomial coefficient is zero, while for $k \geq i$, $Q^{2^i-2-2^k}\xi_1 = 0$ for dimensional reasons. Thus, the only nontrivial operation is

$$P_*^1Q^{2^i-2}\xi_1 = Q^{2^i-3}\xi_1 .$$

For $i = 2$, $Q^{2^i-3}\xi_1 = Q^1\xi_1 = \xi_1^2$. Since $\xi_1 = \chi\xi_1$, the proposition is true for $i = 2$ by Lemma 6.1. For $i > 2$,

$$Q^{2^i-3}\xi_1 = (Q^{2^{i-1}-2}\xi_1)^2$$

$$= (\chi\xi_{i-1})^2 ,$$

by the p-th power lemma and induction. Lemma 6.1 is again sufficient. For $p > 2$, let $i = 1$. Then

$$P_*^1\beta Q^{\rho(1)}\tau_0 = P_*^1\beta Q^1\tau_0 = 1$$

by Lemma 5.5. Thus, $\beta Q^1\tau_0 = -\chi\xi_1$. For $i > 1$,

$$P_*^{p^k}\beta Q^{\rho(i)}\tau_0 = -(p^k,\rho(i)(p-1) - 1)\beta Q^{\rho(i)-p^k}\tau_0$$

$$= -(p^k,p^i-2)\beta Q^{\rho(i)-p^k}\tau_0,$$

by the p-th power lemma and induction. The result follows from Lemma 6.1.

Corollary 6.5. For $p > 2$ and $i > 0$,

$$Q^{\rho(i)}\tau_0 = (-1)^i \chi\tau_1 \ .$$

Proof. We have just shown that $Q^{\rho(i)}\tau_0$ and $(-1)^i\chi\tau_i$ have the same Bockstein. However,

$$P_*^{p^k} Q^{\rho(i)}\tau_0 = -(p^k,\rho(i)(p-1))Q^{\rho(i)-p^k}\tau_0$$

$$= -(p^k,p^i-1)Q^{\rho(i)-p^k}\tau_0 \ .$$

For $k < i$, $(p^k,p^i-1) = 0$, while for $k \geq i$, $Q^{\rho(i)-p^k}\tau_0 = 0$ for dimensional reasons. The result follows from Lemma 6.1.

We wish now to compute the operations on the higher degree generators. By the Nishida relations and Lemma 6.1,

$$P_*^{p^k} Q^s\chi\xi_i = -(p^k,s(p-1))Q^{s-p^k}\chi\xi_i$$

$$+ \sum_{j\geq 1} (-1)^{j+1}(p^k - p\rho(j),s(p-1) + p\rho(j))\cdot Q^{s-p^k+\rho(j)}(-\chi\xi_{i-j}^{p^j}) \ ,$$

and for $p > 2$,

$$P_*^{p^k} \beta Q^s\chi\tau_i = -(p^k,s(p-1) - 1)\beta Q^{s-p^k}\chi\tau_i - (p^k-1,s(p-1))Q^{s-p^k}\chi\xi_i$$

$$+ \sum_{j\geq 1} (-1)^{j+1}(p^k - p\rho(j) - 1,s(p-1) + p\rho(j))Q^{s-p^k+\rho(j)}(-\chi\xi_{i-j}^{p^j}).$$

However, we may simplify this expression considerably.

Lemma 6.6. For $p \geq 2$ and $i > 0$,

$$P_*^{p^k} Q^s\chi\xi_i = -(p^k,s(p-1))Q^{s-p^k}\chi\xi_i - (p^k - p,s(p-1) + p)Q^{s-p^k+1}\chi\xi_{i-1}^p \ .$$

For $p > 2$ and $i \geq 0$,

$$P_*^{p^k} \beta Q^s\chi\tau_i = -(p^k,s(p-1) - 1)\beta Q^{s-p^k}\chi\tau_i - (p^k-1,s(p-1))Q^{s-p^k}\chi\xi_i \ .$$

Moreover, the following additional simplifications hold for particular values of s. For $p > 2$, $s \not\equiv 0 \bmod p$ and $k > 0$,

$$P_*^{p^k} \beta Q^s\chi\tau_i = -(p^k,s(p-1) - 1)\beta Q^{s-p^k}\chi\tau_i \ .$$

For $p \geq 2$, $s \not\equiv -1 \bmod p^2$ and $k > 1$,

$$P_*^{p^k} Q^s \chi\xi_i = -(p^k, s(p-1)) Q^{s-p^k} \chi\xi_i \;.$$

Proof. The assertion is true for $k = 0$ or $k = 1$ because of the left-hand term of the binomial coefficients. We shall assume $k > 1$. If $s \not\equiv -1 \bmod p$ and $j > 0$, then $s - p^k + \rho(j) \not\equiv -1 \bmod p$. By the Cartan formula (or Theorem 1.2(5) if $i = j$), $Q^{s-p^k+\rho(j)} \chi\xi_{i-j}^{p^k} = 0$. If $s \not\equiv -1 \bmod p$, $p > 2$, $k > 0$ and $j \geq 0$, $p^k - p\rho(j) - 1 \equiv -1 \bmod p$, while $s(p-1) + p\rho(j) \not\equiv 0 \bmod p$. Thus,

$$(p^k - p\rho(j) - 1,\; s(p-1) + p\rho(j)) = 0.$$

For $s \equiv -1 \bmod p$, but $s \not\equiv -1 \bmod p^2$ (here $p \geq 2$), $s \equiv tp-1 \bmod p^2$ for $0 < t < p$. Thus

$$s(p-1) + p\rho(j) \equiv (p-t)p+1 \bmod p^2,$$

while

$$p^k - p\rho(j) \equiv (p-1)p \bmod p^2.$$

Thus,

$$(p^k - p\rho(j), s(p-1) + p\rho(j)) = 0.$$

It suffices to assume $s \equiv -1 \bmod p^2$. Here, for $j > 1$ (and $k > 1$),

$$s - p^k + \rho(j) \equiv p \bmod p^2 \;.$$

By the Cartan formula (or Theorem 1.2(5) if $i = j$),

$$Q^{s-p^k+\rho(j)} \chi\xi_{i-j}^{p^j} = 0.$$

Proposition 6.7. For $p = 2$, $i > 0$ and $s > 0$,

$$Q^s \chi\xi_i = \begin{cases} Q^{s+2^i-2}\xi_1 & \text{if } s \equiv 0 \text{ or } -1 \bmod 2^i \\ 0 & \text{otherwise} \;. \end{cases}$$

For $p > 2$, $i > 0$ and $s > 0$,

$$Q^s \chi\xi_i = \begin{cases} (-1)^i \beta Q^{s+\rho(i)} \tau_0 & \text{if } s \equiv -1 \bmod p^i \\ (-1)^{i+1} \beta Q^{s+\rho(i)} \tau_0 & \text{if } j \equiv 0 \bmod p^i \\ 0 & \text{otherwise} \;. \end{cases}$$

Proof. We argue by induction on s and i. For p = 2, the assertion is trivial for $i = 1$. For $p \geq 2$, and $0 < s \leq p^i-1$ the assertion holds by dimensional reasons and the p-th power lemma. Of course, we shall show that both sides of the equations agree under $P_*^{p^k}$ for $k \geq 0$ and under β when $p > 2$. Clearly both sides agree under P_*^1, and when $p > 2$, Lemma 6.1 implies that $\beta Q^s \chi\xi_i = 0$ for all i and s by induction and the Nishida relations. Thus, it suffices to check $P_*^{p^k}$ for $k > 0$.

Case 1. $s \equiv 0 \bmod p$, but $s \not\equiv 0 \bmod p^i$.

By the preceding lemma,

$$P_*^{p^k} Q^s \chi\xi_i = -(p^k, s(p-1)) Q^{s-p^k} \chi\xi_i \ .$$

By induction $Q^{s-p^k}\chi\xi_i = 0$ unless $s - p^k \equiv 0 \bmod p^i$. Since $s \not\equiv 0 \bmod p^i$, this means $k < i$ and $s \equiv p^k \bmod p^i$. Here $(p^k, s(p-1)) = (p^k, p^k(p-1)) = 0$. Thus $Q^s\chi\xi_i = 0$.

Case 2. $s \equiv 0 \bmod p^i$.

Again

$$P_*^{p^k} Q^s \chi\xi_i = -(p^k, s(p-1)) Q^{s-p^k} \chi\xi_i$$

$$= \begin{cases} 0 & \text{if } k < i \text{ or } p^k \geq s \\ (-1)^i (p^k, s(p-1)) \beta Q^{s-p+\rho(i)} \tau_0 & \text{if } s > p^k \geq p^i,\ p > 2 \\ (2^k, s) Q^{s+2^i-2-2^k} \xi_1 & \text{if } s > 2^k \geq 2^i,\ p = 2 \end{cases}$$

by induction. On the other hand,

$$P_*^{p^k} \beta Q^{s+\rho(i)} \tau_0 = -(p^k, s(p-1) + p^i - 2) \beta Q^{s+\rho(i)-p^k} \tau_0 \quad \text{if } p > 2,$$

and

$$P_*^{2^k} Q^{s+2^i-2} \xi_1 = (2^k, s+2^i - 2) Q^{s+2^i-2-2^k} \xi_1 \qquad \text{if } p = 2.$$

Since $s \equiv 0 \bmod p^i$,

$$(p^k, s(p-1) + p^i - 2) = \begin{cases} 0 & \text{for } 1 \leq k < i \\ (p^k, s(p-1)) & \text{for } k \geq i \end{cases}$$

It suffices to show that $P_*^{p^k} \beta Q^{s+\rho(i)} \tau_0 = 0$ for $s \leq p^k < s + \rho(i)$, when $p > 2$, and that $P_*^{2^k} Q^{s+2^i-2} \xi_1 = 0$ for $s \leq 2^k < s+2^i-2$. These inequalities imply that $s = p^k$, so that $(p^k, s(p-1)) = 0$.

Case 3. $s \not\equiv 0$ or $-1 \bmod p$.

Again,

$$P_*^{p^k} Q^s \chi\xi_i = -(p^k, s(p-1)) Q^{s-p^k} \chi\xi_i = 0$$

by induction.

Case 4. $s \equiv -1 \bmod p^i$

Here,

$$P_*^{p^k} Q^s \chi\xi_i = -(p^k, s(p-1)) Q^{s-p^k} \chi\xi_i - (p^k - p, s(p-1) + p)(Q^{((s+1)/p)-p^{k-1}} \chi\xi_{i-1})^p$$

by Lemma 6.6 and the Cartan formula.

For $1 \leq k < i$, $Q^{s-p^k}\chi\xi_i = 0$ by induction. Since $\frac{s+1}{p} - p^{k-1} \equiv -p^{k-1} \bmod p^{i-1}$, $Q^{((s+1)/p)-p^{k-1}} \chi\xi_{i-1} = 0$ for $1 < k < i$. For $k = 1 < i$,

$$P_*^p Q^s \chi\xi_i = \begin{cases} (-1)^i (\beta Q^{((s+1)/p)-1+\rho(i-1)} \tau_0)^p = (-1)^i \beta Q^{s-p+\rho(i)} \tau_0 & \text{for } p > 2 \\ (Q^{((s+1)/2)-1+2^{i-1}-2} \xi_1)^2 = Q^{s+2^i-4} \xi_1 & \text{for } p = 2 \end{cases}$$

by induction and the p-th power lemma. On the other hand, for $p^k < s + \rho(i)$ and $p > 2$,

$$P_*^{p^k} \beta Q^{s+\rho(i)} \tau_0 = -(p^k, s(p-1) + p^i - 2) \beta Q^{s+\rho(i)-p^k} \tau_0$$

and for $p = 2$ and $2^k < s + 2^i - 2$,

$$P_*^{2^k} Q^{s+2^i-2} \xi_1 = (2^k, s+2^i-2) Q^{s+2^i-2-2^k} \xi_1 \; .$$

Since $s \equiv -1 \bmod p^i$, the right-hand side of the binomial coefficient is congruent to $p^i - p - 1 \bmod p^i$. Thus, if $1 < k < i$, the coefficient is zero and if $k = 1$, the coefficient is -1.

For $s > p^k \geq p^i$ and $i > 1$,

$$P_*^{p^k} Q^s \chi\xi_i = \begin{cases} -[(p^k, s(p-1)) + (p^k - p, s(p-1)+p](-1)^i \beta Q^{s+\rho(i)-p^k} \tau_0 & \text{for } p > 2 \\ [(2^k, s) + (2^k - 2, s+2)] Q^{s-2^k+2^i-2} \xi_1 & \text{for } p = 2, \end{cases}$$

by induction and the p-th power lemma. Thus, for these values of k, it suffices to check that

$$(p^k,s(p-1)) + (p^k - p,s(p-1) + p) = (p^k,s(p-1) + p^i - 2),$$

which the reader may verify (or c.f. [101, p.54]).

For $p > 2$, $i = 1$ and $s > p^k$,

$$P_*^{p^k} Q^s \chi\xi_1 = -(p^k,s(p-1))(-\beta Q^{s+1-p^k}\tau_0)$$

by induction, while

$$P_*^{p^k} \beta Q^{s+1}\tau_0 = -(p^k,(s+1)(p-1))\beta Q^{s+1-p^k}\tau_0 ,$$

and the binomial coefficients here are equal.

For $s < p^k \leq s+\rho(i)$, when $p > 2$, or for $s < 2^k \leq s + 2^i-2$, when $p=2$, a simple calculation shows that $s = p^k-1$. Here

$$P_*^{p^k} \beta Q^{p^k-1+\rho(i)}\tau_0 = -(p^k,p^k(p-1) + p(p^{i-1}-1))\beta Q^{\rho(i)-1}\tau_0 \qquad \text{for } p > 2$$

$$P_*^{2^k} Q^{2^k-1+2^i-2}\xi_1 = (2^k,2^k+2^i-3)Q^{2^i-3}\xi_1 \qquad \text{for } p = 2$$

Since $k \geq i > 1$, the binomial coefficient is zero.

<u>Case 5</u>. $s \equiv -1 \bmod p$, but $s \not\equiv -1 \bmod p^2$, $i > 1$ and $k > 1$.

Here,

$$P_*^{p^k} Q^s \chi\xi_i = -(p^k,s(p-1))Q^{s-p^k}\chi\xi_i$$

by Lemma 6.6. But $s-p^k \not\equiv -1 \bmod p^2$, so that $Q^{s-p^k}\chi\xi_i = 0$.

<u>Case 6</u>. $s \equiv -1 \bmod p^2$, but $s \not\equiv -1 \bmod p^i$; or $s \equiv -1 \bmod p$ but $s \not\equiv -1 \bmod p^2$, $k = 1$ and $i > 1$.

Here,

$$P_*^{p^k} Q^s \chi\xi_i = -(p^k,s(p-1))Q^{s-p^k}\chi\xi_i - (p^k-p,s(p-1)+p)(Q^{((s+1)/p)-p^{k-1}}\chi\xi_{i-1})^p.$$

Now $s - p^k \equiv -1 \bmod p^i$ if and only if $\frac{s+1}{p} - p^{k-1} \equiv 0 \bmod p^{i-1}$. Since $\frac{s+1}{p} \not\equiv 0 \bmod p^{i-1}$ either $Q^{s-p^k}\chi\xi_i$ and $(Q^{((s+1)/p)-p^{k-1}}\chi\xi_{i-1})^p$ are both zero or they are both equal to the appropriate operation on τ_0 if $p > 2$ or ξ_1 if $p = 2$. In the latter case, the coefficients cancel as $k < i$ and $s \equiv p^k - 1 \bmod p^i$.

Lemma 6.8. For $p > 2$, $i \geq 0$ and $s > 0$,

$$\beta Q^s \chi\tau_i = \begin{cases} Q^s \chi\xi_i = (-1)^{i+1} \beta Q^{s+\rho(i)} \tau_0 & \text{if } s \equiv 0 \bmod p^i \\ 0 & \text{otherwise} . \end{cases}$$

Proof. We argue by induction on s and i. The lemma is trivial for $i = 1$ or for $0 < s < p^i$. Again, both sides agree under β and P^1_*. We shall show that both sides agree under $P^{p^k}_*$ for $k > 0$.

Case 1: $s \equiv 0 \bmod p$.

Here $\beta Q^{s-p^k} \chi\tau_i = Q^{s-p^k} \chi\xi_i$ by induction. By Lemma 6.6,

$$\begin{aligned} P^{p^k}_* \beta Q^s \chi\tau_i &= -[(p^k, s(p-1) - 1) + (p^k - 1, s(p-1))] Q^{s-p^k} \chi\xi_i \\ &= -(p^k, s(p-1)) Q^{s-p^k} \chi\xi_i \\ &= P^{p^k}_* Q^s \chi\xi_i . \end{aligned}$$

Therefore, $\beta Q^s \chi\tau_i = Q^s \chi\xi_i$.

Case 2. $s \not\equiv 0 \bmod p$.

Here, by Lemma 6.6,

$$P^{p^k}_* \beta Q^s \chi\tau_i = -(p^k, s(p-1) - 1) \beta Q^{s-p^k} \chi\tau_i ,$$

but $\beta Q^{s-p^k} \chi\tau_i = 0$ by induction.

Proposition 6.9. For $p > 2$, $s > 0$ and $i \geq 0$,

$$Q^s \chi\tau_i = \begin{cases} (-1)^{i+1} Q^{s+\rho(i)} \tau_0 & \text{if } s \equiv 0 \bmod p^i \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have shown that both sides of the prospective equation agree under the Bockstein. By Lemma 6.1,

$$P^{p^k}_* Q^s \chi\tau_i = -(p^k, s(p-1)) Q^{s-p^k} \chi\tau_i .$$

For fixed i, we argue by induction on s that $P_*^{p^k}$ agree on both sides of the prospective equation. Again the assertion is a triviality for $i = 0$, for $k = 0$, or for $0 < s < p^i$.

Case 1: $s \not\equiv 0 \bmod p$.

Here, $Q^{s-p^k}\chi\tau_i = 0$ by induction.

Case 2: $s \equiv 0 \bmod p$ but $s \not\equiv 0 \bmod p^i$.

By induction, $Q^{s-p^k}\tau_i = 0$ unless $k < i$ and $s \equiv p^k \bmod p^i$. Here

$$(p^k, s(p-1)) = (p^k, p^k(p-1)) = 0.$$

Case 3: $s \equiv 0 \bmod p^i$.

Here $Q^{s-p^k}\tau_i = 0$ by induction for $k < i$. Again by induction,

$$P_*^{p^k} Q^s \chi\tau_i = -(p^k, s(p-1))(-1)^{i+1} Q^{s-p^k+\rho(i)}\tau_0 ,$$

for $i \leq k < s$. We have

$$P_*^{p^k} Q^{s+\rho(i)}\tau_0 = -(p^k, s(p-1) + p^i - 1) Q^{s-p^k+\rho(i)}\tau_0 .$$

Since $s \equiv 0 \bmod p^i$,

$$(p^k, s(p-1) + p^i - 1) = \begin{cases} 0 & \text{for } 0 < k < i \\ (p^k, s(p-1)) & \text{for } k \geq i. \end{cases}$$

For $s \leq p^k < s+\rho(i)$, $s = p^k$ and

$$P_*^{p^k} Q^{s+\rho(i)}\tau_0 = -(p^k, p^k(p-1)) Q^{\rho(i)}\tau_0$$

$$= 0 .$$