CHAPTER II

MISCELLANEOUS APPLICATIONS IN STABLE HOMOTOPY THEORY

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A number of important results in stable homotopy theory are very easy consequences of quite superficial properties of extended powers of spectra. We give several such applications here.

The preservation properties of equivariant half-smash products (e.g. in I.1.2) do not directly imply such properties for extended powers since the j^{th} power functor from spectra to Σ_j-spectra tends not to enjoy such properties. We illustrate the point in section 1 by analyzing the structure of extended powers of wedges and deriving useful consequences about extended powers of sums of maps. These results are largely spectrum level analogs of results of Nishida [90] about extended powers of spaces, but the connection with transfer was suggested by ideas of Segal [96].

Reinterpreting Nishida's proof [90], we show in section 2 that the nilpotency of the ring π_*S of stable homotopy groups of spheres (or "stable stems") is an immediate consequence of the Kahn-Priddy theorem and our analysis of extended powers of wedges. The implication depends only on the fact that the sphere spectrum is an H_∞ ring spectrum. This proof gives a very poor estimate of the order of nilpotency. Nishida also gave a different proof [90] which applies only to elements of order p but gives a much better estimate of the order of nilpotency. In section 6, we show that this too results by specialization to S of a result valid for general H_∞ ring spectra. Here the key step is an application of a splitting theorem that Steinberger will prove by use of homology operations in the next chapter. His theorem will make clear to what extent this method of proof applies to elements of order p^i with $i > 1$.

The material discussed so far dates to 1976-77 (and was described in [72]). The material of sections 3-5 is much more recent, dating from 1982-83. The ideas here are entirely due to Miller, Jones, and Wegmann, who saw applications of extended powers that we had not envisaged. (However, all of the information about extended powers needed to carry out their ideas was already explicit or implicit in [72] and the 1977 theses [23, 101] of Bruner and Steinberger.) Jones and Wegmann [44] constructed new homology and cohomology theories from old ones by use of systems of extended powers and showed that theorems of Lin [53] and Gunawardena [38] imply that these theories specialize to give exotic descriptions of stable homotopy and stable cohomotopy. Jones [43] later gave a remarkably ingenious proof of the Kahn-Priddy theorem in terms of these theories. The papers [43, 44] only treated the case $p = 2$, and we give the details for all primes in sections 3 and 4. (In fact, much of the work goes through for non-prime integers.) The idea for the Jones-Wegmann theories grew out of Haynes Miller's unpublished observation that systems of extended powers can be used to realize cohomologically a basic algebraic construction introduced by Singer [52, 98]. We explain this fact and its relationship to the cited theorems of Lin and Gunawardena in section 5.

§1. Extended powers of wedges and transfer maps

Fix positive integers j and k and spectra Y_i for $1 \leq i \leq k$. Let $Y = Y_1 \vee \cdots \vee Y_k$ and let $\nu_i : Y_i \to Y$ be the inclusion. For a partition $J = (j_1, \ldots, j_k)$ of j, $j_i \geq 0$ and $j_1 + \cdots j_k = j$, write $\alpha_J = \alpha_{j_1, \ldots, j_k}$ and let f_J denote the composite

$$D_{j_1} Y_1 \wedge \cdots \wedge D_{j_k} Y_k \xrightarrow{D_{j_1}\nu_1 \wedge \cdots \wedge D_{j_k}\nu_k} D_{j_1} Y \wedge \cdots \wedge D_{j_k} Y \xrightarrow{\alpha_J} D_j Y .$$

For later use, note that permutations $\sigma \in \Sigma_k$ act on partitions and that I.2.8 implies the equivariance formula $f_J = f_{\sigma J} \circ \sigma$. Note too that, for maps $h_i : Y_i \to E$ with wedge sum $h : Y \to E$, the following diagram commutes by the naturality of α_J.

$$\begin{array}{ccc} D_{j_1} Y_1 \wedge \cdots \wedge D_{j_k} Y_k & \xrightarrow{f_J} & D_j Y \\ {\scriptstyle D_{j_1} h_1 \wedge \cdots \wedge D_{j_k} h_k} \downarrow & & \downarrow {\scriptstyle D_j h} \\ D_{j_1} E \wedge \cdots \wedge D_{j_k} E & \xrightarrow{\alpha_J} & D_j E \end{array}$$

Theorem 1.1. Let $Y = Y_1 \vee \cdots \vee Y_k$. Then the wedge sum

$$f_j : \bigvee_J D_{j_1} Y_1 \wedge \cdots \wedge D_{j_k} Y_k \longrightarrow D_j Y$$

of the maps f_J is an equivalence of spectra.

Proof. By the distributivity of smash products over wedges,

$$Y^{(j)} \cong \bigvee_I Y_{i_1} \wedge \cdots \wedge Y_{i_j} ,$$

where I runs over all sequences $(i_1,\ldots,i_j)$ such that $1 \le i_r \le k$. Say that $I \in J$ if there are exactly j_s entries i_r equal to s for each s from 1 to k. For each partition J of j, let $\Sigma_J = \Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$ and define

$$Y_J = \bigvee_{I\in J} Y_{i_1} \wedge \cdots \wedge Y_{i_j} \cong \Sigma_j \ltimes_{\Sigma_J} (Y_1^{(j_1)} \wedge \cdots \wedge Y_k^{(j_k)}).$$

(Here the isomorphism would be obvious on the space level and holds on the spectrum level by direct inspection of the definitions in [Equiv. II §§3-4].) Then Y_J is a Σ_j-subspectrum of $Y^{(j)}$ and $Y^{(j)} = \bigvee_J Y_J$. Now

$$D_jY = \bigvee_J E\Sigma_j \ltimes_{\Sigma_j} Y_J \quad \text{and} \quad E\Sigma_j \ltimes_{\Sigma_j} Y_J = E\Sigma_j \ltimes_{\Sigma_J} (Y^{(j_1)} \wedge \cdots \wedge Y^{(j_k)})$$

by I.1.2(i) and I.1.4. Clearly f_J has image in $E\Sigma_j \ltimes_{\Sigma_j} Y_J$ and factors as the composite

$$\begin{array}{c}
(E\Sigma_{j_1} \ltimes_{\Sigma_{j_1}} Y_1^{(j_1)}) \wedge \cdots \wedge (E\Sigma_{j_k} \ltimes_{\Sigma_{j_k}} Y_k^{(j_k)}) \\
\downarrow \alpha \\
(E\Sigma_{j_1} \times \cdots \times E\Sigma_{j_k}) \ltimes_{\Sigma_J} (Y^{(j_1)} \wedge \cdots \wedge Y^{(j_k)}) \\
\downarrow i \ltimes 1 \\
E\Sigma_j \ltimes_{\Sigma_J} (Y^{(j_1)} \wedge \cdots \wedge Y^{(j_k)}) .
\end{array}$$

Here α is an isomorphism. (Technically, the smash product in its domain is "internal" while that in its range is "external"; see [Equiv, II§3].) The map $i: E\Sigma_{j_1} \times \cdots \times E\Sigma_{j_k} \longrightarrow E\Sigma_j$ is given by the commutation with products and naturality of the functor E and is a Σ_J-equivalence. Therefore $i \ltimes 1$ is an equivalence (by [Equiv, VI.1.15]). The conclusion follows.

Our interest is mainly in finite wedges, but precisely the same argument applies to give an analog for infinite wedges.

<u>Theorem 1.2</u>. Let $\{Y_i\}$ be a set of spectra indexed on a totally ordered set of indices and let $Y = \bigvee_i Y_i$. For a strictly increasing sequence $I = \{i_1,\ldots,i_k\}$ of indices and a partition $J = (j_1,\ldots,j_k)$ of j with each $j_i > 0$ (hence $k \le j$), let

$$f_{J,I}: D_{j_1}Y_{i_1} \wedge \cdots \wedge D_{j_k}Y_{i_k} \to D_jY$$

be the composite of f_J and the evident inclusion. Then the wedge sum

$$f_j : \bigvee_{J,I} D_{j_1} Y_{i_1} \wedge \cdots \wedge D_{j_k} Y_{i_k} \longrightarrow D_j Y$$

of the maps $f_{J,I}$ is an equivalence of spectra.

Parenthetically, this leads to an attractive alternative version of the definition, I.4.3, of an H_∞^d ring spectrum.

Proposition 1.3. An H_∞^d ring structure on E determines and is determined by an H_∞ ring structure on the wedge $\bigvee_i \Sigma^{di}E$.

Proof. If $\bigvee_i \Sigma^{di}E$ is an H_∞ ring spectrum with structural maps ξ_j, then the evident composites

$$\xi_{j,i} : D_j \Sigma^{di} E \longrightarrow D_j(\bigvee_h \Sigma^{dh}E) \xrightarrow{\xi_j} \bigvee_h \Sigma^{dh}E \longrightarrow \Sigma^{di}E$$

give E an H_∞^d ring structure. If E is an H_∞^d ring spectrum with structural maps $\xi_{j,i}$, then the maps

$$\xi_j : D_j(\bigvee_i \Sigma^{di}E) \xrightarrow{f_j^{-1}} \bigvee_{J,I} D_{j_1}\Sigma^{di_1}E \wedge \cdots \wedge D_{j_k}\Sigma^{di_k}E \longrightarrow \bigvee_i \Sigma^{di}E$$

determined by the composites

$$D_{j_1}\Sigma^{di_1}E \wedge \cdots \wedge D_{j_k}\Sigma^{di_k}E \xrightarrow{\xi_{j_1,i_1} \wedge \cdots \wedge \xi_{j_k,i_k}} \Sigma^{dj_1i_1}E \wedge \cdots \wedge \Sigma^{dj_ki_k}E \xrightarrow{\phi} \Sigma^{dr}E,$$

$r = \sum_{a=1}^{k} j_a i_a$, give $\bigvee_i \Sigma^{di}E$ on H_∞ ring structure. These correspondences are inverse to one another.

Returning to the context of Theorem 1.1, let

$$g_J : D_j(Y_1 \vee \cdots \vee Y_k) \longrightarrow D_{j_1}Y_1 \wedge \cdots \wedge D_{j_k}Y_k$$

denote the J^{th} component of f_j^{-1}. Thus g_J is the composite of the projection to $E\Sigma_j \ltimes_{\Sigma_J} Y_J$ and the inverse of the equivalence $(i \ltimes 1)\alpha$ in the proof of the theorem.

The theorem is of particular interest when $Y_1 = \cdots = Y_k$, hence we change notations and consider a spectrum Y and its k-fold wedge sum, which we denote by $^{(k)}Y$. Recall that finite wedges are finite products in the stable category and let

$$\Delta : Y \longrightarrow {}^{(k)}Y \quad \text{and} \quad \nabla : {}^{(k)}Y \longrightarrow Y$$

denote the diagonal and folding maps.

Definition 1.4. Define $\tau_J: D_jY \to D_{j_1}Y \wedge \cdots \wedge D_{j_k}Y$ to be the composite

$$D_jY \xrightarrow{D_j\Delta} D_j({}^{(k)}Y) \xrightarrow{g_J} D_{j_1}Y \wedge \cdots \wedge D_{j_k}Y .$$

Explicitly, let $\pi_J: ({}^{(k)}Y)^{(j)} \to \bigvee_{I \,\varepsilon\, J} Y^{(j)}$ be the projection and let τ_J also denote the map

$$E\Sigma_j \ltimes_{\Sigma_j} (\pi_J \Delta^{(j)}): E\Sigma_j \ltimes_{\Sigma_j} Y^{(j)} \longrightarrow E\Sigma_j \ltimes_{\Sigma_j} (\bigvee_{I \,\varepsilon\, J} Y^{(j)}) = E\Sigma_j \ltimes_{\Sigma_J} Y^{(j)}.$$

Our original map τ_J is the composite of this map and the equivalence $[(i \ltimes 1)\alpha]^{-1}$. We write τ_j for $\tau_J: D_jY \to Y^{(j)}$ when $k = j$ and each $j_s = 1$.

We think of τ_J as a kind of spectrum level transfer map. When $Y = \Sigma^\infty X^+$ for a space X and $\pi \subset \Sigma_j$, we have

$$E\Sigma_j \ltimes_\pi Y^{(j)} \cong \Sigma^\infty(E\Sigma_j \ltimes_\pi (X^+)^{(j)}) = \Sigma^\infty(E\Sigma_j \times_\pi X^j)^+$$

by I.1.1. We shall prove the following result in the sequel.

Theorem 1.5. When $Y = \Sigma^\infty X^+$, the map

$$\tau_J: E\Sigma_j \ltimes_{\Sigma_j} Y^{(j)} \to E\Sigma_j \ltimes_{\Sigma_J} Y^{(j)}$$

is the transfer associated to the natural cover

$$E\Sigma_j \times_{\Sigma_J} X^j \to E\Sigma_j \times_{\Sigma_j} X^j .$$

We do not wish to overemphasize this result. As we shall see, the spectrum level maps τ_J, for general Y, are quite easily studied directly.

The importance of these maps is that they measure the deviation from additivity of the functor D_jY.

For maps $h_i: Y \to E$, $h_1 + \cdots + h_k$ is defined to be $\nabla(h_1 \vee \cdots \vee h_k)\Delta$. Thinking now in cohomological terms, consider the h_i as elements of the Abelian group $E^0Y = [Y,E]$ of maps $Y \to E$ in $\overline{h}\mathscr{S}$.

Corollary 1.6. $D_j(h_1 + \cdots + h_k) = \sum_J \tau_J^*(\alpha_J(D_{j_1}h_1 \wedge \cdots \wedge D_{j_k}h_k))$. Moreover, the following equivariance formula holds for $\sigma \,\varepsilon\, \Sigma_k$.

$$\tau_J^*(\alpha_J(D_{j_1}h_1 \wedge \cdots \wedge D_{j_k}h_k)) = \tau_{\sigma J}^*(\alpha_{\sigma J}(D_{j_{\sigma^{-1}(1)}}h_{\sigma^{-1}(1)} \wedge \cdots \wedge D_{j_{\sigma^{-1}(k)}}h_{\sigma^{-1}(k)})).$$

Proof. By Theorem 1.1 and the naturality diagram preceding it, the following diagram commutes.

$$\begin{tikzcd}
D_jY \arrow[r, "D_j\Delta"] \arrow[d, "\Delta"'] & D_j({}^{(k)}Y) \arrow[rr, "D_j(h_1 \vee \cdots \vee h_k)"] \arrow[d, "(g_J)"] & & D_j({}^{(k)}E) \arrow[r, "D_j\nabla"] \arrow[d, "(g_J)"] & D_jE \\
\bigvee_J D_jY \arrow[r, "\vee\tau_J"] & \bigvee_J D_{j_1}Y \wedge \cdots \wedge D_{j_k}Y \arrow[rr, "\vee D_{j_1}h_1 \wedge \cdots \wedge D_{j_k}h_k"] & & \bigvee_J D_{j_1}E \wedge \cdots \wedge D_{j_k}E \arrow[r, "\vee\alpha_J"] & \bigvee_J D_jE \arrow[u, "\nabla"']
\end{tikzcd}$$

The equivariance follows from I.2.8, the formula $f_J = f_{\sigma J} \circ \sigma$, and the fact that $\sigma\Delta = \Delta$.

Taking each h_i to be the identity map, we obtain the following special case.

Corollary 1.7. $D_j(k) = \sum_J \tau_J^*(\alpha_J)$, and $\tau_J^*(\alpha_J)$ depends only on the conjugacy class of J under the action of Σ_j.

When j is a prime number p and $k = p^iq$ with $i \geq 1$ and q prime to p, a simple combinatorial argument demonstrates that every conjugacy class of partitions has p^is elements for some $s \geq 1$ except for the conjugacy class of the partition $J(k) = (1,\ldots,1,0,\ldots,0)$, p values 1, which has $(p,k-p)$ elements. Of course, p^{i-1} but not p^i divides this binomial coefficient. A trivial diagram chase based on use of the projection ${}^{(k)}Y \to {}^{(p)}Y$ shows that $\tau_{J(k)}$ coincides with $\tau_{J(p)} = \tau_p: D_pY \to Y^{(p)}$. Also, by I.2.7 and I.2.11, $\alpha_{J(p)} = \iota_p: E^{(p)} \to D_pE$. Putting these observations together, we obtain the following result.

Corollary 1.8. If $k = p^iq$ with p prime, $i \geq 1$, and q prime to p, then $D_pk: D_pY \to D_pY$ can be expressed in the form $p^i\lambda + (p,k-p)\iota_p\tau_p$ for some map λ.

In favorable cases, the following three lemmas will lead to a more precise calculation of D_p on general sums.

Lemma 1.9. The following diagram commutes for all Y, j, and k and all partitions J of j.

$$\begin{tikzcd}
D_jY \arrow[r, "\tau_J"] \arrow[d, "\tau_j"'] & D_{j_1}Y \wedge \cdots \wedge D_{j_k}Y \arrow[d, "\tau_{j_1} \wedge \cdots \wedge \tau_{j_k}"] \\
Y^{(j)} \arrow[r, equal] & Y^{(j_1)} \wedge \cdots \wedge Y^{(j_k)}
\end{tikzcd}$$

Proof. This follows from a straightforward diagram chase which boils down to the factorization of $\Delta: Y \to {}^{(j)}Y$ as the composite

$$Y \xrightarrow{\Delta} {}^{(k)}Y \xrightarrow{\Delta \vee \cdots \vee \Delta} {}^{(j_1)}Y \vee \cdots \vee {}^{(j_k)}Y$$

(where $\Delta: Y \to {}^{(0)}Y = S$ is interpreted as the zero map if any $j_r = 0$).

Lemma 1.10. The composite $\tau_j \iota_j : Y^{(j)} \to Y^{(j)}$ is the sum over $\sigma \in \Sigma_j$ of the permutation maps $\sigma: Y^{(j)} \to Y^{(j)}$.

Proof. This is an easy direct inspection of definitions and may be viewed as a particularly trivial case of the double coset formula.

Lemma 1.11 For any ordinary homology theory H_*, the composite

$$H_* D_j Y \xrightarrow{\tau_{J*}} H_*(D_{j_1} Y \wedge \cdots \wedge D_{j_k} Y) \xrightarrow{\alpha_{J*}} H_* D_j Y$$

is multiplication by the multinomial coefficient $(j_1, \ldots, j_k)$. In particular, $\iota_{j*}\tau_{j*}$ is multiplication by $j!$.

Proof. We may assume that Y is a CW-spectrum and exploit I.2.1. Since $\pi_i \Delta \simeq 1: Y \to Y$, where $\pi_i: {}^{(k)}Y \to Y$ is the i^{th} projection, $\Delta_*: C_*Y \to C_*({}^{(k)}Y) = C_*Y \oplus \cdots \oplus C_*Y$ is chain homotopy equivalent to the algebraic diagonal. With $Y_1 = \cdots = Y_k = Y$, the composite $(i \ltimes 1)\alpha$ in the proof of Theorem 1.1 induces α_J upon passage to orbits over Σ_j (rather than over $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$). Therefore $\alpha_J \circ \tau_J$ is just the composite

$$W_j \ltimes_{\Sigma_j} Y^{(j)} \xrightarrow{1 \ltimes \Delta^{(j)}} W_j \ltimes_{\Sigma_j} ({}^{(k)}Y)^{(j)} \xrightarrow{1 \ltimes \pi_J} W_j \ltimes_{\Sigma_j} \Big(\bigvee_{I \in J} Y^{(j)}\Big) \xrightarrow{1 \ltimes \nabla} W_j \ltimes_{\Sigma_j} Y^{(j)}.$$

Since there are $(j_1, \ldots, j_k)$ sequences $I \in J$ and thus $(j_1, \ldots, j_k)$ wedge summands here, the conclusion clearly holds on the level of cellular chains.

§2. Power operations and Nishida's nilpotency theorem

Let E be an H_∞ ring spectrum and Y be any spectrum. Recall from I.4.1 that we have power operations $\mathcal{P}_j: E^0 Y \to E^0 D_j Y$ specified by $\mathcal{P}_j(h) = \xi_j D_j(h)$. We use the results of section 1 to derive additivity formulas for these operations and apply these formulas to derive the nilpotency of $\pi_* S$.

Lemma 2.1. For $h_i \in E^0Y$, $\mathcal{P}_j(h_1 + \cdots + h_k) = \sum_J \tau_J^*(\mathcal{P}_{j_1}(h_1) \wedge \cdots \wedge \mathcal{P}_{j_k}(h_k))$, where the product $\wedge$ is the external product in E-cohomology and the sum extends over all partitions $J = (j_1,\ldots,j_k)$ of j.

Proof. This is immediate from Corollary 1.6 and the commutative diagram

$$\begin{tikzcd} D_{j_1}E \wedge \cdots \wedge D_{j_k}E \arrow[r, "\alpha_J"] \arrow[d, "\xi_{j_1} \wedge \cdots \wedge \xi_{j_k}"'] & D_jE \arrow[d, "\xi"] \\ E \wedge \cdots \wedge E \arrow[r, "\phi"] & E \end{tikzcd}$$

Here the terms with one $j_i = j$ and the rest zero give the sum of the $\mathcal{P}_j(h_i)$. When j is a prime number p, the remaining error term simplifies. The full generality of the following result is due to McClure.

Proposition 2.2. Let $h_i \in E^0Y$. If $p = 2$, then

$$\mathcal{P}_2(h_1 + \cdots + h_k) = \mathcal{P}_2(h_1) + \cdots + \mathcal{P}_2(h_k) + \sum_{1\leq i<j\leq k} \tau_2^*(h_i \wedge h_j).$$

If p is an odd prime and Y and E are p-local, then

$$\mathcal{P}_p(h_1+\cdots+h_k) = \mathcal{P}_p(h_1) +\cdots+ \mathcal{P}_p(h_k) + \tau_p^*(\tfrac{1}{p!}[(h_1+\cdots+h_k)^p - (h_1^p+\cdots+h_k^p)]).$$

In particular, $\mathcal{P}_p(kh) = k\,\mathcal{P}_p(h) + \frac{1}{p!}(k^p - k)\tau_p^*(h^p)$ in both cases.

Proof. We must show that

$$j_1! \cdots j_k!\ \tau_J^*(\mathcal{P}_{j_1}(h_1) \wedge \cdots \wedge \mathcal{P}_{j_k}(h_k)) = \tau_p^*(h_1^{j_1} \wedge \cdots \wedge h_k^{j_k})$$

for a partition $J = (j_1,\ldots,j_k)$ of p with no $j_i = p$. By Lemma 1.9, $\tau_p^* = \tau_J^*(\tau_{j_1} \wedge \cdots \wedge \tau_{j_k})^*$. Thus it suffices to show that

$$j!\,\mathcal{P}_j(h) = \tau_j^*(h^j)$$

for any $j \geq 0$ and $h \in E^0(Y)$. If $j = 0$, $h^{(0)}$ and $D_0(h)$ are to be interpreted as the identity map of S and the conclusion is trivial. If $j = 1$, the conclusion is also trivial. There are no more cases if $p = 2$, so assume that $p > 2$ and $1 < j < p$. By Lemma 1.11, the composite

$$D_jY \xrightarrow{\tau_j} Y^{(j)} \xrightarrow{\iota_j} D_jY$$

induces multiplication by $j!$ in ordinary homology. It is thus an equivalence since Y and hence also D_jY is p-local. Therefore $\iota_j^*:E^*(D_jY) \to E^*(Y^{(j)})$ is a monomorphism and we need only check that

$$j!\iota_j^*\mathcal{P}_j(h) = \iota_j^*\tau_j^*(h^j).$$

The left side is $j!h^j$. By Lemma 1.10, the right side is the sum over $\sigma \in \Sigma_p$ of $\sigma_*(h^j)$. The commutativity of E implies that $\sigma_*(h^j) = h^j$ for all σ, j, and h, and the conclusion follows.

Now recall from I.4.2 that elements $\alpha \in E_r(D_pS^q)$ determine homotopy operations $\tilde{\alpha}:\pi_qE \to \pi_rE$ via the formula $\tilde{\alpha}(h) = \alpha/\mathcal{P}_p(h)$.

Corollary 2.3. Let $\alpha \in E_r(D_pS^q)$ and $h \in \pi_qE$, where q is even and E is p-local if p is odd. Then

$$\tilde{\alpha}(kh) = k\tilde{\alpha}(h) + \frac{1}{p!}(k^p - k)(\Sigma^{-pq}\tau_{p*}(\alpha))h^p,$$

where the product is the multiplication in π_*E.

Proof. The following diagram is easily seen to commute.

$$\begin{tikzcd}
S^r \arrow[r, "\alpha"] \arrow[d, phantom, "\cong"] & D_pS^q \wedge E \arrow[r, "\tau_p \wedge 1"] & S^{pq} \wedge E \arrow[r, "h^p \wedge 1"] & E \wedge E \arrow[dr, "\phi"] & \\
\Sigma^{-pq}S^r \wedge S^{pq} \arrow[rr, "\Sigma^{-pq}((\tau_p \wedge 1)\alpha) \wedge h^p"] & & \Sigma^{-pq}(S^{pq} \wedge E) \wedge E \cong E \wedge E \arrow[rr, "\phi"] & & E
\end{tikzcd}$$

Thus $\alpha/\tau_p^*(h^p) = (\Sigma^{-pq}\tau_{p*}(\alpha))h^p$. The conclusion follows from the last statement of the previous proposition.

Assuming that E is p-local (when $p = 2$ as well as when p is odd), we obtain the following immediate corollaries.

Corollary 2.4. If $p^ih = 0$, then $p^{i-1}(\Sigma^{-pq}\tau_{p*}(\alpha))h^{p+1} = 0$ for all α.

Here we have multiplied by h to kill $p^i\tilde{\alpha}(h)$. Of course, this may not be necessary.

Corollary 2.5. If both $p^ih = 0$ and $p^i\alpha = 0$, then $p^{i-1}(\Sigma^{-pq}\tau_{p*}(\alpha))h^p = 0$.

One can also arrive at the last two corollaries by direct diagram chases from Corollary 1.8 and the definition of an H_∞ ring spectrum, without bothering with additivity formulae. (That approach was taken in [72], following Nishida [90, §8]).

These relations specialize to give nilpotency assertions, the sharpest estimate being as follows.

Corollary 2.6. Let $x \in \pi_q E$ satisfy $p^i x = 0$, where $i > 0$ and q is even if $p > 2$. Suppose that $x = \Sigma^{-pq}\tau_{p*}(\alpha)$ for some $\alpha \in E_{pq+q}(D_p S^q)$. Then $p^{i-1}x^{p+2} = 0$. Moreover, if $p^i\alpha = 0$, then $p^{i-1}x^{p+1} = 0$.

The problem, of course, is to study $E_*(D_p S^q)$ and τ_{p*}. Everything above applies to an arbitrary H_∞ ring spectrum E, but to compute τ_{p*} we must specialize. If E = MO, for example, then every element of $\pi_* E$ has order 2 and no element is nilpotent, hence $\tau_{2*}: MO_*(D_2 S^q) \to MO_*(S^{2q})$ must be the zero homomorphism for all q. This does not contradict the following assertion.

Conjecture 2.7. Any element of finite order in the kernel of the (integral) Hurewicz homomorphism $\pi_* E \to H_* E$ is nilpotent.

We shall prove the conjecture for elements of order exactly p in section 6, but the methods there fail for general elements of order p^i with $i > 1$.

When we specialize to E = S, we find that the Kahn-Priddy theorem gives appropriate input for application of the results above.

Theorem 2.8. If p = 2, let $\phi(k)$ be the number of integers j such that $0 < j \leq k$ and $j \equiv 0,1,2,$ or $4 \bmod 8$. If $p > 2$, let $\phi(k) = [k/2(p-1)]$. Let q be an integer such that $q \equiv 0 \bmod p^{\phi(k)}$, where q is even if $p > 2$. Then $\tau_{p*}: \pi_r D_p S^q \to \pi_r S^{pq}$ is a (split) epimorphism for $pq < r < pq+k(p-1)$.

We shall prove this in section 4. Actually, the purely stable methods we use will give surjectivity without giving a splitting. For this reason, we are really only entitled to use Corollary 2.4, rather than Corollary 2.5. This doesn't change the heuristic picture, but to give the correct estimate of the order of nilpotency, we assume the splitting (from [46, 95, or 27]) in the discussion to follow.

Theorem 2.9. Let $x \in \pi_n S$ satisfy $p^i x = 0$, where $i > 0$ and n is even if $p > 2$. Let m be minimal such that $mn \equiv 0 \bmod p^{\phi([n/p-1]+1)}$. Then $p^{i-1}x^{mp+1} = 0$. Inductively, some power of x is zero.

Proof. Let q = mn. Since $n < ([n/p-1]+1)(p-1)$, there exists $\alpha \in \pi_{pq+n} D_p S^q$ such that $\Sigma^{-pq}\tau_{p*}(\alpha) = x$. With $h = x^m$, Corollary 2.4 gives $p^{i-1}x^{mp+2} = 0$. Using $p^i\alpha = 0$, Corollary 2.5 gives $p^{i-1}x^{mp+1} = 0$.

Unfortunately, m increases rapidly with n (although our estimate for $p > 2$ is sharper than Nishida's since he only knew Theorem 2.8 for $r < pq+k$). For example, the first stem in which an interesting element x of order 2 occurs is the 14-stem ("interesting" meaning that x is neither in π_*J nor a product of Hopf maps). Here $m = 64$ and we can only conclude that $x^{129} = 0$, a truly stratospheric estimate. So far, and granting that our stemwise calculations still extend through only a very small range, we have no reason to disbelieve that $x^4 = 0$ if $2x = 0$. Corollary 2.6 seems to suggest that this answer might be correct. However, as pointed out to me by Bruner, $\tau_{2*}:\pi_*D_2S^q \to \pi_*S^{2q}$ is not always an epimorphism and thus Corollary 2.6 cannot be used to prove this answer.

§3. The Jones-Wegmann homology and cohomology theories

The next three sections will all make heavy use of certain twisted diagonal maps implicit in the general properties of extended powers.

Definition 3.1. Let π be a subgroup of Σ_j and let W be a free π-CW complex. For a based CW complex X and a CW spectrum Y, define a map of spectra

$$\Delta:(W \ltimes_\pi Y^{(j)}) \wedge X \to W \ltimes_\pi (Y \wedge X)^{(j)}$$

by passage to orbits over π from the π-map

$$(W \ltimes Y^{(j)}) \wedge X \xrightarrow{1\wedge\Delta} (W \ltimes Y^{(j)}) \wedge X^{(j)} \cong W \ltimes (Y\wedge X)^{(j)}.$$

Here the isomorphism is given by I.1.2(ii) and the shuffle π-isomorphism $Y^{(j)} \wedge X^{(j)} \cong (Y\wedge X)^{(j)}$. Note that Δ is the identity map when $X = S^0$ and that the following transitivity and commutativity diagram commutes, where X' is another based CW complex.

$$\begin{tikzcd}
 & (W \ltimes_\pi Y^{(j)}) \wedge X \wedge X' \arrow[r, "1\wedge\tau"] \arrow[dl, "\Delta\wedge 1"'] \arrow[dd, "\Delta"] & (W \ltimes_\pi Y^{(j)}) \wedge X' \wedge X \arrow[dd, "\Delta"] \\
W \ltimes_\pi (Y\wedge X)^{(j)} \wedge X' \arrow[dr, "\Delta"'] & & \\
 & W \ltimes_\pi (Y\wedge X\wedge X')^{(j)} \arrow[r, "1\ltimes(1\wedge\tau)^{(j)}"] & W \ltimes_\pi (Y\wedge X'\wedge X)^{(j)}
\end{tikzcd}$$

With $\pi = \Sigma_j$ and $W = E\Sigma_j$, we obtain

$$\Delta:(D_jY)\wedge X \to D_j(Y\wedge X).$$

Although not strictly relevant to the business at hand, we record the relationship between these maps and the maps ι_j, $\alpha_{j,k}$, $\beta_{j,k}$, and δ_j of I§2 and use them to construct new examples of H_∞ ring spectra.

Lemma 3.2. The following diagrams commute for spectra Y and Z and spaces X. The unlabeled arrows are obvious composites of shuffle maps and the diagonal on X.

$$\begin{tikzcd} Y^{(j)}\wedge X \arrow[r, "\iota_j\wedge 1"] \arrow[d] & D_jY\wedge X \arrow[d, "\Delta"] \\ (Y\wedge X)^{(j)} \arrow[r, "\iota_j"] & D_j(Y\wedge X) \end{tikzcd} \qquad \begin{tikzcd} D_j(Y\wedge Z)\wedge X \arrow[r, "\delta_j\wedge 1"] \arrow[d, "\Delta"] & D_jY\wedge D_jZ\wedge X \arrow[d] \\ D_j(Y\wedge Z\wedge X) \arrow[d] & D_jY\wedge X\wedge D_jZ\wedge X \arrow[d, "\Delta\wedge\Delta"] \\ D_j(Y\wedge X\wedge Z\wedge X) \arrow[r, "\delta_j"] & D_j(Y\wedge X)\wedge D_j(Z\wedge X) \end{tikzcd}$$

$$\begin{tikzcd} D_jY\wedge D_kY\wedge X \arrow[r, "\alpha_{j,k}\wedge 1"] \arrow[d] & D_{j+k}Y\wedge X \arrow[dd, "\Delta"] \\ D_jY\wedge X\wedge D_kY\wedge X \arrow[d, "\Delta\wedge\Delta"] & \\ D_j(Y\wedge X)\wedge D_k(Y\wedge X) \arrow[r, "\alpha_{j,k}"] & D_{j+k}(Y\wedge X) \end{tikzcd} \qquad \begin{tikzcd} D_jD_kY\wedge X \arrow[r, "\beta_{j,k}\wedge 1"] \arrow[d, "\Delta"] & D_{jk}Y\wedge X \arrow[dd, "\Delta"] \\ D_j(D_kY\wedge X) \arrow[d, "D_j\Delta"] & \\ D_jD_k(Y\wedge X) \arrow[r, "\beta_{j,k}"] & D_{jk}(Y\wedge X) \end{tikzcd}$$

I learned the following lemma from Miller and McClure.

Lemma 3.3. Let X be an unbased space and E be an H_∞ ring spectrum. Then the function spectrum $F(X^+,E)$ is an H_∞ ring spectrum with structural maps the adjoints of the composites

$$D_jF(X^+,E)\wedge X \xrightarrow{\Delta} D_j(F(X^+,E)\wedge X^+) \xrightarrow{D_j\varepsilon} D_jE \xrightarrow{\xi_j} E,$$

where ε is the evaluation map. In particular, the dual $F(X^+,S)$ of $\Sigma^\infty X^+$ is an H_∞ ring spectrum.

Proof. If $j = 0$, $\Delta: S\wedge X^+ \simeq \Sigma^\infty X^+ \longrightarrow \Sigma^\infty S^0 = S$ is to be interpreted as $\Sigma^\infty\delta$, where $\delta: X^+ \to S^0$ is the discretization map sending X to the non-basepoint. The diagrams of I.3.1 are easily checked to commute by use of the diagrams of the previous lemma.

Returning to the business at hand, observe that, with $X = S^1$, we obtain a natural map $\Delta:\Sigma D_jY \to D_j\Sigma Y$. Thus, for any integer n (positive or negative), we have the map

$$\Sigma^n\Delta:\Sigma^{n+1}D_j\Sigma^{-n-1}Y = \Sigma^n\Sigma D_j\Sigma^{-1}\Sigma^{-n}Y \longrightarrow \Sigma^nD_j\Sigma^{-n}Y.$$

We shall be interested in the resulting inverse system

$$\cdots \longrightarrow \Sigma^n D_j \Sigma^{-n} Y \longrightarrow \cdots \longrightarrow \Sigma^1 D_j \Sigma^{-1} Y \longrightarrow D_j Y \longrightarrow \Sigma^{-1} D_j \Sigma Y \longrightarrow \cdots \longrightarrow \Sigma^{-n} D_j \Sigma^n Y \longrightarrow \cdots$$

(where $n \geq 0$). By the diagram in Definition 3.1, the maps

$$\Sigma^n \Delta : (\Sigma^n D_j S^{-n}) \wedge X \cong \Sigma^n (D_j S^{-n} \wedge X) \longrightarrow \Sigma^n D_j (S^{-n} \wedge X) \cong \Sigma^n D_j \Sigma^{-n} \Sigma^\infty X$$

specify a morphism of systems, again denoted Δ,

$$\{(\Sigma^n D_j S^{-n}) \wedge X\} \longrightarrow \{\Sigma^n D_j \Sigma^{-n} \Sigma^\infty X\}.$$

We shall study the homological and homotopical properties of these systems. In this section, we consider any $j \geq 2$. We shall obtain calculational results when j is a prime in the following two sections.

Let E_* and E^* denote the homology and cohomology theories represented by a spectrum E. For spectra Y, define

$$E_*^{(j)} Y = \lim E_*(\Sigma^n D_j \Sigma^{-n} Y) \quad \text{and} \quad E^*_{(j)} Y = \operatorname{colim} E^*(\Sigma^n D_j \Sigma^{-n} Y)$$

$$F_*^{(j)} Y = \lim E_*(\Sigma^n D_j S^{-n} \wedge Y) \quad \text{and} \quad F^*_{(j)} Y = \operatorname{colim} E^*(\Sigma^n D_j S^{-n} \wedge Y).$$

Upon restriction to spaces (that is, to $Y = \Sigma^\infty X$), we obtain induced natural transformations

$$\Delta_* : F_*^{(j)} X \longrightarrow E_*^{(j)} X \quad \text{and} \quad \Delta^* : E^*_{(j)} X \longrightarrow F^*_{(j)} X,$$

and these reduce to identity homomorphisms when $X = S^0$. It is clear that $F_*^{(j)}$ is a homology theory and $F^*_{(j)}$ is a cohomology theory on finite CW spectra. Passage to colimits from the homomorphisms

$$(\Sigma^{n-1}\Delta)^* : E^{i+1}(\Sigma^n D_j \Sigma^{-n} \Sigma Y) \cong E^i(\Sigma^{n-1} D_j \Sigma^{-n+1} Y) \longrightarrow E^i(\Sigma^n D_j \Sigma^{-n} Y)$$

yields suspension isomorphisms

$$E^{i+1}_{(j)} \Sigma Y \longrightarrow E^i_{(j)} Y,$$

and Δ^* is easily seen to commute with suspension. The analogous assertions hold for $E_*^{(j)}$. With these notations, the main theorems of Jones and Wegmann [44] read as follows (although they only consider primes j and only provide proofs when $j = 2$).

<u>Theorem 3.4.</u> The functor $E^*_{(j)}$ is a cohomology theory on finite CW spectra, hence $\Delta^* : E^*_{(j)} X \to F^*_{(j)} X$ is an isomorphism for all finite CW complexes X.

Theorem 3.5. Let E be connective and j-adically complete, with $\pi_* E$ of finite type over the j-adic integers $\hat{Z}_j = \underset{p|j}{\times} \hat{Z}_p$. Then $E_*^{(j)}$ is a homology theory on finite CW spectra, hence $\Delta_*: F_*^{(j)}X \to E_*^{(j)}X$ is an isomorphism for all finite CW complexes X.

We defer the proofs for a moment. As Jones and Wegmann point out, these results are no longer valid for infinite CW complexes.

Recall that $D_j S^0 = \Sigma^\infty B\Sigma_j^+$ and the discretization map $B\Sigma_j^+ \to S^0$ induces $\xi_j: D_j S^0 \to S^0$. Upon smashing with Y, the composites

$$\Sigma^n D_j S^{-n} \xrightarrow{\Delta} D_j S^0 \xrightarrow{\xi_j} S^0$$

give a morphism from the system $\{\Sigma^n D_j S^{-n} \wedge Y\}$ to the constant system at Y. We call this map of systems ξ_j and obtain a map of cohomology theories

$$\xi_j^*: E^*Y \longrightarrow F_{(j)}^*Y,$$

commutation with the suspension isomorphisms being easily checked. We shall shortly prove a complement to this observation.

Proposition 3.6. Let E be an H_∞ ring spectrum. Then the composites of the functions

$$\mathcal{P}_j: E^nY = [\Sigma^{-n}Y, E] \longrightarrow [D_j\Sigma^{-n}Y, E] = E^n(\Sigma^n D_j \Sigma^{-n} Y)$$

and the natural homomorphisms $E^n(\Sigma^n D_j \Sigma^{-n}Y) \to E^n_{(j)}Y$ specify a map of cohomology theories

$$\mathcal{P}_j: E^*Y \longrightarrow E_{(j)}^*Y.$$

We thus have the triangle of cohomology theories

$$\begin{tikzcd} & E^*X \arrow[dl, "\mathcal{P}_j"'] \arrow[dr, "\xi_j^*"] & \\ E_{(j)}^*X \arrow[rr, "\Delta^*"] & & F_{(j)}^*X \end{tikzcd}$$

on finite CW complexes X. Since $\mathcal{P}_j(x) = \xi_j \circ D_j(x)$, we see immediately that $\Delta^* \mathcal{P}_j(1) = \xi_j^*(1)$, where $1 \in E^0(S^0)$ is the identity element. It does not follow that $\Delta^* \mathcal{P}_j = \xi_j^*$ in general. As we shall see in the next section, this fails, for example when E = MO. However, as observed by Jones and Wegmann [44], this implication does hold for E = S.

Proposition 3.7. The following diagram commutes for any finite CW complex X.

$$\begin{tikzcd} & \pi^* X \arrow[dl, "\mathcal{P}_j"'] \arrow[dr, "\xi_j^*"] & \\ \operatorname{colim} \pi^*(\Sigma^n D_j \Sigma^{-n} X) \arrow[rr, "\Delta^*"] & & \operatorname{colim} \pi^*(\Sigma^n D_j S^{-n} \wedge X) \end{tikzcd}$$

Proof. Since $\Delta^* \mathcal{P}_j$ and ξ_j^* are morphisms of cohomology theories, they are equal for all X if they are equal for $X = S^0$. Any morphism $\phi: E^*X \to F^+X$ of cohomology theories is given by morphisms of $\pi^* S^0$-modules. When $E^* = \pi^*$ and $X = S^0$, $\phi(x) = \phi(1 \cdot x) = \phi(1) \cdot x$, so that ϕ is determined by its behavior on the unit $1 \in \pi^0(S^0)$.

For general E and $X = S^0$, it is obvious that $\xi_j^*(x) = \xi_j^*(1)x$. It is not at all obvious that $(\Delta^* \mathcal{P}_j)(x) = \Delta^* \mathcal{P}_j(1) \cdot x$ We now have this relation for E = S, and we shall use it to prove the Kahn-Priddy theorem in the next section. As we shall explain in section 5, theorems of Lin when p = 2 and of Gunawardena when p > 2 imply that ξ_p^* and thus $\mathcal{P}_p$ in Proposition 3.7 are actually isomorphisms. We complete this section by giving the deferred proofs, starting with that of Proposition 3.6. We need two lemmas.

Lemma 3.8. The following diagram commutes for any partition $J = (j_1, \ldots, j_k)$ of j.

$$\begin{tikzcd} D_j Y \wedge X \arrow[r, "\tau_J \wedge 1"] \arrow[d, "\Delta"'] & D_{j_1} Y \wedge \cdots \wedge D_{j_k} Y \wedge X \arrow[r, "(\text{shuffle})(1 \wedge \Delta)"] & D_{j_1} Y \wedge X \wedge \cdots \wedge D_{j_k} Y \wedge X \arrow[d, "\Delta \wedge \cdots \wedge \Delta"] \\ D_j(Y \wedge X) \arrow[rr, "\tau_J"] & & D_{j_1}(Y \wedge X) \wedge \cdots \wedge D_{j_k}(Y \wedge X) \end{tikzcd}$$

Proof. The "transfer" τ_J is specified in Definition 1.4, and the proof is an easy naturality argument.

Lemma 3.9. For an H_∞ ring spectrum E, the composite

$$[Y,E] \xrightarrow{\mathcal{P}_j} [D_j Y, E] \xrightarrow{\Delta^*} [\Sigma D_j \Sigma^{-1} Y, E]$$

is a homomorphism.

Proof. By Lemma 2.1, we have the formula

$$\mathcal{P}_j(x + y) = \mathcal{P}_j(x) + \mathcal{P}_j(y) + \sum_{i=1}^{p-1} \tau^*_{i,p-i}(\mathcal{P}_i(x) \wedge \mathcal{P}_{j-i}(x)).$$

With $X = S^1$, Lemma 3.8 and the fact that $\Delta : S^1 \to S^1 \wedge S^1$ is null homotopic imply that $\tau_{i,j-i}\Delta$ is null homotopic.

Thus $\mathcal{P}_j$ in Proposition 3.6 is a natural homomorphism. It is easily checked that $\mathcal{P}_j$ commutes with suspension and this proves the proposition.

Finally, we turn to the proofs of Theorems 3.4 and 3.5. Clearly it only remains to show that $E^*_{(j)}$ and $E^{(j)}_*$ satisfy the exactness axiom on finite CW pairs (Y,B). Although not strictly necessary, we insert a general observation which helps explain the idea and will be used later.

Lemma 3.10. Let $f: B \to Y$ be a map of CW spectra with cofibre Cf. There is a map $\psi: CD_j f \to D_j Cf$, natural in f, such that the diagram

$$\begin{tikzcd} D_j Y \arrow[r, "i"] \arrow[d, equal] & CD_j f \arrow[r, "\partial"] \arrow[d, "\psi"] & \Sigma D_j B \arrow[d, "\Delta"] \\ D_j Y \arrow[r, "D_j i"] & D_j Cf \arrow[r, "D_j \partial"] & D_j \Sigma B \end{tikzcd}$$

commutes, where $i: Y \to Cf$ and $\partial: Cf \to \Sigma B$ are the canonical maps. If f is the inclusion of a subcomplex in a CW spectrum, then the diagram

$$\begin{tikzcd} CD_j f \arrow[r, "\psi"] \arrow[d, "\pi"'] & D_j Cf \arrow[d, "D_j \pi"] \\ D_j Y / D_j B \arrow[r, "\psi"] & D_j (Y/B) \end{tikzcd}$$

also commutes, where the maps π are the canonical (quotient) equivalences and the bottom map ψ is induced by the quotient map $Y \to Y/B$.

Proof. $CD_j f = D_j Y \cup_{D_j f} CD_j B$ and $D_j Cf = D_j (Y \cup_f CB)$; ψ is induced by the inclusion $D_j Y \to D_j Cf$ and the composite of $\Delta: CD_j B \to D_j CB$ and the inclusion $D_j CB \to D_j Cf$. The diagrams are easily checked.

Of course, the bottom row in the first diagram is not a cofibre sequence and ψ is not an equivalence. Now let (Y,B) be a finite CW pair. For notational simplicity, set

$$D_j(Y,B) = D_j Y / D_j B \quad \text{and} \quad Z = Y/B.$$

As n varies, the maps

$$\Sigma^n \psi : \Sigma^n D_j (\Sigma^{-n} Y, \Sigma^{-n} B) \longrightarrow \Sigma^n D_j \Sigma^{-n} Z$$

specify a map of inverse systems, again denoted ψ, and we shall prove the following result.

Proposition 3.11. For any pair (Y,B) of finite CW spectra,

$$\psi^*: E^*_{(j)}Z \longrightarrow \operatorname{Colim} E^*\Sigma^n D_j(\Sigma^{-n}Y, \Sigma^{-n}B)$$

and, under the hypotheses of Theorem 3.5,

$$\psi_*: \lim E_*\Sigma^n D_j(\Sigma^{-n}Y, \Sigma^{-n}B) \longrightarrow E_*^{(j)}Z$$

are isomorphisms.

Note that the assumptions on E in Theorem 3.5 imply that all groups in sight are finitely generated $\hat{Z}_j$-modules and thus that all inverse limits in sight preserve exact sequences. Given the proposition, the required $E^*_{(j)}$ and $E_*^{(j)}$ exact sequences of the pair (Y,B) are obtained by passage to colimits and limits from the E^* and E_* exact sequences of the pairs $(\Sigma^n D_j\Sigma^{-n}Y, \Sigma^n D_j\Sigma^{-n}B)$.

Following ideas of Bruner (which he uses in a much deeper way in chapters V and VI), we prove Proposition 3.11 by filtering $Y^{(j)}$. For $0 \le s \le j$, define

$$\Gamma_s = \Gamma_s(Y,B) = \bigcup Y_1 \wedge \cdots \wedge Y_j,$$

where $Y_r = Y$ or $Y_r = B$ and s of the Y_r are equal to B. We have

$$B^{(j)} = \Gamma_j \subset \Gamma_{j-1} \subset \cdots \subset \Gamma_0 = Y^{(j)}.$$

Each inclusion is a Σ_j-equivariant cofibration, and we define

$$\Pi_s = \Pi_s(Y,B) = \Gamma_s(Y,B)/\Gamma_{s+1}(Y,B).$$

Then $\Pi_0 = Z^{(j)}$ and, for $0 < s < j$, Π_s breaks up as the wedge of its $(s,j-s)$ distinct subspectra of the form $Z_1 \wedge \cdots \wedge Z_j$, where $Z_r = Z$ or $Z_r = B$ and s of the Z_r are equal to B. It follows that Π_s is the free Σ_j-spectrum generated by the $(\Sigma_s \times \Sigma_{j-s})$-spectrum $B^{(s)} \wedge Z^{(j-s)}$. That is,

$$\Pi_s \cong \Sigma_j \ltimes_{\Sigma_s \times \Sigma_{j-s}} B^{(s)} \wedge Z^{(j-s)}.$$

The functor $E\Sigma_j \ltimes_{\Sigma_j} (?)$ converts Σ_j-cofibrations to cofibrations and commutes with quotients, hence we have cofibre sequences

$$(*) \qquad E\Sigma_j \ltimes_{\Sigma_j} \Gamma_s/\Gamma_t \longrightarrow E\Sigma_j \ltimes_{\Sigma_j} \Gamma_r/\Gamma_t \longrightarrow E\Sigma_j \ltimes_{\Sigma_j} \Gamma_r/\Gamma_s$$

for $0 \le r < s < t \le j$. For a based space X, the map $\Delta: D_jY \wedge X \to D_j(Y \wedge X)$ induces compatible maps

$$\Delta : [E\Sigma_j \ltimes_{\Sigma_j} \Gamma_s(Y,B)] \wedge X \longrightarrow E\Sigma_j \ltimes_{\Sigma_j} \Gamma_s(Y \wedge X, B \wedge X)$$

and similarly for Π_s on passage to quotients. The following simple observation is the crux of the matter.

Lemma 3.12. For $0 < s < j$, there is a natural equivalence

$$\alpha : D_s B \wedge D_{j-s} Z \to E\Sigma_j \ltimes_{\Sigma_j} \Pi_s(Y,B)$$

such that the following diagram commutes for any X.

$$\begin{array}{ccccc}
(D_s B \wedge D_{j-s} Z) \wedge X & \xrightarrow{(1\wedge\tau\wedge 1)\Delta} & D_s B \wedge X \wedge D_{j-s} Z \wedge X & \xrightarrow{\Delta\wedge\Delta} & D_s(B\wedge X) \wedge D_{j-s}(Z \wedge X) \\
\downarrow{\scriptstyle \alpha\wedge 1} & & & & \downarrow{\scriptstyle \alpha} \\
[E\Sigma_j \ltimes_{\Sigma_j} \Pi_s(Y,B)] \wedge X & & \xrightarrow{\Delta} & & E\Sigma_j \ltimes_{\Sigma_j} \Pi_s(Y\wedge X, B \wedge X)
\end{array}$$

In particular, the bottom map Δ is null homotopic when $X = S^1$.

Proof. By I.1.4 and the description of $\Pi_s(Y,B)$ above, we have

$$E\Sigma_j \ltimes_{\Sigma_j} \Pi_s(Y,B) \cong E\Sigma_j \ltimes_{\Sigma_s \times \Sigma_{j-s}} B^{(s)} \wedge Z^{(j-s)}.$$

As in the proof of Theorem 1.1, we may replace $E\Sigma_j$ by $E\Sigma_s \times E\Sigma_{j-s}$ on the right side, and it then becomes isomorphic to $D_s B \wedge D_{j-s} Z$. The diagram is easily checked.

Now apply Σ^n to the cofibre sequence (*) for the pair $(\Sigma^{-n}Y, \Sigma^{-n}B)$ with quotient $\Sigma^{-n}Z$. We obtain an inverse system of cofibre sequences for $0 \le r < s < t \le j$. On passage to E^* and then to colimits (or to E_* and then to limits), there results a long exact sequence. For $0 < s < j$, the maps between terms of the system

$$\{\Sigma^n E\Sigma_j \ltimes_{\Sigma_j} \Pi_s(\Sigma^{-n}Y, \Sigma^{-n}B)\}$$

are null homotopic, hence its colimit of cohomologies is zero. Inductively, we conclude from the long exact sequences that the colimits of cohomologies associated to the quotients Γ_s/Γ_t with $s > 0$ are all zero and that the maps of colimits of cohomologies associated to the quotient maps $\Gamma_0/\Gamma_t \to \Gamma_0/\Gamma_s$ are all isomorphisms. With $s = 1$ and $t = p$, this proves Proposition 3.11.

§4. Jones' proof of the Kahn-Priddy theorem

We prove Theorem 2.8 here. The proof for $p = 2$ is due to Jones [43] and we have adapted his idea to the case $p > 2$. We begin more generally than necessary by relating the cofibre sequences (*) above Lemma 3.12 to the maps $\tau_j : D_j Y \to Y^{(j)}$ of Definiton 1.4. The idea here is again due to Bruner. Thus let (Y,B) be a pair of finite CW spectra with quotient $Z = Y/B$. The map τ_j is obtained by applying the functor $E\Sigma_j \ltimes_{\Sigma_j} (?)$ to the composite

$$Y^{(j)} \xrightarrow{\Delta^{(j)}} ({}^{(j)}Y)^{(j)} \xrightarrow{\pi_J} \Sigma_j \ltimes Y^{(j)},$$

$J = (1,\dots,1)$, and using the equivalence $E\Sigma_j \ltimes Y^{(j)} \simeq Y^{(j)}$ of nonequivariant spectra (where, technically, the smash product is external on the left and internal on the right; see [Equiv. II §3]). The spectrum $\Sigma_j \ltimes Y^{(j)}$ is a wedge of isomorphic copies of $Y^{(j)}$ indexed on the elements of Σ_j, and $\pi_J \Delta^{(j)}$ is just the sum of the $j!$ permutation maps. It follows that $\pi_J \Delta^{(j)}$ restricts to a Σ_j-equivariant map $\Gamma_s \to \Sigma_j \times \Gamma_s$ for $0 \le s \le j$. Upon passage to subquotients and application of the functor $E\Sigma_j \ltimes_{\Sigma_j} (?)$, we obtain maps of cofibre sequences

$$\begin{array}{ccccc}
E\Sigma_j \ltimes_{\Sigma_j} \Gamma_s/\Gamma_t & \longrightarrow & E\Sigma_j \ltimes_{\Sigma_j} \Gamma_r/\Gamma_t & \longrightarrow & E\Sigma_j \ltimes_{\Sigma_j} \Gamma_r/\Gamma_s \\
\downarrow{\scriptstyle \tau_j} & & \downarrow{\scriptstyle \tau_j} & & \downarrow{\scriptstyle \tau_j} \\
\Gamma_s/\Gamma_t & \longrightarrow & \Gamma_r/\Gamma_t & \longrightarrow & \Gamma_r/\Gamma_s
\end{array}$$

for $0 \le r < s < t \le j$. With $t = s+1$, the left map τ_j is nicely related to the equivalence α of Lemma 3.12, as can easily be checked by inspection of definitions.

Lemma 4.1. The following diagram commutes for $0 < s < j$, where ρ is the projection onto the unpermuted wedge summand.

$$\begin{array}{ccc}
D_s B \wedge D_{j-s} Z & \xrightarrow{\quad\alpha\quad} & E\Sigma_j \ltimes_{\Sigma_j} \Pi_s(Y,B) \\
\downarrow{\scriptstyle \tau_s \wedge \tau_{j-s}} & & \downarrow{\scriptstyle \tau_j} \\
B^{(s)} \wedge Z^{(j-s)} & \xleftarrow{\ \rho\ } \Sigma_j \ltimes_{\Sigma_s \times \Sigma_{j-s}} B^{(s)} \wedge Z^{(j-s)} \cong & \Pi_s(Y,B)
\end{array}$$

When $j = 2$, there is only one map of cofibre sequences above, and we obtain the following conclusion.

Proposition 4.2. For CW pair (Y,B) with quotient $Z = Y/B$,

$$B \wedge Z \xrightarrow{\iota_2'} D_2Y/D_2B \xrightarrow{\psi} D_2 Z \xrightarrow{\tau_2'} \Sigma B \wedge Z$$

is a cofibre sequence, where ψ is induced by the quotient map $Y \to Z$, ι_2' is the composite

$$B \wedge Z = (B \wedge Y)/(B \wedge B) \xrightarrow{i \wedge 1} (Y \wedge Y)/(B \wedge B) \xrightarrow{\iota_2} D_2Y/D_2B,$$

and τ_2' is the composite

$$D_2Z \xrightarrow{\tau_2} Z \wedge Z \simeq (Y \cup CB) \wedge Z \xrightarrow{\partial \wedge 1} \Sigma B \wedge Z.$$

Proof. Combine the cofibre sequence

$$E\Sigma_2 \ltimes_{\Sigma_2} \Pi_1(Y,B) \longrightarrow D_2Y/D_2B \longrightarrow D_2Z \longrightarrow \Sigma E\Sigma_2 \ltimes_{\Sigma_2} \Pi_1(Y,B)$$

with the equivalence $\alpha: B \wedge Z \to E\Sigma_2 \ltimes_{\Sigma_2} \Pi_1(Y,B)$ and check that the resulting maps are those specified.

Our main interest is in the pair (CY,Y).

Corollary 4.3. The following is a cofibre sequence.

$$\Sigma(Y \wedge Y) \xrightarrow{\Sigma\iota_2} \Sigma D_2Y \xrightarrow{\Delta} D_2\Sigma Y \xrightarrow{\tau_2} \Sigma Y \wedge \Sigma Y.$$

Proof. Use the evident equivalence $D_2CY/D_2Y \simeq \Sigma D_2Y$ and check the maps, using Lemma 3.10 for the middle one.

For $j > 2$, we have too many cofibre sequences in sight. Henceforward, let p be a prime and localize all spaces and spectra at p without change of notation. We shall show that, for odd primes p and pairs (CS^q,S^q), our system of cofibre sequences collapses to a single one like that in the previous corollary. Recall from Lemma 1.10 that $\tau_r\iota_r:Y^{(r)} \to Y^{(r)}$ is the sum of permutations map and $\iota_r\tau_r:D_rY \to D_rY$ induces multiplication by $r!$ on ordinary homology. In particular, for $1 < r < p$, D_rY is a wedge summand of $Y^{(r)}$.

Lemma 4.4. For $1 < r < p$, D_rS^{2q+1} is equivalent to the trivial spectrum and $\iota_r:S^{2qr} \to D_rS^{2q}$ is an equivalence with inverse $\frac{1}{r!}\tau_r$.

Proof. When $Y = S^{2q}$, $\tau_r\iota_r$ induces multiplication by $r!$ on homology; when $Y = S^{2q+1}$, it induces zero. The conclusions follow.

Thus, when Y is a sphere spectrum, most of the spectra

$$E\Sigma_p \ltimes_{\Sigma_p} \Pi_s(CY,Y) \simeq D_sY \wedge D_{p-s}\Sigma Y$$

are trivial.

Corollary 4.5. Let $p > 2$ and let q be an even integer. Then there are cofibre sequences

$$S^{pq-1} \longrightarrow \Sigma D_p S^{q-1} \xrightarrow{\Delta} D_p S^q \xrightarrow{\tau_p} S^{pq}$$

and

$$S^{pq+1} \longrightarrow \Sigma D_p S^q \xrightarrow{\Delta} D_p S^{q+1} \longrightarrow S^{pq+2} .$$

Proof. Let $\Gamma_s = \Gamma_s(CY,Y)$ and $\Pi_s = \Gamma_s/\Gamma_{s+1}$. If $Y = S^{q-1}$, then $E\Sigma_p \ltimes_{\Sigma_p} \Pi_s$ is trivial for $2 \leq s < p$, hence $E\Sigma_p \ltimes_{\Sigma_p} \Gamma_r/\Gamma_s$ is trivial for $2 \leq r < s \leq p$. Thus $\Gamma_1/\Gamma_p \to \Pi_1$ and $\Gamma_0/\Gamma_p \to \Gamma_0/\Gamma_2$ induce equivalences upon application of $E\Sigma_p \ltimes_{\Sigma_p}(?)$ and there results a cofibre sequence

$$E\Sigma_p \ltimes_{\Sigma_p} \Pi_1 \longrightarrow E\Sigma_p \ltimes_{\Sigma_p} \Gamma_0/\Gamma_p \longrightarrow E\Sigma_p \ltimes_{\Sigma_p} \Pi_0 \longrightarrow \Sigma E\Sigma_p \ltimes_{\Sigma_p} \Pi_1 .$$

This gives the first sequence upon interpreting the terms and maps (by use of Lemmas 3.10, 3.12, 4.1, and 4.4). Similarly, if $Y = S^q$, then $E\Sigma_p \ltimes_{\Sigma_p} \Pi_s$ is trivial for $1 \leq s < p-1$, hence $E\Sigma_p \ltimes_{\Sigma_p} \Gamma_r/\Gamma_s$ is trivial for $1 \leq r < s \leq p-1$. Thus $\Gamma_0/\Gamma_{p-1} \to \Pi_0$ and $\Pi_{p-1} \to \Gamma_1/\Gamma_p$ induces equivalences upon application of $E\Sigma_p \ltimes_{\Sigma_p}(?)$ and there results a cofibre sequence

$$E\Sigma_p \ltimes_{\Sigma_p} \Pi_{p-1} \longrightarrow E\Sigma_p \ltimes_{\Sigma_p} \Gamma_0/\Gamma_p \longrightarrow E\Sigma_p \ltimes_{\Sigma_p} \Pi_0 \longrightarrow \Sigma E\Sigma_p \ltimes_{\Sigma_p} \Pi_{p-1} .$$

This gives the second sequence.

One can also check these cofibre sequences by direct homological calculation; compare Lemma 5.6 below. We need some further information about the spectra $\Sigma^n D_p S^{-n}$ in order to use these sequences to prove Theorem 2.8. Proofs of the claims to follow will be given by Bruner in V§2.

If $p = 2$, let $L = \Sigma^\infty RP^\infty$ with its standard cell structure. (We write L rather than the usual P for uniformity with the case $p > 2$.) If $p > 2$, let L be a CW spectrum of the p-local homotopy type of $\Sigma^\infty B\Sigma_p$ such that L has one cell in each positive dimension $q \equiv 0$ or $-1 \bmod 2(p-1)$. The existence and essential uniqueness of such an L was pointed out by Adams [7,2.2]. Let L^k be the k-skeleton of L and let $L_n = L/L^{n-1}$ and $L_n^{n+k} = L^{n+k}/L^{n-1}$ for $k \geq 0$. Let $\phi(k)$ be as in Theorem 2.8 (and recall that it depends on p). If $p = 2$, then

$$L_n^{n+k} \simeq \Sigma^{n-m} L_m^{m+k} \quad \text{for } m \equiv n \bmod 2^{\phi(k)}.$$

If $p > 2$, $\varepsilon = 0$ or 1, and $k \geq \varepsilon$, then

$$L_{2n+\varepsilon}^{2n+k} \simeq \Sigma^{2(n-m)} L_{2m+\varepsilon}^{2m+k} \quad \text{for } m \equiv n \bmod p^{\phi(k)}.$$

We use this periodicity to define spectra L_n^{n+k} for non-positive n, so that these equivalences hold for all integers m and n. We then have that

$$L_n^{n+k} \text{ is } (-1)\text{-dual to } L_{-n-1-k}^{-n-1} .$$

Our interest in these spectra comes from the following result (proven by Bruner in V§2).

Theorem 4.6. For any integer n, $\Sigma^{-n}D_pS^n$ is p-locally equivalent to $L_{n(p-1)}$.

We define $D_p^kS^n = \Sigma^n L_{n(p-1)}^{n(p-1)+k}$. If p = 2, we may view $D_p^kS^n$ as $S^k \ltimes_{\Sigma_2} S^{2n}$. If p > 2, no model for $E\Sigma_p$ has few enough cells to give as convenient a filtration of D_pS^n. We shall shortly prove the following result.

Proposition 4.7. If $\rho: L_{-k}^0 \to S^0$ is the projection onto the top cell, then

$$\rho^*: \pi^{-q}(S^0) \to \pi^{-q}(L_{-k}^0)$$

is zero for $0 < q < k(p-1)$.

Since ρ is (-1)-dual to the inclusion $\iota: S^{-1} \to L_{-1}^{k-1}$ of the bottom cell, $\iota_*: \pi_q(S^{-1}) \to \pi_q(L_{-1}^{k-1})$ is zero for $0 \le q < k(p-1)-1$. The cofibre sequences of Corollaries 4.3 and 4.5 restrict to give cofibre sequences

$$S^{-1} \xrightarrow{\iota} L_{-1}^{k-1} \xrightarrow{\Delta} L_0^{k-1} \xrightarrow{\tau_p} S^0 .$$

Thus, $\tau_{p*}: \pi_q(L_0^{k-1}) \to \pi_q(S^0)$ is an epimorphism for $0 < q < k(p-1)$. Now let k go to infinity. Of course, $L_0 \simeq \Sigma^\infty B\Sigma_p^+$ splits as the wedge $\Sigma^\infty B\Sigma_p \vee S^0$. Since $\tau_p\iota_p: S^0 \to S^0$ has degree p!, the finiteness of π_*S^0 allows us to deduce the following version of the Kahn-Priddy Theorem.

Theorem 4.8. The restriction $\tau_p: \Sigma^\infty B\Sigma_p \to S^0$ induces an epimorphism $\pi_q(\Sigma^\infty B\Sigma_p) \to \pi_q(S^0) \otimes Z_{(p)}$ for $q > 0$.

To prove Theorem 2.8, consider the following diagram, where $q \equiv 0 \bmod p^{\phi(k)}$ and q is even if p > 2.

$$\begin{tikzcd}
S^{pq-1} \arrow[r, "\Sigma^{pq}\iota"] \arrow[d, equal] & \Sigma^{pq}L_{-1}^{k-1} \arrow[r, "\Sigma^{pq}\Delta"] \arrow[d, "\nu"] & \Sigma^{pq}L_0^{k-1} \arrow[r, "\Sigma^{pq}\tau_p"] \arrow[d, "\omega"] & S^{pq} \arrow[d, equal] \\
S^{pq-1} \arrow[r] & \Sigma^q L_{(p-1)q-1}^{(p-1)q+k-1} \arrow[r, "\Delta"] & \Sigma^q L_{(p-1)q}^{(p-1)q+k-1} \arrow[r, "\tau_p"] & S^{pq}
\end{tikzcd}$$

The bottom cofibre sequence is obtained by restriction from sequences in Corollaries 4.3 and 4.5. Periodicity gives an equivalence ν such that the left square commutes. Standard cofibration sequence arguments then give an equivalence ω such that the remaining squares commute. The bottom map τ_p factors through $\tau_p : D_pS^q \to S^{pq}$ and is an epimorphism in the range stated in Theorem 2.8.

It remains to prove Proposition 4.7. For amusement, we proceed a bit more generally. Recall the not necessarily commutative diagram

$$\begin{tikzcd}
 & E^*X \arrow[dl, "\mathcal{P}_j"'] \arrow[dr, "\xi_j^*"] & \\
E^*_{(j)}X \arrow[rr, "\Delta^*"] & & F^*_{(j)}X
\end{tikzcd}$$

below Proposition 3.6, where E is an H_∞ ring spectrum. With $E = S$ and $X = S^0$, the following result is Proposition 4.7.

Proposition 4.9. Let X be a finite CW complex of dimension less than $k(p-1)-q$, where $0 < q < k(p-1)$. Then

$$(\rho\wedge 1)^* : E^{-q}X = E^{-q}(S^0\wedge X) \longrightarrow E^{-q}(L^0_{-k}\wedge X)$$

is zero if E is a connective H_∞ ring spectrum such that $\Delta^*\mathcal{P}_p = \xi_p^*$.

Proof. For $n \geq k$, the cofibre of $\Delta : \Sigma^{n+1}D_pS^{-n-1} \to \Sigma^nD_pS^{-n}$ has dimension at most $-k(p-1)$, and it follows that the colimit $F^{-q}_{(p)}X$ is attained as $E^{-q}(\Sigma^kD_pS^{-k}\wedge X)$. Let $\iota : L^0_{-k} \to L_{-k} \simeq \Sigma^kD_pS^{-k}$ be the inclusion and consider the following diagram, where x is any map $X \to \Sigma^{-q}E$.

$$\begin{tikzcd}
 & & L^0_{-k}X \arrow[dl, "0"'] \arrow[d, "\iota\wedge 1"] \arrow[ddrr, "\rho\wedge 1"] & & \\
 & \Sigma^{-q}D_pS^q\wedge X \arrow[dl, "\Delta"'] & \Sigma^kD_pS^{-k}\wedge X \arrow[l, "\Delta\wedge 1"'] \arrow[dl, "\Delta"'] \arrow[dr, "\Delta\wedge 1"] & & \\
\Sigma^{-q}D_p\Sigma^qX \arrow[d, "\Sigma^{-q}D_p\Sigma^q x"'] & \Sigma^kD_p\Sigma^{-k}X \arrow[l, "\Delta"'] & & D_pS^0\wedge X \arrow[r, "\xi_p\wedge 1"] & S^0\wedge X \arrow[d, equal] \\
\Sigma^{-q}D_pE \arrow[rr, "\Sigma^{-q}\xi_p"] & & \Sigma^{-q}E & & X \arrow[ll, "x"']
\end{tikzcd}$$

Since $\Delta^*\mathcal{P}_p = \xi_p^*$, the bottom part commutes. We have

$$\xi_p \Delta \iota \simeq \rho : L_{-k}^0 \to S^0$$

since the composite is obviously null homotopic on L_{-k}^{-1} and of degree one on the top cell. We have

$$\Delta \iota \simeq 0 : L_{-k}^0 \to \Sigma^{-q} D_p S^q$$

since $\Sigma^{-q} D_p S^q$ is 0-connected. The conclusion follows.

Replacing S by E in the deductions from Proposition 4.7 and using the results of section 2, we conclude that, for q > 0, all p-torsion elements of $\pi_p E$ are nilpotent if $\Delta^*\mathcal{P}_p = \xi_p^*$. This implies our earlier claim that $\Delta^*\mathcal{P}_2 \neq \xi_2^*$ when E = MO.

§5. The Singer construction and theorems of Lin and Gunawardena

Singer introduced a remarkable algebraic functor R_+ from A-modules to A-modules, where A is the mod p Steenrod algebra, and Miller began the study of the cohomology theories in section 3 by making the following basic observation. All homology and cohomology is to be taken with mod p coefficients.

Theorem 5.1. Let Y be a spectrum such that H_*Y is bounded below and of finite type. Then colim $H^*(\Sigma^n D_p \Sigma^{-n} Y)$ is isomorphic to $\Sigma^{-1} R_+ H^* Y$.

We shall prove this and some related observations after explaining its relationship to the following theorems of Lin [53, 54] and Gunawardena [38, 39]. Let $\hat{\pi}^*$ and $\hat{\pi}_*$ denote the p-adic completions of stable cohomotopy and stable homotopy.

Theorem 5.2. The map $\xi_p^* : \hat{\pi}^* Y \to \operatorname{colim} \hat{\pi}^*(\Sigma^n D_p S^{-n} \wedge Y)$ is an isomorphism for all finite CW spectra Y.

As we shall explain shortly, $\lim \hat{\pi}_{-1}(\Sigma^n D_p S^{-n}) = \hat{Z}_p$. Realizing the unit by a compatible system of maps $\xi^p : S^{-1} \to \Sigma^n D_p S^{-n}$ and smashing with Y, we obtain a compatible system of maps

$$\xi^p : \Sigma^{-1} Y \cong S^{-1} \wedge Y \to \Sigma^n D_p S^{-n} \wedge Y.$$

Theorem 5.3. The map $\xi_*^p : \hat{\pi}_* \Sigma^{-1} Y \to \lim \hat{\pi}_*(\Sigma^n D_p S^{-n} \wedge Y)$ is an isomorphism for all finite CW spectra Y.

Since ξ_p^* is a map of cohomology theories and ξ_*^p is a map of homology theories, it suffices to prove these isomorphisms for $Y = S^0$. Since

$$\Sigma^n D_p^{k(p-1)-1} S^{-n} \text{ is } (-1)\text{-dual to } \Sigma^{n+k} D_p^{k(p-1)-1} S^{-n-k},$$

the theorems are esentially dual to one another. Indeed, using the $\lim^1$ exact sequence and waving one's hands at certain compatibility questions, one finds the following chain of isomorphisms, where $m(p-1) > q$.

$$\begin{aligned}
\operatorname*{colim}_n \hat{\pi}^q(\Sigma^n D_p S^{-n}) &= \hat{\pi}^q(\Sigma^m D_p S^{-m}) \\
&= \lim_k \hat{\pi}^q(\Sigma^m D_p^{k(p-1)-1} S^{-m}) \\
&= \lim_k \hat{\pi}_{-q-1}(\Sigma^{m+k} D_p^{k(p-1)-1} S^{-m-k}) \\
&= \lim_n \hat{\pi}_{-q-1}(\Sigma^n D_p S^{-n})
\end{aligned}$$

There is a map of A-modules $\varepsilon: R_+ Z_p \to Z_p$, and the main point of the work of Lin and Gunawardena can be reformulated as follows; see Adams, Gunawardena, and Miller [9].

<u>Theorem 5.4</u>. $\varepsilon^*: \mathrm{Ext}_A(Z_p, Z_p) \to \mathrm{Ext}_A(R_+ Z_p, Z_p)$ is an isomorphism.

An inverse system $\{Y_n\}$ of bounded below spectra Y_n of finite type gives rise to an inverse limit

$$\{E_r\} = \lim \{E_r Y_n\}$$

of Adams spectral sequences, where $\{E_r Y\}$ denotes the classical Adams spectral sequence for the computation of $\hat{\pi}_* Y$. Clearly

$$E_2 \cong \mathrm{Ext}_A(\operatorname{colim} H^* Y_n, Z_p).$$

As pointed out in [74], $\{E_r\}$ converges strongly to $\lim \hat{\pi}_* Y_n$. We apply this with $Y_n = \Sigma^n D_p S^{-n}$. Here Theorems 5.1 and 5.4 give

$$E_2 \cong \mathrm{Ext}_A(\Sigma^{-1} Z_p, Z_p).$$

From this and convergence, it is easy to check that $\lim \hat{\pi}_{-1}(\Sigma^n D_p S^{-n}) = \hat{Z}_p$. The compatible system of maps $\xi^p: S^{-1} \to \Sigma^n D_p S^{-n}$ then induces a map of spectral sequences

$$\{E_r \xi^p\}: \{E_r S^{-1}\} \to \{E_r\}.$$

By Theorem 5.4 again, $E_2 \xi^p$ is an isomorphism, and Theorem 5.3 follows by convergence. Theorem 5.2 can be obtained by a similar Adams spectral sequence argument (as in Lin [53] and Gunawardena [38]) or by dualization.

The crux of the proof of Theorem 5.1 is the following result of Steinberger, which is proven in VIII.3.2 of the sequel. For spaces, it is due to Nishida [89]; see also [68, 9.4]. Let π be the cyclic group of order p. We assume familiarity with the mod p homology $H_*D_\pi Y$, its determination being a standard exercise in the homology of groups in view of I.2.3 (see e.g. [68, §1]). Suffice it to say that $H_*D_\pi Y$ has a basis consisting of elements of the form $e_0 \otimes x_1 \otimes \cdots \otimes x_p$ and $e_i \otimes x^p$, $i \geq 0$. Here the x_i and x run through basis elements of H_*Y, the x_i are not all equal, and the $x_1 \otimes \cdots \otimes x_p$ and x^p together run through a set of π-generators for $(H_*Y)^p$. Restricting to those i of the form $(2s-q)(p-1)-\varepsilon$, where $q = \deg(x)$ and $\varepsilon = 0$ or 1, and to a set of Σ_p-generators for $(H_*Y)^p$, we obtain a basis for H_*D_pY. At least if H_*Y is bounded below and of finite type, we have analogous dual bases for $H^*D_\pi Y$ and H^*D_pY with typical elements denoted $w_0 \otimes y_1 \otimes \cdots \otimes y_p$ and $w_i \otimes y^p$.

<u>Theorem 5.5</u>. Assume that H_*Y is bounded below and of finite type. The subspace of $H^*D_\pi Y$ spanned by $\{w_0 \otimes y_1 \otimes \cdots \otimes y_p\}$ is closed under Steenrod operations and, modulo this subspace, the following relations hold for $y \in H^qY$.

(i) For p = 2,

$$Sq^s(w_j \otimes y^2) = \sum_i \binom{j+q-i}{s-2i} w_{j+s-2i} \otimes (Sq^i y)^2.$$

(ii) For p > 2, let $\delta(2n+\varepsilon) = \varepsilon$, $m = \frac{1}{2}(p-1)$, and $\alpha(q) = -(-1)^{mq} m!$; then

$$\begin{aligned} P^s(w_j \otimes y^p) = &\sum_i \binom{[j/2]+qm-(p-1)i}{s-pi} w_{j+2(s-pi)(p-1)} \otimes (P^i y)^p \\ &+ \delta(j-1)\alpha(q) \sum_i \binom{[j/2]+qm-(p-1)i-1}{s-pi-1} w_{j-p+2(s-pi)(p-1)} \otimes (\beta P^i y)^p. \end{aligned}$$

(iii) For p > 2, $\Delta(w_{2j-1} \otimes y^p) = w_{2j} \otimes y^p$.

We also need to know $\Delta^*: H^*D_\pi Y \to H^*(\Sigma D_\pi \Sigma^{-1} Y)$. Let $\Sigma^n: H^q(Y) \to H^{q+n}(\Sigma^n Y)$ denote the iterated suspension isomorphism for any integer n.

<u>Lemma 5.6</u>. For $y \in H^qY$,

$$\Delta^*(w_j \otimes y^p) = (-1)^{j+1}\alpha(q)\Sigma(w_{j+p-1} \otimes (\Sigma^{-1}y)^p).$$

<u>Proof</u>. We first compute $\Delta_*: H_*(\Sigma D_\pi Y) \to H_*(D_\pi \Sigma Y)$. Take f to be the identity map of Y and replace D_p by D_π in Lemma 3.10. We find that the composite of Δ_* and the homology suspension Σ_* is the suspension associated to the zero sequence

$$C_*(D_\pi Y) \longrightarrow C_*(D_\pi CY) \longrightarrow C_*(D_\pi \Sigma X).$$

By I.2.3 and [68,§1], we may instead use the zero sequence

$$W \otimes_\pi C_*(Y)^p \longrightarrow W \otimes_\pi C_*(CY)^p \longrightarrow W \otimes_\pi C_*(\Sigma Y)^p,$$

where W is the standard π-free resolution of Z_p. A direct chain level computation, details of which are in [68,p. 166-167], gives the formula

$$\Delta_* \Sigma_* e_{j+p-1} \otimes x^p = (-1)^{j+1} \alpha(q) e_j \otimes (\Sigma_* x)^p$$

for $x \in H_{q-1}(Y)$. Clearly $\Delta_* \Sigma_* (e_0 \otimes x_1 \otimes \cdots \otimes x_p) = 0$ for all x_i. The conclusion follows upon dualization (and a careful check of signs).

The results above determine colim $H^*(\Sigma^n D_\pi \Sigma^{-n} Y)$ as an A-module, and similarly with D_π replaced by D_p. To compare the answer to the Singer construction, we must first recall the definition of the latter [98,52]. When $p = 2$, $\Sigma^{-1} R_+ M$ is additively isomorphic to $\Lambda \otimes M$, where Λ is the Laurent series ring $Z_2[v,v^{-1}]$, deg $v = 1$. Its Steenrod operations are specified by

$$Sq^s(v^r \otimes x) = \sum_i \binom{r-i}{s-2i} v^{r+s-i} \otimes Sq^i x.$$

When $p > 2$, $\Sigma^{-1} R_+ M$ is additively isomorphic to $\Lambda \otimes M$, where $\Lambda = E\{u\} \times Z_p[v,v^{-1}]$, deg $u = 2p-3$ and deg $v = 2p-2$. Its Steenrod operations are specified by

$$P^s(u^\varepsilon v^{r-\varepsilon} \otimes x) = \sum_i (-1)^{s+i} \binom{(p-1)(r-i)-\varepsilon}{s-pi} u^\varepsilon v^{r+s-i-\varepsilon} \otimes P^i x$$

$$+ (1-\varepsilon) \sum_i (-1)^{s+i} \binom{(p-1)(r-i)-1}{s-pi-1} u v^{r+s-i-1} \otimes \beta P^i x$$

and

$$\beta(u^\varepsilon v^{r-\varepsilon} \otimes x) = \varepsilon(v^r \otimes x).$$

We can now prove Theorem 5.1. We define an isomorphism

$$\omega: \text{colim } H^*(\Sigma^n D_p \Sigma^{-n} Y) \to \Sigma^{-1} R_+ H^* Y$$

as follows. For $p = 2$ and $y \in H^q(Y)$, let

$$\omega(\Sigma^n(w_{r-q+n} \otimes (\Sigma^{-n} y)^2 = v^r \otimes y.$$

For $p > 2$ and $y \in H^q(Y)$, let

$$\omega(\Sigma^n(w_{(2r+n-q)(p-1)-\varepsilon} \otimes (\Sigma^{-n} y)^p) = (-1)^{r+q+(\varepsilon+1)n} \nu(q-n)^{-1} u^\varepsilon v^{r-\varepsilon} \otimes y,$$

where $\nu(2j+\varepsilon) = (-1)^j(m!)^\varepsilon$. Note that

$$\alpha(q)\nu(q-1)^{-1} = \nu(q)^{-1} \quad\text{and}\quad (-1)^q\nu(q)^{-1} = (-1)^{mq}\nu(q).$$

By Lemma 5.6, these ω induce a well-defined isomorphism on passage to colimits. by Theorem 5.5, we see that our constants have been so chosen that ω is an isomorphism of A-modules.

Remark 5.7. When $p > 2$, there are two variants of the Singer construction. We are using the smaller one appropriate to D_p. This is a summand of the larger variant, for which Theorem 5.1 is true with D_p replaced by D_π. See Gunawardena [39,9] for details (but note that his signs don't quite agree with ours).

With $Y = S^0$, Theorem 5.1 specializes to an isomorphism

$$\Lambda = \Sigma^{-1}R_+Z_p \cong \operatorname{colim} H^*(\Sigma^n D_p S^{-n}).$$

Since Λ is an A-module, $\Lambda \otimes M$ admits the diagonal A action, which is evidently quite different from that originally specified on $\Sigma^{-1}R_+M$. For finite CW complexes X, we have the isomorphism

$$\Delta^*\colon \operatorname{colim} H^*(\Sigma^n D_p \Sigma^{-n} X) \longrightarrow \operatorname{colim} H^*(\Sigma^n D_p S^{-n} \wedge X)$$

of Theorem 3.2. We next obtain an explicit description of the resulting isomorphism

$$\Delta^*\colon \Sigma^{-1}R_+\tilde{H}^*X \to \Lambda \otimes \tilde{H}^*X.$$

Thus consider $\Delta\colon D_\pi Y \wedge X \to D_\pi(Y \wedge X)$. When $X = S^1$, we computed Δ_* in the proof of Lemma 5.6. When $Y = S$, $D_\pi Y = \Sigma^\infty B\pi^+$ and the effect of Δ_* is implicit in the definition of the Steenrod operations; see Steenrod and Eptein [100] (or, for correct signs, [68, 9.1]). The following result is a common generalization of these calculations.

Propsition 5.8. Let $x \in \tilde{H}_k(X)$ and $y \in H_q(Y)$. If $p = 2$,

$$\Delta_*(e_r \otimes y^2 \otimes x) = \sum_i e_{r+2i-k} \otimes (y \otimes Sq_*^i x)^2.$$

if $p > 2$, let $\nu(2j+1) = (-1)^j(m!)^\varepsilon$ and $\varepsilon(2j+\varepsilon) = \varepsilon$; then

$$\Delta_*(e_r \otimes y^p \otimes x) = (-1)^{mkq}\nu(k) \sum_i (-1)^i e_{r+(2pi-k)(p-1)} \otimes (y \otimes P_*^i x)^p$$
$$-(-1)^{q+m(k-1)q}\delta(r)\nu(k-1) \sum_i (-1)^i e_{r+p+(2pi-k)(p-1)} \otimes (y \otimes P_*^i \beta x)^p.$$

Proof. Modulo shuffling in $C_*(Y)^p$, which introduces the signs depending on q when $p > 2$, Δ_* is computable from the map obtained by quotienting out the action of π from the π-map

$$\Phi \otimes 1 : C_*(W) \otimes \tilde{C}_*(X) \otimes C_*(Y)^p \longrightarrow C_*(W) \otimes \tilde{C}_*(X)^p \otimes C_*(Y)^p$$

induced by a π-equivariant approximation Φ of $1 \otimes \Delta'_*$, where Δ' is a cellular approximation of the diagonal $X \to X^p$; see e.g. [100, V§3] or [68,7.1]. The essential point is that Y acts like a dummy variable, so that the standard calculation for $Y = S^0$ of [68, 9.1] implies the general result.

Dualizing, and paying careful attention to signs, we obtain the following version in cohomology.

Proposition 5.9. Assume that H_*X and H_*Y are of finite type and that H_*Y is bounded below. Let $x \in \tilde{H}^k(X)$ and $y \in H^q(Y)$. If $p = 2$,

$$\Delta^*(w_j \otimes (y \otimes x)^2) = \sum_i w_{j+k-i} \otimes y^2 \otimes Sq^i x.$$

If $p > 2$,

$$\Delta^*(w_j \otimes (y \otimes x)^p) = (-1)^{mk(q+1)}\nu(k) \sum_i (-1)^i w_{j+(k-2i)(p-1)} \otimes y^p \otimes P^i x$$

$$-(-1)^{q+mk(q+1)}\delta(j-1)\nu(k) \sum_i (-1)^i w_{j+(k-2i)(p-1)-1} \otimes y^p \otimes \beta P^i x.$$

A check of constants gives the following consequence.

Corollary 5.10. For $M = \tilde{H}^*X$, the formula

$$\Delta^*(v^r \otimes x) = \sum_i v^{r-i} \otimes Sq^i x$$

if $p = 2$ and

$$\Delta^*(u^\varepsilon v^{r-\varepsilon} \otimes x) = \sum_i u^\varepsilon v^{r-i-\varepsilon} \otimes P^i x - (1-\varepsilon) \sum_i uv^{r-i-1} \otimes \beta P^i x$$

if $p > 2$ specifies a morphism of A-modules $\Delta^* : \Sigma^{-1}R_+M \to \Lambda \otimes M$.

The same formulae give a morphism of A-modules for all A-modules M which are either unstable or bounded above, either assumption ensuring that the relevant sums are finite. In the bounded above case, but not in general in the unstable case, this morphism is an isomorphism. See [98, 52, 82].

Define $\varepsilon: R_+M \to M$ by the formulas

$$\varepsilon\Sigma(v^{r-1} \otimes x) = Sq^r x$$

if $p = 2$ (where $Sq^r(x) = 0$ if $r < 0$) and

$$\varepsilon\Sigma(uv^{r-1} \otimes x) = P^r x \quad \text{and} \quad \varepsilon\Sigma(v^r \otimes x) = -\beta P^r x$$

if $p > 2$. By [98,3.4] and [52,3.5], ε is a well-defined morphism of A-modules. When Δ^* is defined, ε is the composite

$$R_+M \xrightarrow{\Sigma\Delta^*} \Sigma(\Sigma^{-1}R_+Z_p \otimes M) \xrightarrow{\Sigma(\varepsilon \otimes 1)} \Sigma(\Sigma^{-1}Z_p \otimes M) = M.$$

Generalizing Theorem 5.4, Adams, Gunawardena, and Miller [9] proved that ε is an Ext-isomorphism for any M. This leads to a generalization of Theorem 5.3 to a version appropriate to $(Z_p)^k$ for any $k \geq 1$, and this generalization is the heart of the proof of the Segal conjecture for elementary Abelian p-groups. See [9,74].

§6. Nishida's second nilpotency theorem.

If $x \in \pi_n E$ has order p, then x extends over the Moore spectrum $M^n = S^n \cup_p CS^n$. The idea of Nishida's second nilpotency theorem is to exploit this extension by showing that $D_j M^n$ splits as a wedge of Eilenberg-MacLane specta in a range of dimensions. The relevant splitting is a special case of the following result which, as we shall explain shortly, is in turn a special case of the general splitting theorem to be proven by Steinberger in the next chapter.

Theorem 6.1. Let Y be a spectrum obtained from S^n by attaching cells of dimension greater than n. Assume that $\pi_n Y$ is Z or Z_{p^i} and let $\nu \in H^n(Y;Z_p)$ be a generator. Assume one of the following further hypotheses.

(a) $p = 2$ and either n is odd or $\beta(\nu) \neq 0$.

(b) $p > 2$, n is even, and $\beta(\nu) \neq 0$.

(c) $p = 2$ and $Sq^3(\nu) \neq 0$.

(d) $p > 2$, n is even, and $\beta P^1(\nu) \neq 0$.

Then $D_j Y$ splits p-locally as a wedge of suspensions of Eilenberg-MacLane spectra through dimensions $r < nj + \frac{1}{p}(2p-3)(j+1)-1$. In cases (a) and (b), only suspensions of HZ_p are needed.

Before discussing the proof, we explain how to use these splittings to obtain relations in the homotopy groups of H_∞ ring spectra. Let Y and ν be as in the theorem above and localize all spectra at p.

Theorem 6.2. Let E be an H_∞ ring spectrum, let F be a connective spectrum, and let $\phi:E \;\; F \to E$ be any map (for example, the product when $F = E$ or the identity when $F = S$). Let $x \in \pi_n E$ and assume one of the following hypotheses.

(a) $p = 2$ and n is odd; here let $Y = S^n$.

(b) $p \geq 2$, n is even, and x has order 2; here let $Y = M^n$.

(c) $p = 2$, n is even, and x extends over some Y with $Sq^3(\nu) \neq 0$.

(d) $p > 2$, n is even, and x extends over some Y with $\beta P^1(\nu) \neq 0$.

Let $R = Z_p$ in cases (a) and (b) and $R = \pi_n Y$ in cases (c) and (d) and let $y \in \pi_n F$ be in the kernel of the Hurewicz homomorphism $\pi_q F \to H_q(F;R)$. Then $x^j y = 0$ if $q < \frac{1}{p}(2p-3)(j+1)-1$.

Proof. Our hypotheses ensure that $H^{nj}(D_j Y;R) \cong R$. We can choose a generator μ such that the composite

$$S^{nj} \xrightarrow{\iota_j} D_j S^n \xrightarrow{D_j f} D_j Y \xrightarrow{\mu} \Sigma^{nj} HR$$

is $\Sigma^{nj} e$, where $f:S^n \to Y$ is the inclusion of the bottom cell and $e:S \to HR$ is the unit. Choose $\tilde{x}:Y \to E$ such that $\tilde{x} f = x$. Then the solid arrow part of the following diagram commutes and the top composite is $x^j y$.

$$\begin{tikzcd}
S^{nj} \wedge S^q \arrow[r, "1 \wedge y"] \arrow[dd, "\Sigma^{nj} e \wedge y"'] & S^{nj} \wedge F \arrow[r, "x^{(j)} \wedge 1"] \arrow[d, "\iota_j \wedge 1"] & E^{(j)} \wedge F \arrow[r, "\phi \wedge 1"] \arrow[d, "\iota_j \wedge 1"] & E \wedge F \arrow[r, "\phi"] & E \\
 & D_j S^n \wedge F \arrow[r, "D_j x \wedge 1"] \arrow[d, "D_j f \wedge 1"] & D_j E \wedge F \arrow[ur, "\xi_j \wedge 1"'] & & \\
\Sigma^{nj} HR \wedge F \arrow[dr, dashed, "\kappa \wedge 1"'] & D_j Y \wedge F \arrow[l, "\mu \wedge 1"'] \arrow[ur, "D_j \tilde{x} \wedge 1"'] \arrow[d, "\omega \wedge 1"] & & & \\
 & (D_j Y)_r \wedge F \arrow[ul, "\rho \wedge 1"'] & & &
\end{tikzcd}$$

Here $r = nj+q$, $\omega:D_j Y \to (D_j Y)_r$ is the r^{th} stage of a Postnikov decomposition of $D_j Y$, and $\rho:(D_j Y)_r \to \Sigma^{nj} HR$ is the unique cohomology class such that $\rho\omega = \mu$. The previous theorem gives $\kappa:\Sigma^{nj} HR \to (D_j Y)_r$ such that $\rho\kappa = 1$. The complementary wedge summand of $\Sigma^{nj} HR$ in $(D_j Y)_r$ is (nj)-connected, and it follows that $\kappa \cdot \Sigma^{nj} e = \omega \cdot D_j f \cdot \iota_j$. Since F is connective, $\omega \wedge 1$ induces an isomorphism on π_{nj+q}. Since y is in the kernel of the Hurewicz homomorphism and the latter is induced by $e\wedge 1:F = S\wedge F \to HR\wedge F$, $\Sigma^{nj}e\wedge y = 0$. Chasing the diagram, we conclude that $x^j y = 0$.

In particular, with $F = E$, $q = n$, and $y = x$, we obtain $x^{j+1} = 0$. With $E = S$ and $n > 0$, case (b) applies to any even degree element of order p. As observed by Steinberger, when $p = 2$ case (a) applies to any odd degree element and gives a better estimate of the order of nilpotency than that obtained by applying case (b) to x^2. While this result gives a much better estimate of the order of nilpotency of elements of order p in π_*S than does Theorem 2.9, the estimate is presumably still far from best possible. For example, if $p = 2$ and $n = 14$, the estimate is now $x^{30} = 0$. Cases (c) and (d) apply to some elements of order p^i with $i > 1$. The idea is to add further cells to S^n, or to $S^n \cup_{p^i} CS^n$, so as to obtain a spectrum Y for which the relevant Steenrod operation is non-zero. However, a given element x need not extend over any such Y. (Conceivably some power of x must so extend.) This explains why Nishida's second method fails to give the full nilpotency theorem and why we cannot yet prove Conjecture 2.7.

We must still explain how to prove Theorem 6.1. The idea is to approximate D_j through the specified range by a spectrum with additional structure and then use homology operations to split the latter. The approximation is based on the following observation about mod p homology.

Proposition 6.3. Let Y be an (n-1)-connected spectrum with $H_nY = Z_p$, where n is even if $p > 2$. Let $f:S^n \to Y$ induce an isomorphism on H_n. Then the homomorphism $H_i\Sigma^n D_qY \to H_iD_{q+1}Y$ induced by the composite

$$D_qY\wedge S^n \xrightarrow{1\wedge f} D_qY\wedge Y \xrightarrow{\alpha_{q,1}} D_{q+1}Y$$

is a monomorphism for all i and is an isomorphism if $i < n(q+1) + \frac{1}{p}(2p-3)(q+1)$.

For spaces X, a self-contained calculation of H_*D_qX for all q is given in [28,I§4-5]. The generalization to spectra is given by McClure in Chapter IX, and the conclusion is easily read off from these calculations.

With the proposition as a hint, we construct the approximating spectra as follows.

Definition 6.4. Let (Y,f) be a spectrum together with a map $f:S^n \to Y$ for some integer n and define $D(Y,f) = \mathrm{tel}\ \Sigma^{-nq}D_qY$, where the n^{th} map of the system is obtained by applying $\Sigma^{-n(q+1)}$ to the composite

$$D_qY\wedge S^n \xrightarrow{1\wedge f} D_qY\wedge Y \xrightarrow{\alpha_{q,1}} D_{q+1}Y.$$

Now the previous proposition has the following consequence.

Corollary 6.5. With Y and f as in the proposition, assume further that Y is p-local of finite type. Then the natural map $D_jY \to \Sigma^{nj}D(Y,f)$ is an equivalence through dimensions less than $nj + \frac{1}{p}(2p-3)(j+1) - 1$.

Proof. By the proposition, the maps $\Sigma^{-n(q+1)}(\alpha_{q,1} \circ 1 \wedge f)$ used to construct $D(Y,f)$ induce isomorphisms in mod p homology and thus in p-local homology in degrees less than $\frac{1}{p}(2p-3)(q+1)$. This fact for $q \geq j$ implies the conclusion (with the usual loss of a dimension as one passes from homology to homotopy).

Thus, to prove Theorem 6.1, we need only split $D(Y,f)$.

The following ad hoc definition, which generalizes Nishida's notion of a Γ-spectrum [90,1.5], allows us to describe the structure present on the spectra $D(Y,f)$. In the rest of this section we shall refer to weak maps and weakly commutative diagrams when the domain is a telescope and phantom maps are to be ignored.

Definition 6.6. A spectrum E is a pseudo H_∞ ring spectrum if

(i) E is the telescope of a sequence of connective spectra E_q, $q \geq 0$;

(ii) E is a weak ring spectrum with unit induced from a map $S \to E_0$ and product induced from a unital, associative, and commutative system of compatible maps $E_q \wedge E_r \to E_{q+r}$; and

(iii) For each $j \geq 0$ and $q \geq 0$, there exists an integer $d = d(j,q)$ and a map $\xi_j : D_j\Sigma^{dq}E_q \to \Sigma^{djq}E_{jq}$ whose composite with $\iota_j : \Sigma^{djq}E_q^{(j)} \cong (\Sigma^{dq}E_q)^{(j)} \to D_j\Sigma^{dq}E_q$ is the $(djq)^{\text{th}}$ suspension of the interated product $E_q^{(j)} \to E_{jq}$.

Examples 6.7. (i) With each $E_q = E$ and each $d(j,q) = 0$, a connective H_∞ ring spectrum may be viewed as a pseudo H_∞ ring spectrum.

(ii) With each $E_q = E$ and each $d(j,q) = d$, a connective H_∞^{d*} ring spectrum may be viewed as a pseudo H_∞ ring spectrum; since E has structural maps ξ_j for all q, negative as well as positive, we could obtain a different pseudo structure with each $d(j,q) = -d$.

(iii) For an $(n-1)$-connected spectrum Y and map $f:S^n \to Y$ such that either $2 = 0:Y \to Y$ or n is even, $D(Y,f)$ is a pseudo H_∞ ring spectrum with q^{th} term $\Sigma^{-nq}D_qY$. Its product is induced by the maps

$$\Sigma^{-nq}D_qY \wedge \Sigma^{-nr}D_rY \cong \Sigma^{-n(q+r)}(D_qY \wedge D_rY) \xrightarrow{\Sigma^{-n(q+r)}\alpha_{q,r}} \Sigma^{-n(q+r)}D_{q+r}Y ,$$

these forming a unital, associative, commutative, and compatible system by I.2.6 and I.2.8 and our added hypothesis, which serves to eliminate signs coming from permutations of spheres. With all $d(j,q) = n$, its structural maps are

$$\xi_j = \beta_{j,q}: D_j\Sigma^{nq}(\Sigma^{-nq}D_qY) = D_jD_qY \to D_{jq}Y = \Sigma^{njq}(\Sigma^{-njq}D_{jq}Y).$$

The following analog of I.3.6 and I.4.5 admits precisely the same simple cohomological proof.

Proposition 6.8. Let E be a pseudo H_∞ ring spectrum with char $\pi_0E = 2$ or all $d(j,q)$ even. Assume that $\pi_0E = \pi_0E_q$ for all $q \geq q_0$ and, for such q, let $i: E_q \to H(\pi_0E)$ be the unique map which induces the identity homomorphism on π_0. Then the following diagrams commute, where $d = d(j,q)$:

$$\begin{array}{ccc} D_j\Sigma^{dq}E_q & \xrightarrow{D_j\Sigma^{dq}i} & D_j\Sigma^{dq}H(\pi_0E) \\ \downarrow{\scriptstyle \xi_j} & & \downarrow{\scriptstyle \xi_j} \\ \Sigma^{djq}E_{jq} & \xrightarrow{\Sigma^{djq}i} & \Sigma^{djq}H(\pi_0E) \end{array}$$

In the next chapter, Steinberger will use a computation of the homology operations of the H_∞ ring spectrum $\bigvee_q \Sigma^{dq}HZ_p$ to prove the following generalization of Nishida's result [90,3.2].

Theorem 6.9. Let E be a p-local pseudo H_∞ ring spectrum. If $\pi_0E = Z_p$, then E splits as a wedge of suspensions of HZ_p. If $\pi_0E = Z_{p^r}$, $r > 1$, or $\pi_0E = Z_{(p)}$ and if $p = 2$ and $Sq^3i \neq 0$ or $p > 2$ and $\beta P^1i \neq 0$, where i generates $H^0(E;Z_p)$, then E splits as a wedge of suspensions of HZ_{p^s}, $s \geq 1$, and $HZ_{(p)}$.

Considering the natural map $\Sigma^{-n}Y \to D(Y,f)$, and using the formula $\beta(w_0 \otimes \nu^2) = nw_1 \otimes \nu^2$ of Theorem 5.5 for case (a), we easily check that the theorem applies to split $D(Y,f)$ for Y as in Theorem 6.1.

We complete this section with some remarks about the role played by Definition 6.4 in the general theory of H_∞ ring spectra.

Remarks 6.10. Let (E,e) be a spectrum with unit $e: S \to E$. Let $DE = D(E,e)$ and let $\eta: E = D_1E \to DE$ be the natural inclusion. By I.2.7, I.2.9, and I.2.13, the maps $\beta_{j,k}: D_jD_kE \to D_{jk}E$ induce a natural weak map $\mu_k: DD_kE \to DE$ such that the following diagrams (weakly) commute:

$$\begin{tikzcd} D_kE \arrow[r, "\eta"] \arrow[dr] & DD_kE \arrow[d, "\mu_k"] \\ & DE \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} DD_jD_kE \arrow[r, "D\beta_{j,k}"] \arrow[d, "\mu_j"'] & DD_{jk}E \arrow[d, "\mu_{jk}"] \\ DD_kE \arrow[r, "\mu_k"] & DE \end{tikzcd}$$

If E is an H_∞ ring spectrum, then, by Proposition 1.3, the maps $\xi_j:D_jE \to E$ determine a weak map $\xi:DE \to E$ such that the following diagrams (weakly) commute.

$$\begin{tikzcd} E \arrow[r, "\eta"] \arrow[dr, equal] & DE \arrow[d, "\xi"] \\ & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} DD_kE \arrow[r, "D\xi_k"] \arrow[d, "\mu_k"'] & DE \arrow[d, "\xi"] \\ DE \arrow[r, "\xi"] & E \end{tikzcd}$$

Conversely, by the same result, if $\psi:DE \to E$ makes these diagrams weakly commute, then its restrictions $\xi_j:D_jE \to E$ give E a structure of H_∞ ring spectrum. These assertions are analogous to, but weaker than, the assertions that D is a monad and that an H_∞ ring spectrum is an algebra over this monad (compare [69, §2]). The point is that the μ_k fail to satisfy the requisite compatibility to determine a weak map $\mu:DDE \to DE$. By I.2.11 and I.2.15, the compatibility they do have is described by the weakly commutative diagram

$$\begin{tikzcd} DD_kE \wedge S \arrow[r] \arrow[d, equal] & DD_kE \wedge DS \arrow[r] \arrow[d, "\nu_k"] & DD_kE \arrow[d, "D(\alpha_{k,1} \circ 1 \wedge e)"] \\ DD_kE \arrow[r] & DE & DD_{k+1}E \arrow[l] \end{tikzcd}$$

where ν_k is induced by the composites

$$D_jD_kE \wedge D_jS \xrightarrow{\beta_{j,k} \wedge D_je} D_{jk}E \wedge D_jE \xrightarrow{\alpha_{jk,j}} D_{jk+j}E$$

and $\delta:DF \to DF \wedge DS$ is induced by the maps $\delta_j:D_j(F \wedge S) \to D_jF \wedge D_jS$.