CHAPTER IX

THE MOD p K-THEORY OF QX

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In this chapter we use the theory of H_∞ ring spectra to construct and analyze Dyer-Lashof operations in the complex K-theory of infinite loop spaces analogous to the usual Dyer-Lashof operations in ordinary homology. As an application we compute $K_*(QX;Z_p)$ in terms of the K-theory Bockstein spectral sequence of X.

Dyer-Lashof operations in K-theory were first considered by Hodgkin, whose calculation of $K_*(QS^0;Z_p)$ [41] led him to conjecture the existence of a single operation analogous to the sequence of operations in ordinary homology. He constructed such an operation, denoted by Q, for odd primes [42]; a similar construction for p = 2 was given independently by Snaith, who later refined Hodgkin's construction for odd primes and analyzed the properties of Q. The construction of Hodgkin and Snaith was based on the E^∞ term of a certain spectral sequence (namely the spectral sequence of I.2.4) and therefore had indeterminacy, and Hodgkin showed that in fact any useful operation in the mod p K-homology of infinite loop spaces must have indeterminacy. He also observed that the Dyer-Lashof method for calculating $H_*(QX;Z_p)$ by use of the Serre spectral sequence completely failed to generalize to K-theory. The indeterminacy was a considerable inconvenience, but the operation was still found to have applications, notably in the calculation of $K_*(Q\mathbb{R}P^n;Z_2)$ given by Miller and Snaith [84]. This result, which was proved by using the Eilenberg-Moore spectral sequence starting from Hodgkin's calculation of $K_*(QS^0;Z_p)$, was the first indication that $K_*(QX;Z_p)$ might be tractable in the presence of torsion in X. The main technical difficulty in the proof was in determining exactly how many times Q could be iterated on a given element, since Q could be defined only on the kernel of the Bockstein β. (Incidentally, a joint paper of Snaith and the present author showed that the odd-primary construction of Q contained an error and that in this case as well Q could only be defined on the kernel of β.) The answer for $\mathbb{R}P^n$ was that Q could be iterated on an element exactly as many times as the element survived in the Bockstein spectral sequence. Unfortunately, the methods used in this case did not extend to spaces more complicated than $\mathbb{R}P^n$.

In view of these facts, it is rather surprising that there is in fact a theory of primary Dyer-Lashof operations in K-theory for which practically every statement about ordinary Dyer-Lashof operations, including the calculation of $H_*(QX;Z_p)$, has a precise analog. We shall remove the indeterminacy of Q by constructing it as an operation from mod p^2 to mod p K-theory, and more generally from mod p^{r+1} to mod p^r K-theory. It follows that Q can be iterated on any element precisely as often as the element survives in the Bockstein spectral sequence. There are also operations $\mathcal{Q}$ and R taking mod p^r to mod p^{r+1} K-theory in even and odd dimensions respectively ($\mathcal{Q}$ is the K-theory analog of the Pontrjagin p-th power [57, 28], while R has no analog in ordinary homology). These will play a key role in determining the properties of the Q-operation and in our calculation of $K_*(QX;Z_p)$. They also give indecomposable generators in the K-theory Bockstein spectral sequence for QX.[1] The operations Q, $\mathcal{Q}$ and R form a complete set of Dyer-Lashof operations in the sense that they exhaust the possibilities in a certain universal case; see Section 8. The key to defining primary operations in higher torsion is the machinery of stable extended powers, which gives a very satisfactory replacement for the chain-level machinery in ordinary homology; more precisely, it allows questions about the operations to be reduced to a universal case in the same way that chain-level arguments allow reduction to $B\Sigma_p$. In applying this machinery to K-theory we make essential use of the fact that periodic K-theory is an H_∞ ring spectrum, as shown in VII §7 and VIII §4, and the fact that the Adams operations are p-local H_∞ maps as shown in VIII §7.

This chapter is largely self-contained, and in particular it does not depend logically on the earlier work of Hodgkin, Snaith, Miller and the author. The organization is as follows. In section 1 we give a very general definition of Dyer-Lashof operations in E-homology for an H_∞ ring spectrum E. When E is HZ_p we recover the ordinary Dyer-Lashof operations. In section 2 we use some of the properties developed in section 1 to give a new way of computing $H_*(QX;Z_p)$ for connected X without use of the Serre spectral sequence, the Kudo transgression theorem, or even the equivalence $\Omega Q\Sigma X \simeq QX$; instead the basic ingredients are the approximation theorem and the transfer. In section 3 we give the properties of Q, $\mathcal{Q}$ and R and the statement of our calculation of $K_*(QX;Z_p)$; up to isomorphism the result depends only on the K-theory Bockstein spectral sequence of X, but for functoriality we need a more precise description. Section 4 contains the calculation of $K_*(QX;Z_p)$, which is modeled on that in section 2. Sections 5 through 8 give the construction and properties of Q, $\mathcal{Q}$, and R. In section 5 we lay the groundwork by giving very precise descriptions of the groups $K_*(D_pS^n;Z_{p^r})$. Section 6 gives enough information about Q to calculate $K_*(D_pX;Z_p)$, a result needed in section 4. The argument differs from that in [77] in three ways: it is shorter (but less elementary), it gives a more precise result, and it applies to the case p = 2. Sections 7 and 8 complete

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[1] It was asserted in the original version of this work ([76, Theorem 5]) that certain composites of Q and R gave indecomposable generators in $K_*(QX;Z_p)$. Doug Ravenel has since pointed out to the author that this is incorrect: his argument is given in Remark (ii) following Theorem 3.6 below. The corrected versions of [76, Theorems 5 and 6] are also given in Section 3. (The mistake in the original version was in the proof of Lemma 4.7 for $M = \Sigma M_r$, where it was asserted that the r > 1 and r = 1 cases are similar. They are not.)

the construction of Q, $\mathcal{Q}$, and R. In section 9 we prove a purely algebraic fact needed in section 4; this fact is considerably more difficult than its analog in homology because of the nonadditivity of the operations.

I would like to thank Vic Snaith for introducing me to this subject and for the many insights I have gotten from his book and his papers with Haynes Miller. I would also like to thank Doug Ravenel for pointing out the mistake mentioned above. I owe Gaunce Lewis many commutative diagrams, as well as the first version of Definition 1.7. Finally, I would like to thank Peter May for encouragement and for his careful reading of the manuscript.

1. Generalized Homology Operations

Let E be a fixed H_∞ ring spectrum. In this section we shall construct generalized Dyer-Lashof operations in the E-homology of H_∞ ring spectra X. When E is HZ_p these are (up to reindexing) the ordinary Dyer-Lashof operations defined by Steinberger in chapter III, and for E = S they are Bruner's homotopy operations. When E is the spectrum K representing integral K-theory we obtain the operations referred to in the introduction which will be studied in detail in sections 3-9.

For simplicity, we shall begin by defining operations in E_*X, although ultimately (for the application to K-theory) we must introduce torsion coefficients. Fix a prime p. For each $n \in Z$ the operations defined on E_nX will be indexed by $E_*(D_pS^n)$, i.e., for each $e \in E_m(D_pS^n)$ we shall define a natural operation

$$Q_e : E_nX \to E_mX$$

in the E-homology of H_∞ ring spectra called the <u>internal Dyer-Lashof operation determined by</u> e. As usual, Q_e will be the composite of the structural map

$$(\xi_p)_* : E_mD_pX \to E_m X$$

with an <u>external</u> operation

$$Q_e : E_nX \to E_mD_pX$$

which is defined for arbitrary spectra X and is natural for arbitrary maps $X \to Y$. Throughout this chapter we shall use the same symbol for corresponding internal and external Dyer-Lashof operations, with the context indicating which is intended. In this section we shall be concerned only with the external operations, and thus X and Y will always denote arbitrary spectra.

In order to motivate the definition of the external operation Q_e we give it in stages. Fix $m,n \in Z$ and $e \in E_mD_pS^n$. Let $u \in E_0S$ denote the unit element. We define Q_e first on the element $\Sigma^n u \in E_nS^n$ by $Q_e(\Sigma^n u) = e$. If $x \in E_nX$ happens to be spherical, then there is a map $g:S^n \to X$ with $g_*(\Sigma^n u) = x$, and naturality requires us to define $Q_e x = (D_p g)_* e$. Now any element $x \in E_n X$ is represented by a map $f:S^n \to E \wedge X$, and to complete the definition of Q_e it suffices to give an analog for general x of the homomorphism $(D_p g)_*$ which exists when x is spherical. It is useful to do this in a somewhat more general context, so let Y be any spectrum and let $f:Y \to E \wedge X$ be any map. First we define f_{**} to be the composite

$$E_*Y = \pi_*(E \wedge Y) \xrightarrow{(1 \wedge f)_*} \pi_*(E \wedge E \wedge X) \xrightarrow{(\phi \wedge 1)_*} \pi_*(E \wedge X) = E_*X,$$

where ϕ is the product on E. Note that $f_{**}\Sigma^n u = x$ if $f:S^n \to E \wedge X$ represents x. Next define $\overline{D}_\pi f$ for any $\pi \subset \Sigma_j$ to be the composite

$$D_\pi Y \xrightarrow{D_\pi f} D_\pi(E \wedge X) \xrightarrow{\delta} D_\pi E \wedge D_\pi X \xrightarrow{\xi \wedge 1} E \wedge D_\pi X,$$

where ξ comes from the H_∞ structure of E. Combining these definitions we obtain a map

$$(\overline{D}_\pi f)_{**}:E_*D_\pi Y \longrightarrow E_*D_\pi X .$$

<u>Definition 1.1</u>. If x $E_n X$ is represented by $f:S^n \to E \wedge X$ and e is an element of $E_m D_p S^n$ then

$$Q_e x = (\overline{D}_p f)_{**}(e) \in E_m D_p X.$$

Of course, this agrees with the definition given earlier when x is spherical, and in particular when E = S we recover the external version of Bruner's operation. Next let $E = HZ_p$. The standard external operation (as defined by Steinberger) is denoted $e_i \otimes x^p$, where e_i is the generator of $H_i(\Sigma_p;Z_p(n))$ defined in [68,section 1] (recall that $Z_p(n)$ is Z_p with Σ_p acting trivially if n is even and via the sign representation if n is odd). Now it is easy to see that the map

$$\Phi:H_i(\Sigma_p;Z_p(n)) \longrightarrow H_{i+2pn}(D_p S^n;Z_p)$$

given by $e_i \longmapsto e_i \otimes (\Sigma^n u)^p$ is an isomorphism, and we have

<u>Proposition 1.2</u>. If $e = \Phi(e_i)$ then $Q_e x = e_i \otimes x^p$ for all x.

The proof of 1.2 will be given later in this section.

It is possible to put Definition 1.1 in a more categorical context. Let $\mathcal{C}_E$ be the category in which objects are spectra and the morphisms from X to Y are the stable maps from X to $E \wedge Y$. The composite in $\mathcal{C}_E$ of $f:X \to E \wedge Y$ and $g:Y \to E \wedge Z$ is the following composite of stable maps

$$X \xrightarrow{f} E \wedge Y \xrightarrow{1 \wedge g} E \wedge E \wedge Z \xrightarrow{\phi \wedge 1} E \wedge Z .$$

The construction $\overline{D}_\pi$ on morphisms, combined with D_π on objects, gives a functor $\overline{D}_\pi : \mathcal{C}_E \to \mathcal{C}_E$, and we can also define a smash product $\overline{\wedge}$ on E by letting $f_1 \overline{\wedge} f_2$ be the composite

$$X_1 \wedge X_2 \xrightarrow{f_1 \wedge f_2} E \wedge X_1 \wedge E \wedge X_2 \simeq E \wedge E \wedge X_1 \wedge X_2 \longrightarrow E \wedge X_1 \wedge X_2 .$$

Finally, E homology is a functor on $\mathcal{C}_E$ which takes f to f_{**}, and the following lemma shows that both Q_e and the external product in E-homology are natural transformations.

Lemma 1.3. (i) $(\overline{D}_p f)_{**} Q_e y = Q_e f_{**} y$ for any $y \in E_* Y$ and any $f: Y \to E \wedge X$.

(ii) $(f_{1**} y_1) \otimes (f_{2**} y_2) = (f_1 \overline{\wedge} f_1)_{**} (y_1 \otimes y_2)$.

As one would expect, the maps ι, α, β and δ of I§1 also give natural transformations.

Lemma 1.4. (i) $\iota_* (\overline{D}_\pi f)_{**} = (\overline{D}_\rho f)_{**} \iota_*$ if $\pi \subset \rho$.

(ii) $\alpha_* (\overline{D}_\pi f \ \overline{\wedge}\, \overline{D}_\rho f)_{**} = (\overline{D}_{\pi \times \rho} f)_{**} \alpha_*$.

(iii) $\beta_* (\overline{D}_\pi \overline{D}_\rho f)_{**} = (\overline{D}_{\pi \int \rho} f)_{**} \beta_*$.

(iv) $\delta_* (\overline{D}_\pi (f_1 \overline{\wedge} f_2))_{**} = (\overline{D}_\pi f_1 \overline{\wedge}\, \overline{D}_\pi f_2)_{**} \delta_*$.

We shall need two further transformations, namely the "diagonal" $\Delta : \Sigma D_\pi X \to D_\pi \Sigma X$ and the transfer $\tau : D_\rho X \to D_\pi X$. The first of these was constructed in II§3. The transfer was defined in II§1 for certain special cases, and will be defined in IV§3 of the sequel whenever $\pi \subset \rho$.

Lemma 1.5. (i) $(\overline{D}_\pi \Sigma f)_{**} \Delta_* = \Delta_* (\Sigma \overline{D}_\pi f)_{**}$.

(ii) $\tau_* (\overline{D}_\rho f)_{**} = (\overline{D}_\pi f)_{**} \tau_*$.

The proofs of 1.3, 1.4 and 1.5 are routine diagram chases (using [Equi.,VI.3.9] for 1.4(ii) and (iii) and [Equi.,IV.§3] for 1.5(ii)).

Next we would like to define Dyer-Lashof operations in E-homology with torsion coefficients. We shall always abbreviate $E_*(X; Z_{p^r})$ by $E_*(X;r)$. If M_r denotes the Moore spectrum $S^{-1} \cup_{p^r} S^0$ and E_r denotes $E \wedge \Sigma M_r$ then by definition we have $E_n(X;r) = \pi_n(E_r \wedge X)$. Thus if E_r is an H_∞ ring spectrum (for example, if E is ordinary integral homology) we can apply Definition 1.1 directly to E_r. However, it is a melancholy fact that in general E_r is not an H_∞ ring spectrum, as shown by the following, which will be proved at the end of section 7.

<u>Proposition 1.6</u>. K_r is not an H_∞ ring spectrum for any r.

Thus we must generalize 1.1. First of all, if $f:Y \to E\wedge X$ is any map we define f_{**} to be the composite

$$E_*(Y;r) = \pi_*(E_r\wedge Y) \xrightarrow{(1\wedge f)_*} \pi_*(E_r\wedge E\wedge X) \longrightarrow \pi_*(E_r\wedge X) = E_*(X;r).$$

Next observe that the Spanier-Whitehead dual of ΣM_r is M_r, so that there is a natural isomorphism

$$E_n(X;r) \cong [\Sigma^n M_r, E\wedge X].$$

In particular, any $x\in E_n(X;r)$ is represented by a map $f:\Sigma^n M_r \to E\wedge X$ and there results a homomorphism

$$(\overline{D}_p f)_{**}:E_*(D_p\Sigma^n M_r;s) \to E_*(D_pX;s)$$

for any $s \geq 1$. Note that $f_{**}\Sigma^n u_r = x$, where u_r is the composite $M_r = S\wedge M_r \xrightarrow{u\wedge 1} E\wedge M_r$. We shall call u_r the <u>fundamental class</u> of M_r.

<u>Definition 1.7</u>. Let $e\in E_m(D_p\Sigma^n M_r;s)$. Then

$$Q_e:E_n(X;r) \to E_m(D_pX;s)$$

is defined by $Q_e x = (\overline{D}_p f)_{**}(e)$, where $f:\Sigma^n M_r \to E\wedge X$ is a map representing x.

Lemmas 1.3, 1.4, and 1.5 remain valid in this generality.

When E is integral homology and r = s = 1 Definition 1.7 provides another way of constructing ordinary Dyer-Lashof operations, which are of course the same as those given by Definition 1.1. However, even in this case 1.7 has certain technical advantages; for example, it gives the relation between the Bockstein and the Dyer-Lashof operations, and by allowing r and s to be greater than 1 one obtains the Pontryagin p-th powers.

We conclude with the proof of 1.2. We write E for HZ_p. The result holds by definition when $x = \Sigma^n u\in E_nS^n$, so it suffices to show that

$$(\overline{D}_p f)_{**}(e_i\otimes y^p) = e_i\otimes (f_{**}y)^p$$

for all $f:Y \to E\wedge X$. We shall do this by a direct comparison with the mod p chain level. If A_* is any chain complex over Z_p we write D_pA_* for $W\otimes_{\Sigma_p}(A_*)^{\otimes p}$, where W is a fixed resolution of Z_p by free $Z_p[\Sigma_p]$-modules. We let C_* denote the mod p cellular chains functor on CW-spectra, and we have a natural equivalence $D_pC_* \simeq C_*D_p$ by I.2.1. If Γ_* denotes the trivial chain complex with Z_p in dimension zero then there is a natural equivalence between E^0X and the chain-homotopy classes of degree zero maps from C_*X to Γ_*. In particular, we obtain chain maps $\theta: C_*E \to \Gamma_*$ and $\theta': D_pC_*E \to \Gamma_*$ representing the identity $E \to E$ and the structural map $D_pE \to E$. If ε denotes the composite $D_p\Gamma_* = W/\Sigma_p \to \Gamma_*$ (in which the second map is the augmentation) then $\varepsilon \circ D_p\theta$ is a chain map which, like θ', represents an element of $E^0(D_pE)$ extending the product map $E^{(p)} \to E$. But the proof of I.3.6 shows that there is only one such element, hence we have we have $\varepsilon \circ D_p\theta \simeq \theta'$. Next, observe that f_{**} is equal to the composite

$$E_*Y \longrightarrow E_*(E \wedge X) \longrightarrow E_*X,$$

where the second map is the slant product with the identity class in E^0E. Hence f_{**} is represented on the chain level by the composite

$$h: C_*Y \longrightarrow C_*(E \wedge X) \simeq C_*E \otimes C_*X \xrightarrow{\theta \otimes 1} \Gamma_* \otimes C_*X \simeq C_*X.$$

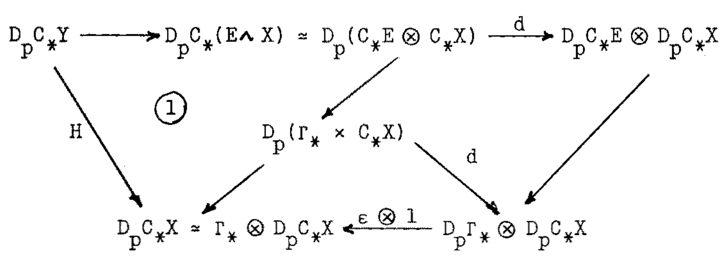
Since h is a chain map we have

$$(D_ph)_*(e_i \otimes y^p) = e_i \otimes (h_*y)^p = e_i \otimes (f_{**}\, y)^p,$$

so it suffices to show $(\overline{D}_pf)_{**} = (D_ph)_*$. Now $(\overline{D}_pf)_{**}$ is equal to the composite

$$E_*D_pY \longrightarrow E_*(D_p(E \wedge X)) \xrightarrow{\delta_*} E_*(D_pE \wedge D_pX) \longrightarrow E_*D_pX\ ,$$

where the last map is the slant product with the structural map in E^0D_pE. Hence $(\overline{D}_pf)_{**}$ is represented on the chain level by the composite H around the outside of the following diagram



Here d is the evident diagonal transformation and the diagram clearly commutes. Inspection of the piece marked (1) shows that $H \simeq D_ph$ as required.

2. The Homology of CX

Our main aim in this chapter is the computation of $K_*(CX;1)$. In this section we illustrate the basic method in a simpler and more familiar situation, namely the computation of the ordinary mod p homology of CX. (All homology in this section is to be taken with mod p coefficients for an odd prime p; the p = 2 case is similar.) This result is of course well-known, but in fact our method gives some additional generality, since both the construction CX and our computation of H_*CX generalize to the situation where X is a (unital) spectrum, while the usual method of computation does not.

We begin by listing the relevant properties of this spectrum-level construction (which is due to Steinberger); a complete treatment will be given in [Equi., chapter VII]. By a unital spectrum we simply mean a spectrum X with an assigned map $S \to X$ called the unit. For any unital spectrum X one can construct an E_∞ ring spectrum CX, and this construction is functorial for unit-preserving maps. In particular, X might be $\Sigma^\infty Y^+$ for some based space Y, and there is then an equivalence $CX \simeq \Sigma^\infty(CY)^+$ relating the space-level and spectrum-level constructions. There is a natural filtration F_kCX of CX and natural equivalences $F_1CX \simeq X$ and

$$F_kCX/F_{k-1}CX \simeq D_k(X/S).$$

Finally, there are natural maps $F_jCX \wedge F_kCX \to F_{j+k}CX$ and $D_jF_kCX \to F_{jk}CX$ for which the following diagrams commute.

$$\begin{array}{ccc} CX \wedge CX & \longrightarrow & CX \\ \uparrow & & \uparrow \\ F_jCX \wedge F_kCX & \longrightarrow & F_{j+k}CX \\ \downarrow & & \downarrow \\ D_j(X/S) \wedge D_k(X/S) & \xrightarrow{\alpha} & D_{j+k}(X/S) \end{array} \qquad \begin{array}{ccc} D_jCX & \longrightarrow & CX \\ \uparrow & & \uparrow \\ D_jF_kCX & \longrightarrow & F_{jk}CX \\ \downarrow & & \downarrow \\ D_jD_k(X/S) & \xrightarrow{\beta} & D_{jk}(X/S) \end{array}$$

Now let X be a unital spectrum and assume the element $\eta \in H_0X$ induced by the unit map is nonzero. We can then choose a set $A \subset H_*X$ such that $A \cup \{\eta\}$ is a basis for H_*X. Let CA be the free commutative algebra generated by the set

$$\{Q^Ix \mid x \in A, \text{ I is admissible and } e(I) + b(I) > |x|\}$$

(here $|x|$ denotes the degree of x; see [28, I.2] for the definitions of admissibility, e(I) and b(I)). The elements of this set, which will be called the standard indecomposables for CA, are to be regarded simply as indeterminates since the Q^I do not act on H_*X. The basis for CA consisting of products of standard indecomposables will be called the standard basis for CA. Using the inclusion $X \to CX$ and the fact that CX is an E_∞ ring spectrum we obtain a ring map

$$\lambda : CA \to H_*CX$$

and we shall show

<u>Theorem 2.1</u>. λ is an isomorphism.

We shall derive this theorem from an analogous fact about extended powers. Let Y be any spectrum and let A be a basis for H_*Y. CA is defined as before, and we make it a filtered ring by giving $Q^I x$ filtration $p^{\ell(I)}$. Let $D_kA = F_kCA/F_{k-1}CA$ for $k \geq 1$; this has a standard basis consisting of the standard basis elements in $F_kCA - F_{k-1}CA$. There is an additive map

$$\lambda_k : D_kA \to H_*D_kY$$

defined as follows. If all Dyer-Lashof operations and products are interpreted externally then a standard basis element of D_kA represents an element of $H_*((D_p)^{j_1}Y \wedge \cdots \wedge (D_p)^{j_s}Y)$ with $p^{j_1} + \cdots + p^{j_s} = k$; here $(D_p)^j$ denotes the j-th iterate of D_p. Applying the natural maps α_* and β_* gives an element H_*D_kY which by definition is the value of λ_k for the original basis element. We then have

<u>Theorem 2.2</u>. λ_k is an isomorphism for all $k \geq 1$.

Assuming 2.2 for the moment, we give the proof of 2.1. Let X be a unital spectrum and let $A \cup \{\eta\}$ be a basis for H_*X. Let $Y = X/S$. Then A projects to a basis for H_*Y which we also denote by A. For each $k \geq 1$ the map $\lambda | F_kCA$ lifts to a map $\lambda^{(k)} : F_kCA \to H_*F_kCX$ and the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{k-1}CA & \longrightarrow & F_kCA & \longrightarrow & D_kA & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda^{(k-1)}} & & \downarrow{\scriptstyle \lambda^{(k)}} & & \downarrow{\scriptstyle \lambda_k} & & \\
 & & H_*F_{k-1}CX & \longrightarrow & H_*F_kCX & \xrightarrow{\gamma} & H_*D_kY & &
\end{array}$$

Since λ_k is an isomorphism, the map γ is onto and hence the bottom row is short exact. It now follows by induction and the five lemma that $\lambda^{(k)}$ is an isomorphism for all k, and 2.1 follows by passage to colimits.

We begin the proof of 2.2 with a special case

<u>Lemma 2.3</u>. λ_p is an isomorphism for all Y.

The proof of the lemma is a standard chain-level calculation which will not be given here (see [68, section 1]). It is interesting to note, however, that one can prove 2.3 without any reference to the chain-level using the methods of section 6 below.

Next we use the machinery of section 1 to reduce to the case where Y is a wedge of spheres. For each $x \in A$ choose a map $f_x : S^{|x|} \to H \wedge Y$ representing x. Let $Z = \bigvee S^{|x|}$ and let $f : Z \to H \wedge Y$ be the wedge of the f_x. Then $f_{**} : H_*Z \to H_*Y$ is an isomorphism. We claim that 2.2 will hold for Y if it holds for Z (where H_*Z is given the basis B consisting of the fundamental classes of the $S^{|x|}$). To see this, consider the following diagram

$$\begin{array}{ccc} D_k B & \xrightarrow{D_k(f_{**})} & D_k A \\ \downarrow{\scriptstyle \lambda_k} & & \downarrow{\scriptstyle \lambda_k} \\ H_* D_k Z & \xrightarrow{(\overline{D}_k f)_{**}} & H_* D_k Y \end{array}$$

The map $D_k(f_{**})$ is induced by f_{**}, which clearly takes B to A. Thus $D_k(f_{**})$ is an isomorphism. The diagram commutes by 1.3 and 1.4(ii) and (iii). The claim now follows from

<u>Lemma 2.4.</u> Let $h : W \to H \wedge X$ be any map. If h_{**} is an isomorphism, so is $(\overline{D}_k h)_{**}$ for all k.

<u>Proof</u>. The proof is by induction on k. First suppose that $k = jp$. Since the case $k = p$ of 2.4 follows from 2.3 we may assume $j > 1$. Let $\pi = \Sigma_j \int \Sigma_p$ and consider the following diagram

$$\begin{array}{ccccccc} H_* D_k W & \xrightarrow{\tau_*} & H_* D_\pi W & \xleftarrow{\beta_*} & H_* D_j D_p W & \xrightarrow{\beta_{jp*}} & H_* D_k W \\ \downarrow{\scriptstyle (\overline{D}_k h)_{**}} & & \downarrow{\scriptstyle (\overline{D}_\pi h)_{**}} & & \downarrow{\scriptstyle (\overline{D}_j \overline{D}_p h)_{**}} & & \downarrow{\scriptstyle (\overline{D}_k h)_{**}} \\ H_* D_k X & \xrightarrow{\tau_*} & H_* D_\pi X & \xleftarrow{\beta_*} & H_* D_j D_p X & \xrightarrow{\beta_{jp*}} & H_* D_k W \end{array}$$

The diagram commutes by 1.4(i) and (iii) and 1.5(ii). The map β_* is an isomorphism. The map $(\overline{D}_p h)_{**}$ is an isomorphism by the case $k = p$, hence so is $(\overline{D}_j \overline{D}_p h)_{**}$ by inductive hypothesis. Our assumption on k implies that τ_* is monic and β_{jp*} is onto, hence $(\overline{D}_k h)_{**}$ is monic by inspection of the first square and onto by inspection of the third. The proof is the same when k is prime to p, except that we let π be $\Sigma_{k-1} \times \Sigma_1$.

Next we reduce to the case of a single sphere. To simplify the notation we assume that Z is a wedge of two spheres $S^m \vee S^n$; the argument is the same in the general case. Let B_1 and B_2 be the bases for H_*S^m and H_*S^n consisting of the fundamental classes, so that $B = B_1 \cup B_2$. There is an evident map $CB_1 \otimes CB_2 \to CB$ and passing to the associated graded gives a map

$$\varphi\colon \sum_{i=0}^{k} (D_i B_1 \otimes D_{k-i} B_2) \longrightarrow D_k B.$$

Recall the equivalence

$$\bigvee_{i=0}^{k} (D_i S^m \wedge D_{k-i} S^n) \simeq D_k(S^m \vee S^n) = D_k Z$$

constructed in II§1.

<u>Lemma 2.5</u>. φ is an isomorphism, and the diagram

$$\begin{array}{ccc}
\sum_{i=0}^{k} (D_i B_1 \otimes D_{k-i} B_2) & \xrightarrow{\ \varphi\ } & D_k B \\
\Big\downarrow {\scriptstyle \sum (\lambda_i \otimes \lambda_{k-i})} & & \Big\downarrow {\scriptstyle \lambda_k} \\
\sum_{i=0}^{k} (H_* D_i S^m \otimes H_* D_{k-i} S^n) & \longrightarrow & H_* D_k Z
\end{array}$$

commutes.

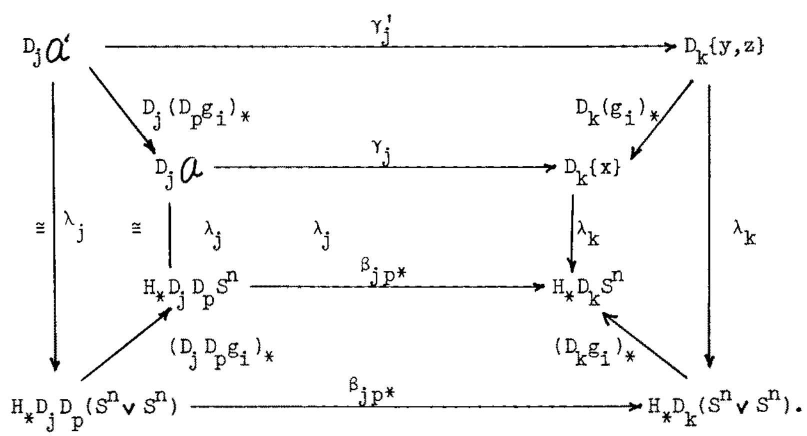
<u>Proof</u>. φ is an isomorphism since it takes the standard basis on the left to that on the right. The commutativity of the diagram is immediate from the definitions.

By Lemma 2.5 we see that 2.2 will hold for Z once we have shown the following. Let $x \in H_n S^n$ be the fundamental class.

<u>Lemma 2.6</u>. $\lambda_k\colon D_k\{x\} \to H_* D_k S^n$ is an isomorphism for all $k \geq 1$ and all integers n.

<u>Proof</u>. By induction on k. First assume that $k = jp$ for some $j > 1$. For the proof in this case we use the following diagram, which will be denoted by (*).

(*)



Here $y,z \in H_n(S^n \vee S^n)$ are the fundamental classes of the first and second summands. The set $\mathcal{A} \subset H_*D_pS^r$ is $\{\beta^\varepsilon Q^s x \mid 2s-\varepsilon \geq n\}$. (The reader is warned as this point to distinguish carefully between the Bockstein β and the natural map β of section I.1. This is made easier by the fact that we never use the latter map per se, only the homomorphism β_* induced by it.) The set $\mathcal{A}' \subset H_*D_p(S^n \vee S^n)$ is $\{\beta^\varepsilon Q^s y, \beta^\varepsilon Q^s z \mid 2s-\varepsilon \geq n\}$ if n is odd and is the union of this set with $\{y^i z^{p-i} \mid 1 \leq i \leq p-1\}$ when n is even. Lemma 2.3 implies that $\mathcal{A}$ and $\mathcal{A}'$ are bases, and hence the maps λ_j are isomorphisms by inductive hypothesis. The maps $g_i : S^n \vee S^n \to S^n$ are defined for i = 0,1 and 2 by $g_0 = 1 \vee 1$, $g_1 = 1 \vee *$ and $g_2 = * \vee 1$, where 1 and * denote the identity map and the trivial map of S^n. To complete the construction of the diagram we require

<u>Lemma 2.7</u>. There exist maps γ_j and γ_j', independent of i, such that diagram (*) commutes for i = 0,1 and 2.

The proof of 2.7 is given at the end of this section; all that is involved is to "simplify" expressions in $D_j\mathcal{A}'$ and $D_j\mathcal{A}$ using the Adem relations and the Cartan formula in a sufficiently systematic way.

Now consider the inner square of diagram (*). By assumption on k we see that $\beta_{jp*} \circ \tau_*$ is an isomorphism, hence λ_k is onto. Let $\theta : D_k\{x\} \to D_k\{x\}$ be the composite $\gamma_j \circ \lambda_j^{-1} \circ \tau_* \circ \lambda_k$. Clearly λ_k will be monic if θ is. In fact we shall show that θ is an isomorphism. We claim first of all that θ takes the subspace $\mathcal{B} \subset D_k\{x\}$ generated by the decomposable standard basis elements isomorphically into itself. To see this we use the outer square of diagram (*). Let $\theta' : D_k\{y,z\} \to D_k\{y,z\}$ be the composite $\gamma_j' \circ \lambda_j^{-1} \circ \tau_* \circ \lambda_k$. Let $\mathcal{B}' \subset D_k\{y,z\}$ be the image of $\sum_{i=1}^{k-1} (D_i\{y\} \otimes D_{k-i}\{z\})$ under the map φ of Lemma 2.5. Then $\mathcal{B}'$ is the kernel of the map

$$D_k(g_1)_* \oplus D_k(g_2)_* : D_k\{y,z\} \longrightarrow D_k\{x\} \oplus D_k\{x\}$$

and hence θ' takes $\mathcal{B}'$ into itself. But $D_k(g_0)_*(\mathcal{B}') = \mathcal{B}$ and $D_k(g_0)_* \circ \theta' = \theta \circ D_k(g_0)_*$, hence θ takes $\mathcal{B}$ into itself and we have the commutative diagram

$$\begin{array}{ccccc}
\mathcal{B}' & \xrightarrow{D_k(g_0)_*} & \mathcal{B} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta'} & & \downarrow & & \\
\mathcal{B}' & \xrightarrow{D_k(g_0)_*} & \mathcal{B} & \longrightarrow & 0 .
\end{array}$$

Since both $\mathcal{B}$ and $\mathcal{B}'$ have finite type $\theta : \mathcal{B} \to \mathcal{B}$ will be an isomorphism if $\theta' : \mathcal{B}' \to \mathcal{B}'$ is monic. But λ_k is monic on $\mathcal{B}'$ by 2.5 and the inductive hypothesis, hence θ' is also monic on $\mathcal{B}'$ since $\lambda_k \circ \theta' = (\beta_{jp*} \circ \tau_*) \circ \lambda_k$.

Now let $\mathcal{J} = D_k\{x\}/\mathcal{B}$. This has the basis $\{Q^I x \mid I$ admissible, $p^{\ell(I)} = k$, $e(I) + b(I) > n\}$. We wish to show that the map $\bar{\theta}: \mathcal{J} \to \mathcal{J}$ induced by θ is an isomorphism. The basic idea is to use the homology suspension, or rather its external analog which is the map $\Delta_*\Sigma: H_i D_p S^n \to H_{i+1} D_p S^{n+1}$, to detect elements of $\mathcal{J}$. Let $\tilde{x} \in H_{n+1} S^{n+1}$ be the fundamental class. We define $\Gamma: \mathcal{J} \to D_k\{\tilde{x}\}$ by $\Gamma(Q^I x) = Q^I \tilde{x}$, where we interpret $Q^I \tilde{x}$ as zero if $e(I) < n+1$ and as a p-th power in the usual way if $e(I) = n+1$ and $b(I) = 0$. The key fact is the following, which will be proved at the end of this section.

Lemma 2.8. The diagram

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{\ \bar{\theta}\ } & \mathcal{J} \\ \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\ D_k\{\tilde{x}\} & \xrightarrow{\ \theta\ } & D_k\{\tilde{x}\} \end{array}$$

commutes.

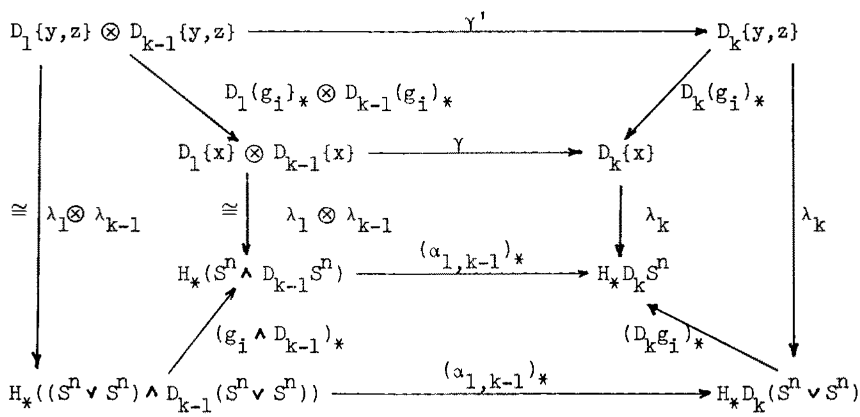
We also need the fact that the evident action of the Bockstein on $\mathcal{J}$ commutes with θ; this will be clear from the proof of 2.7.

Now let $\mathcal{J}_n$ be the subspace of $\mathcal{J}$ spanned by the set $\{Q^I x \mid I$ admissible, $p^{\ell(I)} = k$, $e(I) + b(I) \le n+m\}$. We shall show first that $\bar{\theta}$ is monic on $\mathcal{J}_1$. Let $\mathcal{J}_1'$ be the subspace of $\mathcal{J}_1$ spanned by the set $\{Q^I x \mid I$ admissible, $p^{\ell(I)} = k$, $e(I) = n+1$, $b(I) = 0\}$. Then $\mathcal{J}_1 = \mathcal{J}_1' \oplus \beta \mathcal{J}_1'$. From the definition of Γ we see that $\beta \mathcal{J}_1'$ is the kernel of Γ, that Γ is monic on $\mathcal{J}_1'$ and that $\Gamma(\mathcal{J}_1') = \Gamma(\mathcal{J}) \cap \mathcal{B}$. Let w be a nonzero element of $\mathcal{J}_1'$. We claim that $\bar{\theta} w$ lies in $\mathcal{J}_1$, so that it can be written uniquely in the form $w' + \beta w''$ with $w', w'' \mathcal{J}_1'$, and furthermore we claim that $w' \neq 0$. To see this note that Γw is a nonzero decomposable, hence $\theta \Gamma w$ is also a nonzero decomposable, hence $\Gamma \bar{\theta} w = \theta \Gamma w$ is a nonzero element of $\Gamma(\mathcal{J}) \cap \mathcal{B} = \Gamma(\mathcal{J}_1')$. Thus there is a nonzero element w' of $\mathcal{J}_1'$ with $\Gamma w' = \Gamma \bar{\theta} w$, so that $\bar{\theta} w - w'$ is in $\ker \Gamma = \beta \mathcal{J}_1'$ as required. Now let w_1, w_2 be any elements of $\mathcal{J}_1'$ with $\bar{\theta} w_1 = w_1' + \beta w_1''$ and $\bar{\theta} w_2 = w_2' + \beta w_2''$. Suppose that $v = w_1 + \beta w_2$ is the kernel of $\bar{\theta}$. Then $0 = \bar{\theta} v = w_1' + \beta w_1'' + \beta w_2'$, hence $w_1' = 0$ and $w_1'' + w_2' = 0$. But $w_1' = 0$ implies $w_1 = 0$, hence $w_1'' = 0$. Thus $w_2' = 0$, whence $w_2 = 0$ and $v = 0$, showing that $\bar{\theta}$ is monic on $\mathcal{J}_1$.

Next we claim that $\bar{\theta}$ is monic on $\mathcal{J}_m$ for all $m \ge 1$. Let $w \in \mathcal{J}_m$ with $\bar{\theta} w = 0$. Let $\tilde{\mathcal{J}} = D_k\{\tilde{x}\}/\mathcal{B}$ and let $\bar{\Gamma}$ be the composite $\mathcal{J} \to D_k\{\tilde{x}\} \to \tilde{\mathcal{J}}$. Then $\bar{\Gamma} w$ is in the subspace $\tilde{\mathcal{J}}_{m-1}$ generated by $Q^I \tilde{x}$ with I admissible, $p^{\ell(I)} = k$ and $e(I) + b(I) \,\tilde{-}\, (n+1) \le m-1$. Since $\bar{\theta}\,\bar{\Gamma} w = \bar{\Gamma}\,\bar{\theta} w = 0$ and since (by induction on m) θ is monic on $\tilde{\mathcal{J}}_{m-1}$ we see that $\bar{\Gamma} w = 0$. Now the kernel of $\bar{\Gamma}$ is precisely $\mathcal{J}_1$, and we have shown already that $\bar{\theta}$ is monic on $\mathcal{J}_1$, hence $w = 0$ as required. Thus $\bar{\theta}: \mathcal{J} \to \mathcal{J}$

is monic, and since has finite type Ψ is an isomorphism. This completes the proof of 2.6 for the case $k = jp$.

Now suppose k is prime to p and consider the following diagram



Here γ and γ' are obtained from the products in $C\{x\}$ and $C\{y,z\}$ by passage to the associated graded. The diagram clearly commutes. The analysis of this diagram proceeds as before, except that in this case the map $D_k(g_0)_*$ takes the kernel of $D_k(g_1)_* \oplus D_k(g_2)_*$ onto all of $D_k\{x\}$, so that we can conclude at once that λ_k is an isomorphism without having to consider indecomposables.

This completes the proof of 2.6, and thereby of 2.2, except that we must still verify 2.7 and 2.8. For these we need certain properties of the external Q^s. First of all these operations are additive, and $Q^s x = \iota_*(x^{(p)})$ if $2s = |x|$. The external Cartan formula is

$$\delta_* Q^s(x \otimes y) = \sum_{i=0}^{s} Q^i x \otimes Q^{s-i} y.$$

The external Adem relations are obtained by prefixing β_{pp*} to both sides of the standard Adem relations. All of these relations can be obtained directly from the definitions of section 1, without any use of internal operations (compare sections 7 and 8 below). They can also be derived from the corresponding properties for internal operations by means of the equivalence

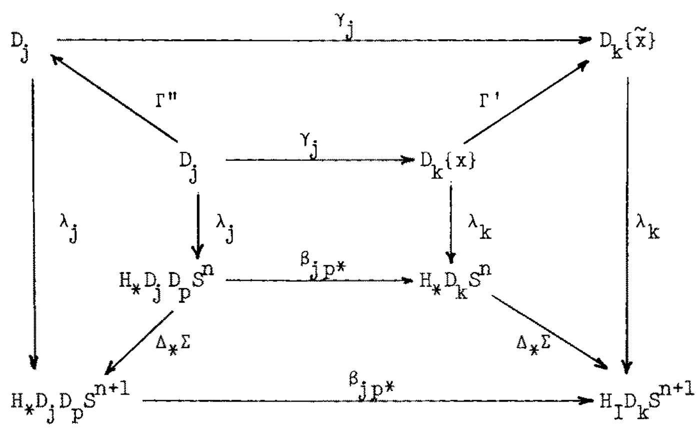
$$C(X \vee S^0) \simeq \bigvee_{k>0} D_k X$$

proved in [Equi., VII§5].

<u>Proof of 2.7.</u> Every standard indecomposable in $C\mathcal{A}$ has the form $Q^I(\beta^\varepsilon Q^s x)$. We can formally simplify such an expression by means of the Adem relations into a sum of admissible sequences acting on x (for definiteness we assume that at each step the

Adem relations are applied at a position in the sequence as far to the right as possible). The result is an element of $C\{x\}$, where we agree to interpret all sequences with excess less than $|x|$ as zero, and we extend multiplcatively to get a map $F_j C\mathcal{A} \to F_k C\{x\}$. The map γ_j is obtained by passage to quotients. The map γ_j' is obtained in the same way except that we use the Cartan formula to simplify expressions of the form $Q^I(y^i z^{p-i})$ with $0 < i < p$. The inner and outer squares of diagram (*) commute as a consequence of the external Cartan formula and Adem relations, and the upper trapezoid clearly commutes when i is 1 or 2. When i is zero the element $y^i z^{p-i}$ of $\mathcal{A}'$ goes to $Q^{n/2}x$, and so it is necessary to check that the result of simplifying $Q^I Q^{n/2}x$ with the Adem relations is the same as using the Cartan formula on $Q^I x^p$; the result in each case is zero unless all entries of I are divisible by p, in which case it is $(Q^{I/p}x)^p$.

Finally, we give the proof of 2.8. We need two facts about $\Delta_*: H_*(\Sigma D_k X) \to H_*(D_k \Sigma X)$, namely that $\Delta_* \Sigma Q^s x = Q^s \Sigma x$ if $k = p$ and that $\Delta_* \Sigma(\alpha_{i,k-i})_*(x \otimes y)$ is zero for $0 < i < k$. The first of these, which is the external version of the stability of Q^s, was proved in II.5.6. For the second, which is the external analog of the fact that the homology suspension annihilates decomposables, we use the third diagram of II.3.1 with $X = S^1$, noting that the diagonal $\Delta: S^1 \to S^1 \wedge S^1$ is nullhomotopic. Now 2.8 is immediate from the commutativity of the following diagram.



Here γ_j is the map constructed in the proof of 2.7 and Γ' is the composite $D_k\{x\} \longrightarrow \mathcal{J} \xrightarrow{\Gamma} D_k\{\tilde{x}\}$. We define Γ'' to take decomposables to zero and $Q^I(\beta^\varepsilon Q^s x)$ to $Q^I(\beta^\varepsilon Q^s \tilde{x})$. Commutativity of the left and right trapezoids follow from the two formulas given above. Commutativity of the upper trapezoid is obvious except on elements of the form $Q^I(\beta^\varepsilon Q^s x)$ with $e(I) = n+1 + 2s(p-1) - \varepsilon$ and $b(I) = 0$, and it follows in this case from a simple calculation.

3. Dyer-Lashof Operations in K-Theory

In this section we give our main results about K-theory Dyer-Lashof operations. We begin by fixing notations. We shall work in the stable category, so that X will always denote a spectrum. Homology operations are to be interpreted as internal rather than external. We use Z_2-graded K-theory, with $|x|$ denoting the mod 2 degree of x. There are evident natural maps

$$\pi : K_\alpha(X;r) \longrightarrow K_\alpha(X;r-1) \quad \text{if } r \geq 2$$

$$p_*^s : K_\alpha(X;r) \longrightarrow K_\alpha(X;r+s) \quad \text{if } s \geq 1$$

$$\beta_r : K_\alpha(X;r) \longrightarrow K_{\alpha+1}(X;r)$$

$$\Sigma : K_\alpha(X;r) \longrightarrow K_{\alpha+1}(\Sigma X;r) .$$

(Recall that ΣX means $S^1 \wedge X$ in this chapter, not $X \wedge S^1$ as in chapters I-VII.) β_1 will usually be written simply as β. We write π^s for the s-th iterate of π. It will often be convenient to denote the identity map either by π^0 or p_*^0. We write π^∞ for the reduction map $K_\alpha(X;Z) \to K_\alpha(X;r)$. Our first two results give some useful elementary facts about mod p^r K-theory; the proofs may be found in [13] (except for 3.2(iii), which is Lemma 6.4 of [63], and 3.2(iv), which will be proved in section 7).

Proposition 3.1. (i) $K_*(X;r)$ is a Z_{p^r}-module.

(ii) If $s \geq 1$ then $\pi^s \beta_{r+s} p_*^s = \beta_r$.

(iii) πp_* and $p_* \pi$ are multiplication by p.

(iv) $\beta_r \beta_r = 0$.

Proposition 3.2. For each $r \geq 1$ there is an external product

$$K_\alpha(X;r) \otimes K_{\alpha'}(Y;r) \to K_{\alpha+\alpha'}(X \wedge Y;r),$$

denoted by $x \otimes y$, which has the following properties.

(i) $\otimes$ is natural, bilinear and associative.

(ii) If u K_0S is the unit then $x \otimes \pi^\infty u = \pi^\infty u \otimes x = x$.

(iii) $\pi(x \otimes y) = \pi x \otimes \pi y$ and $\pi^\infty(x \otimes y) = \pi^\infty x \otimes \pi^\infty y$.

(iv) $p_*(x \otimes \pi y) = (p_* x) \otimes y$.

(v) $\beta_r(x \otimes y) = \beta_r x \otimes y + (-1)^{|x|} x \otimes \beta_r y$.

(vi) $\Sigma(x \otimes y) = \Sigma x \otimes y = (-1)^{|x|} x \otimes \Sigma y$.

If p is odd then the following also holds, where $T: X \wedge Y \to Y \wedge X$ switches the factors.

(vii) $T_*(x \otimes y) = (-1)^{|y||x|} y \otimes x$

If p = 2 there are two external products for each r satisfying (i), (ii), (v) and (vi). If these are denoted by $\otimes$ and $\otimes'$ the relation

(viii) $x \otimes y = x \otimes' y + 2^{r-1} \beta_r x \otimes \beta_r y$

holds. Relations (iii) and (iv) hold when either mod 2^r product is paired with either mod 2^{r-1} product. If $r \geq 2$ then (vii) holds for both $\otimes$ and $\otimes'$, while if r = 1 then the following holds.

(vii)' $T_*(x \otimes y) = y \otimes' x = y \otimes x + \beta y \otimes \beta x.$

We shall actually give a canonical choice of mod 2^r multiplications in Remark 3.4(iv) below. When X is a ring spectrum we obtain an internal product denoted xy. We write $\eta \in K_0(X;r)$ for the unit in this case, reserving the letter u for the unit of K_0S.

Our next result gives the properties of our first operation, which is denoted by Q. In order to relate Q to the K-homology suspension we must restrict to the space level, and we fix notations for dealing with this case. If Y is any space we write $K_*(Y;r)$ for $K_*(\Sigma^\infty Y^+;r)$ and, if Y is based, we write $\tilde{K}_r(Y;r)$ for $K_*(\Sigma^\infty Y;r)$. The homology suspension σ is the composite

$$\tilde{K}_\alpha(\Omega Y;r) \xrightarrow{\Sigma} \tilde{K}_{\alpha+1}(\Sigma\Omega Y;r) \longrightarrow \tilde{K}_{\alpha+1}(Y;r) \subset K_{\alpha+1}(Y;r).$$

If Y is an H_∞ space then ΩY is also an H_∞ space and $\Sigma^\infty Y^+$ is an H_∞ ring spectrum; see I.3.7 and I.3.8.

<u>Theorem 3.3</u>. Let X be an H_∞ ring spectrum. For each $r \geq 2$ and $\alpha \in Z_2$ there is an operation

$$Q: K_\alpha(X;r) \to K_\alpha(X;r-1)$$

with the following properties, where x,y $K_*(X;r)$.

(i) Q is natural for H_∞ maps of X.

(ii) $Q\eta = 0$.

(iii) $Q\pi x = \pi Q x$ if $r \geq 3$.

$$\text{(iv)}\quad Qp_*x = \begin{cases} x^p & \text{if } |x| = 0 \text{ and } r = 1 \\ p_*Qx - (p^{p-1} - 1)x^p & \text{if } |x| = 0 \text{ and } r \geq 2 \\ 0 & \text{if } |x| = 1 \text{ and } r = 1 \\ p_*Qx & \text{if } |x| = 1 \text{ and } r \geq 2 \end{cases}$$

$$\text{(v)}\quad \beta_{r-1}Qx = \begin{cases} Q\beta_r x - p\pi(x^{p-1}\beta_r x) & \text{if } |x| = 0 \\ (\pi\beta_r x)^p + pQ\beta_r x & \text{if } |x| = 1. \end{cases}$$

$$\text{(vi)}\quad Q(x+y) = \begin{cases} Qx + Qy - \pi[\sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i y^{p-i}] & \text{if } p \text{ is odd and } |x| = |y| = 0 \\ Qx + Qy - \pi(xy) + 2^{r-2}(\pi\beta_r x)(\pi\beta_r y) & \text{if } p = 2 \text{ and } |x| = |y| = 0 \\ Qx + Qy & \text{if } |x| = |y| = 1. \end{cases}$$

$$Q(kx) = kQx - \frac{1}{p}(k^p - k)(\pi x)^p \quad \text{if } k \quad Z,\ |x| = 0.$$

(vii) Let $|x| = |y| = 0$. Then

$$Q(xy) = \begin{cases} Qx\cdot\pi(y^p) + (x^p)\cdot Qy + p(Qx)(Qy) & \text{if } p \text{ is odd} \\ Qx\cdot\pi(y^2) + \pi(x^2)\cdot Qy + 2(Qx)(Qy) + 2^{r-2}\pi(x\beta_r x)\pi(y\beta_r y) & \\ \quad + 2^{2r-4}(Q\beta_r x)(Q\beta_r y) & \text{if } p = 2. \end{cases}$$

Let $|x| = 1$, $|y| = 0$. Then

$$Q(xy) = \begin{cases} Qx\cdot\pi(y^p) + p(Qx)(Qy) & \text{if } p \text{ is odd} \\ Qx\cdot\pi(y^2) + 2(Qx)(Qy) + 2^{2r-4}(\pi\beta_r x)^2(Q\beta_r y) & \text{if } p = 2. \end{cases}$$

Let $|x| = |y| = 1$. Then

$$Q(xy) = \begin{cases} (Qx)(Qy) & \text{if } p \text{ is odd} \\ (Qx)(Qy) + 2^{r-2}\pi(x\beta_r x)\pi(y\beta_r y) + 2^{2r-4}(\pi\beta_r x)^2(Q\beta_r y) & \\ \quad + 2^{2r-4}(Q\beta_r x)(\pi\beta_r y)^2 & \text{if } p = 2. \end{cases}$$

(viii) If Y is an H_∞ space and $x \in \tilde{K}_\alpha(\Omega Y;r)$ then $Qx \in \tilde{K}_\alpha(\Omega Y;r-1)$ and

$$\sigma Qx = \begin{cases} Q\sigma x & \text{if } |x| = 0 \\ (\pi\sigma x)^p + pQ\sigma x & \text{if } |x| = 1. \end{cases}$$

(ix) If k is prime to p then $\psi^k Qx = Q\psi^k x$, where ψ^k is the k-th Adams operation.

(x) If p = 2 and $|x| = 1$ then

$$x^2 = \begin{cases} Q\beta_2 2_* x & \text{if } r = 1 \\ 2^{r-2}\beta_r 2_* Qx & \text{if } r \geq 2. \end{cases}$$

In particular $(\pi^{r-1}x)^2 \in K_0(X;1)$ is zero if $r \geq 3$ and is equal to $(\pi\beta_2 x)^2$ if r = 2.

Remarks 3.4. (i) There are no analogs for the Adem relations.

(ii) We shall write $Q^s: K_\alpha(X;r) \to K_\alpha(X;r-s)$ for the s-th iterate of Q when r > s (and similarly for the operations R and $\mathcal{Q}$ to be introduced later).

(iii) If $x \in K_*(X;1)$ has $\beta x = 0$ then x lifts to $y \in K_*(X;2)$. Thus one can define a secondary operation $\overline{Q}$ on the kernel of β by $\overline{Q}x = Qy$. The element y is well-defined modulo the image of p_* and thus 3.3(iv) shows that $\overline{Q}x$ is well-defined modulo p-th powers if $|x| = 0$ and has no indeterminacy if $|x| = 1$. This is essentially the operation defined by Hodgkin and Snaith [42,99] (although their construction is incorrect when p is odd, as shown in [77]).

(iv) When p = 2, parts (vi) and (vii) are corrected versions of the corresponding formulas in [76]. Note that $2^{2r-4} = 0 \bmod 2^{r-1}$ unless r = 2. The formula for Q(xy) with $|x| = |y| = 1$ and p = 2 implicitly assumes that the mod 2^r multiplications for $r \geq 2$ have been suitably chosen, since the evaluation of $Q(xy + 2^{r-1}(\beta_r x)(\beta_r y))$ by means of 3.3(vi) and (vii) gives a different formula. Thus we may (inductively) fix a canonical choice of mod 2^r multiplications by choosing the mod 2 multiplication arbitrarily and requiring the formula to hold as stated for $r \geq 2$. From now on we shall always use this choice of multiplications.

Our next result shows that, in contrast to ordinary homology, $K_*(X;1)$ will in general have nilpotent elements.

Corollary 3.5. If X is an H_∞ ring spectrum and x $K_1(X;r)$ then $(\pi^{r-1}\beta_r x)^{p^r} = 0$ in $K_0(X;1)$.

Proof of 3.5. (By induction on r). If r = 1 then

$$(\beta x)^p = (\pi\beta_2 p_* x)^p = \beta Q p_* x = 0$$

by 3.1(ii), 3.3(v) and 3.3(iv). If $r \geq 2$ then

$$(\pi^{r-1}\beta_r x)^{p^r} = [(\pi^{r-1}\beta_r x)^p]^{p^{r-1}} = (\pi^{r-2}\beta_{r-1}Qx)^{p^{r-1}} = 0$$

by 3.3(v) and the inductive hypothesis.

It turns out that iterated Q-operations on r-th Bocksteins are also nilpotent. In order to see this we must make use of the operation R described in our next theorem.

Theorem 3.6. Let X be an H_∞ ring spectrum. For each $r \geq 1$ there is an operation

$$R: K_1(X;r) \to K_1(X;r+1)$$

with the following properties, where $x, y \in K_1(X;r)$.

(i) R is natural for H_∞ maps of X

(ii) $\pi R x = Q p_* x - x(\beta_r x)^{p-1}$, and if $r \geq 2$ then $R\pi x = Q p_* x - p^{p-1}x(\beta_r x)^{p-1}$

(iii) $p_* R x = R p_* x$

(iv) $\beta_{r+1} R x = Q\beta_{r+2} p_*^2 x$

(v) $R(x+y) = Rx + Ry - \sum_{i=1}^{p-1} [\frac{1}{p}\binom{p}{i}(p_* x)(\beta_{r+1}p_* x)^{i-1}(\beta_{r+1}p_* y)^{p-i}$

$$+ \binom{p-1}{i}\beta_{r+1}p_*(xy)(\beta_{r+1}p_* x)^{i-1}(\beta_{r+1}p_* y)^{p-i-1}]$$

(vi) If Y is an H_∞ space and $x \in \tilde{K}_1(Y;r)$ then

$$\sigma R x = \begin{cases} p_*[(\sigma x)^p] & \text{if } r = 1 \\ p_*[(\sigma x)^p] + p_*^2 Q\sigma x & \text{if } r \geq 2. \end{cases}$$

(vii) If k is prime to p then $\psi^k R x = R\psi^k x$.

(viii) If $r \geq 2$ then $QRx = RQx$. If $r = 1$ then $QRx = 0$.

Remarks (i) Let $x \in K_1(X;r)$ and let $s \geq 1$. By 3.3(v) we have $(\pi^{r+s-1}\beta_{r+s}R^s x)^{p^r} = \pi^{s-1}\beta_s Q^r R^s x$. But $Q^r R^s x = R^{s-1}QR(Q^{r-1}x) = 0$ by 3.6(viii). We therefore have the following nilpotency relation.

$$(\pi^{r+s-1}\beta_{r+s}R^s x)^{p^r} = 0.$$

Note that this is a smaller exponent than would be given by 3.5. In terms of the Q-operation this relation may be written $(\pi^{r-s-1}Q^s\beta_r x)^{p^r} = 0$ for $s < r$ and $(Q^s\beta_{s+1}p_*^{s-r+1}x)^{p^r} = 0$ for $s \geq r$.

(ii) The second statement of 3.6(viii) was not in the original version of this work (cf. [76, Theorem 3(iv)]). The decomposability of QRx when r = 1 (which actually implies its vanishing, as we shall see in Section 8) had been asserted by Snaith when p = 2 ([99, Proposition 5.2(ii)]), but was not included in [76] because the author erroneously thought he could prove QRx to be indecomposable in $K_1(QX;1)$ whenever $x \in K_1(X;1)$ had nonzero Bockstein (cf. [76, Theorem 4]). This point was recently settled by Doug Ravenel, who observed that if one starts with the description of $K_*(Q(S^1 \cup_p e^2);1)$ given in [76, Theorem 4] and applies the Rothenberg-Steenrod spectral sequence (which collapses) then one can see that the only indecomposable in $K_1(Q(S^2 \cup_p e^3);1)$ is the generator of $K_1(S^2 \cup_p e^3;1)$, and in particular QR of this generator is decomposable. This contradicts part of [76, Theorem 4] and a corrected version of that result will be given later in this section. We shall give a completely different argument in Section 8 to show that QRx is decomposable, and in fact vanishes, for all $x \in K_1(X;1)$.

We next introduce an operation $\mathcal{Q}$ which is the K-theoretic analog of the Pontrjagin p-th power [57, 28]. This operation is a necessary tool in our calculation of $K_*(QX;1)$ and will also be used to give generators for the higher terms of the Bockstein spectral sequence.

<u>Theorem 3.7</u>. Let X be an H_∞ ring spectrum. For each $r \geq 1$ there is an operation

$$\mathcal{Q}: K_0(X;r) \to K_0(X;r+1)$$

with the following properties, where $x,y \in K_*(X;r)$.

(i) $\mathcal{Q}$ is natural for H_∞ maps of X.

(ii) $\pi\mathcal{Q}x = x^p$, and if $r \geq 2$ then $\pi x = x^p$.

(iii) $\mathcal{Q}p_*x = p^{p-1}p_*\mathcal{Q}x$.

(iv) $\pi\beta_{r+1}\mathcal{Q}x = x^{p-1}\beta_r x$

(v) $$\mathcal{Q}(x+y) = \begin{cases} \mathcal{Q}x + \mathcal{Q}y + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} p_*(x^i y^{p-i}) & \text{if } p \text{ is odd or } r \geq 2 \\ \mathcal{Q}x + \mathcal{Q}y + 2_*(xy) + (\beta_2 2_* x)(\beta_2 2_* y) & \text{if } p = 2 \text{ and } r = 1. \end{cases}$$

(vi) Let $|x| = |y| = 0$. Then $\mathcal{Q}(xy) = (\mathcal{Q}x)(\mathcal{Q}y)$ if p is odd, while if p = 2 there is a constant $\varepsilon_r \in Z_2$, independent of x and y, with

$$\mathfrak{Q}(xy) = \begin{cases} (\mathfrak{Q}x)(\mathfrak{Q}y) + (1+2\varepsilon_1)(\beta_2\mathfrak{Q}x)(\beta_2\mathfrak{Q}y) & \text{if } r = 1 \\ (\mathfrak{Q}x)(\mathfrak{Q}y) + 2^r\varepsilon_r(\beta_{r+1}\mathfrak{Q}x)(\beta_{r+1}\mathfrak{Q}y) & \text{if } r \geq 2. \end{cases}$$

Let $|x| = 1$, $|y| = 0$. Then

$$R(xy) = \begin{cases} (Rx)(\mathfrak{Q}y) & \text{if } p \text{ is odd and } r = 1 \\ (Rx)(\mathfrak{Q}y) + p_*^2[(Qx)(Qy)] & \text{if } p \text{ is odd and } r \geq 2 \\ (Rx)(\mathfrak{Q}y) - (1+2\varepsilon_1)(\beta_2 Rx)(\beta_2\mathfrak{Q}y) & \text{if } p = 2 \text{ and } r = 1 \\ (Rx)(\mathfrak{Q}y) + 4_*[(Qx)(Qy)] + 2^{r-2}(\beta_{r+1}4_*Qx)(\beta_{r+1}\mathfrak{Q}y) & \\ \qquad + 2^r\varepsilon_r(\beta_{r+1}Rx)(\beta_{r+1}\mathfrak{Q}y) & \text{if } p = 2 \text{ and } r \geq 2, \end{cases}$$

and $R(yx) = (\mathfrak{Q}y)(Rx) + (1+2\varepsilon_1)(\beta_2\mathfrak{Q}y)(\beta_2 Rx)$ if $p = 2$ and $r = 1$. Let $|x| = |y| = 1$. Then there is a constant $\varepsilon_r' \in Z_p$, independent of x and y, with

$$\mathfrak{Q}(xy) = \begin{cases} p^r\varepsilon_r'(Rx)(Ry) & \text{if } p \text{ is odd} \\ (1+2\varepsilon_1')(Rx)(Ry) - (1+2\varepsilon_1+2\varepsilon_1')(\beta_2 Rx)(\beta_2 Ry) & \text{if } p = 2 \text{ and } r = 1 \\ 2^r\varepsilon_r'(Rx)(Ry) + 2^{r-2}(Rx)(4_*Qy) + 2^{r-2}(4_*Qx)(Ry) & \\ \qquad + 2^{2r-4}(\beta_{r+1}4_*Qx)(\beta_{r+1}4_*Qy) & \text{if } p = 2 \text{ and } r \geq 2. \end{cases}$$

(vii) Let Y be an H_∞ space and let $x \in \tilde{K}_0(Y;r)$. If $p = 2$ then $\sigma\mathfrak{Q}x = 2^r R(\sigma x)$, while if p is odd there is a constant ε_r'', independent of x, with $\sigma\mathfrak{Q}x = p^r\varepsilon_r'' R(\sigma x)$.

(viii) If k is prime to p then $\psi^k\mathfrak{Q}x = \mathfrak{Q}\psi^k x$.

(ix) $$Q\mathfrak{Q}x = \begin{cases} 0 & \text{if } r = 1 \\ \sum_{i=1}^{p} \binom{p}{i} p^{i-2} x^{p^2-ip} p_*[(Qx)^i] & \text{if } r \geq 2\,. \end{cases}$$

The undetermined constants ε_r in part (vi) depend on the choice of multiplications; they can be made equal to zero for a suitable choice but it is not clear what their values are for our canonical choice. It is quite possible that the ε_r, ε'_r and ε''_r are all zero.

Next we shall use the operations Q and R to describe $K_*(CX;1)$ for an arbitrary unital spectrum X. If Y is a based space then the homology equivalence of [28, Theorem I.5.10] is also a K-theory equivalence (by the Atiyah-Hirzebruch spectral sequence), hence

$$K_*(QY;1) \cong (\pi_0 Y)^{-1} K_*(CY;1) = (\pi_0 Y)^{-1} K_*(C\Sigma^\infty(Y^+);1)$$

so that our calculation will also give $K_*(QY;1)$.

First recall the K-theory Bockstein spectral sequence $E^r_* X$ (abbreviated BSS) from [13, section 11]. X was assumed to be a finite complex in [13] but we wish to work in greater generality. The finiteness assumption is necessary for those results which deal with the E^∞ term, since in general there is no useful relation between $E^\infty_* X$ and $K_* X$ (for example, $E^\infty_* RP^\infty$ is concentrated in dimension zero, while $K_* RP^\infty$ is concentrated in dimension one). On the other hand, the results of [13] which deal with E^r for r finite remain valid for arbitrary spectra X. In particular, any (r-1)-cycle x can be lifted to an element $y \in K_*(X;r)$ and we have $d_r x = \pi^{r-1}\beta_r y$. The element y has order p^r if and only if x is nonzero in E^r. If we write $K_*(X;\infty)$ for the inverse limit of the $K_*(X;r)$ then an infinite cycle always lifts to $K_*(X;\infty)$; we shall frequently use this notation. Our next definition gives the kind of data necessary for the description of $K_*(CX;1)$.

<u>Definition 3.8</u>. Let $1 \le n \le \infty$. A set $A = \bigcup_{1 \le r \le n} A_r$ with $A_r \subset K_*(X;r)$ is called a <u>subbasis of height</u> n for X if for each $s \le n$ the set

$$\{\pi^{r-1}x \mid x \in A_r,\ s \le r \le n\} \cup \{\pi^{r-1}\beta_r x \mid x \in A_r,\ s \le r < n\}$$

projects to a basis for $E^s_* X$.

If the height of a subbasis is not specified, it will always be assumed to be infinite. Subbases with finite height will occur only in sections 7 and 8. It is not hard to see that any spectrum has a subbasis of any given height. The term subbasis is motivated by our next result, which is an easy consequence of the results of [13,§11]. Recall that a subset S of an abelian group G is a <u>basis</u> for G if G is the direct sum of the cyclic subgroups generated by the elements of S.

<u>Proposition 3.9</u>. If $A = \bigcup_{1 \le r \le n} A_r$ is a subbasis of height n for X and if $s \le n$ (with $s < \infty$ if $n = \infty$) then the set

$$\{\pi^{r-s}x \mid x \in A_r,\ s \leq r \leq n\} \cup \{\pi^{r-s}\beta_r x \mid x \in A_r,\ s \leq r < n\}$$
$$\cup \{p_*^{s-r}x \mid x \in A_r,\ r < s\} \cup \{\beta_s p_*^{s-r}x \mid x \in A_r,\ r < s\}$$

is a basis for $K_*(X;s)$. The elements of the form $p_*^{s-r}x$ and $\beta_s p_*^{s-r}x$ have order p^r and the remaining basis elements have order p^s.

Now let X be a unital spectrum. Let $\eta \in K_0(X;\infty)$ be the unit and suppose that $\pi^\infty\eta$ is nonzero in $K_0(X;1)$. Then we may choose a set $A = \bigcup_{1 \leq r \leq \infty} A_r$ such that $A \cup \{\eta\}$ is a subbasis for X. We write $A_{r,0}$ and $A_{r,1}$ for the zero- and one-dimensional subsets of A_r. Let p be odd, and let CA be the quotient of the free commutative algebra generated by the three sets

$$\{\pi^{r-s-1}Q^s x \mid x \in A_r,\ 0 \leq s < r \leq \infty\}$$
$$\{\pi^{r-s-1}\beta_{r-s}Q^s x \mid x \in A_{r,0},\ 0 \leq s < r < \infty\}$$
and
$$\{\pi^{r+s-1}\beta_{r+s}R^s x \mid x \in A_{r,1},\ r < \infty,\ 0 \leq s < \infty\}$$

by the ideal generated by the set

$$\{(\pi^{r+s-1}\beta_{r+s}R^s x)^{p^r} \mid x \in A_{r,1},\ r < \infty,\ 0 \leq s < \infty\}.$$

The elements of the first three sets will be called the <u>standard indecomposables</u> of CA. Here symbols like $\pi^{r-s-1}Q^s x$ are simply indeterminates, since the Dyer-Lashof operations are not defined on $K_*(X;r)$. However, by means of the inclusion $X \to CX$ we may interpret these symbols as elements of $K_*(CX;1)$. Thus we obtain a ring map

$$\lambda: CA \to K_*(CX;1).$$

Our main theorem is

<u>Theorem 3.10</u>. λ is an isomorphism.

We could have defined CA in terms of the Q-operation alone, without using R, since the third generating set is equal to

$$\{\pi^{r-s-1}Q^s\beta_r x \mid x \in A_{r,1},\ r < \infty,\ 0 \leq s \leq r\} \cup \{Q^s\beta_{s+1}p_*^{s-r+1}x \mid x \in A_{r,1},\ r < \infty,\ s > r\}$$

The definition we have given is more convenient for our purposes, however, since it allows us to treat the cases $s \leq r$ and $s > r$ in a unified way.

Theorem 3.10 also holds for $p = 2$, but the definition of CA in this case is more complicated since mod 2 K-theory is not commutative. Recall from 3.2(vii)' that the commutator of two elements is the product of their Bocksteins. To build this into the definition of CA we define the modified tensor product $C_1 \tilde{\otimes} C_1$ of two Z_2-graded differential algbebras over Z_2 to be their Z_2-graded tensor product with multiplication given by

$$(x \otimes y)(x' \otimes y') = xx' \otimes yy' + x(dx') \otimes (dy)y'.$$

We can define the modified tensor product of finitely many C_i similarly and of infinitely many C_i by passage to direct limits. Now for each $x \in A_{r,0}$ we define C_x to be the free strictly commutative algebra generated by $\{\pi^{r-s-1}Q^s x | 0 \leq s \leq r\}$ and if $r < \infty$, $\{\pi^{r-s-1}\beta_{r-s}Q^s x | 0 \leq s < r\}$. Give this the differential which takes $Q^{r-1}x$ to $\beta Q^{r-1}x$ and all other generators to zero. For each $x \in A_{r,1}$ we define C_x to be the commutative algebra generated by the sets $\{\pi^{r-s-1}Q^s x | 0 \leq s < r\}$ and, if $r < \infty$, $\{\pi^{r+s-1}\beta_{r+s}R^s x | 0 \leq s < r\}$, with the relations

(i) $(\pi^{r+s-1}\beta_{r+s}R^s x)^{2^r} = 0$

and

$$\text{(ii)} \quad (\pi^{r-s-1}Q^s x)^2 = \begin{cases} 0 & \text{if } 0 \leq s < r-2 \\ (\pi^{r-1}\beta_r x)^{2^{r-1}} & \text{if } s = r-2 \\ (\pi^r \beta_{r+1} Rx)^{2^{r-1}} & \text{if } s = r-1. \end{cases}$$

(Relation (ii) is motivated by 3.3(x)). Give C_x the differential which takes $Q^{r-1}x$ to $(\pi^{r-1}\beta_r x)^{2^{r-1}}$ and all other generators to zero. Finally, we define CA to be the modified tensor product $\tilde{\bigotimes}_{x \in A} C_x$. There is an evident ring map $\lambda: CA \to K_*(CX;1)$ and with these definitions Theorem 3.10 and its proof are valid.

Remarks 3.11. (i) When $X = S^0$, or when $p = 2$ and X is a sphere or a real projective space, we recover the calculations of Hodgkin [41] and Miller and Snaith [83,84].

(ii) We can describe the additive structure of CA more explicitly as follows. When $p = 2$ we define the standard indecomposables of CA to be the same three sets as in the odd-primary case. If we give these some fixed total ordering then CA has an additive basis consisting of all ordered products of standard indecomposables in which each of the odd-dimensional indecomposables occurs no more than once and each $\pi^{r+s-1}\beta_{r+s}R^s x$ occurs less than 2^r times. This basis will be called the standard basis for CA. We define the standard basis in the same way when p is odd.

Next we discuss the functoriality of the description given by 3.10. If X and X' are unital spectra with subbases $A \cup \{\eta\}$ and $A' \cup \{\eta\}$ then a unit-preserving map $f: X \to X'$ will be called based if $f_* A_r \subset A'_r \cup \{0\}$ for all $r \geq 1$. Such a map clearly induces a map $f_*: CA \to CA'$, and we have $\lambda \circ f_* = (Cf)_* \circ \lambda$. If f is not based, it is still possible in principle to determine $(Cf)_*$ on $K_*(CX;1)$ by using 3.3, 3.6 and 3.9 (although in practice the formulas may become complicated). For example, if $f:S^2 \to S^2$ is the degree p map and $x \in K_0(S^2;2)$ is the generator then

$$(Cf)_*Qx = Q(f_*x) = Q(px) = \pi(x^p) \neq 0$$

in $K_0(CS^2;1)$. Since $f_*:K_*(S^2;1) \to K_*(S^2;1)$ is zero this gives another proof of Hodgkin's result that $K_*(CX;1)$ cannot be an algebraic functor of $K_*(X;1)$. A similar calculation for the degree p^r map shows that $K_*(CX;1)$ is not a functor of $K_*(X;r)$ for any $r < \infty$. Finally, the projection $S^1 \cup_p e^2 \to S^2$ onto the top cell induces the zero map in integral K-homology but is nonzero on $K_*(C(S^1 \cup_p e^2);1)$ so that $K_*(CX;1)$ is not a functor of $K_*(X;Z)$. Thus it seems that the use of subbases cannot be avoided.

We conclude this section by determining the BSS for CX.

Theorem 3.12. For $1 \leq m < \infty$, E^m_* CX is additively isomorphic to the quotient of the free strictly commutative algebra generated by the six sets

$$\{\pi^{r-s-1}Q^s x \mid x \in A_r,\ m \leq r-s,\ 0 \leq s < r\}$$

$$\{\pi^{r-s-1}\beta_{r-s}Q^s x \mid x \in A_{r,0},\ m \leq r-s < \infty,\ 0 \leq s < r\}$$

$$\{\pi^{m-1}\mathfrak{Q}^{m-r+s}Q^s x \mid x \in A_{r,0},\ 1 \leq r-s < m\}$$

$$\{\pi^{m-1}\beta_m\mathfrak{Q}^{m-r+s}Q^s x \mid x \in A_{r,0},\ 1 \leq r-s < m\}$$

$$\{\pi^{m-1}R^{m-r+s}Q^s x \mid x \in A_{r,1},\ 1 \leq r-s < m\}$$

and

$$\{\pi^{r+s-1}\beta_{r+s}R^s x \mid x \in A_{r,1},\ m \leq r+s < \infty\}$$

by the ideal generated by the set

$$\{(\pi^{r+s-1}\beta_{r+s}R^s x)^{p^t} \mid x \in A_{r,1},\ m \leq r+s < \infty,\ t = \min(r, r+s+1-m)\}.$$

If p is odd or $m \geq 3$ the isomorphism is multiplicative.

The proof of 3.12 is the usual counting argument, and is left to the reader. In order to determine the differential in E^m_*CX one needs the formula

$$\pi^{r-s+t-1}\beta_{r-s+t}R^tQ^s x = (\pi^{r+t-1}\beta_{r+t}R^t x)^{p^s}$$

for $x \in A_{r,1}$, $0 \leq s < r < \infty$, $t \geq 0$; this is is a consequence of 3.3(viii) and 3.3(v).

4. Calculation of $K_*(CX;Z_p)$

In this section we give the proof of Theorem 3.10, except for two lemmas which will be dealt with in Sections 6 and 9. The argument is very similar to that given in Section 2 for ordinary homology, and in several places we shall simply refer to that section.

First we reformulate 3.10 as a result about extended powers. Let Y be any spectrum and let A be a subbasis for Y. We define CA with its standard indecomposables and standard basis as in Section 3. We make CA a filtered ring by giving elements of A filtration 1 and requiring Q and R to multiply filtration by p. Let $D_kA = F_kCA/F_{k-1}CA$ for $k \geq 1$; this has a standard basis consisting of the standard basis elements in $F_kCA - F_{k-1}CA$. There is an additive map

$$\lambda_k : D_kA \to K_*(D_kY;1)$$

defined as in Section 2 by interpreting Q,R and the multiplication externally and then applying α_* and β_*. We shall prove

Theorem 4.1. λ_k is an isomorphism for all $k \geq 1$.

Remark 4.2. Using 4.1 and the external versions of 3.3(v), 3.6(iv) and 3.7(iv) (which will be proved in sections 7 and 8) one can determine the BSS for D_kY as follows. If $m \geq 1$ let C^mA denote the algebra whose generators and relations are given in 3.12. We make C^mA a filtered ring by giving elements of A filtration 1 and requiring R, Q and $\mathcal{Q}$ to multiply filtration by p. If D_k^mA is the k-th subquotient of C^mA there is an isomorphism $D_k^mA \to E_*^mD_kX$. The proof is similar to that for 3.12 and is left to the reader.

The derivation of 3.10 from 4.1 is the same as that given for 2.1 in section 2. We therefore turn to the proof of 4.1. We need the following special case, which will be proved in section 6.

Lemma 4.3. λ_p is an isomorphism for all Y.

We shall reduce the proof of 4.1 to the case where Y is a wedge of Moore spectra. First we need some notation. As in section 1 we write M_r for $S^{-1} \cup_{p^r} e^0$. The set $\{u_r\}$ is a subbasis for M_r. We write M_∞ for the colimit of the M_r with respect to the maps $M_r \to M_{r+1}$ having degree p on the bottom cell. Then $K_1(M_\infty;r) = 0$ for all r and $K_0(M_\infty;r)$ is a copy of Z_{p^r} generated by the image of u_r. Let $u_\infty \in K_0(M_\infty;\infty)$ be the element which projects to the image of u_r for all r. Then $\{u_\infty\}$ is a subbasis for M_∞.

For each $x \in A_r$ we can choose a map $f_x : \Sigma^{|x|}M_r \to K \wedge Y$ representing x. (If $r = \infty$ we let f_x be any map which restricts on each $\Sigma^{|x|}M_r$ to a representaive for the mod p^r reduction of x.) Let $Z = \bigvee_{1 \leq r \leq \infty} \bigvee_{x \in A_r} \Sigma^{|x|}M_r$ and let $f: Z \to K \wedge Y$ be the wedge of the f_x. We give Z the subbasis B consisting of the fundamental classes of the

$\Sigma^{|x|}M_r$. Then $f_{**}:K_*(Z;r) \to K_*(Y;r)$ gives a one-to-one correspondence between B_r and A_r, and in particular it is an isomorphism for all r. Now consider the diagram

$$\begin{array}{ccc} D_kB & \xrightarrow{D_k(f_{**})} & D_kA \\ \downarrow{\lambda_k} & & \downarrow{\lambda_k} \\ K_*(D_kZ;1) & \xrightarrow{(\overline{D}_kf)_{**}} & K_*(D_kY;1) \end{array},$$

which commutes by 1.3 and 1.4(ii) and (iii). If 4.1 holds for Z, its validity for Y will be immediate from the diagram and the following lemma.

Lemma 4.4. Let $h:W \to K \wedge X$ be any map. If $h_{**}:K_*(W;1) \to K_*(X;1)$ is an isomorphism, then

(i) $f_{**}:K_*(W;r) \to K_*(X;r)$ is an isomorphism for all r, and

(ii) $(\overline{D}_kf)_{**}:K_*(D_kW;1) \to K_*(D_kX;1)$ is an isomorphism for all k.

Proof. (i) By induction on r. Suppose the result is true for some $r \geq 1$ and consider the short exact sequence

$$0 \longrightarrow Z_p \longrightarrow Z_{p^{r+1}} \longrightarrow Z_{p^r} \longrightarrow 0 .$$

This gives rise to the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} K_{\alpha+1}(W;r) & \longrightarrow & K_\alpha(W;1) & \longrightarrow & K_\alpha(W;r+1) & \longrightarrow & K_\alpha(W;r) & \longrightarrow & K_{\alpha-1}(W;1) \\ \downarrow{f_{**}} & & \downarrow{f_{**}} & & \downarrow{f_{**}} & & \downarrow{f_{**}} & & \downarrow{f_{**}} \\ K_{\alpha+1}(X;r) & \longrightarrow & K_\alpha(X;1) & \longrightarrow & K_\alpha(X;r+1) & \longrightarrow & K_\alpha(X;r) & \longrightarrow & K_{\alpha-1}(X;1) \end{array}$$

Part (i) follows by the five lemma. The proof of part (ii) is now completely parallel to that of Lemma 2.4.

Next we reduce to the case of a single Moore spectrum. We assume for simplicity that Z is a wedge of two Moore spectra $\Sigma^mM_r \vee \Sigma^nM_s$; the argument is the same in the general case. Let B_1 and B_2 be the subbases $\{\Sigma^mu_r\}$ and $\{\Sigma^nu_s\}$, so that $B = B_1 \cup B_2$. There is an evident map $CB_1 \otimes CB_2 \to CB$ which on passage to the associated graded gives a map

$$\varphi: \sum_{i=0}^{k} (D_iB_1 \otimes D_{k-i}B_2) \to D_kB.$$

Lemma 4.5. φ is an isomorphism, and the diagram

$$\begin{array}{ccc}
\sum_{i=0}^{k} (D_iB_1 \otimes D_{k-i}B_2) & \xrightarrow{\quad\varphi\quad} & D_kB \\
\Big\downarrow {\scriptstyle \Sigma(\lambda_i \otimes \lambda_{k-i})} & & \searrow {\scriptstyle \lambda_k} \\
\sum_{i=0}^{k} (K_*(D_i\Sigma^m M_r;1) \otimes K_*(D_{p-i}\Sigma^n M_s;1)) & \xrightarrow{\quad\cong\quad} & K_*(D_kZ;1)
\end{array}$$

commutes

The proof is the same as for 2.5. The lemma implies that 4.1 will hold for Z once we have shown the following. We write x for $\Sigma^n u_r \in K(\Sigma^n M_r;r)$.

Lemma 4.6. $\lambda_k : D_k\{x\} \to K_*(D_k\Sigma^n M_r;1)$ is an isomorphism for all $k \geq 1$ and all n.

Proof. By induction on k. First let $k = jp$ with $j > 1$. We need the commutativity of the following diagram for i = 0,1 and 2.

$$(*)\qquad \begin{array}{ccccccc}
D_j\mathfrak{a}' & & & \xrightarrow{\qquad\gamma'_j\qquad} & & & D_k\{y,z\} \\
 & \searrow {\scriptstyle F_i} & & & & \swarrow {\scriptstyle D_k(g_i)_*} & \\
 & & D_j\mathfrak{a} & \xrightarrow{\quad\gamma_j\quad} & D_k\{x\} & & \\
\cong\Big\downarrow{\scriptstyle\lambda_j} & & \cong\Big\downarrow{\scriptstyle\lambda_j} & & \Big\downarrow{\scriptstyle\lambda_k} & & \Big\downarrow{\scriptstyle\lambda_k} \\
 & & K_*(D_jD_pM;1) & \xrightarrow{\quad\beta_{jp*}\quad} & K_*(D_kM;1) & & \\
 & \nearrow {\scriptstyle (D_jD_pg_i)_*} & & & & \nwarrow {\scriptstyle (D_kg_i)_*} & \\
K_*(D_jD_p(M\vee M);1) & & & \xrightarrow{\qquad\beta_{jp*}\qquad} & & & K_*(D_k(M\vee M);1)
\end{array}$$

Here M denotes $\Sigma^n M_r$ and $y,z \in K_*(M\ \ M;r)$ are the fundamental classes of the first and second summands. The sets $\mathfrak{a}$ and $\mathfrak{a}'$ are subbases for D_pM and $D_p(M\vee M)$ which will be specified later. The maps $g_i : M\vee M \to M$ are defined by $g_0 = 1\vee 1$, $g_1 = 1\vee *$, and $g_2 = *\vee 1$, and the F_i are determined uniquely by the requirement that the left-hand trapezoid commute. To complete the diagram we need

Lemma 4.7. There exist $\mathfrak{a}$, $\mathfrak{a}'$, γ_j and γ'_j independent of i such that diagram (*) commutes for i = 0,1 and 2.

The proof will be given in Section 9. Like the proof of 2.7, it consists of systematic simplifications of the elements of $D_j\mathcal{A}$ and $D_j\mathcal{A}'$. The details are much more complicated, however, because of the nonadditivity of the operations.

Now consider the inner square of the diagram. Since $\beta_{jp*} \circ \tau_*$ is an isomorphism, we see that λ_k is onto. Letting $\theta = \gamma_j \circ \lambda_j^{-1} \circ \tau_* \circ \lambda_k$, we see as in section 2 that θ induces an isomorphism of the subspace $\mathcal{D}$ of $D_k\{x\}$ spanned by the decomposable standard basis elements. In particular, λ_k is monic on $\mathcal{D}$.

The remainder of the proof differs from that in Section 2, and is in fact considerably simpler since there are only a few indecomposables. It suffices to show the following.

<u>Lemma 4.8.</u> Let $w \in \mathcal{D}$. If $n = 1$ then

(i) $\lambda_k(\pi^{r-s-1}Q^s x - w) \neq 0$, where $k = p^s$, $2 \leq s < r \leq \infty$

(ii) $\lambda_k(\pi^{r+s-1}\beta_{r+s}R^s x - w) \neq 0$, where $k = p^s$, $r < \infty$, $2 \leq s < \infty$.

If $n = 0$ then

(iii) $\lambda_k(\pi^{r-s-1}Q^s x - w) \neq 0$, where $k = p^s$, $2 \leq s < r < \infty$

(iv) $\lambda_k(\pi^{r-s-1}\beta_{r-s}Q^s x - w) \neq 0$, where $k = p^s$, $2 \leq s < r < \infty$.

<u>Proof.</u> We need two facts about the map $\Delta_*:K_*(\Sigma D_k X;r) \to K_*(D_k\Sigma X;r)$, namely that $\Delta_*\Sigma(\alpha_{i,k-1})_*(x \otimes y) = 0$ for $0 < i < k$ and that, when $k = p$,

$$\Delta_*\Sigma Qx = \begin{cases} Q(\Sigma x) & \text{if } |x| = 0 \\ \pi\iota_*(\Sigma x)^{(p)} + pQ\Sigma x & \text{if } |x| = 1 . \end{cases}$$

The first fact is shown as in the proof of 2.8, while the second, which is the external version of 3.3(viii), will be shown in section 7.

Now consider part (i). We have $\Delta_*\Sigma w = 0$ and

$$\Delta_*\Sigma\pi^{r-s-1}Q^s x = \pi^{r-1}\iota_*(\Sigma x)^{p^s}.$$

But $\pi^{r-1}\iota_*(\Sigma x)^{p^s}$ is nonzero since λ_k is monic on decomposables.

Combining part (i) with the fact that λ_k is onto and is monic on decomposables, we see that

$$\lambda_k:D_k\{x\} \to K_*(D_k\Sigma M_r;1)$$

is an isomorphism in degree 1 and is onto in degree zero. It is monic in degree 0 if and only if part (ii) holds. But if not then $K_0(D_k\Sigma M_r;1)$ and $K_1(D_k\Sigma M_r;1)$ would have different dimensions as vector spaces, and therefore the Bockstein spectral sequence $E^m_*(D_k\Sigma M_r)$ would be nonzero for all m. But the transfer embeds $E^m_* D_k \Sigma M_r$ in $E^m_* D_j D_p \Sigma M_r$, and the latter is zero for $p^{m-r-1} > j$ by Remark 4.2 and the inductive hypothesis of 4.6.

Finally, part (iii) follows from (i) and the equation

$$\Delta_* \Sigma \pi^{r-s-1} Q^s x = \pi^{r-s-1} Q^s \Sigma x,$$

while (iv) follows from (iii) using the argument given for (ii).

This completes the proof of 4.6 for the case k = jp. The remaining case, when k is prime to p, is handled exactly as in Section 2.

5. Calculation of $\tilde{K}_*(D_p S^n; Z_{p^r})$

In order to construct and analyze the Q-operation we shall need a precise description of $K_*(D_p \Sigma^n M_r; r-1)$. In this section we give some facts about $K_*(D_p S^n; r)$ which will be used in Sections 6 and 7 to obtain such a description. We work with K-theory on spaces in this section.

If X is a space there is a relative Thom isomorphism

$$\Phi : \tilde{K}_*(D_p X; r) \xrightarrow{\cong} \tilde{K}_*(D_p \Sigma^2 X; r)$$

corresponding to the bundle

$$E\Sigma_p \times_{\Sigma_p} (X^{(p)} \times R^{2p}) \to E\Sigma_p \times_{\Sigma_p} X^{(p)}$$

and the inclusion

$$E\Sigma_p \times_{\Sigma_p} (*) \to E\Sigma_p \times_{\Sigma_p} X^{(p)}.$$

As we have seen in VII§3 and VII§8, this isomorphism can in fact be defined for an arbitrary spectrum X. In calculating $\tilde{K}_*(D_p S^n; r)$ we may therefore assume n = 0 or n = 1; in the former case we have $D_p S^0 = B\Sigma_p^+$.

Lemma 5.1. $K_\alpha(B\Sigma_p; 1)$ is zero if $\alpha = 1$ and $Z_p \oplus Z_p$ if $\alpha = 0$. $\tilde{K}_\alpha(D_p S^1; 1)$ is zero if $\alpha = 0$ and Z_p if $\alpha = 1$.

Proof. We use the Atiyah-Hirzebruch spectral sequence for mod p K-homology. By [40, III.1.2] the differentials d_i vanish for $i < 2p-1$ and d_{2p-1} is $\beta P^1_* - P^1_* \beta$ (here P^1 denotes Sq^2 if p = 2). For spaces of the form $D_p X$, a basis for the E^2-term consisting of external Dyer-Lashof operations is given in [68, 1.3 and 1.4]. The differential d_{2p-1} can be evaluated using the external form of the Nishida relations

[68, 9.4]; the explicit result is that $d_{2p-1}(e_i \otimes y^p)$ is a nonzero multiple of

$$(\beta e_{i+2-2p}) \otimes y^p - e_{i+1-p} \otimes (\beta y)^p$$

for any $y \in H_*(X;1)$. Letting $X = S^0$ or S^1 we see that E^{2p} is generated by $e_0 \otimes u^p$ and $e_{2p-2} \otimes u^p$ in the former case and by $e_{p-1} \otimes (\Sigma u)^p$ in the latter. Then $E^{2p} = E^\infty$ for dimensional reasons and the result follows.

Using 5.1 and the K-theory BSS we conclude that $K_*(B\Sigma_p;r)$ is free over Z_{p^r} on two generators in dimension zero and that $\tilde{K}_*(D_pS^1;r)$ is free over Z_{p^r} on one generator in dimension one. We wish to give explicit bases. It is convenient to work in K-cohomology, as we may by the following.

Lemma 5.2. The natural map

$$\tilde{K}^*(D_pS^n;r) \to \mathrm{Hom}(\tilde{K}_*(D_pS^n;r), Z_{p^r})$$

is an isomorphism for all $r < \infty$.

Proof. When $r = 1$ a cell-by-cell induction and passage to limits gives the results for an arbitrary space; in particular it holds for D_pS^n. The result for general r follows from the BSS.

Next we give a basis for $K^0(B\Sigma_p;r)$. We write 1 for the unit in this group and $1_{(e)}$ for the unit of $K^0(pt.;r)$. Let τ be the transfer $\Sigma^\infty(B\Sigma_p^+) \to \Sigma^\infty(Be^+) = S$.

Proposition 5.3. $K^*(B\Sigma_p;r)$ is freely generated over Z_{p^r} by 1 and $\tau^*1_{(e)}$.

Proof. Let $\pi = Z_p$ and denote the inclusion $\pi \subset \Sigma_p$ by ι. Then $K^1(B\pi;r) = 0$ and the natural map

$$R\pi \otimes Z_{p^r} \to K^0(B\pi;r)$$

is an isomorphism. If ρ is the group of automorphisms of π then a standard transfer argument shows that the restriction

$$\iota^*: K^*(B\Sigma_p;r) \to K^*(B\pi;r)$$

is a monomorphism whose image is contained in the invariant subring $K^*(B\pi;r)^\rho$. Now ι^*1 is the unit 1_π of $K^0(B\pi;r)$, while the double coset formula gives $\iota^*\tau^*1_{(e)} = (p-1)!(\tau')^*1_{(e)}$, where τ' is the transfer $\Sigma^\infty(B\pi^+) \to S$. Since 1_π and $\tau'1_{(e)}$ form a basis for $K^*(B\pi;r)^\rho$ the result follows.

In order to give a specific generator for $\widetilde{K}^*(D_pS^1;r)$ we consider the map

$$\Delta^*:\widetilde{K}^*(D_pS^{n+1};r) \to \widetilde{K}^*(\Sigma D_pS^n;r).$$

Lemma 5.4. The composite

$$\widetilde{K}^0(D_pS^2;r) \xrightarrow{\Delta^*} \widetilde{K}^0(\Sigma D_pS^1;r) \xrightarrow{(\Sigma\Delta)^*} \widetilde{K}^0(\Sigma^2 D_pS^0;r) \cong K^0(B\Sigma_p;r)$$

takes $\Phi(1)$ to $\frac{1}{(p-1)!}(p! - \tau^* 1_{(e)})$ and $\Phi(\tau^* 1_{(e)})$ to zero.

As an immediate consequence we have

Corollary 5.5. $\Sigma\Delta^*\Phi(1)$ generates $\widetilde{K}^*(D_pS^1;r)$.

Before proving 5.4 we give the desired bases for $K_*(B\Sigma_p;r)$ and $\widetilde{K}_*(D_pS^1;r)$.

Definition 5.6. The canonical basis for $K_*(B\Sigma_p;r)$ is the dual of the basis $\{1, \frac{1}{(p-1)!}(p! - \tau^* 1_{(e)})\}$. The canonical basis for $\widetilde{K}_*(D_pS^1;r)$ is the dual of $\{\Sigma\Delta^*\Phi(1)\}$.

Note that the unit η in $K_0(B\Sigma_p;r)$ is the first element of the canonical basis for this group. We shall always write v for the remaining element and v' for the basis element in $K_1(D_pS^1;r)$.

Proof of 5.4. Consider the subset of $E\Sigma_p \times_{\Sigma_p} (\mathbb{R}^2)^p$ consisting of points for which the sum of the $\mathbb{R}^2$-coordinates is zero. The projection to $B\Sigma_p$ makes this subset the total space of a bundle ξ over $B\Sigma_p$. Now D_pS^2 is homeomorphic to the second suspension of the Thom complex $T\xi$ of ξ, and under this homeomorphism the map $\Delta \circ \Sigma\Delta:\Sigma^2 D_pS^0 \to D_pS^2$ is the second suspension of the inclusion $B\Sigma_p^+ \subset T\xi$, while $\Phi(1)$ agrees with the Aityah-Bott-Shapiro orientation for ξ. Thus it suffices to show that the Euler class of ξ is $\frac{1}{(p-1)!}(p! - \tau^* 1_{(e)})$. If $\pi = Z_p$ and $\iota: \pi \subset \Sigma_p$ is the inclusion it suffices to show that the pullback $(B\iota)^*\xi$ has Euler class $p - (\tau')^* 1_{(e)}$ in $K^0(B\pi) \cong R\pi \otimes Z_p^\wedge$, where τ' is the transfer $\Sigma^\infty(B\pi^+) \to S$. Let $x \in R\pi$ be any nontrivial irreducible. Then $(B\iota)^*\xi$ is the sum of the bundles over $B\pi$ induced by $x, x^2, \ldots, x^{p-1}$. These bundles have Euler classes $1-x, \ldots, 1-x^{p-1}$, hence $(B\iota)^*\xi$ has Euler class $(1-x)\cdots(1-x^{p-1})$. Evaluation of characters shows that

$$(1-x) \cdots (1-x^{p-1}) = p - (1 + x + \cdots + x^{p-1})$$

and the result follows.

Next we collect some information about the elements η, v and v' for use in section 7.

Proposition 5.7. (i) $\pi:\tilde{K}_*(D_pS^n;r) \to \tilde{K}_*(D_pS^n;r-1)$ takes v to v and v' to v'.

(ii) $\Delta_*:\tilde{K}_1(\Sigma(B\Sigma_p^+);r) \to \tilde{K}_1(D_pS^1;r)$ takes $\Sigma\eta$ to zero and Σv to v'.

(iii) $\Delta_*:\tilde{K}_0(\Sigma D_pS^1;r) \to \tilde{K}_0(D_pS^2;r)$ takes $\Sigma v'$ to $\Phi(\eta + pv)$.

(iv) $\tau_*:\tilde{K}_*(D_pS^n;r) \to \tilde{K}_*((S^n)^{(p)};r)$ takes η to $p!u$ and v to $-(p-1)!u$ when $n = 0$ and takes v' to zero when $n = 1$.

(v) $\delta_*:K_0(B\Sigma_p;r) \to K_0(B\Sigma_p \times B\Sigma_p;r)$ takes η to $\eta \otimes \eta$ and v to $v \otimes \eta + \eta \otimes v + p(v \otimes v)$.

(vi) $\delta_*:\tilde{K}_1(D_pS^1;r) \to \tilde{K}_1(D_pS^1 \wedge B\Sigma_p^+;r)$ takes v' to $v' \otimes \eta + p(v' \otimes v)$.

(vii) $\delta_*:\tilde{K}_0(D_pS^2;r) \to \tilde{K}_0(D_pS^1 \wedge D_pS^1;r)$ takes $\Phi(\eta)$ to zero and $\Phi(v)$ to $v' \otimes v'$.

For the proof we need a preliminary result.

Lemma 5.8. (i) If X is a spectrum with $E^1 = E^r$ in the K-theory BSS and if Y is any spectrum then the external product map

$$K_*(X;r) \otimes K_*(Y;r) \to K_*(X \wedge Y;r)$$

is an isomorphism, where the tensor product is taken in the Z_2-graded sense.

(ii) If in addition $K_*(X;1)$ and $K_*(Y;1)$ are finitely generated then the external product map

$$K^*(X;r) \otimes K^*(Y;r) \to K^*(X \wedge Y;r)$$

is an isomorphism.

Proof When $r = 1$ the first statement is well-known (see [13, Theorem 6.2], for example). It follows that the external product induces an isomorphism of K-theory Bockstein spectral sequences. Hence if B is a basis for $K_*(X;r)$ and A is a subbasis of height r for Y then the set $\{\pi^{r-s}x \otimes y \mid x \in B,\ y \in A_s\}$ is a subbasis of height r for $X \wedge Y$ and part (i) follows. The case $r = 1$ of part (ii) follows from part (i) by duality, and the general case follows from it as in part (i).

Next we turn to the proof of 5.7, which will conclude this section. In each case it suffices by 5.8 to show the dual. Then (i) is immediate and (ii) and (iii) follow from 5.4. The first and second statements of part (iv) are trivial, as is the third when $p = 2$. When p is odd we observe that τ_*v' must be invariant under the Σ_p action on $\tilde{K}_*((S^1)^{(p)};r)$. Clearly zero is the only invariant element.

For part (v) we observe that $\tau^*1_{(e)} \otimes \tau^*1_{(e)}$ is $\tau^*(\iota^*\tau^*1_{(e)})$ by Frobenius reciprocity. Now $\iota^*\tau^*1_{(e)} = p!1_{(e)}$, and thus

$$\left[\frac{1}{(p-1)!}(p! - \tau^*1_{(e)})\right]^2 = \frac{p}{(p-1)!}(p! - \tau^*1_{(e)})$$

in $K^0(B\Sigma_p;r)$; the result follows by duality.

For part (vi), consider the composite

$$\tilde{K}_1(\Sigma(B\Sigma_p^+);r) \xrightarrow{\Delta_*} \tilde{K}_1(D_pS^1;r) \xrightarrow{\delta_*} \tilde{K}_1(D_pS^1 \wedge B\Sigma_p^+;r) .$$

We have $\Delta_*\Sigma v = v'$, and

$$\begin{aligned} \delta_*\Delta_*\Sigma v &= (\Delta \wedge 1)_*\Sigma\delta_* v \\ &= (\Delta_*\Sigma v) \otimes \eta + (\Delta_*\Sigma\eta) \otimes v + p(\Delta_*\Sigma v) \otimes v \\ &= v' \otimes \eta + p(v' \otimes v). \end{aligned}$$

For part (vii) observe that part (iii) implies that the map

$$(\Delta \wedge 1)_*:\tilde{K}_1(\Sigma D_pS^1 \wedge D_pS^1;r) \to \tilde{K}_1(D_pS^2 \wedge D_pS^1;r)$$

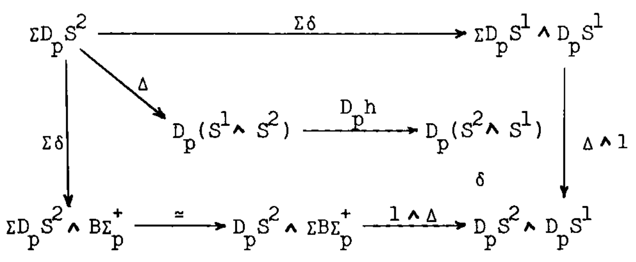
is monic and that $(\Delta \wedge 1)_*(\Sigma v' \otimes v') = \Phi(\eta) \otimes v' + p\Phi(v) \otimes v'$. Hence it suffices to show that $(\Delta \wedge 1)_*(\Sigma\delta_*\Phi(\eta))$ is zero and that

$$(\Delta \wedge 1)_*\Sigma\delta_*\Phi(v) = \Phi(\eta) \times v' + p\Phi(v) \otimes v'.$$

Now let

$$h:S^1 \quad S^2 = S^1 \wedge (S^1 \wedge S^1) \simeq (S^1 \wedge S^1) \wedge S^1 = S^2 \wedge S^1$$

be the associativity transformation and consider the diagram



The upper part clearly commutes, and the lower part also commutes since h is homotopic to the map switching the factors S^1 and S^2. Now

$$\delta_*:\tilde{K}_0(D_pS^2;r) \to \tilde{K}_0(D_pS^2 \wedge B\Sigma_p^+;r)$$

clearly takes $\Phi(\eta)$ to $\Phi(\eta) \otimes \eta$ and $\Phi(v)$ to

$$\Phi(\eta) \otimes v + \Phi(v) \otimes \eta + p\Phi(v) \otimes v.$$

Hence

$$(\Delta \wedge 1)_*(\Sigma\delta_*\Phi(\eta)) = (1 \wedge \Delta)_*(\Phi(\eta) \otimes \Sigma\eta) = 0$$

by the diagram and part (ii), while

$$(\Delta \wedge 1)_*(\Sigma\delta_*\Phi(v)) = (1 \wedge \Delta)_*[\Phi(\eta) \otimes \Sigma v + \Phi(v) \otimes \Sigma\eta + p\Phi(v) \otimes \Sigma v]$$

$$= \Phi(\eta) \otimes v' + p\Phi(v) \otimes v' .$$

6. Calculation of $\tilde{K}_*(D_pX;Z_p)$

In this section we define Q on $K_*(X;2)$ and prove Lemma 4.3. We work with K-theory on spectra in this section.

Our first result collects the information about $K_*(D_p\Sigma^nM_r;1)$ which will be used in this and later sections. We let i and j respectively denote the inclusion of the bottom cell of Σ^nM_r and the projection onto the top cell. Note that $j_*\Sigma^nu_r = \Sigma^nu$ and $i_*\Sigma^{n-1}u = \beta_r\Sigma^nu_r$, where u_r and u are the fundamental classes of M_r and S^0.

Lemma 6.1. (i) For any $n \in Z$ and $\alpha \in Z_2$, $K_\alpha(D_p\Sigma^nM_1;1)$ has dimension 1 over Z_p.

(ii) For any $n \in Z$, $\alpha \in Z_2$ and $r \geq 2$, $K_\alpha(D_p\Sigma^nM_r;1)$ has dimension 2 over Z_p.

(iii) $(D_pj)_*:K_0(D_pM_r;1) \to K_0(D_pS^0;1)$ is monic, and if $r \geq 2$ it is an isomorphism.

(iv) $(D_pj)_* \oplus \tau_*:K_1(D_p\Sigma M_r;1) \to K_1(D_pS^1;1) \oplus K_1((\Sigma M_r)^{(p)};1)^{\Sigma_p}$ is monic, and is an isomorphism if $r \geq 2$.

(v) $(D_pi)_*:K_0(D_pS^0;1) \to K_0(D_p\Sigma M_r;1)$ is onto. If $r = 1$ it has kernel generated by η and if $r \geq 2$ it is an isomorphism.

(vi) The sequence

$$K_1(D_pS^{-1};1) \xrightarrow{(D_pi)_*} K_1(D_pM_r;1) \xrightarrow{\tau_*} K_1((M_r)^{(p)};1)^{\Sigma_p} \longrightarrow 0$$

is exact, and if $r \geq 2$, $(D_pi)_*$ is a monomorphism.

In parts (iv) and (vi), $K_1((\Sigma^nM_r)^{(p)};1)^{\Sigma_p}$ denotes the subgroup invariant under the evident Σ_p-action; this subgroup can easily be calculated using 5.8(i). The proof of 6.1 is similar to that of 5.1 and is left to the reader.

We can now define elements $v_1 \in K_0(D_pM_2;1)$ and $v_1' \in K_1(D_p\Sigma M_2;1)$ by the equations $(D_pj)_*v_1 = v$, $(D_pj)_*v_1' = v'$, and $\tau_*v_1' = 0$. We use definition 1.6 to construct Q.

Definition 6.2. $Q:K_\alpha(X;2) \to K_\alpha(D_pX;1)$ is the generalized Dyer-Lashof operation Q_{v_1} if $\alpha = 0$ and $Q_{v_1'}$ if $\alpha = 1$.

Observe that $v_1 = Qu_2$ and $v_1' = Q\Sigma u_2$.

Next we turn to the proof of 4.3. We use the spectral sequence of I.2.4 with π equal to Z_p or Σ_p and $E = X$. This spectral sequence will be denoted by $E^r_{q,\alpha}(\pi;X)$; by Bott periodicity it is $Z \times Z_2$-graded, so that $\alpha \in Z_2$.

We can describe $E^2_{q,*}(\pi;X) = H_q(\pi;K_*(X;1)^{\otimes p})$ as follows. When $q = 0$ it is just the coinvariant quotient of $K_*(X;1)^{\otimes p}$. Let $\pi = Z_p$ with p odd. If $x \in K_\alpha(X;1)$ then x^p $K_*(X;1)^{\otimes p}$ generates a trivial π-submodule and we write $e_q \otimes x^p$ for the image of $e_q \in H_q(B\pi;1)$ under the inclusion of this submodule. Now $K_*(X;1)^{\otimes p}$ can be written as a direct sum of trivial π-modules of this kind and free π-modules generated by $x_1 \otimes \cdots \otimes x_p$ with not all x_i's equal. Hence the map

$$K_\alpha(X;1) \to E^2_{q,\alpha}(Z_p;X)$$

taking x to $e_q \otimes x^p$ is an isomorphism if $q > 0$ and p is odd. We continue to write $e_q \otimes x^p$ for the image of this element under the natural map

$$E^2_{q,\alpha}(Z_p;X) \to E^2_{q,\alpha}(\Sigma_p;X).$$

By [68,1.4] we see that this map is onto in all bidegrees, is an isomorphism when $q = (2i-\alpha)(p-1)$ or $(2i-\alpha)(p-1)-1$ for some $i \geq 1$, and is zero in all other bidegrees with $q > 0$. Finally, if $p = 2$ then by 3.2(vii)' the Z_2-action on $K_*(X;1)^{\otimes 2}$ is given by $x \otimes y \longmapsto y \otimes x + \beta y \otimes \beta x$; in particular, x^2 is invariant if and only if $\beta x = 0$. Using this it is easy to see that the map taking x to $e_q \otimes x^2$ induces an isomorphism from $\ker \beta/\operatorname{im} \beta$ to $E_{q,0}(Z_2;X)$ if $q > 0$, while $E_{q,1}(Z_2;X) = 0$ for $q > 0$.

Our next two results describe the groups $E^\infty_{q,\alpha}(\Sigma_p;X)$. Let A be a subbasis for X and let $\overline{A}_2 \subset K_*(X;2)$ be the set

$$\{\pi^{r-2}x \mid x \in A_r,\ 2 \leq r \leq \infty\} \cup \{\pi^{r-2}\beta_r x \mid x \in A_r,\ 2 \leq r < \infty\}.$$

Let $\overline{A}_{2,0}$ and $\overline{A}_{2,1}$ be the zero- and one-dimensional subsets of $\overline{A}_2$.

Proposition 6.3. (i) The kernel of the epimorphism $E^2_{0,*}(\Sigma_p;X) \to E^\infty_{0,*}(\Sigma_p;X)$ is generated by the set $\{(\beta x)^p \mid x \in K_1(X;1)\}$ if p is odd and by

$$\{(\pi\beta_2 x)^2 + (\pi x)^2 \mid x \in K_1(X;2)\} \text{ if } p = 2.$$

(ii) The terms $E^{\infty}_{q,\alpha}(\Sigma_p;X)$ with $q > 0$ are freely generated by the sets

$$\{e_{2p-2} \otimes (\pi x)^p | x \in \overline{A}_{2,0}\}$$

$$\{e_{p-1} \otimes (\pi x)^p \mid x \in \overline{A}_{2,1}\}$$

and, if p is odd, $$\{e_{p-2} \otimes x^p \mid x \in A_{1,1}\}$$

Proposition 6.4. (i) If $x \in \overline{A}_{2,0}$ then Qx is represented in $E^{\infty}_{**}(\Sigma_p;X)$ by a nonzero multiple of $e_{2p-2} \times (\pi x)^p$.

(ii) If $x \in \overline{A}_{2,1}$ then Qx is represented by a nonzero multiple of $e_{p-1} \otimes (\pi x)^p$.

(iii) If $x \in A_{1,1}$ then $Q\beta_2 p_* x$ is represented by a nonzero multiple of $e_{p-2} \otimes x^p$.

Note that Lemma 4.3 is an immediate consequence of 6.3, 6.4 and the external versions of 3.3(iii), 3.3(v), and 3.6(iv).

When p is odd, Proposition 6.3 is Corollary 3.2 of [77]. We shall give a different proof, using the methods of Section 1, which also works for $p = 2$. First observe that there are two equivalent ways of constructing the spectral sequence $E^r_{**}(\pi;X)$; one can either apply mod p K-theory to the filtration of D_pX given in Section I.2 or one can apply mod p stable homotopy to the corresponding filtration of $K \wedge D_pX$. The latter procedure has the advantage that the map

$$\overline{D}_\pi f : D_\pi Y \to K \wedge D_\pi X$$

induced by any map $f:Y \to K \wedge X$ clearly gives rise to a homomorphism

$$(\overline{D}_\pi f)_{**} : E^r_{**}(\pi;Y) \to E^r_{**}(\pi;X)$$

of spectral sequences.

Lemma 6.5. If $\pi = Z_p$ or Σ_p and $y \in K_*(Y;1)$ (with $\beta y = 0$ if $p = 2$) then $(\overline{D}_\pi f)_{**}(e_q \otimes y^p) = e_q \otimes (f_{**}y)^p$.

Proof of 6.5. It suffices to consider the case $\pi = Z_p$. The composite

$$D_\pi X = D_\pi(X \wedge S^0) \xrightarrow{\delta} D_\pi X \wedge D_\pi S^0$$

induces a coproduct

$$\Psi : E^r_{**}(\pi;X) \to E^r_{**}(\pi;X) \otimes E^r_{**}(\pi;S^0)$$

and we have

$$\Psi \circ (\overline{D}_\pi f)_{**} = [(\overline{D}_\pi f)_{**} \otimes 1] \circ \Psi.$$

The lemma clearly holds for $q = 0$, and it follows for all q since the component of $\Psi(e_q \otimes y^p)$ in $E^2_{0*}(\pi;Y) \otimes E^2_{q*}(\pi;S^0)$ is $(e_0 \otimes y^p) \otimes e_q$.

Proof of 6.4. (i) Let x be represented by $f: M_2 \to K \wedge X$. Then $f_{**}u_2 = x$, $(\overline{D}_p f)_{**}Qu_2 = Qx$, and $(\overline{D}_p f)_{**}(e_{p-2} \otimes u_2^p) = e_{2p-2} \otimes x^p$. Hence we may assume that $X = M_2$ and $x = u_2$, and it suffices to show that $v_1 = Qu_2$ is not in the image of

$$K_0(M_2^{(p)};1) \to K_0(D_p M_2;1).$$

But this is clear since $(D_p j)_* v_1 = v$.

Part (ii) is similar. For part (iii) we may assume that $X = \Sigma M_1$ and $x = \Sigma u_1$. In this case it suffices to show that $Q\beta_2 p_* u_1$ is nonzero. But $\beta_2 p_* u_1 = i_* u$, where $u \in K_0(S^0;2)$ is the unit, and $Qu = v$. Hence $Q\beta_2 p_* u_1 = (D_p i)_* v$ is nonzero by 6.1(iii).

Proof of 6.3. First let $p = 2$. Since every element of $\ker \beta$ lifts to $K_*(X;2)$, Proposition 6.3 will be a consequence of the following facts.

(a) $d_2 = 0$

(b) $d_3(e_{2q-\alpha-1} \otimes (\pi x)^2) = e_{2q-\alpha-4} \otimes (\pi_{\beta_2} x)^2$

(c) $d_3(e_{2q-\alpha} \otimes (\pi x)^2) = e_{2q-\alpha-3} \otimes [(\pi x)^2 + (\pi\beta_2 x)^2]$.

Note that, when $\beta_2 x \neq 0$, formulas (b) and (c) differ from those given in [99, 3.8(a)(ii)].

First consider the case $X = S^0$. Then the spectral sequence of I.2.4 is isomorphic to the Atiyah-Hirzebruch spectral sequence, so that (a), (b) and (c) hold in this case by 5.1.

Next we need the coproduct Ψ defined in the proof of 6.5. this has the form

$$\Psi(e_q \otimes x^2) = \sum_{i=0}^{q} (e_i \otimes x^2) \otimes e_{q-i},$$

and it follows that if x and y satisfy

$$d_3(e_3 \otimes x^2) = e_0 \otimes y^2$$

then we also have

$$d_3(e_{2s+1} \otimes x^2) = e_{2s-2} \otimes y^2$$

and

$$d_3(e_{2s+2} \otimes x^2) = e_{2s-1} \otimes [y^2 + x^2]$$

for all $s \geq 1$.

Now let $X = S^1$. In this case $d_2 = 0$ for dimensional reasons, and there are only two possibilities for d_3 consistent with the coproduct, namely

$$d_3(e_{2q} \otimes (\Sigma u)^2) = e_{2q-3} \otimes (\Sigma u)^2$$

or
$$d_3(e_{2q-1} \otimes (\Sigma u)^2) = e_{2q-4} \otimes (\Sigma u)^2.$$

Only the second is consistent with 5.1, and hence (b) and (c) hold in this case.

Next observe that, by 6.5, d_2 vanishes in general if it does for M_2 and ΣM_2. In each of these cases, d_2 is zero for dimensional reasons except on $E^2_{2,0}$, and the only element that could be hit is $(\pi\Sigma^\alpha u_2)(\pi\beta_2\Sigma^\alpha u_2)$ in $E^2_{0,1}$. But the corresponding element of $K_1(D_2\Sigma^\alpha M_2;1)$ is nonzero since its transfer it nonzero in $K_1((\Sigma^\alpha M_2)^{(2)})$. Hence $d_2 = 0$.

Finally, (b) and (c) will hold for all x if they hold for $x = u_2$ and $x = \Sigma u_2$. First consider Σu_2. It suffices to show that

$$d_3(e_3 \otimes (\pi u_2)^2) = (\pi u_2)^2 + (\pi\beta_2 u_2)^2.$$

From inspection of the maps

$$E^3_{**}(Z_2;S^0) \to E^3_{**}(Z_2;\Sigma M_2)$$

and
$$E^3_{**}(Z_2;\Sigma M_2) \to E^3_{**}(Z_2;S^1)$$

we see that $d_3(e_3 \otimes (\pi\beta_2\Sigma u_2)^2)$ is zero and that $d_3(e_3 \otimes (\pi\Sigma u_2)^2)$ projects to $(\Sigma u)^2$ in $E^3_{0,0}(Z_2;S^1)$. Hence

$$d_3(e_3 \otimes (\pi\Sigma u_2)^2) = (\pi\Sigma u_2)^2 + \varepsilon(\pi\beta_2\Sigma u_2)^2$$

for some $\varepsilon \in Z_2$ and there are no further differentials. But by the external version of 3.3(x) we have $\iota_*(\pi\Sigma u_2)^{(2)} = \iota_*(\pi\beta_2\Sigma u_2)^{(2)}$ in $K_0(D_2\Sigma M_2;1)$, hence $\varepsilon = 1$ as required.

It remains to show that

$$d_3(e_3 \otimes (\pi u_2)^2) = (\pi\beta_2 u_2)^2.$$

For this we use the map

$$\Psi':E^r_{**}(Z_2;\Sigma M_2) \to E^r_{**}(Z_2;S^1) \otimes E^r_{**}(Z_2;M_2)$$

induced by

$$\delta:D_2\Sigma M_2 \to D_2S^1 \wedge D_2M_2 .$$

We have

$$\Psi'(e_q \otimes (\pi\Sigma u_2)^2) = \sum_{i=0}^{q} (e_i \otimes (\pi\Sigma u)^2) \otimes (e_{q-i} \times (\pi u_2)^2)$$

and therefore

$$d_3\Psi'(e_3 \otimes (\pi\Sigma u)^2) = (e_0 \otimes (\pi\Sigma u)^2) \otimes [d_3(e_3 \otimes (\pi u_2)^2) + e_0 \otimes (\pi u_2)^2]$$

while $\Psi' d_3(e_3 \otimes (\pi\Sigma u)^2) = (e_0 \otimes (\pi\Sigma u)^2) \otimes [e_0 \otimes (\pi u_2)^2 + e_0 \otimes (\pi\beta_2 u_2)^2]$

and the result follows.

Next let p be odd. We must show the following

(a) $d_i = 0$ for $i \leq p-2$

(b) $d_{p-1}(e_q \otimes x^p) = e_{q+1-p} \otimes (\beta x)^p$

(c) $d_i = 0$ for $p \leq i \leq 2p-2$

(d) $d_{2p-1}(e_q \otimes x^p) = e_{q+1-2p} \otimes x^p$

(e) $d_i = 0$ for $i \geq 2p$.

As before, when $X = S^0$ the spectral sequence is isomorphic to the Atiyah-Hirzebruch spectral sequence so that (a)-(e) hold for 5.1. They also hold for $X = S^1$ by 5.1 and the coproduct. Now 6.5 implies that (a) and (b) will hold for all X if they do for $X = M_1$ and $X = \Sigma M_1$. Inspection of the maps

$$E^r_{**}(\Sigma_p; S^{\alpha-1}) \to E^r_{**}(\Sigma_p; \Sigma^\alpha M_1)$$

and

$$E^r_{**}(\Sigma_p, \Sigma^\alpha M_1) \to E^r_{**}(\Sigma_p; S^\alpha)$$

and the coproduct shows in each case that <u>either</u> (a) and (b) hold <u>or</u> (a),(c),(d), and (e) hold with $d_{p-1} = 0$. Only the former gives an E_∞ term compatible with 6.1(i). Hence (a) and (b) hold for all x.

Now applying 6.5 again we see that (c), (d) and (e) will hold in general if they hold for M_2 and ΣM_2. But one can see that they do by inspection of the maps

$$E^r_{**}(\Sigma_p; S^{\alpha-1}) \to E^r_{**}(\Sigma_p; \Sigma^\alpha M_2)$$

and

$$E^r_{**}(\Sigma_p; \Sigma^\alpha M_2) \to E^r_{**}(\Sigma_p; S^\alpha),$$

and the proof is complete.

7. <u>Construction and properties of Q</u>.

In this section we complete the construction of Q and prove external and internal versions of Theorem 3.3.

As in section 6, we shall construct Q by specifying elements $v_{r-1} \in K_0(D_p M_r; r-1)$ and $v'_r \in K_1(D_p \Sigma M_r; r-1)$. In order to do this we need a stronger version of 6.1.

<u>Lemma 7.1</u>. Let $r \geq 2$. The maps

$$(D_p j)_* : K_0(D_p M_r; r-1) \to K_0(D_p S^0; r-1)$$

$$(D_pj)_* \oplus \tau_*:K_1(D_p\Sigma M_r;r-1) \to K_1(D_pS^1;r-1) \oplus K_1((\Sigma M_r)^{(p)};r-1)^{\Sigma_p}$$

and $$(D_pi)_*:K_0(D_pS^0;r-1) \to K_0(D_p\Sigma M_r;r-1)$$

are isomorphisms, and the sequence

$$0 \longrightarrow K_1(D_pS^1;r-1) \xrightarrow{(D_pi)_*} K_1(D_pM_r;r-1) \xrightarrow{\tau_*} K_1((M_r)^{(p)};r-1)^{\Sigma_p} \longrightarrow 0$$

is exact.

Note that the terms in 7.1 which involve iterated smash products may be calculated by using 5.8. Assuming 7.1 for the moment we may define v_{r-1} and v'_{r-1} by the equations $(D_pj)_*v_{r-1} = v$, $(D_pj)_*v'_{r-1} = v'$, and $\tau_*v'_{r-1} = 0$.

Definition 7.2. $Q:K_\alpha(X;r) \to K_\alpha(D_pX;r-1)$ is the operation $Q_{v_{r-1}}$ if $\alpha = 0$ and $Q_{v'_{r-1}}$ if $\alpha = 1$.

Observe that v_{r-1}, v, v'_{r-1} and v' are equal respectively to Qu_r, Qu, $Q\Sigma u_r$, and $Q\Sigma u$. From now on we shall always use the latter notations for these elements.

We shall prove 7.1 by showing that $E^1 = E^{r-1}$ in the K-theory BSS for $D_p\Sigma^nM_r$ when $r \geq 2$. For this we shall require a formula for the Bockstein of the external Q-operation, and this in turn depends on the other formulas collected in the following lemma.

Lemma 7.3. Let $x,y \in K_\alpha(X;r)$ with $r \geq 2$.

(i) $$\tau_*Qx = \begin{cases} 0 & \text{if } \alpha = 1 \\ -(p-1)!\pi x^{(p)} & \text{if } \alpha = 0 \text{ and } p \text{ is odd} \\ -\pi x^{(2)} + \omega 2^{r-2}\pi(\beta_r x)^{(2)} & \text{if } \alpha = 0 \text{ and } p = 2 . \end{cases}$$

Here ω Z_2 is independent of x.

(ii) $\pi Qx = Q\pi x$ if $r \geq 3$.

(iii) $$Q(x+y) = \begin{cases} Qx + Qy - \pi\iota_*[\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i}x^{(i)} \otimes y^{(p-i)}] & \text{if } \alpha = 0 \text{ and } p \text{ is odd} \\ Qx + Qy - \pi\iota_*(x \otimes y) + \omega 2^{r-2}\pi\iota_*[(\beta_r x) \otimes \beta_r y)] & \text{if } \alpha = 0 \text{ and } p = 2 \\ Qx + Qy & \text{if } \alpha = 1. \end{cases}$$

(iv) Let $k \in Z$. Then

$$Q(kx) = \begin{cases} kQx - \frac{1}{p}\,(k^p - k)\pi\iota_* x^{(p)} & \text{if } \alpha = 0 \\ kQx & \text{if } \alpha = 1. \end{cases}$$

(v) $$\Delta_*\Sigma Qx = \begin{cases} Q\Sigma x & \text{if } \alpha = 0 \\ \pi\iota_*(\Sigma x)^{(p)} + pQ\Sigma x & \text{if } \alpha = 1. \end{cases}$$

(vi) $$\beta_{r-1}Qx = \begin{cases} Q\beta_r x - p\pi\iota_*(x^{(p-1)} \otimes \beta_r x) & \text{if } \alpha = 0 \\ \pi\iota_*(\beta_r x)^{(p)} + pQ\beta_r x & \text{if } \alpha = 1. \end{cases}$$

The constant ω in parts (i) and (iii) will turn out to be 1, as required for 3.3(vi). In order to avoid circularity, we shall prove 7.1 and 7.3 by a simultaneous induction. More precisely, we shall assume that 7.1 holds for $r \leq r_0$ and that 7.3 holds for $r < r_0$ (vacuously if $r_0 = 2$) and then prove 7.3 for $r = r_0$ and 7.1 for $r = r_0 + 1$. Before beginning, we need two technical lemmas.

<u>Lemma 7.4</u>. Let $Y \xrightarrow{f} Z \xrightarrow{g} Cf \xrightarrow{h} \Sigma Y$ be a cofiber sequence in $\overline{h}\mathcal{S}$ and let $r \geq 2$. Suppose that β_{r-1} vanish on $K_1(Z;r-1)$. Let $y \in K_1(\Sigma Y;2r-2)$, $z \in K_0(Z;r-1)$ and $w \in K_1(Cf;r-1)$ be any elements satisfying $\pi^{r-1}y = h_*w$ and $p_*^{r-1}(\Sigma z) = f_*y$. Then $\beta_{r-1}w = g_*z$.

<u>Proof</u> Consider the following diagram in $\overline{h}\mathcal{S}$.

$$\begin{array}{ccccccc} K \wedge Cf & \xrightarrow{1 \wedge h} & K \wedge \Sigma Y & \xrightarrow{1 \wedge \Sigma f} & K \wedge \Sigma Z & \xrightarrow{1 \wedge \Sigma g} & K \wedge \Sigma Cf \\ \uparrow{\scriptstyle w} & & \uparrow{\scriptstyle \Sigma y} & & \vdots\,{\scriptstyle \zeta} & & \uparrow{\scriptstyle \Sigma w} \\ \Sigma M_{r-1} & \longrightarrow & \Sigma M_{2r-2} & \longrightarrow & \Sigma M_{r-1} & \longrightarrow & \Sigma^2 M_r \end{array}$$

Here the bottom row is the evident cofiber sequence, with the first map induced by the inclusion $Z_{p^{r-1}} \subset Z_{p^{2r-2}}$ and the second by the projection $Z_{p^{2r-2}} \to Z_{p^{r-1}}$. Precomposition with the first, second, and third maps in this sequence induces the transformations π^{r-1}, p_*^{r-1} and (because of the suspension) $-\beta_{r+1}$, respectively. The left-hand square commutes up to homotopy since $\pi^{r-1}y = h_*w$. Hence there exists an element ζ making the other two squares commute, and we have $-\beta_{r-1}\Sigma w = (\Sigma g)_*\zeta$. Now the map

$$\Sigma z : \Sigma M_{r-1} \to K \wedge \Sigma Z$$

makes the middle square commute, hence $\zeta - \Sigma z$ restricts trivially to ΣM_{2r-2}. Thus $\zeta - \Sigma z$ extends to a map

$$\xi : \Sigma^2 M_r \to K \wedge \Sigma Z$$

with $\beta_{r-1}\xi = \zeta - \Sigma z$. Since β_{r-1} vanishes on $K_0(\Sigma Z;r-1)$ we have $\zeta = \Sigma z$. Thus $-\beta_{r-1}\Sigma w = \Sigma(g_* z)$ and the result follows.

Lemma 7.5. If $f:X \to K \wedge Y$ is any map then f_{**} commutes with π, β_r, p_* and Σ.

The proof of 7.5 is trivial. Before proceeding we use 7.5 to dispose of 3.2(iv).

Proof of 3.2(iv). For any $x \in K_*(X;r-1)$ and y $K_*(Y;r)$ there exist maps $f:\Sigma^{|x|}M_{r-1} \to K \wedge X$ and $g:\Sigma^{|y|}M_r \to K \wedge Y$ with $f_{**}\Sigma^{|x|}u_{r-1} = x$ and $g_{**}\Sigma^{|y|}u_r = y$. Thus by 7.5 and 1.3(ii) we may assume $X = \Sigma^{|x|}M_{r-1}$ and $Y = \Sigma^{|y|}M_r$ with $x = \Sigma^{|x|}u_{r-1}$ and $y = \Sigma^{|y|}u_r$. By 3.2(vi) we may assume $|x| = |y| = 0$. Clearly the set

$$\{u_{r-1} \otimes \pi u_r,\ u_{r-1} \otimes \pi\beta_r u_r\}$$

is a subbasis for $M_{r-1} \wedge M_r$. Hence by 3.9 we have

$$(1) \qquad (p_* u_{r-1}) \otimes u_r = a_1 p_*(u_{r-1} \otimes \pi u_r) + a_2 \beta_r p_*(u_{r-1} \otimes \pi\beta_r u_r)$$

for some $a_1, a_2 \in Z_{p^{r-1}}$. Applying π to each side gives

$$\begin{aligned} pu_{r-1} \otimes \pi u_r &= a_1 pu_{r-1} \otimes \pi u_r + a_2\beta_{r-1}(u_{r-1} \otimes \pi\beta u_r) \\ &= a_1 pu_{r-1} \otimes \pi u_r + a_2\beta_{r-1}u_{r-1} \otimes \pi\beta_r u_r \ . \end{aligned}$$

Hence $a_2 = 0$. Now applying $(j \wedge j)_*$ to each side of equation (1) gives

$$p(u \otimes u) = a_1 p_*(u \otimes u) = a_1 p(u \otimes u)$$

in $K_0(D_pS \wedge D_pS;r) \cong Z_{p^r}$. Hence $a_1 = 1$ in $Z_{p^{r-1}}$.

Next we give the proof of 7.3 for $r = r_0$. The proof of each part will be quite similar to that just given for 3.2(iv). First we observe that by 1.3, 1.4, 1.5 and 7.5 we may assume in each part except (iii) that X is $\Sigma^\alpha M_r$ and that x is the fundamental class $\Sigma^\alpha u_r$.

(i). If $\alpha = 1$ the result holds by Definition 7.2. Suppose $\alpha = 0$ and consider the map

$$j_*^{(p)}:K_0(M_r^{(p)};r-1)^{\Sigma_p} \to K_0(S^0;r-1).$$

This is monic when p is odd and has kernel generated by $2^{r-2}\pi(\beta_r u_r)^{(2)}$ when $p = 2$. The result follows since $j_*^{(p)}u_r^{(p)} = u \in K_0(S^0;r)$ and

$$j_*^{(p)}\tau_* Qu_r = \tau_*(D_p j)_* Qu_r = \tau_* Qu = -(p-1)!u;$$

the last equality is 5.7(iv).

(ii). Let $\alpha = 1$. By 7.1 it suffices to show that

$$(D_p j)_* \pi Q\Sigma u_r = (D_p j)_* Q\pi\Sigma u_r$$

and that

$$\tau_* \pi Q\Sigma u_r = \tau_* Q\pi\Sigma u_r .$$

This second equation follows from part (i) and the first from 5.7(i). The case $\alpha = 0$ is similar.

(iii). Let $\alpha = 0$ with p odd. By 1.3, 1.4 and 7.5 we may assume that X is $M_r \vee M_r$ with x and y being the fundamental classes of the two summands. Let

$$F: \bigvee_{i=0}^{p} D_i M_r \wedge D_{p-i} M_r \to D_p(M_r \vee M_r)$$

be the equivalence of II.1.1 and let $f: M_r \to M_r \vee M_r$ be the pinch map. Then $(D_p f)_* Qu_r = Q(x + y)$, and it suffices to show that

$$F_*^{-1}(D_p f)_* Qu_r = Qu_r \otimes u + u \otimes Qu_r - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} \pi\iota_* u_r^{(i)} \otimes \pi\iota_* u_r^{(p-i)}$$

since F_* applied to the right side of this equation clearly gives the right side of the desired formula. Now the projection of $F^{-1} \circ D_p f$ on the i-th wedge summand is the transfer

$$\tau_{i,p-i} : D_p M_r \to D_i M_r \wedge D_{p-i} M_r .$$

When i is 0 or p this transfer is the evident natural equivalence, hence it suffices to show

$$(2) \qquad (\tau_{i,p-i})_* Qu_r = -\frac{1}{p}\binom{p}{i} \pi\iota_* u_r^{(i)} \otimes \pi\iota_* u_r^{(p-i)}$$

for $0 < i < p$. Now the transfer

$$\tau'_{i,p-i}: D_i M_r \wedge D_{p-i} M_r \to M_r^{(p)}$$

induces a monomorphism since the order of $\Sigma_i \times \Sigma_{p-i}$ is prime to p for $0 < i < p$. We have

$$(\tau'_{i,p-i})_*(\tau_{i,p-i})_* Qu_r = \tau_* Qu_r = -(p-1)!u_r^{(p)}$$

by part (i) while

$$(\tau'_{i,p-i})_*[\pi\iota_* u_r^{(i)} \otimes \pi\iota_* u_r^{(p-i)}] = i!(p-i)!u_r^{(p)}$$

by the double coset formula. Equation (2) follows. The proof when $p = 2$ or $\alpha = 1$ is similar.

Part (iv) follows from (iii) by induction on k. When $p = 2$ and $\alpha = 0$ we need to know that $2^{r-2}\pi\iota_*(\beta_r x)^{(2)} = 0$. If $r > 2$ this is evident since $\iota_*(\beta_r x)^{(2)}$ has order 2 by 3.2(viii). If $r = 2$ then by 6.4(iii) we have

$$\iota_*(\pi\beta_2 x)^{(2)} = Q\beta_2 2_*\pi\beta_2 x = 0.$$

(v). Let $\alpha = 0$. By 7.1 is suffices to show

$$(\Sigma D_p j)_*\Delta_*\Sigma Qu_r = Q\Sigma u$$

and

$$\tau_*\Delta_*\Sigma Qu_r = 0.$$

The first equation is immediate from 7.2 and 5.7(ii). For the second, consider the diagram

$$\begin{array}{ccc} S^1 \wedge D_p M_r & \xrightarrow{\Delta} & D_p(S^1 \wedge M_r) \\ \downarrow{\scriptstyle 1\wedge\tau} & & \downarrow{\scriptstyle t} \\ S^1 \wedge M_r^{(p)} & \xrightarrow{\Delta'} & (S^1 \wedge M_r)^{(p)} \end{array} .$$

Here the map Δ' is induced by the diagonal of S^1. By definition, the map Δ is obtained by aplying the functor $E\Sigma_p^+ \wedge_{\Sigma_p} (\ \ )$ to the map of Σ_p-spectra

$$S_1 \wedge (M_r)^{(p)} \to (S_1 \wedge M_r)^{(p)}$$

induced by the diagonal of S^1. Hence the diagram commutes by naturality of τ. But the diagonal map of S^1 is nonequivariantly trivial, hence $\tau_*\Delta_* \Sigma Qu_r = 0$ as required. The proof when $\alpha = 1$ is similar.

(vi). Suppose first that $\alpha = 1$. Consider the following diagram

$$\begin{array}{ccccccc} D_pS & \xrightarrow{D_pf} & D_pS & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma D_pS \\ & & & \searrow^{D_pi} & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Delta} \\ & & & & D_p\Sigma M_r & \xrightarrow{D_pj} & D_pS^1 \end{array}$$

Here $f:S \to S$ has degree p^r and the top row is the cofiber sequence of D_pf. The map γ is that constructed in II.3.8, where it was called ψ, and the diagram commutes. For any $s \geq 1$ the map

$$(D_pf)_*:K_0(D_pS;s) \to K_0(D_pS;s)$$

is given by the formula $(D_pf)_*\eta = p^{pr}\eta$ and

$$(D_pf)_*Qu = Q(p^r u) = p^r Qu - (p^{p^{r-1}} - p^{r-1})\eta$$

In particular, when $s = r-1$ the map $(D_pf)_*$ is zero, and since $K_1(D_ps;r-1) = 0$ we see that

$$h_*:K_1(C;r-1) \to K_1(\Sigma D_pS;r-1)$$

is an isomorphism. Thus there is a unique $w \in K_1(C;r-1)$ with $h_*w = \Sigma Qu$. Letting

$$y = \Sigma Qu \in K_1(*\Sigma D_pS;2r-2)$$

and $$z = pQu + \eta \in K_0(D_pS;r-1)$$

we have $\pi^{r-1}y = h_*w$ and $p_*^{r-1}\Sigma z = (D_pf)_*y$, hence by Lemma 7.4 we conclude that $\beta_{r-1}w = g_*z$ in $K_1(C;r-1)$.

Next we shall show that $\gamma_*w = Q\Sigma u_r$. Assuming this for the moment, we have

$$\beta_{r-1}Q\Sigma u_r = \gamma_*\beta_{r-1}w = \gamma_*g_*zx = (D_pi)_*z = pQ\beta_ru_r + \pi\iota_*(\beta_ru_r)^{(p)}$$

which gives (vi) when $\alpha = 1$. To show $\gamma_*w = Q\Sigma u_r$, we must show that $(D_pj)_*\gamma_*w = Q\Sigma u$ and $\tau_*\gamma_*w = 0$. The first equation is immediate from the diagram and part (v). For the second, we observe that D_pf and γ are obtained by applying $E\Sigma_p^+ \wedge_{\Sigma_p} (\ \ )$ to certain Σ_p-equivariant maps F and Γ, so that by naturality of τ we have the following commutative diagram of nonequivariant spectra.

$$\begin{array}{ccccc}
C & = & E\Sigma_p^+ \wedge_{\Sigma_p} CF & \xrightarrow{\ \tau\ } & CF \\
\Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle E\Sigma_p^+ \wedge_{\Sigma_p} \Gamma} & & \Big\downarrow{\scriptstyle\Gamma} \\
D_p\Sigma M_r & = & E\Sigma_p^+ \wedge_{\Sigma_p} (\Sigma M_r)^{(p)} & \xrightarrow{\ \tau\ } & (\Sigma M_r)^{(p)}
\end{array}$$

Thus it suffices to show $\Gamma_*\tau_* = 0$ on $K_1(C;r-1)$. As a nonequivariant map F is the map $S \to S$ of degree p^{pr}, hence the cofiber CF is nonequivariantly equivalent to ΣM_{pr}. The resulting Σ_p-action is clearly trivial on $K_0(\Sigma M_{pr};pr)$, hence also on $K_1(\Sigma M_{pr};pr)$ since the Bockstein β_{pr} is an isomorphism between these two groups. Thus

$$\Gamma_*:K_1(\Sigma M_{pr};pr) \to K_1((\Sigma M_r)^{(p)};pr)$$

lands in the Σ_p-invariant subgroup. We claim that this subgroup is generated by the element

$$p_*^{pr-r}[(\Sigma u_r) \otimes (\beta_r\Sigma u_r)^{p-1}]$$

when p is odd and by this element together with

$$2^{r-1}\beta_{2r}2_*^r[(\Sigma u_r) \otimes (\Sigma u_r)]$$

when $p = 2$. From this it will follow that π^{pr-r+1} vanishes on this subgroup and therefore that Γ_* vanishes on $K_1(\Sigma M_{pr};r-1)$, since π^{pr-r+1} maps onto the latter group; thus we will have shown $\Gamma_*\tau_* w = 0$ as required. To verify the claim we observe that the set

$$\{\Sigma u_r \otimes x_2 \otimes \cdots \otimes x_p \mid x_i = \Sigma u_r \text{ or } \beta_r \Sigma u_r\}$$

is a subbasis for $(\Sigma M_r)^{(p)}$. Using the basis for $K_1((\Sigma M_r)^{(p)};pr)$ given by 3.9, we see at once that the elements

$$z_1 = p_*^{pr-r}[(\Sigma u_r) \otimes (\beta_r \Sigma u_r)^{(p-1)}]$$

and $$z_2 = \beta_{pr} p_*^{pr-r}[\Sigma u_r \otimes [\sum_{i=1}^{p-1} (\beta_r \Sigma u_r)^{(i)} \otimes \Sigma u_r \otimes (\beta_r \Sigma u_r)^{(p-i-1)}]]$$

are a basis for the $\Sigma_1 \times \Sigma_{p-1}$ invariant subgroup. Now if T is the map switching the first two factors of $(\Sigma M_r)^{(p)}$ we have $T_* z_1 = z_1$ and

$$T_* z_2 = z_2 - 2\beta_{pr} p_*^{pr-r}[(\Sigma u_r)^{(2)} \otimes (\beta_r \Sigma u_r)^{(p-2)}];$$

the claim follows.

Finally, we must prove part (vi) with $\alpha = 0$. By 7.1 we have

$$(3) \qquad \beta_{r-1} Q u_r = a_1 Q \beta_r u_r + a_2 \pi \iota_*(u_r^{(p-1)} \otimes \beta_r u_r)$$

for some $a_1, a_2 \in Z_{p^{r-1}}$. Applying $\Delta_* \Sigma$ and using part (v) gives

$$\beta_{r-1} Q \Sigma u_r = a_1[\pi \iota_*(\beta_r \Sigma u_r)^{(p)} + pQ\beta_r \Sigma u_r].$$

Comparing this with the case $\alpha = 1$ of (vi) gives $a_1 = 1$. Now applying τ_* to (3) and using part (i) gives

$$-(p-1)!(\beta_{r-1}\pi(u_r^{(p)}) = a_2 (p-1)!\pi[\sum_{i=0}^{p-1} u_r^{(i)} \otimes \beta_r u_r \otimes u_r^{(p-i-1)}].$$

But $\beta_{r-1}\pi(u_r^{(p)}) = p\pi\beta_r(u_r^{(p)})$ and it follows that $a_2 = -p$ as required.

This completes the case $r = r_0$ of 7.3. Next we must show 7.1 for $r = r_0+1 \geq 3$. It suffices to show that $E^1 = E^{r-1}$ in the K-theory BSS for $D_p M_r$ and $D_p \Sigma M_r$. We shall give the proof for $D_p M_r$, the other case being similar. Let x and y denote the elements πu_r and $\pi \beta_r u_r$. by 6.1,7.2 and 7.3(ii) we see that the set

$$\{\pi^{r-2}\iota_* x^{(p)},\ \pi^{r-3} Qx,\ \pi^{r-2}\iota_*(x^{(p-1)} \otimes y),\ \pi^{r-3} Qy\}$$

is a basis for $K_*(D_p M_r;1)$. Since all elements of this basis lift to $K_*(D_p M_r;r-2)$ we have $E^1 = E^{r-2}$ in the BSS. The elements $\pi^{r-2} x^{(p)}$ and $\pi^{r-2}(x^{(p-1)} \otimes y)$ are (r-2)-

cycles since they clearly lift to $K_0(D_pM_r;r-1)$. Next we have

$$d_{r-2}\pi^{r-3}Qx = \pi^{r-3}\beta_{r-2}Qx = \pi^{r-3}Q\beta_{r-1}x = \pi^{r-3}Qpy = 0,$$

where the 2nd and 4th equalities follow from 7.3(vi) and 7.3(iv) respectively. Similarly,

$$d_{r-2}\pi^{r-3}Qy = \pi^{r-3}\beta_{r-2}Qy = \pi^{r-2}(\beta_{r-1}y)^{(p)} = 0.$$

This completes the inductive proof of 7.1 and 7.3.

Next we shall prove the external version of 3.3. Rather than write out the complete list of external properties, we give rules for changing the internal statements to their external analogs. All internal products and Dyer-Lashof operations are to be changed to external ones, with the map ι_* prefixed to any p-fold product which is to lie in $K_*(D_pX;r)$. The map δ_* is to be prefixed to the left-hand side of each Cartan formula. In the stability formulas, σ is to be changed to Σ and Δ_* prefixed to the left-hand side. These conventions give the correct external analog for each part of 3.3 except for part (ii) which has no external analog.

Proposition 7.6. The external Q-operation satisfies the external versions of each part of Theorem 3.3 except part (ii).

Before beginning the proof we need a lemma to deal with the prime 2. (See II.4.3 for another proof of this lemma.)

Lemma 7.7. Let X be any spectrum. The sequence

$$\Sigma D_2X \xrightarrow{\Delta} D_2\Sigma X \xrightarrow{\tau} \Sigma^2(X\wedge X) \xrightarrow{\Sigma^2\iota} \Sigma^2 D_2X$$

is a cofibering.

Proof. Consider the cofiber sequence

(4) $$S^1 \xrightarrow{\Delta} S^1\wedge S^1 \longrightarrow S^2\wedge S^2 \longrightarrow S^2$$

of Z_2-spaces. Here Z_2 acts trivially on the first and fourth terms and by switching factors (respectively, wedge summands) in the second and third terms. Now $S^1\wedge S^1$ is the one-point compactification S^V of the regular representation V of Z_2, and it is easy to see that the second map in the sequence (4) stabilizes to the transfer $S^V \to Z_2^+\wedge S^V$. The sequence of the lemma is obtained by applying the functor $EZ_2^+\wedge_{Z_2}(?\wedge X\wedge X)$ to the sequence (4).

Next we turn to the proof of 7.6. Part (i) is trivial and parts (iii), (v) and (viii) are contained in 7.3.

(iv). We may assume $X = \Sigma^{\alpha}M_r$, $x = \Sigma^{\alpha}u_r$. Suppose $\alpha = 1$. By 7.1 and 7.3(vi) we see that the set

$$\{Q\Sigma u_r,\ \iota_*[(\Sigma u_r)\otimes(\beta_r\Sigma u_r)^{(p-1)}], Q\beta_{r+1}p_*\Sigma u_r\}$$

is a subbasis of height r for $D_p\Sigma M_r$, hence the set

$$\{p_*Q\Sigma u_r,\ \iota_*[(\Sigma u_r)\otimes(\beta_r\Sigma u_r)^{(p-1)}]\}$$

is a basis for $K_1(D_p\Sigma M_r;r)$. It follows that the map

$$(D_pj)_*\oplus\tau_* : K_1(D_p\Sigma M_r;r) \longrightarrow K_1(D_pS^1;r) + K_1((\Sigma M_r)^{(p)};r)$$

is monic. Now

$$(D_pj)_*Qp_*\Sigma u_r = Q(p_*j_*\Sigma u_r) = Q(p\Sigma u) = pQ\Sigma u$$

$$= \begin{cases} 0 & \text{if } r = 1 \\ (D_pj)_*p_*Q\Sigma u_r & \text{if } r \geq 2, \end{cases}$$

and $\tau_*Qp_*\Sigma u_r = 0$ for all r. The result follows, and the case $\alpha = 0$ is similar.

Next we prove part (x). The proof is by induction on r. If r = 1 we have $\iota_*x^{(2)} = Q\beta_2 2_*x$ by 6.4(iii). Suppose $r \geq 2$. We may assume $x = \Sigma u_r$. The set

$$\{Q\Sigma u_r, \iota_*(\Sigma u_r\otimes\beta_r\Sigma u_r), Q\beta_{r+1}2_*\Sigma u_r\}$$

is a subbasis of height r for $D_2\Sigma M_r$, hence by 3.9 we have

$$(5)\qquad \iota_*(\Sigma u_r)^{(2)} = a_1\beta_r 2_*Q\Sigma u_r + a_2Q\beta_{r+1}2_*\Sigma u_r$$

with $a_1 \in Z_{2^{r-1}}$ and $a_2 \in Z_{2^r}$. Applying τ_* to (5) gives

$$0 = -a_2(\beta_r\Sigma u_r)^{(2)}$$

hence $a_2 = 0$. Now applying π to (5) gives

$$(6)\qquad \iota_*(\pi\Sigma u_r)^{(2)} = a_1\beta_{r-1}Q\Sigma u_r.$$

If r = 2 the inductive hypothesis gives

$$\iota_*(\pi\Sigma u_2)^{(2)} = Q\beta_2 2_*(\pi\Sigma u_2) = Q(2\beta_2\Sigma u_2) = \pi\iota_*(\beta_2\Sigma u_2)^{(2)} = \beta Q\Sigma u_2$$

(where the third and fourth equalities follow from 7.3(iv) and 7.3(vi)) and we conclude that $a_1 = 1$ as required. If $r \geq 3$ the inductive hypothesis gives

$$\iota_*(\pi\Sigma u_r)^{(2)} = 2^{r-3}\beta_{r-1}2_*Q(\pi\Sigma u_r) = 2^{r-2}\beta_{r-1}Q\Sigma u_r$$

and comparing with (6) gives $a_1 = 2^{r-2}$ as required.

Next we show part (vi). This will follow immediately from 7.3(iii) and 7.3(iv) once we show that $\omega = 1$ in 7.3(i). Letting $X = \Sigma M_r$ in 7.7, we have

$$\begin{aligned} 0 &= (\Sigma^2\iota)_*\tau_*Q\Sigma^2 u_r \\ &= (\Sigma^2\iota)_*\Pi[-(\Sigma^2 u_r)^{(2)} + \omega 2^{r-2}(\beta_r\Sigma u_r)^{(2)}] \\ &= \Sigma^2\pi\iota_*[(\Sigma u_r)^{(2)} + \omega 2^{r-2}(\beta_r\Sigma u_r)^{(2)}]. \end{aligned}$$

By part (ix), we have

$$\pi\iota_*(\Sigma u_r)^{(2)} = 2^{r-2}\beta_{r-1}Q\Sigma u_r = 2^{r-2}\iota_*(\pi\beta_r\Sigma u_r)^{(2)} \neq 0.$$

Hence $\omega \neq 0$ as required.

(vii) Let $p = 2$; the odd primary case is similar and somewhat easier. First let $|x| = |y| = 1$. We may assume $x = \Sigma u_r$, $y = \Sigma u_r$. We assume by induction on r that we have chosen mod 2^s multiplications for $s < r$ such that the desired formula holds. We begin by giving a basis for

$$K_0(D_2\Sigma M_r \wedge D_2\Sigma M_r; r-1).$$

The set

$$\{\pi\iota_*(\Sigma u_r \otimes \beta_r\Sigma u_r), \pi\iota_*(\beta_r\Sigma u_r)^{(2)}, Q\Sigma u_r, Q\beta_r\Sigma u_r\}$$

is a subbasis of height r-1 for $D_2\Sigma M_r$ and in particular it is a basis for $K_*(D_2\Sigma M_r; r-1)$. By 5.8 we have

$$K_*(D_2\Sigma M_r \wedge D_2\Sigma M_r; r-1) \cong K_*(D_2\Sigma M_r; r-1) \otimes K_*(D_2\Sigma M_r; r-1)$$

with the tensor product taken in the Z_2-graded sense. We therefore obtain a basis for $K_*(D_2\Sigma M_r \wedge D_2\Sigma M_r; r-1)$ by taking all 16 external products of the elements in the set given above. It will be convenient to denote Σu_r by x in the first factor and by y in the second factor. Let $a_1, \ldots, a_8 \in Z_{2^{r-1}}$ be the coefficients of $\delta_*Q(x \otimes y)$ with respect to this basis, so that we have

$$\begin{aligned} (7) \qquad \delta_*Q(x \otimes y) &= a_1\pi\iota_*(x \otimes \beta_r x) \otimes \pi\iota_*(y \otimes \beta_r y) + a_2 Qx \otimes \pi\iota_*(y \otimes \beta_r y) \\ &+ a_3\pi\iota_*(x \otimes \beta_r x) \otimes Qy + a_4 Qx \otimes Qy + a_5\pi\iota_*(\beta_r x)^{(2)} \otimes \pi\iota_*(\beta_r y)^{(2)} \\ &+ a_6\pi\iota_*(\beta_r x)^{(2)} \otimes Q\beta_r y + a_7 Q\beta_r x \otimes \pi\iota_*(\beta_r y)^{(2)} + a_8 Q\beta_r x \otimes Q\beta_r y. \end{aligned}$$

We claim first that $2a_5 = 0$, so that a_5 is either 2^{r-2} or 0. When $r = 2$ this is trivial, while for $r \geq 3$ it follows from the inductive hypothesis and the equation $\pi Q(x \otimes y) = Q(\pi x \otimes \pi y)$. Now as in Remark 3.4(iv) we see that changing the choice of mod 2^r multiplication changes the value of a_5 without changing the other a_i. We can therefore choose the mod 2^r multiplication for which $a_5 = 0$. (When p is odd the commutativity of the multiplications gives $a_5 = 0$.)

It remains to determine the other coefficients in equation (7). If we apply the map $(D_2 j \wedge D_2 j)_*$ to this equation, the left side becomes $Q\Sigma u \otimes Q\Sigma u$ by 5.7(vii) while the right side becomes $a_4 Q\Sigma u \otimes Q\Sigma u$. Hence $a_4 = 1$. Next consider the following diagram

$$\begin{array}{ccccc}
D_2(X \wedge Y) & & \xrightarrow{\quad\delta\quad} & & D_2X \wedge D_2Y \\
\downarrow{\scriptstyle \tau} & & & & \downarrow{\scriptstyle \tau \wedge 1} \\
X \wedge Y \wedge X \wedge Y & \xrightarrow{1 \wedge T \wedge 1} & X \wedge X \wedge Y \wedge Y & \xrightarrow{1 \wedge \iota} & X \wedge X \wedge D_2Y
\end{array}$$

The commutativity of this diagram will be proved in VI.3.10 of the sequel. With $X = Y = \Sigma M_r$ we obtain

$$(\tau \wedge 1)_* \delta_* Q(x \otimes y) = (1 \wedge \iota)_* (1 \wedge T \wedge 1)_* \tau_* Q(x \otimes y)$$

$$= (1 \wedge \iota)_* (1 \wedge T \wedge 1)_* \pi[-x \otimes y \otimes x \otimes y + 2^{r-2} \beta_r(x \otimes y) \otimes \beta_r(x \otimes y)]$$

$$= (1 \wedge \iota)_* \pi[x^{(2)} \otimes y^{(2)} + 2^{r-2} x^{(2)} \otimes (\beta_r y)^{(2)}$$

$$+ 2^{r-2} \beta_r x \otimes x \otimes y \otimes \beta_r y + 2^{r-2} x \otimes \beta_r x \otimes \beta_r y \otimes y + 2^{r-2} (\beta_r x)^{(2)} \otimes y^{(2)}]$$

$$= \pi x^{(2)} \otimes \pi \iota_* y^{(2)} + 2^{r-2} \pi x^{(2)} \otimes \pi \iota_* (\beta_r y)^{(2)}$$

$$+ 2^{r-2} \pi \tau_* \iota_* (x \otimes \beta_r x) \otimes \pi \iota_* (y \otimes \beta_r y) + 2^{r-2} \pi (\beta_r x)^{(2)} \otimes \pi \iota_* y^{(2)}$$

$$= 2^{r-2} \pi \tau_* \iota_* (x \otimes \beta_r x) \otimes \pi \iota_* (y \otimes \beta_r y) + 2^{2r-4} \pi (\beta_r x)^{(2)} \otimes \pi \iota_* (\beta_r y)^{(2)},$$

with the last equation following from part (x). Now applying $(\tau \wedge 1)_*$ to the right side of (7) and comparing coefficients gives $a_1 = 2^{r-2}$, $a_3 = 0$, $a_7 = 2^{2r-4}$ and $a_8 = 2a_6$. Similarly, applying $(1 \wedge \tau)_*$ to equation (7) gives $a_2 = 0$ and $a_6 = 2^{2r-4}$, whence $a_8 = 2a_6 = 0$. This completes the proof of part (vii) when $|x| = |y| = 1$.

Next let $|x| = 1$, $|y| = 0$. Consider the following commutative diagram

$$\begin{array}{ccc}
\Sigma D_2(X \wedge Y) & \xrightarrow{\Sigma\delta} & \Sigma D_2 X \wedge D_2 Y \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle T \wedge 1} \\
D_2(\Sigma X \wedge Y) & & D_2 X \wedge \Sigma D_2 Y \\
\downarrow{\scriptstyle D_2(T \wedge 1)} & & \downarrow{\scriptstyle 1 \wedge \Delta} \\
D_2(X \wedge \Sigma Y) & \xrightarrow{\delta} & D_2 X \wedge D_2 \Sigma Y
\end{array}$$

If we let $X = M_r$, $Y = \Sigma^{-1}M_r$ we obtain

(10) $$\delta_*[D_2(T \wedge 1)]_*\Delta_*\Sigma Q(-\Sigma u_r \otimes \Sigma^{-1}u_r) = (1 \wedge \Delta)_*(T \wedge 1)_*(\Sigma\delta)_*Q(-\Sigma u_r \otimes \Sigma^{-1}u_r)$$

We can evaluate the left side of (10) using 7.3(v); the result is $\delta_*Q(\Sigma u_r \otimes u_r)$. On the other hand we can evaluate the right side of (10) by using 7.3(v) and the part of 7.6(vii) just shown; the result is

$$Q\Sigma u_r \otimes \pi\iota_* u_r^{(21)} + 2Q\Sigma u_r \otimes Qu_r + 2^{2r-4}\pi\iota_*(\beta_r \Sigma u_r)^{(2)} \otimes Q\beta_r u_r.$$

Thus equation (10) gives the desired formula when $x = \Sigma u_r$ and $y = u_r$, and therfore this formula holds in general.

Finally, let $|x| = |y| = 0$. We may assume $x = u_r$, $y = u_r$. The set

$$\{\pi\iota_* x^{(p)} \otimes \pi\iota_* y^{(p)}, Qx \otimes \pi\iota_* y^{(p)}, \pi\iota_* x^{(p)} \otimes Qy, Qx \otimes Qy,$$
$$\pi\iota_*(x \otimes \beta_r x) \otimes \pi\iota_*(y \otimes \beta_r y), Q\beta_r x \otimes \pi\iota_*(y \otimes \beta_r y),$$
$$\pi\iota_*(x \otimes \beta_r x) \otimes Q\beta_r y, Q\beta_r x \otimes Q\beta_r y\}$$

is a basis for $K_0(D_2M_r \wedge D_2M_r;r-1)$. Let $a_1,\dots,a_8$ be the coefficients of $\delta_*Q(x \otimes y)$ in this basis. By 5.7(v) we have

$$(D_2 j \wedge D_2 j)_*\delta_*Q(x \otimes y) = \delta_*Q(u \otimes u) = Qu \otimes \eta + \eta \otimes Qu + pQu \otimes Qu,$$

hence $a_1 = 0$, $a_2 = a_3 = 1$ and $a_4 = 2$. Diagram (8) gives

$$(\tau \wedge 1)_*\delta_*Q(x \otimes y) = (1 \wedge \iota)_*\delta_*\tau_*Q(x \otimes y)$$

and it follows that $a_5 = 2^{r-2}$ and $a_6 = 0$. Similarly,

$$(1 \wedge \tau)_*\delta_*Q(x \otimes y) = (\iota \wedge 1)_*\delta_*\tau_*Q(x \otimes y)$$

and hence $a_7 = 0$. Thus we have

$$(11)\qquad \delta_*Q(x\otimes y) = Qx\otimes \pi\iota_*y^{(2)} + \pi\iota_*x^{(2)}\otimes Qy + 2Qx\otimes Qy$$
$$+\ 2^{r-2}\pi\iota_*(x\otimes\beta_r x)\otimes\pi\iota_*(y\otimes\beta_r y) + a_8 Q\beta_r x\otimes Q\beta_r y$$

and it remains to determine a_8. Consider the following commutative diagram

$$(12)\qquad \begin{array}{ccc} \Sigma D_2(X\wedge Y) & \xrightarrow{\ \Sigma\delta\ } & \Sigma D_2X\wedge D_2Y \\ \Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\Delta\wedge 1} \\ D_2(\Sigma X\wedge Y) & \xrightarrow{\ \delta\ } & D_2\Sigma X\wedge D_2Y \end{array}$$

With $X = Y = M_r$ we have

$$(13)\qquad (\Delta\wedge 1)_*\Sigma\delta_*Q(x\times y) = \delta_*\Delta_*\Sigma Q(x\otimes y).$$

We evaluate the left side of (13) using 7.3(v) and equation (11); the result is

$$Q\Sigma x\otimes\pi\iota_*y^{(2)} + 2Q\Sigma x\otimes Qy + a_8\pi\iota_*(\beta_r\Sigma x)^{(2)}\otimes Q\beta_r y + 2a_8 Q\beta_r\Sigma x\otimes Q\beta_r y.$$

Evaluating the right side of (13) using 7.34(v) and the part of 7.6(vii) already shown gives

$$Q\Sigma x\otimes\pi\iota_*y^{(2)} + 2Q\Sigma x\otimes Qy + 2^{2r-4}\pi\iota_*(\beta_r\Sigma x)^{(2)}\otimes Q\beta_r y.$$

Hence $a_8 = 2^{2r-4}$ as required.

(ix) We have seen in VIII.7.4 that ψ^k is an H_∞ ring map of $K_{(p)}$ for k prime to p. Hence we have

$$(\bar{D}_pf)_{**}\psi^k = \psi^k(\bar{D}_pf)_{**}:K_*(D_pY;r-1)\to K_*(D_pX;r-1)$$

for any map $f:Y\to K\wedge X$. Thus we may assume $x = \Sigma^\alpha u_r$ with $\alpha = 0$ or 1. First let $\alpha = 0$. Since the map

$$(D_pj)_*:K_0(D_pM_r;r-1)\to K_0(D_pS;r-1)$$

is monic and since $\psi^k u = u$, it suffices to show $\psi^k Qu = Qu$. Dually, it suffices to show that ψ^k is the identity on $K^0(B\Sigma_p;r-1)$. But this is immediate from 5.3 since ψ^k commutes with τ^*. Now, if $\alpha = 1$ we have

$$\psi^kQ\Sigma u_r = \psi^k\Delta_*\Sigma Qu_r = \Delta_*\Sigma\psi^kQu_r = \Delta_*\Sigma Qu_r = Q\Sigma u_r.$$

This completes the proof of 7.6.

Next we must prove 3.3. Each part of this theorem is in fact an easy consequence of the corresponding external formula except for parts (ii) and (viii). For part (ii) we may clearly assume $X = S$, and it suffices to show that Qu goes to zero under the nontrivial map from $B\Sigma_p^+$ to S^0. But the induced map

$$\tilde{K}^0(S^0;r) \to \tilde{K}^0(B\Sigma_p;r)$$

takes 1 to 1, and $\langle 1,Qu\rangle = 0$ by Definition 5.6, whence the result follows.

The proof of part (viii) is more difficult. First recall that if X is any nondegenerately based space and $\lambda:X^+ \to X$ is the identity on X then the cofiber sequence

$$\Sigma^\infty S^0 \xrightarrow{\Sigma^\infty \eta} \Sigma^\infty X^+ \xrightarrow{\Sigma^\infty \lambda} \Sigma^\infty X$$

is naturally split by the evident retraction $\mu:X^+ \to S^0$. In particular, there is a natural transformation

$$\nu:\Sigma^\infty X \to \Sigma^\infty X^+$$

and the inclusion

$$\tilde{K}_*(X;r) \subset K_*(X;r)$$

can be identified with ν_*. Now let Y be an H_∞ space, let $Z = \Omega Y$, and let $\varepsilon:\Sigma Z \to Y$ be the counit. Then

$$\sigma:\tilde{K}_\alpha(\Omega Y;r) \to K_{\alpha+1}(Y;r)$$

is the composite $\nu_*\varepsilon_*\Sigma$.

Let $x \in \tilde{K}_0(\Omega Y;r)$; the case $|x| = 1$ is similar. First we must show that Qx is in $\tilde{K}_\alpha(\Omega Y;r-1)$, i.e., that $\mu_* Qx = 0$. But $\mu:\Sigma^\infty(\Omega Y)^+ \to \Sigma^\infty S^0$ is clearly an H_∞ ring map, and therefore $\mu_* Qx = Q\mu_* x = 0$. Next we state the required formula more precisely as follows:

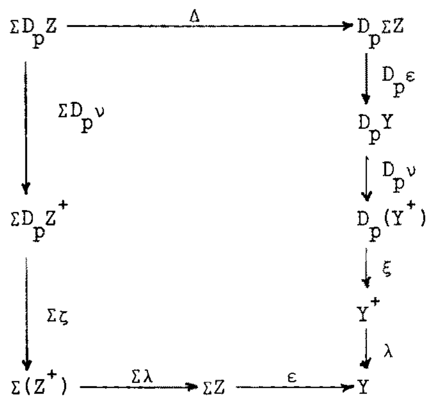
(14) $$\sigma\lambda_* Q\nu_* x = Q\sigma x.$$

Since μ_* applied to each side of (14) gives zero, it suffices to show that λ_* makes the two sides of (14) equal, i.e., that

$$\varepsilon_*\Sigma\lambda_* Q\nu_* x = \lambda_* Q\nu_*\varepsilon_*\Sigma x.$$

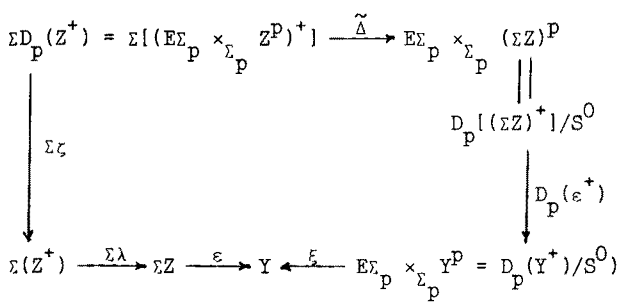
This in turn follows at once from 7.3(v) and the commutativity of the following diagram in $h\bar{\mathcal{J}}$ (where we suppress Σ^∞ to simplify the notation).

(15)



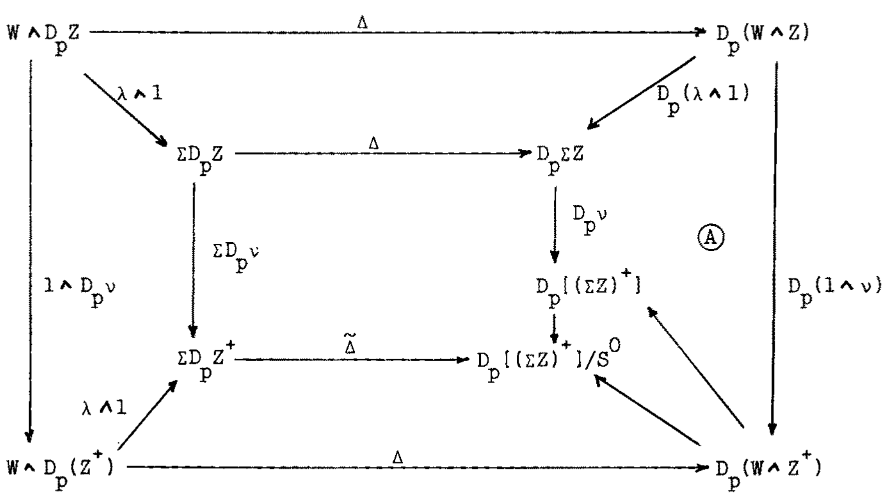
Here ζ and ξ are the H_∞ structural maps for Z^+ and Y^+ respectively. In order to see that (15) commutes we need two further diagrams. The first is the following in the catgory of spaces.

(16)



Here $\tilde{\Delta}$ is the evident diagonal map. This diagram commutes by definition of ζ; see [69, Lemma 1.5]. Next we have the following diagram in $\overline{h}\mathscr{s}$ (where we again suppress Σ^∞).

(17)



Here $W = (S^1)^+$ and the unlabeled arrows are the evident quotient maps. It suffices to show that the inner square of this diagram commutes, since combining it with diagram (16) gives diagram (15). Since

$$\lambda \wedge 1 : W \wedge D_p Z \to \Sigma D_p Z$$

is a split surjection, the commutativity of the inner square will be a consequence of the commutativity of the rest of the diagram. Each of the remaining parts clearly commutes except that marked Ⓐ. To show that Ⓐ commutes it suffices to show that the composites

$$W \wedge Z \xrightarrow{1 \wedge \nu} W \wedge Z^+ = (S^1 \times Z)^+ \longrightarrow (S^1 \wedge Z)^+$$

and
$$W \wedge Z \xrightarrow{\lambda \wedge 1} S^1 \wedge Z \xrightarrow{\nu} (S^1 \wedge Z)^+$$

are equal. But is is easy to see that these composites agree when composed with either of the maps $\lambda : (S^1 \wedge Z)^+ \to S^1 \wedge Z$ and $\mu : (S^1 \wedge Z)^+ \to S^0$; they are therefore equal since wedges are products in $\overline{h\mathcal{S}}$. This completes the proof of 3.3.

We conclude this section with the proof of 1.6. First we calculate

$$\beta_r p_* Q \Sigma u_r = \beta_r Q p_* \Sigma u_r = \iota_*(\beta_r \Sigma u_r)^{(p)} + pQ\beta_{r+1} p_* \Sigma u_r$$

in $K_0(D_p \Sigma M_r; r)$. Multiplying by p^{r-1} gives

$$0 = p^{r-1} \beta_r p_* Q \Sigma u_r = p^{r-1} \iota_*(\beta_r \Sigma u_r)^{(p)},$$

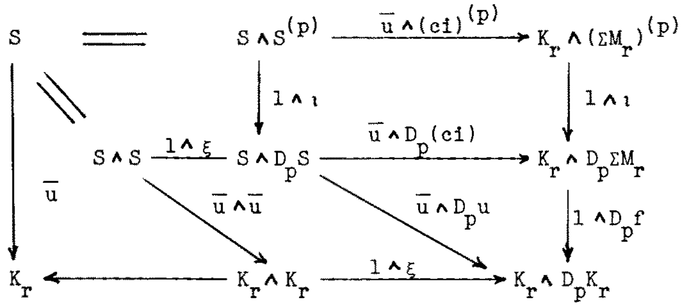
hence $\iota_*(\beta_r \Sigma u_r)^{(p)}$ has order $\le p^{r-1}$. Now suppose $\overline{K}_r$ has an H_∞ structure. Let $\overline{u} : S \to K_r$ be the unit map for this structure. Then $\overline{u} = cu \in K_0(S;r)$ for some c prime to p. Let f be the composite

$$\Sigma M_r = S \wedge \Sigma M_r \xrightarrow{u \wedge 1} K \wedge \Sigma M_r = K_r$$

and let F be the composite

$$K_0(D_p \Sigma M_r; r) \xrightarrow{(D_p f)_*} K_0(D_p K_r; r) \xrightarrow{\xi_*} K_0(K_r; r) \longrightarrow K_0(S;r),$$

where the last map is induced by the product for K_r. We claim $c^{p+1} F \iota_*(\beta_r \Sigma u_r)^{(p)} = \overline{u}$, which contradicts the fact that $\iota_*(\beta_r \Sigma u_r)^{(p)}$ has order $\le p^{r-1}$. The claim is a consequence of the commutativity of the following diagram



Here the composite $(1 \wedge \iota) \circ [\bar{u} \wedge (ci)^{(p)}]$ represents $c\iota_*(c\beta_r \Sigma u_r)^{(p)}$ and the diagram commutes since $\bar{u}$ is an H_∞ ring map.

8. Construction and properties of R and $\mathcal{Q}$.

In this section we construct R and $\mathcal{Q}$ and prove the external and internal versions of 3.6 and 3.7.

We begin with the construction.

Lemma 8.1. The map

$$\beta_{r+1}: K_1(D_p\Sigma M_r; r+1) \longrightarrow K_0(D_p\Sigma M_r; r+1)$$

is an isomorphism.

Lemma 8.2. The map

$$(D_p j)_*: K_0(D_p M_r; s) \longrightarrow K_0(D_p S; s)$$

is monic if $s = r$ or $s = r+1$, and $\eta \in K_0(D_p S; r+1)$ is in the image of $(D_p j)_*$.

Definition 8.3. Let $e \in K_1(D_p\Sigma M_r; r+1)$ be the unique element with $\beta_{r+1} e = Q\beta_{r+2} p_*^2 \Sigma u_r$. Let $e' \in K_0(D_p M_r; r+1)$ be the unique element with $(D_p j)_* e' = \eta$. Then

$$R: K_1(X; r) \to K_1(D_p X; r+1)$$

and

$$\mathcal{Q}: K_0(X; r) \to K_0(D_p X; r+1)$$

are the operations Q_e and $Q_{e'}$.

Note that e and e' are equal to $R\Sigma u_r$ and $\mathcal{Q} u_r$ respectively. We shall always use the latter notations for these elements. Also note that $\mathcal{Q}u = \eta$ in $K_0(B\Sigma_p; r+1)$.

Proof of 8.1. Let $r \geq 2$; the case $r = 1$ is similar. Consider the K-theory BSS for $D_p\Sigma M_r$. By 6.1 the set

$$\{\pi^{r-2}Q\Sigma u_r, \pi^{r-2}Q\beta_r\Sigma u_r, \pi^{r-1}\iota_*[\Sigma u_r \otimes (\beta_r\Sigma u_r)^{(p-1)}], \pi^{r-1}\iota_*(\beta_r\Sigma u_r)^{(p)}\}$$

is a basis for E^1. By 7.6(v) we have

$$d_{r-1}\pi^{r-2}Q\Sigma u_r = \pi^{r-1}\iota_*(\beta_r\Sigma u_r)^{(p)}, \tag{1}$$

while clearly $d_{r-1}\pi^{r-2}Q\beta_r\Sigma u_r = 0$ and

$$d_{r-1}\pi^{r-1}\iota_*[\Sigma u_r \otimes (\beta_r\Sigma u_r)^{(p-1)}] = 0;$$

hence the set

$$\{\pi^{r-2}Q\beta_r\Sigma u_r, \pi^{r-1}\iota_*[\Sigma u_r \otimes (\beta_r\Sigma u_r)^{p-1}]\}$$

is a basis for E^r. Now $d_r\pi^{r-2}Q\beta_r\Sigma u_r = 0$ by 7.6(v), and

$$d_r\pi^{r-1}\iota_*[\Sigma u_r \otimes (\beta_r u_r)^{(p-1)}] = \pi^{r-1}\iota_*(\beta_r\Sigma u_r)^{(p)} ,$$

which is zero in E^r. Thus there is an element x in $K_1(D_p\Sigma M_r;r+1)$ with

$$\pi^r x = \pi^{r-1}\iota_*[\Sigma u_r \otimes (\beta_r\Sigma u_r)^{(p-1)}],$$

and the set $\{Q\Sigma u_r, x, Q\beta_{r+2}p_*^2\Sigma u_r\}$ is a subbasis of height r+1 for $D_p\Sigma M_r$. In particular the group $K_\alpha(D_p\Sigma M_r;r+1)$ has the same order p^{2r} for $\alpha = 0$ and $\alpha = 1$. The lemma will follow if we show that $\beta_{r+1} \otimes Z_p$ maps onto $K_0(D_p\Sigma M_r;r+1) \otimes Z_p$. But the map

$$\pi^r \otimes Z_p : K_0(D_p\Sigma M_r;r+1) \otimes Z_p \to K_0(D_p\Sigma M_r;1) \otimes Z_p = K_0(D_p\Sigma M_r;1)$$

is an isomorphism, hence it suffices to show that $\pi^r\beta_{r+1}$ maps onto $K_0(D_p\Sigma M_r;1)$. Now equation (1) shows that $\pi^{r-1}\iota_*(\beta_r\Sigma u_r)^{(p)}$ is in the image of $\pi^r\beta_{r+1}$, and it remains to consider $\pi^{r-2}Q\beta_r\Sigma u_r$. By the exact sequence

$$K_1(D_p\Sigma M_r;r+1) \xrightarrow{\pi^r\beta_{r+1}} K_0(D_p\Sigma M_r;1) \xrightarrow{p_*^{r+1}} K_0(D_p\Sigma M_r;r+2)$$

it suffices to show $p_*^{r+1}\pi^{r-2}Q\beta_r\Sigma u_r = 0$. But 7.6(vi) gives

$$\begin{aligned} 0 = Qp^{r+1}\beta_{r+3}p_*^3\Sigma u_r &= p^{r+1}Q\beta_{r+3}p_*^3\Sigma u_r - (p^{pr+p-1} - p^r)\iota_*(\beta_{r+2}p_*^2\Sigma u_r)^{(p)} \\ &= p^{r+1}Q\beta_{r+3}p_*^3\Sigma u_r = p_*^{r+1}\pi^{r-2}Q\beta_r\Sigma u_r \end{aligned}$$

which completes the proof.

Proof of 8.2. It is easy to see that $\pi^{r-1}\beta_r\iota_*u_r^{(p)}$ and $\pi^{r-1}\beta_r\iota_*[u_r^{(p-1)} \otimes \beta_r u_r]$ are zero, hence by the exact sequence

$$K_\alpha(D_pM_r;r+1) \xrightarrow{\pi} K_\alpha(D_pM_r;r) \xrightarrow{\pi^{r-1}\beta_r} K_{\alpha-1}(D_pM_r;1)$$

there exist elements x and y with $\pi x = \iota_* u_r^{(p)}$ and $\pi y = \iota_*[u_r^{(p-1)} \otimes \beta_r u_r]$. Clearly the set $\{x,y,Qu_r\}$ is a subbasis of height r+1 for D_pM_r. In particular the set $\{x,p_*^2Qu_r\}$ is a basis for $K_0(D_pM_r;r+1)$. Since $\{\eta,Qu\}$ is a basis for $K_0(D_pS;r+1)$ we have

(2) $$(D_pj)_*x = a_1\eta + a_2Qu.$$

where a_1, a_2 $Z_{p^{r+1}}$. Applying π to both sides of (2) gives

$$\eta = (D_pj)_*\iota_*u_r^{(p)} = a_1\eta + a_2Qu$$

in $K_0(D_pS;r)$, hence $a_1 = 1 + a_1'p^r$ and $a_2 = a_2'p^r$ for some $a_1,a_2 \in Z_p$. This fact, together with the equation $(D_pj)_*p_*^2Qu_r = p^2Qu$, shows that $(D_pj)_*$ is monic on $K_0(D_pM_r;r+1)$. A similar argument shows that $(D_pj)_*$ is monic on $K_0(D_pM_r;r)$. If $r \geq 2$ we have

$$(D_pj)_*[x - a_1'p^{r-1}p_*\iota_*u_r^{(p)} - a_2'p^{r-2}p_*^2Qu_r] = \eta$$

so that $\eta \in K_0(D_pS;r+1)$ is in the image of $(D_pj)_*$ as required. If r = 1 we must show $a_2' = 0$. For this we need the map j': $M_1 \to M_2$ induced by the inclusion $Z_p \subset Z_{p^2}$. We have $j' \circ j = j:M_1 \to S$, hence

$$(D_pj)_*(D_pj')_*(x) = (1 + a_1'p)\eta + a_2'pQu$$
$$= (D_pj)_*[(1 + a_1'p)u_2^{(p)} + a_2'p_*Qu_2] .$$

Since $(D_pj)_*$ is monic we conclude

$$(D_pj')_*(x) = (1 + pa_1')u_2^{(p)} + a_2'p_*Qu_2 .$$

Hence

(3) $$\pi\beta_2(D_pj')_*(x) = a_2'\beta Qu_2 = a_2'Q\beta_2u_2 .$$

On the other hand, 6.1(vi) implies that $\iota_*[u_1^{(p-1)} \otimes \beta u_1]$ generates $K_1(D_pM_1;1)$, hence $\pi\beta_2x = c\iota_*(u_1^{(p-1)} \otimes \beta u_1)$ for some $c \in Z_p$ and

(4) $$\pi\beta_2(D_pj')_*(x) = (D_pj')_*(\pi\beta_2x) = c\iota_*[(j_*'u_1)^{(p-1)} \otimes j_*'\beta u_1] = 0$$

since $j_*'\beta u_1 = 0$. Comparing (3) and (4) gives $a_2' = 0$ and thus

$$(D_pj)_*[x - a_1'p_*\iota_*u_1^{(p)}] = \eta$$

which completes the proof.

Next we shall prove the external analogs of 3.6 and 3.7. The conventions preceding 7.6 give the correct external version of each statement except for 3.6(viii) and 3.7(ix). For 3.6(viii) we must prefix $(\beta_{p,p})_*$ to both sides, where

$\beta_{p,p}$ is the natural map $D_pD_pX \to D_{p^2}X$ defined in I.2, and for 3.7(ix) we prefix $(\beta_{p,p})_*$ to the left and $(\alpha_{p,p,\ldots,p})_*$ to the right.

Proposition 8.4. The operation

$$R: K_1(X;r) \longrightarrow K_1(D_pX;r+1)$$

satisfies the external analog of each part of 3.6.

Proposition 8.5. The operation

$$\mathcal{Q}: K_0(X;r) \to K_0(D_pX;r+1)$$

satisfies the external analog of each part of 3.7.

Theorems 3.6 and 3.7 will follow at once from 8.4 and 8.5 by the same proof given for 3.3. The rest of this section is devoted to the proofs of 8.4 and 8.5.

Proof of 8.4. Part (i) is trivial. In each of the remaining parts except (v) we may assume $X = \Sigma M_r$ with $x = \Sigma u_r$; part (iv) now follows from Definition 8.3. Observe that by the proof of 8.1 the set $\{Q\Sigma u_r, R\Sigma u_r\}$ is a subbasis for $D_p\Sigma M_r$ if $r \geq 2$ while $\{R\Sigma u_1\}$ is a subbasis for $D_p\Sigma M_1$.

(iii). The map

$$\pi\beta_{r+2}: K_1(D_p\Sigma M_r;r+2) \to K_0(D_p\Sigma M_r;r+1)$$

is an isomorphism since it takes the basis for the first group to that for the second. Now

$$\pi\beta_{r+2}Rp_*\Sigma u_r = \pi Q\beta_{r+3}p_*^3\Sigma u_r = Q\beta_{r+2}p_*^2\Sigma u_r$$

$$= \beta_{r+1}R\Sigma u_r = \pi\beta_{r+2}p_*R\Sigma u_r$$

and the result follows.

(iv). The map

$$\beta_{r+1}p_*: K_1(D_p\Sigma M_r;r) \to K_0(D_p\Sigma M_r;r+1)$$

is monic since it takes the basis elements $\pi R\Sigma u_r$ and (when $r \geq 2$) $p_*Q\Sigma u_r$ to $p\beta_{r+1}R\Sigma u_r$ and $\beta_{r+1}p_*^2Q\Sigma u_r$ respectively. We have

$$\beta_{r+1}p_*\pi R\Sigma u_r = p\beta_{r+1}R\Sigma u_r = pQ\beta_{r+2}p_*^2\Sigma u_r$$

$$= \beta_{r+1}Qp_*^2\Sigma u_r - \iota_*(\beta_{r+1}p_*\Sigma u_r)^{(p)}$$

$$= \beta_{r+1}p_*[Qp_*\Sigma u_r - \iota_*(\Sigma u_r \otimes (\beta_r\Sigma u_r)^{(p-1)})]$$

which gives the first formula. For the second formula, we have

$$\begin{aligned}\beta_{r+1}p_*R\pi\Sigma u_r &= \beta_{r+1}Rp_*\pi\Sigma u_r = \beta_{r+1}Rp\Sigma u_r\\ &= Q\beta_{r+2}p_*^2(p\Sigma u_r) = Qp\beta_{r+2}p_*^2\Sigma u_r\\ &= pQ\beta_{r+2}p_*^2\Sigma u_r - (p^{p-1}-1)\iota_*(\beta_{r+1}p_*\Sigma u_r)^{(p)}\\ &= \beta_{r+1}Qp_*^2\Sigma u_r - p^{p-1}\iota_*(\beta_{r+1}p_*\Sigma u_r)^{(p)}\\ &= \beta_{r+1}p_*[Qp_*\Sigma u_r - p^{p-1}\iota_*(\Sigma u_r\otimes(\beta_r\Sigma u_r)^{(p-1)})]\end{aligned}$$

and the result follows.

(v). Let z denote Σu_r and fix i with $0 < i < p$. As in the proof of 7.3(iii) it suffices to show that the equation

$$(5)\qquad \begin{aligned}(\tau_{i,p-i})_*Rx &= a_1\iota_*[p_*z\otimes(\beta_{r+1}p_*z)^{(i-1)}]\otimes\iota_*(\beta_{r+1}p_*z)^{(p-i)}\\ &\quad + a_2\beta_{r+1}p_*[\iota_*(z\otimes(\beta_r z)^{(i-1)})\otimes\iota_*(z\otimes(\beta_r z)^{(p-i-1)})]\end{aligned}$$

holds in $K_1(D_i\Sigma M_r\wedge D_{p-i}\Sigma M_r;r+1)$ with $a_1 = -\frac{1}{p}\binom{p}{i}$ and $a_2 = \binom{p-1}{i}$. First observe that the group $K_*(D_i\Sigma M_r;1)$ is the Σ_i-coinvariant quotient of $K_*((\Sigma M_r)^{(i)};1) = K_*(\Sigma M_r;1)^{\otimes i}$, so that the set $\{\iota_*(z\otimes(\beta_r z)^{(i-1)}\}$ is a subbasis for $D_i\Sigma M_r$. Thus the set

$$\{\iota_*[z\otimes(\beta_r z)^{(i-1)}]\otimes\iota_*(\beta_r z)^{(p-i)},\iota_*[z\otimes(\beta_r z)^{(i-1)}]\otimes\iota_*[z\otimes(\beta_r z)^{(p-i-1)}]\}$$

is a subbasis for $D_i\Sigma M_r\wedge D_{p-i}\Sigma M_r$ and we see that equation (5) holds for <u>some</u> $a_1,a_2\in Z_{p^r}$. Now applying $(\tau'_{i,p-i})_*\beta_{r+1}$ to both sides of (5) gives

$$\tau_*\beta_{r+1}Rz = i!(p-i)!a_1(\beta_{r+1}p_*z)^{(p)}.$$

On the other hand we have

$$\tau_*\beta_{r+1}Rz = \tau_*Q\beta_{r+2}p_*^2z = -(p-1)!(\beta_{r+1}p_*z)^{(p)};$$

hence $a_1 = -\frac{(p-1)!}{i!(p-i)!} = -\frac{1}{p}\binom{p}{i}$. Next we apply π to (5) to get

$$(6)\qquad \begin{aligned}(\tau_{i,p-i})_*\pi Rz &= -\binom{p}{i}\iota_*[z\otimes(\beta_r z)^{(i-1)}]\otimes\iota_*(\beta_r z)^{(p-i)}\\ &\quad + a_2\iota_*(\beta_r z)^{(i)}\otimes\iota_*[z\otimes(\beta_r z)^{(p-i-1)}]\\ &\quad - a_2\iota_*[z\otimes(\beta_r z)^{(i-1)}]\otimes\iota_*(\beta_r z)^{(p-i)}.\end{aligned}$$

But we have

$$(\tau_{i,p-i})_*\pi Rz = (\tau_{i,p-i})_*[Qp_*z - \iota_*(z\otimes(\beta_r z)^{(p-1)}]$$
$$= -(\tau_{i,p-i})_*\iota_*(z\otimes(\beta_r z)^{(p-i)})$$
$$= -\binom{p-1}{i-1}\iota_*(z\otimes(\beta_r z)^{(i-1)})\otimes\iota_*(\beta_r z)^{(p-i)}$$
$$-\binom{p-1}{i}\iota_*(\beta_r z)^{(i)}\otimes\iota_*(z\otimes(\beta_r z)^{(p-i-1)}),$$

where the last equality follows from the double-coset formula; comparing with (6) gives $a_2 = -\binom{p-1}{i}$ as required.

(vi). Let $r \geq 2$; the case $r = 1$ is similar. Let f be the composite

$$\Sigma^{-1}M_r = S^{-2}\wedge \Sigma M_r \xrightarrow{\Sigma^{-2}u\wedge 1} \Sigma^{-2}K\wedge\Sigma M_r \xrightarrow{B\wedge 1} K\wedge\Sigma M_r ,$$

where B is the Bott equivalence. We have $f_{**}\Sigma^{-1}u_r = \Sigma u_r$, hence it suffices to prove

$$\Delta_*\Sigma R(\Sigma^{-1}u_r) = p_*\iota_*u_r^{(p)} + p_*^2 Qu_r.$$

Now

$$(D_p j)_*\Delta_* R(\Sigma^{-1}u_r) = \Delta_*\Sigma R(\Sigma^{-1}u) = \Delta_*\Sigma R(\pi\Sigma^{-1}u)$$
$$= \Delta_*\Sigma Qp\Sigma^{-1}u = p\Delta_*\Sigma Q\Sigma^{-1}u$$
$$= p\iota_*u^{(p)} + p^2 Qu$$
$$= (D_p j)_*(p_*\iota_*u_r^{(p)} + p_*^2 Qu_r);$$

the result follows since $(D_p j)_*$ is monic by 8.2.

(vii) $$\beta_{r+1}\psi^k R\Sigma u_r = \psi^k\beta_{r+1}R\Sigma u_r = \psi^k Q\beta_{r+2}p_*^2\Sigma u_r$$
$$= Q\beta_{r+2}p_*^2\Sigma\psi^k u_r = \beta_{r+1}R\Sigma u_r ,$$

the last equality following from the fact that $\psi^k u_r = u_r$. The result now follows by 8.1.

(viii). Let z denote Σu_r, and abbreviate $(\beta_{p,p})_*$ by β_* and $(\alpha_{p,\ldots,p})_*$ by α_* (the reader is requested to remember that β_* is <u>not</u> a Bockstein). We must show

$$\beta_* QRx = \begin{cases} 0 & \text{if } r = 1 \\ \beta_* RQz & \text{if } r \geq 2 \end{cases}$$

in $K_1(D_{p^2}\Sigma M_r;r)$. We shall need the equation

(7) $$\delta_* Qx^{(n)} = \sum_{i=1}^{n}\binom{n}{i}p^{i-1}(\pi\iota_* x^{(p)})^{(n-i)}\otimes(Qx)^{(i)}$$

which holds in $K_0((D_pX)^{(n)};r-1)$ for each $x \in K_0(X;r)$ provided that p is odd (the proof is by induction on n from 7.6(ii)).

First let r = 1. The set {QRz,RRz} is a subbasis for $D_pD_p\Sigma M_1$, and it follows easily from Proposition 3.9 that the map

$$\beta_3 p_*^2 : K_1(D_pD_p\Sigma M_1;1) \longrightarrow K_0(D_pD_p\Sigma M_1;3)$$

is a monomorphism. Since $K_1(D_{p^2}\Sigma M_1;1)$ is imbedded in $K_1(D_pD_p\Sigma M_1;1)$ by the transfer we see that

$$\beta_3 p_*^2 : K_1(D_{p^2}\Sigma M_1;1) \longrightarrow K_0(D_{p^2}\Sigma M_1;3)$$

is a monomorphism. It therefore suffices to show that $\beta_*\beta_3 p_*^2 QRz$ is zero. We have

$$\begin{aligned}\beta_*\beta_3 p_*^2 QRz &= \beta_*\beta_3 Qp_*(Rp_*z) \quad \text{by 7.6(iv) and 8.4(iii)}\\ &= \beta_*\beta_3[R\pi Rp_*z + p^{p-1}\iota_*(Rp_*z \otimes (\beta_3 Rp_*z)^{(p-1)})]\\ &= \beta_*\beta_3 R[Qp_*^2 z - \iota_*(p_*z \otimes (\beta_2 p_*z)^{(p-1)})] + p^{p-1}\beta_*\iota_*(\beta_3 Rp_*z)^{(p)},\end{aligned}$$

where the last two equalities follow from the second and first parts of 8.4(ii). Now $Qp_*^2 z = 0$ by 7.6(iv), and

$$\begin{aligned}\beta_*\beta_3 R\iota_*(p_*z \otimes (\beta_2 p_*z)^{(p-1)}) &= \alpha_*\delta_*\beta_3 R(p_*z \otimes (\beta_2 p_*z)^{(p-1)}) \quad \text{by I.2.12}\\ &= \alpha_*\delta_* Q((\beta_4 p_*^3 z)^{(p)}) \quad \text{by 8.4(iv)}\\ &= p^{p-1}\alpha_*(Q\beta_4 p_*^3 z)^{(p)} \quad \text{by 7.6(vii) when } p = 2 \text{ and equation (7) when } p \text{ is odd}\\ &= p^{p-1}\beta_*\iota_*(\beta_3 Rp_*z)^{(p)} \quad \text{by 8.4(iv) and I.2.11.}\end{aligned}$$

We conclude that $\beta_*\beta_3 p_*^2 QRz = 0$ as required, which concludes the case r = 1.

Next let r = 2. We have

$$\begin{aligned}\pi\beta_*(QRz - RQz) &= \beta_*[Q[Qp_*z - \iota_*(z \otimes (\beta_r z)^{(p-1)})]\\ &\qquad - Qp_*Qz + \iota_*(Qz \otimes (\beta_{r-1}Qz)^{(p-1)})]\\ &= \beta_*[-Q\iota_*(z \otimes (\beta_r z)^{(p-1)}) + \iota_*(Qz \otimes (\beta_{r-1}Qz)^{(p-1)})]\\ &= \alpha_*[-\delta_* Q(z \otimes \beta_r z)^{(p-1)}) + Qz \otimes (\beta_{r-1}Qz)^{(p-1)}] \quad \text{by I.2.11 and I.2.12.}\\ &= \alpha_*[-Qz \otimes (\pi\iota_*(\beta_r z)^{(p)})^{p-1} - pQz \otimes \delta_* Q(\beta_r z)^{(p-1)}\\ &\qquad + Qz \otimes (\iota_*(\pi\beta_r z)^{(p)} + pQ\beta_r z)^{p-1}].\end{aligned}$$

When p = 2 the last expression is clearly zero, while if p is odd it is zero by (7). Hence we have

$$\pi\beta_*(QRz - RQz) = 0. \tag{8}$$

A similar calculation gives

$$\beta_{r+2}p_*^2\beta_*(QRz - RQz) = 0. \tag{9}$$

To proceed further we need the case $k = p^2$ of 4.1. First we must check that the argument is not circular, since the present result is certainly used in the proof of 4.1. However, it enters only through the proof of 4.7, to be given in Section 9. An inspection of Section 9 will show that only the case r = 1 of the present result is used in proving the case $k = p^2$ of 4.7. Thus we may proceed. We suppose $r \geq 3$; the case r = 2 differs only slightly. By Remark 4.2 we obtain a subbasis

$$A = A_{r-2} \cup A_{r-1} \cup A_r \cup A_{r+2}$$

for $D_{p^2}\Sigma M_r$ with $A_{r-2} = \{\beta_*QQz\}$,

$$A_{r-1,1} = \{\alpha_*[Qz \otimes (\beta_{r-1}Qz)^{(i)} \otimes (\pi^2\beta_{r+1}Rz)^{(p-i-1)}] \mid 0 \leq i \leq p-2\},$$

$$A_{r-1,0} = \{\alpha_*[Qz \otimes \pi^2 Rz \otimes (\beta_{r-1}Qz)^{(i-1)} \otimes (\pi^2\beta_{r+1}Rz)^{(p-i-1)}] \mid 1 \leq i \leq p-2\},$$

$A_r = \{\beta_*RQz\}$ and $A_{r+2} = \{\beta_*RRz\}$. Therefore the set

$$\{\pi^{r-3}\beta_*QQz, \pi^{r-1}\beta_*RQz, \pi^{r+1}\beta_*RRz\} \cup \pi^{r-2}A_{r-1,1} \cup \pi^{r-2}\beta_{r-1}A_{r-1,0}$$

is a basis for $K_1(D_p\Sigma M_r;1)$, and the subset $\pi^{r-2}\beta_{r-1}A_{r-1,0}$ is a basis for the image of $\pi^{r-2}\beta_{r-1}$, hence for the kernel of p_*^{r-1}. By (8) we see that $\beta_*(QRz - RQz)$ is in the image of p_*^{r-1}, hence there exist constants $a,b,c,d_0,\ldots,d_{p-2}$ Z_p with

$$\beta_*(QRz - RQz) = p_*^{r-1}[a\pi^{r-3}\beta_*QQz + b\pi^{r-1}\beta_*RQz + c\pi^{r+1}\beta_*RRz + \alpha_*\pi^{r-2}\sum_{i=0}^{p-2}(d_iQz \otimes (\beta_{r-1}Qz)^{(i)} \otimes (\pi^2\beta_{r+1}Rz)^{(p-i-1)})]. \tag{10}$$

If we apply $\beta_{r+2}p_*^2$ to both sides of (10) then the left side becomes zero by (9), hence we have

$$0 = ap^{r-3}\beta_{r+2}p_*^4\beta_*QQz + bp^{r-1}\,p_*^2\beta_*RQz + cp^{r+1}\beta_{r+2}\beta_*RRz + \sum_{i=0}^{p-2} d_ip^{r-2}\beta_{r+2}p_*^3\alpha_*[Qz \otimes (\beta_{r-1}Qz)^{(i)} \otimes (\pi^2\beta_{r+1}Rz)^{(p-i-1)}].$$

Since the set A is a subbasis this gives $a = b = c = d_0 = \cdots d_{p-2} = 0$ as required. This completes the proof of 8.4.

Proof of 8.5. Part (i) is trivial.

(ii) We may assume $x = u_r$. We have

$$(D_pj)_*\pi\mathcal{Q}u_r = \eta = \iota_*u^{(p)} = (D_pj)_*\iota_*u_r^{(p)}$$

hence $\pi\mathcal{Q}u_r = \iota_*u_r^{(p)}$ by 8.2. If $r \geq 2$ then

$$(D_pj)_*\mathcal{Q}\pi u_r = \mathcal{Q}\pi u = \mathcal{Q}u = (D_pj)_*\iota_*u_r^{(p)} ,$$

hence $\mathcal{Q}\pi u_r = i_*u_r^{(p)}$ by 8.2.

(v) As in the proof of 7.3(iii) it suffices to show

$$\tau_*\mathcal{Q}u_r = \begin{cases} (p-1)!p_*u_r^{(p)} & \text{if } p \text{ is odd or } r \geq 2 \\ 2_*u_1^{(2)} + (\beta_2 2_*u_r)^{(2)} & \text{if } p = 2 \text{ and } r = 1. \end{cases}$$

We prove this when $p = 2$; the odd primary case is similar. The element $\tau_*\mathcal{Q}u_r$ is in the Σ_2-invariant subgroup of $K_0(M_r^{(2)};r+1)$, and this subgroup has a basis consisting of $2_*u_r^{(2)}$ with order 2^r and $2^{r-1}(\beta_{r+1}2_*u_r)^{(2)}$ with order 2. Thus we have

(11) $$\tau_*\mathcal{Q}u_r = a_1 2_*u_r^{(2)} + a_2 2^{r-1}(\beta_{r+1}2_*u_r)^{(2)}$$

with $a_1 \in Z_{2^r}$ and $a_2 \in Z_2$. Now

$$j_*^{(2)}\tau_*\mathcal{Q}u_r = \tau_*(D_2j)_*\mathcal{Q}u_r = \tau_*\eta = 2u;$$

thus applying $j_*^{(2)}$ to both sides of (11) gives $2u = 2a_1u$ in $K_0(S;r+1)$ so that $a_1 = 1$. Next we have

$$\pi\tau_*\mathcal{Q}u_r = \tau_*\iota_*u_r^{(2)} = \begin{cases} 2u_r^{(2)} & \text{if } r \geq 2 \\ (\beta u_1)^{(2)} & \text{if } r = 1, \end{cases}$$

hence applying π to (11) gives $a_2 = 0$ if $r \geq 2$ and $a_2 = 1$ if $r = 1$.

(iv) We may assume $x = u_r$. Let $r \geq 2$; the case $r = 1$ is similar. The set

$$\{Qu_r, \iota_*u_r^{(p)}, i_*(u_r^{(p-1)} \otimes b_ru_r)\}$$

is a subbasis of height r for D_pM_r, hence we have

(12) $$\pi\beta_{r+1}\mathcal{Q}u_r = a_1\iota_*(u_r^{(p-1)} \otimes \beta_ru_r) + a_2\beta_rp_*Qu_r$$

with $a_1 \in Z_{p^r}$, $a_2 \in Z_{p^{r-1}}$. Let $j':M_r \to M_{r+1}$ be the map induced by the inclusion $Z_{p^r} \subset Z_{p^{r+1}}$. Then $j \circ j' = j:M_r \to S$, hence $(j')_*u_r = \pi u_{r+1}$ and $(j')_*\beta_ru_r =$

$p\pi\beta_{r+1}u_{r+1}$. Thus

$$(D_p j')_*\pi\beta_{r+1}\mathfrak{Q}u_r = \pi\beta_{r+1}\mathfrak{Q}\pi u_{r+1} = \pi\beta_{r+1}\iota_* u_{r+1}^{(p)}$$
$$= p\pi\iota_*(u_{r+1}^{(p-1)}\otimes\beta_{r+1}u_{r+1})$$

and comparing with (12) gives $a_2 = 0$. Next we have

$$\tau_*\pi\beta_{r+1}\mathfrak{Q}u_r = \pi\beta_{r+1}(p-1)!p_*u_r^{(p)} = (p-1)!\beta_r u_r^{(p)} = \tau_*\iota_*(u_r^{(p-1)}\otimes\beta_r u_r)$$

and comparing with (12) gives $a_1 = 1$.

(iii) By part (iv) we see that the set $\{Qu_r, \mathfrak{Q}u_r\}$ is a subbasis for D_pM_r if $r \geq 2$, while $\{\mathfrak{Q}u_r\}$ is a subbasis for D_pM_1. It follows that the map

$$(D_pj)_*: K_0(D_pM_r; r+2) \to K_0(D_pS; r+2)$$

is monic. But

$$(D_pj)_*\mathfrak{Q}p_*u_r = \mathfrak{Q}(pu) = \mathfrak{Q}(\pi pu) = \iota_*(pu)^{(p)} = p^{p-1}p_*\eta = (D_pj)_*p^{p-1}p_*\mathfrak{Q}u_r$$

and the result follows.

(vi). Let $p = 2$; the odd primary case is similar. First let $|x| = |y| = 0$ with $r \geq 2$. We may assume $x = u_r$, $y = u_r$. The set

$$\{\mathfrak{Q}x\otimes\mathfrak{Q}y, \pi\iota_*x^{(2)}\otimes Qy, Qx\otimes\pi\iota_*y^{(2)}, Qx\otimes Qy, \mathfrak{Q}x\otimes\beta_{r+1}\mathfrak{Q}y,$$
$$\mathfrak{Q}x\otimes\beta_{r+1}4_*Qy, Qx\otimes\pi^2\beta_{r+1}\mathfrak{Q}y, Qx\otimes\beta_{r-1}Qy\}$$

is a subbasis for $D_2M_r \wedge D_2M_r$, hence we have

$$(13)\qquad \delta_*\mathfrak{Q}(x\otimes y) = a_1\mathfrak{Q}x\otimes\mathfrak{Q}y + a_2\mathfrak{Q}x\otimes 4_*Qy + a_3 4_*Qx\otimes\mathfrak{Q}y$$
$$+ a_4 4_*(Qx\otimes Qy) + a_5\beta_{r+1}\mathfrak{Q}x\otimes\beta_{r+1}\mathfrak{Q}y$$
$$+ a_6\beta_{r+1}\mathfrak{Q}x\otimes\beta_{r+1}4_*Qy + a_7\beta_{r+1}4_*Qx\otimes\beta_{r+1}\mathfrak{Q}y$$
$$+ a_8\beta_{r+1}4_*Qx\otimes\beta_{r+1}4_*Qy$$

with $a_1, a_5 \in Z_{2^{r+1}}$ and $a_2, a_3, a_4, a_6, a_7, a_8 \in Z_{2^{r-1}}$. Since

$$\pi\delta_*\mathfrak{Q}(x\otimes y) = \delta_*\iota_*(x\otimes y)^{(2)} = \iota_*x^{(2)}\otimes\iota_*y^{(2)}$$

we have $a_6 = a_7 = a_8 = 0$. The equation

$$(D_2j\wedge D_2j)_*\delta_*\mathfrak{Q}(x\otimes y) = \delta_*\mathfrak{Q}u = \delta_*\eta = \eta\otimes\eta$$

implies $a_1 = 1$ and $a_2 = a_3 = a_4 = 0$. Hence we have

(14) $$\delta_* \mathfrak{Q}(x \otimes y) = \mathfrak{Q}x \otimes \mathfrak{Q}y + a_5\beta_{r+1}\mathfrak{Q}x \otimes \beta_{r+1}\mathfrak{Q}y$$

with a_5 depending on r. A similar argument shows that (14) holds also when r = 1. Now let T_1 and T_2 switch the factors of $M_r \wedge M_r$ and $D_2M_r \wedge D_2M,_r$. Then

$$\delta_* \mathfrak{Q}(T_{1*}(x \otimes y)) = T_{2*}\delta_* \mathfrak{Q}(x \otimes y) = \mathfrak{Q}y \otimes \mathfrak{Q}x - a_5\beta_{r+1}\mathfrak{Q}y \otimes \beta_{r+1}\mathfrak{Q}x.$$

On the other hand, if $r \geq 2$ then

$$\delta_* \mathfrak{Q}(T_{1*}(x \otimes y)) = \delta_* \mathfrak{Q}(y \otimes x) = \mathfrak{Q}y \otimes \mathfrak{Q}x + a_5\beta_{r+1}\mathfrak{Q}y \otimes \beta_{r+1}\mathfrak{Q}x,$$

hence $2a_5 = 0$ as required. If r = 1 then

$$\begin{aligned}\delta_* \mathfrak{Q}(T_{1*}(x \otimes y)) &= \delta_* \mathfrak{Q}(y \otimes x + \beta y \otimes \beta x) \\ &= \delta_* \mathfrak{Q}(y \otimes x) + 2\beta_2\mathfrak{Q}y \otimes \beta_2\mathfrak{Q}x.\end{aligned}$$

Hence in this case $-a_5 = a_5 + 2 \bmod 4$, so that $a_5 \equiv 1 \bmod 2$ as required.

Next let $|x| = 1$, $|y| = 0$ with $r \geq 2$ we may assume $x = \Sigma u_r$, $y = u_r$. Choosing a subbasis for $D_2\Sigma M_r \wedge D_2M_r$ as in the preceeding case, we see that

(15) $$\begin{aligned}\delta_* R(x \otimes y) = {} & a_1 Rx \otimes \mathfrak{Q}y + a_2 Rx \otimes 4_*Qy + a_3 4_*Qx \otimes \mathfrak{Q}y \\ & + a_4 4_*(Qx \otimes Qy) + a_5\beta_{r+1}Rx \otimes \beta_{r+1}\mathfrak{Q}y \\ & + a_6\beta_{r+1}Rx \otimes \beta_{r+1}4_*Qy + a_7\beta_{r+1}4_*Qx \otimes \beta_{r+1}\mathfrak{Q}y \\ & + a_8\beta_{r+1}4_*Qx \otimes \beta_{r+1}4_*Qy\end{aligned}$$

with $a_1, a_5 \in Z_{2^{r+1}}$ and the remaining a_i in $Z_{2^{r-1}}$. If f denotes the composite

$$D_2\Sigma M_r \wedge D_2M_r \xrightarrow{1 \wedge D_2 j} D_2\Sigma M_r \wedge D_2S^0 \xrightarrow{1 \wedge \xi} D_2\Sigma M_r \wedge S^0 = D_2\Sigma M_r$$

then the diagram

$$\begin{array}{ccc} D_1(\Sigma M_r \wedge M_r) & \xrightarrow{\delta} & D_2\Sigma M_r \wedge D_2M_r \\ \big\downarrow D_2(1 \wedge j) & & \big\downarrow f \\ D_2(\Sigma M_r \wedge S^0) & = & D_2\Sigma M_r \end{array}$$

commutes. Applying f_* to (15) and using the equation $\xi_* Qu = 0$ (which was shown in the proof of 3.3(ii)) gives

$$Rx = a_1 Rx + a_3 4_*Qx,$$

hence $a_1 = 1$ and $a_3 = 0$. To determine a_2 and a_4 we calculate

$$\pi^2\beta_{r+1}\delta_* R(x \otimes y) = \delta_* Q\beta_r(x \otimes y) = \pi^2\beta_{r+1}[Rx \otimes \mathcal{Q}y + 4_*(Qx \otimes Qy)],$$

hence $a_2 = 0$ and $a_4 = 1$. Next we calculate

$$\begin{aligned} \pi\delta_* R(x \otimes y) &= \delta_* \pi R(x \otimes y) \\ &= \pi Rx \otimes \pi\mathcal{Q}y + \pi 4_*(Qx \otimes Qy) + 2^{r-2}\beta_r 2_* Qx \otimes \pi\beta_{r+1}\mathcal{Q}y \\ &\quad + 2^{2r-3}\iota_*(\beta_r x)^{(2)} \otimes Q\beta_{r+1}\mathcal{Q}y. \end{aligned}$$

Now the element $2^{2r-3}\iota_*(\beta_r x)^{(2)}$ is zero when $r \geq 3$ since $2r-3 \geq r$ while when $r = 2$ we have

$$0 = 2\beta_2 2_* Qx = 2\beta_2 Q 2_* x = 2\iota_*(\beta_2 x)^{(2)}.$$

Thus applying π to both sides of (15) gives $2a_5 = a_6 = a_8 = 0$ and $a_7 = 2^{r-2}$. It remains to show $a_5 = 2^r\varepsilon_r$, where $\varepsilon_r \in Z_2$ is the constant in the formula for $\delta_*\mathcal{Q}(x \otimes y)$. But this follows from the equation

(16) $$(\delta \wedge 1)_*\delta_* R((\Sigma u_r \otimes u_r) \otimes u_r) = (1 \wedge \delta)_*\delta_* R(\Sigma u_r \otimes (u_r \otimes u_r))$$

if we expand both sides using the formulas already shown.

Next let $x = \Sigma u_1$, $y = u_1$. A suitable choice of subbasis for $D_2\Sigma M_1 \wedge D_2 M_1$ gives

$$\delta_* R(x \otimes y) = a_1 Rx \otimes \mathcal{Q}y + a_2\beta_2 Rx \otimes \beta_2\mathcal{Q}y$$

and we see as before that $a_1 = 1$. Evaluating both sides of equation (16) in this case gives $a_2 = -(1 + 2\varepsilon_1)$. Finally, we have

$$\begin{aligned} \delta_* R(y \otimes x) &= \delta_* R(T_{1*}(x \otimes y + \beta x \otimes \beta y)) \\ &= T_{2*}\delta_* R(x \otimes y + \beta x \otimes \beta y) \\ &= \mathcal{Q}y \otimes Rx + (1 + 2\varepsilon_1)\beta_2\mathcal{Q}y \otimes \beta_2 Rx \end{aligned}$$

as required.

Now let $x = \Sigma u_r$ and $y = \Sigma u_r$, with $r \geq 2$. We have

(17) $$\begin{aligned} \delta_* \ (x \otimes y) &= a_1 Rx \otimes Ry + a_2 Rx \otimes 4_* Qy + a_3 4_* Qx \otimes Ry \\ &\quad + a_4 4_*(Qx \otimes Qy) + a_5\beta_{r+1}Rx \otimes \beta_{r+1}Ry + a_6\beta_{r+1}Rx \otimes \beta_{r+1}4_* Qy \\ &\quad + a_7\beta_{r+1}4_* Qx \otimes \beta_{r+1}Ry + a_8\beta_{r+1}4_* Qx \otimes \beta_{r+1}4_* Qy \end{aligned}$$

with $a_1, a_2 \in Z_{2^{r+1}}$ and the remaining a_i in $Z_{2^{r-1}}$. The equation

(18) $$\pi\delta_* \ (x \otimes y) = \delta_* \iota_*(x \otimes y)^{(2)} = \iota_* x^{(2)} \otimes \iota_* y^{(2)} = 2^{2r-4}\beta_r 2_* Qx \otimes \beta_r 2_* Qy$$

shows that $a_6 = a_7 = 0$, $a_8 = 2^{2r-4}$, and also that $a_1 \equiv 0 \bmod 2^r$ and that $a_2 \equiv a_3 \equiv a_4 = 0 \bmod 2^{r-2}$. Next we apply $(D_2 j \wedge D_2 j)_*$ to both sides of (17). The left side becomes

$$(D_2 j \wedge D_2 j)_* \delta_* \mathcal{Q}(x \otimes y) = \delta_* \mathcal{Q}(\Sigma u \otimes \Sigma u) = \pi\delta_* \mathcal{Q}(\Sigma u \otimes \Sigma u),$$

which is zero by (18). By 8.4(ii) we have

$$(D_2 j)_* R\Sigma u_r = R\Sigma u = R\pi\Sigma u = 2Q\Sigma u,$$

hence (since $8a_1 \equiv 8a_2 \equiv 8a_3 \equiv 0 \bmod 2^{r+1}$) the right side of (17) becomes $4a_4 Q\Sigma y \times Q\Sigma u$, so that $a_4 = 0$ in $Z_{2^{r-1}}$. Next we calculate

$$\pi\beta_{r+1}\delta_* \mathcal{Q}(x \otimes y) = 2^{r-2}\pi\beta_{r+1}[Rx \otimes 4_* Qy + 4_* Qx \otimes Ry],$$

hence $a_2 = a_3 = 2^{r-2}$. Finally, if we expand both sides of the equation

$$(\delta \wedge 1)_* \delta_* \mathcal{Q}((\Sigma u_r \otimes \Sigma u_r) \otimes u_r) = (1 \wedge \delta)_* \delta_* \mathcal{Q}(\Sigma u_r \otimes (\Sigma u_r \otimes u_r))$$

using the formulas already shown, it follows that $a_5 = 0$. The proof when $r = 1$ is similar.

(vii). We may assume $x = u_r$. Let $r \geq 2$; the case $r = 1$ is similar. Then

(19) $$\Delta_* \Sigma \mathcal{Q} u_r = a_1 R\Sigma u_r + a_2 p_*^2 Q\Sigma u_r$$

with $a_1 \in Z_{p^{r+1}}$ and $a_2 \in Z_{p^{r-1}}$. Applying π to (19) shows that $a_1 \equiv 0 \bmod p^r$, hence applying $(D_p j)_*$ to (19) gives $a_2 = 0$. It only remains to show that $\Delta_* \Sigma \mathcal{Q} u_r \neq 0$ when $p = 2$. But Lemma 7.7 gives the exact sequence

$$K_1(\Sigma M_r \wedge M_r; r+1) \xrightarrow{(\Sigma\iota)_*} K_1(\Sigma D_2 M_r; r+1) \xrightarrow{\Delta_*} K_1(D_2 \Sigma M_r; r+1).$$

Since $\Sigma \mathcal{Q} u_r$ has order 2^{r+1}, it cannot be in the image of $(\Sigma\iota)_*$ and the result follows.

(viii). We may assume $x = u_r$. We have

$$(D_p j)_* \psi^k \mathcal{Q} u_r = \psi^k \mathcal{Q} u = \psi^k \eta = \eta = (D_p j)_* \mathcal{Q} \psi^k u_r$$

since $\psi^k u_r = u_r$; the result follows by 8.2.

(ix) By equation (7) in the proof of 8.4(viii) and I.2.14 we have the following equation in $K_0(D_{p^2} X; r-1)$ when p is odd and $r \geq 2$.

$$(20) \qquad \beta_* Q \iota_* x^{(p)} = \iota_* \delta_* Q x^{(p)} = \iota_* [\sum_{i=1}^{p} \binom{p}{i} p^{i-1} (\pi \iota_* x^{(p)})^{(p-i)} \otimes (Qx)^{(i)}].$$

When $p = 2$ this equation follows from 7.6(vii) since $\iota_*(x \otimes \beta_r x \otimes x \otimes \beta_r x)$ and $\iota_*(Q\beta_r x \otimes Q\beta_r x)$ are zero by 7.6(x).

Let $r = 1$, $x = u_1$. The set $\{\mathcal{Q} u_1\}$ is a subbasis for $D_p M_1$, hence by 4.3 the set $\{Q \mathcal{Q} u_1, \iota_* u_1^{(p^2)}\}$ is a basis for $K_0(D_p D_p M_1;1)$. Lemma 4.3 also implies that the set

$$\{Q \mathcal{Q} u, \iota_* u^{(p^2)}\} \subset K_0(D_p D_p S;1)$$

is linearly independent. Hence $(D_p D_p j)_*$ is monic on $K_0(D_p D_p M_r;1)$. Since the transfer

$$\tau_* : K_0(D_{p^2} M_1;1) \to K_0(D_p D_p M_1;1)$$

is monic and $(D_p D_p j)_* \circ \tau = \tau_* \circ (D_{p^2} j)_*$, it follows that $(D_{p^2} j)_*$ is monic on $K_0(D_{p^2} M_1;1)$. But

$$(D_{p^2} j)_* \beta_* Q \mathcal{Q} u_1 = \beta_* Q \mathcal{Q} u = \beta_* Q i_* u^{(p)},$$

which is zero by (20), hence $\beta_* Q \quad u_1 = 0$ as required.

Next let $r \geq 2$ and let y denote the element

$$\beta_* Q \mathcal{Q} u_r - \iota_* \sum_{i=1}^{p} \binom{p}{i} p^{i-2} [\iota_* u_r^{(p)}]^{(p-i)} \otimes p_*[(Qu_r)^{(i)}]$$

in $K_0(D_{p^2} M_r;r)$. Then (20) implies that $\pi y = 0$ and $(D_{p^2} j)_* y = 0$, and we must show $y = 0$. Since $\pi y = 0$ we see that y is in the image of p_*^{r-1}. To proceed further we need the case $k = p^2$ of 4.1; we may use this result without circularity since only the case $r = 1$ of the present result is used in proving it (see section 9). Now as in the proof of 8.4(viii) we see that the union of the sets

$$\{\iota_*[(\pi^{r-1} \iota_* u_r^{(p)}]^{(i)} \otimes (\pi^{r-2} Q u_r)^{(p-i)}] \mid 0 \leq i \leq p\}$$

$$\{\iota_*[(\pi^{r-1} \iota_* u_r^{(p)})^{(i-1)} \otimes \pi^{r-1} \iota_*(u_r^{(p-1)} \otimes \beta_r u_r)$$

$$\otimes (\pi^{r-2} Q u_r)^{(p-i-1)} \otimes \pi^{r-2} \beta_{r-1} Q u_r] \mid 1 \leq i \leq p-1\}$$

and, if $r \geq 3$, $\{\pi^{r-3} \beta_* Q Q u_r\}$, is a basis for $K_0(D_{p^2} M_r;1)$. The second of these sets generates the kernel of p_*^{r-1} and also the kernel of $(D_{p^2} j)_*$, and it follows that $(D_{p^2} j)_*$ is monic on the image of p_*^{r-1}. Since $(D_{p^2} j)_* y = 0$ we conclude $y = 0$ as required.

9. Cartan formulas

In this section we shall prove Lemma 4.7. As in the proof of 2.7, the basic idea is to "simplify" each expression in $C\mathcal{A}$ (respectively $C\mathcal{A}'$) to obtain an expression in $C\{x\}$ (respectively $C\{y,z\}$). We shall refer to the simplified expression as a Cartan formula for the original one. Some explicit examples of such formulas will be given below. However, some of the formulas we need are too complicated to give explicitly, and instead we shall use an inductive argument to establish their existence.

In order to do so it is convenient to work in a suitable formal context. Let $\xi_1,\dots,\xi_t$ be indeterminates and suppose that to each has been assigned a mod 2 dimension denoted $|\xi_i|$ and two positive integers called the height and filtration and denoted $\|\xi_i\|$ and $\nu\xi_i$. Intuitively, ξ_i should be thought of as an element of $K_{|\xi_i|}(D_{\nu\xi_i}X;\|\xi_i\|)$ for some spectrum X. We wish to consider certain finite formal combinations $E(\xi_1,\dots,\xi_t)$ involving the ξ_i and the operations of section 3, namely those combinations which would represent elements in one of the groups $K_\alpha(D_jX;r)$ when interpreted "externally" as in section 4. More precisely, we define the allowable expressions $E(\xi_1,\dots,\xi_t)$ and assign them dimensions, heights and filtration by induction on their length as follows.

Definition 9.1. (i) Each indeterminate ξ_i is an expression of length 1. For each $\alpha\in Z_2$, $r\geq 1$, $j\geq 1$ there is an expression $0_{\alpha,r,j}$ (called zero sub α,r,j) having length 1, dimension α, height r and filtration j. These are the only expressions of length 1.

(ii) Suppose that the expressions of length $\leq \ell$ have been defined and assigned dimensions, heights and filtrations. The expressions of length $\ell+1$ are the following, where E ranges over the expressions of length ℓ.

(a) p_*E. We define $|p_*E| = |E|$, $\|p_*E\| = \|E\| + 1$ and $\nu(p_*E) = \nu E$.

(b) β_rE if $\|E\| = r$. We define $|\beta_rE| = |E|-1$, $\|\beta_rE\| = \|E\|$ and $\nu(\beta_rE) = \nu E$.

(c) πE if $2\leq\|E\|$. We define $|\pi E| = |E|$, $\|\pi E\| = \|E\|-1$ and $\nu(\pi E) = \nu E$.

(d) $E_1 + E_2$, where E_1 and E_2 are any expressions whose lengths add up to $\ell+1$ and which satisfy $|E_1| = |E_2|$, $\|E_1\| = \|E_2\|$, and $\nu E_1 = \nu E_2$. We define $|E_1 + E_2| = |E_1|$, $\|E_1 + E_2\| = \|E_1\|$ and $\nu(E_1 + E_2) = \nu E_1$.

(e) $E_1\cdot E_2$ (the formal product) where E_1 and E_2 are any expressions whose lengths add up to $\ell+1$ and which satisfy $\|E_1\| = \|E_2\|$. We define $|E_1\cdot E_2| = |E_1| + |E_2|$, $\|E_1\cdot E_2\| = \|E_1\|$, and $\nu(E_1\cdot E_2) = \nu E_1 + \nu E_2$.

(f) QE if $2\leq\|E\|$. We define $|QE| = |E|$, $\|QE\| = \|E\|-1$ and $\nu QE = p\nu E$.

(g) $\mathcal{Q}E$ if $|E| = 0$. We define $|\mathcal{Q}E| = 0$, $\|\mathcal{Q}E\| = \|E\|+1$, and $\nu\mathcal{Q}E = p\nu E$.

(h) RE if $|E| = 1$. We define $|RE| = 1$, $\|RE\| = \|E\|+1$, and $\nu RE = p\nu E$.

Note that we have not required formal addition and multiplication to satisfy commutativity, associativity or other properties. However, in writing down particular expressions we shall often omit some of the necessary parentheses, since their precise position will usually be irrelevant. We shall also abbreviate $0_{\alpha,r,j}$ by 0.

We have given Definition 9.1 in complete detail as a pattern for other inductive definitions about which we will not be so scrupulous. For example, let E be an expression in the indeterminates $\xi_1,\ldots,\xi_t$. If $E_1,\ldots,E_t$ are expressions in another set of indeterminates $\eta_1,\ldots,\eta_s$ with $|E_i| = |\xi_i|$, $\|E_i\| = \|\xi_i\|$, and $\nu E_i = \nu u_i$ for $1 \leq i \leq t$ then we may (inductively) define the composite expression $E(E_1,\ldots,E_t)$ in $\eta_1,\ldots,\eta_s$. Again, if X is any spectrum and $x_i \in K_{|\xi_i|}(D_{\nu\xi_i}X;\|\xi_i\|)$ for $1 \leq i \leq t$ then we can define

$$E(x_1,\ldots,x_t) \in K_{|E|}(D_{\nu E}X;\|E\|)$$

as in section 4 by interpreting Q, $\mathcal{Q}$, R and the multiplication externally and applying α_* and β_* to formal products and composites.

<u>Definition 9.2</u>. Let $\xi_1,\ldots,\xi_t$ be a fixed set of indeterminates. <u>Equivalence</u>, denoted by $\sim$, is the smallest equivalence relation on the set of expressions in $\xi_1,\ldots,\xi_t$ which satisfies the following.

(1) $\sim$ is preserved by left composition with Q, $\mathcal{Q}$, R, π, p_* and β_r and by formal addition and multiplication.

(2) For each $r \geq 1$ the equivalence classes of expressions of height r, graded by dimension and filtration, form a $Z_2 \times Z$ graded ring (without unit) with the $0_{\alpha,r,j}$ as zero elements. The relation $E_1 \cdot E_2 = (-1)^{|E_1||E_2|} E_2 \cdot E_1$ is satisfied and left composition with π, β_r or p_* is additive.

(3) If x and y denote expressions E_1 and E_2 having height r and the required dimensions then the following hold with = replaced by $\sim$: 3.1; 3.2(iii),(iv) and (v); 3.3(iii), (iv), (v), (vi), (vii) and (x); 3.6(ii), (iii), (iv), (v) and (viii); 3.7(ii), (iii), (iv), (v), (vi) and (ix).

Roughly speaking, two expressions are equivalent if one can be transformed into the other by using the relations of Section 3.

It is easy to see that equivalent expressions must have the same dimension, height, and filtration but not necessarily the same length. An inductive argument shows that $E(E_1,\ldots,E_t)$ and $E'(E_1',\ldots,E_t')$ are equivalent if $E \sim E'$ and $E_i \sim E_i'$

for $1 \leq i \leq t$. A similar inductive argument using 3.1, 3.2, 7.6, 8.4 and 8.5 gives the following.

Lemma 9.3. Let E and E' be equivalent expressions in $\xi_1,\ldots,\xi_t$. Let X be any spectrum and let x_i be an element of $K_{|\xi_i|}(D_{\nu\xi_i}X;\|\xi_i\|)$, for $1 \leq i \leq t$. Then $E(x_1,\ldots,x_t) = E'(x_1,\ldots,x_t)$.

If $A = \{\xi_1,\ldots,\xi_t\}$ is any set of indeterminates we can define the filtered algebra CA and the subquotient groups D_jA with their standard bases exactly as in sections 3 and 4. If A' is another set of indeterminates and $f:A \to A' \cup \{0\}$ preserves degree, height and filtration we say that f is subbasic. Clearly, the constructions CA and D_jA are functorial with respect to subbasic maps. We can think of the elements of D_jA as expressions in $\xi_1,\ldots,\xi_t$ by inserting parentheses so that addition and multiplication are treated as binary operations. (Of course, up to equivalence it doesn't matter how the parentheses are inserted.) This identifies D_jA with a subset of the expressions of height 1 and filtration j in $\xi_1,\ldots,\xi_t$. By a Cartan formula for an expression E of height 1 we mean simply an equivalent expression in $D_{\nu E}A$. The next result, which will be proved later in this section, provides some examples which will be useful in the proof of 4.7. We say that two expressions E_1 and E_2 are equivalent mod p if there is an expression E' with $E_1 \sim E_2 + pE'$; in particular this implies $\pi^{\|E_1\|-1}E_1 \sim \pi^{\|E_1\|-1}E_2$.

Proposition 9.4. Let ξ_1, ξ_2, ξ_3, ξ_4 be indeterminates of height r with dimensions 0, 0, 1, 1 respectively. Let $1 \leq s < r$ and let $t \geq 1$.

(i) $\beta_{r-s}Q^s\xi_1 \sim Q^s\beta_r\xi_1$ mod p.

(ii) $\beta_{r-s}Q^s\xi_3 \sim (\pi^s\beta_r\xi_3)^{p^s}$ mod p.

(iii) $Q^s(\xi_1\xi_3) \sim (\pi^s\xi_1)^{p^s}Q^s\xi_3$ mod p if p is odd or $r \geq 3$.

(iv) $Q^s(\xi_3\xi_4)$ is equivalent to $(Q^s\xi_3)(Q^s\xi_4)$ if p is odd and to

$$(Q^s\xi_3)(Q^s\xi_4) + 2^{r-s-1}(\pi Q^{s-1}\xi_3)(\pi^s\beta_r\xi_3)^{2^{s-1}}(\pi Q^{s-1}\xi_4)(\pi^s\beta_r\xi_4)^{2^{s-1}}$$

if $p = 2$ and $r \geq 3$.

(v) $Q^s(\xi_1\xi_3\xi_4) \sim (\pi^s\xi_1)^{p^s}(Q^s\xi_3)(Q^s\xi_4)$ if p is odd.

(vi) If $1 \leq i \leq p-1$ then

$$\beta_{r-s}Q^s(\xi_1^i\xi_2^{p-i}) \sim i(\beta_{r-s}Q^s\xi_1)(\pi^s\xi_1)^{p^s(i-1)}(\pi^s\xi_2)^{p^s(p-i)}$$
$$- i(\pi^s\xi_1)^{ip^s}(\beta_{r-s}Q\xi_2)(\pi^s\xi_2)^{p^s(p-i-1)} \text{ mod } p$$

(vii) If $1 \leq i \leq p-1$ then $\pi^{r+t-1}\beta_{r+t}R^t[(\beta_r\xi_1)\xi_1^{i-1}\xi_2^{p-i}]$ is equivalent to

$$i(\pi^{r-1}\xi_1)^{(i-1)p^t}(\pi^{r-t-1}\beta_{r-t}Q^t\xi_1)(\pi^{r-1}\xi_2)^{(p-i-1)p^t}(\pi^{r-t-1}\beta_{r-t}Q^t\xi_2)$$

if $t < r$ and to zero otherwise.

(viii) $\beta Q^{r}\mathfrak{2}\,\xi_1 \sim 0$.

(ix) If $s \leq t$ then $Q^s p^t \xi_1$ is equivalent mod p^{t-s+2} to

$$p^{t-s+1}(\pi^{s-1}Q\xi_1)^{p^{s-1}} + c_1 p^{t-s}(\pi^s\xi_1)^{p^s},$$

where

$$c_1 = \begin{cases} 1 \text{ if } p \text{ is odd or } s < t \\ -1 \text{ if } p = 2 \text{ and } s = t. \end{cases}$$

$Q^s p^{s-1}\xi_1$ is equivalent mod p to

$$(\pi^{s-1}Q\xi_1)^{p^{s-1}} + c_2(\pi^s\xi_1)^{p^s},$$

where

$$c_2 = \begin{cases} 0 \text{ if } p \text{ is odd} \\ 1 \text{ if } p = 2. \end{cases}$$

There remain expressions, such that $Q^{r}\mathfrak{2}\,\xi_1$, for which the Cartan formula is too complicated to give explicitly. Our next result will guarantee the existence of such formulas. Let $A = \{\xi_1,\ldots,\xi_t\}$. We say that an element of D_jA is homogeneous if it is a sum of standard basis elements each of which involves every ξ_i. Note that such elements are in the kernel of D_jf whenever $f:A \to A'\cup\{0\}$ takes at least one ξ_i to 0.

Proposition 9.5. Any expression E of height 1 in $\xi_1,\ldots,\xi_t$ is equivalent to an expression in D_jA for some j. If the ξ_i have height r and degree 0 then the expression $\pi^{r-s-1}Q^s(\xi_1\cdots\xi_t)$ is equivalent to a homogeneous expression in D_jA for each $s < r$. If the ξ_i have height r and degree 1 then $\pi^{r+s-1}\beta_{r+s}R^s(\xi_1(\beta_r\xi_2)\cdots(\beta_r\xi_t))$ is equivalent to a homogeneous expression in D_jA for each $t \geq 0$.

The proof of 9.5 will be given at the end of this section. Unfortunately, there seems to be no direct algebraic proof that the Cartan formulas provided by 9.5 are unique, that is, that distinct elements of D_jA cannot be equivalent as expressions. If we had uniqueness in this sense then Lemma 4.7 would be an immediate consequence of 9.5. Instead we shall have to give a much more elaborate construction of γ_j and γ_j', making use of the explicit formulas of 9.4 in order to avoid appealing to uniqueness. (A similar difficulty in ordinary homology is implicit in our proof of 2.7). On the other hand, it is easy to see from 4.1 and 9.3 that uniqueness does hold, but of course such an argument cannot be used in proving 4.7. However, we can and shall use uniqueness in filtrations less than k in the following inductive proof of 4.7.

Proof of 4.7. We shall give the proof for $r < \infty$. The case $r = \infty$, which is similar and somewhat easier, requires some straightforward modifications in Definition 9.1 to allow for infinite heights; details are left to the reader.

First let $M = M_r$ with $r \geq 2$ (the $r = 1$ case is similar and easier). We define $\mathcal{A}$ to be $\{Qx, \mathcal{Q}x\}$. Let u_m and v_m respectively denote $y^m z^{p-m}$ and $(\beta_r y)y^{m-1}z^{p-m}$ for $1 \leq m \leq p-1$ and define $\mathcal{A}'$ to be

$$\{Qy, Qz, \mathcal{Q}y, \mathcal{Q}z\} \cup \{u_m \mid 1 \leq m \leq p-1\} \cup \{v_m \mid 1 \leq m \leq p-1\}.$$

Lemma 4.3 implies that $\mathcal{A}$ and $\mathcal{A}'$ are in fact subbases for $D_p M_r$ and $D_p(M_r \vee M_r)$. Note that $(D_p g_1)_*$ takes Qy and $\mathcal{Q}y$ to Qx and $\mathcal{Q}x$ and takes all other elements of $\mathcal{A}'$ to zero. In particular $(D_p g_1)_*\colon \mathcal{A}' \to \mathcal{A} \cup \{0\}$ is a subbasic map and hence $F_1 = D_j(D_p g_1)_*$. Similarly, $F_2 = D_j(D_p g_2)_*$. On the other hand, $(D_p g_0)_*$ is not subbasic since it takes u_m to $\pi\mathcal{Q}x$ and v_m to $\pi\beta_{r+1}\mathcal{Q}x$, hence F_0 is not induced by functoriality from $(D_p g_0)_*$. It is determined by $(D_p g_0)_*$, however, in the following way. If

$$E(Qy, Qz, \mathcal{Q}y, \mathcal{Q}z, u_1, \ldots, u_{p-1}, v_1, \ldots, v_{p-1})$$

is any expression in $D_j\mathcal{A}'$ and E' is an expression in $D_j\mathcal{A}$ equivalent to

$$E(Qx, Qx, \mathcal{Q}x, \mathcal{Q}x, \pi\mathcal{Q}x, \ldots, \pi\mathcal{Q}x, \pi\beta_{r+1}\mathcal{Q}x, \ldots, \pi\beta_{r+1}\mathcal{Q}x)$$

then by 9.3 we have $\lambda_j(F_0(E)) = \lambda_j(E')$, hence $F_0 E = E'$.

Next we shall construct γ_j and γ_j'. We assume inductively that γ_ℓ and γ_ℓ' with the required properties have been constructed for all $\ell < j$. By using the values of γ_ℓ and γ_ℓ' on indecomposables and extending multiplicatively, we can define γ_j and γ_j' on the decomposables of $D_j\mathcal{A}$ and $D_j\mathcal{A}'$ so that the diagram commutes when restricted to decomposables. It remains to define γ_j and γ_j' on the standard indecomposables of $D_j\mathcal{A}$ and $D_j\mathcal{A}'$. We may assume that $j = p^s$ for some s, since otherwise there are no indecomposables in filtration j.

Let $\xi_1, \ldots, \xi_p$ be indeterminates with dimension zero, height r, and filtration 1. If $s < r$ we use 9.5 to choose a homogeneous expression E in $D_k\{\xi_1, \ldots, \xi_p\}$ equivalent to $\pi^{r-s-1}Q^s(\xi_1 \cdots \xi_p)$. If $s = r$, let E be an expression in $D_k\{\xi_1, \ldots, \xi_p\}$ equivalent to $Q^r\mathcal{Q}\xi_1$. We define subbasic maps

$$f_m:\{\xi_1,\ldots,\xi_p\} \to A' \cup \{0\}$$

for $0 \leq m \leq p$ by

$$f_m(\xi_\ell) = \begin{cases} y & \text{for } \ell < m \\ z & \text{for } \ell > m\ . \end{cases}$$

Finally, we define $h:\{\xi_1,\ldots,\xi_p\} \to A$ by $h(\xi_\ell) = x$ for all ℓ. Note that $(g_0)_* \circ f_m = h$ for all m.

We define γ_j and $\gamma_j^!$ on indecomposables in table 1. The first column lists the standard indecomposables in $D_j\mathcal{A}'$, and the second column (we claim) gives the value of F_0 on each. The first four entries in column 2 are precisely the standard indecomposables in $D_j\mathcal{A}$, and the corresponding entries in column 3 define γ_j on each. The remaining entries in column 3 then give the resulting values of γ_j on the other entries of column 2. Finally, column 4 defines $\gamma_j^!$ on each entry in column 1. Note that we have denoted iterates of π in the table simply by π; the precise iterate intended can easily be determined since all entries in the table are to have height 1.

The values of F_0 claimed in column 2 are either obviously correct or follow easily from 9.4 or the formulas of section 3. For example, in line 10 we have

$$\pi^{r-s}\beta_{r-s+1}Q^s\pi\mathfrak{L}x \sim \pi^{r-s}\beta_{r-s+1}\pi Q^s\mathfrak{L}x \sim p\pi^{r-s+1}\beta_{r-s+2}Q^s\mathfrak{L}x \sim 0$$

and in line 12 we have

$$\pi^{r+s-1}\beta_{r+s}R^s\pi\beta_{r+1}\mathfrak{L}x \sim \pi^{r+s-1}Q^s\beta_{r+2s}P_*^{2s}\pi\beta_{r+1}\mathfrak{L}x \sim 0.$$

Table 1

		F_0	$\gamma_j \circ F_0$	γ_j'
1.	$\pi Q^s(\mathcal{Q} y)$	$\pi Q^s(\mathcal{Q} x)$	$(D_k h)(E)$	$(D_k f_p)(E)$
2.	$\pi\beta_{r-s+1}Q^s(\mathcal{Q} y)$	$\pi\beta_{r-s+1}Q^s(\mathcal{Q} x)$	$\begin{cases}(\pi x)^{(p-m-1)j}\,\pi\beta_{r-s}Q^s x & \text{if } s < r\\ 0 & \text{if } s = r\end{cases}$	$\begin{cases}(\pi y)^{(p-m-1)j}\,\pi\beta_{r-s}Q^s y & \text{if } s < r\\ 0 & \text{if } s = r\end{cases}$
3.	$\pi Q^s(Qy)$	$\pi Q^s(Qx)$	$\pi Q^{s+1}x$	$\pi Q^{s+1}y$
4.	$\pi\beta_{r-s-1}Q^s(Qy)$	$\pi\beta_{r-s-1}Q^s(Qx)$	$\pi\beta_{r-s-1}Q^{s+1}x$	$\pi\beta_{r-s-1}Q^{s+1}y$
5.	$\pi Q^s(\mathcal{Q} z)$	$\pi Q^s(\mathcal{Q} x)$	$(D_k h)(E)$	$(D_k f_0)(E)$
6.	$\pi\beta_{r-s+1}Q^s(\mathcal{Q} z)$	$\pi\beta_{r-s+1}Q^s(\mathcal{Q} x)$	same as line 2	$\begin{cases}(\pi z)^{(p-m-1)j}\,\pi\beta_{r-s}Q^s z & \text{if } s < r\\ 0 & \text{if } s = r\end{cases}$
7.	$\pi Q^s(Qz)$	$\pi Q^s(Qx)$	$\pi Q^{s+1}x$	$\pi Q^{s+1}z$
8.	$\pi\beta_{r-s-1}Q^s(Qz)$	$\pi\beta_{r-s-1}Q^s(Qx)$	$\pi\beta_{r-s-1}Q^{s+1}x$	$\pi\beta_{r-s-1}Q^{s+1}z$
9.	$\pi Q^s u_m$	$\pi Q^s(\mathcal{Q} x)$	$(D_k h)(E)$	$(D_k f_m)(E)$
10.	$\pi\beta_{r-s}Q^s u_m$	0	0	$m(\beta_{r-s}Q^s y)(\pi y)^{(m-1)j}(\pi z)^{(p-m)j}$ $-m(\pi y)^{mj}(\beta_{r-s}Q^s z)(\pi z)^{(p-m-1)j}$
11.	$\pi Q^s v_m$	$\pi\beta_{r-s+1}Q^s(\mathcal{Q} x)$	$(\pi x)^{(p-m-1)j}\,\pi\beta_{r-s}Q^s x$	$(\pi y)^{(m-1)j}(\pi z)^{(p-m)j}\,\pi\beta_{r-s}Q^s y$
12.	$\pi\beta_{r+s}R^s v_m$	0	0	$m(\pi y)^{(m-1)j}(\pi z)^{(p-m-1)j}(\pi\beta_{r-s}Q^s y)(\pi\beta_{r-s}Q^s z)$ if $s < r$, 0 otherwise

The listed generators occur only for certain values of s. In lines 1,2,5 and 6 we require $s \leq r$; in lines 9, 10, and 11, $s \leq r-1$; and in lines 3,4,7 and 8, $s \leq r-2$.

To complete the proof of 4.7 for $M = M_r$ it remains to show that diagram (*) of section 4 commutes for i = 0, 1, 2. In order to see that the inner square commutes it suffices, by Lemma 9.3, to show that the first four entries in columns 2 and 3 are equivalent as expressions in x. This is clear for lines 1, 3 and 4 and for line 2 if s = r (by 9.4(viii)). If s < r in line 2 we have

$$\pi^{r-s}\beta_{r-s+1}Q^s(\mathcal{Q}x) \sim \pi^{r-s-1}Q^s\pi\beta_{r+1}\mathcal{Q}x \sim \pi^{r-s-1}Q^s(x^{p-1}\beta_r x)$$

which is equivalent to the required formula by 9.4(iii).

To see that the outer square commutes, we must show that the entries in columns 1 and 4 are equivalent as expressions in y and z. The first eight cases follow as in the preceding paragraph. Line 9 follows from the definition of E, line 10 from 9.4(vi), line 11 from 9.4(iii), and line 12 from 9.4(vii).

For commutativity of the upper trapezoid when i = 1, we must show that $D_k(g_1)_*$ takes the first four entries in column 4 to the corresponding entries in column 3 (which is obvious) and takes the remaining entries in column 4 to zero. This follows in line 9 from the fact that E is homogeneous (since $(g_1)_* \circ f_m$ takes at least one ξ_ℓ to zero if $1 \le m \le p-1$) and the remaining cases are clear. Similarly, we see that the upper trapezoid commutes when i = 2. Finally, we observe that each entry of column 4 goes to the corresponding entry of column 3 under $D_k(g_0)_*$, and hence the upper trapezoid commutes when i = 0. This completes the proof of 4.7 for $M = M_r$.

Next suppose $M = \Sigma M_r$. We define $\mathcal{A} = \{Rx\}$ when r = 1 and $\mathcal{A} = \{Qx, Rx\}$ when $r \ge 2$. Let $u_m = y(\beta_r y)^{m-1}(\beta_r z)^{p-m}$ and $v_m = y(\beta_r y)^{m-1}z(\beta_r z)^{p-m-1}$ for $1 \le m \le p-1$. We define

$$\mathcal{A}' = \{Ry, Rz\} \cup \{u_m | 1 \le m \le p-1\} \cup \{v_m | 1 \le m \le p-1\}$$

when r =1 and

$$\mathcal{A}' = \{Qy, Qz, Ry, Rz\} \cup \{u_m | 1 \le m \le p-1\} \cup \{v_m | 1 \le m \le p-1\}$$

when $r \ge 2$.

Then $(D_p g_1)_*$ and $(D_p g_2)_*$ induce subbasic maps from $\mathcal{A}'$ to $\mathcal{A}$ and we therefore have $F_i = D_j(D_p g_i)_*$ if i = 1 or 2. The map $(D_p g_0)_*$ takes u_m to $-\pi Rx$ when r = 1 and to $p_*Qx - \pi Rx$ when $r \ge 2$. it takes v_m to zero when p is odd. When p = 2, 3.3(x) implies

$$(D_p g_0)_* v_m = \begin{cases} Q\beta_2 2_* x & \text{uf } r = 1 \\ 2^{r-2}\beta_r 2_* Qx & \text{if } r \ge 2 \end{cases}$$

We begin with the case r = 1. We define γ_j and γ_j' on decomposables by inductive hypothesis as in the $M = M_r$ case. To define γ_j and γ_j' on indecomposables we use Table 2.

Table 2

		F_0	$\gamma_j \circ F_0$	γ_j'
1.	$Q(Ry)$	$Q(Rx)$	0	0
2.	$\pi\beta_{s+2}R^s(Ry)$	$\pi\beta_{s+2}R^s(Rx)$	$\pi\beta_{s+2}R^{s+1}x$	$\pi\beta_{s+2}R^{s+1}y$
3.	$Q(Rz)$	$Q(Rx)$	0	0
4.	$\pi\beta_{s+2}R^s(Rz)$	$\pi\beta_{s+2}R^s(Rx)$	$\pi\beta_{s+2}R^{s+1}x$	$\pi\beta_{s+2}R^{s+1}z$
5.	$\pi\beta_{s+1}R^s u_m$	$F_0(\pi\beta_{s+1}R^s u_m)$	0	0

Here the first column lists the indecomposables of $D_j\mathcal{A}'$ and the second column (we claim) gives the value of F_0 each (note that lines 1 and 3 are relevant only when s = 1, i.e., when $k = p^2$). The first two entries in column 2 are the indecomposables of $D_j\mathcal{A}$, and the corresponding entries in column 3 give our definition of γ_j on each, while the remaining entries in column 3 are claimed to be values of γ_j determined by the definition we have just given. The entries in column 4 define γ_j' on indecomposables. The necessary verifications are similar to those in the case $M = M_r$, and they are straightforward except in line 5. Here we must show that that $\gamma_j F_0(\pi^s\beta_{s+1}R^s u_m)$ is equal to zero and that $\pi^s\beta_{s+1}R^s(y(\beta y)^{m-1}(\beta z)^{p-m})$ is equivalent to zero as an expression in y and z. For simplicity we assume that p is odd -- the case p = 2 differs only slightly. First recall that to calculate $F_0(\pi^s\beta_{s+1}R^s u_m)$ we need only find an element of $D_j\mathcal{A}$ which is equivalent to $-\pi\beta_{s+1}R^s\pi(Rx)$ as an expression in the indeterminate Rx. Now

$$-\pi^s\beta_{s+1}R^s\pi(Rx) \sim -\pi^s Q^s\beta_{2s+1}p_*^{2s}\pi(Rx) \qquad \text{by 3.6(iv)}$$

$$\sim -Q^s p(\beta_{s+1}p_*^{s-1}(Rx)).$$

We see by induction on t using (3.3(vi) and 3.3(vii) that Q^t of a multiple of p is equivalent to a sum of terms each of which has either p or a p-th power as a factor. Hence $F_0(\pi^s\beta_{s+1}R^s u_m)$ is a sum of terms each of which has a p-th power factor, and the same is true for the element $\gamma_j F_0(\pi^s\beta_{s+1}R^s u_m)$ of $D_k\{x\}$. But by definition all p-th powers in $C\{x\}$ are zero when r = 1, so that $\gamma_j F_0(\pi^s\beta_{s+1}R^s u_m) = 0$ as required. The proof that $\pi^s\beta_{s+1}R^s(y(\beta y)^{m-1}(\beta z)^{p-m})$ is equivalent to zero is similar. We have

$$\pi^s\beta_{s+1}R^s(y(\beta y)^{m-1}(\beta z)^{p-m}) \sim \pi^s Q^s\beta_{2s+1}p_*^{2s}(y(\beta y)^{m-1}(\beta z)^{p-m})$$

$$\sim Q^s((\beta_{s+1}p_*^s y)^m(\beta_{s+1}p_*^s z)^{p-m}),$$

and 3.3(vi) and 3.3(vii) show that Q^t of a product of elements of degree zero is equivalent to a sum of terms each of which has either p or a p-th power as a factor. But again p-th powers in $C\{y,z\}$ are zero and we see that $\pi^s\beta_{s+1}R^s(y(\beta y)^{m-1}(\beta z)^{p-m}) \sim 0$ as required. This completes the proof of Lemma 4.7 for $M = \Sigma M_1$.

Next let $r \geq 2$. We can define γ_j and γ_j' on decomposables precisely as before. In defining γ_j and γ_j' on indecomposables when $r \geq 2$, it will be convenient to modify the standard basis we have been using as follows. Let η_1 and η_2 be indeterminates with dimension 1, filtration p and heights $\|\eta_1\| = r-1$, $\|\eta_2\| = r+1$. We use 9.5 to obtain an expression $E(\eta_1,\eta_2)$ in $D_j\{\eta_1,\eta_2\}$ equivalent to $\pi^{r+s-1}\beta_{r+s}R^s(p_*\eta_1 - \pi\eta_2)$. We claim that the coefficient of $\pi^{r+s-2}\beta_{r+s-1}R^s\eta_1$ in $E(\eta_1,\eta_2)$ is 1. To see this, write $E(\eta_1,\eta_2)$ as $E_1 + E_2$, where E_1 involves only η_1 and every standard basis element in E_2 involves η_2. If $f:\{\eta_1,\eta_2\} \to \{\eta_1\} \cup \{0\}$ takes η_1 to itself and η_2 to zero then $(D_jf)(E(\eta_1,\eta_2)) = E_1$. On the other hand,

$$(D_jf)(E(\eta_1,\eta_2)) \sim E(\eta_1,0) \sim \pi^{r+s-1}\beta_{r+s}R^s p_*\eta_1 \sim \pi^{r+s-2}\beta_{r+s-1}R^s\eta_1.$$

Since uniqueness holds (by inductive hypothesis) in filtration j we have

$$E_1 = \pi^{r+s-2}\beta_{r+s-1}R^s\eta_1 ,$$

proving the claim. We can therefore give new bases for the indecomposables of D_j and $D_j\mathcal{A}'$ when $r \geq 2$ by replacing $\pi^{r+s-2}\beta_{r+s-1}R^s(Qx)$, $\pi^{r+s-2}\beta_{r+s-1}R^s(Qy)$ and $\pi^{r+s-2}\beta_{r+s-1}R^s(Qz)$ in the standard bases by $E(Qx,Rx)$, $E(Qy,Ry)$ and $E(Qx,Rz)$ respectively.

Next let $\xi_1,\dots,\xi_p$ be indeterminates with dimension 1, height r and filtration 1. We use 9.5 to choose a homogeneous expression $E'(\xi_1,\dots,\xi_p)$ in $D_k\{\xi_1,\dots,\xi_p\}$ equivalent to

$$\pi_{r+s-1}\beta_{r+s}R^s(\xi_1(\beta_r\xi_2)\cdots(\beta_r\xi_p)).$$

Finally, we define the subbasic maps f_m and h exactly as in the case $M = M_r$.

We can now define γ_j and γ_j' on indecomposables by means of Table 3. The first column lists the new basis for the indecomposables of $D_j\mathcal{A}'$. The second column (we claim) gives the values of F_0 on each basis element.

Table 3

		F_0	$\gamma_j \circ F_0$	γ_j'
1.	$\pi Q^s(Ry)$	$\pi Q^s(Rx)$	$-(\pi Q^s x)(\pi\beta_r x)^{(p-1)j}$ if $s < r$; 0 if $s = r$	$-(\pi Q^s y)(\pi\beta_r y)^{(p-1)j}$ if $s < r$; 0 if $s = r$
2.	$\pi\beta_{r+s+1}R^s(Ry)$	$\pi\beta_{r+s+1}R^s(Rx)$	$\pi\beta_{r+s+1}R^{s+1}x$	$\pi\beta_{r+s+1}R^{s+1}y$
3.	$\pi Q^s(Qy)$	$\pi Q^s(Qx)$	$\pi Q^{s+1}x$	$\pi Q^{s+1}z$
4.	$E(Qy,Ry)$	$E(Qx,Rx)$	$(D_k h)(E')$	$(D_k f_0)(E')$
5.	$\pi Q^s(Rz)$	$\pi Q^s(Rx)$	$-(\pi Q^s x)(\pi\beta_r x)^{(p-1)j}$ if $s < r$; 0 if $s = r$	$-(\pi Q^s z)(\pi\beta_r z)^{(p-1)j}$ if $s < r$; 0 if $s = r$
6.	$\pi\beta_{r+s+1}R^s(Rz)$	$\pi\beta_{r+s+1}R^s(Rx)$	$\pi\beta_{r+s+1}R^{s+1}x$	$\pi\beta_{r+s+1}R^{s+1}z$
7.	$\pi Q^s(Qz)$	$\pi Q^s(Qx)$	$\pi Q^{s+1}x$	$\pi Q^{s+1}z$
8.	$E(Qz,Rz)$	$E(Qx,Rx)$	$(D_k h)(E')$	$(D_k f_0)(E')$
9.	$\pi Q^s u_m$	$-\pi Q^s(Rx)$	$(\pi Q^s x)(\pi\beta_r x)^{(p-1)j}$	$(\pi Q^s y)(\pi\beta_r y)^{(m-1)j}(\pi\beta_r z)^{(p-m)j}$
10.	$\pi\beta_{r+s}R^s u_m$	$E(Qx,Rx)$	$(D_k h)(E')$	$(D_k f_m)(E')$
11.	$\pi Q^s v_m$	$(\pi\beta_{r-1}(Qx))^{2^s}$ if $p = 2$, $s = r-2$	$(\pi\beta_r x)^{2^{s+1}}$	
		$(\pi\beta_r R(Qx))^{2^{s-1}}$ if $p = 2$, $s = r-1$	$(\pi\beta_{r+1}Rx)^{2^s}$	$(\pi Q^s y)(\pi\beta_r y)^{(m-1)j}(\pi Q^s z)(\pi\beta_r z)^{(p-m-1)j}$
		0 otherwise	0	
12.	$\pi\beta_{r-s}Q^s v_m$	0	0	$(\pi\beta_r y)^{mj}(\pi Q^s z)(\pi\beta_r z)^{(p-m-1)j}$ $-(\pi Q^s y)(\pi\beta_r y)^{(m-1)j}(\pi\beta_r z)^{(p-m)j}$

The elements listed in lines 3 and 7 occur only when $s \le r-2$. In lines 9, 11, and 12 they occur only when $s \le r-1$, and in lines 1 and 5 only when $s \le r$.

The first six entries in this column are the new basis for the indecomposables of $D_j\mathcal{A}$, and the first six entries in column 3 define γ_j, while the remaining entries in column 3 give the values of γ_j on the remaining entries in column 2. The entries in column 4 define γ_j'. The verifications necessary to prove 4.7 in this case are again similar to those in the case $M = M_r$. The less obvious ones are the following. If $s < r$ we have

$$\pi^{r-s}Q^sRx \sim \pi^{r-s-1}Q^s\pi Rx \sim \pi^{r-s-1}Q^{s+1}P_*x - \pi^{r-s-1}Q^s(x(\beta_r x)^{p-1})$$
$$\sim -(\pi^{r-s-1}Q^s x)(\pi^{r-1}\beta_r x)^{(p-1)j}$$

in lines 1,5 and 9 by 9.4(iii). (In particular we observe, as claimed in the proof of 8.4(viii), that the relation 3.6(viii) is not used in the present proof when $s = 1$ and $r \geq 2$.) If $s = r$ we have

$$Q^sRx \sim QRQ^{s-1}x \sim 0$$

in lines 1 and 5 by 3.6(viii). In line 11 with $p = 2$ we apply 9.4(ix) to show

$$F_0(\pi^{r-s-1}Q^s v_m) \sim \pi^{r-s-1}Q^s(2^{r-2}\beta_r 2_* Qx)$$
$$\sim \begin{cases} 0 & \text{if } s < r-2 \\ (\pi^{r-1}\beta_r 2_* Qx)^{2^{r-2}} & \text{if } s = r-2 \\ (\pi^{r-2}Q\beta_r 2_* Qx)^{2^{r-2}} + (\pi^{r-1}\beta_r 2_* Qx)^{2^{r-2}} & \text{if } s = r-1 \end{cases}$$

and the claimed values of F_0 follow from 3.1(ii), 3.5 and 3.6(iii) and (iv). This concludes the proof of 4.7.

<u>Proof of 9.4</u>. Let $\approx$ denote mod p equivalence. Parts (i), (ii), (iii), and (iv) follow easily by induction from 3.3(v) and 3.3(vii). For part (v) we have

$$Q^s((\xi_1\xi_2)\xi_4) \approx Q^s(\xi_1\xi_3)Q^s(\xi_4) \approx (\pi^s\xi_1)^{p^s}(Q^s\xi_3)(Q^s\xi_4)$$

by (iii) and (iv). For part (vi) we have

$$\begin{aligned}\beta_{r-s}Q^s(\xi_1^i\xi_2^{p-i}) &\approx Q^s\beta_r(\xi_1^i\xi_2^{p-i}) \\ &\approx Q^s[i(\beta_r\xi_1)\xi_1^{i-1}\xi_2^{p-i} = (p-i)\xi_1^i(\beta_r\xi_2)^{p-i-1}] \\ &\approx iQ^s[(\beta_r\xi_1)\xi_1^{i-1}\xi_2^{p-i}] - iQ^s[\xi_1^i(\beta_r\xi_2)\xi_2^{p-i-1}] \\ &\approx i(Q^s\beta_r\xi_1)(\pi^s\xi_1)^{(i-1)p^s}(\pi^s\xi_2)^{(p-i)p^s} \\ &\quad - i\,(\pi^s\xi_1)^{p^s}(Q^s\beta_r\xi_2)(\pi^s\xi_2)^{(p-i-1)p^s}\end{aligned}$$

and the result follows by part (i).

(vii) First we claim

$$(*) \qquad Q^r\beta_{r+1}p_*\xi_1 \sim 0.$$

This is true when r = 1 by 3.3(iv) and 3.3(v). If $r \geq 2$ we have

$$\begin{aligned} Q^r\beta_{r+1}p_*\xi_1 &\sim Q^{r-1}\beta_r Qp_*\xi_1 \\ &\sim Q^{r-1}\beta_r[p_*Q\xi_1 - (p^{p-1}-1)\xi_1^p] \\ &\sim Q^{r-1}\beta_r p_*Q\xi_1 \end{aligned}$$

and the claim follows by induction on r.

Now we have

$$\begin{aligned} \pi^{r+t-1}\beta_{r+1}R^t[(\beta_r\xi_1)\xi_1^{i-1}\xi_2^{p-i}] &\sim \pi^{r+t-1}Q^t\beta_{r+2t}p_*^{2t}[(\beta_r\xi_1)\xi_1^{i-1}\xi_2^{p-i}] \\ &\sim \pi^{r+t-1}Q^t\beta_{r+2t}[(\beta_{r+2t}p_*^{2t}\xi_1)p_*^{2t}(\xi_1^{i-1}\xi_2^{p-i})] \\ &\sim -(\pi^{r+t-1}Q^t\beta_{r+2t}p_*^{2t}\xi_1)[\pi^{r+t-1}Q^t\beta_{r+2t}p_*^{2t}(\xi_1^{i-1}\xi_2^{p-i})] \end{aligned}$$

If $t \geq r$ then

$$\pi^{r+t-1}Q^t\beta_{r+2t}p_*^{2t}\xi_1 \sim Q^t\beta_{t+1}p_*(p_*^{t-r}\xi_1),$$

which is equivalent to 0 by (*). Otherwise we have

$$\begin{aligned} (\pi^{r+t-1}Q^t\beta_{r+2t}p_*^{2t}\xi_1)[\pi^{r+t-1}Q^t\beta_{r+2t}p_*^{2t}(\xi_1^i\xi_2^{p-i})] \\ \sim (\pi^{r-t-1}Q^t\beta_r\xi_1)[\pi^{r-t-1}Q^t\beta_r(\xi_1^i\xi_2^{p-i})] \end{aligned}$$

and the result follows from part (iii).

For (viii), we have

$$\begin{aligned} \beta Q^r \boldsymbol{2}\,\xi_1 &\sim Q^{r-1}\beta Q\boldsymbol{2}\,\xi_1 \\ &\sim \begin{cases} \frac{1}{p}\binom{p}{i}(\pi^{r-1}x)^{(p^2-p)(r-1)}Q^{r-1}\beta_r p_*Qx & \text{if } r \geq 2 \\ 0 \quad \text{if } r = 1, \end{cases} \end{aligned}$$

but the expression for $r \geq 2$ is also equivalent to zero by (*).

Finally, part (ix) follows from 3.3(vi) by induction on s.

It remains to prove 9.5. In order to keep track of when an element of $D_j\{\xi_1,\ldots,\xi_t\}$ is homogeneous, we make the following definition. Let S be a fixed set and suppose that we have assigned to each ξ_i a subset $h(\xi_i)$ of S called the homogeneity of ξ_i. Then we define the homogeneity of an arbitrary expression in $\xi_1,\ldots,\xi_t$ by requiring that $0_{\alpha,r,j}$ have homogeneity S, that $p_*,\beta_r,\pi,Q,\mathcal{Q}$ and R commute with h and that $h(E + E') = h(E) \cap h(E')$ and $h(E \cdot E') = h(E) \cup h(E')$. We say that an expression $E(\xi_1,\ldots,\xi_t)$ of height 1 is reducible with respect to h if there is an E' $D_j\{\xi_1,\ldots,\xi_t\}$ with $E' \sim E$ and $h(E') \supset h(E)$.

Proposition 9.6. If S is any set and $h(\xi_1),\ldots,h(\xi_t)$ are any subsets of S then every expression of height 1 in $\xi_1,\ldots,\xi_t$ is reducible with respect to h.

If $S = \{\xi_1,\ldots,\xi_t\}$ and $h(\xi_i) = \{\xi_i\}$ for $1 \leq i \leq t$ then the expressions listed in 9.5 have homogeneity S, while an expression in $D_j\{\xi_1,\ldots,\xi_t\}$ has homogeneity S if and only if it is homogeneous. Thus 9.5 follows from this case of 9.6. The extra generality allowed for S and h is technically useful in proving 9.6.

In the remainder of this section we prove 9.6. We fix a set S and assume from now on that any indeterminates mentioned have been assigned homogeneities contained in S as well as dimensions, heights and filtrations. It will be convenient to let ξ, η and θ denote indeterminates and to let E, F, G and H denote expressions. We say that two expressions (possibly involving different sets of indeterminates) match if they have the same dimension, height, filtration and homogeneity. We shall frequently use the fact that a sum or product of reducible expressions is reducible and that homogeneity is preserved by substitution, i.e., if F is any expression in $\eta_1,\ldots,\eta_s$ and $E_1,\ldots,E_s$ matching $\eta_1,\ldots,\eta_s$ respectively then $h(F(E_1,\ldots,E_s)) = h(F)$. Note, however, that equivalent expressions generally have different homogeneities; for example, $p\xi$ is equivalent to 0 if $\|\xi\| = 1$ but $h(\xi)$ is not necessarly equal to S.

For our next two results we fix a set $\{\eta_1,\ldots,\eta_s,\eta_1',\ldots,\eta_s',\eta_1'',\ldots,\eta_s''\}$ of indeterminates such that each η_i' matches $Q\eta_i$ and each η_i'' matches $R\eta_i$. Here and elsewhere we shall interpret $Q\eta_i$ as $0_{1,1,1}$ if $\|\eta_i\| = 1$ and $R\eta_i$ as $0_{1,1,1}$ if $|\eta_i| = 0$. We say that an expression is elementary if it does not involve Q or R.

Lemma 9.7. Let G be an elementary expression of length 2 in $\eta_1,\ldots,\eta_s$ and let θ match G.

(i) If F is $\pi^{\|\theta\|-1}\theta$ or $\pi^{\|\theta\|-1}\beta_{\|\theta\|}\theta$ then there is an elementary expression G' $D_{\nu G}\{\eta_1,\ldots,\eta_s\}$ with $G' \sim F(G)$ and $hG' \supset hF$.

(ii) If $F = Q\theta$ or $F = R\theta$ then there is an elementary expression $G'(\eta_1,\ldots,\eta_s,\eta_1',\ldots,\eta_s',\eta_1'',\ldots,\eta_s'')$ with $hG' \supset hF$ and

$$F(G) \sim G'(\eta_1,\ldots,\eta_s,Q\eta_1,\ldots,Q\eta_s,R\eta_1,\ldots,R\eta_s).$$

Proof. The possibilities for G are $\pi\eta_i, p_*\eta_i, \beta_r\eta_i, \eta_i+\eta_j, \eta_i\eta_j$ and η_i. The result can be checked in each case from the formulas of section 3.

Next we define the complexity $c(E)$ of a standard indecomposable E in $D_j\{\eta_1,\ldots,\eta_s\}$ to be the total number of Q's and R's that appear in it. We define $c(E)$ for an arbitrary expression E in $D_j\{\eta_1,\ldots,\eta_s\}$ to be the maximum of the complexities of the indecomposables that appear as factors in the terms of E.

Lemma 9.8. Let $H \in D_j\{\eta_1,\ldots,\eta_s,\eta_1',\ldots,\eta_s',\eta_1'',\ldots,\eta_s''\}$. Then there is an $H' \in D_j\{\eta_1,\ldots,\eta_s\}$ such that $h(H') \supset h(H)$, $c(H') \leq c(H) + 1$ and H' is equivalent to

$$H(\eta_1,\ldots,\eta_s,Q\eta_1,\ldots,Q\eta_s,R\eta_1,\ldots,R\eta_s).$$

In particular, the latter expression is reducible.

Proof. We may assume that H is a standard indecomposable and hence that it involves only one of the indeterminates. If it involves one of the η_i the result is trivial. Otherwise H has one of the forms

$\pi^{\|\eta_i\|-t-2}Q^t\eta_i'$, $\pi^{\|\eta_i\|-t-2}\beta_{\|\eta_i\|-t-2}Q^t\eta_i'$, $\pi^{\|\eta_i\|-t}Q^t\eta_i''$, $\pi^{\|\eta_i\|+t-2}\beta_{\|\eta_i\|+t-1}R^t\eta_i'$, or $\pi^{\|\eta_i\|+t}\beta_{\|\eta_i\|+t+1}R^t\eta_i''$. In each case the result follows either trivially or from the formulas of section 3.

Lemma 9.9. Let $E_1,\ldots,E_r$ be elementary expressions in $\xi_1,\ldots,\xi_t$ and let $\theta_1,\ldots,\theta_r$ match $E_1,\ldots,E_r$ respectively. Let $F \in D_j\{\theta_1,\ldots,\theta_r\}$. Then there is an $H \in D_j\{\xi_1,\ldots,\xi_t\}$ such that $c(H) \leq c(F)$, $h(H) \supset h(F)$ and $H \sim F(E_1,\ldots,E_r)$. In particular, $F(E_1,\ldots,E_r)$ is reducible.

Proof. Let ℓ be the maximum of the lengths of the E_i. If $\ell = 1$ the result is trivial. We shall prove the result in general by induction on $c(F)$ with a subsidiary induction on ℓ. We may assume that F is a standard indecomposable, and hence that it involves only one of the θ_i, say θ_1. Now by Definition 9.1, E_1 can be written in the form $G(E_{11},E_{12})$, where $E_{11}(\xi_1,\ldots,\xi_t)$ and $E_{12}(\xi_1,\ldots,\xi_t)$ are elementary with lengths less than ℓ and $G(\eta_1,\eta_2)$ is elementary with length 2. If $c(F) = 0$ then F has the form $\pi^{\|\theta_1\|-1}\theta_1$ or $\pi^{\|\theta_1\|-1}\beta_{\|\theta_1\|-1}\theta_1$ and the result follows by 9.7(i) and the subsidiary inductive hypothesis. Otherwise F has the form

$F'(F'')$, where $F'' = Q\theta_1$ or $R\theta_1$ and $c(F') = c(F) - 1$. Thus $F(E_1) = F'(F''(G(E_{11}, E_{12})))$. If $\eta_1', \eta_2', \eta_1'', \eta_2''$ are as in 9.7 then by 9.7(ii) there is an elementary expression $G'(\eta_1, \eta_2, \eta_1', \eta_2', \eta_1'', \eta_2'')$ such that $h(G') \quad h(F'')$ and $G'(\eta_1, \eta_2, Q\eta_1, Q\eta_2, R\eta_1, R\eta_2) \sim F''(G(\eta_1, \eta_2))$. Thus

$$F(G(\eta_1, \eta_2)) \sim F'(G'(\eta_1, \eta_2, Q\eta_1, Q\eta_2, R\eta_1, R\eta_2)).$$

Now since $c(F') < c(F)$ the inductive hypothesis gives an expression $H \in D_j\{\eta_1, \eta_2, \eta_1', \eta_2', \eta_1'', \eta_2''\}$ with $c(H) \leq c(F') < c(F)$, $h(H) \supset f(F') \supset h(F)$, and

$$H \sim F'(G'(\eta_1, \eta_2, \eta_1', \eta_2', \eta_1'', \eta_2''))$$

So that

$$F(G(\eta_1, \eta_2)) \sim H(\eta_1, \eta_2, Q\eta_1, Q\eta_2, R\eta_1, R\eta_2).$$

Now by Lemma 9.8 there is an expression $H' \in D_j\{\eta_1, \eta_2\}$ such that $c(H') \leq c(H) + 1 \leq c(F)$ and $h(H') \supset h(H) \supset h(F)$ with $H' \sim F(G(\eta_1, \eta_2))$. Hence $F(E_1) \sim H'(E_{11}, E_{12})$. Since E_{11} and E_{12} both have lengths less than ℓ, the result now follows by the subsidiary inductive hypothesis.

Finally, we complete the proof of 9.6. Let $G(\xi_1, \ldots, \xi_t)$ be any expression of height 1. The proof is by induction on the length of G, which we may assume is ≥ 2. It is easy to see from definition 9.1 (by another induction on the length of G) that G can be written in the form $G'(\xi_1, \ldots, \xi_t, E)$, where $G'(\xi_1, \ldots, \xi_t, \eta)$ has length less than ℓ and E has length 2. Then G' has height 1 and $h(G') = h(G)$. By inductive hypothesis we may assume $G' \in D_{\nu G}\{\xi_1, \ldots, \xi_t, \eta\}$. If E is elementary the result now follows by 9.9, while if E is $Q\eta$ or $R\eta$ the result follows by 9.8. This concludes the proof.