DEFORMATION THEORY AND PARTITION LIE ALGEBRAS

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Abstract. A theorem of Lurie and Pridham establishes a correspondence between formal moduli problems and differential graded Lie algebras in characteristic zero, thereby formalising a well-known principle in deformation theory. We introduce a variant of differential graded Lie algebras, called partition Lie algebras, in arbitrary characteristic. We then explicitly compute the homotopy groups of free algebras, which parametrise operations. Finally, we prove generalisations of the Lurie-Pridham correspondence classifying formal moduli problems via partition Lie algebras over an arbitrary field, as well as over a complete local base.

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1. Introduction

A well-known principle in deformation theory postulates that the infinitesimal structure of any moduli space in characteristic zero is controlled by a differential graded Lie algebra.

This heuristic can be traced back to Deligne [Del], Drinfeld [Dri], and Feigin, and was explored further in the work of Goldman-Millson [GM88], Hinich [Hin01], Kontsevich-Soibelman [KS02], Manetti [Man09], and many others. Eventually, it was articulated as a precise correspondence by Lurie [Lur11a] and Pridham [Pri10], who constructed an equivalence between formal moduli problems and differential graded Lie algebras in characteristic zero.

Our principal aim is to generalise this equivalence to finite and mixed characteristic, thereby giving a Lie algebraic description of formal deformations in these contexts.

1.1. Background.

Before delving into any technical details, we shall recall a classical example.

Example. Given a smooth and proper variety Z over the field $\mathbb{C}$ of complex numbers, we can study its infinitesimal deformations over local Artinian $\mathbb{C}$-algebras. It is well-known that these deformations are closely related to the lower cohomology groups of the tangent bundle T_Z:

a) $\mathrm{H}^0(Z, T_Z)$ classifies infinitesimal automorphisms of the trivial deformation $Z \times_{\mathrm{Spec}\mathbb{C}} \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$; infinitesimal automorphisms therefore correspond to vector fields.
b) $\mathrm{H}^1(Z, T_Z)$ classifies isomorphism classes of first-order deformations; every such deformation $\mathcal{Z} \to \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ gives rise to a Kodaira-Spencer class $x_{\mathcal{Z}}$ in $H^1(Z, T_Z)$, cf. [KS58].

To formulate a criterion for when a given first order deformation $\mathcal{Z} \to \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ extends to higher order, observe that the classical Dolbeault complex

$$C^*(Z, T_Z) = (\ \mathcal{A}^{0,0}(T_Z) \to \mathcal{A}^{0,1}(T_Z) \to \mathcal{A}^{0,2}(T_Z) \to \ \dots\ )$$

computing $\mathrm{H}^*(Z, T_Z)$ admits the structure of a *differential graded Lie algebra*. Its Lie bracket combines the wedge product on differential forms with the commutator bracket on vector fields.

c) $\mathrm{H}^2(Z, T_Z)$ contains the obstructions to extending first-order deformations: a first-order deformation $\mathcal{Z} \to \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ extends to $\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^3)$ precisely if the self-bracket $[x_{\mathcal{Z}}, x_{\mathcal{Z}}]$ vanishes.

It turns out that the differential graded Lie algebra $C^*(Z, T_Z)$ records all the formal deformation theory of Z. More precisely, given any local Artinian $\mathbb{C}$-algebra A, deformations of Z to A correspond to Maurer-Cartan elements in $\mathfrak{m}_A \otimes C^1(Z, T_Z)$, considered up to gauge equivalence. A key insight of Drinfeld [Dri] was that $C^*(Z, T_Z)$ does not just remember the infinitesimal deformations, but also the *derived* infinitesimal deformations of Z to simplicial local Artinian $\mathbb{C}$-algebras (via a refined Maurer-Cartan construction, cf. [Hin01, Section 1.3]). In fact, the differential graded Lie algebra $C^*(Z, T_Z)$ is uniquely determined by this property (up to equivalence).

This illustrates the general principle we alluded to in the very beginning: the derived infinitesimal deformations of an object in characteristic zero are controlled by a differential graded Lie algebra. A precise formulation was given by Lurie [Lur11a] and Pridham [Pri10] using the language of *formal moduli problems* (cf. Definition 1.4 below); roughly speaking, any reasonably geometric deformation problem (such as deformations of schemes, of complexes, or the formal completion of a suitably geometric stack) gives rise to a formal moduli problem. For example, there is a formal moduli problem corresponding to deformations of a variety Z as above.

We then have the following correspondence:

Theorem 1.1 (Lurie, Pridham)**.** *If k is a field of characteristic zero, then there is an equivalence of ∞-categories between formal moduli problems and differential graded Lie algebras over k.*

This result has been extended in various directions in characteristic zero, but an analogue in positive or mixed characteristic has not appeared. Note that derived deformation theory in such settings has attracted interest in number theory, for example in the work of Galatius-Venkatesh [GV18] on derived deformations of Galois representations.

In this paper, we formulate and prove a generalisation of the Lurie-Pridham theorem in positive and mixed characteristic. For this purpose, we introduce the notion of a *partition Lie algebra*, which is the correct generalisation of the notion of a differential graded Lie algebra in this context. Partition Lie algebras are subtle homotopical objects controlled by the equivariant topology of the partition complex. By studying this simplicial complex, we compute the homotopy groups of free partition Lie algebras. These parametrise the natural operations acting on the homotopy groups of any partition Lie algebra.

1.2. **Statement of Results.** Away from characteristic zero, classical algebraic geometry can be generalised in two inequivalent ways (cf. [TV08] and [Lur16] for detailed treatments):

a) derived algebraic geometry is based on simplicial commutative rings.
b) spectral algebraic geometry is based on (connective) $\mathbb{E}_\infty$-rings.

We will prove variants of our main results in both settings, and will begin with the former. Here, affine schemes over a given field k correspond to simplicial commutative k-algebras, and infinitesimal thickenings of $\mathrm{Spec}(k)$ (over k) correspond to the following kind of objects:

Definition 1.2 (Derived Artinian algebras)**.** A simplicial commutative k-algebra A is called *Artinian* if

(1) $\pi_0(A)$ is a local Artinian ring with residue field k.
(2) $\pi_*(A)$ is a finite-dimensional k-vector space.

Let $\mathrm{SCR}_k^{\mathrm{art}}$ denote the ∞-category of Artinian simplicial commutative k-algebras, defined as a full subcategory of the ∞-category of simplicial commutative k-algebras (cf. Construction 5.36).

Notation 1.3. Write $\mathcal{S}$ for the ∞-category of spaces (cf. [Lur09, Definition 1.2.16.1]). Unless stated otherwise, limits and colimits will be computed in an ∞-categorical sense (cf. [Lur09, Chapter 4]).

Derived infinitesimal deformations will be described by the following kind of functors:

Definition 1.4 (Formal moduli problems)**.** A *derived formal moduli problem over a field k* is given by a functor $X : \mathrm{SCR}_k^{\mathrm{art}} \to \mathcal{S}$ satisfying the following two properties:

(1) The space $X(k)$ is contractible.
(2) Given a pullback square

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & A' \arrow[d] \\ A \arrow[r] & A'' \end{tikzcd}$$

in $\mathrm{SCR}_k^{\mathrm{art}}$ in which $\pi_0(A') \to \pi_0(A'')$ and $\pi_0(A) \to \pi_0(A'')$ are surjective, the square

$$\begin{tikzcd} X(\widetilde{A}) \arrow[r] \arrow[d] & X(A') \arrow[d] \\ X(A) \arrow[r] & X(A'') \end{tikzcd}$$

is a (homotopy) pullback of spaces.

Let $\mathrm{Moduli}_{k,\Delta}$ be the full subcategory of $\mathrm{Fun}(\mathrm{SCR}_k^{\mathrm{art}}, \mathcal{S})$ spanned by all formal moduli problems.

Formal moduli problems exist in abundance. Indeed, the formal neighbourhood of any point in a (suitably geometric) derived stack, as well as various natural deformation problems (such as deformations of varieties or vector bundles), provide a vast supply of examples. We refer to [Lur16, Chapter 16] or [Toë14] for a discussion of some of them.

A guiding goal in the subject is to give an "algebraic" classification of formal moduli problems. Theorem 1.1 realises this aim when k is a field of characteristic zero by constructing an equivalence

$$\mathrm{Moduli}_{k,\Delta} \xrightarrow{\simeq} \mathrm{Alg}_{\mathrm{Lie}_k^{\mathrm{dg}}}$$

between $\mathrm{Moduli}_{k,\Delta}$ and the ∞-category of differential graded Lie algebras over k. Intuitively, this correspondence arises as follows. Given a formal moduli problem $X \in \mathrm{Moduli}_{k,\Delta}$, one first constructs its tangent complex T_X. This chain complex is a derived version of the tangent space, and is determined by the values of F on trivial square-zero extensions. The correspondence then equips the shift $T_X[-1]$ with the structure of a differential graded Lie algebra over k, and moreover provides a method to functorially recover X from $T_X[-1]$. Hence, it can be interpreted as a variant of formal Lie theory (cf. [GR17, Chapter 7]).

The Lurie-Pridham correspondence has been extended in several directions, for example by Gaitsgory-Rozenblyum [GR17], Hennion [Hen18], and Nuiten [Nui17]. These generalisations treat the case of formal moduli problems relative to a base (rather than a point) in characteristic zero.

To generalise Theorem 1.1 to general base fields, we will introduce the new algebraic and homotopy-theoretic structure of a *partition Lie algebra*. In characteristic zero, partition Lie algebras are equivalent to differential graded Lie algebras; in finite characteristic, this is no longer true. Partition Lie algebras are closely related to the genuine equivariant topology of the following spaces:

Definition 1.5 (Partition complexes)**.** Given an integer $n \geq 1$, the n^{th} *partition complex* $|\Pi_n|$ is the genuine Σ_n-space given by the geometric realisation of the following simplicial Σ_n-set:

$$\Pi_n := \mathrm{N}_\bullet\left(\{\text{Poset of partitions of } \{1,\ldots,n\}\} - \{\hat{0},\hat{1}\}\right).$$

Here $\mathrm{N}_\bullet$ denotes the nerve construction, $\hat{0} = \boxed{1}\boxed{2}\ldots\boxed{\mathrm{n}}$ is the discrete partition, $\hat{1} = \boxed{12\ldots\mathrm{n}}$ is the indiscrete partition, and all partitions are ordered under refinement.

Partition complexes were linked to ordinary Lie algebras in the work of Barcelo [Bar90], Hanlon [Han81], Joyal [Joy86], and Stanley [Sta82], who constructed and examined an isomorphism of Σ_n-representations $\widetilde{\mathrm{H}}^{n-1}(\Sigma|\Pi_n|^\diamond,\mathbb{Z}) \cong \mathrm{Lie}_n \otimes \mathrm{sgn}_n$. Here $\Sigma|\Pi_n|^\diamond$ is the unreduced-reduced suspension of $|\Pi_n|$ and sgn_n denotes the sign representation of Σ_n. Moreover, Lie_n is the quotient of the free abelian group on all Lie words in letters $x_1,\ldots,x_n$ involving each x_i exactly once by the usual antisymmetry and Jacobi relations.

This above isomorphism between representations of the symmetric group was later refined in work of Salvatore [Sal98] and Ching [Chi05], who constructed the Lie operad in the ∞-category of spectra; algebras over this operad are called *spectral Lie algebras*.

Remark 1.6. Spectral Lie algebras offer excellent computational and conceptual opportunities in unstable chromatic homotopy theory, as is exploited, for example, in [BR15], [Bra17], [Heu18],[BHK].

Over a field of characteristic zero, spectral Lie algebras are equivalent to differential graded Lie algebras. In contrast, they are *not* the correct structure for the purposes of deformation theory in characteristic p, where it will in fact not be possible to define partition Lie algebras as algebras over any operad. Instead, we will need to use the language of monads, which we briefly recall:

Notation 1.7. Write Mod_k for the derived ∞-category of k; its objects are chain complexes of k-vector spaces, or equivalently k-module spectra (cf. [Lur17, Definition 1.3.5.8; Remark 7.1.1.16]).

Recall that a *monad* on Mod_k consists of an endofunctor T together with natural transformations $\mathrm{id} \to T$, $T \circ T \to T$, and an infinite set of coherence data. Every monad T on Mod_k gives rise to an ∞-category Alg_T of T-algebras (sometimes also called T-modules); an object in Alg_T is informally given by a chain complex $M \in \mathrm{Mod}_k$, a natural transformation $T(M) \to M$, and an infinite set of coherence data. We refer to [Lur17, Section 4.7] for precise definitions.

Example 1.8. If k is a field of characteristic zero, then the ∞-category $\mathrm{Alg}_{\mathrm{Lie}_k^{\mathrm{dg}}}$ of differential graded Lie algebras can be described as algebras over a certain monad $\mathrm{Lie}_k^{\mathrm{dg}}$ on Mod_k, which sends a chain complex V to the chain complex $\mathrm{Lie}_k^{\mathrm{dg}}(V) = \bigoplus_n (\mathrm{Lie}_n \otimes V^{\otimes n})_{\Sigma_n}$. Here the tensor product is computed in complexes, and $(-)_{\Sigma_n}$ denotes Σ_n-orbits (which are equivalent to Σ_n-homotopy orbits).

In Definition 5.47 below, we will construct the *partition Lie algebra monad* $\mathrm{Lie}^{\pi}_{k,\Delta}$ on Mod_k.

Construction 1.9 (Partition Lie algebras)**.** The monad $\mathrm{Lie}^{\pi}_{k,\Delta}$ satisfies the following properties:

(1) If V is a finite-dimensional k-vector space (considered as a discrete k-module spectrum), then $\mathrm{Lie}^{\pi}_{k,\Delta}(V)$ is the linear dual of the (algebraic) cotangent fibre of $k \oplus V^{\vee}$, the trivial square-zero extension of k by $V^{\vee}$. In fact, this remains true for any coconnective k-module spectrum V for which $\pi_i(V)$ is finite-dimensional for all i.

(2) If $V \simeq \mathrm{Tot}(V^{\bullet}) \in \mathrm{Mod}_{k,\leq 0}$ is represented by a cosimplicial k-vector space $V^{\bullet}$, then
$$\mathrm{Lie}^{\pi}_{k,\Delta}(V) \simeq \bigoplus_n \mathrm{Tot}\left(\widetilde{C}^{\bullet}(\Sigma|\Pi_n|^{\diamond}, k) \otimes (V^{\bullet})^{\otimes n}\right)^{\Sigma_n}.$$
Here $\widetilde{C}^{\bullet}(\Sigma|\Pi_n|^{\diamond}, k)$ denotes the k-valued cosimplices on the space $\Sigma|\Pi_n|^{\diamond}$, the functor $(-)^{\Sigma_n}$ takes strict fixed points, and the tensor product is computed in cosimplicial k-modules.

(3) The functor $\mathrm{Lie}^{\pi}_{k,\Delta}$ commutes with filtered colimits and geometric realisations.

(4) The tangent fibre T_X of any $X \in \mathrm{Moduli}_{k,\Delta}$ has the structure of a $\mathrm{Lie}^{\pi}_{k,\Delta}$-algebra.

We write $\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k,\Delta}}$ for the ∞-category of $\mathrm{Lie}^{\pi}_{k,\Delta}$-algebras in Mod_k.

Remark 1.10. Given any partition Lie algebra $\mathfrak{g} \in \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k,\Delta}}$, the homotopy groups $\pi_*(\mathfrak{g})$ form a graded Lie algebra in the shifted sense. This means that given $x \in \pi_i(\mathfrak{g})$ and $y \in \pi_j(\mathfrak{g})$, we have a bracket $[x, y] \in \pi_{i+j-1}(\mathfrak{g})$. This shift is merely a matter of convention, but we have decided to adopt it as it seems more natural for our applications.

When k has characteristic zero, $\mathrm{Lie}^{\pi}_{k,\Delta}$ can be identified with the shifted differential graded Lie algebra monad (cf. Proposition 5.48). For general fields, partition Lie algebras provide a new generalisation of differential graded Lie algebras. While $\mathrm{Lie}^{\pi}_{k,\Delta}$ looks somewhat similar to the restricted Lie algebra monad (cf. e.g. [Fre00]), it behaves in a substantially different way; for example, $\mathrm{Lie}^{\pi}_{k,\Delta}$ does not preserve modules concentrated in any particular homological degree.

Most importantly, partition Lie algebras have the following application in deformation theory:

Theorem 1.11 (Main theorem)**.** *If k is a field, there is an equivalence of ∞-categories*
$$\mathrm{Moduli}_{k,\Delta} \simeq \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k,\Delta}}$$
between formal moduli problems and partition Lie algebras over k. It sends a formal moduli problem $X \in \mathrm{Moduli}_{k,\Delta}$ to its tangent fibre T_X.

This means that locally, moduli spaces are still governed by an appropriate Lie algebraic structure.

As in [Lur11a, Pri10], the correspondence between formal moduli problems and partition Lie algebras arises from a form of Koszul duality for algebras, which we shall formulate next.

Write $\mathrm{SCR}_k^{\mathrm{aug}}$ for the ∞-category of augmented simplicial commutative k-algebras. We will construct a Koszul duality functor

$$\mathfrak{D} : \mathrm{SCR}_k^{\mathrm{aug}} \to \mathrm{Alg}^{op}_{\mathrm{Lie}^\pi_{k,\Delta}},$$

which sends an augmented simplicial commutative k-algebra A to the dual of its (algebraic) tangent fibre $(k \otimes_A L^{\Delta}_{A/k})^\vee$, equipped with its natural $\mathrm{Lie}^\pi_{k,\Delta}$-algebra structure. A key step in the proof of Theorem 1.11 is to show that $\mathfrak{D}$ restricts to an equivalence on the following subcategory of $\mathrm{SCR}_k^{\mathrm{aug}}$:

Definition 1.12. An augmented simplicial commutative k-algebra $A \in \mathrm{SCR}_k^{\mathrm{aug}}$ is *complete local Noetherian* if

1. $\pi_0(A)$ is a complete local Noetherian ring.
2. Each $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module.

Let $\mathrm{SCR}_k^{\mathrm{cN}}$ be the full subcategory spanned by all complete local Noetherian k-algebras. Then:

Theorem 1.13. *The Koszul duality functor $\mathfrak{D}$ restricts to a contravariant equivalence between $\mathrm{SCR}_k^{\mathrm{cN}}$ and the full subcategory of $\mathrm{Alg}_{\mathrm{Lie}^\pi_{k,\Delta}}$ spanned by those partition Lie algebras $\mathfrak{g}$ for which $\pi_i(\mathfrak{g})$ is finite-dimensional for each i and vanishes for $i > 0$.*

We prove Theorem 1.13 "by hand" at the level of simplicial commutative rings, by working carefully with filtered objects and exploiting the fact that all rings that one encounters in this way are Noetherian. To deduce Theorem 1.11, one also needs to prove that the Koszul duality functor carries appropriate pullbacks of simplicial commutative rings to pushouts of $\mathrm{Lie}^\pi_{k,\Delta}$-algebras.

Partition Lie algebras are subtle homotopical objects, and we therefore need tools to study them. For example, one can consider the natural operations acting on their homotopy groups. These are parametrised by the homotopy groups of *free* partition Lie algebras, which we will compute by using techniques from the work of the first author and Arone [AB18], which rely on discrete Morse theory and an argument inspired by earlier work of Arone and Mahowald [AM99].

To state our result, we need the following classical notion (cf. [Shi58], [CFL58]):

Definition 1.14. A word w in letters $x_1, \ldots, x_k$ is said to be a *Lyndon word* if it is smaller than any of its cyclic rotations in the lexicographic order with $x_1 < \cdots < x_k$. Write $B(n_1, \ldots, n_k)$ for the set of Lyndon words which involve the letter x_i precisely n_i times.

Remark 1.15. It is well-known that the set of Lyndon words in letters $x_1, \ldots x_k$ forms a basis for the free (ungraded) Lie algebra over $\mathbb{Z}$ on n letters (cf. e.g. [Reu03]).

Theorem 1.16. *The $\mathbb{F}_p$-vector space $\pi_*(\mathrm{Lie}^\pi_{k,\Delta}(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_m}\mathbb{F}_p))$ has a basis indexed by sequences $(i_1, \ldots, i_k, e, w)$. Here $w \in B(n_1, \ldots, n_m)$ is a Lyndon word. We have $e \in \{0, \epsilon\}$, where $\epsilon = 1$ if p is odd and $\deg(w) := \sum_i (\ell_i - 1)n_i + 1$ is even. Otherwise, $\epsilon = 0$.*

The integers $i_1, \ldots, i_k$ satisfy:

1. *Each $|i_j|$ is congruent to 0 or 1 modulo $2(p-1)$.*
2. *For all $1 \leq j < k$, we have $pi_{j+1} < i_j < -1$ or $0 \leq i_j < pi_{j+1}$*
3. *We have $(p-1)(1+e)\deg(w) - \epsilon \leq i_k < -1$ or $0 \leq i_k \leq (p-1)(1+e)\deg(w) - \epsilon$.*

The sequence $(i_1, \ldots, i_k, e, w)$ sits in homological degree $((1+e)\deg(w) - e) + i_1 + \cdots + i_k - k$ and multi-weight $(n_1 p^k(1+e), \ldots, n_m p^k(1+e))$.

The input of this computation is the homotopy of free simplicial and cosimplicial commutative rings as computed by Dold [Dol58], Nakaoka [Nak57] [Nak59], Milgram [Mil69], and Priddy [Pri73]. The case where $\ell_i \leq 0$ for all i follows immediately from [AB18, Theorem 8.14]. For $p = 2$ and $\ell_i \leq 0$ for all i, our result can also be read off from the work of Goerss [Goe90], who computed the algebraic André-Quillen homology of trivial square-zero extensions at $p = 2$.

Remark 1.17. Note that Pridham (cf. [Pri10, Section 5.3]) also considers the operations acting on the tangent spaces of formal moduli problems. He abstractly identifies the operations on the coconnective part with the operations on André-Quillen homology.

Up to now, we have stated our results in the context of simplicial commutative rings, which was indicated by the subscript "Δ". We can obtain parallel results in the context of spectral algebraic geometry, hence describing deformations parametrised by connective $\mathbb{E}_\infty$-rings over a given field k.

More precisely, define the ∞-category $\mathrm{CAlg}_k^{\mathrm{art}}$ of spectral Artinian k-algebras and the ∞-category $\mathrm{Moduli}_{k,\mathbb{E}_\infty}$ of spectral formal moduli problems by replacing the term "simplicial commutative k-algebra" by the term "connective $\mathbb{E}_\infty$-k-algebra" in Definition 1.2 and Definition 1.4, respectively. In Definition 5.32 below, we construct the *spectral partition Lie algebra monad* $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$ on Mod_k.

Construction 1.18 (Spectral partition Lie algebras)**.** The monad $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$ satisfies the following:

(1) If V is a finite-dimensional k-vector space (considered as a discrete k-module spectrum), then $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}(V)$ is the linear dual of the topological cotangent fibre of $k \oplus V^\vee$, the trivial square-zero extension of k by $V^\vee$. In fact, this remains true for any coconnective k-module spectrum V for which $\pi_i(V)$ is finite-dimensional for all i.

(2) If $V \in \mathrm{Mod}_{k,\leq N}$ is truncated above, then
$$\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}(V) \simeq \bigoplus_n \left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes V^{\otimes n}\right)^{h\Sigma_n},$$
where $(-)^{h\Sigma_n}$ denotes homotopy fixed points and the other notation is as above.

(3) The functor $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$ commutes with filtered colimits and geometric realisations.

(4) The tangent fibre T_X of any $X \in \mathrm{Moduli}_{k,\mathbb{E}_\infty}$ has the structure of a $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$-algebra.

We write $\mathrm{Alg}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}}$ for the ∞-category of $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$-algebras in Mod_k.

We then have a variant of the previous equivalence:

Theorem 1.19. *If k is a field, then there is an equivalence of ∞-categories*
$$\mathrm{Moduli}_{k,\mathbb{E}_\infty} \simeq \mathrm{Alg}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}}$$
between spectral formal moduli problems and spectral partition Lie algebras over k, sending a formal moduli problem $X \in \mathrm{Moduli}_{k,\mathbb{E}_\infty}$ to its tangent fibre T_X.

Proving this equivalence again requires constructing a Koszul duality functor, and showing that it restricts to an equivalence between complete local Noetherian $\mathbb{E}_\infty$-k-algebras and spectral partition Lie algebras $\mathfrak{g}$ which are coconnective and have degreewise finite-dimensional homotopy groups. Theorem 1.11 and Theorem 1.19 are proven with similar methods, which is why we present much of the argument in an axiomatic way (cf. Section 4) applying to both of these contexts at once.

The natural operations on $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$-algebras are parametrised by the homotopy groups of free spectral partition Lie algebras, which we compute by a similar method as before:

Theorem 1.20. *The $\mathbb{F}_p$-vector space $\pi_*(\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_m}\mathbb{F}_p))$ has a basis indexed by sequences $(i_1, \ldots, i_k, e, w)$. Here $w \in B(n_1, \ldots, n_m)$ is a Lyndon word. We have $e \in \{0, \epsilon\}$, where $\epsilon = 1$ if p is odd and $\deg(w) := \sum_i(\ell_i - 1)n_i + 1$ is even. Otherwise, $\epsilon = 0$.*

The integers $i_1, \ldots, i_k$ satisfy:

(1)' Each i_j is congruent to 0 or 1 modulo $2(p-1)$.

(2)' For all $1 \leq j < k$, we have $i_j < pi_{j+1}$.

(3)' We have $i_k \leq (p-1)(1+e)\deg(w) - \epsilon$.

The homological degree of $(i_1, \ldots, i_k, e, w)$ is $((1+e)\deg(w) - e) + i_1 + \cdots + i_k - k$ and its multi-weight is $(n_1 p^k(1+e), \ldots, n_m p^k(1+e))$.

The input to this computation is the homotopy of free $\mathbb{E}_\infty$-rings computed by Adem [Ade52], Serre [Ser53], Cartan [Car54] [Car55], Dyer-Lashof [DL62], May, and Steinberger [BMMS86]. It is again inspired by Arone-Mahowald's classical work [AM99].

Finally, we prove variants of Theorem 1.11 and Theorem 1.19 in mixed characteristic. More precisely, let A be complete local Noetherian with residue field k, either in simplicial commutative rings or in $\mathbb{E}_\infty$-rings. There is a natural notion of formal moduli problems in these mixed contexts (cf. Definition 6.2 below); we write $\mathrm{Moduli}_{A//k,\Delta}$ and $\mathrm{Moduli}_{A//k,\mathbb{E}_\infty}$ for the respective ∞-categories. For example, we can describe the formal neighbourhood of a k-point inside a (suitably geometric) stack defined over $\mathrm{Spec}(A)$ by one of these "mixed" formal moduli problems.

In Theorem 6.15 and Construction 6.20, we construct relative versions of partition Lie algebras and spectral partition Lie algebras. The resulting ∞-categories are denoted by $\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{A,\Delta}}$ and $\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}}$, respectively. Finally, we prove:

Theorem 1.21. *Let k be a field.*

1. *If A is a complete local Noetherian simplicial commutative ring with residue field k, there is an equivalence of ∞-categories $\mathrm{Moduli}_{A//k,\Delta} \simeq \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{A,\Delta}}$.*
2. *If A is a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k, then there is an equivalence of ∞-categories $\mathrm{Moduli}_{A//k,\mathbb{E}_\infty} \simeq \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}}$.*

In both cases, these equivalences send a formal moduli problem to its tangent fibre.

Hence the various variants of partition Lie algebras provide an algebraic description of formal deformation theory in finite and mixed characteristic.

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2. Preliminaries

Let $\mathcal{C}$ be a presentable stable ∞-category. In this section, we will briefly review various preliminaries involving filtered and graded objects in $\mathcal{C}$, and moreover fix some notation for the remainder of this paper. A convenient reference for this material (with slightly different notation) is [GP18].

For notational convenience, we will usually work with filtrations and gradings concentrated in degrees 1 and above. This choice reflects that in the sequel (in particular in Section 4), we will often work with nonunital commutative algebras. We will use the notation Fil and Gr in this context.

When we discuss unital commutative algebras in Section 5, we will use the notation Fil^+ and Gr^+ for filtrations and gradings that start in degree zero.

Definition 2.1 (Filtered objects)**.** Consider the nerve $\mathrm{N}(\mathbb{Z}_{\geq 1})$ of the partially ordered set $\mathbb{Z}_{\geq 1}$ and its opposite $\mathrm{N}(\mathbb{Z}_{\geq 1})^{op}$. We define the ∞-category $\mathrm{Fil}(\mathcal{C})$ of *filtered objects* of $\mathcal{C}$ as

$$\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}(N(\mathbb{Z}_{\geq 1})^{op}, \mathcal{C}).$$

We will often write a filtered object $X \in \mathrm{Fil}(\mathcal{C})$ as a system $\{F^i X\}_{i \geq 1}$ of objects in $\mathcal{C}$, i.e. a sequence $\cdots \to F^i X \to F^{i-1} X \to \cdots \to F^1 X$. We call $F^1 X$ the *underlying object* of X, and obtain a functor

$$\mathrm{und} : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}.$$

Similarly, we define $\mathrm{Fil}^+(\mathcal{C})$ as

$$\mathrm{Fil}^+(\mathcal{C}) := \mathrm{Fun}(N(\mathbb{Z}_{\geq 0})^{op}, \mathcal{C}).$$

The objects of $\mathrm{Fil}^+(\mathcal{C})$ are filtered objects where the filtration starts in degree zero instead.

Example 2.2. The functor $\mathrm{und} : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ sending $\{F^i X\}$ to $F^1 X \in \mathcal{C}$ admits a left adjoint, which sends an object $Y \in \mathcal{C}$ to the filtered object $(\cdots \to 0 \to 0 \to Y)$.

Definition 2.3 (Graded objects)**.** Let $\mathbb{Z}^{\mathrm{ds}}_{\geq 1}$ denote the category with one object for every nonnegative integer and only identity morphisms. Define the ∞-category $\mathrm{Gr}(\mathcal{C})$ of *graded objects* of $\mathcal{C}$ as

$$\mathrm{Gr}(\mathcal{C}) = \mathrm{Fun}(\mathrm{N}(\mathbb{Z}^{\mathrm{ds}}_{\geq 1}), \mathcal{C}).$$

We write objects of $\mathrm{Gr}(\mathcal{C})$ as $X_\star$ whenever we want to emphasize the grading. Given a graded object $X_\star$, the direct sum $\bigoplus_{i \geq 1} X_i$. is referred to as the *underlying object* of $X_\star$.

Similarly as for filtered objects, we define a variant $\mathrm{Gr}^+(\mathcal{C})$ as

$$\mathrm{Gr}^+(\mathcal{C}) = \mathrm{Fun}(\mathrm{N}(\mathbb{Z}^{\mathrm{ds}}_{\geq 0}), \mathcal{C}),$$

where $\mathbb{Z}^{\mathrm{ds}}_{\geq 0}$ denotes the discrete category on $\mathbb{Z}_{\geq 0}$.

Definition 2.4 (Associated gradeds)**.** We have a functor

$$\mathrm{Gr} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C}),$$

which sends $X = \{F^i X\}_{i \geq 1}$ to the associated graded object $\mathrm{Gr}X$ satisfying $(\mathrm{Gr}X)_i = F^i X / F^{i+1} X$ for all $i \geq 1$. Similarly, we have a natural functor

$$\mathrm{Gr} : \mathrm{Fil}^+(\mathcal{C}) \to \mathrm{Gr}^+(\mathcal{C}).$$

Definition 2.5 (Symmetric monoidal structures)**.** Suppose that $\mathcal{C}$ is nonunital presentably symmetric monoidal, by which we mean that $\mathcal{C}$ is presentable and the tensor product preserves colimits in each variable. Using Day convolution (cf. [Gla16]), one can equip both $\mathrm{Fil}(\mathcal{C})$ and $\mathrm{Gr}(\mathcal{C})$ with the structure of presentably nonunital symmetric monoidal ∞-categories. Furthermore, the associated graded functor $\mathrm{Gr} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ is (nonunital) symmetric monoidal (cf. [GP18, Sec. 2.23]).

Definition 2.6 (Gradings in degree $\geq a$)**.** Given an integer $a \geq 1$,

(1) let $\mathrm{Gr}_{\geq a}(\mathcal{C})$ be the full subcategory of $\mathrm{Gr}(\mathcal{C})$ spanned by all $X_\star$ with $X_j \simeq 0$ for $j < a$.
(2) let $\mathrm{tr}_{\leq a} : \mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ denote the functor which sends a graded object $X_\star$ to the graded object $Y_\star$ with $Y_j = X_j$ for $j \leq a$ and $Y_j = 0$ for $j > a$.

Next, we will review the process of completion.

Definition 2.7 (Complete and constant filtered objects)**.**

(1) A filtered object $Y = \{F^i Y\}_{i\geq 1}$ is said to be *constant* if the maps $F^{i+1}Y \to F^i Y$ are equivalences for all $i \geq 1$, or equivalently, if $\mathrm{Gr}(Y) = 0$.
(2) A filtered object $Z = \{F^i Z\}_{i\geq 1}$ is *complete* if $\varprojlim_i F^i Z = 0$, i.e. if for each constant filtered object $Y \in \mathrm{Fil}(\mathcal{C})$, we have $\mathrm{Hom}_{\mathrm{Fil}(\mathcal{C})}(Y, Z) = 0$. Let $\mathrm{Fil}_{\mathrm{cpl}}(\mathcal{C}) \subset \mathrm{Fil}(\mathcal{C})$ denote the full subcategory of complete objects. We have a similar notion for objects of $\mathrm{Fil}^+(\mathcal{C})$.

Definition 2.8 (Completions)**.** The general theory implies that the inclusion $\mathrm{Fil}_{\mathrm{cpl}}(\mathcal{C}) \subset \mathrm{Fil}(\mathcal{C})$ is the right adjoint of a Bousfield localisation, which we will refer to as the *completion* functor $\mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}_{\mathrm{cpl}}(\mathcal{C})$. Given a filtered object $X = \{F^i X\}_{i\geq 1}$, we can form its *completion* $\hat{X}$. The canonical map $X \to \widehat{X}$ induces an equivalence on $\mathrm{Gr}(-)$.

Remark 2.9. We can detect whether a given morphism $X \xrightarrow{f} Y$ induces an equivalence after completion by passing to associated gradeds. Indeed, if $\mathrm{Gr}(f) : \mathrm{Gr}(X) \to \mathrm{Gr}(Y)$ is an equivalence, then $\mathrm{Gr}(\mathrm{cofib}(f)) \simeq \mathrm{cofib}(\mathrm{Gr}(f)) \simeq 0$. This implies that $\mathrm{cofib}(\widehat{f}) \simeq \widehat{\mathrm{cofib}(f)}$ is both constant and complete, and therefore vanishes.

In general, $\mathrm{Fil}_{\mathrm{cpl}}(\mathcal{C}) \subset \mathrm{Fil}(\mathcal{C})$ is not closed under colimits. However completions are preserved by geometric realisations under suitable connectivity hypotheses. More precisely, let R be a connective $\mathbb{E}_\infty$-ring. Write $\mathrm{Mod}_{R,\geq 0} \subset \mathrm{Mod}_R$ for the full subcategory of connective R-modules.

Definition 2.10 (Connective filtered and graded objects)**.** We define:

(1) $\mathrm{FilMod}_{R,\geq 0} \subset \mathrm{FilMod}_R$ is the subcategory spanned by all $X = \{F^i X\}_{i\geq 1}$ for which each $F^i X$ belongs to $\mathrm{Mod}_{R,\geq 0}$; we have an analogous subcategory $\mathrm{Fil}^+\mathrm{Mod}_{R,\geq 0} \subset \mathrm{Fil}^+\mathrm{Mod}_R$.
(2) $\mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{R,\geq 0} \subset \mathrm{FilMod}_{R,\geq 0}$ is the subcategory of complete objects (and similarly for Fil^+).
(3) $\mathrm{GrMod}_{R,\geq 0} \subset \mathrm{GrMod}_R$ is the subcategory of objects $X_\star$ with X_i connective for each $i \geq 1$.

In the sequel, we will make frequent use of the following observation:

Proposition 2.11. *The subcategory* $\mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{R,\geq 0} \subset \mathrm{FilMod}_R$ *is closed under geometric realisations.*

Proof. Let $X_\bullet$ be a simplicial object in $\mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{R,\geq 0}$ and let $Y = |X_\bullet|$ denote its geometric realisation (computed in FilMod_R). We need to see that Y is complete, i.e. $\varprojlim_i F^i Y = 0$ in Mod_R. By the Milnor short exact sequence, this is equivalent to the assertion that for each j, we have

$$\varprojlim_i \pi_j(F^i Y) = \varprojlim{}^1_i \pi_j(F^i Y) = 0, \tag{1}$$

in the category of abelian groups.

We observe that $F^i Y = |F^i X_\bullet|$. Since all modules in question are connective, we have $\pi_j(F^i Y) \cong \pi_j(\mathrm{sk}_n|F^i X_\bullet|)$ for $n > j$ and all i. Thus, in verifying (1) for a given j, we may replace Y with the filtered object $\mathrm{sk}_{j+1}|X_\bullet| \in \mathrm{FilMod}_R$. This can be expressed as a *finite* colimit of a diagram in the X_s, and since $\mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_R \subset \mathrm{FilMod}_R$ is closed under finite colimits, we deduce $\mathrm{sk}_{j+1}|X_\bullet| \in \mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_R$. Applying the Milnor exact sequence again to $\mathrm{sk}_{j+1}|X_\bullet| \in \mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_R$ shows (1). ☐

In the sequel, it will be important to understand how functors F interact with the internal grading. To this end, we will use the following natural definition.

Definition 2.12 (Increasing functors)**.** We say that a functor $F : \mathrm{GrMod}_{k,\geq 0} \longrightarrow \mathrm{GrMod}_{k,\geq 0}$ is *i-increasing* if:

(1) The functor $\mathrm{tr}_{\leq n}F$ factors through $\mathrm{tr}_{\leq n-i+1} : \mathrm{GrMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$.
(2) Given any $X \in \mathrm{GrMod}_{k,\geq 0}$, we have $F(X)_j = 0$ for all $j < i$. That is, F takes values in objects which have contractible components in internal grading less than i.

Example 2.13. The functor $V \mapsto V^{\otimes i}, \mathrm{GrMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$ is i-increasing. Similarly, the functor $V \mapsto (V^{\otimes i})_{h\Sigma_i}$ (which appears in the expression for the free $\mathbb{E}_\infty$-algebra) is i-increasing.

Next, we record several finiteness conditions which will be useful in the sequel. Mostly the following serves to record some notation.

Definition 2.14 (Finiteness conditions)**.** Let k be a field.

(1) Let $\mathrm{Mod}_k^{\mathrm{ft}} \subset \mathrm{Mod}_k$ denote the subcategory spanned by those objects $X \in \mathrm{Mod}_k$ such that each homotopy group $\pi_i(X), i \in \mathbb{Z}$ is a finite-dimensional vector space. We say that these objects are of *finite type.* Define $\mathrm{Mod}_{k,\geq 0}^{\mathrm{ft}}, \mathrm{Mod}_{k,\leq 0}^{\mathrm{ft}} \subset \mathrm{Mod}_k^{\mathrm{ft}}$ as the subcategories spanned by connective and coconnective objects, respectively.
(2) Let $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_k \subset \mathrm{GrMod}_k$ denote the subcategory spanned by all objects $X_\star \in \mathrm{GrMod}_k$ for which $\bigoplus_{i\geq 1} X_i$ belongs to $\mathrm{Mod}_k^{\mathrm{ft}}$. We define $\mathrm{Gr}^{+}\mathrm{Mod}_k^{\mathrm{ft}}$ in a similar way. We let $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0} \subset \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_k$ be the subcategory spanned by connective objects.
(3) Let $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_k \subset \mathrm{FilMod}_k$ denote the subcategory of filtered objects $X = \{F^iX\}_{i\geq 1}$ which are complete and such that $\mathrm{Gr}(X) \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_k$. Similarly, we denote the full subcategory of connective objects by $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0} \subset \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_k$. (As an example, we could take a (discrete) finite-dimensional k-vector space with a finite classical filtration by subspaces).

3. Functors of k-modules

Let k be a field and write Mod_k for the ∞-category of k-module spectra. In this section, we will discuss functors $\mathrm{Mod}_k \to \mathrm{Mod}_k$ which preserve sifted colimits, which is equivalent to preserving filtered colimits and geometric realisations.

Below, we will need to construct various such functors $\mathrm{Mod}_k \to \mathrm{Mod}_k$, e.g. the free partition Lie algebra functor. One typically cannot write down such a functor easily by hand on all of Mod_k. However, it will be easy to describe functor on a suitable subcategory on Mod_k, often in particular the *coconnective* perfect k-module spectra. For the general theory we will need our functors on all of Mod_k, though, and the primary purpose of this section is to discuss some abstract homological algebra which will enable us to construct the extension to all of Mod_k.

3.1. **Extending Functors.** In the following, we will freely use the theory of Kan extensions along fully faithful inclusions, as in [Lur09, Section 4.3.2].

Notation 3.1. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories admitting sifted colimits. We write $\mathrm{Fun}_\Sigma(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by all functors which preserve sifted colimits.

The first observation is that it is easy to describe sifted-colimit-preserving functors out of the full subcategory $\mathrm{Mod}_{k,\geq 0} \subset \mathrm{Mod}_k$ of connective k-module spectra. In fact, $\mathrm{Mod}_{k,\geq 0}$ can be characterised by a universal property (cf. [Lur17, Sec. 7.2.2]):

Proposition 3.2 (The universal property of $\mathrm{Mod}_{k,\geq 0}$)**.** *Given any ∞-category $\mathcal{D}$ with sifted colimits, restriction induces an equivalence of ∞-categories*

$$\mathrm{Fun}_\Sigma(\mathrm{Mod}_{k,\geq 0}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{Vect}_k^\omega, \mathcal{D})$$

whose inverse is given by left Kan extension. Here Vect_k^ω denotes the full subcategory spanned by all finite-dimensional discrete k-module spectra, which is equivalent to the (nerve of the) usual category of finite-dimensional k-vector spaces.

One therefore has the following construction (which goes back to the work of Dold-Puppe [DP61]):

Construction 3.3 (Nonabelian derived functors)**.** Fix a functor $F : \mathrm{Vect}_k^\omega \to \mathrm{Vect}_k$ from the category Vect_k^ω of finite-dimensional k-vector spaces to the category Vect_k of all k-vector spaces. Using that Vect_k is equivalent to the full subcategory of Mod_k spanned by all discrete k-module spectra, we can extend F to a sifted-colimit-preserving functor $\mathbb{L}F : \mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$. The functor $\mathbb{L}F$ is often called the *nonabelian derived functor* of F_0.

Example 3.4. We recall the following classical examples:

(1) Given an integer $n \geq 0$, consider the functor $\bigotimes^n : \mathrm{Vect}_k^\omega \to \mathrm{Vect}_k^\omega$ which sends a vector space V_0 to $V_0^{\otimes n}$. This canonically extends to a functor $\mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$, which is necessarily just the iterated tensor power functor $\bigotimes^n : \mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$ coming from the symmetric monoidal structure on $\mathrm{Mod}_{k,\geq 0}$.

(2) Given $n \geq 0$, we consider the functors Sym^n, $\bigwedge^n$, $\Gamma^n : \mathrm{Vect}_k^\omega \to \mathrm{Vect}_k$ which send a finite-dimensional vector space V to its n^{th} symmetric, exterior, or divided power, respectively. The nonabelian derived functor Construction 3.3 again allows us to define canonical extensions $\mathbb{L}\mathrm{Sym}^i, \mathbb{L}\bigwedge^i, \mathbb{L}\Gamma^i : \mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$ of these three functors.

There is a basic asymmetry between these two examples. The extended functor on $\mathrm{Mod}_k^{\geq 0}$ arose naturally from the symmetric monoidal structure; consequently, the functor $\bigotimes^i$ is naturally defined on all of Mod_k, not only on the connective k-module spectra. By contrast, if k is of characteristic $p > 0$, the functors $\mathbb{L}\mathrm{Sym}^i$ generally cannot be described directly in terms of the symmetric monoidal structure on $\mathrm{Mod}_{k,\geq 0}$. It is correspondingly less clear that $\mathbb{L}\mathrm{Sym}^i$ and $\mathbb{L}\bigwedge^i$ naturally extend to all of Mod_k, though this was first shown in work of Illusie [Ill71, Sec. I-4].

In this section, we will establish two generalisations of Proposition 3.2 to functors defined on *all* k-module spectra, and help bridge the above asymmetry: in particular, we will describe functors such as $\mathbb{L}\mathrm{Sym}^i, \mathbb{L}\bigwedge^i$ on all of Mod_k.

We begin by reviewing some basic facts about compact generation and perfect modules, and refer to [Lur17, Sec. 7.2.4] for a detailed treatment.

Notation 3.5 (Perfect modules)**.** We write $\mathrm{Perf}_k \subset \mathrm{Mod}_k$ for the full subcategory of Mod_k spanned by all *perfect* k-module spectra, i.e. k-module spectra M with $\dim_k(\pi_*(M)) < \infty$. Let $\mathrm{Perf}_{k,[n_1,n_2]}$ be the full subcategory of Perf_k spanned by k-module spectra whose homotopy groups are concentrated between degrees n_1 and n_2. Set $\mathrm{Perf}_{k,\geq n} := \mathrm{Perf}_{k,[n,\infty]}$ and $\mathrm{Perf}_{k,\leq n} := \mathrm{Perf}_{k,[-\infty,n]}$.

The ∞-category Mod_k is a compactly generated ∞-category, and a module spectrum $M \in \mathrm{Mod}_k$ is compact if and only if it is perfect. The ∞-category Mod_k can therefore be identified with the Ind-completion (cf. [Lur09, Sec. 5.3.5]) of Perf_k. We deduce that for any ∞-category $\mathcal{D}$ with filtered colimits, restriction and left Kan extension give mutually inverse equivalences

$$\mathrm{Fun}_\omega(\mathrm{Mod}_k, \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{Perf}_k, \mathcal{D}) \tag{2}$$

between $\mathrm{Fun}(\mathrm{Perf}_k, \mathcal{D})$ and the ∞-category $\mathrm{Fun}_\omega(\mathrm{Mod}_k, \mathcal{D})$ of functors $\mathrm{Mod}_k \to \mathcal{D}$ which preserve filtered colimits.

Notation 3.6. Given a simplicial diagram $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ in some ∞-category $\mathcal{C}$, we write $|X_\bullet| := \mathrm{colim}_{\Delta^{op}}(X_\bullet)$ for its geometric realisation. The simplicial object $X_\bullet$ is said to be *m-skeletal* if it is the left Kan extension of its restriction to $\Delta^{op}_{\leq m}$, the full subcategory of Δ^{op} spanned by $[0], [1], \ldots, [m]$. We recall that $\Delta^{op}_{\leq m}$ is cofinal to a finite simplicial set (see [Lur17, 1.2.4.17]), so that geometric realisations of m-skeletal simplicial objects behave like finite colimits.

Definition 3.7 (Finite geometric realisations)**.** Let $\mathcal{D}$ be an ∞-category admitting geometric realisations. We say that a functor $F : \mathrm{Perf}_k \to \mathcal{D}$ *preserves finite geometric realisations* if for every simplicial object $X_\bullet$ of Perf_k which is m-skeletal for some m (so that $|X_\bullet|$ belongs to Perf_k), the natural map $|F(X_\bullet)| \to F(|X_\bullet|)$ is an equivalence. We write $\mathrm{Fun}_\sigma(\mathrm{Perf}_k, \mathcal{D}) \subset \mathrm{Fun}(\mathrm{Perf}_k, \mathcal{D})$ for the full subcategory spanned by functors which preserve finite geometric realisations.

We can now state our first generalisation of Proposition 3.2:

Proposition 3.8. *Given any ∞-category $\mathcal{D}$ with sifted colimits, restriction and left Kan extension induce mutually inverse equivalences*

$$\mathrm{Fun}_\Sigma(\mathrm{Mod}_k, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_\sigma(\mathrm{Perf}_k, \mathcal{D}).$$

Proof. This follows from the equivalence $\mathrm{Fun}_\omega(\mathrm{Mod}_k, \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{Perf}_k, \mathcal{D})$ stated in (2). It suffices to show that a functor $F : \mathrm{Mod}_k \to \mathcal{D}$ which preserves filtered colimits additionally preserves sifted colimits if and only if its restriction $F|_{\mathrm{Perf}_k}$ preserves finite geometric realisations. This follows easily from the following facts:

(1) The functor F preserves sifted colimits if and only if it preserves filtered colimits (which it does by assumption) and geometric realisations (cf. [Lur09, Corollary 5.5.8.17]).
(2) Every simplicial object in Mod_k is a filtered colimit of simplicial objects which are m-skeletal for various m (take the left Kan extensions from truncations).

(3) Every m-skeletal simplicial object in Mod_k is a filtered colimit of m-skeletal simplicial objects which take values in Perf_k (this follows as the hom-sets in $\Delta^{op}_{\leq m}$ are finite). $\square$

For later applications, we need to refine the above result from Perf_k to a smaller subcategory.

Definition 3.9 (Finite coconnective geometric realisations)**.** Let $\mathcal{D}$ be an ∞-category admitting geometric realisations. We say that a functor $F : \mathrm{Perf}_{k,\leq 0} \to \mathcal{D}$ *preserves finite coconnective geometric realisations* if for every simplicial object $X_\bullet$ of $\mathrm{Perf}_{k,\leq 0}$ which is m-skeletal for some m and such that $|X_\bullet|$ belongs to $\mathrm{Perf}_{k,\leq 0}$, the natural map $|F(X_\bullet)| \to F(|X_\bullet|)$ is an equivalence. We write $\mathrm{Fun}_\sigma(\mathrm{Perf}_{k,\leq 0}, \mathcal{D}) \subset \mathrm{Fun}(\mathrm{Perf}_{k,\leq 0}, \mathcal{D})$ for the full subcategory spanned by functors which preserve finite coconnective geometric realisations.

We will now interpret the condition in Definition 3.9 in terms of left Kan extensions (albeit with an infinite number of such conditions).

Proposition 3.10. *Let $\mathcal{D}$ be an ∞-category with sifted colimits.*

(1) Let $F \in \mathrm{Fun}(\mathrm{Perf}_k, \mathcal{D})$. Then F preserves finite geometric realisations if and only if, for each $n \geq 0$, the restriction $F|_{\mathrm{Perf}_{k,\geq -n}}$ is left Kan extended from $\mathrm{Vect}^\omega_k[-n]$.

(2) Let $F \in \mathrm{Fun}(\mathrm{Perf}_{k,\leq 0}, \mathcal{D})$. Then F preserves finite coconnective geometric realisations if and only if, for any $n \geq 0$, the restriction $F|_{\mathrm{Perf}_{k,[-n,0]}}$ is left Kan extended from $\mathrm{Vect}^\omega_k[-n]$.

Proof. We shall only prove statement (2); the proof of (1) is similar. Suppose $F \in \mathrm{Fun}_\sigma(\mathrm{Perf}_{k,\leq 0}, \mathcal{D})$. We claim that $F_n = F|_{\mathrm{Perf}_{k,[-n,0]}}$ is left Kan extended from $\mathrm{Vect}^\omega_k[-n]$. Analogously to Proposition 3.2, the left Kan extension of $F^0_n := F|_{\mathrm{Vect}^\omega_k[-n]}$ to $\mathrm{Perf}_{k,[-n,0]}$ (and indeed to all of $\mathrm{Mod}_{k,\geq -n}$) can be computed as follows: given $X \in \mathrm{Perf}_{k,[-n,0]}$, we find an n-skeletal simplicial object $Y_\bullet$ with each $Y_i \in \mathrm{Vect}^\omega_k[-n]$ such that $|Y_\bullet| \simeq X$. The value of the left Kan extension of F^0_n on X is then given by $|F^0_n(Y_\bullet)|$. Since F preserves finite coconnective geometric realisations, it follows that this agrees with $F_n(X)$ and F_n is indeed left Kan extended from $\mathrm{Vect}^\omega_k[-n]$, as desired.

Conversely, suppose $F \in \mathrm{Fun}(\mathrm{Perf}_{k,\leq 0}, \mathcal{D})$ has the property that $F_n = F|_{\mathrm{Perf}_{k,[-n,0]}}$ is left Kan extended from $\mathrm{Vect}^\omega_k[-n]$ for all $n \geq 0$. For each n, it then follows (as in Proposition 3.2) that F preserves geometric realisations of simplicial objects in $\mathrm{Perf}_{k,[-n,0]}$ whose realisation also belongs to $\mathrm{Perf}_{k,[-n,0]}$. Since any m-skeletal simplicial object in $\mathrm{Perf}_{k,\leq 0}$ is a simplicial object in $\mathrm{Perf}_{k,[-n,0]}$ for n sufficiently large, we deduce that F preserves all coconnective finite geometric realisations. $\square$

Corollary 3.11. *If $F : \mathrm{Perf}_k \to \mathcal{D}$ preserves finite geometric realisations, then F is left Kan extended from $\mathrm{Perf}_{k,\leq 0}$.*

Proof. The statement follows from part (1) of Proposition 3.10 by "taking the limit as $n \to \infty$". More precisely, since $F|_{\mathrm{Perf}_{k,\geq -n}}$ is left Kan extended from $\mathrm{Vect}^\omega_k[-n]$, it is also left Kan extended from the larger subcategory $\mathrm{Perf}_{k,[-n,0]}$. The claim then follows from the remark below. $\square$

Remark 3.12. Let $\mathcal{C} = \bigcup_n \mathcal{C}^n$ be the union of an increasing chain $\mathcal{C}^1 \subset \mathcal{C}^2 \subset \dots$ of full subcategories of $\mathcal{C}$ and suppose that $F : \mathcal{C} \to \mathcal{D}$ has the property that $F|_{\mathcal{C}^n}$ is left Kan extended from $\mathcal{C}^n_0 := \mathcal{C}_0 \cap \mathcal{C}^n$ for all n. Then F is left Kan extended from $\mathcal{C}_0$. This follows from the definition of a Kan extension because for any $x \in \mathcal{C}$, we have an equivalence of ∞-categories $(\mathcal{C}_0)_{/x} = \mathrm{colim}_n (\mathcal{C}^n_0)_{/x}$.

We arrive at our second generalisation of Proposition 3.2:

Proposition 3.13. *Given any ∞-category $\mathcal{D}$ with sifted colimits, restriction induces an equivalence*

$$\mathrm{Fun}_\Sigma(\mathrm{Mod}_k, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_\sigma(\mathrm{Perf}_{k,\leq 0}, \mathcal{D})$$

between $\mathrm{Fun}_\Sigma(\mathrm{Mod}_k, \mathcal{D})$ *and the full subcategory of* $\mathrm{Fun}(\mathrm{Perf}_{k,\le 0}, \mathcal{D})$ *spanned by all functors which preserve finite coconnective geometric realisations. The inverse is given by left Kan extension.*

Proof. In view of Proposition 3.8, it suffices to show that the restriction functor $\mathrm{Fun}_\sigma(\mathrm{Perf}_k, \mathcal{D}) \to \mathrm{Fun}_\sigma(\mathrm{Perf}_{k,\le 0}, \mathcal{D})$ is an equivalence whose inverse is given by taking the left Kan extension. By Corollary 3.11, this restriction functor is fully faithful.

For essential surjectivity, we will check that if $G : \mathrm{Perf}_{k,\le 0} \to \mathcal{D}$ preserves finite coconnective geometric realisations, then its left Kan extension $\widetilde{G} : \mathrm{Perf}_k \to \mathcal{D}$ preserves finite geometric realisations. For this, let $\widetilde{G}_n$ denote the left Kan extension of $G|_{\mathrm{Perf}_{k,[-n,0]}}$ to Perf_k. The various functors $\widetilde{G}_n$ are linked by natural transformations $\widetilde{G}_0 \to \widetilde{G}_1 \to \widetilde{G}_2 \to \ldots$. By Proposition 3.10(2), the restriction of $\widetilde{G}_n$ to $\mathrm{Perf}_{k,\ge -n}$ preserves finite geometric realisations, as it is left Kan extended from $\mathrm{Vect}_k^\omega[-n]$. Any simplicial object in Perf_k which is m-skeletal for some m belongs to $\mathrm{Perf}_{k,\ge -n}$ for $n \gg 0$, and thus its geometric realisation is preserved by $\widetilde{G}_n$ for n sufficiently large. The result then follows from the equivalence $\widetilde{G} \simeq \mathrm{colim}_n \widetilde{G}_n$. □

Proposition 3.13 characterises sifted-colimit-preserving functors $F : \mathrm{Mod}_k \to \mathcal{D}$ in terms of their restriction to $\mathrm{Perf}_{k,\le 0}$. Setting $\mathcal{D} = \mathrm{Mod}_k$, we can deduce:

Corollary 3.14. *Let* $\mathrm{End}_\Sigma^{\mathrm{Perf}_{\le 0}}(\mathrm{Mod}_k)$ *be the full subcategory of* $\mathrm{End}_\Sigma(\mathrm{Mod}_k)$ *spanned by those functors which preserve* $\mathrm{Perf}_{k,\le 0}$. *Then the monoidal restriction functor*

$$\mathrm{End}_\Sigma^{\mathrm{Perf}_{\le 0}}(\mathrm{Mod}_k) \to \mathrm{End}_\sigma(\mathrm{Perf}_k^{\le 0})$$

is an equivalence. Here $\mathrm{End}_\sigma(\mathrm{Perf}_{k,\le 0})$ *denotes the* ∞*-category of endofunctors of* $\mathrm{Perf}_{k,\le 0}$ *which preserve finite coconnective geometric realisations.*

Corollary 3.14 allows us to extend functors $F : \mathrm{Perf}_{k,\le 0} \to \mathrm{Perf}_{k,\le 0}$ which preserve suitable colimits to sifted-colimit-preserving endofunctors of Mod_k in a monoidal fashion. However, the functors which we will want to extend later will usually *not* preserve Perf_k. Instead, they will preserve the following larger subcategory of Mod_k. We will now record slight variants of the above results extending from $\mathrm{Mod}_{k,\le 0}^{\mathrm{ft}}$ instead of $\mathrm{Perf}_{k,\le 0}$.

Definition 3.15. Let $\mathcal{D}$ be an ∞-category with sifted colimits. A functor $F : \mathrm{Mod}_{k,\le 0}^{\mathrm{ft}} \to \mathcal{D}$ is said to *preserve finite coconnective geometric realisations* if for every simplicial object $X_\bullet \in \mathrm{Mod}_{k,\le 0}^{\mathrm{ft}}$ which is m-skeletal for some m and with $|X_\bullet| \in \mathrm{Mod}_{k,\le 0}^{\mathrm{ft}}$, the natural map $|F(X_\bullet)| \to F(|X_\bullet|)$ is an equivalence. We say that F is *right complete* if for any $X \in \mathrm{Mod}_{k,\le 0}^{\mathrm{ft}}$, the natural map $\mathrm{colim}_n F(\tau_{\ge -n} X) \to F(X)$ is an equivalence. We write $\mathrm{Fun}'_\sigma(\mathrm{Mod}_{k,\le 0}^{\mathrm{ft}}, \mathcal{D}) \subset \mathrm{Fun}(\mathrm{Mod}_{k,\le 0}^{\mathrm{ft}}, \mathcal{D})$ for the full subcategory spanned by all functors which preserve finite coconnective geometric realisations and which are right complete.

We can now deduce the following "finite type variant" of Proposition 3.13:

Proposition 3.16. *Given any* ∞*-category* $\mathcal{D}$ *with sifted colimits, restriction induces an equivalence*

$$\mathrm{Fun}_\Sigma(\mathrm{Mod}_k, \mathcal{D}) \longrightarrow \mathrm{Fun}'_\sigma(\mathrm{Mod}_{k,\le 0}^{\mathrm{ft}}, \mathcal{D})$$

between $\mathrm{Fun}_\Sigma(\mathrm{Mod}_k, \mathcal{D})$ *and the full subcategory of* $\mathrm{Fun}(\mathrm{Mod}_{k,\le 0}^{\mathrm{ft}}, \mathcal{D})$ *spanned by all functors which preserve finite coconnective geometric realisations and are right complete.*

Proof. Using Proposition 3.13, we observe that it suffices to check that the restriction functor $\mathrm{Fun}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}, \mathcal{D}) \to \mathrm{Fun}_\sigma(\mathrm{Perf}_{k,\leq 0}, \mathcal{D})$ is an equivalence with inverse given by left Kan extension.

Given $F \in \mathrm{Fun}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}, \mathcal{D})$, the restriction $F|_{\mathrm{Perf}_{k,\leq 0}}$ preserves finite coconnective geometric realisations. Proposition 3.13 implies that the left Kan extension of $F|_{\mathrm{Perf}_{k,\leq 0}}$ to Mod_k preserves all sifted colimits, which in turn shows that the left Kan extension $\widetilde{F}$ of $F|_{\mathrm{Perf}_{k,\leq 0}}$ to $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ is right complete. We deduce that $\widetilde{F} \to F$ is a transformation between right-complete functors $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0} \to \mathcal{D}$ which is an equivalence on $\mathrm{Perf}_{k,\leq 0}$. This transformation is therefore an equivalence and F is left Kan extended from $\mathrm{Perf}_{k,\leq 0}$. Hence, the restriction $\mathrm{Fun}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}, \mathcal{D}) \to \mathrm{Fun}_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}, \mathcal{D})$ is fully faithful. It is also essentially surjective since the left Kan extension of any $G \in \mathrm{Fun}_\sigma(\mathrm{Perf}_{k,\leq 0}, \mathcal{D})$ to Mod_k preserves sifted colimits by Proposition 3.13, which implies that the left Kan extension of G to $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ is right complete and preserves finite geometric realisations. □

Setting $\mathcal{D} = \mathrm{Mod}_k$ in Proposition 3.16, we can deduce the following result, which will be crucial in our later applications to extend monads. We let $\mathrm{End}_\Sigma(\mathrm{Mod}_k)$ denote the monoidal ∞-category of functors $\mathrm{Mod}_k \to \mathrm{Mod}_k$ which preserve sifted colimits.

Corollary 3.17. *Let* $\mathrm{End}_\Sigma^{\mathrm{Mod}^{\mathrm{ft}}_{\leq 0}}(\mathrm{Mod}_k)$ *be the subcategory of* $\mathrm{End}_\Sigma(\mathrm{Mod}_k)$ *spanned by all functors which preserve* $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$*. Then the monoidal restriction functor*

$$\mathrm{End}_\Sigma^{\mathrm{Mod}^{\mathrm{ft}}_{\leq 0}}(\mathrm{Mod}_k) \to \mathrm{End}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})$$

is an equivalence. Here $\mathrm{End}'_\sigma(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})$ *denotes the* ∞*-category of endofunctors of* $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ *which preserve finite coconnective geometric realisations and are right-complete.*

3.2. **Right-Left Extension.** We shall now apply the tools developed in the previous subsection and build an array of extended functors.

Remark 3.18. Closely related ideas appear in the work of Illusie [Ill71, Sec. I-4], and more recently in the work of Kaledin [Kal15, Sec. 3].

Throughout this subsection, we fix a stable ∞-category $\mathcal{D}$ admitting all limits and colimits and a field k. Our basic procedure first extends a functor on finite-dimensional k-vector spaces in the coconnective direction and then in the connective direction.

More precisely, let $F : \mathrm{Vect}_k^\omega \to \mathcal{D}$ be a functor. Our goal is to extend F to a sifted-colimit-preserving functor $\mathrm{Mod}_k \to \mathcal{D}$. In a first step, we take the right Kan extension $F^R : \mathrm{Perf}_{k,\leq 0} \to \mathcal{C}$ of F along the inclusion $\mathrm{Vect}_k^\omega \subset \mathrm{Perf}_{k,\leq 0}$.

Remark 3.19. Using linear duality and Proposition 3.2, we see that the right Kan extension F^R of a functor F as above can be computed as follows: given $X \in \mathrm{Perf}_{k,\leq 0}$, we write $X \simeq \mathrm{Tot}(V^\bullet)$ for $V^\bullet$ a cosimplicial object of Vect_k^ω. Then $F^R(X) \simeq \mathrm{Tot}(F(V^\bullet))$.

In order to further extend $F^R : \mathrm{Perf}_{k,\leq 0} \to \mathcal{D}$ to a sifted-colimit-preserving functor $\mathrm{Mod}_k \to \mathcal{D}$ as in Proposition 3.13, we need to assume the following condition:

Definition 3.20. A functor $F : \mathrm{Vect}_k^\omega \to \mathcal{D}$ is said to be *right-extendable* if the right Kan extension $F^R : \mathrm{Perf}_{k,\leq 0} \to \mathcal{D}$ commutes with finite coconnective geometric realisations (cf. Definition 3.9).

Construction 3.21 (Right-left extension)**.** The *right-left extension* $F^{RL} : \mathrm{Mod}_k \to \mathcal{D}$ of a right-extendable functor $F : \mathrm{Vect}_k^\omega \to \mathcal{D}$ is given by the left Kan extension of $F^R : \mathrm{Perf}_{k,\leq 0} \to \mathcal{D}$ to Mod_k.

Remark 3.22. By Proposition 3.13, the right-left Kan extension F^{RL} of any right-extendable functor F preserves sifted colimits. Hence, F^{RL} restricts on $\mathrm{Mod}_{k,\geq 0}$ to the left Kan extension $\mathbb{L}F$.

Let $\mathrm{Vect}_k \subset \mathrm{Mod}_k$ be the subcategory of discrete k-module spectra, i.e. ordinary k-vector spaces.

Proposition 3.23. *Let $F : \mathrm{Vect}_k \to \mathrm{Mod}_{k,\leq 0}$ be a filtered-colimit-preserving functor, and suppose that the restriction $F|_{\mathrm{Vect}_k^\omega}$ admits a right-left extension $\widetilde{F}$ to Mod_k (cf. Construction 3.21).*

If $M^\bullet$ is a cosimplicial k-vector space, then $\widetilde{F}$ is determined by the formula

$$\widetilde{F}(\mathrm{Tot}(M^\bullet)) \simeq \mathrm{Tot}(F(M^\bullet)).$$

Proof. Since F preserves filtered colimits, we obtain a natural equivalence $\widetilde{F}|_{\mathrm{Vect}_k} \simeq F$. Given a cosimplicial k-vector space $M^\bullet$ as above, we can write

$$M^\bullet \simeq \operatorname*{colim}_{i\in I}(M_i^\bullet),$$

where each M_i^n belongs to Vect_k^ω and every $M_i^\bullet$ is a finite cosimplicial diagram for all i.

Since each M_i^n is coconnective, every truncation $\tau_{\geq m}(\mathrm{Tot}(M^\bullet))$ is only affected by the m^{th} coskeleton of the appearing totalisations. We can therefore commute the filtered colimit past the totalisation to obtain a equivalences $\mathrm{Tot}(M^\bullet) \simeq \mathrm{Tot}(\operatorname*{colim}_{i\in I}(M_i^\bullet)) \simeq \operatorname*{colim}_{i\in I}(\mathrm{Tot}(M_i^\bullet))$. The same argument applies to the diagram $F(M_i^\bullet)$.

Combining this observation with Remark 3.22 and the defining property of $\widetilde{F}$, we obtain equivalences $\widetilde{F}(\mathrm{Tot}(M^\bullet)) \simeq \mathrm{colim}_{i\in I}(\mathrm{Tot}(F(M_i^\bullet))) \simeq \mathrm{Tot}(\mathrm{colim}_{i\in I}(F(M_i^\bullet)))$. Since F preserves filtered colimits, this is then equivalent to $\mathrm{Tot}(F(\mathrm{colim}_{i\in I} M_i^\bullet)) \simeq \mathrm{Tot}(F(M^\bullet))$, as desired. □

Let $\mathrm{Fun}_{RL}(\mathrm{Vect}_k^\omega, \mathcal{D}) \subset \mathrm{Fun}(\mathrm{Vect}_k^\omega, \mathcal{D})$ denote the full subcategory of right-extendable functors. Right-left extension establishes a fully faithful embedding $\mathrm{Fun}_{RL}(\mathrm{Vect}_k^\omega, \mathcal{D}) \subset \mathrm{Fun}_\Sigma(\mathrm{Mod}_k, \mathcal{D})$ whose image consists of all functors F whose restriction to $F|_{\mathrm{Perf}_{k,\leq 0}}$ commutes with totalisations which are m-coskeletal for some m (equivalently, $F|_{\mathrm{Perf}_{k,\leq 0}}$ is right Kan extended from Vect_k^ω).

Construction 3.21 will be our basic tool for building sifted-colimit-preserving functors on Mod_k. In order to proceed, we will need a criterion for right-extendability. We begin by recalling the following definition, originally due to Eilenberg-MacLane [EML54]:

Definition 3.24 (Functors of finite degree)**.** A functor $F : \mathrm{Vect}_k^\omega \to \mathcal{D}$ is said to be

(1) *of degree* 0 if F is constant.
(2) *of degree* n with $n \geq 1$ if for any $X \in \mathrm{Vect}_k^\omega$, the difference functor $D_X F : \mathrm{Vect}_k^\omega \to \mathcal{D}$ defined via $D_X F(Y) = \mathrm{fib}(F(X \oplus Y) \to F(Y))$ is of degree $n-1$.
(3) *of finite degree* if it is of degree n for some $n \geq 0$.

Example 3.25. The n^{th} symmetric, exterior, and divided power functors $\mathrm{Sym}^n, \bigwedge^n, \Gamma^n$ on Vect_k^ω are all of degree n.

We can now state the main result of this section:

Theorem 3.26. *Let $F : \mathrm{Vect}_k^\omega \to \mathcal{D}$ be a functor of finite degree. Then F is right-extendable. In particular, we obtain a canonical sifted-colimit-preserving extension $F : \mathrm{Mod}_k \to \mathcal{D}$.*

Our proof of the above result will rely on Goodwillie's calculus of functors. We recall several of the key definitions from Goodwillie's work (cf. [Goo92] [Goo03]) in their ∞-categorical incarnation, which is described in detail in [Lur17, Chapter 6].

For the rest of this section, we fix a pointed ∞-category $\mathcal{A}$ with finite colimits and a stable ∞-category $\mathcal{D}$ with small colimits.

Definition 3.27 (n-excisive functors)**.**

(1) An $(n+1)$-*cube* in $\mathcal{A}$ is a functor $\mathcal{P}(\{0,\dots,n\}) \to \mathcal{A}$, where $\mathcal{P}(\{0,\dots,n\})$ denotes the poset of finite subsets of $\{0,\dots,n\}$. Such a cube is
   - *strongly coCartesian* if it is left Kan extended from subsets of cardinality at most 1.
   - *coCartesian* if it is a colimit diagram, i.e. its value on $\{0,\dots,n\}$ is determined by its values on proper subsets.

(2) A functor $F : \mathcal{A} \to \mathcal{D}$ is n-*excisive* if F carries strongly coCartesian $(n+1)$-cubes to coCartesian cubes (recall that $\mathcal{D}$ is assumed to be stable).

Let $\mathrm{Exc}^n(\mathcal{A},\mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{A},\mathcal{D})$ spanned by all n-excisive functors $\mathcal{A} \to \mathcal{D}$.

Example 3.28. Let $G : \mathcal{A}^n \to \mathcal{D}$ be a functor which preserves finite colimits in each variable. Then the diagonal functor F defined by $F(X) = G(X,\dots,X)$ is n-excisive.

Remark 3.29. The subcategory $\mathrm{Exc}^n(\mathcal{A},\mathcal{D}) \subset \mathrm{Fun}(\mathcal{A},\mathcal{D})$ is closed under arbitrary limits and colimits: that is, limits and colimits of n-excisive functors are n-excisive. Here and in the preceding Example, we have used that $\mathcal{D}$ is stable.

Remark 3.30. Suppose $\mathcal{A}$ is small. Given an n-excisive functor $F : \mathcal{A} \to \mathcal{D}$, we can canonically extend F to a filtered-colimit-preserving functor $\mathrm{Ind}(\mathcal{A}) \to \mathcal{D}$, where $\mathrm{Ind}(\mathcal{A})$ is the Ind-completion of $\mathcal{A}$. It is not hard to see that this functor is also n-excisive (cf. [Lur17, Proposition 6.1.5.4]). In fact, restriction and left Kan extension establish an equivalence $\mathrm{Exc}^n(\mathcal{A},\mathcal{D}) \simeq \mathrm{Exc}^n_c(\mathrm{Ind}(\mathcal{A}),\mathcal{D})$ between n-excisive functors $\mathcal{A} \to \mathcal{D}$ and n-excisive filtered-colimit-preserving functors $\mathrm{Ind}(\mathcal{A}) \to \mathcal{D}$.

A theorem of Goodwillie allows us to universally approximate functors by n-excisive functors:

Proposition 3.31 (The n-excisive approximation)**.** *The inclusion* $\mathrm{Exc}^n(\mathcal{A},\mathcal{D}) \subset \mathrm{Fun}(\mathcal{A},\mathcal{D})$ *admits a left adjoint* $P_n : \mathrm{Fun}(\mathcal{A},\mathcal{D}) \to \mathrm{Exc}^n(\mathcal{A},\mathcal{D})$*.*

Goodwillie has in fact given a more explicit description of the functor $P_n(F)$ as a sequential colimit $P_n(F) = \mathrm{colim}_n\,(F \to T_n(F) \to T_n(T_n(F)) \to \dots)$. Here $G \mapsto T_n(G)$ is a certain construction on functors $G : \mathcal{A} \to \mathcal{D}$ with the property that the value $T_n(G)(X)$ is obtained as a finite limit of copies of G evaluated on various direct sums of suspensions of X. We will not need to know the precise formula for T_n, except that it has the following implications:

Proposition 3.32. *If F preserves filtered colimits, then so does P_nF.*

Proposition 3.33. *Given a right exact functor $\mathcal{A} \to \mathcal{B}$ between two pointed ∞-categories with finite colimits, the following diagram commutes*

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{B},\mathcal{D}) \arrow[r, "P_n"] \arrow[d] & \mathrm{Exc}^n(\mathcal{B},\mathcal{D}) \arrow[d] \\ \mathrm{Fun}(\mathcal{A},\mathcal{D}) \arrow[r, "P_n"] & \mathrm{Exc}^n(\mathcal{A},\mathcal{D}) \end{tikzcd} .$$

Here the vertical maps are simply restrictions.

Proposition 3.34 (Johnson-McCarthy [JM99, Proposition 5.10])**.** *If $F : \mathrm{Vect}^\omega_k \to \mathcal{D}$ is of degree n, then its nonabelian derived functor $\mathbb{L}F : \mathrm{Mod}^{\geq 0}_k \to \mathcal{D}$ is n-excisive.*

Proof. Given a collection of maps $Y \to X_i$ for $0 \leq i \leq n$, we can form a strongly coCartesian cube $c : \mathcal{P}(\{0,\dots,n\}) \to \mathrm{Mod}^{\geq 0}_k$ by left Kan extension. It suffices to show that the functor LF carries every such cube c to a coCartesian cube $\mathbb{L}F \circ c$.

We first assume that each map $Y \to X_i \simeq Y \oplus Z_i$ is a (split) injection between discrete finite-dimensional k-vector spaces. In this case, the fact that F is of degree n immediately implies that $F \circ c$ is a coCartesian cube.

But we can write any collection $\mathfrak{C} = \{Y, X_i, Y \to X_i\}_{0 \leq i \leq n}$ as a geometric realisation of collections $\mathfrak{C}'$ with all $Y', X_i' \in \mathrm{Vect}_k^\omega$ and each $Y' \to X_i'$ (split) injective. Since $\mathbb{L}F$ preserves geometric realisations, the known assertion for each $\mathfrak{C}'$ implies the desired result for $\mathfrak{C}$. $\square$

Theorem 3.35. *Let $\mathcal{A}$ be small and stable. Suppose that $\mathcal{A}_0$ is a full subcategory of $\mathcal{A}$ which is closed under finite colimits such that for any $X \in \mathcal{A}$, we have $\Sigma^m X \in \mathcal{A}_0$ for $m \gg 0$ sufficiently large. The restriction functor induces an equivalence*

$$\mathrm{Exc}^n(\mathcal{A}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Exc}^n(\mathcal{A}_0, \mathcal{D})$$

whose inverse is given by $F \mapsto P_n(\mathrm{Lan}_{\mathcal{A}_0}^{\mathcal{A}}(F))$.

Proof. The composite functor $\mathrm{Exc}^n(\mathcal{A}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{A}_0, \mathcal{D})$ admits a left adjoint $P_n \circ \mathrm{Lan}_{\mathcal{A}_0}^{\mathcal{A}}$ by [Lur09, Proposition 4.3.2.17], [Lur09, Lemma 4.3.2.13], and Proposition 3.31. Given that $\mathrm{Exc}^n(\mathcal{A}_0, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{A}_0, \mathcal{D})$ is fully faithful, this implies that $\mathrm{Exc}^n(\mathcal{A}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{A}_0, \mathcal{D})$ also admits a left adjoint given by $F \mapsto P_n(\mathrm{Lan}_{\mathcal{A}_0}^{\mathcal{A}} F)$.

This left adjoint is fully faithful. Indeed, given an n-excisive functor $F_0 : \mathcal{A}_0 \to \mathcal{D}$, we first observe that $\mathrm{Lan}_{\mathcal{A}_0}^{\mathcal{A}}(F_0)|_{\mathcal{A}_0} \simeq F_0$ since $\mathcal{A}_0 \subset \mathcal{A}$ is a full subcategory. By Proposition 3.33, Goodwillie's explicit construction of the n-excisive approximation allows us to recover F_0 from $P_n(\mathrm{Lan}_{\mathcal{A}_0}^{\mathcal{A}}(F_0))$:

$$P_n(\mathrm{Lan}_{\mathcal{A}_0}^{\mathcal{A}}(F_0))|_{\mathcal{A}_0} \simeq P_n(\mathrm{Lan}_{\mathcal{A}_0}^{\mathcal{A}}(F_0)|_{\mathcal{A}_0}) \simeq P_n(F_0) \simeq F_0.$$

To conclude the proof, it suffices to show that the right adjoint $\mathrm{Exc}^n(\mathcal{A}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{A}_0, \mathcal{D})$ is conservative. After passing to cofibres, it is enough to prove that an n-excisive functor $F : \mathcal{A} \to \mathcal{D}$ which vanishes on $\mathcal{A}_0$ must also vanish on $\mathcal{A}$. For each $r \geq 0$, we consider the statement S_r that $F(\Sigma^r X^{\oplus m}) \simeq 0$ for all $m \geq 0$ and all $X \in \mathcal{A}$. Since F is n-excisive, statement S_r implies statement S_{r-1} as we can use the strongly coCartesian $(n+1)$-cube obtained from the maps $\{X \to 0\}$ to recover $F(X)$ from $F(0), F(\Sigma X), F(\Sigma X \oplus \Sigma X), \ldots$. By assumption, we know that S_r holds true for $r \gg 0$. A descending induction shows that S_0 is true, i.e. that $F \simeq 0$. $\square$

Proposition 3.36. *Let $F : \mathcal{A} \to \mathcal{D}$ be a functor between cocomplete stable ∞-categories which is n-excisive and preserves filtered colimits. Then F preserves totalisations which are m-skeletal for some m and all geometric realisations.*

Proof. We begin with the fibre sequence $D_n(F) \to P_n(F) \to P_{n-1}(F)$. By induction, it suffices to prove the claim for the n-homogeneous functor $D_n(F)$. We can find a symmetric functor $G = \mathrm{cr}_n(F) : \mathcal{A}^n \to \mathcal{D}$ which preserves colimits in each variable such that $D_n(F) \simeq G(X, \ldots, X)_{h\Sigma_n}$ (cf. [Lur17, Proposition 6.1.4.14., Corollary 6.1.4.15]). Since $(-)_{h\Sigma_n} : \mathrm{Fun}(B\Sigma_n, \mathcal{D}) \to \mathcal{D}$ is exact and limits and colimits in functor categories are computed pointwise, it suffices to check that the functor $\mathcal{A} \to \mathcal{D}$ given by $X \mapsto G(X, \ldots, X)$ preserves geometric realisations and finite totalisations.

For realisations, this follows immediately since $\Delta^{op} \to (\Delta^{op})^n$ is left cofinal (cf. [Lur09, Lemma 5.5.8.4]). If $X^\bullet : \Delta \to \mathcal{A}$ is a cosimplicial object which is right Kan extended from $\Delta_{\leq m}$, we first observe that $(X^\bullet, \ldots, X^\bullet) : \Delta^n \to \mathcal{A}^n$ is right Kan extended from $(\Delta_{\leq m})^n$. Since this is a finite limit condition and G is exact in each variable, the multisimplicial object $G(X^\bullet, \ldots, X^\bullet)$ is also right Kan extended from $(\Delta_{\leq m})^n$, which implies that $G(\mathrm{Tot}(X^\bullet), \ldots, \mathrm{Tot}(X^\bullet)) \simeq \lim_{\Delta^n} G(X^\bullet, \ldots, X^\bullet)) \simeq \mathrm{Tot}\,(G(X^\bullet, \ldots, X^\bullet))$. For the final equivalence, we have used that the diagonal map $\Delta \to \Delta^n$ is right cofinal. $\square$

Proof of Theorem 3.26. If $F : \mathrm{Vect}_k^\omega \to \mathcal{D}$ is of degree n, we know by Proposition 3.34 that the sifted-colimit-preserving left Kan extension $\mathbb{L}F : \mathrm{Mod}_k^{\geq 0} \to \mathcal{D}$ is n-excisive.

The functor $\widetilde{F} := P_n(\mathrm{Lan}_{\mathrm{Mod}_k^{\geq 0}}^{\mathrm{Mod}_k} \mathbb{L}F) \simeq P_n(\mathbb{L}F \circ \tau_{\geq 0})$ is evidently n-excisive, and it preserves filtered colimits as the t-structure on Mod_k is compatible with filtered colimits (cf. [Lur17, Proposition 1.3.5.21]). Combining this observation with Proposition 3.33 and Theorem 3.35, we can deduce that the restriction of $\widetilde{F}$ to $\mathrm{Mod}_{k,\geq 0}$ agrees with $\mathbb{L}F$.

Proposition 3.36 then implies that $\widetilde{F}$ preserves geometric realisations and finite totalisations. Hence, $\widetilde{F}|_{\mathrm{Perf}_k^{\leq 0}}$ is right Kan extended from F and preserves finite coconnective realisations. □

Example 3.37. As a consequence, we obtain n-excisive functors

$$\mathbb{L}\mathrm{Sym}^n,\ \mathbb{L}\bigwedge^n,\ \mathbb{L}\Gamma^n : \mathrm{Mod}_k \longrightarrow \mathrm{Mod}_k.$$

extending the ordinary symmetric, exterior, and divided power functors Example 3.4

We have the following duality phenomenon:

Proposition 3.38 (Duality)**.** *Let $F : \mathrm{Vect}_k^\omega \longrightarrow \mathrm{Mod}_k$ be a functor of finite degree. Then the functor $F^\vee : \mathrm{Vect}_k^\omega \longrightarrow \mathrm{Mod}_k$ given by $M \mapsto (F(M)^\vee)^\vee$ is right-extendable and its extension $\widetilde{F^\vee}$ satisfies $\widetilde{F^\vee}(M) \simeq (\widetilde{F}(M^\vee))^\vee$ for all $M \in \mathrm{Perf}_k$.*

Proof. Since the functor $(-)^\vee : \mathrm{Mod}_k \to \mathrm{Mod}_k^{op}$ preserves colimits, it is exact. As $\widetilde{F}$ preserves geometric realisations and finite totalisations, the functor $G = (\widetilde{F}(M^\vee))^\vee$ preserves finite realisations and finite totalisations. This in turn implies that $G|_{\mathrm{Perf}_k^{\leq 0}}$ is right Kan extended from Vect_k^ω and that $G|_{\mathrm{Perf}_k^{\leq 0}}$ preserves finite coconnective geometric realisations. Hence, the functor $F^\vee$ is right-extendable. The second claim follows since $\widetilde{F^\vee}$ and G agree on $\mathrm{Perf}_k^{\leq 0}$ and preserve finite geometric realisations. □

Example 3.39. Since the n^{th} symmetric power functor $M \mapsto (M^{\otimes})_{\Sigma_n}$ is dual to the n^{th} divided power functor $M \mapsto (M^{\otimes})^{\Sigma_n}$ for $M \in \mathrm{Vect}_k^\omega$, we conclude from the above Proposition that the extended functors satisfy $\Gamma^n(M) \simeq (\mathrm{Sym}^n(M^\vee))^\vee$ for all $M \in \mathrm{Perf}_k$.

3.3. **Extended Functors and Bredon Homology.** We shall now attach n-excisive functors $\mathrm{Mod}_k \to \mathrm{Mod}_k$ to genuine Σ_n-spaces and construct a spectral sequence to compute their values.

Given a field k and a finite pointed Σ_n-set (X, x), we write $k[X]$ for the quotient of the free k-vector space on X by the relation $x \simeq 0$. We define a functors $F_X, F_X^h : \mathrm{Vect}_k^\omega \to \mathrm{Mod}_k$ by setting

$$F_X(M) := (k[X] \otimes M^{\otimes n})_{\Sigma_n} \qquad \text{and} \qquad F_X^h(M) := (k[X] \otimes M^{\otimes n})_{h\Sigma_n}.$$

Proposition 3.40. *The functors F_X and F_X^h are of degree n in the sense of Definition 3.24.*

Proof. In order to prove that F_X is of degree n, we prove the more general claim that for any subgroup $H \subset \Sigma_m \times \Sigma_n$ and any Σ_m-vector space V in Vect_k^ω, the functor G given by $Y \mapsto (V \otimes Y^{\otimes n})_H$ is of degree n. The claim is evident for $n = 0$. For $n > 0$, the binomial formula shows that $D_X G(Y) = \mathrm{fib}(G(X \oplus Y) \to G(Y))$ sends $Y \in \mathrm{Vect}_k^\omega$ to $Y \mapsto \bigoplus_{j=0}^{n-1} \left(V \otimes \mathrm{Ind}_{\Sigma_{n-j} \times \Sigma_j}^{\Sigma_n} (X^{\otimes(n-j)} \otimes Y^{\otimes j}) \right)_H$. Using the projection formula, we deduce that the functor $D_X G(Y)$ is in fact given by a sum of functors $Y \mapsto (V \otimes X^{\otimes(n-j)} \otimes Y^{\otimes j})_{H'}$ for H' a subgroup of $(\Sigma_m \times \Sigma_{n-j}) \times \Sigma_j$ with $j \leq n-1$. The claim follows by induction. A similar argument shows that F_X^h is degree n. □

By Theorem 3.26, the functors F_X, F^h_X admit canonical n-excisive sifted-colimit-preserving extensions $\mathrm{Mod}_k \to \mathrm{Mod}_k$, which we will denote by the same names. Extending the resulting assignments $(\mathrm{Set}_*^{\mathrm{Fin}})^{\Sigma_n} \to \mathrm{End}^n_\Sigma(\mathrm{Mod}_k)$ given by $X \mapsto F_X$ and $X \mapsto F^h_X$ in a sifted-colimit-preserving manner, we obtain functors

$$F_{(-)}, F^h_{(-)} \ : \ \mathcal{S}_*^{\Sigma_n} \simeq \mathcal{P}_\Sigma((\mathrm{Set}_*^{\mathrm{Fin}})^{\Sigma_n}) \ \longrightarrow \ \mathrm{End}^n_\Sigma(\mathrm{Mod}_k)$$

from genuine Σ_n-spaces to n-excisive sifted-colimit-preserving endofunctors of Mod_k.

We will now describe a method for computing the value of F_X and F^h_X on a given k-module spectrum M. First, we recall the following notion:

Definition 3.41. Given an additive functor $\mu : (\mathrm{Set}_*^{\mathrm{Fin}})^G \to \mathrm{Mod}_k$ from finite pointed G-sets to k-module spectra, we define the *(reduced) Bredon chains* $\widetilde{C}_*(-,\mu) : \mathcal{S}_*^G \to \mathrm{Mod}_k$ as the left Kan extension of μ to the ∞-category of pointed genuine G-spaces $\mathcal{S}_*^G \simeq \mathcal{P}_\Sigma((\mathrm{Set}_*^{\mathrm{Fin}})^G)$. The *(reduced) Bredon homology groups* of $X \in \mathcal{S}_*^G$ with coefficients in μ are given by $\widetilde{H}^{\mathrm{Br}}_*(X,\mu) := \pi_*(\widetilde{C}_*(X,\mu))$.

If $X \in \mathcal{S}_*^{\Sigma_n}$ is the geometric realisation of a simplicial diagram $X_\bullet$ taking values in the category of finite pointed Σ_n-sets, then we have $F_X(M) \simeq |F_{X_\bullet}(M)|$ and $F^h_X(M) \simeq |F^h_{X_\bullet}(M)|$. Using the skeletal filtration of the simplicial k-module spectra $F_{X_\bullet}(M)$ and $F^h_{X_\bullet}(M)$ gives convergent half-plane spectral sequences

$$E^2_{s,t} = \pi_s\left(\pi_t(F_{X_\bullet}(M))\right) \quad \Rightarrow \quad \pi_{s+t}(F_X(M))$$

$$E^{2,h}_{s,t} = \pi_s\left(\pi_t(F^h_{X_\bullet}(M))\right) \quad \Rightarrow \quad \pi_{s+t}(F^h_X(M)).$$

Thus, we have $E^2_{s,t} = \widetilde{H}^{\mathrm{Br}}_s(X, \mu^M_t)$ and $E^{2,h}_{s,t} = \widetilde{H}^{\mathrm{Br}}_s(X, \mu^{M,h}_t)$, where $\mu^M_t, \mu^{M,h}_t : (\mathrm{Set}_*^{\mathrm{Fin}})^{\Sigma_n} \to \mathrm{Mod}_k$ are given by $X \mapsto \pi_t(F_X(M))$ and $X \mapsto \pi_t(F^h_X(M))$, respectively.

The functors μ^M_t and $\mu^{M,h}_t$ are particularly computable when M is perfect and coconnective. In this case, we can write $M = \mathrm{Tot}(M^\bullet)$ as a finite totalisation of a cosimplicial finite-dimensional k-vector space $M^\bullet$. For $X \in (\mathrm{Set}_*^{\mathrm{Fin}})^{\Sigma_n}$ a finite pointed Σ_n-set, the functor $(F_X)|_{\mathrm{Perf}_k^{\leq 0}}$ is right Kan extended from Vect_k^ω, which in turn implies that $F_X(M) \simeq \mathrm{Tot}(F_X(M^\bullet)) = \mathrm{Tot}(k[X] \otimes (M^\bullet)^{\otimes n})_{\Sigma_n}$ and $F^h_X(M) \simeq \mathrm{Tot}(F^h_X(M^\bullet)) = \mathrm{Tot}(k[X] \otimes (M^\bullet)^{\otimes n})_{h\Sigma_n}$ (by the dual of Proposition 3.2). Dual remarks apply for M connective and of finite type.

In both cases, we can use the standard wrong-way maps to upgrade μ^M_t and $\mu^{M,h}_t$ to *Mackey functors*. We recall that a Mackey functor consists of a pair of functors

$$(\mu_t : (\mathrm{Set}^{\mathrm{Fin}})^G \to \mathrm{Mod}_k \ , \ \mu^\natural : (\mathrm{Set}^{\mathrm{Fin}})^G \to \mathrm{Mod}_k^{op})$$

from finite Σ_n-sets to Mod_k which agree on objects and such that whenever the left square below is a pullback of finite pointed G-sets, then the right hand square commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} \mu(A_+) \arrow[r, "\mu(f)"] & \mu(B) \\ \mu(C) \arrow[u, "\mu^\natural(h)"] \arrow[r, "\mu(k)"'] & \mu(D) \arrow[u, "\mu^\natural(g)"'] \end{tikzcd}$$

We will later use the above spectral sequence to compute the homotopy of free partition Lie algebras.

3.4. **Admissible functors.** The main purpose of this subsection is to isolate a class of functors which preserve certain totalisations in $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. This will play a technical role at various stages in the axiomatic argument in the following section.

We first need the following elementary and classical observation asserting that endofunctors of $\mathrm{Mod}_{k,\geq 0}$ which preserve sifted colimits naturally commute with limits of Postnikov towers.

Proposition 3.42. *Let $F : \mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$ be a functor which preserves sifted colimits. Suppose $V \to V'$ is a map in $\mathrm{Mod}_{k,\geq 0}$ such that $\tau_{\leq n} V \simeq \tau_{\leq n} V'$. Then $\tau_{\leq n} F(V) \simeq \tau_{\leq n} F(V')$.*

Proof. Since the functor $\Omega^\infty : \mathrm{Mod}_{k,\geq 0} \to \mathcal{S}$ preserves products and sifted colimits, the functor $\Omega^\infty \circ F : \mathrm{Mod}_{k,\geq 0} \to \mathcal{S}$ has the same properties. Hence $\widetilde{F} := \Omega^\infty \circ F$ is left Kan extended from a product-preserving functor $\mathrm{Vect}^\omega_k \to \mathcal{S}$, since $\mathrm{Mod}_{k,\geq 0} = \mathcal{P}_\Sigma(\mathrm{Vect}^\omega_k)$. The general theory of $\mathcal{P}_\Sigma$ shows now that any such functor can be written as a geometric realisation of functors of the form

$$h_{V_0}(-) = \mathrm{Hom}_{\mathrm{Mod}_k}(V_0, -) : \mathrm{Mod}_{k,\geq 0} \to \mathcal{S},$$

for $V_0 \in \mathrm{Vect}^\omega_k$. It is clear that h_{V_0} has the desired property: if $V \to V'$ is a map in $\mathrm{Mod}_{k,\geq 0}$ which induces an equivalence on $\tau_{\leq n}$, then $\tau_{\leq n} h_{V_0}(V) \to \tau_{\leq n} h_{V_0}(V')$ is an equivalence. The result follows as the collection of all functors $\mathrm{Mod}_{k,\geq 0} \to \mathcal{S}$ satisfying this property is closed under colimits. $\square$

Definition 3.43 (Admissible functors)**.** Let $F : \mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$ be a functor which preserves sifted colimits. We will say that F is *admissible* if the following hold:

(1) The functor F preserves the subcategory $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} \subset \mathrm{Mod}_{k,\geq 0}$ of finite type k-modules.
(2) If $X^\bullet$ is a cosimplicial object of $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ such that the totalisation $\mathrm{Tot}(X^\bullet)$ (computed in Mod_k) belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$, then $F(\mathrm{Tot}(X^\bullet)) \longrightarrow \mathrm{Tot}(F(X^\bullet))$ is an equivalence. In particular, this means that the right-hand side is connective.

Proposition 3.44. *Let $F : \mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$ be a functor which preserves sifted colimits and preserves the subcategory $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. Then we can define a functor $F^\vee : \mathrm{Perf}_{k,\leq 0} \to \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ by the formula $V \mapsto F(V^\vee)^\vee$. The functor F is admissible if and only if $F^\vee$ preserves finite coconnective geometric realisations (Definition 3.9).*

Proof. If $F^\vee$ preserves finite coconnective geometric realisations, then $F^\vee$ extends uniquely to a sifted-colimit-preserving functor $\widetilde{F^\vee} : \mathrm{Mod}_k \to \mathrm{Mod}_k$. For $V \in \mathrm{Perf}_{k,\leq 0}$, we have a natural equivalence $\widetilde{F^\vee}(V) \simeq F(V^\vee)^\vee$. By Proposition 3.42, this in fact holds for all $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. Now suppose that $W^\bullet$ is an augmented cosimplicial object in $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ which is a limit diagram in Mod_k.

Dualising, we obtain an augmented simplicial object $(W^\vee)_\bullet$ of $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ which is a colimit diagram. Here we have used that linear duality is conservative. Now $\widetilde{F^\vee}((W^\vee)_\bullet)$ is a colimit diagram. Dualising again, we find that $F(W^\bullet)$ is a limit diagram, as desired. The reverse implication follows by a similar argument. $\square$

Proposition 3.45. *Let $F : \mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_k$ be a functor which commutes with sifted colimits. Suppose that F carries $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ into $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ and that F is n-excisive. Then F is admissible.*

Proof. We first observe that F extends uniquely to an n-excisive functor on Mod_k (Theorem 3.35), and that it therefore preserves all finite totalisations by Proposition 3.36. The functor $F^\vee$ on Perf_k given by $V \mapsto F(V^\vee)^\vee$ is also n-excisive and therefore preserves finite geometric realisations. By Proposition 3.44, it follows that F is admissible. $\square$

Proposition 3.45 provides a large supply of examples of admissible functors. However, we will also need to work with functors, e.g. the free $\mathbb{E}_\infty$-algebra functor, which are not admissible. However, they will become admissible in the graded setting, and this observation will be crucial.

Definition 3.46 (Pointwise finite type)**.** Let $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$ be the full subcategory of GrMod_k consisting of those objects $X_\star$ such that $X_i \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ for all $i > 0$. We shall refer to these objects as *pointwise of finite type*. Note that $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ is a subcategory of $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.

Remark 3.47. This is a weaker notion than being of finite type in the sense of Definition 2.14.

We can give the following graded variant of Definition 3.43:

Definition 3.48 (Graded admissible functors)**.** Let $F : \mathrm{GrMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$ be a functor which preserves sifted colimits. We will say that F is *admissible* if the following conditions hold:

1. The functor F preserves the subcategory $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0} \subset \mathrm{GrMod}_{k,\geq 0}$.
2. If $X^\bullet$ is a cosimplicial object of $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$ such that the totalisation $\mathrm{Tot}(X^\bullet)$ (computed in GrMod_k) belongs to $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$, then $F(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(F(X^\bullet))$ is an equivalence. In particular, the right-hand side is connective.

We illustrate the technical advantage of working in the graded setting:

Example 3.49. For some $n > 0$, consider the functor $\mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$ given by $V \mapsto V^{\otimes n}$. Since this is n-excisive, Proposition 3.45 implies that this functor is admissible. However, the functor $V \mapsto \bigoplus_{n>0} V^{\otimes n}$ *is not* admissible as a functor $\mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$.

It is, however, easy to see that the assignment $V \mapsto \bigoplus_{n>0} V^{\otimes n}$ *is* admissible as a functor $\mathrm{GrMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$. This holds because each summand is admissible, and the summands live in higher and higher internal degree.

4. The axiomatic argument

Given an augmented monad acting on Mod_k, we can consider the ∞-category of formal moduli problems based in algebras over this monad (cf. Definition 4.22). In this section, we shall prove that under certain conditions specified in Definition 4.15, this ∞-category admits a "Lie algebraic" description (cf. Theorem 4.23) as algebras over a monad constructed in terms of the monadic bar construction.

4.1. **An informal overview.** We briefly recall some axiomatic aspects of bar-cobar duality, which goes back to the classical work of Moore [Moo71]. For recent modern treatments, we refer to [Chi05, Chapter 4], [FG12, Sections 3,4], or [GR17, Chapter 6], [LV12, Chapters 2,6].

Given an $\mathbb{E}_\infty$-ring spectrum k, we write Mod_k for the ∞-category of k-module spectra. If $\mathcal{O}$ is an ∞-operad in Mod_k (cf. e.g. [Bra17, Definition 4.1.4], then we can consider the ∞-category $\mathrm{Alg}_\mathcal{O}(\mathrm{Mod}_k)$ of $\mathcal{O}$-*algebras* in Mod_k. Suppose now that $\mathcal{O}(0) = 0, \mathcal{O}(1) = k$. Restriction along the canonical map from $\mathcal{O}$ to the trivial operad gives rise to the *square-zero* functor

$$\mathrm{sqz} : \mathrm{Mod}_k \to \mathrm{Alg}_\mathcal{O}(\mathrm{Mod}_k),$$

which turns an object $V \in \mathrm{Mod}_k$ into an $\mathcal{O}$-algebra whose operadic multiplication maps are trivial. This functor admits a left adjoint, which we will denote by

$$\mathrm{cot} : \mathrm{Alg}_\mathcal{O}(\mathrm{Mod}_k) \to \mathrm{Mod}_k.$$

It is often called the *cotangent fibre* or $\mathcal{O}$-*algebra homology* and can be thought of as the derived indecomposables functor.

Remark 4.1. If k is a field of characteristic zero, this was discussed by Hinich [Hin97], who studied operads in chain complexes of k-vector spaces. For general $\mathbb{E}_\infty$-ring spectra R, modelled as commutative algebra objects in symmetric spectra, these functors were also studied in [Har10, Har09, Har15].

Informally speaking, bar-cobar duality aims to recover an $\mathcal{O}$-algebra A from $\mathrm{cot}(A)$ together with some additional structure placed on it. More formally, we observe that $C = \mathrm{cot} \circ \mathrm{sqz}$ defines a comonad on Mod_k, and abstract nonsense gives rise to a functor

$$\mathrm{Alg}_\mathcal{O}(\mathrm{Mod}_k) \to \mathrm{coAlg}_{\mathrm{cot} \circ \mathrm{sqz}}(\mathrm{Mod}_k). \tag{3}$$

Furthermore, the functor $\mathrm{cot} \circ \mathrm{sqz}$ can be identified with $V \mapsto \bigoplus_{n>0} (\mathcal{K}(n) \otimes V^{\otimes n})_{h\Sigma_n}$, where $\mathcal{K}(n) \in \mathrm{Fun}(B\Sigma_n, \mathrm{Mod}_k)$ is a symmetric sequence of k-module spectra. This symmetric sequence is in fact the underlying symmetric sequence of the *Koszul dual* cooperad $\mathcal{K} = \mathrm{Bar}(\mathcal{O})$, which one can often identify explicitly, cf. e.g. [GK94, GK95, GJ94]. Taking k-linear duals, it follows that $\mathrm{Bar}(\mathcal{O})^\vee$ is an operad in Mod_k, and the functor $\mathrm{cot}^\vee$ takes values in algebras over $\mathrm{Bar}(\mathcal{O})^\vee$. Under suitable conditions, one may hope that (3) will restrict to an equivalence on appropriate subcategories.

We mention some well-known results in this direction. First of all, the following general comparison result shows that one has an equivalence under connectivity hypotheses:

Theorem 4.2 (Ching-Harper [CH19])**.** *Let k be a connective commutative symmetric ring spectrum and let $\mathcal{O}$ an operad Mod_k for which $\mathcal{O}(0) \simeq 0, \mathcal{O}(1) \simeq k$, and $\mathcal{O}(i)$ is connective for all $i \geq 0$. Let $\mathcal{K}$ be the associated cooperad on Mod_k. Then the duality functor $\mathrm{Alg}_\mathcal{O}(\mathrm{Mod}_k) \rightleftarrows \mathrm{coAlg}_\mathcal{K}(\mathrm{Mod}_k)$ restricts to an equivalence between 0-connected objects on both sides.*

The hypotheses of 0-connectedness (which means that $\pi_n(X) = 0$ for all $n \leq 0$) is crucial and essential for the convergence of certain filtrations. This result has many predecessors, including the result of Moore [Moo71], who proves Theorem 4.2 for $\mathcal{O}$ the nonunital associative operad (over a discrete ring k) by explicitly constructing the adjunctions on chain complexes. Quillen [Qui69] proves it for $\mathcal{O}$ the Lie operad over $\mathbb{Q}$, where $\mathrm{coAlg}_{\mathcal{K}}$ becomes (up to a shift) the ∞-category of cocommutative coalgebras.

The theorem of Lurie [Lur11a] and Pridham [Pri10] classifying formal moduli problems in characteristic zero can be interpreted as a result in this vein, albeit with some additional hypotheses and conclusions. In particular, one needs to slightly extend the equivalence beyond the assumption of 0-connectedness, and one needs to prove that certain pullbacks are carried to pushouts.

We recall the work of Lurie and Pridham in more detail. Let k be a field of characteristic zero, and write $\mathrm{CAlg}_k^{\mathrm{aug}}$ for the ∞-category of augmented $\mathbb{E}_\infty$-algebras over k. Note that we can also regard this as the ∞-category of algebras over the *nonunital* $\mathbb{E}_\infty$-operad in Mod_k, so this is an instance of the situation described above.

Definition 4.3. We say that $A \in \mathrm{CAlg}_k^{\mathrm{aug}}$ is *complete local Noetherian* if A is connective, $\pi_0(A)$ is a Noetherian ring which is complete with respect to the augmentation ideal, and each $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module.

Let $\mathrm{CAlg}_{\mathrm{k}}^{\mathrm{cN}}$ denote the full subcategory of $\mathrm{CAlg}_k^{\mathrm{aug}}$ spanned by such algebras.

It is well-known that the Koszul dual to the nonunital $\mathbb{E}_\infty$-operad is the shifted Lie operad. It follows that if $A \in \mathrm{CAlg}_k^{\mathrm{aug}}$, then $\cot(A)^\vee[-1]$ is naturally equipped with the structure of a differential graded Lie algebra, and we obtain a functor

$$\mathfrak{D} : (\mathrm{CAlg}_k^{\mathrm{aug}})^{op} \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_k).$$

Here $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_k)$ denotes the ∞-category of Lie algebras in Mod_k, which is presented by the model category differential graded Lie algebras.

The Lurie-Pridham Theorem 1.1 is essentially equivalent to the following result:

Theorem 4.4 (Lurie, Pridham)**.** *The functor $\mathfrak{D}$ restricts to an anti-equivalence*

$$\mathfrak{D} : (\mathrm{CAlg}_{\mathrm{k}}^{\mathrm{cN}})^{op} \simeq \mathrm{Lie}_{k,\leq -1}^{\mathrm{coh}}$$

between $\mathrm{CAlg}_{\mathrm{k}}^{\mathrm{cN}}$ and the ∞-category $\mathrm{Lie}_{k,\leq -1}^{\mathrm{coh}}$ of differential graded Lie algebras whose homotopy groups are finite-dimensional in each degree and concentrated in negative degrees.

Furthermore, given maps $A \to A''$ and $A' \to A''$ in $\mathrm{CAlg}_{\mathrm{k}}^{\mathrm{cN}}$ which induce surjections on π_0, the functor $\mathfrak{D}$ takes the associated pullback diagram to a pushout square of differential graded Lie algebras.

The deduction of Theorem 1.1 from Theorem 4.4 is explained in Sections 6 and 7 of the survey paper [Lur10], or, using the language of deformation theories, in [Lur11a, Theorem 1.3.12].

The main "formal" contribution of this paper is a new method for proving results like Theorem 4.4 for ∞-categories of algebras more general than $\mathrm{CAlg}_k^{\mathrm{aug}}$ with $\mathrm{char}(k) = 0$. We will prove the relevant equivalence using Lurie's higher categorical version of the Barr-Beck comonadicity theorem (cf. Theorem 4.7.3.5 in [Lur17]), which we shall briefly recall for the reader's convenience:

Theorem 4.5 (Barr-Beck-Lurie)**.** *The adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is comonadic if and only if the following conditions hold true:*

(1) The functor F is conservative, i.e. a morphism f in $\mathcal{C}$ is an equivalence if and only if $F(f)$ is an equivalence.

(2) *Given an F-split cosimplicial object $X^\bullet$ in $\mathcal{C}$ (cf.* [Lur17, Section 4.7.2]*), the diagram $X^\bullet$ admits a limit in $\mathcal{C}$, which is preserved by F.*

The Barr-Beck-Lurie theorem is a powerful tool in higher category theory which, just like its classical counterpart, has been used to establish various descent equivalences by checking certain convergence results. We give an instructive example (cf. [Mat16, Sec. 3]):

Example 4.6 (Nilpotent descent)**.** Fix a field k and consider the algebra object $k[\epsilon]/\epsilon^2$, equipped with the obvious augmentation map to k. We claim that the natural functor

$$\mathrm{Mod}_{k[\epsilon]/\epsilon^2} \to \mathrm{Mod}_k, \quad M \mapsto k \otimes_{k[\epsilon]/\epsilon^2} M$$

is comonadic. To see this, we need to check that the functor $k \otimes_{k[\epsilon]/\epsilon^2} (-)$ is (1) conservative and (2) commutes with totalisations of cosimplicial objects in $\mathrm{Mod}_{k[\epsilon]/\epsilon^2}$ which become split after applying $k \otimes_{k[\epsilon]/\epsilon^2} (-)$. Both of these facts follow easily from the fibre sequence of $k[\epsilon]/\epsilon^2$-modules

$$k \to k[\epsilon]/\epsilon^2 \to k.$$

Indeed, if $M \in \mathrm{Mod}_{k[\epsilon]/\epsilon^2}$ satisfies $k \otimes_{k[\epsilon]/\epsilon^2} M = 0$, then the above fibre sequence implies that $M = 0$. This implies (1). Similarly, we can prove statement (2) about commutativity with totalisations by observing that if $M^\bullet$ is a cosimplicial object in $\mathrm{Mod}(k[\epsilon]/\epsilon^2)$ such that $k \otimes_{k[\epsilon]/\epsilon^2} M^\bullet$ is split, then the Tot-tower defined by M is a constant pro-object.

Since we will be working with algebras rather than modules, the arguments are more involved. The basic observation is that the convergence criterion appearing in the Barr-Beck-Lurie Theorem 4.5(2) is far easier to check when working in the context of connected graded objects (compare also the notion of pro-nilpotence in [FG12]). As a consequence, it becomes much easier to establish a Koszul duality statement for connected graded $\mathbb{E}_\infty$-algebras.

Any augmented $\mathbb{E}_\infty$-algebra is automatically endowed with a canonical $\mathfrak{m}$-adic filtration. With some care, we can use this fact to transfer all convergence questions into the simpler connected graded setting. We note that the use of filtrations in this type of argument is standard in the literature on operadic Koszul duality, cf. for example [Kuh04, HH13, KP17, GR17, CH19].

In Definition 4.15 below, we will isolate axiomatic properties of a monad which will guarantee that the strategy outlined above passes through. Indeed, we prove in Theorem 4.20 that if these axioms are satisfied, then a generalised version of Theorem 4.4 holds true.

4.2. **The Axiomatic Setup.**

Let k be a field and suppose that T is a sifted-colimit-preserving monad acting on Mod_k. In order to set up a cotangent formalism for T-algebras and apply it to T-based formal moduli problems, we shall need to assume that T is augmented over the identity monad. In fact, we will adopt an equivalent formalism based in adjunctions rather than monads; this will later facilitate our treatment of filtrations:

Definition 4.7. An *augmented monadic adjunction* consists of a pair of adjunctions

$$\begin{tikzcd} \mathrm{Mod}_{k,\geq 0} \arrow[r, shift left, "\mathrm{free}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "\mathrm{forget}"] \arrow[r, shift left, "\mathrm{cot}"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_{k,\geq 0} \arrow[l, shift left, "\mathrm{sqz}"] \end{tikzcd}$$

whose composite is the identity such that $\mathcal{C}$ is pointed and presentable and (free $\dashv$ forget) is monadic with sifted-colimit-preserving right adjoint.

Remark 4.8. Given an augmented monadic adjunction as above, we can identify $\mathcal{C}$ with the ∞-category of algebras for a monad $T = \mathrm{forget} \circ \mathrm{free}$ on $\mathrm{Mod}_{k,\geq 0}$, and this monad is canonically augmented via $T = \mathrm{forget} \circ \mathrm{free} \longrightarrow \mathrm{forget} \circ \mathrm{sqz} \circ \mathrm{cot} \circ \mathrm{free} = \mathrm{id}$.

Example 4.9. The main example of interest to us is the case where $\mathcal{C}$ is the ∞-category of augmented simplicial commutative k-algebras. In this case, the above adjunctions are defined as expected: free builds the free simplicial commutative k-algebra, forget sends an augmented simplicial commutative k-algebra to its augmentation ideal, cot takes the cotangent fibre, and sqz is the trivial square-zero construction functor.

The augmented monad $\mathbb{L}\mathrm{Sym} = \mathrm{forget} \circ \mathrm{free}$ sends M to $\bigoplus_{i\geq 1} \mathbb{L}\mathrm{Sym}^i(M)$, where $\mathbb{L}\mathrm{Sym}^i$ is the left derived functor of the i^{th} symmetric power functor $(-)^{\otimes i}_{\Sigma_i}$. The functor cot can be computed explicitly as $R \mapsto \mathrm{Bar}(1, \mathbb{L}\mathrm{Sym}, IA)$, where IA is the augmentation ideal of A.

Example 4.10. There is also a variant of the above example when we consider $\mathcal{C}$ to be the ∞-category of connective augmented $\mathbb{E}_\infty$-algebras over k.

Example 4.11. In fact, given any ∞-operad $\mathcal{O}$ in $\mathrm{Mod}_{k,\geq 0}$ with $\mathcal{O}(0) = 0$ and $\mathcal{O}(1) \simeq k$, we can take $\mathcal{C}$ to be the ∞-category of connective $\mathcal{O}$-algebras. For k a field of characteristic zero, we recover the desired adjunctions as in the discussion at the beginning of Section 4.1.

Remark 4.12. We will often suppress the notation forget when it will not cause confusion. For instance, given $A \in \mathcal{C}$, we shall write $\pi_i(A) = \pi_i(\mathrm{forget}(A))$.

In the situation of Definition 4.7, we hope to establish a version of Theorem 4.4. For this, we shall need to identify a full subcategory $\mathcal{C}_{\mathrm{afp}} \subset \mathcal{C}$ of *complete almost finitely presented objects* satisfying the following desiderata:

(1) The adjunction $(\mathrm{cot} \dashv \mathrm{sqz})$ restricts to a comonadic adjunction $\mathcal{C}_{\mathrm{afp}} \underset{\mathrm{sqz}}{\overset{\mathrm{cot}}{\rightleftarrows}} \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$.

(2) The monad $(M \mapsto \mathrm{cot}(\mathrm{sqz}(M^\vee))^\vee)$ on $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ extends uniquely to a sifted-colimit-preserving monad $T^\vee$ on Mod_k (cf. Corollary 3.17).

We write $\mathfrak{D} : \mathcal{C}^{op}_{\mathrm{afp}} \to \mathrm{Alg}_{T^\vee}$ for the fully faithful embedding sending A to $\mathrm{cot}(A)^\vee$.

(3) Given $A, A', A'' \in \mathcal{C}_{\mathrm{afp}}$ and π_0-surjective maps $A \to A''$ and $A' \to A''$, the fibre product $A \times_{A''} A' \in \mathcal{C}$ also belongs to $\mathcal{C}_{\mathrm{afp}}$, and the square

$$\begin{tikzcd} \mathfrak{D}(A'') \arrow[r] \arrow[d] & \mathfrak{D}(A') \arrow[d] \\ \mathfrak{D}(A) \arrow[r] & \mathfrak{D}(A \times_{A''} A') \end{tikzcd}$$

is a pushout in $T^\vee$-algebras.

Remark 4.13. Writing $T = \mathrm{cot} \circ \mathrm{sqz} : \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} \to \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ for the comonad induced by the above adjunction, the above conditions give equivalences $\mathcal{C}_{\mathrm{afp}} \simeq \mathrm{coAlg}_T(\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}) \simeq \mathrm{Alg}_{T^\vee}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})^{op}$. This is a version of bar-cobar duality.

Remark 4.14. In all our Examples 4.9 and 4.10 above, we will be able to explicitly identify the target subcategory $\mathcal{C}_{\mathrm{afp}}$. More precisely, $\mathcal{C}_{\mathrm{afp}}$ will be the ∞-category of augmented simplicial commutative rings (resp. connective $\mathbb{E}_\infty$-rings) A such that $\pi_0(A)$ is complete local Noetherian, and such that $\pi_i(A)$ is finitely generated as a $\pi_0(A)$-module for all $i \geq 0$.

In order to construct a subcategory $\mathcal{C}_{\mathrm{afp}} \subset \mathcal{C}$ satisfying the desirable properties listed above, we will need to work in a more refined context. Thinking of $\mathcal{C}$ as some ∞-category of algebras, we will specify ∞-categories $\mathcal{C}^{\mathrm{Fil}}$ and $\mathcal{C}^{\mathrm{Gr}}$ of filtered and graded algebras, and assume that these categories are suitably linked by several natural functors. Further technical assumptions, which are readily checked in practice, will then allow us to establish a Koszul dual description of $\mathcal{C}$-based formal moduli problems.

We summarise all required data in the following central definition, which is a filtered enhancement of Definition 4.7:

Definition 4.15. A *filtered augmented monadic adjunction* consists of a diagram of left adjoints

$$\begin{tikzcd}
\mathrm{Mod}_{k,\geq 0} \arrow[r, "\mathrm{free}"] \arrow[d, "(-)_1"'] & \mathcal{C} \arrow[r, "\mathrm{cot}"] \arrow[d, "\mathrm{adic}"'] & \mathrm{Mod}_{k,\geq 0} \arrow[d, "(-)_1"'] \\
\mathrm{FilMod}_{k,\geq 0} \arrow[r, "\mathrm{free}"] \arrow[d, "\mathrm{Gr}"'] & \mathcal{C}^{\mathrm{Fil}} \arrow[r, "\mathrm{cot}"] \arrow[d] & \mathrm{FilMod}_{k,\geq 0} \arrow[d, "\mathrm{Gr}"'] \\
\mathrm{GrMod}_{k,\geq 0} \arrow[r, "\mathrm{free}"] & \mathcal{C}^{\mathrm{Gr}} \arrow[r, "\mathrm{cot}"] & \mathrm{GrMod}_{k,\geq 0}
\end{tikzcd}$$

where the vertical arrows $(-)_1 : \mathrm{Mod}_{k,\geq 0} \to \mathrm{FilMod}_{k,\geq 0}$ send V to $(V)_1 = (\cdots \to 0 \to 0 \to V)$ (cf. Example 2.2) and the functor Gr takes the associated graded. Moreover, we shall assume:

(1) Augmented Monadicity. All horizontal composites give the identity, and the adjunctions $(\mathrm{Mod}_{k,\geq 0} \rightleftarrows \mathcal{C})$, $(\mathrm{FilMod}_{k,\geq 0} \rightleftarrows \mathcal{C}^{\mathrm{Fil}})$, $(\mathrm{GrMod}_{k,\geq 0} \rightleftarrows \mathcal{C}^{\mathrm{Gr}})$ are monadic with sifted-colimit-preserving right adjoints.

(2) Adjointability. Taking right adjoints of the top vertical arrows gives commutative squares

$$\begin{tikzcd}
\mathrm{Mod}_{k,\geq 0} \arrow[r, "\mathrm{free}"] & \mathcal{C} \arrow[r, "\mathrm{cot}"] & \mathrm{Mod}_{k,\geq 0} \\
\mathrm{FilMod}_{k,\geq 0} \arrow[r, "\mathrm{free}"] \arrow[u, "\mathrm{F}^1"] & \mathcal{C}^{\mathrm{Fil}} \arrow[r, "\mathrm{cot}"] \arrow[u, "\mathrm{F}^1"] & \mathrm{FilMod}_{k,\geq 0} \arrow[u, "\mathrm{F}^1"]
\end{tikzcd}$$

Taking right adjoints of the middle and lower horizontal maps gives commutative squares

$$\begin{tikzcd}
\mathrm{FilMod}_{k,\geq 0} \arrow[d, "\mathrm{Gr}"'] & \mathcal{C}^{\mathrm{Fil}} \arrow[l, "\mathrm{forget}"'] \arrow[d, "\mathrm{Gr}"'] & \mathrm{FilMod}_{k,\geq 0} \arrow[l, "\mathrm{sqz}"'] \arrow[d, "\mathrm{Gr}"'] \\
\mathrm{GrMod}_{k,\geq 0} & \mathcal{C}^{\mathrm{Gr}} \arrow[l, "\mathrm{forget}"'] & \mathrm{GrMod}_{k,\geq 0} \arrow[l, "\mathrm{sqz}"']
\end{tikzcd}$$

(3) Admissibility. The map $\mathrm{GrMod}_{k,\geq 0} \xrightarrow{\mathrm{forget} \circ \mathrm{free}} \mathrm{GrMod}_{k,\geq 0}$ is admissible (cf. Definition 3.48). Moreover, given $A \in \mathcal{C}^{\mathrm{Gr}}$, there is a functorial tower $\{A^{(i)}\}_{i \geq 1}$ in $\mathcal{C}^{\mathrm{Gr}}$ and compatible maps $A \to A^{(i)}$ satisfying the following three properties:
   (a) There is a natural isomorphism $A^{(1)} \simeq \mathrm{sqz} \circ \mathrm{cot}(A)$ of objects over A in $\mathcal{C}^{\mathrm{Gr}}$.
   (b) For $i > 1$, there is a natural isomorphism $\mathrm{forget}(A^{(i-1)}) / \mathrm{forget}(A^{(i)}) \simeq G_i(\mathrm{cot}(A))$ for an admissible and i-increasing functor $G_i : \mathrm{GrMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$.
   (c) The map $\mathrm{forget}(A) \to \mathrm{forget}(A^{(i)})$ in $\mathrm{GrMod}_{k,\geq 0}$ induces an equivalence on graded pieces of (internal) degree $\leq i$.

Let $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}} \subset \mathcal{C}^{\mathrm{Gr}}$ consist of all A with $\cot(A) \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$. Write $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}} \subset \mathcal{C}^{\mathrm{Fil}}$ for the full subcategory of all complete A with $\mathrm{Gr}(A) \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$. Let $\mathcal{C}_{\mathrm{afp}} \subset \mathcal{C}$ consist of all A with $\mathrm{adic}(A) \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$.

(4) Coherence.
   (a) When $A, A', A'' \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ and we have maps $A \to A''$, $A' \to A''$ which induce surjections on π_0, then the fibre product $A \times_{A''} A' \in \mathcal{C}^{\mathrm{Gr}}$ also belongs to $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$.
   (b) If $V \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$, then $\mathrm{sqz}(V) \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$.

(5) Completeness. If $A \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$, then the following conditions hold true:
   a) The cotangent complex $\cot(A)$ is complete.
   b) The adic filtration $\mathrm{adic}(F^1A)$ on $F^1A \in \mathcal{C}$ is complete.

Remark 4.16. The first part of the adjointability axiom asserts that taking free algebras and taking the cotangent fibre commutes with taking underlying objects. The second part ensures that taking underlying objects or taking a square-zero extension commutes with passage to associated gradeds.

The functor adic is an abstraction of the construction which sends an augmented commutative ring R with augmentation ideal $\mathfrak{m}$ to its (derived) $\mathfrak{m}$-adic filtration.

Remark 4.17. The reader is encouraged to keep in mind the following example (which is discussed in more detail starting in Construction 5.36). Let $\mathcal{C}$ be the ∞-category of augmented simplicial commutative k-algebras, as in Example 4.9. Then we can let $\mathcal{C}^{\mathrm{Fil}}$ to be the ∞-category of augmented filtered simplicial commutative k-algebras and $\mathcal{C}^{\mathrm{Gr}}$ the of augmented graded simplicial commutative k-algebras. In this case, one can show:

(1) $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ consists of those graded simplicial commutative rings A such that the bigraded ring $\pi_*(A)$ has the following property: $\pi_0(A)$ is Noetherian, and $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module for $i \geq 0$.
(2) $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ consists of those complete filtered simplicial commutative rings A whose associated graded $\mathrm{Gr}(A)_*$ is almost finitely presented as the previous item. This in particular implies that $\pi_0(A)$ is a complete local Noetherian ring and each $\pi_i(A)$ is finitely generated as a $\pi_0(A)$-module.

Remark 4.18. Given any augmented ∞-operad $\mathcal{O}$ in $\mathrm{Mod}_{k,\geq 0}$, there is a natural ∞-category of filtered $\mathcal{O}$-algebras (i.e. $\mathcal{O}$-algebras in filtered objects), as well as one of graded $\mathcal{O}$-algebras. If $\mathcal{O}$ satisfies reasonable finiteness properties, then these ∞-categories are linked by a diagram satisfying conditions $(1)-(3)$ of Definition 4.15, where (3) can be handled using the homotopy completion tower for graded k-modules (cf. [HH13]). More generally, it is not hard to check that a similar statement holds for any augmented monad in the category of strict polynomial functors (cf. [FS97]), with the identity functor as weight one component.

Remark 4.19. The admittedly clumsy axiomatisation adopted in Definition 4.15 allows us to simultaneously handle the cases of $\mathbb{E}_\infty$-k-algebras (which are algebras over an operad) and simplicial commutative k-algebras (for which this is not true).

We can state and prove the main formal result in this paper:

Theorem 4.20. *Let $\mathcal{C}, \mathcal{C}^{\mathrm{Fil}}, \ldots$ be part of a filtered augmented monadic adjunction (cf. Definition 4.15).*

(1) The adjunction $(\cot \dashv \mathrm{sqz})$ *restricts to a comonadic adjunction* $\mathcal{C}_{\mathrm{afp}} \underset{\mathrm{sqz}}{\overset{\cot}{\rightleftarrows}} \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$, *where $\mathcal{C}_{\mathrm{afp}}$ is defined as in Definition 4.15 above or Definition 4.43 below.*
(2) The monad $(M \mapsto \cot(\mathrm{sqz}(M^\vee))^\vee)$ *on* $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ *extends uniquely to a sifted-colimit-preserving monad $T^\vee$ on* Mod_k *(cf. Corollary 3.17).*

We write $\mathfrak{D} : \mathcal{C}_{\mathrm{afp}}^{op} \to \mathrm{Alg}_{T^\vee}$ for the fully faithful embedding sending A to $\cot(A)^\vee$.

(3) *Given $A, A', A'' \in \mathcal{C}_{\mathrm{afp}}$ and π_0-surjective maps $A \to A''$ and $A' \to A''$, the fibre product $A \times_{A''} A' \in \mathcal{C}$ also belongs to $\mathcal{C}_{\mathrm{afp}}$, and the following square is a pushout in $T^\vee$-algebras:*

$$\begin{tikzcd} \mathfrak{D}(A'') \arrow[r] \arrow[d] & \mathfrak{D}(A') \arrow[d] \\ \mathfrak{D}(A) \arrow[r] & \mathfrak{D}(A \times_{A''} A') \end{tikzcd}$$

This will allow us to prove that $\mathcal{C}$-based formal deformations are in fact governed by $T^\vee$-algebras. First, we specify the small objects which will parametrise our deformations (cf. [Lur11a, Definition 1.1.8]):

Definition 4.21 (Artinian objects)**.** Let $\mathcal{C}_{\mathrm{art}} \subset \mathcal{C}_{\mathrm{afp}}$ denote the smallest full subcategory satisfying:

(1) $\mathcal{C}_{\mathrm{art}}$ contains the terminal object $*$.

(2) Given $A \in \mathcal{C}_{\mathrm{art}}$ and a morphism $A \to \mathrm{sqz}(k[n])$ with $n > 0$, then the fibre product $A' = A \times_{\mathrm{sqz}(k[n])} (*)$ also belongs to $\mathcal{C}_{\mathrm{art}}$.

Observe that (1) and (2) together imply that $\mathcal{C}_{\mathrm{art}}$ contains $\mathrm{sqz}(k[n])$ for any $n \geq 0$.

We spell out the definition of a $\mathcal{C}$-based formal moduli problem (cf. [Lur11a, Definition 1.1.14]):

Definition 4.22. A *$\mathcal{C}$-based formal moduli problem* is a functor $X : \mathcal{C}_{\mathrm{art}} \to \mathcal{S}$ with the properties:

(1) X carries the terminal object $*$ to a contractible space.

(2) Given $A \in \mathcal{C}_{\mathrm{art}}$ and a morphism $A \to \mathrm{sqz}(k[n])$ with $n > 0$, the functor X sends the fibre product $A' \simeq A \times_{\mathrm{sqz}(k[n])} (*)$ to a fibre product in spaces $X(A') \simeq X(A) \times_{X(\mathrm{sqz}(k[n]))} X(*)$.

Let $\mathrm{Moduli}_{\mathcal{C}} \subset \mathrm{Fun}(\mathcal{C}_{\mathrm{art}}, \mathcal{S})$ be the ∞-category of formal moduli problems.

Given $F \in \mathrm{Moduli}_{\mathcal{C}}$, the functor $F \circ \mathrm{sqz} : \mathrm{Perf}_{k,\geq 0} \to \mathcal{S}$ is excisive. The corresponding k-module $T_F \in \mathrm{Mod}_k$ called the *tangent fibre* to F. Its underlying spectrum satisfies $\Omega^{\infty - n} T_F \simeq F(\mathrm{sqz}(k[n]))$.

We then have the following consequence of Theorem 4.20:

Theorem 4.23. *If $\mathcal{C}$ is part of a filtered augmented monadic adjunction (cf. Definition 4.15), there is an equivalence of ∞-categories $\mathrm{Alg}_{T^\vee} \xrightarrow{\simeq} \mathrm{Moduli}_{\mathcal{C}}$ with $\mathfrak{g} \mapsto (R \mapsto \mathrm{Map}_{\mathrm{Alg}_{T^\vee}}(\mathfrak{D}(R), \mathfrak{g}))$ such that the composite $\mathrm{Moduli}_{\mathcal{C}} \to \mathrm{Alg}_{T^\vee} \to \mathrm{Mod}_k$ is equivalent to the tangent fibre functor $F \mapsto T_F$.*

4.3. **Graded Objects.** Let $\mathcal{C}, \mathcal{C}^{\mathrm{Fil}}, \mathcal{C}^{\mathrm{Gr}}, \ldots$ be part of a filtered augmented monadic adjunction in the sense of Definition 4.15. Our first goal is to study the adjunction $\cot : \mathcal{C}^{\mathrm{Gr}} \rightleftarrows \mathrm{GrMod}_{k,\geq 0} : \mathrm{sqz}$ on graded objects. The admissibility axiom (3) will allow us to argue in a straightforward manner that this adjunction is in fact comonadic on a large class of objects. The rest of the argument required to prove Theorem 4.20 will then amount to reducing everything to this case. We begin with several basic observations on graded objects:

Remark 4.24 (The bar construction)**.** Let $A \in \mathcal{C}^{\mathrm{Gr}}$ (resp. $\mathcal{C}, \mathcal{C}^{\mathrm{Fil}}$). The augmented simplicial object $\mathrm{Bar}_\bullet(\mathrm{free}, \mathrm{free}, A) \to A$ admits an extra degeneracy in $\mathrm{Mod}_{k,\geq 0}$, and therefore induces an equivalence $|\mathrm{Bar}_\bullet(\mathrm{free}, \mathrm{free}, A)| \simeq A$. As the forgetful functor preserves geometric realisations, we deduce that $\mathrm{Bar}_\bullet(\mathrm{free}, \mathrm{free}, A) \to A$ is in fact a colimit diagram in $\mathcal{C}^{\mathrm{Gr}}$ (resp. $\mathcal{C}, \mathcal{C}^{\mathrm{Fil}}$).

Since $\cot(-)$ preserves colimits, it follows that $\cot(A)$ is the geometric realisation of the simplicial object $\mathrm{Bar}_\bullet(\mathrm{id}, \mathrm{free}, A)$ whose value in degree i is $\mathrm{free}^{\circ i}(A)$.

In addition, we have a natural map $A \to \cot(A)$ in $\mathrm{GrMod}_{k,\geq 0}$ (resp. $\mathrm{Mod}_{k,\geq 0}, \mathrm{FilMod}_{k,\geq 0}$), which is obtained by observing that $\mathrm{Bar}_0(A) = A$ and that we have a map $\mathrm{Bar}_0(A) \to |\mathrm{Bar}_\bullet(A)|$.

Note that for each i, we obtain a (degeneracy) map $A \to \mathrm{Bar}_i(A) = \mathrm{free}^{\circ i}(A)$; this is simply a composite of unit maps.

Remark 4.25 (Conservativity of cot in the graded case)**.** Let $A \in \mathcal{C}^{\mathrm{Gr}}$. Then the map $A \to \cot(A)$ induces an equivalence $A_1 \simeq \cot(A)_1$ in degree 1 by assumption $(3a)$ in Definition 4.15.

If a map $A \to B$ in $\mathcal{C}^{\mathrm{Gr}}$ induces an equivalence on $\cot(-)$, then $A \to B$ is an equivalence. Indeed, it follows inductively from assumption (3) of Definition 4.15 that $A^{(i)} \to B^{(i)}$ is an equivalence for all $i \geq 1$. Letting $i \to \infty$ and considering graded pieces, it follows that $A \to B$ is an equivalence.

We will now prove that the associated graded of $\mathrm{adic}(A)$ is canonically a free algebra for any $A \in \mathcal{C}$. Heuristically, we can think of the functor $\mathrm{adic}(-)$ as a method of interpolating between a general algebra and a free algebra.

Proposition 4.26. *There is a canonical isomorphism of functors $\mathcal{C} \to \mathcal{C}^{\mathrm{Gr}}$ given by*

$$\mathrm{Gr}(\mathrm{adic}(A)) \simeq \mathrm{free}([\cot(A)]_1).$$

Here we place $\cot(A) \in \mathrm{Mod}_{k,\geq 0}$ in graded degree one to construct $[\cot(A)]_1 \in \mathrm{GrMod}_{k,\geq 0}$.

Proof. Note that for any $B \in \mathcal{C}^{\mathrm{Gr}}$, there is a natural map from $\mathrm{free}([B_1]_1) \to B$. This map is an equivalence in internal degree 1: by assumption 3 of Definition 4.15, it suffices to check this after applying cot. Here, the map becomes $B_1 \to \cot(B)_1$, which again is an equivalence by assumption 3.

Let $A \in \mathcal{C}$. Then $\cot(\mathrm{adic}(A))$ is the filtered object $\cdots \to 0 \to 0 \to \cot(A)$, and therefore $\cot(\mathrm{Gr}(\mathrm{adic}(A)))$ is the graded object $[\cot(A)]_1$ with $\cot(A)$ concentrated in degree 1. It follows that we obtain a map $\mathrm{free}([\cot(A)]_1) \to \mathrm{Gr}(\mathrm{adic}(A))$ in $\mathcal{C}^{\mathrm{Gr}}$. This map is an equivalence when $A = \mathrm{free}(X)$ for $X \in \mathrm{Mod}_{k,\geq 0}$ as in this case, it is simply the identity map on $\mathrm{free}([X]_1)$. It must therefore be an equivalence in general since both sides preserve geometric realisations. $\square$

We invite the reader to recall the notion of graded modules pointwise of finite type introduced in Definition 3.46. This finiteness property can be detected using the cotangent fibre functor:

Proposition 4.27. *Let $A \in \mathcal{C}^{\mathrm{Gr}}$. Then the following are equivalent:*

(1) $\mathrm{forget}(A) \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.
(2) $\cot(A) \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.

Proof. As before, we will omit the forgetful functor from our notation.

Suppose first that $\cot(A) \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$, i.e. that each graded piece $\cot(A)_i \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ is of finite type. The filtration appearing in assumption (3) of Definition 4.15 shows by induction that each $A^{(i)}$ lies in $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$, because the functors G_j preserve $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$ by assumption. Since $A \to A^{(i)}$ is an equivalence in graded degrees below i, letting $i \to \infty$ implies that $A \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.

Conversely, suppose $A \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$ is a k-module spectrum of pointwise finite type. We can again proceed by induction. By Remark 4.25, we know that $\cot(A)_1 \simeq A_1$ is of finite type. Suppose $\cot(A)_1, \ldots, \cot(A)_{i-1} \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. Since the functors G_j are increasing and preserve $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$ for all $j > 1$, it follows that $G_j(\cot(A))_i = G_j(\mathrm{tr}_{<i}(\cot(A)))_i$ belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ for all $1 < j \leq i$. Then the filtration of assumption 3 shows inductively that the graded piece $A_i = A^{(i)}_i$ of $A^{(i)} \in \mathcal{C}^{\mathrm{Gr}}$ has a finite filtration involving $\cot(A)_i$ and terms in $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ (namely, $G_j(\cot(A))_i$ for $1 < j \leq i$). Since we assumed that $A_i \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$, it follows from this $\cot(A)_i$ lies in $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. $\square$

We can now establish the basic tool for commuting cot and totalisations, which is the heart of the convergence arguments needed in this work. First we need a basic notion.

Definition 4.28. The ∞-categories $\mathcal{C}, \mathcal{C}^{\mathrm{Fil}}, \mathcal{C}^{\mathrm{Gr}}$ are presentable, and hence have all limits. These are computed at the level of objects in $\mathrm{Mod}_{k,\geq 0}$ (or $\mathrm{GrMod}_{k,\geq 0}, \mathrm{FilMod}_{k,\geq 0}$). We will say that a limit in $\mathcal{C}, \mathcal{C}^{\mathrm{Fil}}, \mathcal{C}^{\mathrm{Gr}}$ *connectively exists* if the limit is also preserved in $\mathrm{Mod}_k, \mathrm{GrMod}_k,$ or FilMod_k. In particular, the limit in $\mathrm{Mod}_k, \mathrm{GrMod}_k,$ or FilMod_k is connective.

Proposition 4.29 (Convergence criterion in $\mathcal{C}^{\mathrm{Gr}}$)**.** *Let $A^\bullet$ be a cosimplicial object of $\mathcal{C}^{\mathrm{Gr}}$ such that for each i, we have $\mathrm{forget}(A^i) \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$. Then the following are equivalent:*

(1) The totalisation $\mathrm{Tot}(\mathrm{forget}(A^\bullet))$ (computed in GrMod_k) belongs to $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.
(2) The totalisation $\mathrm{Tot}(\cot(A^\bullet))$ (computed in GrMod_k) belongs to $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.

If these conditions are satisfied, then the limit $\mathrm{Tot}(A^\bullet)$ connectively exists in $\mathcal{C}^{\mathrm{Gr}}$, and the canonical map $\cot(\mathrm{Tot}(A^\bullet)) \to \mathrm{Tot}(\cot(A^\bullet))$ in $\mathrm{GrMod}_{k,\geq 0}$ is an equivalence.

Proof. In fact, we shall prove the following more refined statement:

Let $B^\bullet$ be an augmented cosimplicial object of $\mathcal{C}^{\mathrm{Gr}}$ with $B^j \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$ for all $j \geq 0$. Suppose that $B_1^\bullet, \ldots, B_{i-1}^\bullet$ are limit diagrams in Mod_k with $B_1^{-1}, \ldots, B_{i-1}^{-1} \in \mathrm{Mod}_{k,\geq 0}^{\mathrm{ft}}$. Then the following are equivalent:

(1) $B_i^\bullet$ is a limit diagram with $B_i^{-1} \in \mathrm{Mod}_{k,\geq 0}^{\mathrm{ft}}$.
(2) $\cot(B^\bullet)_i$ is a limit diagram in Mod_k and $\cot(B^{-1})_i \in \mathrm{Mod}_{k,\geq 0}^{\mathrm{ft}}$.

By induction, this implies the equivalence of (1) and (2) in the proposition, as well as the asserted convergence. Here we use that the forgetful functor $\mathcal{C}^{\mathrm{Gr}} \to \mathrm{GrMod}_{k,\geq 0}$ creates limits.

For $i = 1$, our refined claim follows from the equivalence $B_1^\bullet \simeq \cot(B^\bullet)_1$. In general, we observe that $B_i^\bullet$ (considered as an augmented cosimplicial object of Mod_k) admits a finite filtration whose associated graded terms are given by $\cot(B^\bullet)_i$ and $G_j(\cot(B^\bullet))_i$ for $1 < j \leq i$. By assumption, the functor G_j is admissible and increasing, and so the augmented cosimplicial object $G_j(\cot(B^\bullet))_i \simeq G_j(\mathrm{tr}_{i-j+1}\cot(B^\bullet))_i$ is a limit diagram by the hypothesis. It follows that $B_i^\bullet$ is a limit diagram with $B^{-1} \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$ if and only if $\cot(B^\bullet)_i$ is a limit diagram with $\cot(B^{-1}) \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$. $\square$

Let $\mathcal{C}^{\mathrm{Gr}^{\mathrm{pft}}} \subset \mathcal{C}^{\mathrm{Gr}}$ be the full subcategory spanned by objects whose underlying graded module belongs to $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.

Proposition 4.30. *Restriction gives rise to a comonadic adjunction*

$$\cot : \mathcal{C}^{\mathrm{Gr}^{\mathrm{pft}}} \rightleftarrows \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0} : \mathrm{sqz}.$$

Proof. The adjunction is well-defined by Proposition 4.27. It is in fact comonadic by Theorem 4.5, whose conditions are satisfied by Remark 4.25 and Proposition 4.29. $\square$

This comonadicity result for $\mathcal{C}^{\mathrm{Gr}^{\mathrm{pft}}}$ will later allow us to check convergence results in $\mathcal{C}$ by first lifting cosimplicial diagrams to filtered objects and then taking associated gradeds everywhere.

However, $\mathcal{C}^{\mathrm{Gr}^{\mathrm{pft}}}$ is *not* the correct graded analogue of the ∞-category $\mathcal{C}^{\mathrm{afp}}$ of almost finitely presented objects because the cotangent complex need not be finite type (only pointwise finite type). We therefore introduce a more restrictive notion of finiteness.

For this, we first recall Definition 2.14, which introduces $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ as the full subcategory of GrMod_k spanned by all $X_\star$ for which the underlying module $\bigoplus_{i\geq 1} X_i$ is of finite type. Note that $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0} \subset \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$ is a proper inclusion.

We recall the corresponding full subcategory of $\mathcal{C}^{\mathrm{Gr}}$:

Definition 4.31 (The subcategory $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$)**.** The category $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ of *almost finitely presented* objects consists of all $A \in \mathcal{C}^{\mathrm{Gr}}$ whose cotangent complex $\cot(A) \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ has finite type.

We should think of this as a finite generation condition (at least after any truncation). Note that by Proposition 4.27, we have an inclusion $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}} \subset \mathcal{C}^{\mathrm{Gr}^{\mathrm{pft}}}$.

Remark 4.32. While $\mathcal{C}^{\mathrm{Gr}^{\mathrm{pft}}}$ is evidently closed under fibre products of maps which are surjective on π_0 (as these can be computed pointwise), the corresponding claim in $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ only holds by our coherence axiom (4) in Definition 4.15, which will be easy to check in the examples of interest.

For instance, this ensures that if $V, V', V'' \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ and we have maps $V \to V'', V' \to V''$ which induce surjections on π_0, then the fibre product $A = \mathrm{free}(V) \times_{\mathrm{free}(V'')} \mathrm{free}(V')$ has the property that $\cot(A) \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$. Proposition 4.27 only shows that $\cot(A) \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$, so we need to postulate this stronger statement.

The coherence axiom (4) in Definition 4.15 implies:

Construction 4.33. The $(\cot, \mathrm{sqz})$-adjunction between $\mathcal{C}^{\mathrm{Gr}}$ and $\mathrm{GrMod}_{k,\geq 0}$ restricts to an adjunction

$$\cot : \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}} \rightleftarrows \mathrm{GrMod}^{\mathrm{ft}}_{k,\geq 0} : \mathrm{sqz}.$$

4.4. **Filtered objects.** We will now transfer some of the above results to the ∞-category $\mathcal{C}^{\mathrm{Fil}}$. In order to obtain similarly strong statements, we need to restrict attention to *complete* objects. Recall that a filtered connective k-module $(\ldots \to F^2M \to F^1M) \in \mathrm{FilMod}_{k,\geq 0}$ is said to be complete if the inverse limit $\lim_i(F^iM)$ vanishes. The inclusion $\mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{k,\geq 0} \subset \mathrm{FilMod}_{k,\geq 0}$ of complete filtered connective k-modules into all filtered connective k-modules admits a left adjoint called *completion* (cf. Definition 2.8).

In order to lift this completion functor to $\mathcal{C}^{\mathrm{Fil}}$, we need an elementary categorical observation.

Remark 4.34 (Adjunctions and localisations)**.** Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be an adjunction of presentable ∞-categories and suppose that we are given a Bousfield localisation $L_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ of $\mathcal{A}$ with corresponding strongly saturated class of morphisms $\mathcal{W}_{\mathcal{A}}$ in $\mathcal{A}$ (cf. [Lur09, Def. 5.5.4.5]).

To produce a corresponding Bousfield localisation on $\mathcal{B}$, we let $\mathcal{W}_{\mathcal{B}}$ be the class of morphisms $f : B_1 \to B_2$ satisfying $G(f) \in \mathcal{W}_{\mathcal{A}}$. Assume that $\mathcal{W}_{\mathcal{B}}$ is strongly saturated and contains $F(\mathcal{W}_{\mathcal{A}})$. The localisation functor $L_{\mathcal{B}}$ for $\mathcal{W}_{\mathcal{B}}$ (cf. [Lur09, Sec.5.5.4]) sits in a commutative square:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "L_{\mathcal{B}}"] \arrow[d, "G"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "L_{\mathcal{A}}"] & \mathcal{A} \end{tikzcd}$$

Indeed, given any $B \in \mathcal{B}$, the unit $B \to L_{\mathcal{B}}B$ lies in $\mathcal{W}_{\mathcal{B}}$. It follows that if B is $\mathcal{W}_{\mathcal{B}}$-local, then $G(B) \in \mathcal{A}$ is $\mathcal{W}_{\mathcal{A}}$-local. Since G sends $\mathcal{W}_{\mathcal{B}}$ to $\mathcal{W}_{\mathcal{A}}$, the commutativity follows. We say that the adjunction is *compatible with localisations.*

Using this, we can lift the notion of completeness to $\mathcal{C}^{\mathrm{Fil}}$:

Definition 4.35 (Completions in $\mathcal{C}^{\mathrm{Fil}}$)**.** An object $A \in \mathcal{C}^{\mathrm{Fil}}$ is *complete* if $\mathrm{forget}(A) \in \mathrm{FilMod}_{k,\geq 0}$ is complete. We let $\widehat{\mathcal{C}^{\mathrm{Fil}}} \subset \mathcal{C}^{\mathrm{Fil}}$ be the full subcategory spanned by all complete objects.

The strongly saturated class associated with the localisation $\mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{k,\geq 0} \subset \mathrm{FilMod}_{k,\geq 0}$ consists of all maps which induce equivalences on associated gradeds. By the assumptions in Definition 4.15, the free-forgetful adjunction $\mathrm{FilMod}_{k,\geq 0} \rightleftarrows \mathcal{C}^{\mathrm{Fil}}$ is compatible with completions in the sense of Remark 4.34. We therefore obtain a completion functor $\mathcal{C}^{\mathrm{Fil}} \to \widehat{\mathcal{C}^{\mathrm{Fil}}}$ which is the left adjoint of a Bousfield localisation. Any $A \in \mathcal{C}^{\mathrm{Fil}}$ comes equipped with a natural morphism $A \to \widehat{A}$ to its completion, and the underlying object $\mathrm{forget}(\widehat{A}) \in \mathrm{FilMod}_{k,\geq 0}$ of $\widehat{A}$ is simply given by the completion of the filtered object $\mathrm{forget}(A)$.

Note that if a morphism $A \to B$ in $\mathcal{C}^{\mathrm{Fil}}$ induces an equivalence on associated gradeds, then so does $\cot(A) \to \cot(B)$. We deduce that for any $A \in \mathcal{C}$, the associated map on cotangent fibres $\cot(A) \to \cot(\widehat{A})$ is also an equivalence on associated gradeds. We obtain a diagram of left adjoints:

$$\begin{tikzcd} \mathrm{FilMod}_{k,\geq 0} \arrow[r, "\mathrm{free}"] \arrow[d] & \mathcal{C}^{\mathrm{Fil}} \arrow[r, "\cot"] \arrow[d] & \mathrm{FilMod}_{k,\geq 0} \arrow[d] \\ \mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{k,\geq 0} \arrow[r, "\widehat{\mathrm{free}}"] & \widehat{\mathcal{C}^{\mathrm{Fil}}} \arrow[r, "\widehat{\cot}"] & \mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{k,\geq 0} \end{tikzcd}$$

in which the vertical arrows are given by completion functors.

Remark 4.36 (The completed cotangent adjunction)**.** The functor $\widehat{\cot} = \widehat{(-)} \circ \cot$ appearing in the lower right hand part of the above diagram is part of an adjunction $\widehat{\cot} : \widehat{\mathcal{C}}^{\mathrm{Fil}} \rightleftarrows \mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{k,\geq 0} : \mathrm{sqz}$. Its left adjoint sends a complete object A of $\mathcal{C}$ to its completed cotangent fibre $\widehat{\cot(A)}$, whereas the right adjoint sends $V \in \mathrm{Fil}_{\mathrm{cpl}}\mathrm{Mod}_{k,\geq 0}$ to $\mathrm{sqz}(V) \in \widehat{\mathcal{C}} \subset \mathcal{C}$.

We can next make a direct translation of Proposition 4.29 to the filtered setting:

Proposition 4.37 (Convergence criterion in $\mathcal{C}^{\mathrm{Fil}}$)**.** *Let $X^\bullet$ be a cosimplicial complete object of $\mathcal{C}^{\mathrm{Fil}}$ such that for each i, we have $\mathrm{Gr}(\mathrm{forget}(X^i)) \in \mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$. Then the following are equivalent:*

(1) The associated graded $\mathrm{Gr}(\mathrm{Tot}(\mathrm{forget}(X^\bullet)))$ of the totalisation $\mathrm{Tot}(\mathrm{forget}(X^\bullet))$ (computed in FilMod_k) belongs to $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.

(2) The associated graded $\mathrm{Gr}(\mathrm{Tot}(\cot(X^\bullet)))$ of the totalisation $\mathrm{Tot}(\cot(X^\bullet))$ (computed in FilMod_k) belongs to $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.

Under these assumptions, the limit $\mathrm{Tot}(X^\bullet)$ connectively exists, and the map $\widehat{\cot}(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(\widehat{\cot}(X^\bullet))$ is an equivalence.

Proof. The equivalence of (1) and (2) follows immediately from Proposition 4.29 by taking associated gradeds (using the adjointability axiom (2) in Definition 4.15 together with the fact that $\mathrm{Gr} : \mathrm{FilMod}_k \to \mathrm{GrMod}_k$ commutes with totalisations).

The filtered module $\mathrm{Tot}(\mathrm{forget}(X^\bullet))$ (computed in FilMod_k) is complete (as completeness is a limit condition) and has associated graded in $\mathrm{Mod}_{k,\geq 0}$. The Milnor sequence implies that $\mathrm{Tot}(\mathrm{forget}(X^\bullet))$ in fact belongs to $\mathrm{FilMod}_{k,\geq 0}$. It follows that $X^\bullet$ admits a limit in $\mathcal{C}^{\mathrm{Fil}}$. The arrow $\widehat{\cot}(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(\widehat{\cot}(X^\bullet))$ induces an equivalence after passing to associated gradeds by Proposition 4.29. Hence, it is itself an equivalence since both domain and target are complete. ☐

Remark 4.38. Note that both conditions (1) and (2) in Proposition 4.37 contain the nontrivial assertion that the respective totalisations are connective.

We can now formulate a notion of almost finite presentation in the complete filtered context:

Definition 4.39 (The subcategory $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$)**.** Let $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ denote the subcategory of objects $A \in \mathcal{C}^{\mathrm{Fil}}$ which are complete and satisfy $\mathrm{Gr}(A) \in \mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$, i.e. $\mathrm{Gr}(\cot(A)) \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$. We will refer to these objects as *complete almost finitely presented.*

Example 4.40 (Completed-free algebras in $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$)**.** Completed-free algebras on filtered (connective) modules of finite type belong to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$. Indeed, if $V \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$, then $\mathrm{free}(V) \in \mathcal{C}^{\mathrm{Fil}}$ has cotangent fibre V. Since $\mathrm{free}(V) \to \widehat{\mathrm{free}(V)}$ induces an equivalence on associated gradeds, the observations made after Definition 4.35 imply that $\mathrm{Gr}(\mathrm{cot}(\widehat{\mathrm{free}(V)})) \simeq \mathrm{Gr}(\mathrm{cot}(\mathrm{free}(V))) \simeq \mathrm{Gr}(V)$ belongs to $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$, and hence $\widehat{\mathrm{free}(V)} \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$.

In fact, the completeness assumption $(5a)$ in Definition 4.15 implies a stronger assertion. We know that $\mathrm{cot}(\widehat{\mathrm{free}(V)})$ is complete, and we can therefore conclude that $\mathrm{cot}(\widehat{\mathrm{free}(V)}) \simeq V$. In other words, the morphism $\mathrm{free}(V) \to \widehat{\mathrm{free}}(V)$ induces an equivalence on cot. This can be thought of as a generalisation of the classical fact that a polynomial ring and a power series ring on finitely many variables have the same cotangent fibre.

Construction 4.41. If $V \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$, then the coherence axiom (4b) of Definition 4.15 implies that $\mathrm{sqz}(V) \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$. As in Construction 4.33, we obtain the following adjunction by restriction:

$$\widehat{\mathrm{cot}} : \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}} \rightleftarrows \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0} : \mathrm{sqz}$$

Remark 4.42. The full subcategory $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}} \subset \mathcal{C}^{\mathrm{Fil}}$ is closed under geometric realisations. This follows as if $X_\bullet$ is a simplicial diagram in $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$, then the underlying module $\mathrm{forget}(|X_\bullet|) \simeq |\,\mathrm{forget}(X_\bullet)|$ is complete by Proposition 2.11, and moreover $\mathrm{Gr}(\mathrm{cot}(|X_\bullet|)) \simeq |\mathrm{Gr}(\mathrm{cot}(X_\bullet|))|$ lies in $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ since $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} \subset \mathrm{Mod}_k$ is closed under geometric realisations.

Hence any geometric realisation of completed-free objects $\widehat{\mathrm{free}(V)}$ with $V \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ lies in $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$.

We will now pass from filtered to non-filtered objects:

Definition 4.43 (The subcategory $\mathcal{C}_{\mathrm{afp}}$)**.** The full subcategory $\mathcal{C}_{\mathrm{afp}} \subset \mathcal{C}$ of *complete almost finitely presented* objects in $\mathcal{C}$ consists of all A for which $\mathrm{adic}(A) \in \mathcal{C}^{\mathrm{Fil}}$ is complete and $\mathrm{cot}(A) \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$.

In Examples 4.9 and 4.10, the respective subcategories $\mathcal{C}_{\mathrm{afp}}$ will be as expected, i.e. consist of all augmented simplicial commutative k-algebras (resp. connective $\mathbb{E}_\infty$-k-algebras) A for which $\pi_0(A)$ is complete local Noetherian and $\pi_i(A)$ finitely generated over $\pi_0(A)$ for all i.

Remark 4.44. By the completeness axiom (5) in Definition 4.15, we know that if $A \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$, then $F^1A \in \mathcal{C}_{\mathrm{afp}}$. Indeed, assumption $(5b)$ implies that F^1A is complete, and $\mathrm{cot}(F^1A) \simeq F^1(\mathrm{cot}(A))$ lies in $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ since $\mathrm{cot}(A)$ is complete by $(5a)$ and $\mathrm{Gr}(\mathrm{cot}(A))$ lies in $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ by definition.

Example 4.45 (Completed-free algebras in $\mathcal{C}_{\mathrm{afp}}$)**.** Completed-free algebras on (connective) modules of finite type belong to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$. Indeed, if $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$, then we consider $\widetilde{V} = (\cdots \to 0 \to 0 \to V)$ in $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$. We have $\mathrm{free}(\widetilde{V}) \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ by Example 4.40, and this implies that $\mathrm{free}(V) \simeq F^1\mathrm{free}(\widetilde{V})$ lies in $\mathcal{C}_{\mathrm{afp}}$ by Remark 4.44.

Remark 4.46 (Closure properties of $\mathcal{C}_{\mathrm{afp}}$)**.** The subcategory $\mathcal{C}_{\mathrm{afp}} \subset \mathcal{C}$ is closed under geometric realisations. This follows from Remark 4.42 by noting that the left adjoints adic and cot preserve realisations and the subcategory $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} \subset \mathrm{Mod}_k$ is closed under realisations.

Moreover, if $A, A', A'' \in \mathcal{C}_{\mathrm{afp}}$ and we are given maps $A \to A'', A' \to A''$ which induce surjections on π_0, then the pullback $A \times_{A''} A'$ also belongs to $\mathcal{C}_{\mathrm{afp}}$. Indeed, note that $\mathrm{adic}(A), \mathrm{adic}(A'), \mathrm{adic}(A'')$ belong to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$, and that both maps $\mathrm{adic}(A) \to \mathrm{adic}(A'')$ and $\mathrm{adic}(A') \to \mathrm{adic}(A'')$ are surjective on π_0.

By the coherence axiom $(4a)$ in Definition 4.15, we deduce that $\mathrm{Gr}(\mathrm{adic}(A))\times_{\mathrm{Gr}(\mathrm{adic}(A''))}\mathrm{Gr}(\mathrm{adic}(A'))$ belongs to $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$. The canonical arrow

$$\mathrm{Gr}\left(\mathrm{adic}(A)\times_{\mathrm{adic}(A'')}\mathrm{adic}(A')\right)\to \mathrm{Gr}(\mathrm{adic}(A))\times_{\mathrm{Gr}(\mathrm{adic}(A''))}\mathrm{Gr}(\mathrm{adic}(A'))$$

induces an equivalence after applying forget by the adjointability axiom (2) in Definition 4.15. We deduce that $\mathrm{adic}(A)\times_{\mathrm{adic}(A'')}\mathrm{adic}(A')\in\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ (as it is evidently complete). Applying the right adjoint F^1 now shows that $A\times_{A'}A''\in\mathcal{C}_{\mathrm{afp}}$ by Remark 4.44.

Remark 4.47. If $V\in\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$, then $\mathrm{sqz}(V)$ belongs to $\mathcal{C}_{\mathrm{afp}}$. To see this, we lift V to a complete filtered module $\widetilde{V}=(\ldots\to 0\to 0\to V)$ in $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$. We then observe that $\mathrm{sqz}(\widetilde{V})$ is evidently complete and it therefore lies in $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$ by the coherence axiom $(4b)$ in Definition 4.15. By Remark 4.44, this implies that $F^1(\mathrm{sqz}(\widetilde{V}))\simeq\mathrm{sqz}(V)$ lies in $\mathcal{C}_{\mathrm{afp}}$.

Using this observation, we can establish a version of Construction 4.33 in the unfiltered context:

Construction 4.48. The $(\mathrm{cot},\mathrm{sqz})$-adjunction between $\mathcal{C}$ and $\mathrm{Mod}_{k,\geq 0}$ restricts to an adjunction

$$\mathrm{cot}:\mathcal{C}_{\mathrm{afp}}\ \underset{\longleftarrow}{\longrightarrow}\ \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}:\mathrm{sqz}.$$

We now wish to show that this adjunction satisfies the desirable conditions stated in Theorem 4.20. For this, we will need a convergence criterion for cosimplicial objects in $\mathcal{C}_{\mathrm{afp}}$. Indeed, our criterion says that if a cosimplicial object in $\mathcal{C}_{\mathrm{afp}}$ admits a suitable lift to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$, then taking the cotangent fibre commutes with totalisation. More precisely:

Proposition 4.49 (Convergence criterion in $\mathcal{C}$)**.** *Let $X^\bullet$ be a cosimplicial object of $\mathcal{C}_{\mathrm{afp}}$. Suppose that there exists a lift $\widetilde{X}^\bullet$ of $X^\bullet$ to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ which satisfies the equivalent conditions of Proposition 4.37 and moreover has complete almost finitely presented totalisation $\mathrm{Tot}(\widetilde{X}^\bullet)\in\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$.*

Then the limit $\mathrm{Tot}(X^\bullet)$ of $X^\bullet$ connectively exists in $\mathcal{C}$, belongs to $\mathcal{C}_{\mathrm{afp}}$, and the following map is an equivalence:

$$\mathrm{cot}(\mathrm{Tot}(X^\bullet))\to\mathrm{Tot}(\mathrm{cot}(X^\bullet)).$$

Proof. By Proposition 4.37, the totalisation $\widetilde{X}^{-1}:=\mathrm{Tot}(\widetilde{X}^\bullet)$ connectively exists in $\mathcal{C}^{\mathrm{Fil}}$ and we have an equivalence $\widehat{\mathrm{cot}}(\widetilde{X}^{-1})\xrightarrow{\simeq}\mathrm{Tot}(\widehat{\mathrm{cot}}(\widetilde{X}^\bullet))$. The right adjoint $F^1:\mathcal{C}^{\mathrm{Fil}}\to\mathcal{C}$ preserves limits, and so $X^{-1}:=\mathrm{Tot}(X^\bullet)$ connectively exists in $\mathcal{C}$ and we have $X^{-1}\simeq F^1(\widetilde{X}^{-1})$. Our assumption $\widetilde{X}^{-1}=\mathrm{Tot}(\widetilde{X}^\bullet)\in\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ implies that $X^{-1}\simeq F^1(\widetilde{X}^{-1})$ belongs to $\mathcal{C}_{\mathrm{afp}}$ by Remark 4.44.

By the completeness axiom $(5a)$ of Definition 4.15, we know that $\mathrm{cot}(\widetilde{X}^i)\xrightarrow{\simeq}\widehat{\mathrm{cot}}(\widetilde{X}^i)$ is an equivalence for all $i\geq -1$. Proposition 4.37 therefore shows that $\mathrm{cot}(\widetilde{X}^{-1})\simeq\mathrm{Tot}(\mathrm{cot}(\widetilde{X}^\bullet))$. We conclude the proof by applying F^1 and using the adjointability axiom (2) of Definition 4.15. □

We can now proceed to the proof of the main result of this axiomatic section:

Proof of Theorem 4.20. We constructed the adjunction $\mathcal{C}_{\mathrm{afp}}\underset{\mathrm{sqz}}{\overset{\mathrm{cot}}{\rightleftarrows}}\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ in Construction 4.48. To prove that this adjunction is comonadic, we will verify the conditions of Theorem 4.5.

First, we check that the functor cot is conservative. Let $A\to B$ be a map in $\mathcal{C}_{\mathrm{afp}}$ which induces an equivalence $\mathrm{cot}(A)\xrightarrow{\simeq}\mathrm{cot}(B)$ on cotangent fibres. Then $\mathrm{adic}(A)\to\mathrm{adic}(B)$ also induces an equivalence on cotangent fibres, and hence also on associated gradeds by Proposition 4.26. Since $\mathrm{adic}(A)$ and $\mathrm{adic}(B)$ are both complete, it follows from Remark 2.9 that $\mathrm{adic}(A)\xrightarrow{\simeq}\mathrm{adic}(B)$ is an equivalence, and hence the same holds true for $A\xrightarrow{\simeq}B$.

To check the second condition of Theorem 4.5, we fix a cosimplicial object $X^\bullet$ in $\mathcal{C}_{\mathrm{afp}}$ and assume that $\cot(X^\bullet)$ admits a splitting in $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. We pick the filtered lift $\widetilde{X}^\bullet := \mathrm{adic}(X^\bullet)$ of $X^\bullet$, which is a cosimplicial object in $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ by definition. Proposition 4.26, implies that $\mathrm{Gr}(\widetilde{X}^\bullet) \simeq \mathrm{free}([\cot(X^\bullet)]_1)$ admits a splitting in $\mathcal{C}^{\mathrm{Gr}}_{\mathrm{afp}}$. Using the adjointability axiom (2) in Definition 4.15 and Proposition 4.27, we see that $\mathrm{Gr}(\mathrm{Tot}(\mathrm{forget}(\widetilde{X}^\bullet))) \simeq \mathrm{forget}(\mathrm{Tot}(\mathrm{Gr}(\widetilde{X}^\bullet)))$ belongs to $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$. Proposition 4.37 therefore shows that the limit $\widetilde{X}^{-1} := \mathrm{Tot}(\widetilde{X}^\bullet)$ connectively exists in $\mathcal{C}^{\mathrm{Fil}}$, and that the natural map $\widehat{\cot}(\widetilde{X}^{-1}) \xrightarrow{\simeq} \mathrm{Tot}(\widehat{\cot}(\widetilde{X}^\bullet))$ is an equivalence. Since the cosimplicial diagram $\widehat{\cot}(\widetilde{X}^\bullet) \simeq (\cot(X^\bullet))_1$ is split in $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$, it follows that $\widehat{\cot}(\widetilde{X}^{-1})$ belongs to $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ as well. This shows that $\mathrm{Gr}(\cot(\widetilde{X}^{-1}) \simeq \mathrm{Gr}(\widehat{\cot}(\widetilde{X}^{-1}) \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_k$, which allows us to conclude that $\widetilde{X}^{-1}$ belongs to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ (it is evidently complete as this is a limit condition).

The convergence criterion Proposition 4.49 then implies that the limit $X^{-1} := \mathrm{Tot}(X^\bullet)$ belongs to $\mathcal{C}_{\mathrm{afp}}$, and that the canonical map $\cot(X^{-1}) \xrightarrow{\simeq} \mathrm{Tot}(\cot(X^\bullet))$ is an equivalence. This proves comonadicity, i.e. statement (1) of the theorem.

Before proceeding further, we record that comonadicity implies that any $A \in \mathcal{C}_{\mathrm{afp}}$ is the totalisation of its canonical cobar resolution $((\mathrm{sqz} \circ \cot)(A) \rightrightarrows (\mathrm{sqz} \circ \cot) \circ (\mathrm{sqz} \circ \cot)(A) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \dots)$. In particular, A is a totalisation of a cosimplicial object in $\mathcal{C}_{\mathrm{afp}}$ which at each level is square-zero.

We will now verify part (2) of the theorem using Proposition 3.16. For this, let $T = \cot \circ \mathrm{sqz}$ be the comonad on $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ induced by the adjunction and define $T^\vee$ as the monad induced on $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ by linear duality. Let $V^\bullet$ be a cosimplicial object in $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ which is m-coskeletal for some m, and assume that $V^{-1} := \mathrm{Tot}(V^\bullet)$ belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. To prove statement (2) of the theorem, we need to first verify that the following map is an equivalence:

$$T(V^{-1}) \xrightarrow{\ \simeq\ } \mathrm{Tot}(T(V^\bullet)). \tag{4}$$

Via duality, this implies that $T^\vee$ commutes with finite coconnective geometric realisations in $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. To prove the equivalence (4) above, we apply Proposition 4.49 to the cosimplicial object $X^\bullet = \mathrm{sqz}(V^\bullet)$ together with its filtered lift $\widetilde{X}^\bullet = \mathrm{sqz}(\widetilde{V}^\bullet)$, where $\widetilde{V}^\bullet = (\cdots \to 0 \to 0 \to V^\bullet)$. Using axioms (2) and (4b) in Definition 4.15, we see that the filtered lift $\widetilde{X}^\bullet$ is a cosimplicial object in $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ and also satisfies the other assumptions of Proposition 4.49. It follows that $\cot(\mathrm{Tot}(X^\bullet)) \xrightarrow{\simeq} \mathrm{Tot}(\cot(X^\bullet))$ is an equivalence, which proves (4) since sqz preserves limits. Moreover, if $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ is of finite type, then $T(V) \simeq \varprojlim_n T(\tau_{\leq n} V)$, and the inverse limit stabilises in any finite range of homological degrees by Proposition 3.42. We deduce that $T^\vee$ is right complete. Thus, it follows that the criteria of Proposition 3.16 are satisfied, which shows that $T^\vee$ admits the sifted colimit-preserving extension $\mathrm{Mod}_k \to \mathrm{Mod}_k$ asserted in (2).

Finally, we establish part (3) of the theorem. For this, let $A, A', A'' \in \mathcal{C}_{\mathrm{afp}}$ and suppose that we are given maps $A \to A', A \to A''$ which induce surjections on π_0. We need to show that the natural map $\mathfrak{D}(A) \sqcup_{\mathfrak{D}(A'')} \mathfrak{D}(A') \xrightarrow{\simeq} \mathfrak{D}(A \times_{A'} A')$ is an equivalence. This is easy to check if everything is square-zero. That is, if we are given $V, V', V'' \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ together with π_0-surjective maps $V \to V''$ and $V' \to V''$, then the following map of $T^\vee$-algebras is an equivalence:

$$\mathfrak{D}(\mathrm{sqz}(V)) \sqcup_{\mathfrak{D}(\mathrm{sqz}(V''))} \mathfrak{D}(\mathrm{sqz}(V')) \xrightarrow{\ \simeq\ } \mathfrak{D}(\mathrm{sqz}(V \times_{V''} V')).$$

Indeed, the left-hand-side is the pushout of the free $T^\vee$-algebras on $V^\vee, V''^\vee$, and $V'^\vee$, respectively, whereas the right-hand side is the free $T^\vee$-algebra on $(V \times_{V''} V')^\vee$. Our strategy now is to reduce the general case to the square-zero case by using cobar resolutions.

For this, let $X^\bullet, X'^\bullet, X''^\bullet$ be the canonical cobar resolutions of A, A', A'', respectively. For example, we have $X^0 = (\mathrm{sqz} \circ \cot)(A)$ and $X^1 = (\mathrm{sqz} \circ \cot) \circ (\mathrm{sqz} \circ \cot)(A)$. Note that these are cosimplicial objects of $\mathcal{C}_{\mathrm{afp}}$. The maps $X^\bullet \to X''^\bullet, X'^\bullet \to X''^\bullet$ induce surjections on π_0 at each level, and so we can also form the cosimplicial object in $\mathcal{C}_{\mathrm{afp}}$ given by $Y^\bullet := X^\bullet \times_{X''^\bullet} X'^\bullet$ by Remark 4.46. Comonadicity implies that $A \simeq \mathrm{Tot}(X^\bullet), A' \simeq \mathrm{Tot}(X'^\bullet), A'' \simeq \mathrm{Tot}(X''^\bullet)$. Therefore, we have $\mathrm{Tot}(Y^\bullet) \simeq A \times_{A'} A''$. Since we have already verified claim (3) in the case of square-zero extensions, we have the following equivalence of $T^\vee$-algebras for all $i \geq 0$:

$$\mathfrak{D}(X^i) \sqcup_{\mathfrak{D}(X''^i)} \mathfrak{D}(X'^i) \xrightarrow{\simeq} \mathfrak{D}(Y^i).$$

To deduce that $\mathfrak{D}(A) \sqcup_{\mathfrak{D}(A'')} \mathfrak{D}(A') \simeq \mathfrak{D}(A \times_{A'} A'')$, it therefore suffices to verify the following facts:

a) $|\mathfrak{D}(X^\bullet)| \simeq \mathfrak{D}(A)$ and $|\mathfrak{D}(X'^\bullet)| \simeq \mathfrak{D}(A')$ and $|\mathfrak{D}(X''^\bullet)| \simeq \mathfrak{D}(A'')$.
b) $|\mathfrak{D}(Y^\bullet)| \simeq \mathfrak{D}(A \times_{A'} A'')$.

Geometric realisations in $T^\vee$-algebras can be computed in Mod_k as $T^\vee$ preserves sifted colimits.

Claim (a) follows immediately by applying linear duality to the comonadicity established above.

Claim (b) will follow by applying the convergence criterion established in Proposition 4.37.

First, we form the cosimplicial object $\widetilde{Y}^\bullet := \mathrm{adic}(X^\bullet) \times_{\mathrm{adic}(X''^\bullet)} \mathrm{adic}(X'^\bullet)$, which is a filtered lift of $Y^\bullet$ to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ by Remark 4.46.

Second, we will check that the totalisation of $\mathrm{Gr}(\widetilde{Y}^\bullet) \in \mathcal{C}^{\mathrm{Gr}}$ belongs to $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$. For this, we first observe that by construction, the cosimplicial objects $\cot(X^\bullet)$, $\cot(X'^\bullet)$, $\cot(X''^\bullet)$ are all split in $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. Proposition 4.26 then shows that $\mathrm{Gr}(\mathrm{adic}(X^\bullet)), \mathrm{Gr}(\mathrm{adic}(X'^\bullet))$, and $\mathrm{Gr}(\mathrm{adic}(X''^\bullet)$ have splittings and therefore admit totalisations in $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$. Since the maps between these objects are levelwise surjective on π_0, we see that $\mathrm{Gr}(\widetilde{Y}^\bullet)$ also admits a totalisation in $\mathrm{Gr}^{\mathrm{pft}}\mathrm{Mod}_{k,\geq 0}$.

Third, we need to show that $\mathrm{Tot}(\widetilde{Y}^\bullet)$ admits a totalisation in $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$. For this, we first observe that the map $\mathrm{adic}(A) \to \mathrm{Tot}(\mathrm{adic}(X^\bullet))$ induces an equivalence. By completeness, it suffices to check this after applying Gr. Here, it is true because the functor $\mathrm{Gr} \circ \mathrm{adic}$ is equivalent to $\mathrm{free} \circ [-]_1$ by Proposition 4.26, the natural map $\cot(A) \xrightarrow{\simeq} \mathrm{Tot}(\cot(X^\bullet))$ is an equivalence by the construction of the cobar resolution, and the functor $\mathrm{forget} \circ \mathrm{free}$ is admissible by axiom (3) of Definition 4.15. A similar argument gives equivalences $\mathrm{adic}(A') \to \mathrm{Tot}(\mathrm{adic}(X'^\bullet))$ and $\mathrm{adic}(A'') \to \mathrm{Tot}(\mathrm{adic}(X''^\bullet))$. We deduce $\mathrm{Tot}(\widetilde{Y}^\bullet) \simeq \mathrm{Tot}(\mathrm{adic}(X^\bullet)) \times_{\mathrm{Tot}(\mathrm{adic}(X''^\bullet))} \mathrm{Tot}(\mathrm{adic}(X'^\bullet)) \simeq \mathrm{adic}(A) \times_{\mathrm{adic}(A'')} \mathrm{adic}(A'')$, which belongs to $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ by Remark 4.46.

We can now apply Proposition 4.49 to conclude that $\cot(A \times_{A''} A') \cong \cot(\mathrm{Tot}(Y^\bullet)) \xrightarrow{\simeq} \mathrm{Tot}(\cot(Y^\bullet))$ is an equivalence. Using that $\cot(Y) \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ is of finite type and therefore equivalent to its own bidual, we can therefore apply duality and deduce that the natural map $|\mathfrak{D}(Y^\bullet)| \xrightarrow{\simeq} \mathfrak{D}(A \times_{A'} A'')$ is an equivalence. This completes the verification of claim (b) above. $\square$

4.5. **Deformation theories.** We shall now explain how Theorem 4.20 translates into the language of deformation theories studied in [Lur16, Ch. 12] or [Lur11a].

As before, we fix a filtered augmented monadic adjunction (cf. Definition 4.15) throughout. Write $T^\vee$ for the monad on Mod_k constructed in Theorem 4.20 (2), i.e. the unique sifted-colimit-preserving extension of the monad $(M \mapsto (\cot \mathrm{sqz}(M^\vee))^\vee)$ acting on $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. Our main aim in this section is to prove Theorem 4.23, which asserts that $\mathcal{C}$-based formal moduli problems are equivalent to $T^\vee$-algebras.

We begin by constructing the required deformation functor. First, observe that composing the $(\cot \dashv \mathrm{sqz})$-adjunction with linear duality in fact gives rise to an adjunction

$$\cot^\vee \;:\; \mathcal{C} \;\underset{\longleftarrow}{\overset{\longrightarrow}{}}\; \mathrm{Mod}_k^{op} \;:\; \mathrm{sqz}\circ\tau_{\geq 0}\circ(-)^\vee. \tag{5}$$

Its left adjoint sends $A \in \mathcal{C}$ to $\cot(A)^\vee$, i.e. the linear dual of the cotangent fibre, whereas its right adjoint maps $V \in \mathrm{Mod}_k$ to the trivial square-zero extension on $\tau_{\geq 0}(V^\vee)$.

We can extend the functor $\mathfrak{D} : \mathcal{C}_{\mathrm{afp}} \to \mathrm{Alg}_{T^\vee}^{op}$ from Theorem 4.20 to all of $\mathcal{C}$ in such a way that postcomposing with the forgetful functor to Mod_k recovers $\cot^\vee$. Let $\mathcal{C}_{\mathrm{wafp}} \subset \mathcal{C}$ be the subcategory of all A with $\cot(A) \in \mathrm{Mod}_{k,\geq 0}^{\mathrm{ft}}$.

Construction 4.50 (The Koszul duality adjunction)**.** Since the action of $T^\vee$ on $\mathrm{Mod}_{k,\leq 0}^{\mathrm{ft}}$ agrees with the monad induced by adjunction (5), we have a natural functor

$$\mathfrak{D} : \mathcal{C}_{\mathrm{wafp}} \to \mathrm{Alg}_{T^\vee}^{op}$$

which forgets to $\cot^\vee$ in $\mathrm{Mod}_{k,\leq 0}$. Observe that this functor is left Kan extended from the compact objects of $\mathcal{C}$, since $\cot^\vee : \mathcal{C} \to \mathrm{Mod}_k^{op}$ has this property.

We can left Kan extend further to $\mathcal{C}$, to finally obtain the deformation functor

$$\mathcal{D} : \mathcal{C} \longrightarrow \mathrm{Alg}_{T^\vee}^{op}.$$

By [Lur09, Proposition 5.3.5.13], the functor $\mathcal{D}$ admits a right adjoint

$$C^* : \mathrm{Alg}_{T^\vee}^{op} \longrightarrow \mathcal{C}.$$

Remark 4.51. It should not be a surprise that we can extend $\mathfrak{D} : \mathcal{C}_{\mathrm{afp}} \to \mathrm{Alg}_{T^\vee}^{op}$ to all of $\mathcal{C}$. Indeed, writing $\widetilde{T^\vee}$ for the monad on Mod_k associated with the above adjunction (5), the monad $T^\vee$ is defined as the unique sifted-colimit-preserving extension of the restriction $\widetilde{T^\vee}|_{\mathrm{Mod}_{k,\leq 0}^{\mathrm{ft}}}$. Since this extension is obtained by left Kan extension, there is a natural transformation of monads $T^\vee \to \widetilde{T^\vee}$; we may think of $T^\vee$ as an "uncompletion" of $\widetilde{T^\vee}$.

The functor $\mathfrak{D} : \mathcal{C} \to \mathrm{Alg}_{T^\vee}^{op}$ is then simply obtained as the composite $\mathcal{C} \longrightarrow \mathrm{Alg}_{\widetilde{T^\vee}}^{op} \longrightarrow \mathrm{Alg}_{T^\vee}^{op}$. Here the first map comes from adjunction (5), whereas the second uses the map of monads $T^\vee \to \widetilde{T^\vee}$.

Construction 4.50 factors (5) through the free-forgetful adjunction forget : $\mathrm{Alg}_{T^\vee}^{op} \rightleftarrows \mathrm{Mod}_k^{op}$: free. Unwinding the above definitions, we can observe the following natural equivalences:

$$C^*(\mathrm{free}_{T^\vee}(V)) \simeq \mathrm{sqz}(\tau_{\geq 0}V^\vee), \quad V \in \mathrm{Mod}_k \tag{6}$$

$$\mathfrak{D}(\mathrm{free}(W)) \simeq W^\vee, \quad W \in \mathrm{Mod}_{k,\geq 0}, \tag{7}$$

$$\mathfrak{D}(\mathrm{sqz}(W)) \simeq \mathrm{free}_{T^\vee}(W^\vee), \quad W \in \mathrm{Mod}_{k,\geq 0}^{\mathrm{ft}}. \tag{8}$$

The statement (7) is to be interpreted as on the level of objects of Mod_k; informally the $T^\vee$-algebra-structure should be square-zero, but we do not attempt to make this precise.

Combining these basic facts with Theorem 4.20, we can conclude that the adjunction $(\mathfrak{D} \dashv C^*)$ restricts to a pair of inverse equivalences between $\mathcal{C}_{\mathrm{afp}}$ and $\mathrm{Alg}_{T^\vee}(\mathrm{Mod}_{k,\leq 0}^{\mathrm{ft}})^{op}$:

Proposition 4.52. *Let $\mathcal{C}, \mathcal{C}_{\mathrm{afp}}, \mathfrak{D}, C^*, \ldots$ be defined as above.*

(1) Given any $A \in \mathcal{C}_{\mathrm{afp}}$, the natural map $A \to C^*(\mathfrak{D}(A))$ is an equivalence.

(2) *Given any $T^\vee$-algebra $\mathfrak{g}$ such that the underlying k-module belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$, the natural map $\mathfrak{g} \to \mathfrak{D}(C^*(\mathfrak{g}))$ is an equivalence.*

Proof. If $A = \mathrm{sqz}(W) \in \mathcal{C}_{\mathrm{afp}}$ is a trivial square-zero extension on some $W \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$, the first claim follows from observations (6) and (8) above. Given a general $A \in \mathcal{C}_{\mathrm{afp}}$, the comonadicity claim in Theorem 4.20(1) shows that A can be written as a totalisation of a cosimplicial object $A^\bullet$ in $\mathcal{C}$ consisting of square-zero extensions, and that moreover $\mathfrak{D}$ preserves this totalisation (i.e. carries it to a geometric realisation of $T^\vee$-algebras). Of course, the right adjoint C^* also preserves all totalisations. Claim (1) therefore follows.

For claim (2), we use that by Theorem 4.20(1), any $T^\vee$-algebra $\mathfrak{g}$ with underlying k-module in $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ can be written as $\mathfrak{g} = \mathfrak{D}(A)$ for some $A \in \mathcal{C}_{\mathrm{afp}}$. The statement then follows from (1). ☐

Proposition 4.53. *Let $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}'' \in \mathrm{Alg}_{T^\vee}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})$ be $T^\vee$-algebras with underlying module in $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ and suppose that we are given maps $\mathfrak{g}'' \to \mathfrak{g}, \mathfrak{g}'' \to \mathfrak{g}'$ which induce injections on π_0.*

Then the pushout $\mathfrak{g} \sqcup_{\mathfrak{g}''} \mathfrak{g}'$ (computed in $T^\vee$-algebras) has underlying k-module in $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ as well.

Proof. Define $A, A', A'' \in \mathcal{C}_{\mathrm{afp}}$ as $C^*(\mathfrak{g}), C^*(\mathfrak{g}')$, and $C^*(\mathfrak{g}'')$. We now observe that the induced maps $A \to A'', A' \to A''$ induce surjections on π_0. Indeed, this follows immediately from from the adic filtration. For example, A has a complete filtration with associated graded given by $\mathrm{free}([\mathfrak{g}^\vee]_1)$, where $[\mathfrak{g}^\vee]_1$ is the graded k-module with $\mathfrak{g}^\vee$ in internal degree one (cf. Proposition 4.26). The maps $\mathrm{free}([\mathfrak{g}^\vee]_1) \to \mathrm{free}([\mathfrak{g}''^\vee]_1)$ and $\mathrm{free}([\mathfrak{g}'^\vee]_1) \to \mathrm{free}([\mathfrak{g}''^\vee]_1)$ then induce surjections on π_0, and passing to the filtered objects shows that $A \to A'', A' \to A''$ have the same property.

Theorem 4.20(3) now shows that $\mathfrak{D}(A \times_{A''} A') \simeq \mathfrak{g} \sqcup_{\mathfrak{g}''} \mathfrak{g}$ has the asserted properties. ☐

We are now ready to show that one can obtain a deformation theory in the sense of Lurie. Recall the following definition from [Lur11a, Definition 1.3.1, 1.3.9] (see also [CG18, Section 2] for a treatment). Roughly, it expresses the idea of a bar-cobar duality, together with suitable subcategories on which one obtains an inverse equivalence, albeit translated into more abstract language.

Definition 4.54 (Lurie)**.** A *deformation theory* consists of a presentable ∞-category $\mathcal{A}$, a set of objects $\{E_\alpha\}_{\alpha \in T}$ in the stabilisation of $\mathcal{A}$, and an adjunction $\mathfrak{D} : \mathcal{A} \rightleftarrows \mathcal{B}^{op} : C^*$ with $\mathcal{B}$ presentable. Moreover, we require that there exists a full subcategory $\mathcal{B}_0 \subset \mathcal{B}$ satisfying the following conditions:

(1) For $B \in \mathcal{B}_0$, the natural map $B \to \mathfrak{D}C^*(B)$ in $\mathcal{B}$ is an equivalence.
(2) The subcategory $\mathcal{B}_0 \subset \mathcal{B}$ contains the initial object $\emptyset$ of $\mathcal{B}$. Moreover, for any $\alpha \in T$ and $n \geq 1$, there is an object $K_{\alpha,n} \in \mathcal{B}_0$ such that $\Omega^{\infty-n} E_\alpha = C^*(K_{\alpha,n})$.
(3) Given an object $K \in \mathcal{B}_0$ and maps $K_{\alpha,n} \to K$ and $K_{\alpha,n} \to \emptyset$, the pushout $K \sqcup_{K_{\alpha,n}} \emptyset$ (computed in $\mathcal{B}$) is contained in $\mathcal{B}_0$.
(4) For each $\alpha \in T$ and any $n \geq 2$, assumptions (a) through (c) above imply equivalences $\Sigma K_{\alpha,n} \simeq K_{\alpha,n-1}$. Using these equivalences, we can define a functor $f_\alpha : \mathcal{B} \to \mathrm{Sp}$ with
$$\Omega^{\infty-n} f_\alpha(X) = \mathrm{Hom}_{\mathcal{B}}(K_{\alpha,n}, X).$$
We then assume that each functor f_α commutes with sifted colimits.

We now wish to define a deformation theory in the above sense from the filtered augmented monadic adjunction (cf. Definition 4.15) fixed throughout this section.

For this, we consider the adjunction $\mathfrak{D} : \mathcal{C} \rightleftarrows \mathrm{Alg}^{op}_{T^\vee} : C^*$ constructed in the beginning of this subsection. Observe that since sqz is a right adjoint functor $\mathrm{Mod}_{k,\geq 0} \to \mathcal{C}$, it naturally lifts to the stabilisation of $\mathcal{C}$: given any $V \in \mathrm{Mod}_{k,\geq 0}$, the object $\mathrm{sqz}(V)$ defines an object of $\mathrm{Stab}(\mathcal{C})$ corresponding to the sequence $\{\mathrm{sqz}(V[n])\}_{n\geq 0}$.

Proposition 4.55. *The ∞-category $\mathcal{C}$, the infinite loop object $\{\mathrm{sqz}(k[n])\}_{n\geq 0}$ in $\mathrm{Stab}(\mathcal{C})$, and the functor $\mathfrak{D}^{op} : \mathcal{C}^{op} \to \mathrm{Alg}_T$ together define a deformation theory in the sense of Definition 4.54.*

Proof. We define $\mathcal{B}_0 \subset \mathcal{B} = \mathrm{Alg}_{T^\vee}$ as the subcategory of those objects whose underlying k-module belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. We let K_n be the free $T^\vee$-algebra on $k[-n]$; recall that $C^*(K_n) \simeq \mathrm{sqz}(k[n])$ and $\mathfrak{D}(\mathrm{sqz}(k[n])) \simeq K_n$ by construction of these adjunctions. Assumptions (a) through (c) now follow from Proposition 4.52 and Proposition 4.53. Assumption (d) follows because $T^\vee$ commutes with sifted colimits by construction, and so the forgetful functor from $T^\vee$-algebras to Mod_k (which is the functor f_α described above) also commutes with sifted colimits. $\square$

We can now deduce the classification of formal moduli problems through $T^\vee$-algebras which was asserted in the beginning of this section:

Proof of Theorem 4.23. Indeed, this follows from [Lur16, Theorem 12.3.3.5], since we know that we have a deformation theory by Proposition 4.55. $\square$

5. Deformations over a field

In this section we consider some concrete examples of the general argument in the previous section. In particular, we verify that one can apply the argument for connective $\mathbb{E}_\infty$-algebras or simplicial commutative rings augmented over a field.

5.1. $\mathbb{E}_\infty$-algebras.

We begin by examining deformations parametrised by $\mathbb{E}_\infty$-algebras.

Preliminaries on $\mathbb{E}_\infty$-algebras. To set the stage, we will briefly review some of the basic facts about $\mathbb{E}_\infty$-algebras; a comprehensive treatment of this theory can be found in [Lur17].

Let $(\mathcal{A}, \otimes, \mathbf{1})$ be a presentably symmetric monoidal stable ∞-category.

(1) We can associate the ∞-category $\mathrm{CAlg}(\mathcal{A})$ of *$\mathbb{E}_\infty$-algebras in $\mathcal{A}$*, and there is a natural free-forgetful adjunction

$$\mathrm{free}_{\mathbb{E}_\infty} : \mathcal{A} \rightleftarrows \mathrm{CAlg}(\mathcal{A}) : \mathrm{forget}$$

The free $\mathbb{E}_\infty$-algebra on an object $X \in \mathcal{A}$ is given by $\mathrm{free}_{\mathbb{E}_\infty}(X) \simeq \bigoplus_{n \geq 0} (X^{\otimes n})_{h\Sigma_n}$.

(2) We write $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{A}) = \mathrm{CAlg}(\mathcal{A})_{\mathbf{1}//\mathbf{1}}$ for the ∞-category of *augmented $\mathbb{E}_\infty$-algebras in $\mathcal{A}$*; its objects are $\mathbb{E}_\infty$-algebras A in $\mathcal{A}$ equipped with an augmentation map $A \to \mathbf{1}$ to the unit.

(3) Let $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A})$ denote the ∞-category of *nonunital* $\mathbb{E}_\infty$-algebras in $\mathcal{A}$. Since $\mathcal{A}$ is assumed to be stable, we have an equivalence $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{A}) \simeq \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A})$ which sends an augmented $\mathbb{E}_\infty$-algebra A to the fibre $\mathfrak{m}_A$ of the augmentation map $A \to \mathbf{1}$. As expected, there is a free-forgetful adjunction

$$\mathrm{free}_{\mathbb{E}_\infty^{\mathrm{nu}}} : \mathcal{A} \rightleftarrows \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A}) : \mathrm{forget},$$

and the free nonunital algebra on an object $X \in \mathcal{A}$ is given by $\mathrm{free}_{\mathbb{E}_\infty^{\mathrm{nu}}}(X) \simeq \bigoplus_{n > 0} (X^{\otimes n})_{h\Sigma_n}$.

(4) Since the monad $\mathrm{forget}_{\mathbb{E}_\infty^{\mathrm{nu}}} \circ \mathrm{free}_{\mathbb{E}_\infty^{\mathrm{nu}}}$ is naturally augmented over the identity monad, and this gives rise to an adjunction

$$\cot : \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A}) \rightleftarrows \mathcal{A} : \mathrm{sqz},$$

where the functor sqz sends an object of $\mathcal{A}$ to the associated nonunital $\mathbb{E}_\infty$-algebra with square-zero multiplication and the left adjoint cot is called the *cotangent fibre*.

Under the identification $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{A}) \simeq \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A})$, we have an equivalence

$$\cot(A) \simeq \Omega L_{\mathbf{1}/A},$$

where $L_{-/A}$ denotes the cotangent complex of an $\mathbb{E}_\infty$-A-algebra in $\mathcal{A}$ (cf. [Lur17, Section 7.3–7.4] or in the original setting [Bas99a]). Alternatively, we have $\cot(A) \simeq \mathbf{1} \otimes_A L_{A/\mathbf{1}}$.

Remark 5.1. The definition of nonunital $\mathbb{E}_\infty$-rings and the construction of the cotangent fibre adjunction (4) do not require the unit in $\mathcal{A}$; they therefore both make sense for nonunital symmetric monoidal ∞-categories. This is relevant for us as we will later study the ∞-category FilMod_k of k-modules filtered by positive integers (cf. Definition 2.1).

Finally, we will review the adic filtration and how it allows us to approximate every augmented $\mathbb{E}_\infty$-algebra by extended powers of its cotangent fibre. This is also discussed in [GL, Section 4.2], and in fact a special case of the homotopy completion tower studied in [HH13].

Construction 5.2 (The functor adic and the completion tower)**.** As explained in Definition 2.5, the functor $F^1 : \mathrm{Fil}(\mathcal{A}) \to \mathcal{A}$ is (nonunital) symmetric monoidal. It therefore lifts to a functor on algebras $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Fil}(\mathcal{A})) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A})$, and this functor preserves limits (as these are computed on underlying ∞-categories.

Taking the left adjoint now gives rise to a functor

$$\mathrm{adic} : \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A}) \to \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Fil}(\mathcal{A})).$$

This construction refines any nonunital commutative algebra object in $\mathcal{A}$ to a filtered one, and we therefore get a natural tower. More explicitly, we see that adic carries the free nonunital $\mathbb{E}_\infty$-algebra $\bigoplus_{i>0}(V^{\otimes i})_{h\Sigma_i}$ on $V \in \mathcal{A}$ to the free filtered nonunital $\mathbb{E}_\infty$-algebra on the filtered object $(\cdots \to 0 \to 0 \to V)$. Unwinding the definitions of the tensor product in $\mathrm{Fil}(\mathcal{A})$, we see that for each n, there is an equivalence

$$F^n\mathrm{adic}(\mathrm{free}(V)) \simeq \bigoplus_{i\geq n}(V^{\otimes i})_{h\Sigma_i}. \tag{9}$$

Example 5.3. Let $\mathcal{A} = \mathrm{Mod}_k$. Let $I \in \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_k)$ be a connective nonunital $\mathbb{E}_\infty$-algebra, so that $\pi_0(I)$ is an ordinary nonunital k-algebra. Then the image of $\pi_0(F^n\mathrm{adic}(I)) \to \pi_0(I)$ is given by the nth power ideal of $\pi_0(I)$. This is evident in the free case, and the general case follows by taking sifted colimits.

We return to the general case where $\mathcal{A}$ is any presentably symmetric monoidal stable ∞-category.

Remark 5.4 (The cotangent fibre in degree 1)**.** For any $A \in \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Gr}(\mathcal{A}))$, the natural map $A \to \cot(A)$ in $\mathrm{Gr}(\mathcal{A})$ induces an equivalence in (internal) degree 1. Indeed, this follows by considering the free case and then observing that everything commutes with sifted colimits. We obtain a natural map

$$\mathrm{free}^{\mathrm{nu}}([\cot(A)_1]_1) \to A$$

in $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Gr}(\mathcal{A}))$, where $[\cot(A)_1]_1$ denotes the graded object given by placing $\cot(A)_1$ in degree 1.

We will now verify some basic properties of of the construction adic.

Proposition 5.5. *For any nonunital $\mathbb{E}_\infty$-algebra $A \in \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A})$, we have:*

(1) The natural unit map $A \to F^1\mathrm{adic}(A)$ is an equivalence.
(2) There is a natural equivalence $\cot(\mathrm{adic}(A)) \simeq (\cot(A))_1 = (\cdots \to 0 \to \cot(A))$ in $\mathrm{Fil}(\mathcal{A})$.
(3) There is a functorial identification $\mathrm{free}^{\mathrm{nu}}([\cot(A)]_1) \simeq \mathrm{Gr} \circ \mathrm{adic}(A)$ in $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Gr}(\mathcal{A}))$, where $[\cot(A)]_1$ denotes the graded object obtained by placing $\cot(A)$ in degree one.

Proof. If A is free, then (1) follows by our explicit computation in (9). From this, we can deduce the general case by observing that both F^1 and adic preserve geometric realisations.

For (2), we observe that the following square of right adjoints evidently commutes:

$$\begin{tikzcd}
\mathrm{Fil}(\mathcal{A}) \arrow[r, "F^1"] \arrow[d, "\mathrm{sqz}"'] & \mathcal{A} \arrow[d, "\mathrm{sqz}"'] \\
\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Fil}(\mathcal{A})) \arrow[r, "F^1"] & \mathrm{CAlg}^{\mathrm{nu}}(\mathcal{A})
\end{tikzcd}$$

Finally, statement (3) follows (just like Proposition 4.26 above) by directly checking the free case and then taking geometric realisations. ∎

The setup for $\mathbb{E}_\infty$-algebras. We shall now define a filtered augmented monadic adjunction (cf. Definition 4.15) for $\mathbb{E}_\infty$-algebras over a given field k. By our previous work, this will allow us to deduce a version of Theorem 4.20 and thereby give a Lie algebraic description of deformation theory in this context.

We write CAlg_k for the ∞-category of $\mathbb{E}_\infty$-k-algebras and $\mathrm{CAlg}_k^{\mathrm{nu}}$ for its nonunital version. We let $\mathcal{C} = \mathrm{CAlg}^{\mathrm{nu}}_{k,\geq 0}$ denote the full subcategory of connective objects in $\mathrm{CAlg}_k^{\mathrm{nu}} \simeq \mathrm{CAlg}_k^{\mathrm{aug}}$.

Remark 5.6. For simplicity, we will generally state our results in terms of *augmented* (rather than nonunital) algebras in this section.

Construction 5.7 (The setup for $\mathbb{E}_\infty$-algebras)**.** Let k be a field.

a) Let $\mathcal{C} = \mathrm{CAlg}^{\mathrm{aug}}_{k,\geq 0} \simeq \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Mod}_{k,\geq 0})$ be the ∞-category of augmented (or equivalently nonunital) connective $\mathbb{E}_\infty$-algebras over k.
b) Let $\mathcal{C}^{\mathrm{Fil}} = \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{FilMod}_{k,\geq 0})$ be the ∞-category of nonunital $\mathbb{E}_\infty$-algebras in the nonunital symmetric monoidal ∞-category $\mathrm{FilMod}_{k,\geq 0}$. Note that $\mathcal{C}^{\mathrm{Fil}}$ is equivalent to the full subcategory of $\mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Fil}^{+}\mathrm{Mod}_{k,\geq 0})$ spanned by all augmented $\mathbb{E}_\infty$-algebra objects A with $F^0A/F^1A \simeq k$.
c) Let $\mathcal{C}^{\mathrm{Gr}} = \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{GrMod}_{k,\geq 0})$ denote the ∞-category of nonunital $\mathbb{E}_\infty$-algebras in the nonunital symmetric monoidal ∞-category $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{GrMod}_{k,\geq 0})$. Equivalently, $\mathcal{C}^{\mathrm{Gr}}$ is the full subcategory of $\mathrm{CAlg}^{\mathrm{aug}}(\mathrm{GrMod}_{k,\geq 0})$ spanned by those objects $A_\star$ such that $A_0 \simeq k$.
d) We obtain the free-forgetful adjunction $\mathrm{free} : \mathrm{Mod}_{k,\geq 0} \rightleftarrows \mathcal{C} : \mathrm{forget}$ and the cotangent fibre adjunction $\mathrm{cot} : \mathcal{C} \rightleftarrows \mathrm{Mod}_{k,\geq 0} : \mathrm{sqz}$ from (10) and (4) above by restricting to connective objects. Note that if we describe $\mathcal{C}$ as augmented $\mathbb{E}_\infty$-algebras, then the forgetful functor forget sends an augmented $\mathbb{E}_\infty$-algebra to its augmentation ideal.
   Taking $\mathcal{A} = \mathrm{FilMod}_k$ (or $\mathcal{A} = \mathrm{GrMod}_k$) instead, we obtain a similar pair of adjunctions $(\mathrm{cot} \dashv \mathrm{sqz}), (\mathrm{free} \dashv \mathrm{forget})$ between $\mathcal{C}^{\mathrm{Fil}}$ and $\mathrm{FilMod}_{k,\geq 0}$ (or $\mathcal{C}^{\mathrm{Gr}}$ and $\mathrm{GrMod}_{k,\geq 0}$).
e) The functor $F^1 : \mathcal{C}^{\mathrm{Fil}} \to \mathcal{C}$ forgets the filtration on an object; its left adjoint is the functor adic (cf. Construction 5.2).
f) The (nonunital) symmetric monoidal functor $\mathrm{Gr} : \mathrm{FilMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$ induces a functor $\mathrm{Gr} : \mathcal{C}^{\mathrm{Fil}} \to \mathcal{C}^{\mathrm{Gr}}$ on the level of algebras.

Proposition 5.8. *The setup of connective $\mathbb{E}_\infty$-algebras in Construction 5.7 satisfies conditions* $(1)-(3)$ *of Definition 4.15.*

Proof. Conditions (1) and (2) of Definition 4.15 both follow from straightforward formal arguments. For (3), we first observe that the free nonunital $\mathbb{E}_\infty$-algebra functor $\mathrm{GrMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$ is given by $X \mapsto \bigoplus_{i>0} (X^{\otimes i})_{h\Sigma_i}$ and therefore admissible by Example 3.49.

To construct the filtration required in condition (3), we use the adic filtration. Given $X \in \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{GrMod}_{k,\geq 0})$, we form $\mathrm{adic}(X) \in \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Fil}(\mathrm{GrMod}_k))$ and set $X^{(i)} = F^i\mathrm{adic}(X) \in \mathrm{GrMod}_k$. It follows that we have the tower $\{X^{(i)}\}_{i\geq 1}$, naturally in X, and natural isomorphisms

$$X^{(i)}/X^{(i+1)} \simeq (\mathrm{cot}(X)^{\otimes i})_{h\Sigma_i}$$

in GrMod_k. By taking simplicial resolutions and thereby reducing to the free case, it follows that $X^{(i)}$ is concentrated in internal degrees $\geq i$ for any X. We deduce that the tower $\{X^{(i)}\}_{i\geq 1}$ converges in GrMod_k. Setting $A^{(n)} = X/X^{(n+1)}$ gives the required filtration in condition (3). ☐

Finiteness conditions. In order to verify the remaining axioms (4) and (5) of Definition 4.15, we will need to exploit the Noetherian and finiteness properties of $\mathbb{E}_\infty$-ring spectra; we refer to [Lur17, Chapter 7] or [Lur16, Chapter II.4] for a detailed study of these notions.

Definition 5.9 (Cf. [Lur17, Definition 7.2.4.30])**.** An $\mathbb{E}_\infty$-ring spectrum R is *Noetherian* if

(1) R is connective.
(2) $\pi_0(R)$ is Noetherian.
(3) For each $i \geq 0$, $\pi_i(R)$ is a finitely generated $\pi_0(R)$-module.

We will also need the following notion:

Definition 5.10. An $\mathbb{E}_\infty$-k-algebra R is said to be *almost finitely presented* if R is Noetherian and $\pi_0(R)$ is a finitely generated k-algebra.

The almost finitely presented $\mathbb{E}_\infty$-algebras over k are precisely those connective $\mathbb{E}_\infty$-k-algebras R for which the functor $\mathrm{Map}_{\mathrm{CAlg}_k}(R, -)$ commutes with filtered colimits of connective, n-truncated $\mathbb{E}_\infty$-algebras. In fact, this is used as the definition when one works over a non-Noetherian base (cf. [Lur17, Definition 7.2.4.26] and [Lur17, Proposition 7.2.4.31]).

One can define analogous finiteness properties on the level of modules. If $R \in \mathrm{CAlg}_k$ is a connective $\mathbb{E}_\infty$-algebra over k, then there is a notion of an *almost perfect* R-module (cf. [Lur17, Definition 7.2.4.10], [Lur16, Section 2.7]). More generally, for each $n \in \mathbb{Z}$, one has the notion of an R-module which is *perfect to order* n; an R-module M is almost perfect if and only if it is perfect to order n for each n. When R is Noetherian, then this notion simplifies by [Lur17, Proposition 7.2.4.17]. Indeed, an R-module M is then almost perfect if and only if M is bounded below and each homotopy group $\pi_i(M)$ is a finitely generated $\pi_0(R)$-module.

The theory of Noetherian $\mathbb{E}_\infty$-rings is well-behaved and robust; for instance, one has a version of Hilbert's basis theorem (cf. [Lur17, Proposition 7.2.4.31]) which, when combined with [Lur17, Proposition 7.4.3.18], implies:

Proposition 5.11. *The cotangent fibre of any augmented Noetherian k-algebra R lies in* $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$.

One can detect almost finite presentation of k-algebras using the cotangent complex by the following special case of [Lur17, Theorem 7.4.3.18]:

Theorem 5.12. *Let R be a connective $\mathbb{E}_\infty$-algebra over k. Suppose $\pi_0(R)$ is a finitely generated k-algebra. Then the following are equivalent:*

(1) R is almost finitely presented.
(2) The cotangent complex $L_{R/k}$ is almost perfect as an R-module.

We shall now discuss graded versions of the above definitions.

Definition 5.13. Let k be a field.

(1) A graded $\mathbb{E}_\infty$-k-algebra $R_\star \in \mathrm{CAlg}(\mathrm{Gr}^+(\mathrm{Mod}_k))$ is *almost finitely presented* if the underlying $\mathbb{E}_\infty$-k-algebra $\bigoplus_{i\geq 0} R_i$ is almost finitely presented (cf. Definition 5.10).
(2) If $R_\star \in \mathrm{CAlg}(\mathrm{Gr}^+(\mathrm{Mod}_{k,\geq 0}))$ is a connective graded $\mathbb{E}_\infty$-ring and $M_\star$ is a graded $R_\star$-module, then we will say that $M_\star$ is *almost perfect* (respectively *perfect to order n*) if the underlying ungraded $\bigoplus_{i\geq 0} R_i$-module $\bigoplus_{i\geq 0} M_i$ is almost perfect (respectively perfect to order n).

If $R_0 = k$, then we have the following straightforward criterion for almost perfectness:

Proposition 5.14. *Let $R_\star$ be a degreewise connective graded $\mathbb{E}_\infty$-k-algebra with $R_0 = k$. Let $M_\star$ be an $R_\star$-module which is bounded below in each degree. Then the following are equivalent:*

(1) $M_\star$ is almost perfect.

(2) $\pi_i(M_\star \otimes_{R_\star} k)$ is a finite-dimensional k-vector space for all i. We use the augmentation $R_\star \to k$.

Proof. Condition (1) implies (2) since almost perfect modules are preserved under base-change.

For the converse direction, we may assume without restriction that $M_\star$ is connective in each degree. We will show by induction that if (2) holds, then $M_\star$ is perfect to order n for each $n \geq 0$.

The graded k-module $M_\star$ is perfect to order 0 precisely if $\pi_0(M_\star)$ is finitely generated. This follows from the graded variant of Nakayama's lemma since $\pi_0(M_\star \otimes_{R_\star} k) = \pi_0(M_\star) \otimes_{\pi_0(R_\star)} k$ is finite-dimensional.

Now suppose we know that $M_\star$ is perfect to order $n-1$ for some $n \geq 1$. Choose a degreewise connective, finitely generated free graded $R_\star$-module $P_\star$, together with a map $P_\star \to M_\star$ inducing a surjection of graded k-vector spaces on π_0. Let $F_\star$ be the homotopy fibre of $P_\star \to M_\star$. By [Lur16, Proposition 2.7.2.1], $M_\star$ is perfect to order n if and only if $F_\star$ is perfect to order $n-1$. Since $F_\star \otimes_{R_\star} k$ is almost perfect over k, the inductive hypothesis shows that $F_\star$ is perfect to order $n-1$, which in turn implies that $M_\star$ is perfect to order n. □

The finiteness notion for graded $\mathbb{E}_\infty$-rings introduced in Definition 5.13 recovers the "axiomatic" notion of graded finiteness given Definition 4.31:

Proposition 5.15. *Let $R_\star \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Gr}^+(\mathrm{Mod}_{k,\geq 0}))$ be a degreewise connective, augmented graded $\mathbb{E}_\infty$-k-algebra with $R_0 = k$. Then the following are equivalent:*

(1) $R_\star$ is almost finitely presented.

(2) $\cot(R_\star) \in \mathrm{Gr}(\mathrm{Mod}_{k,\geq 0})$ belongs to $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$.

Proof. If $R_\star$ is almost finitely presented, then it follows from Theorem 5.12 that the cotangent complex $L_{R_\star/k}$ is almost perfect as an $R_\star$-module. The cofibre sequence for a triple of rings then implies that the cotangent complex $L_{k/R_\star}$ belongs to $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$.

Conversely, suppose that $\cot(R_\star)$ belongs to $\mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$. Recall that $\pi_0(\cot(R_\star))$ is the module of indecomposables of the augmented graded algebra $\pi_0(R_\star)$. Since $\pi_0(\cot(R_\star))$ is a finite-dimensional vector space, it follows immediately that $\pi_0(R_\star)$ is a finitely generated graded algebra; here we use that all elements in the augmentation ideal sit in positive degrees. Next, we observe that the cotangent complex $L_{R_\star/k} \in \mathrm{Mod}_{R_\star}$ has the property that $L_{R_\star/k} \otimes_{R_\star} k \simeq \cot(R_\star)$ is almost perfect as a graded k-module, which in turn implies that $L_{R_\star/k}$ is almost perfect by Proposition 5.14. Thus, we see that $L_{R_\star/k}$ is almost perfect as an $R_\star$-module. By Theorem 5.12, it follows that $R_\star$ is almost finitely presented. □

Next, we record analogous finiteness notions in the filtered case.

Definition 5.16. Let $R \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Fil}^+(\mathrm{Mod}_{k,\geq 0}))$ be a degreewise connective, filtered augmented $\mathbb{E}_\infty$-algebra over k such that $F^0R/F^1R \simeq k$. We say that R is *complete almost finitely presented* if:

(1) R is complete as a filtered object.

(2) The associated graded algebra $\mathrm{Gr}(R) \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Gr}^+(\mathrm{Mod}_{k,\geq 0}))$ is almost finitely presented in the sense of Definition 5.13.

Remark 5.17. Again by Proposition 5.15, we can see that $R \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Fil}^+(\mathrm{Mod}_{k,\geq 0}))$ is complete almost finitely presented precisely if it belongs to to the ∞-category $\mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ defined in Definition 4.39 of the axiomatic section.

We shall now discuss the completed finiteness condition in the absence of a grading or filtration. This will later allow us to give a concrete description of the subcategory $\mathcal{C}_{\mathrm{afp}} \subset \mathcal{C}$ from Definition 4.43 in our context. For future reference, we will start with a slightly more general notion:

Definition 5.18 (Complete local Noetherian $\mathbb{E}_\infty$-rings)**.** An $\mathbb{E}_\infty$-ring spectrum R is said to be *complete local Noetherian* if R is Noetherian (cf. Definition 5.9) and $\pi_0(R)$ is a complete local ring. We will refer to the residue field of $\pi_0(R)$ as the residue field of R itself. We write $\mathrm{CAlg}^{\mathrm{cN}} \subset \mathrm{CAlg}$ for the subcategory spanned by complete local Noetherian $\mathbb{E}_\infty$-rings; we will use the same notation for simplicial commutative rings.

Remark 5.19 (Topological finite generation)**.** Let $R \in \mathrm{CAlg}_k^{\mathrm{aug}}$ be complete local Noetherian. By assumption, the local ring $\pi_0(R)$ has residue field k. If $x_1, \dots, x_n \in \pi_0(R)$ are generators for the maximal ideal, then we can use the Cohen structure theorem to write $\pi_0(R)$ as a quotient of the formal power series ring $k[[t_1, \dots, t_n]]$ via the map $t_i \mapsto x_i$. Moreover, if $R \to R'$ is a map in $\mathrm{CAlg}_k^{\mathrm{aug}}$ between complete local Noetherian $\mathbb{E}_\infty$-rings, then $\pi_0(R) \to \pi_0(R')$ is a local homomorphism.

Example 5.20 (Completions of almost finitely presented algebras)**.** Let $R \in \mathrm{CAlg}_k$ be a connective $\mathbb{E}_\infty$-k-algebra which is almost finitely presented in the sense of Definition 5.10. Let $\mathfrak{m}$ be a maximal ideal of $\pi_0(R)$ whose residue field is k. Then the completion $\hat{R}_\mathfrak{m}$ of R along $\mathfrak{m}$, together with its canonical augmentation to k, is a complete local Noetherian with residue field k. In fact, $\pi_0(\hat{R}_\mathfrak{m})$ is the (algebraic) completion of $\pi_0(R)$ along $\mathfrak{m}$, and $\pi_i(\hat{R}_\mathfrak{m}) \simeq \pi_i(R) \otimes_{\pi_0(R)} \pi_0(\hat{R}_\mathfrak{m})$. We refer to [Lur16, Section 7.3] for a general reference on completions in the context of $\mathbb{E}_\infty$-ring spectra.

We will now prove that the cotangent fibre functor is conservative on complete local Noetherian $\mathbb{E}_\infty$-algebras augmented over k. We need the following straightforward observation:

Lemma 5.21. *Let R be a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. Let M be an R-module which is almost perfect. If $M \otimes_R k = 0$, then $M = 0$.*

Proof. Suppose that M is nonzero. We then look at the smallest integer n for which $\pi_n(M) \neq 0$. Since $\pi_n(M \otimes_R k) = \pi_n(M) \otimes_{\pi_0(R)} k = 0$ and $\pi_n(M)$ is finitely generated over $\pi_0(R)$, Nakayama's lemma gives a contradiction. $\square$

Proposition 5.22. *Let $f : R \to R'$ be a map between complete local Noetherian $\mathbb{E}_\infty$-k-algebras. If f induces an equivalence after applying* $\cot(-)$*, then f is itself an equivalence.*

Proof. First, we recall that $\pi_0(\cot(R))$ is given by the cotangent space of the augmented k-algebra $\pi_0(R)$, i.e. the quotient of indecomposables of the maximal ideal. A similar statement holds for R'. Since both $\pi_0(R)$ and $\pi_0(R')$ are complete local rings, it follows that $\pi_0(R) \to \pi_0(R')$ is surjective. This in turn implies that the map $R \to R'$ makes R' into an R-module of almost finite presentation.

The triple of maps $R \to R' \to k$ gives rise to a cofibre sequence $k \otimes_{R'} L_{R'/R} \to L_{k/R} \to L_{k/R'}$. By assumption, the second map is an equivalence, which shows that $k \otimes_{R'} L_{R'/R}$ is contractible. Since $L_{R'/R}$ is almost perfect by [Lur17, Theorem 7.4.3.18], we can deduce that $L_{R'/R} \simeq 0$ by Lemma 5.21. By [Lur16, Lemma 4.6.2.4], this in turn proves that the surjection $\pi_0(R) \to \pi_0(R')$ is étale, and hence an isomorphism (cf. e.g. [Sta19, Tag 0257]). The claim now follows from [Lur17, Corollary 7.4.3.4]. $\square$

We now wish to show that $\mathcal{C}_{\mathrm{afp}}$ from Definition 4.43 in the axiomatic setting precisely consists of all complete local Noetherian $\mathbb{E}_\infty$-algebras, and that the remaining axioms of Definition 4.15 are satisfied. To this end, we first show that complete almost finitely presented *filtered* algebras are well-behaved.

Proposition 5.23. *Let R be an augmented filtered $\mathbb{E}_\infty$-algebra over k which is complete almost finitely presented in the sense of Definition 5.16. Then the underlying augmented $\mathbb{E}_\infty$-k-algebra F^0R is a complete local Noetherian $\mathbb{E}_\infty$-ring (cf. Definition 5.18).*

Proof. Choose a system of generators $\overline{x}_1, \ldots, \overline{x}_n$ of $\pi_0(\mathrm{Gr}(R))$ in positive internal degrees. Lifting them to $\pi_0(F^0R)$, we get a system of elements $x_1, \ldots, x_n \in \pi_0(F^0R)$ living in positive filtration.

By the Milnor exact sequence, we know that $\varprojlim_n \pi_0(F^nR)$ vanishes, which in turn implies that the commutative ring $\pi_0(R)$ is $(x_1, \ldots, x_n)$-adically complete. We therefore obtain a map $k[[t_1, \ldots, t_n]] \to \pi_0(F^0R)$, which is readily seen to be surjective by passing to associated gradeds. It follows that $\pi_0(F^0R)$ is a quotient of a formal power series ring in finitely many variables over k. It is therefore complete, local, and Noetherian.

It remains to check that the homotopy groups of F^0R are finitely generated $\pi_0(F^0R)$-modules. To this end, we observe that since the filtration on R is complete, [Lur16, Corollary 7.3.3.3] shows that F^0R is in fact an $(x_1, \ldots, x_n)$-adically complete $\mathbb{E}_\infty$-ring (cf. [Lur16, Definition 7.3.1.1]), which by [Lur16, Theorem 7.3.4.1] implies that $\pi_i(F^0R)$ is a derived complete $\pi_0(F^0R)$-module for the ideal $(x_1, \ldots, x_n)$ (cf. [Lur16, Theorem 7.3.0.5]). By [Sta19, Tag 091N], it therefore suffices to prove that $\pi_i(F^0R)/(x_1, \ldots, x_n)$ is a finitely generated $\pi_0(F^0R)/(x_1, \ldots, x_n)$-module for all i.

For this, we first upgrade the F^0R-module $F^0R/(x_1, \ldots, x_n)$ to a filtered R-module $R/(x_1, \ldots, x_n)$ by taking the iterated cofibre of multiplication by each x_i in filtered R-modules, thereby placing x_i in the appropriate filtration degree. Using that $\mathrm{Gr}(R)$ is almost finitely presented, a standard induction argument shows that the $\mathrm{Gr}(R)$-module $\mathrm{Gr}(R/(x_1, \ldots, x_n)) \simeq \mathrm{Gr}(R)/(\overline{x}_1, \ldots, \overline{x}_n)$ has homotopy groups which are finite-dimensional k-vector space in each degree i. Since $R/(x_1, \ldots, x_n)$ is complete, we deduce that $F^0R/(x_1, \ldots, x_n)$ belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. □

We proceed to verify the completeness axiom (5) in Definition 4.15, indicating that the cotangent complex of a complete almost finitely presented $\mathbb{E}_\infty$-ring is well-behaved. We first examine the filtered setting, and start with the easy 0-connected case where π_0 is simply k.

Proposition 5.24. *Let $R \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Fil}^+(\mathrm{Mod}_{k,\geq 0}))$ be complete almost of finite presentation. Suppose that $\pi_0(\mathrm{Gr}(R)) = k$. Then $\cot(R) \in \mathrm{Fil}(\mathrm{Mod}_{k,\geq 0})$ belongs to $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$.*

Proof. For all i, our assumptions imply that $\pi_i(\mathrm{Gr}(R))$ is a finite-dimensional $k = \pi_0(\mathrm{Gr}(R))$-module. Therefore the augmentation ideal of R, which belongs to $\mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Fil}(\mathrm{Mod}_{k,\geq 0}))$, has underlying object in $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$. We observe that if $V \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$, then the free filtered algebra $\bigoplus_{i\geq 1}(V^{\otimes i})_{h\Sigma_i}$ on V also belongs to $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$; this follows since the i^{th} summand is i-connective. By Remark 4.24, we conclude that $\cot(R)$ is a geometric realisation of objects in $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$, which implies the claim by Proposition 2.11. □

To extend the above result, we will need the following notion:

Definition 5.25. Let $R \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Fil}^+(\mathrm{Mod}_{k,\geq 0}))$ be complete almost finitely presented. Let M be an R-module in $\mathrm{Fil}^+(\mathrm{Mod}_k)$ for which the following conditions hold true:

(1) $\mathrm{Gr}(M)$ is almost perfect as a $\mathrm{Gr}(R)$-module (cf. Definition 5.13).
(2) M is complete.

Then we say that M is *almost perfect.*

The ∞-category of almost perfect R-modules behaves analogously to the ∞-category of almost perfect modules over a connective ring spectrum. Rather than verifying all details (which we leave as an exercise to the reader), we simply observe the following:

Proposition 5.26. *Let $R \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Fil}^+(\mathrm{Mod}_{k,\geq 0}))$ be complete almost finitely presented. If M and N are almost perfect R-modules, then so is $M \otimes_R N$.*

Proof. We first note that almost perfect R-modules are necessarily bounded-below. Since tensor products of almost perfect modules over a connective $\mathbb{E}_\infty$-ring are almost perfect, it only remains to show that $M \otimes_R N$ is complete as a filtered object. Observe that this is clear when M is a free R-module. Suppose we know that $M \otimes_R N$ is complete in (homotopical) degrees $\leq n$ (i.e. that $\pi_i(\varprojlim(M \otimes_R N))$ vanishes for all $i \leq n$). The assumptions imply that there exists a finite free R-module F together with a map $F \to M$ inducing a surjection on π_0. The homotopy fibre F' remains connective and is still almost perfect. The cofibre sequence $F \to M \to \Sigma F'$ and induction then imply that $M \otimes_R N$ is complete in degree n. □

Remark 5.27. An elaboration of the above argument shows that if M is connective, then it arises as the geometric realisation of a simplicial diagram of finite free R-modules. In the unfiltered case, this is established in [Lur17, Proposition 7.2.4.11].

We can now generalise Proposition 5.24:

Proposition 5.28. *Let $R \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Fil}^+(\mathrm{Mod}_{k,\geq 0}))$ be complete almost of finite presentation. Then $\cot(R) \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$.*

Proof. First, $\pi_0(\mathrm{Gr}(R))$ is finitely generated, and we can lift generators to various filtered pieces. It follows that there is a finite-dimensional vector space V equipped with a finite filtration

$$\ldots \subset 0 \subset 0 \subset V_n \subset V_{n-1} \subset \ldots \subset V_1 = V,$$

together with a map of filtered augmented $\mathbb{E}_\infty$-algebras $\mathrm{free}(V) \to R$ inducing a surjection on $\pi_0(\mathrm{Gr}(-))$. Since R is complete, this map factors through a map $\widehat{\mathrm{free}(V)} \to R$, which turns R into an almost perfect $\widehat{\mathrm{free}(V)}$-module.

A direct calculation shows that the cotangent fibre of $\widehat{\mathrm{free}(V)}$ is just $V \in \mathrm{FilMod}_{k,\geq 0}$: that is, the map $\mathrm{free}(V) \to \widehat{\mathrm{free}(V)}$ induces an equivalence on cotangent fibres. This is clearly true on associated gradeds, so it suffices to see this on underlying objects. But on underlying objects, $F^0\widehat{\mathrm{free}(V)} = \prod_{i\geq 0}(V^{\otimes i})_{h\Sigma_i}$ is the completion of the augmented $\mathbb{E}_\infty$-algebra $\bigoplus_{i\geq 0}(V^{\otimes i})_{h\Sigma_i}$ at the augmentation ideal which does not change the cotangent fibre. For this, compare also Proposition 5.23.

By Proposition 5.26, it follows that

$$R' = R \otimes_{\widehat{\mathrm{free}(V)}} k \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Fil}^+(\mathrm{Mod}_{k,\geq 0}))$$

is also an almost perfect $\widehat{\mathrm{free}(V)}$-module, and hence complete. Since we have an equivalence of filtered objects $\cot(R') \simeq \mathrm{cofib}(V \to \cot(R))$, it suffices to show that $\cot(R')$ belongs to $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$. Indeed, this follows from the special case given by Proposition 5.24 since $\pi_0(\mathrm{Gr}(R')) = k$. □

Proposition 5.29. *If $R \in \mathrm{CAlg}^{\mathrm{aug}}_k$ is complete local Noetherian, then $\mathrm{adic}(R)$ is complete.*

Proof. Since $\mathrm{adic}(R) \to \widehat{\mathrm{adic}(R)}$ is an equivalence on associated gradeds, it suffices to check that it also induces an equivalence on underlying objects to conclude that $\mathrm{adic}(R)$ is equivalent to $\widehat{\mathrm{adic}(R)}$ and thus complete.

We begin by observing that by Proposition 5.5, the cotangent fibre $\cot(\mathrm{adic}(R))$ is given by the filtered module $\ldots \to 0 \to 0 \to \cot(R)$. Furthermore, the natural map $\cot(\mathrm{adic}(R)) \to \cot(\widehat{\mathrm{adic}(R)})$ induces an equivalence on associated gradeds.

Now $\widehat{\mathrm{adic}(R)}$ is complete almost finitely presented as a filtered $\mathbb{E}_\infty$-ring, since its associated graded is free on $[\cot(R)]_1$ by Proposition 5.5. We have used that $\cot(R) \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ as R is

Noetherian (cf. Proposition 5.11). By Proposition 5.28, this implies that $\mathrm{cot}(\widehat{\mathrm{adic}(R)})$ belongs to $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$ and it is therefore in particular complete. Hence $\mathrm{cot}(\mathrm{adic}(R)) \to \mathrm{cot}(\widehat{\mathrm{adic}(R)})$ is an equivalence, as it is a map between complete objects inducing an equivalence on associated gradeds. Taking underlying objects everywhere, it follows that $R = F^0\mathrm{adic}(R) \to F^0\widehat{\mathrm{adic}(R)}$ induces an equivalence on cot. Both source and target are complete local Noetherian $\mathbb{E}_\infty$-algebras over k (the latter by Proposition 5.23). Hence $R \to F^0\widehat{\mathrm{adic}(R)}$ is an equivalence by Proposition 5.22. □

We can now show that two notions of smallness coincide:

Corollary 5.30. *Let $R \in \mathrm{CAlg}_k^{\mathrm{aug}}$ be a connective augmented $\mathbb{E}_\infty$-algebra over k. Then the following are equivalent:*

(1) R belongs to $\mathcal{C}_{\mathrm{afp}}$ in the sense of Definition 4.43.

(2) R is complete local Noetherian in the sense of Definition 5.18.

Proof. If R is complete local Noetherian, then $\mathrm{adic}(R)$ is complete by Proposition 5.29 and $\mathrm{cot}(R)$ belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ by Proposition 5.11. Conversely, if R belongs to $\mathcal{C}_{\mathrm{afp}}$, then applying Proposition 5.23 to $\mathrm{adic}(R)$ shows that R is complete local Noetherian. □

We can finally establish the following result:

Proposition 5.31. *The datum of $\mathcal{C} = \mathrm{CAlg}^{\mathrm{aug}}_{k,\geq 0}, \mathcal{C}^{\mathrm{Gr}}, \mathcal{C}^{\mathrm{Fil}}$ specified in Construction 5.7 specifies a filtered augmented monadic adjunction in the sense of Definition 4.15. It therefore satisfies the conditions of Theorem 4.20.*

Proof. We have already verified conditions (1), (2), and (3) of Definition 4.15 in Proposition 5.8.

For part a) of the coherence condition (4), we must check that if A, A', A'' are graded $\mathbb{E}_\infty$-algebras over k (with k as degree zero component) which are almost finitely presented, and if $A \to A'', A' \to A''$ are maps which induce surjections on π_0, then $A \times_{A''} A'$ is almost finitely presented. This follows easily from the algebraic fact that if S, S', S'' are finitely generated k-algebras and $S \to S'', S' \to S''$ are surjections, then $S \times_{S''} S'$ is finitely generated as a k-algebra, cf. e.g. [Sta19, Tag 08KG]. Part b) of the coherence condition (4) follows from Proposition 5.11, as if $V \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$, then $\mathrm{sqz}(V)$ is manifestly Noetherian.

Part a) of the completeness axiom (5) follows from Proposition 5.28. For part b) of axiom (5), we note that if $A \in \mathcal{C}^{\mathrm{Fil}}_{\mathrm{afp}}$ is an augmented complete $\mathbb{E}_\infty$-k-algebra which is complete almost finitely presented, then the underlying augmented $\mathbb{E}_\infty$-k-algebra F^0A is complete local Noetherian by Proposition 5.23. Proposition 5.29 and Corollary 5.30 then imply that $\mathrm{adic}(F^0A)$ is complete. □

Hence, Theorem 4.20 applies to the present setting, and we can perform the following construction:

Definition 5.32 (Spectral partition Lie algebras)**.** Write $\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty} : \mathrm{Mod}_k \to \mathrm{Mod}_k$ for the unique sifted-colimit-preserving monad on Mod_k satisfying $\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}(V) = \mathrm{cot}(\mathrm{sqz}(V^\vee))^\vee$ for all $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. Algebras over $\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}$ will be called *spectral partition Lie algebras.*

In particular, Theorem 4.23 asserts an equivalence between formal moduli problems for $\mathbb{E}_\infty$-algebras and the ∞-category $\mathrm{Alg}_{\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}}$. We postpone the discussion and formulation of this result to the next section (cf. Theorem 6.27), where we will also discuss generalisations to other bases (but still augmented over k).

Instead, we will now give a more explicit description of partition Lie algebras. To begin with, we check that when k is a field of characteristic zero, we recover a familiar notion. Recall that in this case, the ordinary category of differential graded Lie algebras over k carries a model structure whose weak equivalences and fibrations are transported along the forgetful functor to chain complexes. Its underlying ∞-category will be denoted by $\mathrm{Alg}_{\mathrm{Lie}_k^{\mathrm{dg}}}$ (cf. [Lur16, Section 13.1] for a detailed treatment), and we write $\mathrm{Lie}_k^{\mathrm{dg}}$ for the corresponding monad on Mod_k.

Proposition 5.33. *If k is of characteristic zero, then the following monads on Mod_k are equivalent:*

$$\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty} \qquad\qquad \Sigma \circ \mathrm{Lie}_k^{\mathrm{dg}} \circ \Sigma^{-1}.$$

As a result, the ∞-categories of spectral partition Lie algebras and shifted differential graded Lie algebras are all equivalent.

Proof. Writing C^{dg} for the classical Chevalley-Eilenberg complex functor, we have a pair of adjunctions (cf. [Lur16, Section 13.3]) given by

$$\mathrm{CAlg}_k^{\mathrm{aug}} \underset{C^{\mathrm{dg}}}{\overset{\mathfrak{D}^{\mathrm{dg}}}{\rightleftarrows}} (\mathrm{Alg}_{\mathrm{Lie}_k^{\mathrm{dg}}})^{op} \underset{\mathrm{free}^{\mathrm{dg}}}{\overset{\mathrm{forget}^{\mathrm{dg}}}{\rightleftarrows}} \mathrm{Mod}_k^{op}.$$

By [Lur16, Proposition 13.3.1.4], their composite is given by

$$\Sigma^{-1} \circ \cot(-)^\vee \ : \ \mathrm{CAlg}_k^{\mathrm{aug}} \rightleftarrows \mathrm{Mod}_k^{op} \ : \ \mathrm{sqz}(-^\vee) \circ \Sigma.$$

Abstract nonsense therefore gives rise to a natural transformation of monads

$$\mathrm{Lie}_k^{\mathrm{dg}} = \mathrm{forget}^{\mathrm{dg}} \circ \mathrm{free}^{\mathrm{dg}} \longrightarrow \Sigma^{-1} \circ \cot\left(\mathrm{sqz}(-^\vee)\right)^\vee \circ \Sigma \ ,$$

which is obtained by inserting the unit $\mathrm{id} \to \mathfrak{D}^{\mathrm{dg}} \circ C^{\mathrm{dg}}$. The monad $\mathrm{Lie}_k^{\mathrm{dg}}$ preserves $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq -1}$, and we can therefore deduce from [Lur16, Proposition 13.3.1.1] that the above transformation is an equivalence for all $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq -1}$. By construction of $\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}$, we obtain an equivalence of monads $(\Sigma \circ \mathrm{Lie}_k^{\mathrm{dg}} \circ \Sigma^{-1})|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}} \simeq \mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}|_{\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}}$. Since both $\mathrm{Lie}_k^{\mathrm{dg}}$ and $\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}$ preserve sifted colimits (the former by [Lur16, Proposition 13.1.4.4], the latter by construction), we in fact obtain an equivalence of monads $\Sigma \circ \mathrm{Lie}_k^{\mathrm{dg}} \circ \Sigma^{-1} \simeq \mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}$ on Mod_k, applying Corollary 3.17 above. □

Remark 5.34. As ∞-categories, $\mathrm{Alg}_{\mathrm{Lie}_k^{\mathrm{dg}}}$ and $\mathrm{Alg}_{\Sigma\circ\mathrm{Lie}_k^{\mathrm{dg}}\circ\Sigma^{-1}}$ are equivalent via a functor whose effect on underlying k-modules is simply a shift by 1.

For general fields, partition Lie algebras are somewhat more complicated objects. We recall the notion of partition complexes from Definition 1.5 above. Write $\Sigma|\Pi_n|^\diamond$ for the Σ_n-space given by the reduced-unreduced suspension of the n^{th} partition complex. Let $\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k)$ be the cosimplicial k-vector space given by its k-valued (reduced) singular cochains.

Proposition 5.35. *Given any $V \in \mathrm{Mod}_k$, there is an equivalence*

$$\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}(V) \simeq \bigoplus_{n\geq 1} (F^h_{\Sigma|\Pi_n|^\diamond})^\vee.$$

Here $(F^h_{\Sigma|\Pi_n|^\diamond})^\vee$ is the right-left extension (cf. Section 3.2) of the functor $\mathrm{Vect}_k^\omega \to \mathrm{Mod}_k$ given by $V \mapsto (\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes V^{\otimes n})^{h\Sigma_n}$. If $V \in \mathrm{Mod}_{k,\leq N}$ is truncated above, there is an equivalence

$$\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}(V) \simeq \bigoplus_n \left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes V^{\otimes n}\right)^{h\Sigma_n}.$$

Proof. For each $n \geq 0$, we define a simplicial Σ_n-set $T(n)$ by specifying its set of k-simplices as

$$T(n)_k = \left\{ [\hat{0} = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_k = \hat{1}] \;\middle|\; \sigma_i \text{ are partitions of } \{1, \ldots, n\} \right\} \coprod \{*\}$$

Degeneracy maps insert repeated partitions into chains and fix $*$. Face maps delete partitions from chains whenever this yields a "legal" chain starting in $\hat{0}$ and ending in $\hat{1}$; otherwise, they map to $*$.

As cot preserves geometric realisations, we obtain, for any $V \in \mathrm{Vect}_k^{\omega}$, the following equivalence:

$$\cot(\mathrm{sqz}(V)) \simeq |\operatorname{Bar}_\bullet(\mathrm{id}, \mathrm{Free}_{\mathbb{E}_\infty^{\mathrm{nu}}}, V)| \simeq \left| \;\ldots\; \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \bigoplus_{m\geq 1} (\bigoplus_{n \geq 1} V^{\otimes n}_{h\Sigma_n})^{\otimes m}_{h\Sigma_m} \substack{\rightarrow\\\rightarrow\\\rightarrow} \bigoplus_{n\geq 1} V^{\otimes n}_{h\Sigma_n} \rightrightarrows V \right|.$$

For $(X, *)$ a pointed set, we write $k[X]$ for the free k-module on X subject to the relation $0 \simeq *$. Expanding out extended powers binomially, a well-known and elementary combinatorial observation (explained for example in [Chi05]) shows that $\cot(\mathrm{sqz}(V))$ is equivalent to

$$\left| \;\ldots\; \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \bigoplus_{n\geq 1} (k[T(n)_2] \otimes V^{\otimes n})_{h\Sigma_n} \substack{\rightarrow\\\rightarrow\\\rightarrow} \bigoplus_{n\geq 1} (k[T(n)_1] \otimes V^{\otimes n})_{h\Sigma_n} \rightrightarrows \bigoplus_{n\geq 1} (k[T(n)_0] \otimes V^{\otimes n})_{h\Sigma_n} \right|,$$

which is equivalent to $\bigoplus_{n\geq 1} (\widetilde{C}_\bullet(|T(n)|, k) \otimes V^{\otimes n})_{h\Sigma_n}$. Since both functors preserve sifted colimits, we deduce that $\cot(\mathrm{sqz}(V)) \simeq \bigoplus_{n\geq 1} (\widetilde{C}_\bullet(|T(n)|, k) \otimes V^{\otimes n})_{h\Sigma_n}$ for all $V \in \mathrm{Mod}_k$.

We now observe that $T(n)$ can be identified with the quotient of the join $\{\hat{0}\} * \Pi_n * \{\hat{1}\}$ by the simplicial subset spanned by all chains not containing $[\hat{0} \leq \hat{1}]$ as a subchain. The realisation $|T(n)|$ is therefore equivalent to the reduced-unreduced suspension of the n^{th} partition complex $\Sigma|\Pi_n|^\diamond$ (cf. [AB18, Section 2.9]).

If $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$, then $\mathrm{sqz}(V^\vee)$ is Noetherian, and so $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}(V) = \cot(\mathrm{sqz}(V^\vee))^\vee$ belongs to $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ too (cf. Proposition 5.11). We can then identify the appearing infinite product with an infinite sum and compute

$$\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}(V) \simeq (\cot(\mathrm{sqz}(V^\vee)))^\vee \simeq \bigoplus_n \left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes V^{\otimes n}\right)^{h\Sigma_n} \simeq \bigoplus_{n \geq 1} (F^h_{\Sigma|\Pi_n|^\diamond})^\vee(V).$$

Since $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}(V)$ and $\bigoplus_{n\geq 1}(F^h_{\Sigma|\Pi_n|^\diamond})^\vee$ commute with sifted colimits, the first claim follows.

We observe that both $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}(-)$ and $\bigoplus_n (\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes (-)^{\otimes n})^{h\Sigma_n}$ preserve filtered colimits in $\mathrm{Mod}_{k,\leq 0}$; the above formula therefore holds for any $V \in \mathrm{Mod}_{k,\leq 0}$. Both functors also preserve finite geometric realisations, which implies the formula whenever $V \in \mathrm{Mod}_{k,\leq N}$ for some N. $\square$

5.2. **Simplicial commutative rings.** We shall now explain the modifications needed in order to obtain a Lie algebraic description of deformations over simplicial commutative rings over a field k. In particular, we will obtain a setup as in Definition 4.15 of our axiomatic section.

For this, we will need to recall the basic homotopy theory of simplicial commutative rings, as introduced by Quillen. We refer to [Lur16, Chapter 25] for a detailed ∞-categorical treatment of simplicial commutative rings. For our axiomatic setup, we will also need graded and filtered versions. We give a quick summary below:

Construction 5.36 (The setup for simplicial commutative rings)**.**

a) Let $\mathcal{D} = \mathrm{SCR}_k^{\mathrm{aug}}$ be the ∞-category of augmented simplicial commutative k-algebras. Explicitly, $\mathrm{SCR}_k^{\mathrm{aug}}$ can be obtained as the nonabelian derived ∞-category $\mathcal{P}_\Sigma$ (as in [Lur09, Section 5.5.8]) of the category of finitely generated augmented polynomial k-algebras.
b) Let $\mathcal{D}^{\mathrm{Fil}}$ be the ∞-category of filtered, augmented simplicial commutative k-algebras R with $F^0R/F^1R \simeq k$. Specifically, $\mathcal{D}^{\mathrm{Fil}}$ can be obtained by applying $\mathcal{P}_\Sigma$ to the category of filtered, augmented k-algebras which are free on a finite-dimensional vector space V equipped with a finite filtration with $F^0V = F^1V$.
c) Let $\mathcal{D}^{\mathrm{Gr}}$ denote the ∞-category of graded, augmented simplicial commutative k-algebras. More precisely, $\mathcal{D}^{\mathrm{Gr}}$ is obtained as $\mathcal{P}_\Sigma$ of the category of finitely generated, graded augmented polynomial algebras of the form $k[x_1, \ldots, x_r]$ with each x_i homogeneous of *positive* degree.
d) We have free-forgetful adjunctions $\mathrm{Mod}_{k,\geq 0} \rightleftarrows \mathcal{D}$, $\mathrm{Gr}(\mathrm{Mod}_{k,\geq 0}) \rightleftarrows \mathcal{D}^{\mathrm{Gr}}$, and $\mathrm{Fil}(\mathrm{Mod}_{k,\geq 0}) \rightleftarrows \mathcal{D}^{\mathrm{Fil}}$. The forgetful functors act as expected on the polynomial generators of the respective ∞-categories of algebras (i.e. by taking the kernel of the augmentation). Moreover, the forgetful functors are required to commute with sifted colimits (cf. Construction 5.38 below).

   The three evident square-zero functors $\mathrm{sqz} : \mathrm{Mod}_{k,\geq 0} \to \mathcal{D}$, $\mathrm{sqz} : \mathrm{Gr}(\mathrm{Mod}_{k,\geq 0}) \to \mathcal{D}^{\mathrm{Gr}}$, and $\mathrm{sqz} : \mathrm{Fil}(\mathrm{Mod}_{k,\geq 0}) \to \mathcal{D}^{\mathrm{Fil}}$ admit left adjoints
$$\mathrm{cot}_\Delta : \mathcal{D} \to \mathrm{Mod}_{k,\geq 0},\ \mathrm{cot}_\Delta : \mathcal{D}^{\mathrm{Gr}} \to \mathrm{Gr}(\mathrm{Mod}_{k,\geq 0}),\ \mathrm{cot}_\Delta : \mathcal{D}^{\mathrm{Fil}} \to \mathrm{Fil}(\mathrm{Mod}_{k,\geq 0}).$$
   We use the subscript Δ to contrast this with the $\mathbb{E}_\infty$-cotangent fibre construction.
e) The underlying object functor $F^1 : \mathcal{D}^{\mathrm{Fil}} \to \mathcal{D}$ forgets the filtration. On the polynomial generators, it behaves as the name indicates; in general, it is determined by commuting with sifted colimits. The functor $\mathrm{adic} : \mathcal{D} \to \mathcal{D}^{\mathrm{Fil}}$ is the left adjoint of the underlying functor.
f) The associated graded functor $\mathcal{D}^{\mathrm{Fil}} \to \mathcal{D}^{\mathrm{Gr}}$ is constructed similarly by first defining it in the evident way on polynomial generators and then extending in a sifted-colimit-preserving manner.

Remark 5.37. $\mathcal{D}$ contains the (ordinary) category of augmented k-algebras as a full subcategory. Similarly, $\mathcal{D}^{\mathrm{Gr}}$ and $\mathcal{D}^{\mathrm{Fil}}$ contain the categories of graded and filtered augmented k-algebras.

Construction 5.38 (The free functors)**.** We let $\mathbb{L}\mathrm{Sym}^* : \mathrm{Mod}_{k,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$ denote the functor which sends $V \in \mathrm{Mod}_{k,\geq 0}$ to the augmentation ideal of the free simplicial commutative k-algebra on V (with its natural augmentation). Explicitly, if V is a (discrete) k-vector space, then $\mathbb{L}\mathrm{Sym}^*(V) = \bigoplus_{i>0} \mathrm{Sym}^i V$ is the (usual) nonunital symmetric algebra on V; in general $\mathbb{L}\mathrm{Sym}^*$ is defined as the nonabelian (left) derived functor construction (Construction 3.3).

The functors $\mathbb{L}\mathrm{Sym}^* : \mathrm{Gr}(\mathrm{Mod}_{k,\geq 0}) \to \mathrm{Gr}(\mathrm{Mod}_{k,\geq 0})$ and $\mathbb{L}\mathrm{Sym}^* : \mathrm{Fil}(\mathrm{Mod}_{k,\geq 0}) \to \mathrm{Fil}(\mathrm{Mod}_{k,\geq 0})$ are defined in a similar way; they recover $\mathbb{L}\mathrm{Sym}^*$ on underlying k-modules, but keep track of the additional grading and filtration, respectively.

We now observe that $\mathbb{L}\mathrm{Sym}^i$ is a polynomial functor of degree i (as it preserves filtered colimits and is i-excisive by Proposition 3.34). Combining this with the finiteness properties of symmetric powers established in [Lur16, Section 25.2.5], it follows that $\mathbb{L}\mathrm{Sym}^* : \mathrm{GrMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$ is admissible in the sense of Definition 3.43.

Example 5.39 (The adic filtration of a polynomial ring)**.** Unwinding the definitions, we see that applying the functor adic to a free simplicial commutative ring $k[x_1, \ldots, x_n] \in \mathrm{SCR}_k^{\mathrm{aug}} = \mathcal{C}$ recovers the usual $\mathfrak{m}$-adic filtration, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is augmentation ideal. In other words, one obtains the free filtered simplicial commutative ring on $x_1, \ldots, x_n$ in filtration 1.

Very explicitly, the adic filtration can also be defined as follows: on polynomial rings, it is the $\mathfrak{m}$-adic filtration and in general, it is defined via left Kan extension.

Proposition 5.40. *The setup of simplicial commutative k-algebras in Construction 5.36 satisfies conditions* $(1) - (3)$ *of Definition 4.15.*

Proof. Conditions (1) and (2) are straightforward to check.

In Construction 5.38, we saw that $\mathbb{L}\mathrm{Sym}^* : \mathrm{GrMod}_{k,\geq 0} \to \mathrm{GrMod}_{k,\geq 0}$ is an admissible functor. It remains to produce the filtration for a graded, augmented simplicial commutative ring $A \in \mathcal{D}^{\mathrm{Gr}}$; for this, we will follow the discussion in Example 5.39. If A is free with maximal ideal $\mathfrak{m}_A$, the $\mathfrak{m}_A$-adic filtration gives a natural convergent filtration on $\mathfrak{m}_A$; its associated graded is given by the symmetric algebra $\mathrm{Sym}^*(\mathfrak{m}_A/\mathfrak{m}_A^2)$. By taking left Kan extension, we conclude that for any $A \in \mathcal{D}^{\mathrm{Gr}}$, the augmentation ideal $\mathfrak{m}_A$ is equipped with a convergent filtration with associated graded $\mathbb{L}\mathrm{Sym}^*(\mathrm{cot}_\Delta(A))$. This immediately implies that condition (3) of Definition 4.15 is satisfied. □

We will now verify the coherence axiom 4 and the completeness axiom 5 in Definition 4.15. These will both be deduced from the analogous assertions involving $\mathbb{E}_\infty$-rings, which we have already checked in Proposition 5.31 above.

Construction 5.41 (Forgetting to $\mathbb{E}_\infty$-algebras)**.** There is a natural forgetful functor from simplicial commutative rings to $\mathbb{E}_\infty$-rings. It is characterised by the properties of acting as the forgetful functor on ordinary polynomial rings and preserving sifted colimits (cf. [Lur16, Section 25.1.2]). This construction clearly carries over to the augmented, filtered, and graded settings, and we therefore obtain forgetful functors $\mathcal{D} \to \mathcal{C}$, $\mathcal{D}^{\mathrm{Fil}} \to \mathcal{C}^{\mathrm{Fil}}$, and $\mathcal{D}^{\mathrm{Gr}} \to \mathcal{C}^{\mathrm{Gr}}$. Here we use the notation introduced in Construction 5.7 and Construction 5.36.

Definition 5.42. We say that $A \in \mathrm{SCR}_k^{\mathrm{aug}} = \mathcal{D}$ is *Noetherian* (respectively *complete local Noetherian*) if the underlying $\mathbb{E}_\infty$-algebra of A is Noetherian (respectively complete local Noetherian).

The axiomatic Definitions 4.31 and 4.39 give notions of almost finite presentation and complete almost finite presentation for graded and filtered simplicial commutative k-algebras, respectively.

Proposition 5.43. *These notions are compatible with the forgetful functor to $\mathbb{E}_\infty$-k-algebras:*

(1) A graded (augmented) simplicial commutative k-algebra $A \in \mathcal{D}^{\mathrm{Gr}}$ is almost finitely presented if and only if the underlying graded $\mathbb{E}_\infty$-k-algebra (in $\mathcal{C}^{\mathrm{Gr}}$) is almost finitely presented.

(2) A filtered (augmented) simplicial commutative k-algebra $A \in \mathcal{D}^{\mathrm{Fil}}$ is complete almost finitely presented if and only if the underlying filtered $\mathbb{E}_\infty$-k-algebra (in $\mathcal{D}^{\mathrm{Fil}}$) is complete almost finitely presented.

Proof. Both assertions follow straightforwardly from [Lur16, Remark 25.3.3.7]. More explicitly, this remark shows that there is an associative ring spectrum k^+ with an augmentation $k^+ \to k$ such that $\mathrm{cot}(A)$ is a k^+-module and $\mathrm{cot}_\Delta(A) \simeq \mathrm{cot}(A) \otimes_{k^+} k$. Moreover, k^+ is connective with $\pi_0(k^+) = k$, and its homotopy groups are finite-dimensional in each degree. This readily implies that $\mathrm{cot}(A)$ has finite-dimensional homotopy groups in each degree if and only if $\mathrm{cot}_\Delta(A)$ does, hence proving (1). Assertion (2) follows from (1) as completeness is detected on underlying k-module spectra. □

Proposition 5.44. *If $R \in \mathcal{D}^{\mathrm{Fil}}$ is complete almost finitely presented, then* $\mathrm{cot}_\Delta(R) \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$.

Proof. This follows by combining [Lur16, Remark 25.3.3.7] with Proposition 5.28. Namely, we already know that $\cot(R) \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 0}$, and the identification $\cot_\Delta(R) \simeq \cot(R) \otimes_{k^+} k$ (which works in the filtered category too) together with Proposition 2.11 allow us to conclude the claim. □

Proposition 5.45. *If $R \in \mathrm{SCR}_k^{\mathrm{aug}}$ is complete local Noetherian, then the adic filtration converges.*

Proof. This follows by the argument used in the proof of Proposition 5.29, where we simply replace $\mathbb{E}_\infty$-rings by simplicial commutative rings everywhere. □

Corollary 5.46. *The setup of simplicial commutative k-algebras of Construction 5.36 satisfies the axioms of Definition 4.15. Consequently, Theorem 4.20 holds true in this context.*

Proof. We have already checked axioms (1)-(3) in Proposition 5.40. The coherence axiom (4) follows immediately by combining Proposition 5.43 with the corresponding result for $\mathbb{E}_\infty$-algebras, which was established in Proposition 5.31. Part *a*) of the completeness axiom (5) was proven in Proposition 5.44, whereas part *b*) follows by combining Proposition 5.45 with Proposition 5.23. □

In particular, we can again perform the following construction:

Definition 5.47 (Partition Lie algebras)**.** Write $\mathrm{Lie}^\pi_{k,\Delta} : \mathrm{Mod}_k \to \mathrm{Mod}_k$ for the unique sifted-colimit-preserving monad on Mod_k satisfying $\mathrm{Lie}^\pi_{k,\Delta}(V) = \cot_\Delta(\mathrm{sqz}(V^\vee))^\vee$ for all $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. Algebras over $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$ will be called *partition Lie algebras.*

Applying Theorem 4.23 to our setup, we obtain a classification of formal moduli problems for augmented simplicial commutative rings as equivalent to the ∞-category of partition Lie algebras. We again postpone stating the result formally until the next section (cf. Theorem 6.27).

Proposition 5.48. *If k is of characteristic zero, then the monad $\mathrm{Lie}^\pi_{k,\Delta}$ is equivalent to the monads $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$ and $\Sigma \circ \mathrm{Lie}^{\mathrm{dg}}_k \circ \Sigma^{-1}$ building free spectral and free shifted differential graded Lie algebras. As a result, the ∞-categories of partition Lie algebras, spectral partition Lie algebras, and shifted differential graded Lie algebras are equivalent.*

Proof. Since k has characteristic zero, the forgetful functor from simplicial commutative k-algebras to connective $\mathbb{E}_\infty$-k-algebras is an equivalence (cf. [Lur16, Proposition 25.1.2.2]). Together with Proposition 5.48, this implies the claim. □

We proceed to establish a concrete description of $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$. As above, let $\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k)$ denote the k-valued (reduced) singular cochains of the doubly suspended n^{th} partition complex. The following result uses the genuine Σ_n-equivariant structure of this cosimplicial k-module:

Proposition 5.49. *Given any $V \in \mathrm{Mod}_k$, there is an equivalence $\mathrm{Lie}^\pi_{k,\Delta}(V) \simeq \bigoplus_{n\geq 1}(F_{\Sigma|\Pi_n|^\diamond})^\vee$. Here $(F_{\Sigma|\Pi_n|^\diamond})^\vee$ is the right-left extension (cf. Section 3.2) of the functor $\mathrm{Vect}_k^\omega \to \mathrm{Mod}_k$ given by $V \mapsto (\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes V^{\otimes n})^{\Sigma_n}$ (cf. Section 3.3 for a more formal definition).*

If $V \simeq \mathrm{Tot}(V^\bullet) \in \mathrm{Mod}_{k,\leq 0}$ is represented by a cosimplicial k-vector space $V^\bullet$, then

$$\mathrm{Lie}^\pi_{k,\Delta}(V) \simeq \bigoplus_n \mathrm{Tot}\left(\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k) \otimes (V^\bullet)^{\otimes n}\right)^{\Sigma_n}.$$

Here $\widetilde{C}^\bullet(\Sigma|\Pi_n|^\diamond, k)$ denotes the k-valued cosimplices on the space $\Sigma|\Pi_n|^\diamond$, the functor $(-)^{\Sigma_n}$ takes strict fixed points, and the tensor product is computed in cosimplicial k-modules.

Proof. We apply the same argument as in Proposition 5.35, replacing homotopy orbits with strict orbit, to deduce the first statement. The second statement then follows from Proposition 3.23. □

5.3. **Operads.** In this section, we fix a field k and an ∞-operad $\mathcal{O}$ internal to Mod_k (cf. e.g. [Bra17, Definition 4.1.4]) satisfying the following three basic properties:

(1) $\mathcal{O}(0)$ is contractible.
(2) $\mathcal{O}(1)$ is equivalent to k via the unit map $k \to \mathcal{O}(1)$.
(3) $\mathcal{O}(i) \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$ is connective and of finite type for all $i \geq 0$.

The ∞-category $\mathrm{Alg}_{\mathcal{O}}$ of $\mathcal{O}$-algebras in Mod_k, which comes with a free-forgetful adjunction

$$\mathrm{free}_{\mathcal{O}} \;:\; \mathrm{Mod}_k \rightleftarrows \mathrm{Alg}_{\mathcal{O}} \;:\; \mathrm{forget}, \tag{10}$$

The free functor is given by the formula $\mathrm{free}_{\mathcal{O}}(V) \simeq \bigoplus_{i\geq 1}(\mathcal{O}(i)\otimes V^{\otimes i})_{h\Sigma_i}$.

The main result of this section is that when we restrict to formal moduli problems defined on *connected* $\mathcal{O}$-algebras (i.e. $\mathcal{O}$-algebras in $\mathrm{Mod}_{k,\geq 1}$), then our axiomatic Theorem 4.23 implies a classification of formal moduli problems. We stress that this will not quite recover the (harder) main result of Section 5.1 above due to the stronger connectedness assumption; more on this point in Remark 5.61 below. Since the essential features are very similar to the previous examples, we will be brief. Compare also the result of Ching-Harper [CH19], which proves the comonadicity assertion under the connectedness assumption.

Construction 5.50 (The setup for connected $\mathcal{O}$-algebras)**.** Let k be a field.

a) Let $\mathcal{C}$ be the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_{k,\geq 1})$ of connected $\mathcal{O}$-algebras.
b) Let $\mathcal{C}^{\mathrm{Fil}} = \mathrm{Alg}_{\mathcal{O}}(\mathrm{FilMod}_{k,\geq 1})$ be the ∞-category of filtered $\mathcal{O}$-algebras.
c) Let $\mathcal{C}^{\mathrm{Gr}} = \mathrm{Alg}_{\mathcal{O}}(\mathrm{GrMod}_{k,\geq 1})$. Denote the ∞-category of graded $\mathcal{O}$-algebras.
d) We have a free-forgetful adjunction $\mathrm{free}_{\mathcal{O}} : \mathrm{Mod}_{k,\geq 1} \rightleftarrows \mathcal{C} : \mathrm{forget}$. The natural augmentation map from $\mathcal{O}$ to the trivial operad induces an adjunction $\mathrm{cot}_{\mathcal{O}} : \mathcal{C} \rightleftarrows \mathrm{Mod}_{k,\geq 1} : \mathrm{sqz}$. We define free-forgetful and cotangent fibre adjunctions in the filtered and graded context in a similar way.
e) The adic filtration functor $\mathrm{adic} : \mathcal{C} \to \mathcal{C}^{\mathrm{Fil}}$ is right adjoint to the underlying functor $F^1 : \mathcal{C}^{\mathrm{Fil}} \to \mathcal{C}$.
f) The associated graded functor lifts naturally to define a functor $\mathrm{Gr} : \mathcal{C}^{\mathrm{Fil}} \to \mathcal{C}^{\mathrm{Gr}}$.

Because of the connectedness assumption, verifying the hypotheses of Theorem 4.20 turns out to be much simpler than before, as convergence works more nicely. To verify this, we will first show that the adic filtration converges automatically for connected $\mathcal{O}$-algebras. In a second step, we then show that finiteness can be detected by (and is reflected in) $\mathrm{cot}_{\mathcal{O}}$.

Example 5.51 (The adic filtration on a free algebra)**.** The adic filtration on $\mathrm{free}_{\mathcal{O}}(V)$ is given by

$$F^n\mathrm{adic}(\mathrm{free}_{\mathcal{O}}(V)) \simeq \bigoplus_{i\geq n}(\mathcal{O}(i)\otimes V^{\otimes i})_{h\Sigma_i}.$$

In particular, if $V \in \mathrm{Mod}_{k,\geq 1}$, then the filtration converges for connectivity reasons.

Proposition 5.52 (Convergence of the adic filtration)**.** *Let $A \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_{k,\geq 1})$ be a connected $\mathcal{O}$-algebra. Then the adic filtration on A converges.*

Proof. This follows from Example 5.51 and Proposition 2.11, since any connected $\mathcal{O}$-algbera is the geometric realisation of free connected $\mathcal{O}$-algebras. □

The next result, which is essentially already contained in [HH13], shows that finite type conditions can be detected using the cotangent fibre functor $\mathrm{cot}_{\mathcal{O}}$ (under the assumption of connectedness).

Proposition 5.53 (Finiteness, completeness, and $\mathrm{cot}_{\mathcal{O}}$)**.**

(1) Let $A \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_{k,\geq 1})$ be connected. Then $A \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 1}$ if and only if $\mathrm{cot}_{\mathcal{O}}(A) \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 1}$.

(2) *Let* $B \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{GrMod}_{k,\geq 1})$ *be connected and graded. Then* $B \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$ *if and only if* $\cot_{\mathcal{O}}(B) \in \mathrm{Gr}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$.

(3) *Let* $C \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{FilMod}_{k,\geq 1})$ *be a connected, filtered* $\mathcal{O}$*-algebra. Then* $C \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$ *if and only if* C *is complete and* $\cot_{\mathcal{O}}(C) \in \mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$.

Proof. For part (1), let A be a connected $\mathcal{O}$-algebra. Suppose that $\cot_{\mathcal{O}}(A)$ is of finite type. The fact that A is of finite type (as a k-module spectrum) follows from the adic filtration on A. Indeed, this filtration converges by Proposition 5.52, and the terms of its associated graded are each of finite type and become arbitrarily connected. Conversely, if A is of finite type, then the bar construction can be used to express $\cot_{\mathcal{O}}(A)$ as a geometric realisation of a simplicial k-module spectrum whose terms are of the form $\mathrm{free}_{\mathcal{O}} \circ \cdots \circ \mathrm{free}_{\mathcal{O}}(A)$. Each of these is connected and of finite type, so that the geometric realisation $\cot_{\mathcal{O}}(A)$ is connected of finite type as well.

Part (2) follows directly from part (1) by forgetting to underlying ungraded $\mathcal{O}$-algebras.

Part (3) follows from part (2) together with the claim that if C is a connected filtered $\mathcal{O}$-algebra whose underlying object belongs to $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$, then $\cot_{\mathcal{O}}(C)$ is automatically complete. This last claim again follows from the bar construction: the key observation is that $\mathrm{free}_{\mathcal{O}}$ preserves $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$, and that $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$ is closed under geometric realisations. □

We can explicitly identify the subcategories $\mathcal{C}_{\mathrm{afp}}, \mathcal{C}_{\mathrm{afp}}^{\mathrm{Fil}}$, and $\mathcal{C}_{\mathrm{afp}}^{\mathrm{Gr}}$ which appear when applying the axiomatic definitions from Section 4 to the setup specified in Construction 5.50 The following is immediate from Proposition 5.53:

Corollary 5.54.

(1) *An object* $A \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_{k,\geq 1})$ *is complete almost finitely presented if and only if the underlying* k*-module spectrum is finite type.*

(2) *An object* $B \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{GrMod}_{k,\geq 1})$ *is almost finitely presented if and only if the underlying* k*-module spectrum of* B *is of finite type.*

(3) *An object* $C \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{FilMod}_{k,\geq 1})$ *is complete almost finitely presented if and only if the underlying object of* $\mathrm{FilMod}_{k,\geq 1}$ *belongs to* $\mathrm{Fil}^{\mathrm{ft}}\mathrm{Mod}_{k,\geq 1}$.

Corollary 5.55. *The above satisfies the conditions of Definition 4.15.*

Proof. Since the conditions of almost finite presentation, complete almost finite presentation, and so forth are purely module-theoretic in view of Corollary 5.54, the conditions of Definition 4.15 are evidently satisfied. □

Construction 5.56. It follows that we obtain a monad $T_{\mathcal{O}}^{\vee}$ on Mod_k and a Koszul duality functor

$$\mathfrak{D} : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_{k,\geq 1}) \to \mathrm{Alg}_{T_{\mathcal{O}}^{\vee}}^{op}$$

as Theorem 4.20 and Construction 4.50. By construction, the monad $T_{\mathcal{O}}^{\vee}$ preserves sifted colimits and its value on $V \in \mathrm{Mod}_{k,\leq -1}^{\mathrm{ft}}$ is given by $T_{\mathcal{O}}^{\vee}(V) \simeq \cot_{\mathcal{O}}(\mathrm{sqz}(V^{\vee}))^{\vee}$.

Remark 5.57. Let $\mathcal{O}^{\vee} = \mathrm{Bar}(\mathcal{O})$ be the Koszul dual ∞-operad in Mod_k. Then the functor $T_{\mathcal{O}}^{\vee}$ is given, for $V \in \mathrm{Mod}_{k,\leq -1}^{\mathrm{ft}}$, by the formula

$$V \mapsto (\cot_{\mathcal{O}} \circ \mathrm{sqz}(V^{\vee}))^{\vee} = \bigoplus_{i \geq 1} (\mathcal{O}^{\vee}(i) \otimes V^{\otimes i})^{h\Sigma_i}.$$

Here the product could be interchanged with the sum for connectivity reasons. Roughly speaking, we should regard $T_{\mathcal{O}}^{\vee}$-algebras as "divided power" algebras over the Koszul dual operad $\mathcal{O}^{\vee}$.

We introduce the following explicit definition:

Definition 5.58 (Artinian $\mathcal{O}$-algebras and formal moduli problems)**.**

(1) A connected $\mathcal{O}$-algebra A is Artinian if $\pi_*(A)$ is a finite-dimensional k-vector space. Let $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{art}}$ denote the ∞-category of Artinian $\mathcal{O}$-algebras.

(2) An *$\mathcal{O}$-formal moduli problem* is a functor $F : \mathrm{Alg}_{\mathcal{O}}^{\mathrm{art}} \to \mathcal{S}$ such that if $A \to A''$, $A' \to A''$ are maps in $\mathrm{Alg}_{\mathcal{O}}^{\mathrm{art}}$ inducing surjections on π_1, then $F(A \times_{A''} A') \simeq F(A) \times_{F(A')} F(A'')$.

We can then use our results in Section 4:

Corollary 5.59. *There is an equivalence between the ∞-category of $\mathcal{O}$-formal moduli problems and the ∞-category $\mathrm{Alg}_{T_{\mathcal{O}}^{\vee}}$.*

Proof. In order to apply Theorem 4.23, it suffices to verify that Artinian $\mathcal{O}$-algebras in the sense of Definition 5.58(1) are exactly the Artinian objects with respect to the deformation theory given by $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_{k,\geq 1})$, i.e. objects which can be built up inductively by pullbacks of trivial extensions (cf. Definition 4.21). It then also follows that Definition 5.58(2) is an instance of Definition 4.22 in the present context.

Observe that any $\mathcal{O}$-algebra with homotopy groups concentrated in degree 1 is necessarily square-zero by our connectivity assumptions on $\mathcal{O}$. Given an algebra $A \in \mathrm{Alg}_{\mathcal{O}}^{\mathrm{art}}$, we write $\tau_{\leq n}A$ for the n^{th} Postnikov truncation of A (cf. [Lur09, Proposition 5.5.6.18]). Arguing as in [Lur17, Proposition 7.1.3.15], we see that the underlying k-module of $\tau_{\leq n}A$ is simply given by the k-truncation in Mod_k. It then suffices to verify that if $A \in \mathrm{Alg}_{\mathcal{O}}^{\mathrm{art}}$ has top homotopy group in degree n, then there exists a pullback square of $\mathcal{O}$-algebras

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ \tau_{\leq n-1}A \arrow[r] & \mathrm{sqz}((\pi_n A)[n+1]) \end{tikzcd} .$$

In the case of $\mathbb{E}_\infty$-rings, this observation appears in [Kri96] and [Bas99b], and is discussed in modern language in [Lur09, Corollary 7.4.1.28].

To construct the desired pullback square, we first form the pushout $P = \tau_{\leq n-1}A \sqcup_A 0$ in $\mathrm{Alg}_{\mathcal{O}}$ and then apply the functor $\tau_{\leq n+1}$ to it, which implies the claim as $\tau_{\leq n+1}P \simeq \mathrm{sqz}((\pi_n A)[n+1])$. □

Remark 5.60. In characteristic zero, $T_{\mathcal{O}}^{\vee}$ agrees with the free $\mathcal{O}^{\vee}$-algebra monad. In particular, the assertion is that $\mathcal{O}$-formal moduli problems are equivalent to $\mathcal{O}^{\vee}$-algebras under the above assumptions. This fact is well-known to experts, but we are not aware it has appeared explicitly yet.

Remark 5.61 (Comparison with spectral formal moduli problems)**.** One can apply the above results when $\mathcal{O}$ is the (nonunital) commutative operad. This gives rise to a classification of a variant of spectral formal moduli problems (which are only defined on *connected* Artinian $\mathbb{E}_\infty$-k-algebras) in terms of the same ∞-category of spectral partition Lie algebras.

In particular, it follows that spectral formal moduli problems over k are determined by their restriction to connected Artinian $\mathbb{E}_\infty$-k-algebras; they automatically extend to all Artinian $\mathbb{E}_\infty$-k-algebras. However, the arguments are simpler when one restricts to the connected case (as in the present section), and apply to more general operads $\mathcal{O}$.

The fact that, for $\mathcal{O}$ the commutative operad, the theory extends to some connective (rather than connected) objects requires an additional calculation (and does not appear to be purely formal).

6. Deformations over a complete local base

Assume that A is complete local Noetherian with residue field k (cf. Definition 5.18), either in the setting of $\mathbb{E}_\infty$-rings or in the setting of simplicial commutative rings. The following rings will play the role of infinitesimal thickenings of $\mathrm{Spec}(k)$ in this mixed setting:

Definition 6.1 (Artinian rings for $A//k$)**.** An object B of $\mathrm{CAlg}_{A//k}$ is called *Artinian* if

(1) $\pi_0(B)$ is a local Artinian ring (with residue field k)
(2) $\bigoplus_{i\geq 0}\pi_i(B)$ is a finitely generated module over $\pi_0(B)$ (in particular, $\pi_i(B)=0$ for $i\gg 0$).

We let $\mathrm{CAlg}^{\mathrm{art}}_{A//k}$ denote the full subcategory of $\mathrm{CAlg}_{A//k}$ spanned by Artinian objects.

An object of $\mathrm{SCR}_{A//k}$ is *Artinian* if the underlying object of $\mathrm{CAlg}_{A//k}$ is, and we let $\mathrm{SCR}^{\mathrm{art}}_{A//k}$ be the full subcategory of $\mathrm{SCR}_{A//k}$ spanned by all Artinian objects.

We can now generalise the notion of a formal moduli problem to the relative context; note that this notion also appears in [Lur11b, Section 6.1].

Definition 6.2 (Formal moduli problems for $A//k$)**.** A *spectral formal moduli problem for $A//k$* is a functor $F:\mathrm{CAlg}^{\mathrm{art}}_{A//k}\to\mathcal{S}$ such that:

(1) $F(k)$ is contractible.
(2) If $B,B',B''\in\mathrm{CAlg}^{\mathrm{art}}_{A//k}$ and we have maps $B\to B''$ and $B'\to B''$ which induce surjections on π_0, then the canonical map $F(B\times_{B''}B')\xrightarrow{\simeq}F(B)\times_{F(B'')}F(B')$ is an equivalence.

We denote the ∞-category of spectral formal moduli problems by $\mathrm{Moduli}_{A//k,\mathbb{E}_\infty}$.

Similarly, a *derived formal moduli problem* for $A//k$ is a functor $F:\mathrm{SCR}^{\mathrm{art}}_{A//k}\to\mathcal{S}$ satisfying the analogous conditions (1) and (2) above. We denote the ∞-category of derived formal moduli problems for $A//k$ by $\mathrm{Moduli}_{A//k,\Delta}$.

Remark 6.3. Suppose k is a perfect field of characteristic p. Let $W^+(k)$ denote the spherical Witt vectors of k (cf. [Lur18, Example 5.2.7]), so that $W^+(k)$ is a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. Then the ∞-category of Artinian objects of $\mathrm{CAlg}_{A//k}$ is equivalent to a full subcategory of $\mathrm{CAlg}_{/k}$ (namely, those which are Artinian). It therefore follows that we can regard spectral formal moduli problems as defined on all Artinian $\mathbb{E}_\infty$-algebras augmented over k; the map from A is therefore superfluous.

A similar statement holds for derived formal moduli problems and the classical Witt vectors $W(k)$.

The principal goal of this section is to generalise Section 5 to the mixed context. Our main result is Theorem 6.27 below, which gives a Lie algebraic description of spectral and derived formal moduli problems for $A//k$.

6.1. Descent properties of modules.

Let A be a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. We will now establish several convergence results on modules. This will later allow us to reduce the proof of Theorem 6.27 to the case $A=k$, which has been handled in Section 5.

Definition 6.4 (Complete A-modules)**.** An A-module spectrum $M\in\mathrm{Mod}_A$ is *complete* if it is derived $\mathfrak{m}$-complete (cf. [Lur16, Theorem 7.3.4.1]), where $\mathfrak{m}\subset\pi_0(A)$ is the unique maximal ideal.

Proposition 6.5 (Convergence criterion for A-modules)**.** *Let $M^\bullet$ be a cosimplicial object of* $\mathrm{Mod}_{A,\geq 0}$. *Suppose that each M^i is complete and that* $\mathrm{Tot}(k\otimes_A M^\bullet)\in\mathrm{Mod}_k$ *is connective. Then:*

(1) $\mathrm{Tot}(M^\bullet)$ *is connective and complete (as an A-module).*

(2) If N is an almost perfect A-module, then $N \otimes_A \mathrm{Tot}(M^\bullet) \xrightarrow{\simeq} \mathrm{Tot}(N \otimes_A M^\bullet)$ is an equivalence.

Proof. We begin with statement (1). Consider the class $\mathcal{T}$ of connective A-modules N for which $\mathrm{Tot}(N \otimes_A M^\bullet)$ is connective. By assumption, $\mathcal{T}$ contains $k[i]$ for $i \geq 0$. Moreover, $\mathcal{T}$ is closed under extensions: given a cofibre sequence $N_1 \to N_2 \to N_3$ with $N_1, N_3 \in \mathcal{T}$, it follows that $N_2 \in \mathcal{T}$.

Assume that N' is an almost perfect and connective A-module such that the homotopy groups $\pi_i(N')$ are all $\mathfrak{m}$-power torsion. By induction, the above properties of $\mathcal{T}$ show that all the truncations $\tau_{\leq n} N'$ all belong to $\mathcal{T}$. Passage to the limit as $n \to \infty$ now shows that $N' \in \mathcal{T}$ too.

For instance, if $x_1, \ldots, x_r \in \pi_0(A)$ generate the maximal ideal, then if N is any connective, almost perfect A-module, we conclude that the iterated cofibre $N/(x_1, \ldots, x_r)$ belongs to $\mathcal{T}$. It follows that $\mathrm{Tot}(N \otimes_A M^\bullet)/(x_1, \ldots, x_r)$ is a connective A-module. Since each $N \otimes_A M^i$ is complete, it follows that $\mathrm{Tot}(N \otimes_A M^\bullet)$ is a complete A-module. Thus, it also follows that $\mathrm{Tot}(N \otimes_A M^\bullet)$ is connective itself. Therefore, we have shown that $\mathcal{T}$ contains every connective, almost perfect A-module. In particular, taking $N = A$ verifies part (1) of the theorem.

We shall now verify part (2). The claim is clearly true in the case where N is perfect. Suppose that N is an arbitrary almost perfect A-module, and assume without restriction that N is also connective. Both domain and target of the map $N \otimes_A \mathrm{Tot}(M^\bullet) \to \mathrm{Tot}(N \otimes_A M^\bullet)$ are connective (since $N \in \mathcal{T}$ by the previous paragraph). Fix $n > 0$. We can find a perfect A-module P and a map $P \to N$ which induces an equivalence on n-truncations, so we obtain a cofibre sequence $P \to N \to N'$ where $N' \in \mathrm{Mod}_{A, \geq n+1}$. Since $N'[-n-1] \in \mathcal{T}$, it follows that $\mathrm{Tot}(N' \otimes_A M^\bullet) \in \mathrm{Mod}_{A, \geq n+1}$. In particular, in the commutative square

$$\begin{tikzcd} P \otimes_A \mathrm{Tot}(M^\bullet) \arrow[r] \arrow[d] & \mathrm{Tot}(P \otimes_A M^\bullet) \arrow[d] \\ N \otimes_A \mathrm{Tot}(M^\bullet) \arrow[r] & \mathrm{Tot}(N \otimes_A M^\bullet) \end{tikzcd},$$

the vertical maps $P \otimes_A \mathrm{Tot}(M^\bullet) \to N \otimes_A \mathrm{Tot}(M^\bullet)$ and $\mathrm{Tot}(P \otimes_A M^\bullet) \to \mathrm{Tot}(N \otimes_A M^\bullet)$ are equivalences on n-truncations. Since the top horizontal map is an equivalence, it follows that the bottom horizontal map is an equivalence on n-truncations. Since n was arbitrary, we conclude that the bottom horizontal map is an equivalence, which implies (2). $\square$

Next, we show that for connective complete A-modules, the Adams spectral sequence converges. More precisely, consider the Čech nerve of $A \to k$, i.e. the augmented cosimplicial diagram

$$A \longrightarrow (\ k \ \rightrightarrows \ k \otimes_A k \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ \ldots \ ) \tag{11}$$

in the ∞-category of $\mathbb{E}_\infty$-rings. We then have:

Proposition 6.6. *As before, let A be a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. If M is a connective and complete A-module, then the diagram $M \xrightarrow{\simeq} \mathrm{Tot}(\ M \otimes_A k \rightrightarrows M \otimes_A k \otimes_A k \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ \ldots\ )$ (obtained by tensoring (11) with M) is a limit diagram.*

Proof. After base-change along the map $A \to k$, the cosimplicial diagram

$$k \rightrightarrows k \otimes_A k \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ \ldots \tag{12}$$

admits a splitting, since it becomes the Čech nerve of the map $k \to k \otimes_A k$, which has a section. Proposition 6.5 therefore applies to diagram (12), which implies that the totalisation is connective and commutes with base-change with any almost perfect A-module. From this, we deduce that $M \to \mathrm{Tot}\,(M \otimes_A k \rightrightarrows M \otimes_A k \otimes_A k \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ldots)$ is a map of connective, complete A-modules which becomes an equivalence after applying $k \otimes_A -$. Thus, it is an equivalence by Nakayama's lemma. $\square$

As an application of Proposition 6.6, we observe the following descent theorem for complete connective modules. Note that descent for all modules in the faithfully flat case appears in [Lur16, Section D.6.3] and in the proper surjective case in [Lur16, Theorem 5.6.6.1]. We thank Bhargav Bhatt for indicating the following result to us.

Theorem 6.7. *Let A be a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. Writing $\mathrm{Mod}^{\mathrm{cpl}}_{A,\geq 0}$ for the full subcategory of all complete connective A-modules (cf. Definition 6.4), the natural map*

$$\mathrm{Mod}^{\mathrm{cpl}}_{A,\geq 0} \xrightarrow{\simeq} \mathrm{Tot}\,(\,\mathrm{Mod}_{k,\geq 0} \rightrightarrows \mathrm{Mod}_{k\otimes_A k,\geq 0} \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \dots\,)$$

is an equivalence of ∞-categories.

Proof. It suffices to show that the functor $k\otimes_A - : \mathrm{Mod}^{\mathrm{cpl}}_{A,\geq 0} \to \mathrm{Mod}_{k,\geq 0}$ is comonadic as in [Lur16, Lemma D.3.5.7]. First, we observe that $k\otimes_A -$ is conservative on connective and complete A-modules by Nakayama's lemma. Next, let $M^\bullet$ be an object of $\mathrm{Mod}^{\mathrm{cpl}}_{A,\geq 0}$ such that the cosimplicial k-module $k\otimes_A M^\bullet$ admits a splitting. It follows that $\mathrm{Tot}(k\otimes_A M^\bullet)$ (computed in k-modules) is connective. Thus, by Proposition 6.5, $\mathrm{Tot}(M^\bullet)$ is connective and $k\otimes_A \mathrm{Tot}(M^\bullet) \to \mathrm{Tot}(k\otimes_A M^\bullet)$ is an equivalence. This verifies the hypotheses of the comonadicity theorem. □

Notation 6.8. Let APerf_R be the full subcategory of Mod_R spanned by almost perfect R-modules.

As a consequence of Theorem 6.7, we observe also that almost perfect modules satisfy descent, cf. [HLP14, Sec. 4] for closely related results:

Corollary 6.9 (Descent for almost perfect modules)**.** *Let A be a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. Then the diagram*

$$\mathrm{APerf}_A \xrightarrow{\simeq} \mathrm{Tot}\,(\,\mathrm{APerf}_k \rightrightarrows \mathrm{APerf}_{k\otimes_A k} \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \dots\,)$$

is an equivalence of symmetric monoidal ∞-categories. This remains true when we replace APerf *by the corresponding ∞-categories* Perf *of perfect modules.*

Proof. For the first claim, by Theorem 6.7, it suffices to check that if M is any connective complete A-module with $k\otimes_A M \in \mathrm{APerf}_k$, then M belongs to APerf_A, which means that M has finitely generated homotopy groups. We show inductively that the homotopy groups of M are finitely generated.

Indeed, if M is such a module, then $\pi_0(k\otimes_A M) \simeq k\otimes_{\pi_0(A)} M$ is finitely generated. Choose a map $A^r \to M$ which induces a surjection on $\pi_0(k\otimes_A \cdot)$ and let C be the cofibre. Then [Sta19, Lemma 09B9] implies that $\pi_0(A^r) \to \pi_0(M)$ is surjective, and $\pi_0(C) = 0$. Therefore, $\pi_0(M)$ is finitely generated.

Now assume $n > 0$, and that for any connective, complete module M with $k\otimes_A M \in \mathrm{APerf}_k$, we have that $\pi_i(M)$ is finitely generated for $i < n$. We will show additionally that $\pi_n(M)$ is finitely generated, which will establish the claim by induction. Choose the map $A^r \to M$ as above. The inductive hypothesis shows that $\mathrm{fib}(A^r \to M)$ has finitely generated homotopy groups $\pi_i, i < n$. The long exact sequence now shows that $\pi_n(M)$ is finitely generated.

Finally, for the second claim, we observe that perfect modules are characterised as the dualisable objects in APerf, so the second claim follows from the first. □

6.2. **Constructing a deformation theory.** Let A be a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. Write $\mathrm{CAlg}_{A//k,\geq 0}$ for the ∞-category of connective $\mathbb{E}_\infty$-A-algebras B equipped with a map $B \to k$. In this subsection, we use this data to construct a deformation theory in the sense of Lurie (cf. [Lur16, Definition 12.3.3.2]). When $A = k$, this was done in Section 5.1 above, and we will now indicate the necessary modifications.

First, we consider the adjunction

$$\mathrm{cot}_A : \mathrm{CAlg}_{A//k,\geq 0} \rightleftarrows \mathrm{Mod}_{k,\geq 0} : \mathrm{sqz} \tag{13}$$

where:

a) the left adjoint cot_A sends $B \in \mathrm{CAlg}_{A//k,\geq 0}$ to $\mathrm{cot}_A(B) := k \otimes_B L_{B/A}$.
b) the right adjoint sqz sends $V \in \mathrm{Mod}_{k,\geq 0}$ to the object $k \oplus V$, considered as a trivial square-zero k-algebra and equipped with an A-algebra structure via $A \to k$.

Remark 6.10. As $\mathrm{CAlg}_{A//k,\geq 0}$ is not pointed when $A \neq k$, the mixed context does not quite fit into the framework of Section 4. However, it will be possible to deduce all results from the case $A = k$. This is possible because the adjunction (13) above is the composite of (4) with the adjunction

$$\mathrm{CAlg}_{A//k,\geq 0} \rightleftarrows \mathrm{CAlg}_{k//k,\geq 0}$$

given by base-change and forgetting. In particular, observe that for any $B \in \mathrm{CAlg}_{A//k,\geq 0}$, we have

$$\mathrm{cot}_A(B) \simeq \mathrm{cot}(k \otimes_A B). \tag{14}$$

Notation 6.11. Let $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ denote the full subcategory of $\mathrm{CAlg}_{A//k,\geq 0}$ spanned by those objects B which are complete local Noetherian, i.e. such that B is Noetherian (cf. Definition 5.9) and such that $\pi_0(A)$ is a complete local ring. We use similar notation $\mathrm{SCR}^{\mathrm{cN}}_{A//k}$ when A is a simplicial commutative ring which is complete local Noetherian with residue field k.

Example 6.12. The completion $A\widehat{\{x_1, \ldots, x_n\}}$ of a free $\mathbb{E}_\infty$-algebra in variables $x_1, \ldots, x_n$ over A is an object of $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$. Indeed, these are the free objects of $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$: if $B \in \mathrm{CAlg}^{\mathrm{cN}}_{A//k}$, then $\mathrm{Hom}_{\mathrm{CAlg}^{\mathrm{cN}}_{A//k}}(A\widehat{\{x_1, \ldots, x_n\}}, B) \simeq \Omega^\infty \mathfrak{m}_B^n$, where $\mathfrak{m}_B = \mathrm{fib}(B \to k)$ is the augmentation ideal of B.

We begin by observing that any object of $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ can be written as a geometric realisation of such completed-free objects; while this will not be used in the sequel. For convenience, we state the result as well for simplicial commutative rings.

Theorem 6.13. *Let A be a complete local Noetherian $\mathbb{E}_\infty$-algebra (resp. simplicial commutative ring) augmented over k. Then any object of $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ (resp. $\mathrm{SCR}^{\mathrm{cN}}_{A//k}$) can be expressed as the geometric realisation of a simplicial object $X_\bullet$ in $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ (resp. $\mathrm{SCR}^{\mathrm{cN}}_{A//k}$) where each X_i is the formal completion of a free algebra over A on finitely many variables in degree zero.*

Proof. We give the proof for $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$; the simplicial commutative ring case is similar. Here we use the notation and language of Lemma 8.6 below. We take $\mathcal{C} = \mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ and S to be the class of maps $B \to B'$ which induce surjections on π_0. Note that coproducts in $\mathcal{C}$ are given by completed tensor products. Similarly, we take $\mathcal{F}$ to be the class of objects in $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ which are free on a finite set of generators in degree zero. These play the role of free objects in $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ as in Example 6.12, so they have the lifting property with respect to S. Thus, we can apply Lemma 8.6 to produce an $(\mathcal{F}, S)$-hypercover $X_\bullet$ in $\mathcal{C}$ as desired. This is necessarily a colimit diagram, since one can check this after applying Ω^∞, and hypercovers are colimit diagrams in the ∞-category $\mathcal{S}$ [Lur09, Lemma 6.5.3.11]. $\square$

If $B \in \mathrm{CAlg}^{\mathrm{cN}}_{A//k}$, then the augmented $\mathbb{E}_\infty$-k-algebra $k \otimes_A B$ is complete local Noetherian. It therefore has an almost perfect cotangent fibre by Proposition 5.11, which means that $\cot_A(B)$ is almost perfect. We can therefore restrict 13 to obtain an adjunction

$$\cot_A : \mathrm{CAlg}^{\mathrm{cN}}_{A//k} \rightleftarrows \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} : \mathrm{sqz}. \tag{15}$$

with associated comonad $T_A : \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} \to \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}$. Pre- and postcomposing with linear duality as before, we obtain a monad $\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty} : \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0} \to \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ satisfying $\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}(V) = \cot_A(\mathrm{sqz}(V^\vee))^\vee$.

Example 6.14. Given a complete local Noetherian algebra $B \in \mathrm{CAlg}^{\mathrm{cN}}_{A//k}$, we set $\cot(B) = L_{k/B}[-1]$. Note that if B arises from an augmented k-algebra by restriction of scalars along $A \to k$, then $\cot(B)$ agrees with the cotangent fibre considered before. The natural pushout square in $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ given by

$$\begin{tikzcd} B \arrow[r] \arrow[d] & k \otimes_A B \arrow[d] \\ k \arrow[r] & k \otimes_A k \end{tikzcd}$$

induces a basic cofibre sequence

$$\cot(B) \to \cot_A(B) \to \cot(k \otimes_A k).$$

Taking $B = k \oplus V$, we obtain a cofibre sequence

$$\cot(k \otimes_A k)^\vee \to \mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}(V) \to \mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}(V), \quad V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0}. \tag{16}$$

We can use this to establish a relative version of Theorem 4.20 in the context of $\mathbb{E}_\infty$-rings:

Theorem 6.15. *Let A be a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. Then:*

(1) The adjunction (15) *is comonadic.*

(2) The monad $\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty} : \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0} \to \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ from above extends to a sifted-colimit-preserving monad $\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}$ on Mod_k.

(3) The induced functor $\mathfrak{D}_A : (\mathrm{CAlg}^{\mathrm{cN}}_{A//k})^{op} \to \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}}$ carries pullbacks of diagrams $B \to B''$, $B' \to B''$ inducing surjections on π_0 to pushouts of $\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}$-algebras.

Proof. For $A = k$, we have verified the claim in Proposition 5.31. We will now reduce to this case.

For part (1), we verify the hypotheses of the comonadicity theorem (cf. Theorem 4.5 above). First, we observe that $\cot_A$ is conservative. Indeed, $\cot_A$ is the composite of the base-change functor $\mathrm{CAlg}^{\mathrm{cN}}_{A//k} \to \mathrm{CAlg}^{\mathrm{cN}}_{k//k}$ with the cotangent fibre functor $\cot : \mathrm{CAlg}^{\mathrm{cN}}_{k//k} \to \mathrm{Mod}_k$, both of which are conservative by [Sta19, Lemma 09B9] and Proposition 5.22.

Let $B^\bullet$ be a cosimplicial object in $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$ such that $\cot_A(B^\bullet)$ admits a splitting. Using the equivalence $\cot_A(B^\bullet) \simeq \cot(k \otimes_A B^\bullet)$, we conclude that $k \otimes_A B^\bullet$ defines a cosimplicial object of $\mathrm{CAlg}^{\mathrm{cN}}_{k//k}$ such that $\cot(k \otimes_A B^\bullet)$ is split. By the comonadicity already proved when $A = k$, it follows that $\mathrm{Tot}(k \otimes_A B^\bullet)$ is complete local Noetherian), and the natural map

$$\cot(\mathrm{Tot}(k \otimes_A B^\bullet)) \xrightarrow{\simeq} \mathrm{Tot}(\cot(k \otimes_A B^\bullet))$$

is an equivalence. Using Proposition 6.5, we can conclude that $\mathrm{Tot}(B^\bullet)$ is connective and that $k \otimes_A \mathrm{Tot}(B^\bullet) \xrightarrow{\simeq} \mathrm{Tot}(k \otimes_A B^\bullet)$ is an equivalence. Therefore, $\mathrm{Tot}(B^\bullet)$ is also complete local Noetherian and $\cot_A(\mathrm{Tot}(B^\bullet)) \xrightarrow{\simeq} \mathrm{Tot}(\cot_A(B^\bullet))$ is an equivalence.

Part (2) follows from the case $A = k$ and the cofibre sequence (16) constructed above.

Finally, part (3) will be proved in Proposition 6.18 below. □

We can use Theorem 6.15 to generalise Definition 5.32 to the mixed setting:

Definition 6.16 (Mixed spectral partition Lie algebras)**.** Given a complete local Noetherian $\mathbb{E}_\infty$-ring A with residue field k, an (A,k)-*spectral partition Lie algebra* is an algebra over $\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}$.

Construction 6.17 (The forgetful functor from $\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}$-algebras to $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$-algebras)**.** The base-change functor $\mathrm{CAlg}^{\mathrm{cN}}_{A//k} \to \mathrm{CAlg}^{\mathrm{cN}}_{k//k}$ given by $B \mapsto k \otimes_A B$ induces, by using the anti-equivalences established in Theorem 6.15, a functor $U : \mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}) \longrightarrow \mathrm{Alg}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})$. By construction, U preserves the forgetful functors to $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. We extend this to a functor

$$U : \mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}} \to \mathrm{Alg}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}}.$$

which commutes with the forgetful functor to Mod_k. For this, we simply left Kan extend from free $\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}$-algebras on objects of $\mathrm{Perf}_{k,\leq 0}$ (using Example 8.12).

Proposition 6.18. *The forgetful functor* $U : \mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}} \to \mathrm{Alg}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}}$ *from Construction 6.17 commutes with pushouts.*

Proof. Suppose that we are given maps $V'' \to V$ and $V'' \to V'$ with $V, V', V'' \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. Consider the $\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}$-algebras $\mathfrak{g}'' = \mathrm{free}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}}(V'')$, $\mathfrak{g} = \mathrm{free}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}}(V)$, and $\mathfrak{g}' = \mathrm{free}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}}(V')$. The diagram

$$\begin{tikzcd}
U(\mathrm{free}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}A}(V'')) \arrow[r] \arrow[d] & U(\mathrm{free}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}A}(V)) \arrow[d] \\
U(\mathrm{free}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}A}(V')) \arrow[r] & U(\mathrm{free}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}A}(V \sqcup_{V''} V'))
\end{tikzcd}$$

is a pushout of $\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}$-algebras, as it is by construction equivalent to

$$\begin{tikzcd}
\mathfrak{D}(k \otimes_A (k \oplus V''^\vee)) \arrow[r] \arrow[d] & \mathfrak{D}(k \otimes_A (k \oplus V^\vee)) \arrow[d] \\
\mathfrak{D}(k \otimes_A (k \oplus V'^\vee)) \arrow[r] & \mathfrak{D}(k \otimes_A (k \oplus (V \sqcup_{V''} V')^\vee))
\end{tikzcd},$$

which is a pushout by Theorem 4.20 (applied to the case of augmented $\mathbb{E}_\infty$-k-algebras).

Since we can write any object in $\mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}}$ as a sifted colimit of free algebras on objects of $\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$, the claim follows as U preserves sifted colimits. □

We now shift attention to the context of simplicial commutative rings, where we fix a complete local Noetherian simplicial commutative ring A with residue field k. Since the arguments will be precisely analogous to the ones given in the previous paragraphs, we will simply state the results.

Definition 6.19. Let $\mathrm{SCR}_{A//k}$ be the ∞-category of simplicial commutative A-algebras B with a map to k factoring the augmentation $A \to k$. Write $\mathrm{SCR}^{\mathrm{cN}}_{A//k}$ for the full subcategory of $\mathrm{SCR}_{A//k}$ spanned by all complete local Noetherian objects.

Construction 6.20. Consider the adjunction $\cot_{A,\Delta} : \mathrm{SCR}_{A//k} \rightleftarrows \mathrm{Mod}_{k,\geq 0} : \mathrm{sqz}$, whose the right adjoint is the square-zero functor given by $V \mapsto k \oplus V$. This induces a comonad on $\mathrm{Mod}_{k,\geq 0}$ and, by dualisation, a monad $\mathrm{Lie}^\pi_{A,\Delta} : \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0} \to \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ satisfying $\mathrm{Lie}^\pi_{A,\Delta}(V) = \cot_{A,\Delta}(\mathrm{sqz}(V^\vee))^\vee$.

Theorem 6.21. *Let A be a complete local Noetherian simplicial commutative ring with residue field k. Then:*

1. *The adjunction* $\cot_{A,\Delta} : \mathrm{SCR}^{\mathrm{cN}}_{A//k} \rightleftarrows \mathrm{Mod}^{\mathrm{ft}}_{k,\geq 0} : \mathrm{sqz}$ *is comonadic.*
2. *The induced monad* $\mathrm{Lie}^{\pi}_{A,\Delta} : \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0} \to \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$ *extends to a sifted-colimit-preserving monad on* Mod_k*.*
3. *The induced functor* $\mathfrak{D}_{A,\Delta} : \mathrm{SCR}^{\mathrm{cN}}_{A//k} \to \mathrm{Alg}^{op}_{\mathrm{Lie}^{\pi}_{A,\Delta}}$ *carries pullbacks of diagrams* $B \to B''$, $B' \to B''$ *inducing surjections on* π_0 *to pushouts of* $\mathrm{Lie}^{\pi}_{A,\Delta}$*-algebras.*

We can therefore generalise Definition 5.47 as follows:

Definition 6.22 (Mixed partition Lie algebras)**.** Given a complete local Noetherian simplicial commutative ring A with residue field k, an (A,k)*-partition Lie algebra* is an algebra over $\mathrm{Lie}^{\pi}_{A,\Delta}$.

6.3. **Formal moduli problems in mixed characteristic.** Finally, we can prove that formal moduli problems in mixed characteristic are governed by (possibly spectral) partition Lie algebras. For this, we fix a complete local Noetherian $\mathbb{E}_\infty$-ring (respectively simplicial commutative ring) A with residue field k. We define a version of the Chevalley-Eilenberg cochains functor in this context:

Construction 6.23 (The adjunction $(\mathfrak{D}_A, C^*_A)$)**.** The colimit-preserving functor

$$\mathfrak{D}_A : \mathrm{CAlg}_{A//k} \to \mathrm{Alg}^{op}_{\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}}$$

defined by $\mathfrak{D}_A(B) := \cot_A(B)^\vee$ admits a right adjoint $C^*_A : \mathrm{Alg}^{op}_{\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}} \to \mathrm{CAlg}_{A//k}$.

Similarly, if A is a simplicial commutative ring, the cocontinuous functor

$$\mathfrak{D}_{A,\Delta} : \mathrm{SCR}_{A//k} \to \mathrm{Alg}^{op}_{\mathrm{Lie}^{\pi}_{A,\Delta}}$$

defined by $\mathfrak{D}_{A,\Delta}(B) := \cot_{A,\Delta}(B)^\vee$ admits a right adjoint $C^*_{A,\Delta} : \mathrm{Alg}^{op}_{\mathrm{Lie}^{\pi}_{A,\Delta}} \to \mathrm{SCR}_{A//k}$.

Theorem 6.24. *Under the above assumptions, we have:*

1. *The adjunction* $(\mathfrak{D}_A, C^*_A)$ *restricts to an equivalence* $\mathrm{CAlg}^{\mathrm{cN}}_{A//k} \simeq \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})^{op}$.
2. *If A is additionally a simplicial commutative ring, the adjunction* $(\mathfrak{D}_{A,\Delta}, C^*_{\Delta,A})$ *restricts to an equivalence* $\mathrm{SCR}^{\mathrm{cN}}_{A//k} \simeq \mathrm{Alg}_{\mathrm{Lie}^{\pi}_{A,\Delta}}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0})^{op}$.

Proof. In both cases we will follow the argument in Proposition 4.52. We will only prove (1) in detail; assertion (2) can be established by a parallel argument.

The claim essentially follows from the comonadicity established in Theorem 6.15. To show that $\mathfrak{D}_A$ is fully faithful on $\mathrm{CAlg}^{\mathrm{cN}}_{A//k}$, it suffices to check that for any $B \in \mathrm{CAlg}^{\mathrm{cN}}_{A//k}$, the unit

$$B \to C^*_A(\mathfrak{D}_A(B)) \tag{17}$$

is an equivalence. We first observe that by construction, we have $C^*_A(\mathrm{free}_{\mathrm{Lie}^{\pi}_{A,\mathbb{E}_\infty}}(V)) = \mathrm{sqz}(V^\vee)$ for all $V \in \mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}$. Thus, the map (17) is an equivalence for $B = \mathrm{sqz}(V)$. Using the comonadicity claim in Theorem 6.15, we can write any B as a totalisation of a cosimplicial diagram of square-zero extensions via the cobar resolution. It follows that (17) is an equivalence in general. This shows that $\mathfrak{D}_A$ is fully faithful; since it is also essentially surjective by Theorem 6.15(1), it follows that $\mathfrak{D}_A$ is an equivalence. □

We can now prove that one obtains a deformation theory in the sense of Lurie (cf. [Lur16, Definition 12.3.3.2]) for simplicial commutative rings and $\mathbb{E}_\infty$-rings with respect to a complete base:

Theorem 6.25.

(1) *Let A be a complete local Noetherian $\mathbb{E}_\infty$-ring with residue field k. The ∞-category $\mathrm{CAlg}_{A//k}$, the infinite loop object $\{\mathrm{sqz}(k[n]) \in \mathrm{Stab}(\mathrm{CAlg}_{A//k})\}_{n\geq 0}$, the adjunction $(\mathfrak{D}_A, C^*_A)$, and the subcategory $\mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}) \subset \mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}}$ form a deformation theory.*

(2) *Let A be a complete local Noetherian simplicial commutative ring, The ∞-category $\mathrm{SCR}_{A//k}$, the infinite loop object $\{\mathrm{sqz}(k[n]) \in \mathrm{Stab}(\mathrm{SCR}_{A//k})\}_{n\geq 0}$, the adjunction $(\mathfrak{D}_{A,\Delta}, C^*_{A,\Delta})$, and the subcategory $\mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\Delta}}(\mathrm{Mod}^{\mathrm{ft}}_{k,\leq 0}) \subset \mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\Delta}}$ form a deformation theory.*

Proof. Combine Theorem 6.24 and Theorem 6.15 (or Theorem 6.21). □

Using the argument of [Lur16, Proposition 12.1.2.9], it follows that the Artinian objects of $\mathrm{CAlg}_{A//k}$ (respectively $\mathrm{SCR}_{A//k}$) from Definition 6.1 are exactly the ones which are Artinian in the axiomatic deformation theory setup of [Lur16, Definition 12.1.2.4]. In order words, they are those which can be built from a point by taking iterated fibres of maps to square-zero extensions $\mathrm{sqz}(k[n])$ with $n > 0$. Arguing as in [Lur, Proposition 6.1.4], we see that a morphism between two Artinian objects is small in the axiomatic sense of [Lur16, Definition 12.1.2.4] if and only if it is surjective on π_0. This allows us to conclude that Definition 6.2 agrees with the axiomatic notion of a formal moduli problem attached to the above deformation problem (cf. [Lur16, Definition 12.1.3.1, Proposition 12.1.3.2(3)]).

Construction 6.26 (The tangent complex)**.** Given a formal moduli problem X, we can construct its tangent complex $T_X \in \mathrm{Mod}_k$ (cf. [Lur16, Definition 12.2.2.1]); its underlying spectrum satisfies $\Omega^{\infty-n}T_X = X(\mathrm{sqz}(k[n]))$ for all $n \geq 0$.

Combining Theorem 6.25 with Lurie's axiomatic [Lur16, Theorem 12.3.3.5], we can finally deduce:

Theorem 6.27.

(1) *Let A be a complete local Noetherian $\mathbb{E}_\infty$-ring. There is an equivalence of ∞-categories $\mathrm{Moduli}_{A//k,\mathbb{E}_\infty} \simeq \mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\mathbb{E}_\infty}}$.*

(2) *Let A be a complete local Noetherian simplicial commutative ring. There is an equivalence of ∞-categories $\mathrm{Moduli}_{A//k,\Delta} \simeq \mathrm{Alg}_{\mathrm{Lie}^\pi_{A,\Delta}}$.*

On underlying objects in Mod_k, both equivalences send a formal moduli problem $X \in \mathrm{Moduli}_{A//k}$ to its tangent complex T_X.

7. The homology of partition Lie algebras

Away from characteristic zero, partition Lie algebras display additional subtleties:

Example 7.1. For $A \in \mathrm{SCR}^{\mathrm{aug}}_{\mathbb{F}_p}$ complete local Noetherian, the Frobenius ($x \mapsto x^p$) on A induces an endomorphism ϕ on the partition Lie algebra $\cot(A)^\vee$. While ϕ is zero as a map of $\mathbb{F}_p$-modules (as $px^{p-1} = 0$), Theorem 6.24 shows that ϕ is generally nonzero as a map of partition Lie algebras.

To get a better handle on our Lie algebras, we may wish to consider Dyer-Lashof-like operations on their homotopy groups. These are parametrised by the homotopy groups of free Lie algebras:

Construction 7.2. Given a class $\alpha \in \pi_j\left(\mathrm{Lie}^\pi_{k,\Delta}(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_n}\mathbb{F}_p)\right)$, we define a universal n-ary operation acting on the homotopy groups of any partition Lie algebra $\mathfrak{g}$. For this, we send a tuple $(x_1 \in \pi_{\ell_1}(\mathfrak{g}), \ldots, x_n \in \pi_{\ell_n}(\mathfrak{g}))$ to the element $\alpha(x_1, \ldots, x_n) \in \pi_j(\mathfrak{g})$ represented by

$$\Sigma^j\mathbb{F}_p \xrightarrow{\alpha} \mathrm{Lie}^\pi_{k,\Delta}(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_n}\mathbb{F}_p) \xrightarrow{\mathrm{Lie}^\pi_{k,\Delta}(x_1,\ldots,x_n)} \mathrm{Lie}^\pi_{k,\Delta}(\mathfrak{g}) \to \mathfrak{g}.$$

There is a similar construction for spectral partition Lie algebras.

We will now compute the homotopy groups of free (possibly spectral) partition Lie algebras over $\mathbb{F}_p$. Write $B(n_1, \ldots, n_m)$ for the set of Lyndon words in m letters involving the i^{th} letter n_i times (cf. Definition 1.14). Given integers $\ell_1, \ldots, \ell_m$, we have the following results:

Theorem 7.3. *The $\mathbb{F}_p$-vector space $\pi_*(\mathrm{Lie}^\pi_{k,\Delta}(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_m}\mathbb{F}_p))$ has a basis indexed by sequences $(i_1, \ldots, i_k, e, w)$. Here $w \in B(n_1, \ldots, n_m)$ is a Lyndon word. We have $e \in \{0, \epsilon\}$, where $\epsilon = 1$ if p is odd and $\deg(w) := \sum_i(\ell_i - 1)n_i + 1$ is even. Otherwise, $\epsilon = 0$. The integers $i_1, \ldots, i_k$ satisfy:*

(1) Each $|i_j|$ is congruent to 0 or 1 modulo $2(p-1)$.
(2) For all $1 \leq j < k$, we have $pi_{j+1} < i_j < -1$ or $0 \leq i_j < pi_{j+1}$
(3) We have $(p-1)(1+e)\deg(w) - \epsilon \leq i_k < -1$ or $0 \leq i_k \leq (p-1)(1+e)\deg(w) - \epsilon$.

The sequence $(i_1, \ldots, i_k, e, w)$ sits in homological degree $((1+e)\deg(w) - e) + i_1 + \cdots + i_k - k$ and multi-weight $(n_1p^k(1+e), \ldots, n_mp^k(1+e))$.

Theorem 7.4. *The $\mathbb{F}_p$-vector space $\pi_*(\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_m}\mathbb{F}_p))$ has a basis indexed by sequences $(i_1, \ldots, i_k, e, w)$. Here $w \in B(n_1, \ldots, n_m)$ is a Lyndon word. We have $e \in \{0, \epsilon\}$, where $\epsilon = 1$ if p is odd and $\deg(w) := \sum_i(\ell_i - 1)n_i + 1$ is even. Otherwise, $\epsilon = 0$. The integers $i_1, \ldots, i_k$ satisfy:*

(1)' Each i_j is congruent to 0 or 1 modulo $2(p-1)$.
(2)' For all $1 \leq j < k$, we have $i_j < pi_{j+1}$.
(3)' We have $i_k \leq (p-1)(1+e)\deg(w) - \epsilon$.

The homological degree of $(i_1, \ldots, i_k, e, w)$ is $((1+e)\deg(w) - e) + i_1 + \cdots + i_k - k$ and its multi-weight is $(n_1p^k(1+e), \ldots, n_mp^k(1+e))$.

Our strategy will closely follow the proof of [AB18, Theorem 8.14], which essentially computes the homotopy groups of free coconnective partition Lie algebras (for $p = 2$, [AB18, Theorem 8.14] also follows from [Goe90]). Our computation relies on many classical ingredients and insights, which we will reference in detail below. Broadly speaking, we will proceed in three steps:

(1) First, we compute the homotopy groups of a free Lie algebra on an odd class. We use a bar spectral sequence and the known homotopy groups of symmetric or extended powers.
(2) In a second step, we express the homotopy groups of a free Lie algebra on an even class in terms of the odd case (1). We rely on the Takayasu cofibration sequence and its strict cousin.
(3) Finally, we give a Hilton-Milnor decomposition for free Lie algebras on many generators, thereby reducing the computation of their homotopy groups to the cases (1) and (2). We rely on a certain splitting of the restriction of partition complexes to Young subgroups.

7.1. **Free Partition Lie Algebras on an Odd Generator.** The principal aim of this subsection is to compute the homotopy groups of free Lie algebras on a single odd class. We will establish the following results:

Theorem 7.5. *Let ℓ be an integer, assumed to be odd if p is. Then $\mathrm{Lie}^{\pi}_{k,\Delta}(\Sigma^{\ell}\mathbb{F}_p)$ has a basis given by all sequences $(i_1, i_2, \ldots i_k)$ satisfying the following conditions:*

(1) Each $|i_j|$ is congruent to 0 or 1 modulo $2(p-1)$.
(2) For all $1 \leq j < k$ we have $pi_{j+1} < i_j < -1$ or $0 \leq i_j < pi_{j+1}$.
(3) We have $(p-1)\ell \leq i_k < -1$ or $0 \leq i_k \leq (p-1)\ell$.

The sequence $(i_1, i_2 \ldots i_k)$ lies in homological degree $\ell + i_1 + i_2 - \ldots + i_k - k$ and weight p^k.

Theorem 7.6. *Let ℓ be an integer, assumed to be odd if p is. Then $\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}(\Sigma^{\ell}\mathbb{F}_p)$ has a basis given by all sequences $(i_1, i_2, \ldots i_k)$ satisfying the following conditions:*

(1)' Each i_j is congruent to 0 or 1 modulo $2(p-1)$.
(2)' For all $1 \leq j < k$ we have $i_j < pi_{j+1}$.
(3)' We have $i_k \leq (p-1)\ell$.

The sequence $(i_1, i_2 \ldots i_k)$ lies in homological degree $\ell + i_1 + i_2 - \ldots + i_k - k$ and weight p^k.

We carry out these two parallel computations "weight-by-weight" by generalising the argument provided in [AB18, Section 9], which is inspired by [AM99, Section 3]. We outline the main steps:

a) By duality, it suffices to compute the homotopy groups $\pi_*(F_{\Sigma|\Pi_n|^\diamond}(\Sigma^{\ell}\mathbb{F}_p))$ and $\pi_*(F^h_{\Sigma|\Pi_n|^\diamond}(\Sigma^{\ell}\mathbb{F}_p))$. The functors $F_{\Sigma|\Pi_n|^\diamond}$ and $F^h_{\Sigma|\Pi_n|^\diamond}$ were constructed in Section 3.3.
b) In a second step, we show that whenever ℓ is odd or $p = 2$, the Bredon spectral sequences for $\pi_*(F_{\Sigma|\Pi_n|^\diamond}(\Sigma^{\ell}\mathbb{F}_p))$ and $\pi_*(F^h_{\Sigma|\Pi_n|^\diamond}(\Sigma^{\ell}\mathbb{F}_p))$ degenerate. This follows by applying a result of Arone, Dwyer, and Lesh (cf. [ADL16, Theorem 1.1]).
c) We then use the known homotopy of symmetric and extended powers to describe $\pi_*(F_{\Sigma_n/H_+}(\Sigma^{\ell}\mathbb{F}_p))$ and $\pi_*(F_{\Sigma_n/H_+}(\Sigma^{\ell}\mathbb{F}_p))$ for all subgroups $H \subset \Sigma_n$ arising as stabilisers of points in $|\Pi_n|^\diamond$.
d) This allows us to compute the above E^2-page by applying a combinatorial matching argument.

We will now provide the details of our computation.

Duality. Recall that given a genuine pointed Σ_n-space X, we have defined functors

$$F_X, F^h_X : \mathrm{Mod}_{\mathbb{F}_p} \to \mathrm{Mod}_{\mathbb{F}_p}$$

which extend the assignments $M \mapsto (\mathbb{F}_p[X] \otimes M^{\otimes n})_{\Sigma_n}$ and $M \mapsto (\mathbb{F}_p[X] \otimes M^{\otimes n})_{h\Sigma_n}$ from discrete $\mathbb{F}_p$-modules to all $\mathbb{F}_p$-modules in a sifted-colimit-preserving way (cf. Section 3.3).

Combining Proposition 5.35, Proposition 5.49, and Proposition 3.38, we see that in order to prove Theorem 7.5 and Theorem 7.6, it suffices to compute $\pi_*(F_{\Sigma|\Pi_n|^\diamond}(\Sigma^{\ell}\mathbb{F}_p))$ and $\pi_*(F^h_{\Sigma|\Pi_n|^\diamond}(\Sigma^{\ell}\mathbb{F}_p))$ for all n, where ℓ is odd or $p = 2$.

Degeneration of the Bredon Spectral Sequence. As explained in Section 3.3, the skeletal filtration on the pointed simplicial Σ_n-set $|\Pi_n|^\diamond$ gives rise to spectral sequences converging to $\pi_*(F_{|\Pi_n|^\diamond}(M))$ and $\pi_*(F^h_{|\Pi_n|^\diamond}(M))$. Their E^2-pages are given by the reduced Bredon homology groups of $|\Pi_n|^\diamond$ with respect to the graded Mackey functors

$$\mu^M_*, \mu^{M,h}_* : \mathcal{S}^{\Sigma_n}_* \to \mathrm{Mod}_{\mathbb{F}_p}$$

sending a discrete G-set X to $\pi_*(F_X(M))$ and $\pi_*(F^h_X(M))$, respectively.

To establish degeneration of the Bredon spectral sequence, we will apply Arone-Dwyer-Lesh's [ADL16, Theorem 1.1]. We begin by checking that the conditions of this theorem are satisfied:

Proposition 7.7. *For $M = \Sigma^\ell \mathbb{F}_p$ with ℓ odd or $p = 2$, the functors $\mu_*^M, \mu_*^{M,h}$ satisfy:*

(1) For any Sylow p-subgroup $P \subset \Sigma_n$, projection induces split epimorphisms

$$\mu_*^M(\Sigma_n/P \times -) \to \mu_*^M(-) \quad \text{and} \quad \mu_*^{M,h}(\Sigma_n/P \times -) \to \mu_*^{M,h}(-)\ .$$

(2) If $D \subset \Sigma_n$ is an elementary abelian p-subgroup acting freely and non-transitively, then $\ker\big(C_{\Sigma_n}(D) \to \pi_0 C_{\mathrm{GL}_n(\mathbb{R})}(D)\big)$ acts trivially on $\mu_*^M(\Sigma_n/D)$ and $\mu_*^{M,h}(\Sigma_n/D)$, respectively.

(3) If p is odd, then odd involution in $C_{\Sigma_n}(D)$ acts as (-1) on $\mu_*^M(\Sigma_n/D)$ and $\mu_*^{M,h}(\Sigma_n/D)$.

Proof. We follow [AB18, Proposition 9.3][ADL16, Example 11.5], which imply the result for $\ell \geq 0$.

If X is a finite pointed Σ_n-set, then the transformations of functors $\mathrm{Vect}_{\mathbb{F}_p}^\omega \to \mathrm{Mod}_{\mathbb{F}_p}$ given by

$$(\mathbb{F}_p[X] \otimes (-)^{\otimes n})_{\Sigma_n} \xrightarrow{(\mathrm{tr}\otimes \mathrm{id}^{\otimes n})_{\Sigma_n}} (\mathbb{F}_p[\Sigma_n/P \times X] \otimes (-)^{\otimes n})_{\Sigma_n} \longrightarrow (\mathbb{F}_p[X] \otimes (-)^{\otimes n})_{\Sigma_n}$$

$$(\mathbb{F}_p[X] \otimes (-)^{\otimes n})_{h\Sigma_n} \xrightarrow{(\mathrm{tr}\otimes \mathrm{id}^{\otimes n})_{h\Sigma_n}} (\mathbb{F}_p[\Sigma_n/P \times X] \otimes (-)^{\otimes n})_{h\Sigma_n} \longrightarrow (\mathbb{F}_p[X] \otimes (-)^{\otimes n})_{h\Sigma_n}$$

induce multiplication by $|\Sigma_n/P|$ on homotopy groups. They are therefore equivalences.

Taking right-left extensions of these degree n functors (cf. Theorem 3.26), we obtain transformations $F_X \to F_{\Sigma_n/P\times X}$ and $F_X^h \to F_{\Sigma_n/P\times X}^h$ such that the two composites $F_X \to F_{\Sigma_n/P\times X} \to F_X$ and $F^h_X \to F^h_{\Sigma_n/P\times X} \to F^h_X$ are equivalences, which clearly implies (1).

For (2), we begin with the diagram drawn on the left. Its rightmost arrow takes D-orbits or D-homotopy orbits, respectively. Freely adding sifted colimits, we obtain the diagram on the right.

$$\begin{tikzcd}
\mathrm{Set}_*^{\mathrm{Fin}} \arrow[r, "A\mapsto A^{\wedge n}"] \arrow[d, "A\mapsto \mathbb{F}_p[A]"'] & (\mathrm{Set}_*^{\mathrm{Fin}})^D \arrow[d, "A\mapsto \mathbb{F}_p[A]"] & \\
\mathrm{Vect}_{\mathbb{F}_p}^\omega \arrow[r, "V\mapsto V^{\otimes n}"] & (\mathrm{Vect}_{\mathbb{F}_p}^\omega)^D \arrow[r] & \mathrm{Mod}_{\mathbb{F}_p}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S}_* \arrow[r, "A\mapsto A^{\wedge n}"] \arrow[d, "A\mapsto \widetilde{C}_*(A)"'] & P_\Sigma((\mathrm{Set}_*^{\mathrm{Fin}})^D) \arrow[d, "A\mapsto \widetilde{C}_*(A)"] & \\
\mathrm{Mod}_{k,\geq 0} \arrow[r, "V\mapsto V^{\otimes n}"] & P_\Sigma((\mathrm{Vect}_{\mathbb{F}_p}^\omega)^D) \arrow[r] & \mathrm{Mod}_{\mathbb{F}_p}
\end{tikzcd}$$

The lower composite is equivalent to $F_{\Sigma_n/D}(-)$ or $F^h_{\Sigma_n/D}(-)$, respectively. Hence, the assignment $A \mapsto F_{\Sigma_n/D}(\widetilde{C}_*(A, \mathbb{F}_p))$ factors through the functor sending a space A to the genuine D-space $A^{\wedge n}$. Replacing $A \mapsto \mathbb{F}_p[A]$ by the functor $A \mapsto \mathbb{F}_p[A]^\vee$ and using Proposition 3.23, a similar argument shows that $A \mapsto F_{\Sigma_n/D}(\widetilde{C}^*(A, \mathbb{F}_p))$ factors through the functor sending A to the genuine D-space $A^{\wedge n}$.

We can write each $M = \Sigma^\ell \mathbb{F}_p$ as the singular chains or the singular cochains of a sphere $X = S^\ell$ (depending on whether ℓ is positive or negative). The above observations therefore show that in order to prove (2), it suffices to check that any $\sigma \in \ker(C_{\Sigma_n}(D) \to \pi_0 C_{\mathrm{GL}_n(R)}(D))$ acts on the genuine Σ_n-space $(S^\ell)^{\wedge n}$ by a map that is D-equivariantly homotopic to the identity. This is clear because any such σ lies in the connected component of the identity in $C_{\mathrm{GL}_n(R)}(D)$.

For (3), we first recall that if X is a spectrum with 2 invertible in $\pi_*(X)$, then $\tau : X \to X$ acts as (-1) on $\pi_*(X)$ if and only if $\tau - 1$ is an equivalence (cf. [ADL16, Proposition 11.4] and its proof). Observe that if p is an odd prime and $\tau \in C_{\Sigma_n}(D)$ is an odd permutation of order 2 centralising D, then τ acts by (-1) on $\widetilde{H}_*\left((S^{n\ell})^A, k\right)$ for any subgroup $A \subset D$. This implies that $\tau - 1$ induces a quasi-isomorphism on the $\mathbb{F}_p$-modules $\widetilde{C}_*((S^{n\ell})^A, \mathbb{F}_p)$ and $\widetilde{C}^*((S^{n\ell})^A, \mathbb{F}_p)$. Elmendorf's theorem expresses the D-space $S^{n\ell}$ as a homotopy colimit of D-spaces of the form $(D/B)_+ \wedge ((S^{n\ell})^A$. By the functoriality established in the proof of (2), we can therefore express $F_{\Sigma_n/D}(M)$ and $F^h_{\Sigma_n/D}(M)$ as homotopy colimits of $\mathbb{F}_p$-modules $\widetilde{C}_*((S^{n\ell})^A, \mathbb{F}_p)$ if $\ell > 0$ or of $\mathbb{F}_p$-modules $\widetilde{C}^*((S^{n\ell})^A, \mathbb{F}_p)$ if $\ell < 0$. Hence $\tau - 1$ acts as an equivalence on $F_{\Sigma/D}(M)$ and $F^h_{\Sigma/D}(M)$, which implies the thrid claim. $\square$

Hence, we can apply [ADL16, Theorem 1.1., Corollary 1.2] to conclude:

Corollary 7.8. *For $M = \Sigma^\ell \mathbb{F}_p$ with ℓ even or $p = 2$, the Bredon homology groups*

$$E^2_{s,t} = \mathrm{H}^{\mathrm{Br}}_s(|\Pi_n|^\diamond, \mu^M_t) \qquad E^{2,h}_{s,t} = \mathrm{H}^{\mathrm{Br}}_s(\Sigma|\Pi_n|^\diamond, \mu^{M,h}_t)$$

vanish unless $n = p^k$ for some k and $s = k-1$. In particular, the spectral sequence degenerates and

$$\pi_*(F_{\Sigma|\Pi_n|^\diamond}(M)) = \pi_{*-1}(F_{|\Pi_n|^\diamond}(M)) = \begin{cases} \widetilde{\mathrm{H}}^{\mathrm{Br}}_{k-1}(|\Pi_{p^k}|^\diamond; \mu^M_{*-k}) & \text{if } n = p^k \\ 0 & \text{else} \end{cases}$$

$$\pi_*(F^h_{\Sigma|\Pi_n|^\diamond}(M)) = \pi_{*-1}(F_{|\Pi_n|^\diamond}(M)) = \begin{cases} \widetilde{\mathrm{H}}^{\mathrm{Br}}_{k-1}(|\Pi_{p^k}|^\diamond; \mu^{M,h}_{*-k}) & \text{if } n = p^k \\ 0 & \text{else} \end{cases}.$$

Hence, it suffices to compute these Bredon homology groups to establish Theorems 7.5 and 7.6.

The Bredon Homology of Stabilisers. Next, we compute $\pi_*(F_{\Sigma_n/H}(M))$ and $\pi_*(F^h_{\Sigma_n/H}(M))$ for H the stabiliser of a point in the partition complex $|\Pi_n|$ and $M \in \mathrm{Mod}_{\mathbb{F}_p}$ any $\mathbb{F}_p$-module. This extends [AB18, Section 9.2], which is inspired by [AM99, Section 3], to the coconnective setting.

We need several auxiliary additive functors:

Definition 7.9. Given $k \geq 0$, the functor $\mathcal{F}_k$ sends a graded $\mathbb{F}_p$-vector space V to the graded $\mathbb{F}_p$-vector space $\mathcal{F}_k(V)$ generated by symbols $(i_1, ..., i_k; v)$, where v is a homogeneous element of V and $i_1, \ldots, i_k$ are integers satisfying the following conditions:

(1) Each $|i_j|$ is congruent to 0 or 1 modulo $2(p-1)$.
(2) $i_j \geq p i_{j+1} > p$ *or* $i_j \leq p i_{j+1} \leq 0$ for all $1 \leq j < k$,
(3) If p is odd, then $1 < i_1 < (p-1)(|v| + i_2 + \ldots + i_k)$ *or* $0 \geq i_1 > (p-1)(|v| + i_2 + \ldots + i_k)$.
If p is even, then $1 < i_1 \leq (p-1)(|v| + i_2 + \ldots + i_k)$ *or* $0 \geq i_1 \geq (p-1)(|v| + i_2 + \ldots + i_k)$.

We divide out by the relation $(i_1, ..., i_k; u) + (i_1, ..., i_k; v) = (i_1, ..., i_k; u + v)$. There is a homological grading on $\mathcal{F}_k(V)$, which puts $(i_1, \ldots, i_k; v)$ in degree $|v| + i_1 + \ldots + i_k$ whenever v is homogeneous of degree $|v|$. Moreover, there is a weight grading putting $(i_1, \ldots, i_k; v)$ in weight p^k.

Remark 7.10. Observe that either all i_j are strictly larger than 1 or all i_j are nonpositive.

Definition 7.11. For $k \geq 0$, the functor $\mathcal{F}^h_k$ sends a graded $\mathbb{F}_p$-vector space V to the graded $\mathbb{F}_p$-vector space $\mathcal{F}^h_k(V)$ generated by symbols $(i_1, ..., i_k; v)$, where v is a homogeneous element of V and $i_1, \ldots, i_k$ are integers satisfying the following conditions:

(1)' Each i_j is congruent to 0 or -1 modulo $2(p-1)$.
(2)' $i_j \leq p i_{j+1}$ for all $1 \leq j < k$,
(3)' If p is odd, then $i_1 > (p-1)(|v| + i_2 + \ldots + i_k)$.
If p is even, then $i_1 \geq (p-1)(|v| + i_2 + \ldots + i_k)$.

We again divide out by the relation $(i_1, ..., i_k; u) + (i_1, ..., i_k; v) = (i_1, ..., i_k; u + v)$. The homological grading and the weight grading are as in Definition 7.9.

Definition 7.12. Given a homologically graded $\mathbb{F}_p$-vector space, let $S(V) = \bigoplus_{n \geq 0} S_n(V)$ be the free graded-commutative algebra on V if p is odd and for the free exterior algebra on V if $p = 2$.

Observe that if V is equipped with an additional weight grading, then $S(V)$ is naturally bigraded.

We will now use the functors $\mathcal{F}_k$, $\mathcal{F}_k^h$, and S to give a simple formula for homotopy groups of the symmetric and exterior powers of a given $M \in \mathrm{Mod}_k$, thereby summarising computations of Dold [Dol58], Nakaoka [Nak57] [Nak59], Milgram [Mil69], and Priddy [Pri73] in the "strict" case, as well as of Adem [Ade52], Serre [Ser53], Cartan [Car54] [Car55], Dyer-Lashof [DL62], May, and Steinberger [BMMS86] in the "homotopy orbits" case:

Proposition 7.13. *For any $M \in \mathrm{Mod}_k$, there are (unnatural) isomorphisms*

$$\pi_*\left(\bigoplus_n F_{\Sigma_n/\Sigma_n}(M)\right) \cong S\left(\bigoplus_k \mathcal{F}_k(\pi_*(M))\right) \qquad \pi_*\left(\bigoplus_n F^h_{\Sigma_n/\Sigma_n}(M)\right) \cong S\left(\bigoplus_k \mathcal{F}^h_k(\pi_*(M))\right)$$

which respect the homological and the weight grading. Here $\mathcal{F}_k(\pi_*(M)), \mathcal{F}_k^h(\pi_*(M))$ sit in weight p^k.

Remark 7.14. Note that the functor $F_{\Sigma_n/\Sigma_n}(M)$ computes the (suitably derived) n^{th} symmetric power of M, whereas $F^h_{\Sigma_n/\Sigma_n}(M)$ computes its n^{th} extended power.

Warning 7.15. *These are isomorphisms of bigraded vector spaces; they do not respect the multiplicative structure. In fact, they are not even functorial in M, as we should really use divided power functors on the left. However, this will not cause any problems for us, since we will only need a dimension count of the weighted pieces. We therefore adopt this simpler approach for notational convenience.*

Proof of Proposition 7.13. After picking a basis, we may identify M with a direct sum of shifts of $\mathbb{F}_p$. Since all functors commute with filtered colimits and send finite direct sums to tensor products, it suffices to check the claim for $\mathbb{F}_p$-module spectra of the form $M = \Sigma^\ell \mathbb{F}_p$, where ℓ is any integer.

For $\ell > 0$, the vector space $\mathcal{F}_k(\Sigma^\ell \mathbb{F}_p)$ is the k^{th} summand of the free simplicial commutative $\mathbb{F}_p$-algebra on one generator in degree ℓ. The work of Nakaoka (cf. [Nak57] [Nak59]) therefore shows that it has a basis given by all sequences $(i_1, \ldots, i_k)$ with

(1) Each i_j is congruent to 0 or 1 modulo $2(p-1)$,
(2) $i_j \geq p i_{j+1} > p$,
(3) If p is odd, then $1 < i_1 < (p-1)(v + i_2 + \ldots + i_k)$.
If p is even, then $1 < i_1 \leq (p-1)(v + i_2 + \ldots + i_k)$.

For $\ell \leq 0$, the $\mathbb{F}_p$-vector space $\mathcal{F}_k(\Sigma^\ell \mathbb{F}_p)$ agrees with the k^{th} summand in the free cosimplicial $\mathbb{F}_p$-vector space on a generator in degree ℓ. The work of Priddy (cf. [Pri73, Theorem 4.1]) therefore shows that $\mathcal{F}_k(\Sigma^\ell \mathbb{F}_p)$ has a basis given by all sequences $(i_1, \ldots, i_k)$ with

(1) Each i_j is congruent to 0 or -1 modulo $2(p-1)$,
(2) $i_j \leq p i_{j+1} \leq 0$ for all $1 \leq j < k$,
(3) If p is odd, then $0 \geq i_1 > (p-1)(v + i_2 + \ldots + i_k)$.
If p is even, then $0 \geq i_1 \geq (p-1)(v + i_2 + \ldots + i_k)$.

Corresponding statements for G^h are described on p.298 of [BMMS86] and p.16 of [CLM76]. $\square$

Remark 7.16. Note that for $p = 2$, the cited sources state the result in a slightly different, yet equivalent, form, which uses a strict inequality for the excess and the free symmetric algebra functor.

Write $P(n)$ for the set of sequences $(a_0, a_1, \ldots)$ of natural numbers satisfying $n = \sum_{k \geq 0} a_k p^k$. Restricting attention to a specific weight and expanding Proposition 7.13 binomially, we deduce:

Corollary 7.17. *For each $n \geq 0$ and any $M \in \mathrm{Mod}_k$, there are isomorphisms*

$$\pi_* \left(F_{\Sigma_n/\Sigma_n}(M)\right) \cong \bigoplus_{(a_0,a_1,\dots)\in P(n)} \left(\bigotimes_{k\geq 0} S_{a_k}\left(\mathcal{F}_k(\pi_*(M))\right)\right)$$

$$\pi_* \left(F^h_{\Sigma_n/\Sigma_n}(M)\right) \cong \bigoplus_{(a_0,a_1,\dots)\in P(n)} \left(\bigotimes_{k\geq 0} S_{a_k}\left(\mathcal{F}^h_k(\pi_*(M))\right)\right).$$

We compute $\pi_* \left(F_{\Sigma_n/K_\sigma}(M)\right)$ and $\pi_* \left(F^h_{\Sigma_n/K_\sigma}(M)\right)$ for K_σ the stabiliser of any simplex

$$\sigma = [\ \hat{0} < x_1 < \ldots < x_i < \hat{1}\ ]$$

in the doubly suspended partition complex $\Sigma|\Pi_n|^\diamond$, where the integer $n \geq 1$ is fixed throughout. We will use [AB18, Definition 9.12], which is a variant of [AM99, Definition 1.10]:

Definition 7.18. A p-*enhancement* of a chain of partitions $\sigma = [\ \hat{0} < x_1 < \ldots < x_i < \hat{1}\ ]$ consists of a refining chain

$$\Theta = [\ \hat{0} \leq e_1 \leq x_1 \leq \ldots \leq e_i \leq x_i \leq e_{i+1} \leq \hat{1}\ ]$$

such that the following two conditions hold true:

(1) The number of x_a-classes contained in a given e_{a+1}-class is a power of p.
(2) Given x_a-classes S_1 and S_2, we can define chains of partitions of S_1 and S_2 by restricting Θ. If S_1, S_2 lie in the same e_{a+1}-class, then these restricted chains are isomorphic, by which we mean that they lie in the same Σ_n-orbit.

Two p-enhancements are said to be *isomorphic* if they lie in the same Σ_n-orbit. We can then define endofunctors from enhancements as follows (cf. [AB18, Definition 9.13]):

Definition 7.19. Assume we are given a chain $\sigma = [\ \hat{0} < x_1 < \ldots < x_i < \hat{1}\ ]$ and an isomorphism class of p-enhancements of σ represented by $\Theta = [\ \hat{0} \leq e_1 \leq x_1 \leq \ldots \leq e_i \leq x_i \leq e_{i+1} \leq \hat{1}\ ]$.

We define endofunctors $[\Theta]$ and $[\Theta]^h$ on graded $\mathbb{F}_p$-vector spaces by the following rules:

- If $i = 0$ and $[\Theta] = [\ \hat{0} \leq e_1 \leq \hat{1}\ ]$ with e_1 having a_j classes of size p^k, we set

$$[\Theta](C) := \bigotimes_j S_{a_j}(\mathcal{F}_j(V)) \qquad\qquad [\Theta]^h(C) := \bigotimes_j S_{a_j}(\mathcal{F}^h_j(V)).$$

- If $i > 0$, assume that restricting the chain Θ to the classes of e_{i+1} gives a_1 chains of isomorphism type 1, a_2 chains of isomorphism type 2, etc.... Suppose that each e_{i+1}-class of type t contains p^{b_t} many x_i-classes, and write Θ_t for the restriction of Θ to any x_i-class contained in an e_{i+1}-class of type t. We then define

$$[\Theta](V) := \bigotimes_t S_{a_t}(\mathcal{F}_{b_t}([\Theta_t](V))) \qquad\qquad [\Theta]^h(V) := \bigotimes_t S_{a_t}(\mathcal{F}^h_{b_t}([\Theta_t]^h(V))).$$

The functors $[\Theta]$ and $[\Theta]^h$ are well-defined as the above construction only depends on the isomorphism class of the p-enhancement Θ.

Using this notation, we can generalise [AB18, Proposition 9.14] and describe the Bredon homology of stabilisers in the partition complex:

Proposition 7.20. *Let $K_\sigma \subset \Sigma_n$ be the stabiliser of a simplex $\sigma = [\ \hat{0} < x_1 < \cdots < x_i < \hat{1}\ ]$ in $\Sigma|\Pi_n|^\diamond$ and write $E[\sigma]$ for the set of isomorphism classes of its p-enhancements.*

For any $M \in \mathrm{Mod}_k$, there are isomorphisms

$$\pi_*(F_{\Sigma_n/K_\sigma}(M)) = \bigoplus_{[\Theta]\in E[\sigma]} [\Theta](\pi_*(M)) \qquad\qquad \pi_*(F^h_{\Sigma_n/K_\sigma}(M)) = \bigoplus_{[\Theta]\in E[\sigma]} [\Theta]^h(\pi_*(M))$$

Proof. This follows formally from Corollary 7.17 by precisely the same argument as used in the proof of [AB18, Proposition 9.14]. □

The Bredon Homology of the Partition Complex. Let $\mathcal{P}_n$ be the poset of partitions of $\{1,\dots,n\}$. Observe that $|\Pi_n|^\diamond$ is Σ_n-equivariantly equivalent to the realisation of the pointed simplicial set

$$T_\bullet := N_\bullet(\mathcal{P}_n - \{\hat{0}\})/N_\bullet(\mathcal{P}_n - \{\hat{0},\hat{1}\}).$$

Its nondegenerate i-simplices are either the basepoint or correspond to chains of partitions

$$\sigma = [\ \hat{0} < x_1 < \ldots < x_i < \hat{1}\ ].$$

The groups $\widetilde{\mathrm{H}}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^M_t)$ and $\widetilde{\mathrm{H}}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^{M,h}_t)$ are given by the homology of the normalised chain complexes $\widetilde{C}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^M_t)$, $\widetilde{C}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^{M,h}_t)$ of the simplicial abelian groups $\mu^M_t(T_\bullet)$, $\mu^{M,h}_t(T_\bullet)$.

The i^{th} degree of the chain complexes $\widetilde{C}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^M_t)$ and $\widetilde{C}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^{M,h}_t)$ can be decomposed with the help of Proposition 7.20 as a direct sum indexed by isomorphism classes of p-enhancements

$$\Theta = [\ \hat{0} \le e_1 \le x_1 \le \ldots \le e_i \le x_i \le e_{i+1} \le \hat{1}\ ].$$

In fact, we can discard most p-enhancements. Let us call a p-enhancement as above *pure* if $e_j = x_j$ for all $1 \le j \le i+1$. Extending [AB18, Proposition 9.19] to our setting, we have:

Proposition 7.21. *For each t, the Euler characteristics of $\widetilde{\mathrm{H}}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^M_t)$ and $\widetilde{\mathrm{H}}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^{M,h}_t)$ agrees with the Euler characteristic of the submodules of $\widetilde{C}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^M_t)$ and $\widetilde{C}^{\mathrm{Br}}_*(|\Pi_n|^\diamond, \mu^{M,h}_t)$ spanned by all summands corresponding to pure p-enhancements.*

We can now establish the main results of this section:

Proof of Theorem 7.5 and Theorem 7.6. By the observations on duality made in the beginning of this section on page 68, it suffices to compute $\pi_t(F_{\Sigma|\Pi_n|^\diamond}(\Sigma^l\mathbb{F}_p))$ and $\pi_t(F^h_{\Sigma|\Pi_n|^\diamond}(\Sigma^l\mathbb{F}_p))$ for $l = -\ell$.

By Corollary 7.8, these groups vanish if n is not a power of p. If $n = p^k$, then the dimension of these groups is given by the Euler characteristics of $H^{\mathrm{Br}}_*(|\Pi_{p^k}|^\diamond, \mu^M_{t-k})$ and $\mathrm{H}^{\mathrm{Br}}_*(|\Pi_{p^k}|^\diamond, \mu^{M,h}_{t-k})$.

Proposition 7.21 shows that these dimensions agree with the Euler characteristics of the *submodules* of $\widetilde{C}^{\mathrm{Br}}_*(|\Pi_{p^k}|^\diamond, \mu^M_{t-a})$ and $\widetilde{C}^{\mathrm{Br}}_*(|\Pi_{p^k}|^\diamond, \mu^{M,h}_{t-a})$ spanned by all summands corresponding to pure p-enhancements. As in [AB18, Theorem 9.1], we are therefore reduced to computing the Euler characteristics of the following bigraded abelian groups in "Bredon-direction".

$$\textbf{(1)} \qquad \mathcal{F}_k(\Sigma^l\mathbb{F}_p) \qquad \bigoplus_{k_1+k_2=k} \mathcal{F}_{k_1}\mathcal{F}_{k_2}(\Sigma^l\mathbb{F}_p) \qquad \ldots \qquad \mathcal{F}_1\ldots\mathcal{F}_1(\Sigma^l\mathbb{F}_p)$$

$$\textbf{(2)} \qquad \mathcal{F}^h_k(\Sigma^l\mathbb{F}_p) \qquad \bigoplus_{k_1+k_2=k} \mathcal{F}^h_{k_1}\mathcal{F}^h_{k_2}(\Sigma^l\mathbb{F}_p) \qquad \ldots \qquad \mathcal{F}^h_1\ldots\mathcal{F}^h_1(\Sigma^l\mathbb{F}_p)$$

To begin with, we use Definition 7.9 to see that the summand $\mathcal{F}_{k_1}\ldots\mathcal{F}_{k_r}(\Sigma^l\mathbb{F}_p)$ in (**1**) has a basis consisting of all sequences $(i_1,\ldots,i_k)$ satisfying the following properties:

1. Each $|i_j|$ is congruent to 0 or 1 modulo $2(p-1)$.
2. $i_j \ge p i_{j+1} > p$ *or* $i_j \le p i_{j+1} \le 0$ for all $1 \le j < k$ with $j \ne k_1, k_1+k_2, \ldots$.

(3) If p is odd, then for $t = 0, \dots, r-1$, we have
$1 < i_{k_1+\ldots+k_t+1} < (p-1)(l + i_{k_1+\ldots+k_t+2} + \ldots + i_{k_1+\ldots+k_r})$
or $0 \geq i_{k_1+\ldots+k_t+1} > (p-1)(l + i_{k_1+\ldots+k_t+2} + \ldots + i_{k_1+\ldots+k_r})$.
If p is even, then for $t = 0, \dots, r-1$, we have
$1 < i_{k_1+\ldots+k_t+1} \leq (p-1)(l + i_{k_1+\ldots+k_t+2} + \ldots + i_{k_1+\ldots+k_r})$
or $0 \geq i_{k_1+\ldots+k_t+1} \geq (p-1)(l + i_{k_1+\ldots+k_t+2} + \ldots + i_{k_1+\ldots+k_r})$.

Observe that if $l > 0$, then $i_j > 1$ for all j, whereas if $l \leq 0$, then $i_j \leq 0$ for all j.

Fix a sequence $(i_1, \dots, i_k)$ satisfying the conditions above, but such that for $j = k_1, k_1 + k_2, \dots$, we have $1 < i_j < pi_{j+1}$ *or* $0 \geq i_j > pi_{j+1}$. Informally speaking, (2) is violated whenever possible.

This sequence $(i_1, \dots, i_k)$ appears exactly once as a basis element in $\mathcal{F}_{m_1} \ldots \mathcal{F}_{m_s}(\Sigma^l \mathbb{F}_p)$ for any ordered partition $m_1 + \ldots + m_s = k$ of the number k refining the ordered partition $k_1 + \ldots + k_r = k$. Counting the number of such refinements, we see that the total contribution of $(i_1, \dots, i_k)$ to the Euler characteristic in "Bredon direction" is

$$\sum_{s=r}^{k} (-1)^{s-1} \binom{k-r}{s-r}.$$

This alternating sum is zero for $k \neq r$. For $k = r$, it is equal to $(-1)^{k-1}$. In this case, $(i_1, \dots, i_k)$ indexes a basis element in $\mathcal{F}_1 \ldots \mathcal{F}_1(\Sigma^l \mathbb{F}_p)$. Hence, the absolute value of the Euler characteristic "in Bredon direction" is equal to the number of sequences $(i_1, \dots, i_k)$ satisfying the following:

(1) Each $|i_j|$ is congruent to 0 or 1 modulo $2(p-1)$.
(2) $1 < i_j < pi_{j+1}$ *or* $0 \geq i_j > pi_{j+1}$ for all $1 \leq j < k$.
(3) If p is odd, then for $t = 0, \dots, k-1$, we have
$1 < i_{t+1} < (p-1)(l + i_{t+2} + \ldots + i_k)$ *or* $0 \geq i_{t+1} > (p-1)(l + i_{t+2} + \ldots + i_k)$.
If p is even, then for $t = 0, \dots, k-1$, we have
$1 < i_{t+1} \leq (p-1)(l + i_{t+2} + \ldots + i_k)$ *or* $0 \geq i_{t+1} \geq (p-1)(l + i_{t+2} + \ldots + i_k)$.

We observe that these conditions can be rephrased as follows:

(1) Each $|i_j|$ is congruent to 0 or 1 modulo $2(p-1)$.
(2) For all $1 \leq j < k$ we have $1 < i_j < pi_{j+1}$ or $pi_{j+1} < i_j \leq 0$.
(3) We have $1 < i_k \leq (p-1)l$ or $(p-1)l \leq i_k \leq 0$.

Theorem 7.5 follows by replacing each i_j by its inverse for the sake of notational convenience.

We compute the "Bredon Euler characteristic" of the complex (**2**) above by a similar method. Using Definition 7.11, we see that the summand $\mathcal{F}^h_{k_1} \ldots \mathcal{F}^h_{k_r}(\Sigma^l \mathbb{F}_p)$ has a basis consisting of all sequences $(i_1, \dots, i_k)$ satisfying the following properties:

(1)' Each i_j is congruent to 0 or -1 modulo $2(p-1)$.
(2)' $i_j \leq pi_{j+1}$ for all $1 \leq j < k$ with $j \neq k_1, k_1 + k_2, \ldots$.
(3)' If p is odd, then for $t = 0, \dots, r-1$, we have
$i_{k_1+\ldots+k_t+1} > (p-1)(l + i_{k_1+\ldots+k_t+2} + \ldots + i_{k_1+\ldots+k_r})$.
If p is even, then for $t = 0, \dots, r-1$, we have
$i_{k_1+\ldots+k_t+1} \geq (p-1)(l + i_{k_1+\ldots+k_t+2} + \ldots + i_{k_1+\ldots+k_r})$.

We fix a sequence $(i_1, \dots, i_k)$ satisfying the four conditions above, such that for $j = k_1, k_1 + k_2, \dots$, we have $i_j > pi_{j+1}$. Again, the sequence appears exactly once as a basis element in $\mathcal{F}^h_{m_1} \ldots \mathcal{F}^h_{m_s}(\Sigma^l \mathbb{F}_p)$ for any ordered partition $m_1 + \ldots + m_s = k$ of k refining $k_1 + \ldots + k_r = k$

of k. As above, we see that these copies have vanishing contribution to the Euler characteristic unless $k = r$, in which case they contribute $(-1)^{k-1}$. Hence, the absolute value of the Euler characteristic in Bredon direction is equal to the number of sequences $(i_1, \ldots, i_k)$ satisfying:

(1)' Each i_j is congruent to 0 or -1 modulo $2(p-1)$.
(2)' $i_j > pi_{j+1}$ for all $1 \leq j < k$.
(3)' If p is odd, then for $t = 0, \ldots, k-1$, we have $i_{t+1} > (p-1)(l + i_{t+2} + \ldots + i_k)$.
If p is even, then for $t = 0, \ldots, k-1$, we have $i_{t+1} \geq (p-1)(l + i_{t+2} + \ldots + i_k)$.

To conclude the proof of Theorem 7.6, we check that these conditions are equivalent to the following:

(1)' Each i_j is congruent to 0 or -1 modulo $2(p-1)$.
(2)' For all $1 \leq j < k$ we have $pi_{j+1} < i_j$.
(3)' We have $(p-1)l \leq i_k$.

□

7.2. **Free Partition Lie Algebras on an Even Generator.** In the last section, we have computed the homotopy groups of free partition Lie algebras on an odd generator (cf. Theorems 7.5, 7.6).

We will now shift attention to the even degree case. For this, recall that given a pointed space X and a positive integer d, Theorem 8.5. in [AB18] constructs a natural sequence of spaces

$$\Sigma^2|\Pi_{\frac{d}{2}}|^\diamond \underset{\Sigma_{\frac{d}{2}}}{\wedge} (\Sigma X^{\wedge 2})^{\wedge \frac{d}{2}} \to \Sigma^2|\Pi_d|^\diamond \underset{\Sigma_d}{\wedge} X^{\wedge d} \to \Sigma|\Pi_d|^\diamond \underset{\Sigma_d}{\wedge} (\Sigma X)^{\wedge d} \tag{18}$$

which varies naturally in X. If $X = S^n$ is an even-dimensional sphere, then this sequence is in fact a *cofibration sequence*. Applying $\mathbb{F}_p$-valued cohomology to this sequence, we can decompose the free partition Lie algebra on a class in negative even degree $-n$ in terms of free partition Lie algebras on odd classes $-n-1$ and $-2n-1$ (using Proposition 5.49).

To extend this decomposition to *all* even integers, we will need to mildly generalise the above sequence (18) and construct it naturally in topological vector spaces rather than just spaces. A minor modification of our argument will also allow us to decompose the free *spectral* partition Lie algebras on an even class (using Proposition 5.35), thereby reproving the classical Takayasu cofibration sequence (cf. [Tak99]) and its "dual" (cf [Aro06, Theorem 3.2]) by a new argument.

Topological Vector Spaces. Let **Top** be the category of compactly generated topological spaces (henceforth simply called "spaces") with its Quillen model structure. This is a well-fibred topological cartesian closed category over sets in the sense of [AHS06, Definitions 21.7., 27.20]. By [Sea05, Proposition 2.2, Proposition 4.6], we can therefore lift the usual tensor product on $\mathbb{F}_p$-vector spaces to a closed symmetric monoidal structure $\otimes$ on the category $\mathbf{tMod}_{\mathbb{F}_p}$ of (compactly generated) topological $\mathbb{F}_p$-vector spaces. This topological tensor product satisfies the expected universal property with respect to continuous bilinear maps.

Given a pointed completely regular space (X, x), we can form the *free topological* $\mathbb{F}_p$*-vector space* $\mathbb{F}_p\{X\}$ on X with $x = 0$ satisfying the obvious universal property and containing X as a closed subset (cf. [AGM96, Theorem 6.2.2]). The underlying $\mathbb{F}_p$-vector space of $\mathbb{F}_p\{X\}$ is simply given by the free $\mathbb{F}_p$-vector space on X with $x = 0$.

Gradings. For J (commutative) indexing monoid in sets, work of Schwänzl-Vogt [SV91] shows that the category $\mathbf{tMod}^J_{\mathbb{F}_p}$ of functors from J to (compactly generated) topological $\mathbb{F}_p$-vector spaces carries a cofibrantly generated model structure. Its underlying fibrations and weak equivalences are given by pointwise fibrations and weak equivalences on underlying spaces.

Moreover, $\mathbf{tMod}^J_{\mathbb{F}_p}$ has a symmetric monoidal structure given by Day convolution. It sends V, W to the J-graded topological $\mathbb{F}_p$-vector space with $(V \otimes W)_k := \bigoplus_{a+b=k}(V_a \otimes W_b)$. For (X, x) a pointed completely regular space, we equip $\mathbb{F}_p\{X\}$ with a grading concentrated in degree 0.

Topological Algebras. Write $\mathbf{tAlg}^J_{\mathbb{F}_p}$ for the category of commutative algebra objects in the symmetric monoidal category $\mathbf{tMod}^J_{\mathbb{F}_p}$; these are J-graded (compactly generated) topological commutative $\mathbb{F}_p$-algebras. Again, the work of Schwänzl-Vogt [SV91] equips $\mathbf{tAlg}^J_{\mathbb{F}_p}$ with a cofibrantly generated model structure in which a map is a fibration or weak equivalence if the underlying map in $\mathbf{tMod}^J_{\mathbb{F}_p}$ has the corresponding property. We denote the augmented variant by $\mathbf{tAlg}^{J,aug}_{\mathbb{F}_p}$. There is a natural functor $\mathbf{tMod}^J_{\mathbb{F}_p} \to \mathbf{tAlg}^{aug,J}_{\mathbb{F}_p}$ sending V to the trivial square-zero extension $\mathbb{F}_p \oplus V$ on V.

Simplicial Variants. We can also define model categories $\mathbf{sMod}^J_{\mathbb{F}_p}$ and $\mathbf{sAlg}^J_{\mathbb{F}_p}$ of J-graded simplicial $\mathbb{F}_p$-modules and simplicial commutative $\mathbb{F}_p$-algebras, respectively. The standard Quillen equivalence $|-| : \mathbf{sSet} \leftrightarrows \mathbf{Top} : \mathrm{Sing}$ preserves finite products and therefore induces Quillen equivalences

$$|-| : \mathbf{sMod}^J_{\mathbb{F}_p} \leftrightarrows \mathbf{tMod}^J_{\mathbb{F}_p} : \mathrm{Sing} \quad \text{ and } \quad |-| : \mathbf{sAlg}^J_{\mathbb{F}_p} \leftrightarrows \mathbf{tAlg}^J_{\mathbb{F}_p} : \mathrm{Sing}\,.$$

Given two J-graded simplicial $\mathbb{F}_p$-vector spaces $V_\bullet$ and $W_\bullet$, it is straightforward to check that there is an isomorphism $|V_\bullet| \otimes |W_\bullet| \cong |V_\bullet \otimes W_\bullet|$. Geometric realisation intertwines square-zero extensions in $\mathbf{sAlg}^{J,aug}_{\mathbb{F}_p}$ with the corresponding construction in $\mathbf{tAlg}^{J,aug}_{\mathbb{F}_p}$. Finally, the functor $|-|$ sends free simplicial $\mathbb{F}_p$-modules to free topological $\mathbb{F}_p$-modules.

Homotopy Pushouts of Algebras. As expected, pushouts in $\mathbf{tAlg}^J_{\mathbb{F}_p}$ are simply computed by relative tensor products. More precisely, given a span $B \leftarrow A \to C$ of J-graded topological $\mathbb{F}_p$-algebras, the pushout is given by the coequaliser $B \otimes_A C := \mathrm{coequ}(B \otimes A \otimes C \rightrightarrows B \otimes C)$.

Construction 7.22. We describe an explicit model for the *homotopy pushout* of $B \leftarrow A \to C$ in $\mathbf{tAlg}^J_{\mathbb{F}_p}$. For $k \geq 0$, the topological $\mathbb{F}_p$-vector space $\mathbb{F}_p\{\Delta^k\} \otimes B \otimes A^{\otimes n} \otimes C$ is generated by symbols

$$\begin{pmatrix} 0 & \leq & t_1 & \leq & t_2 & \leq & \dots & \leq & t_k & \leq & 1 \\ b & \otimes & a_1 & \otimes & a_2 & \otimes & \dots & \otimes & a_k & \otimes & c \end{pmatrix}$$

with $t_i \in [0,1]$, $a_i \in A$, $b \in B$, and $c \in C$, subject to the standard multilinear relations.

Define an object in $\mathbf{tMod}^J_{\mathbb{F}_p}$ by $B \otimes^{\mathrm{h}}_A C = |\operatorname{Bar}_\bullet(B, A, C)| = \left(\bigoplus_{k \geq 0} \mathbb{F}_p\{\Delta^k\} \otimes B \otimes A^{\otimes n} \otimes C \right) \Big/ \sim$, where $\sim$ denotes the quotient by the $\mathbb{F}_p$-linear subspace generated by the following relations:

$$\begin{pmatrix} 0 & \leq & \dots & \leq & t_i & = & t_{i+1} & \leq & \dots \\ b & \otimes & \dots & \otimes & a_i & \otimes & a_{i+1} & \otimes & \dots \end{pmatrix} \sim \begin{pmatrix} 0 & \leq & \dots & \leq & t_i & \leq & t_{i+2} & \leq & \dots \\ b & \otimes & \dots & \otimes & a_i \cdot a_{i+1} & \otimes & a_{i+2} & \otimes & \dots \end{pmatrix}$$

$$\begin{pmatrix} 0 & \leq & \dots & \leq & t_i & \leq & t_{i+1} & \leq & \dots \\ b & \otimes & \dots & \otimes & a_i & \otimes & 1 & \otimes & \dots \end{pmatrix} \sim \begin{pmatrix} 0 & \leq & \dots & \leq & t_i & \leq & t_{i+2} & \leq & \dots \\ b & \otimes & \dots & \otimes & a_i & \otimes & a_{i+2} & \otimes & \dots \end{pmatrix}.$$

We endow $B \otimes^{\mathrm{h}}_A C$ with the unique multiplication $(B \otimes^{\mathrm{h}}_A C) \otimes (B \otimes^{\mathrm{h}}_A C) \to (B \otimes^{\mathrm{h}}_A C)$ satisfying

$$\begin{pmatrix} 0 & \leq & t_{i_1} & \leq & \dots & \leq & t_{i_n} & \leq & 1 \\ b_1 & \otimes & a_{i_1} & \otimes & \dots & \otimes & a_{i_n} & \otimes & c_1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \leq & t_{j_1} & \leq & \dots & \leq & t_{j_m} & \leq & 1 \\ b_2 & \otimes & a_{j_1} & \otimes & \dots & \otimes & a_{j_m} & \otimes & c_2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \leq & t_1 & \leq & \dots & \leq & t_{n+m} & \leq & 1 \\ b_1 b_2 & \otimes & a_1 & \otimes & \dots & \otimes & a_{n+m} & \otimes & c_1 c_2 \end{pmatrix}$$

for any disjoint union $\{1 < \dots < m+n\} = \{i_1 < \dots < i_n\} \coprod \{j_1 < \dots < j_m\}$. Simple checks show that this multiplication is well-defined, commutative and associative.

The following is proven by an argument entirely parallel to the proof of [AB18, Proposition 7.15]:

Proposition 7.23. *If the unit $\mathbb{F}_p \to A$ is a cofibration, then $B \otimes^h_A C$ is a homotopy pushout of the span $B \leftarrow A \to C$ of topological J-graded $\mathbb{F}_p$-algebras.*

Definition 7.24. The *suspension* $\Sigma^{\otimes} A$ of some $A \in \mathbf{tAlg}^{J,aug}_{\mathbb{F}_p}$ is given by $\mathbb{F}_p \otimes^{\mathrm{h}}_A \mathbb{F}_p$.

Homotopy Pullbacks of Algebras. A much simpler construction gives us explicit models for homotopy pullbacks of J-graded topological $\mathbb{F}_p$-algebras. For this, let $D^I = \mathrm{Map}_{\mathbf{Top}}([0,1], D)$ denote the space of paths in a given space D.

Definition 7.25. If $B \xrightarrow{f} A \xleftarrow{g} C$ is a diagram of J-graded topological commutative $\mathbb{F}_p$-algebras, we equip the J-graded space $B \times^{\mathrm{h}}_A C$ determined by

$$(B \overset{\mathrm{h}}{\underset{A}{\times}} C)_j := \big\{(b, \alpha, c) \in B_j \times A^I_j \times C_j \mid \alpha(0) = f(b), \alpha(1) = g(c)\big\}$$

with an $\mathbb{F}_p$-algebra structure by setting

$$\lambda_1(b_1, \alpha_1, c_1) + \lambda_2(b_2, \alpha_2, c_2) = (\lambda_1 b_1 + \lambda_2 b_2, \lambda_1\alpha_1 + \lambda_2\alpha_2, \lambda_1 c_1 + \lambda_2 c_2)$$
$$(b_1, \alpha_1, c_1) \cdot (b_2, \alpha_2, c_2) = (b_1 b_2, \alpha_1\alpha_2, c_1 c_2),$$

where the paths $\lambda_1\alpha_1 + \lambda_2\alpha_2$ and $\alpha_1\alpha_2$ are defined using pointwise operations.

An entirely parallel argument to the proof of [AB18, Proposition 7.24] then shows:

Proposition 7.26. *The homotopy pullback of a diagram $B \xrightarrow{f} A \xleftarrow{g} C$ in $\mathbf{tAlg}^J_{\mathbb{F}_p}$ is given by $B \times^h_A C$.*

Definition 7.27. The *loop space* of some $A \in \mathbf{tAlg}^J_{\mathbb{F}_p}$ is given by $\Omega^{\otimes} A := \mathbb{F}_p \times^{\mathrm{h}}_A \mathbb{F}_p$.

Remark 7.28. The underlying space of $\Omega^{\otimes} A$ is given by the space of all paths $[0,1] \to A$ which start and end at the same point in $\mathbb{F}_p \subset A$.

Suspension-Loops Adjunction. We can link the two constructions above by setting up an adjunction

$$\Sigma^{\otimes} : \mathbf{tAlg}^J_{\mathbb{F}_p} \leftrightarrows \mathbf{tAlg}^J_{\mathbb{F}_p} : \Omega^{\otimes}.$$

Its unit η is defined by by the following explicit formula:

$$A \xrightarrow{\eta_A} \Omega^{\otimes}\Sigma^{\otimes} A \quad \text{sends} \quad a \in A \quad \text{to} \quad \alpha_a := \left((0 \leq s \leq 1) \mapsto \begin{pmatrix} 0 & \leq & s & \leq & 1 \\ 1 & \otimes & a & \otimes & 1 \end{pmatrix}\right).$$

Here $s \in [0,1]$ denotes a parameter for a loop, and it is not hard to check that the map η_A respects the grading and is both linear and multiplicative. The unit ϵ is specified as follows:

$$\Sigma^{\otimes}\Omega^{\otimes} A \xrightarrow{\epsilon_A} A \text{ sends } \begin{pmatrix} 0 & \leq & t_1 & \leq & \dots & \leq & t_n & \leq & 1 \\ \lambda & \otimes & \alpha_1 & \otimes & \dots & \otimes & \alpha_n & \otimes & \mu \end{pmatrix} \in \Sigma^{\otimes}\Omega^{\otimes} A \text{ to } \lambda \cdot \alpha_1(t_1) \cdot \ldots \cdot \alpha_n(t_n) \cdot \mu.$$

Here $\alpha_1, \ldots, \alpha_n \in \Omega^{\otimes} A$ are given paths, and we it is again straightforward to check that this assignment gives a map in $\mathbf{tAlg}^J_{\mathbb{F}_p}$. Observe that $\epsilon_{\Sigma^{\otimes}} \circ \Sigma^{\otimes}\eta$ and $\Omega^{\otimes}\epsilon \circ \eta_{\Omega^{\otimes}}$ are indeed given by the identity transformations, and we have therefore defined an adjunction.

The following result is proven by an argument parallel to the proof of [AB18, Proposition 7.28]:

Lemma 7.29. *The adjunction $\Sigma^{\otimes} : \mathbf{tAlg}^J_{\mathbb{F}_p} \leftrightarrows \mathbf{tAlg}^J_{\mathbb{F}_p} : \Omega^{\otimes}$ is Quillen.*

The EHP*-sequence for Topological Vector Spaces.* We proceed to generalise the EHP-sequence for (strictly commutative) monoid spaces (cf. [AB18, Definition 7.42]) to the setting of topological $\mathbb{F}_p$-algebras. We begin with the following observation, which is immediate from Definition 7.24:

Proposition 7.30. *Given a J-graded topological $\mathbb{F}_p$-vector space V, we let $\mathbb{F}_p \oplus V \in \mathbf{tAlg}^J_{\mathbb{F}_p}$ denote the trivial square-zero extension of $\mathbb{F}_p$ by V. There is an of J-graded topological $\mathbb{F}_p$-vector spaces*

$$\Sigma^{\otimes}(\mathbb{F}_p \oplus V) \simeq \bigoplus_k \mathbb{F}_p\{S^k\} \otimes V^{\otimes k}.$$

Using this splitting, we define a natural transformation of functors $\mathbf{tMod}^J_{\mathbb{F}_p} \to \mathbf{tAlg}^J_{\mathbb{F}_p}$ as follows:

Definition 7.31. Given $V \in \mathbf{tMod}^J_{\mathbb{F}_p}$, the *Einhängung* $E_V : \Sigma^{\otimes}(\mathbb{F}_p \oplus V) \to \mathbb{F}_p \oplus \big(\mathbb{F}_p\{S^1\} \otimes V\big)$ is the map of J-graded topological $\mathbb{F}_p$-algebras obtained by projecting to the first two summands.

The construction of the Hopf map is somewhat more interesting. To this date, we do not know of a definition staying in the realm of higher category theory, and this is in fact the reason why we had to work with strict models.

First, given any $V \in \mathbf{tMod}^J_{\mathbb{F}_p}$, we define a map of J-graded topological $\mathbb{F}_p$-vector spaces

$$\begin{array}{rcl} \Phi : \mathbb{F}_p\{S^1\} \otimes V^{\otimes 2} & \longrightarrow & \Omega^{\otimes}\Sigma^{\otimes}(\mathbb{F}_p \oplus V) \\ (0 \leq t \leq 1) \otimes v \otimes w & \mapsto & \left(s \mapsto \begin{pmatrix} 0 & \leq & ts & \leq & s & \leq & 1 \\ 1 & \otimes & v & \otimes & w & \otimes & 1 \end{pmatrix} \right). \end{array}$$

Since $\Phi(x) \cdot \Phi(y) = 0$ for all x, y by inspection, we obtain a map of J-graded topological $\mathbb{F}_p$-algebras

$$\mathbb{F}_p \oplus (\mathbb{F}_p\{S^1\} \otimes V^{\otimes 2}) \longrightarrow \Omega^{\otimes}\Sigma^{\otimes}(\mathbb{F}_p \oplus V).$$

Definition 7.32. For $V \in \mathbf{tMod}^J_{\mathbb{F}_p}$, the *Hopf map* $H_V : \Sigma^{\otimes}(\mathbb{F}_p \oplus (\mathbb{F}_p\{S^1\} \otimes V^{\otimes 2})) \longrightarrow \Sigma^{\otimes}(\mathbb{F}_p \oplus V)$ is adjoint to the map $\mathbb{F}_p \oplus (\mathbb{F}_p\{S^1\} \otimes V^{\otimes 2}) \longrightarrow \Omega^{\otimes}\Sigma^{\otimes}(\mathbb{F}_p \oplus V)$ specified above.

The Hopf map varies naturally in the $\mathbb{F}_p$-module V.

We now fix the indexing monoid of nonnegative integers $J = \mathbb{N}$, considered under addition. Observe that the ∞-category $\mathcal{D}^{\mathrm{Gr}}$ from Construction 5.36 arises as a full subcategory of the underlying ∞-category of $\mathbf{tAlg}^{\mathbb{N}}_{\mathbb{F}_p}$. Namely, it is spanned by all algebras which are equal to $\mathbb{F}_p$ in weight 0.

Given a finite-dimensional discrete $\mathbb{F}_p$-vector space $V \in \mathrm{Vect}^{\omega}_{\mathbb{F}_p}$, we write V_1 for the $\mathbb{N}$-graded topological $\mathbb{F}_p$-vector space consisting of V concentrated in degree 1. Combining Definition 7.31 and Definition 7.32, we obtain a sequence of $\mathbb{N}$-graded topological $\mathbb{F}_p$-algebras

$$\Sigma^{\otimes}(\mathbb{F}_p \oplus (\mathbb{F}_p\{S^1\} \otimes V_1^{\otimes 2})) \xrightarrow{H} \Sigma^{\otimes}(\mathbb{F}_p \oplus V_1) \xrightarrow{E} \mathbb{F}_p \oplus \big(\mathbb{F}_p\{S^1\} \otimes V_1\big). \tag{19}$$

which varies naturally in V. Inverting weak equivalences in $\mathbf{tMod}^{\mathbb{N}}_{\mathbb{F}_p}$, we obtain:

Proposition 7.33. *There is a natural sequence of functors* $\mathrm{Vect}^{\omega}_{\mathbb{F}_p} \to \mathcal{D}^{\mathrm{Gr}}$ *(cf. Construction 5.36) sending* $V \in \mathrm{Vect}^{\omega}_k$ *to a sequence*

$$\Sigma^{\otimes}(\mathrm{sqz}(\Sigma V_1^{\otimes 2})) \xrightarrow{H} \Sigma^{\otimes}(\mathrm{sqz}(V_1)) \xrightarrow{E} \mathrm{sqz}(\Sigma V_1)$$

Here $\Sigma^{\otimes}$ *denotes the suspension functor in the pointed* ∞*-category* $\mathcal{D}^{\mathrm{Gr}}$*, whereas* Σ *denotes the suspension functor in* Mod_k*, i.e. the shift in* Mod_k*.*

Proof. Writing $\mathbb{F}_p\{S^1_\bullet\}$ for the free simplicial $\mathbb{F}_p$-module on the simplicial circle $S^1_\bullet$ with the basepoint equal to 0, we verify the universal property to deduce that $|\mathbb{F}_p\{S^1_\bullet\}| \cong \mathbb{F}_p\{S^1\}$. Since the geometric realisation functor $|-| : \mathbf{sMod}_{\mathbb{F}_p} \to \mathbf{tMod}_{\mathbb{F}_p}$ also respects tensor products, we deduce that $|\mathbb{F}_p\{S^1_\bullet\} \otimes V_1| \cong \mathbb{F}_p\{S^1\} \otimes V_1$ is equivalent to the chain complex $\Sigma V \in \mathrm{Mod}_k$, concentrated in weight 1. The claim follows from Proposition 7.23, since all appearing units are cofibrations. $\square$

Decomposing Lie Algebras on an Even Class. The preceding section allows us to decompose even Lie algebras in terms of odd ones. In the terminology of Section 3.3 and Definition 1.5, we obtain:

Proposition 7.34. *For* $w \geq 0$*, there is a natural sequence of functors* $\mathrm{Vect}^{\omega}_k \to \mathrm{Mod}_k$ *sending* V *to*

$$\Sigma F_{\Sigma|\Pi_{\frac{w}{2}}|^{\diamond}}(\Sigma V^{\otimes 2}) \longrightarrow \Sigma F_{\Sigma|\Pi_w|^{\diamond}}(V) \longrightarrow F_{\Sigma|\Pi_w|^{\diamond}}(\Sigma V),$$

where the leftmost module is interpreted as zero whenever w *is odd.*

Proof. First, we apply the cotangent fibre functor cot (for simplicial commutative rings) to the sequence appearing in Proposition 7.33. In a second step, we note that since the left adjoint cot_Δ preserves colimits, there is a natural equivalence $\Sigma \circ \mathrm{cot}_\Delta \simeq \mathrm{cot}_\Delta \circ \Sigma^{\otimes}$. Finally, we proceed as in the proof of Proposition 5.49 to evaluate the functor cot_Δ on a trivial square-zero extension, thereby keeping track of the weights. $\square$

By Theorem 3.26, we can in fact take the right-left Kan extension and obtain a sequence of w-excisive functors $\mathrm{Mod}_k \to \mathrm{Mod}_k$ sending V to

$$\Sigma F_{\Sigma|\Pi_{\frac{w}{2}}|^{\diamond}}(\Sigma V^{\otimes 2}) \to \Sigma F_{\Sigma|\Pi_w|^{\diamond}}(V) \to F_{\Sigma|\Pi_w|^{\diamond}}(\Sigma V). \tag{20}$$

Applying the reduced singular chains functor $\widetilde{C}_*(-, \mathbb{F}_p)$ to [AB18, Theorem 8.5], we see that (20) is a cofibre sequence when evaluated on modules $V = \Sigma^n \mathbb{F}_p$ with $n \geq 0$ even. By [AM99, Proposition 4.6], this implies that (20) is in fact a cofibre sequence on all modules of the form $V = \Sigma^n \mathbb{F}_p$ with n an even integer. Applying linear duality and using Proposition 5.49, we deduce:

Theorem 7.35. *For all even integers* n *and all weights* $w \geq 0$*, there is a cofibre sequence in* Mod_k

$$\Sigma\, \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\Delta}}[w](\Sigma^{n-1}\mathbb{F}_p) \to \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\Delta}}[w](\Sigma^{n}\mathbb{F}_p) \to \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\Delta}}\left[\frac{w}{2}\right](\Sigma^{2n-1}\mathbb{F}_p).$$

The forgetful functor $\mathcal{D}^{\mathrm{Gr}} \to \mathcal{C}^{\mathrm{Gr}}$ from graded simplicial algebras to graded $\mathbb{E}_\infty$-algebras described in Construction 5.41 preserves pushouts and trivial square-zero extensions. We may therefore interpret Proposition 7.33 as a natural sequence of $\mathbb{E}_\infty$-algebras. Repeating the argument in the proof of Proposition 7.33 in this context, we conclude:

Theorem 7.36. *For all even integers* n *and all weights* $w \geq 0$*, there is a cofibre sequence in* Mod_k

$$\Sigma\, \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}}[w](\Sigma^{n-1}\mathbb{F}_p) \to \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}}[w](\Sigma^{n}\mathbb{F}_p) \to \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}}\left[\frac{w}{2}\right](\Sigma^{2n-1}\mathbb{F}_p).$$

7.3. **Free Partition Lie Algebras on Many Generators.** We can express free Lie algebras on many classes in terms of free Lie algebras on a single generator. Recall the following terminology:

Definition 7.37. A *Lyndon word* in letters $x_1, \dots, x_k$ is a word which is lexicographically (strictly) minimal among all its cyclic rotations. Write B_k for the set of Lyndon words in k letters and let $B(m_1, \dots, m_k) \subset B_k$ be the subset consisting of all words involving each x_i precisely m_i times.

Given a Lyndon word $w \in B_k$, we write $|w|_i$ for the number of occurrences of the letter x_i in w.

Our decomposition will follow from [AB18, Theorem 5.10], which we will now recall:

Theorem 7.38. *Given a decomposition $n = n_1 + \ldots + n_k$, there is a $\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$-equivariant (simple) homotopy equivalence*

$$\Sigma|\Pi_n|^\diamond \xrightarrow{\simeq} \bigvee_{\substack{d|\gcd(n_1,\dots,n_k)\\ w\in B(\frac{n_1}{d},\dots,\frac{n_k}{d})}} \mathrm{Ind}_{\Sigma_d}^{\Sigma_{n_1}\times\ldots\times\Sigma_{n_k}} \left((S^{\frac{n}{d}-1})^{\wedge d} \wedge \Sigma|\Pi_d|^\diamond \right).$$

From this, we can obtain the following decomposition:

Proposition 7.39. *Given integers $\ell_1, \dots, \ell_m$, there are isomorphisms of $\mathbb{N}^m$-graded $\mathbb{F}_p$-modules*

$$\bigoplus_{w\in B_m} \mathrm{Free}_{\mathrm{Lie}^\pi_{k,\Delta}} \left(\Sigma^{1+\sum_i(\ell_i-1)|w|_i}(\mathbb{F}_p) \right) \cong \mathrm{Free}_{\mathrm{Lie}^\pi_{k,\Delta}} \left(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_m}\mathbb{F}_p \right)$$

$$\bigoplus_{w\in B_m} \mathrm{Free}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}} \left(\Sigma^{1+\sum_i(\ell_i-1)|w|_i}(\mathbb{F}_p) \right) \cong \mathrm{Free}_{\mathrm{Lie}^\pi_{k,\mathbb{E}_\infty}} \left(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_m}\mathbb{F}_p \right)$$

The "multinomial" grading by $\mathbb{N}^m$ will be constructed in the course of the proof.

Proof. Recall the colimit-preserving functors $F_{(-)}, F^h_{(-)} : \mathcal{S}_*^{\Sigma_n} \to \mathrm{End}^n_\Sigma(\mathrm{Mod}_{\mathbb{F}_p})$ from Section 3.3. For $X \in \mathrm{Set}_*^{\mathrm{Fin}}$, $V \in \mathrm{Mod}_k^\omega$, and $\ell_1, \dots, \ell_m \in \mathbb{Z}$, expanding "binomially" gives a natural equivalence

$$\mathbb{F}_p[X] \underset{\Sigma_n}{\otimes} (\Sigma^{-\ell_1}V \oplus \ldots \oplus \Sigma^{-\ell_m}V)^{\otimes n} \simeq \bigoplus_{n_1+\ldots+n_k=n} \Sigma^{-\ell_1 n_1 - \ldots - \ell_m n_m} \left(\mathbb{F}_p[X] \underset{\Sigma_{n_1}\times\ldots\times\Sigma_{n_m}}{\otimes} V^{\otimes n} \right).$$

Taking the right-left extension of these degree n functors (cf. Theorem 3.26), we obtain

$$F_X(\Sigma^{-\ell_1}V \oplus \ldots \oplus \Sigma^{-\ell_m}V) \simeq \bigoplus_{n_1+\ldots+n_k=n} \Sigma^{-\ell_1 n_1 - \ldots - \ell_m n_m} F_{\mathrm{Ind}^{\Sigma_n}_{\Sigma_{n_1}\times\ldots\times\Sigma_{n_k}}(X)}(V). \tag{21}$$

Since $F_{(-)} : \mathcal{S}_*^{\Sigma_n} \to \mathrm{End}^n_\Sigma(\mathrm{Mod}_{\mathbb{F}_p})$ was defined by freely extending from finite Σ_n-sets to genuine Σ_n-spaces under sifted colimits, this equivalence in fact holds for general Σ_n-spaces X in $\mathcal{S}_*^{\Sigma_n}$.

We will further analyse the right hand side of the above equivalence in the case $X = \Sigma|\Pi_n|^\diamond$. Here, Theorem 7.38 gives rise to an equivalence of Σ_n-spaces

$$\mathrm{Ind}^{\Sigma_n}_{\Sigma_{n_1}\times\ldots\times\Sigma_{n_k}}(\Sigma|\Pi_n|^\diamond) \simeq \bigvee_{\substack{d|\gcd(n_1,\dots,n_k)\\ w\in B(\frac{n_1}{d},\dots,\frac{n_k}{d})}} \mathrm{Ind}^{\Sigma_n}_{\Sigma_d} \left((S^{\frac{n}{d}-1})^{\wedge d} \wedge \Sigma|\Pi_d|^\diamond \right)$$

Plugging this equivalence into (21), we obtain an identification

$$\mathrm{F}_{\Sigma|\Pi_n|^\diamond}(\Sigma^{-\ell_1}V \oplus \ldots \oplus \Sigma^{-\ell_m}V) \simeq \bigoplus_{\substack{n_1+\ldots+n_m=n\\ d|\gcd(n_1,\dots,n_k)\\ w\in B(\frac{n_1}{d},\dots,\frac{n_k}{d})}} F_{\Sigma|\Pi_d|^\diamond} \left(\Sigma^{1+(1-\ell_1)\frac{n_1}{d}+\ldots+(1-\ell_m)\frac{n_m}{d}} V \right).$$

Combining this with Proposition 5.49, we can deduce the first claim: the weight n piece of $\mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\Delta}}\left(\Sigma^{\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{\ell_m}\mathbb{F}_p\right)$ is $\left(F_{\Sigma|\Pi_n|^{\diamond}}(\Sigma^{-\ell_1}\mathbb{F}_p \oplus \ldots \oplus \Sigma^{-\ell_m}\mathbb{F}_p)\right)^{\vee}$, whereas the weight n piece of $\bigoplus_{w\in B_m} \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\Delta}}\left(\Sigma^{1+\sum_i(\ell_i-1)|w|_i}(\mathbb{F}_p)\right)$ is $\bigoplus_{\substack{n_1+\ldots+n_m=n\\ d|\gcd(n_1,\ldots,n_k)\\ w\in B(\frac{n_1}{d},\ldots,\frac{n_k}{d})}} \left(F_{\Sigma|\Pi_d|^{\diamond}}\left(\Sigma^{1+(1-\ell_1)\frac{n_1}{d}+\ldots+(1-\ell_m)\frac{n_m}{d}}\mathbb{F}_p\right)\right)^{\vee}$.

The second claim follows by a parallel argument using the construction $F^h_{(-)}$ instead of $F_{(-)}$. $\square$

We combine our results to prove the main claim of this section.

Proof of Theorem 7.3 and Theorem 7.4. We first consider the case $m=1$.
If $p=2$ or ℓ_1 odd, both statements can be read off from Theorem 7.5 and Theorem 7.6, respectively.

If p is odd and ℓ_1 is even, we recall the two cofibre sequences of weight graded $\mathbb{F}_p$-module spectra established in Theorem 7.35 and Theorem 7.36:

$$\Sigma\,\mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\Delta}}[w](\Sigma^{\ell_1-1}\mathbb{F}_p) \to \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\Delta}}[w](\Sigma^{\ell_1}\mathbb{F}_p) \to \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\Delta}}\left[\frac{w}{2}\right](\Sigma^{2\ell_1-1}\mathbb{F}_p).$$

$$\Sigma\,\mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}}[w](\Sigma^{\ell_1-1}\mathbb{F}_p) \to \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}}[w](\Sigma^{\ell_1}\mathbb{F}_p) \to \mathrm{Free}_{\mathrm{Lie}^{\pi}_{k,\mathbb{E}_\infty}}\left[\frac{w}{2}\right](\Sigma^{2\ell_1-1}\mathbb{F}_p).$$

If $w=p^k$ for some k, then the right terms vanish. This implies by the "odd case" that in both cases, the middle terms have a basis consisting of all sequences $(i_1,\ldots,i_k)$ satisfying conditions $(1),(2)$ and $(1)',(2)'$ respectively, together with the respective conditions

(3) $(p-1)(\ell_1-1)\leq i_k<-1$ or $0\leq i_k\leq (p-1)(\ell_1-1)$

(3)' $i_k\leq (p-1)(\ell_1-1)$.

Since ℓ_1 is even, these conditions are (in light of the congruences (1) or $(1)'$) equivalent to

(3) $(p-1)\ell_1-1\leq i_k<-1$ or $0\leq i_k\leq (p-1)\ell_1-1$

(3)' $i_k\leq (p-1)\ell_1-1$.

This agrees with the assertions made in the two theorems (where $\epsilon=1$ and $e=0$ in this case).

If $w=2p^k$ for some k, then the respective left terms in the above cofibre sequences vanish. By the "odd cases", the middle terms have a basis consisting of all sequences $(i_1,\ldots,i_k)$ satisfying conditions $(1),(2)$ or $(1)',(2)'$, together with the respective conditions

(3) $(p-1)(2\ell_1-1)\leq i_k<-1$ or $0\leq i_k\leq (p-1)(2\ell_1-1)$.

(3)' $i_k\leq (p-1)(2\ell_1-1)$.

In light of the congruence conditions (1) or $(1)'$, these conditions are in turn equivalent to

(3) $(p-1)(2\ell_1)-1\leq i_k<-1$ or $0\leq i_k\leq (p-1)(2\ell_1)-1$.

(3)' $i_k\leq (p-1)(2\ell_1)-1$.

Again, this agrees with the assertions made in the two theorems (with $\epsilon=1$ and $e=1$ in this case).

If $w\neq p^k,2p^k$ for all k, then the outer summands in the above cofibre sequences vanish, which implies that the middle term must also vanish. This verifies the two claims in these weights. We have finally verified the two theorems whenever there is just a single generator.

The statement for $m>1$ follows immediately from the single generator case by the direct sum decomposition established in Proposition 7.39. $\square$

8. Appendix: Hypercoverings and Kan extensions

In higher algebra, simplicial resolution arguments often proceed by writing a given object X (which we want to control) as a geometric realisation of a simplicial diagram $X_\bullet$ consisting of simpler objects (which we can control). The theory of hypercoverings gives a general tool for building such simplicial resolutions. It goes back to Verdier's Exposé V in SGA 4 (cf. [AGV72]), and was studied in a higher categorical context by Dugger-Hollander-Isaksen [DHI04], Toën-Vezzosi [TV05, Section 3.2], Lurie [Lur09, Section 6.5.3] [Lur17, Section Prop. 7.2.1], and many others.

In this appendix, we will develop a variant of these ideas which will allow us to construct completed-free resolutions of complete Noetherian algebras in Theorem 6.13 above. Moreover, we will use hypercoverings to explicitly describe certain left Kan extensions of algebras; this technical result is needed in Construction 6.17 in the main body of this article.

8.1. **Construction of hypercoverings.** We begin by recalling the following classical definition:

Definition 8.1 (Matching and latching objects)**.** Let $X_\bullet$ be a simplicial object in an ∞-category $\mathcal{C}$.

(1) The nth *matching object* $M_n(X_\bullet)$ is given by the limit $M_n(X_\bullet) = \varprojlim_{[m]\to[n], m<n} X_m$, if this limit exists. The limit is taken over the opposite of the subcategory of $\Delta_{/[n]}$ spanned by arrows $[m] \to [n]$ with $m < n$. Equivalently, by a classical cofinality argument, we can take the limit over the opposite of the poset of proper subsets of $[n]$.

(2) The nth *latching object* $L_n(X_\bullet)$ is given by the colimit $\varinjlim_{[n]\to[m], m<n} X_m$, if it exists. By cofinality, we can also take the colimit over the poset of surjections $[n] \twoheadrightarrow [m]$ with $m < n$.

For each n, we have natural maps $L_n(X_\bullet) \to X_n \to M_n(X_\bullet)$.

We recall a criterion for contractibility, together with its relative variant:

Example 8.2. Let $X_\bullet$ be a simplicial space. Suppose the map $X_n \to M_n(X_\bullet)$ induces a surjection on π_0 for all $n \geq 0$. Then $|X_\bullet|$ is contractible. This is proven in [Lur09, Lemma 6.5.3.11].

Example 8.3. Let $X_\bullet$ be a simplicial space augmented over a space Z, i.e. a simplicial object of $\mathcal{S}_{/Z}$. Consider the nth mapping object $M_n(X_\bullet)$ in $\mathcal{S}_{/Z}$ (by computing the relevant limit internal to $\mathcal{S}_{/Z}$). If the map $X_n \to M_n(X_\bullet)$ induces a surjection on π_0 for all $n \geq 0$, then $|X_\bullet| \simeq Z$. This reduces to the previous example by taking homotopy fibre products over points of Z.

The theory of hypercoverings provides a generalisation of the last example: we will look for (possibly augmented) simplicial objects such that the map $X_n \to M_n(X_\bullet)$ has some type of surjectivity. We will study hypercoverings in the following general context:

Definition 8.4. Let $\mathcal{C}$ be an ∞-category which admits (finite nonempty) coproducts and a terminal object $*$, S a class of morphisms in $\mathcal{C}$, and $\mathcal{F} \subset \mathcal{C}$ a class of objects. We say that $(\mathcal{F}, S)$ forms a *weakly orthogonal pair* if:

(1) S is closed under composition and contains all equivalences. Moreover, we have the following two-out-of-three property: given composable arrows g, f with $g \circ f \in S$, we have $g \in S$ too.
(2) Pullbacks of morphisms in S along morphisms in S exist and belong to S.
(3) $\mathcal{F}$ is closed under coproducts.
(4) For each $F \in \mathcal{F}$, the map $F \to *$ belongs to S.
(5) Given $F \in \mathcal{F}$ and a morphism $f : Y \to Y'$ in S, the map $\mathrm{Map}_{\mathcal{C}}(F, Y) \to \mathrm{Map}_{\mathcal{C}}(F, Y')$ is surjective on π_0. Hence objects in F have the left lifting property with respect to S.
(6) Given an object $Y \in \mathcal{C}$, there exists a map $f : F \to Y$ in S with $F \in \mathcal{F}$.

Remark 8.5. Let $\mathcal{C}, \mathcal{F}, S$ be as in Definition 8.4 and fix an object $Z \in \mathcal{C}$. Consider the full subcategory $(\mathcal{C}_Z)' \subset \mathcal{C}_{/Z}$ consisting of those maps $Y \to Z$ which belong to S. Our assumptions imply that $(\mathcal{C}_Z)'$ contains a terminal object as well as finite nonempty coproducts. Then $(\mathcal{C}_Z)'$ admits a weakly orthogonal pair $(\mathcal{F}_Z, S_Z)$ as follows. The class $\mathcal{F}_Z$ consists of those objects in $(\mathcal{C}_{/Z})'$ whose underlying object of $\mathcal{C}$ belongs to $\mathcal{F}$. The class $S_{/Z}$ consists of those morphisms whose underlying morphism in $\mathcal{C}$ belongs to S.

We will now define the notion of a hypercovering and prove an existence statement. This is essentially a classical result from SGA4; the ∞-categorical treatment is a slight modification of [Lur17, Proposition 7.2.1.5], except that we do not assume the existence of finite limits.

Lemma 8.6 (General hypercovering lemma)**.** *Assume that $(\mathcal{F}, S)$ is a weakly orthogonal pair in an ∞-category $\mathcal{C}$ which admits finite nonempty coproducts and a terminal object $*$. Then there exists a simplicial object $X_\bullet$ such that for all $n \geq 0$, we have:*

(1) The object X_n belongs to $\mathcal{F}$.
(2) The matching object $M_n(X_\bullet)$ exists in $\mathcal{C}$.
(3) The latching object $L_n(X_\bullet)$ exists in $\mathcal{C}$.
(4) The map $X_n \to M_n(X_\bullet)$ belongs to S. (When $n = 0$, this is the map $X_0 \to *$.)
(5) The map $L_n(X_\bullet) \to X_n$ expresses X_n as a coproduct of the source and an object in $\mathcal{F}$.

Definition 8.7 ($(\mathcal{F}, S)$-hypercoverings)**.** Fix a weakly orthogonal pair $(\mathcal{F}, S)$ on an ∞-category $\mathcal{C}$.

(1) A simplicial object $X_\bullet$ is said to be an $(\mathcal{F}, S)$*-hypercovering* of the terminal object $*$ if it satisfies conditions (1)–(5) of Lemma 8.6.
(2) An augmented simplicial object $X_\bullet \to Z$ is called an $(\mathcal{F}, S)$ *hypercovering* of Z if each $X_i \to Z$ belongs to S, and, when considered as a simplicial object of $(\mathcal{C}_{/Z})'$, it is an $(\mathcal{F}_Z, S_Z)$ hypercovering of the terminal object.

To prove Lemma 8.6, we will need the following technical result:

Proposition 8.8. *Let $\mathcal{P}$ be a finite poset. Let $\mathcal{D}$ be an ∞-category containing an initial object and suppose that T is a class of morphisms in $\mathcal{D}$ which is closed under composition and contains all equivalences. Let $G : \mathcal{P} \to \mathcal{D}$ be any functor. Suppose that:*

(1) Pushouts of morphisms in T along morphisms in T exist, and remain in T.
(2) For any $x \in \mathcal{P}$, the functor $G|_{\mathcal{P}_{<x}} : \mathcal{P}_{<x} \to \mathcal{D}$ admits a colimit in $\mathcal{D}$.
(3) For any $x \in \mathcal{P}$, the morphism $\varinjlim_{y \in \mathcal{P}_{<x}} G(y) \to G(x)$ belongs to T.

Then G admits a colimit in $\mathcal{D}$, and the canonical map from the initial object to $\varinjlim_{\mathcal{P}} G$ belongs to T.

Proof. Let $\mathcal{Q} \subset \mathcal{P}$ be an arbitrary downward-closed subset; this means that if $x \in \mathcal{Q}$ and $y \in \mathcal{P}$ satisfies $y \leq x$, then $y \in \mathcal{Q}$. We claim that if $\mathcal{Q}' \subset \mathcal{Q}$ is a downward closed subset of $\mathcal{Q}$, then the colimits of G over $\mathcal{Q}, \mathcal{Q}'$ exist, and the morphism $\varinjlim_{\mathcal{Q}'} G \to \varinjlim_{\mathcal{Q}} G$ belongs to T. Taking $\mathcal{Q} = \mathcal{P}$ (and $\mathcal{Q}' = \emptyset$) will then imply the result.

Suppose $\mathcal{Q}$ is maximal among downward closed subsets for which the above claim holds true. If $\mathcal{Q} \neq \mathcal{P}$, let $z \in \mathcal{P}$ be an element minimal subject to the condition that $z \notin \mathcal{Q}$; this means that any z' with $z' < z$ belongs to $\mathcal{Q}$. In particular, $\widetilde{\mathcal{Q}} := \mathcal{Q} \cup \{z\}$ is a downward closed subset as well. The poset $\widetilde{\mathcal{Q}}$ is the union of $\mathcal{Q}$ and $\mathcal{P}_{\leq z}$, with common intersection being given by $\mathcal{P}_{<z}$. Moreover, we have a pushout, and in fact a homotopy pushout in the Joyal model structure, of simplicial sets

$$N(\widetilde{\mathcal{Q}}) = N(\mathcal{Q}) \sqcup_{N(\mathcal{P}_{<z})} N(\mathcal{P}_{\leq z}).$$

By assumption, the colimit $\varinjlim_{y\in\mathcal{P}_{<z}} G(y)$ exists, and $\varinjlim_{y\in\mathcal{P}_{<z}} G(y) \to G(z) = \varinjlim_{y\in\mathcal{P}_{\leq z}} G(y)$ belongs to T. By the defining hypothesis on $\mathcal{Q}$, we know that $\varinjlim_{y\in\mathcal{P}_{<z}} G(y) \to \varinjlim_{y\in\mathcal{Q}} G(y)$ belongs to T as well. Using [Lur09, Corollary 4.2.3.10], we deduce that $G|_{\widetilde{\mathcal{Q}}}$ admits a colimit as desired, which is given as the pushout of the restricted colimits. It follows from (1) that both $\varinjlim_{y\in\mathcal{Q}} G(y) \to \varinjlim_{y\in\widetilde{\mathcal{Q}}} G(y)$ and $\varinjlim_{y\in\mathcal{P}_{\leq z}} G(y) \to \varinjlim_{y\in\widetilde{\mathcal{Q}}} G(y)$ belong to T. Since any proper subposet $\mathcal{Q}''$ of $\widetilde{\mathcal{Q}}$ is contained in either $\mathcal{P}_{\leq z}$ or $\mathcal{Q}$, we conclude that $\varinjlim_{y\in\mathcal{Q}''} G(y) \to \varinjlim_{y\in\widetilde{\mathcal{Q}}} G(y)$ belongs to T by using the defining hypothesis of $\mathcal{Q}$ and the fact that T is closed under composition. □

We are now in a position to construct hypercoverings:

Proof of Lemma 8.6. Using condition (5) in Definition 8.4, we can chose an object $X_0 \in \mathcal{F}$ such that $X_0 \to *$ belongs to S. We will now construct a simplicial object by a recursive construction.

Suppose that we have defined X on $\Delta^{op}_{\leq r}$ so that it satisfies conditions (1)–(5) of Lemma 8.6, for all $n \leq r$. In order to extend X to $\Delta^{op}_{\leq r+1}$, we first observe that the colimit $\varinjlim_{[r+1]\to[m], m<r+1} X_m$ (i.e. the latching object, which is already defined for the r-truncated simplicial object) and the limit $\varprojlim_{[m]\to[r+1], m<r+1} X_m$ (i.e. the corresponding matching object) both exist. Furthermore, we claim that the latching object belongs to $\mathcal{F}$. To verify these claims, we apply Proposition 8.8 as follows:

1. The matching object $M_{r+1}(X_\bullet)$ (if it exists) can be computed as the limit $\varprojlim_{U\subsetneq[r+1]} X_U$, taken over the opposite of the poset of proper subsets $U \subsetneq [r+1]$. Given a proper subset $U \subsetneq [r+1]$, say $U = [m]$, the limit $\varprojlim_{U'\subsetneq[m]} X_{U'}$ exists and $X_m \to \varprojlim_{U'\subsetneq[m]} X_{U'}$ belongs to S by the inductive hypothesis. Therefore, the matching object exists by Proposition 8.8.
2. We apply a dual argument for the latching object. Indeed, define T to be the class of morphisms which are equivalent to $Y \to Y \sqcup X$ with $X \in \mathcal{F}$. By Proposition 8.8, it then follows that the latching object exists and belongs to $\mathcal{F}$.

To construct X on $\Delta^{op}_{\leq r+1}$, by [Lur09, Proposition A.2.9.15] and the surrounding discussion, it suffices to provide an object X_{r+1} and a factorisation

$$\varinjlim_{[r+1]\to[m], m<r+1} X_m \longrightarrow X_{r+1} \longrightarrow \varprojlim_{[m]\to[n], m<n} X_m.$$

We define X_{r+1} as the coproduct of the left-hand-side with an object $F \in \mathcal{F}$ with a map $F \to \varprojlim_{[m]\to[n], m<n} X_m$ that belongs to S. Then, $X_{r+1} \to \varprojlim_{[m]\to[n], m<n} X_m$ belongs to S by the two-out-of-three property of S. This extends X to $\Delta^{op}_{\leq r+1}$, and it is not hard to check that the conditions (1)–(5) are satisfied for all $n \leq r+1$. □

8.2. **Kan extensions.** Hypercoverings will allow us to compute certain left Kan extensions via geometric realisations. For this, we will need a general way of producing weakly orthogonal pairs:

Construction 8.9. Let $\mathcal{C}$ be a presentable ∞-category, and assume that $\mathcal{F}^0$ is a set of objects which is closed under finite coproducts.

a) Let $\mathcal{F}$ denote the class of objects of $\mathcal{C}$ which are (possibly infinite) coproducts of objects in $\mathcal{F}^0$.
b) Let S denote the class of morphisms $f : X \to Y$ in $\mathcal{C}$ such that for all $F \in \mathcal{F}^0$, the map of sets $\pi_0 \operatorname{Map}_{\mathcal{C}}(F, X) \to \pi_0 \operatorname{Map}_{\mathcal{C}}(F, Y)$ is surjective.

It is then straightforward to check that $(\mathcal{F}, S)$ forms a weakly orthogonal pair in $\mathcal{C}$. Part 6 of Definition 8.4 follows from a compactness argument.

Proposition 8.10. *Suppose $\mathcal{C}$ and $(\mathcal{F}, S)$ are specified as in Construction 8.9. Let $\mathcal{D}$ be a presentable ∞-category and assume that $G : \mathcal{C} \to \mathcal{D}$ is a functor which is left Kan extended from $\mathcal{F}^0$. Given any $(\mathcal{F}, S)$-hypercovering $X_\bullet$ of an object $Y \in \mathcal{C}$, we have*

$$|G(X_\bullet)| \simeq G(Y).$$

Proof. Let $\mathcal{F}^1$ be a small subcategory with $\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}$ such that the image of $X_\bullet$ is contained in $\mathcal{F}^1$. By assumption, the functor G is left Kan extended from $\mathcal{F}^1$ too; in fact, the sole purpose of introducing $\mathcal{F}^1$ is to avoid discussing Kan extensions from non-small subcategories.

Recall (cf. [Lur09, Section 4.3.2]) that the left Kan extension can be computed by the formula

$$G(Y) \simeq \varinjlim_{Z \in \mathcal{F}^1_{/Y}} G(Z).$$

We have a natural functor $\Delta^{op} \to \mathcal{F}^1_{/Y}$ given by the simplicial object $X_\bullet$, and it therefore suffices to check that this functor is left cofinal.

Using the ∞-categorical version of Quillen's Theorem A [Lur09, Theorem 4.1.3.1], we are reduced to proving that for any $Y_1 \in \mathcal{F}^1_{/Y}$, the homotopy pullback $\Delta^{op} \times_{\mathcal{F}^1_{/Y}} \mathcal{F}^1_{Y_1//Y}$ has a weakly contractible nerve. By the Grothendieck construction, it in fact suffices to show that the geometric realisation of the simplicial space $\mathrm{Hom}_{\mathcal{F}^1_{/Y}}(Y_1, X_\bullet)$ is weakly contractible. This is true because $X_\bullet$ being an $(\mathcal{F}, S)$-hypercovering of Y implies that $\mathrm{Hom}_{\mathcal{F}^1_{/Y}}(Y_1, X_\bullet)$ satisfies the conditions of Example 8.2. $\square$

We now illustrate Proposition 8.10 in two examples of interest:

Example 8.11 (Left Kan extension from $\mathrm{Perf}_{k,\leq 0}$)**.** Suppose that k is a field. We can then take $\mathcal{C}$ to be the ∞-category Mod_k and $\mathcal{F}_0$ to be the subcategory $\mathrm{Perf}_{k,\leq 0}$. It is then not hard to check that S becomes the class of morphisms in $\mathrm{Perf}_{k,\leq 0}$ which induce surjections on π_i for all $i \leq 0$.

We now claim that any $(\mathcal{F}, S)$-hypercovering $X_\bullet$ of $Y \in \mathrm{Mod}_k$ is a colimit diagram. Indeed, applying the functors $\mathrm{Hom}_{\mathrm{Mod}_k}(k[-n], -)$, and combining condition (4) of Lemma 8.6 with Example 8.3, we see that $|\Omega^{\infty-n} X_\bullet| \simeq \Omega^{\infty-n} Y$ is an equivalence for all $n \geq 0$. As we can write any spectrum Z as a canonical colimit $Z \simeq \varinjlim_n \Sigma^{\infty-n}\Omega^{\infty-n} Z$, we deduce our claim.

We can therefore explicitly describe the procedure of left Kan extension along the inclusion $\mathrm{Perf}_{k,\leq 0} \to \mathrm{Mod}_k$, even for functors which do not preserve finite coconnective geometric realisations. For this, let $G_0 : \mathrm{Perf}_{k,\leq 0} \to \mathcal{D}$ be any functor. To compute its left Kan extension $G : \mathrm{Mod}_k \to \mathcal{D}$, we first extend G_0 in a filtered-colimit-preserving way to a functor $G_1 : \mathrm{Mod}_{k,\leq 0} \to \mathcal{D}$.

Given an arbitrary k-module $Y \in \mathrm{Mod}_k$, we can pick an $(\mathcal{F}, S)$-hypercovering $X_\bullet \to Y$ by applying Lemma 8.6. By construction this means that each X_i belongs to $\mathrm{Mod}_{k,\leq 0}$. By Proposition 8.10, we obtain an equivalence $G(Y) \simeq |G_1(X_\bullet)|$. Note in particular that while G need not preserve all geometric realisations, it can still be computed in this fashion.

We conclude by generalising the preceding application of Proposition 8.10 to algebras:

Example 8.12 (Left Kan extension for ∞-categories of algebras)**.** Let $T : \mathrm{Mod}_k \to \mathrm{Mod}_k$ be a monad which preserves sifted colimits. Write $\mathcal{F}_0 \subset \mathrm{Alg}_T$ for the full subcategory spanned by all free T-algebras of the form $T(V)$ with $V \in \mathrm{Perf}_{k,\leq 0}$. It is again not difficult to check that the associated class S consists those maps of T-algebras which induce surjections on π_i for all $i \leq 0$.

Let now $\mathcal{D}$ be a presentable ∞-category, and suppose that we are given a functor $G_0 : \mathcal{F}_0 \to \mathcal{C}$. We can then ask: what is the left Kan extension $G : \mathrm{Alg}_T \to \mathcal{C}$ of G_0 to all of Alg_T?

We observe that the associated ∞-category $\mathcal{F}$ is spanned by all free T-algebras of the form $T(W)$ with $W \in \mathrm{Mod}_{k,\leq 0}$. Since G is left Kan extended from its values on compact objects, it follows that G commutes with filtered colimits, which determines its values on all objects in $\mathcal{F}$.

Given an arbitrary T-algebra A, we can use Lemma 8.6 to find an $(\mathcal{F}, S)$-hypercovering $X_\bullet$ of A. It follows as in Example 8.11 that in T-algebras, we have $|X_\bullet| \simeq A$, and Proposition 8.10 gives rise to an equivalence

$$|G(X_\bullet)| \simeq G(A).$$

For each i, the value $G(X_i)$ is determined since X_i is free on a coconnective k-module spectrum.

As a simple consequence, we deduce that any sifted-colimit-preserving functor $\mathrm{Alg}_T \to \mathcal{C}$ is left Kan extended from $\mathcal{F}_0$. The forgetful functor $\mathrm{Alg}_T \to \mathrm{Mod}_k$ is therefore left Kan extended from $\mathcal{F}_0$.

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