THE p-LOCAL STABLE ADAMS CONJECTURE AND HIGHER ASSOCIATIVE STRUCTURES ON MOORE SPECTRA

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ABSTRACT. Fix a prime p and consider the homotopy coequalizer of the identity map and any Adams operation ψ_q for any other prime $q \neq p$, acting on p-local connective complex K-theory. In this article we establish a p-local version of the stable Adams conjecture that constructs a canonical extension of the stable J-homomorphism through this coequalizer, and taking values in the infinite delooping of the group like E_∞-space of stable self homotopy equivalences of the p-local sphere. Our result can be seen as a p-local version of Friedlander's work on the infinite loop space Adams conjecture, which is a key ingredient in the proof of our result. As a powerful application, we settle the question on the height of higher associative structures on the mod p^k Moore spectrum $M_p(k)$ induced by spherical fibrations 1.1, where p is any odd prime. More precisely, for any odd prime p, we show that $M_p(k)$ admits a A_n-structure induced by a spherical fibration if and only if $n < p^k$. We also prove a weaker result for $p = 2$.

CONTENTS

1. INTRODUCTION.

The celebrated result commonly referred to as "The Adams conjecture" establishes the fact that for a given odd prime p, and for any integer q serving as a topological generator of the p-adic units $\mathbb{Z}_p^\times$, the following composite map of spaces is null homotopic (when localized at p):

$$J \circ (1 - \psi_q) : \mathrm{BU} \longrightarrow \mathrm{BU} \longrightarrow \mathrm{BG}\mathbb{S},$$

where ψ_q represents the corresponding Adams operation, and J represents the J homomorphism from the infinite unitary group to the grouplike E_∞-space $\mathrm{G}\mathbb{S}$ of stable self homotopy equivalences of the sphere. A similar statement holds for the prime $p = 2$, with BU replaced by BO and q is taken to be 3.

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The host of important consequences ensuing from this result are well known to any practitioner of Algebraic Topology (see [Ad] and subsequent articles).

Notice that all the spaces and maps involved in the statement of the Adams conjecture are infinite loop spaces and infinite loop maps (when localized at odd primes p). However, the original conjecture did not demand that the composite be null as an infinite loop map. Demanding that the composite be null as an infinite loop map is known as the *infinite loop Adams conjecture* or the *stable Adams conjecture*. This stable enhancement of the Adams conjecture has its own set of important consequences [MP].

In [F], Friedlander proved a completed version of the stable Adams conjecture. More precisely, he shows in [F] (Theorem 10.4) that for any integer q prime to p, the following composite map is null:

$$J \circ (1 - \psi_q) : \mathrm{kU}_{(p)} \longrightarrow \mathrm{kU}_{(p)} \longrightarrow \mathrm{bSG}\mathbb{S}_p,$$

where $\mathrm{kU}_{(p)}$ denotes the p-local connective complex K-theory spectrum. The spectrum $\mathrm{bSG}\mathbb{S}_p$ is an infinite delooping of the space $\mathbb{Z} \times \mathrm{BSG}\mathbb{S}_p$, where $\mathrm{BSG}\mathbb{S}_p$ is the classifying space of the identity component $\mathrm{SG}\mathbb{S}$ of the grouplike E_∞-space $\mathrm{G}\mathbb{S}_p$ of stable self homotopy equivalences of the p-complete sphere.

Let $\mathrm{hCE}(\psi_\mathrm{q})$ denote the homotopy coeqalizer of ψ_q and the identity self map of $\mathrm{kU}_{(p)}$, then Friedlander's theorem yields a (possibly non-unique) extension of the J-homomorphism:

$$J_q : \mathrm{hCE}(\psi_\mathrm{q}) \longrightarrow \mathrm{bSG}\mathbb{S}_p \, .$$

By construction, the above map is trivial on π_1 since $\pi_1(\mathrm{hCE}(\psi_\mathrm{q})) = \mathbb{Z}$ and $\pi_1(\mathrm{bSG}\mathbb{S}_p) = 0$.

For our purposes, we would like a p-local version of the above statement. More precisely, we would like to replace $\mathrm{bSG}\mathbb{S}_p$ by the spectrum $\mathrm{bG}\mathbb{S}_{(p)}$ given by the infinite delooping of the space $\mathbb{Z} \times \mathrm{BG}\mathbb{S}_{(p)}$, where $\mathrm{BG}\mathbb{S}_{(p)}$ denotes the classifying space of the group like E_∞-space of stable self homotopy equivalences of the p-local sphere. Notice that $\pi_1(\mathrm{bSG}\mathbb{S}_{(p)}) = \mathbb{Z}^\times_{(p)}$. We also demand the existence of an extension of the map J to $\mathrm{hCE}(\psi_\mathrm{q})$, which has a prescribed behavior on the first homotopy group π_1. We call this the *p-local stable Adams conjecture*, which we shall prove in this article:

Theorem. *Given a prime p, let $q \neq p$ be any other prime. Then the J-homomorphism, seen as a stable map from $\mathrm{kU}_{(p)}$ to $\mathrm{bG}\mathbb{S}_{(p)}$, extends to a canonical map:*

$$J_q : \mathrm{hCE}(\psi_\mathrm{q}) \longrightarrow \mathrm{bG}\mathbb{S}_{(p)} \, .$$

Moreover, the map J_q has the property that it sends a generator of $\pi_1(\mathrm{hCE}(\psi_\mathrm{q})) = \mathbb{Z}$ to the element $q \in \mathbb{Z}^\times_{(p)} = \pi_1(\mathrm{bG}\mathbb{S}_{(p)})$.

Remark 1. *In [FS] R. Seymour announced a version of the p-local stable Adams conjecture where the target of the J-homomorphism was lifted from $\mathrm{bSG}\mathbb{S}_p$ to the whole local automorphism group $\mathrm{bG}\mathbb{S}_{(p)}$. However, that proof was later retracted due to a fatal flaw in the argument.*

Remark 2. *Notice the curious requirement in the statement of our solution of the p-local stable Adams conjecture that requires that the integer q be a prime (and not just prime to p). One may trace this requirement to the technique we use of approximating the spaces $\mathrm{BGl}_i(\mathbb{C})$, at the prime p, by their discrete approximations $\mathrm{BGl}_i(\overline{\mathbb{F}}_q)$, so as to identify the Adams operations ψ_q with the Frobenius automorphism on $\overline{\mathbb{F}}_q$.*

Having established the p-local stable Adams conjecture, we will also demonstrate that it has far-reaching applications. In particular, we will use the above theorem to explore the existence of higher associative structures (as defined by Stasheff [St]) on the mod p^k Moore spectrum $M_p(k)$, which is defined as the cofiber of the degree p^k-map on the sphere spectrum $\mathbb{S}$, where p is a prime number.

Let us consider some examples of the question of higher associative structures on Moore spectra. Using squaring operations, it can be easily seen that $M_2(1)$ do not admit a unital multiplication i.e. an A_2-structure. The obstruction class is precisely $\eta \in \pi_1(M_2(1))$.

The history of the question of higher A_n-structures on the Moore spectra goes back at least 50 years. In 1968, Toda [T] proved that $M_3(1)$ admits a unital multiplication, but there does not exist a homotopy associative multiplication, i.e. an A_3-structure. Work of Kochman [K] and Kraines [Kr] can be combined to prove that, for any odd prime p, $M_p(1)$ admits an A_{p-1}-structure and not an A_p-structure, the Greek letter element $\alpha_1 \in \pi_{2p-3}(M_p(1))$ being the obstruction (see [A, Example 3.3]). In 1982, Oka [O] proved that $M_2(k)$ admits an A_3-structure for $k > 1$. Aside from these sporadic results, the question of A_n-structures on $M_p(k)$ has proved to be intractable and has remained largely open.

More recently, first author in his thesis work [Bh], obtained a lower bound on k, dependant on p and n, which guarantees an A_n-structure on $M_p(k)$. Let $\mathrm{BG}\mathbb{S}_p$ be the classifying space of $\mathrm{G}\mathbb{S}_p$, the group like E_∞-space of stable self-homotopy equivalences of the p-adic sphere and let $\lambda \in \mathbb{Z}_p^\times$. The key idea in [Bh] was to obtain $M_p(i)$ as the Thom spectrum associated to the map

$$\tau : S^1 \longrightarrow \mathrm{BG}\mathbb{S}_p$$

which sends $1 \mapsto 1 + p^k\lambda$ on π_1. The Moore spectrum $M_p(k)$ admits an A_n-structure if the map τ is an A_n-map (see [LMS, §IX]). Motivated by this we define:

Definition 1.1. *The Moore spectrum $M_p(k)$ is said to admit an A_n-structure induced by a spherical fibration if there exists an A_n-map*

$$\tau : S^1 \longrightarrow \mathrm{BG}\mathbb{S}_p,$$

which has degree $1 + p^k\lambda$ for some $\lambda \in \mathbb{Z}_p^\times$. As mentioned above, the Thom spectrum of τ is equivalent to $M_p(k)$ with the induced A_n-structure.

By a theorem due to Stasheff [St, Theorem 8.4], τ is an A_n-map if and only if it extends to a stable map of the form

$$\tau : S^1 \longrightarrow \Sigma^{-1}\mathbb{CP}^n \longrightarrow \mathrm{bG}\mathbb{S}_p,$$

where $\mathrm{bG}\mathbb{S}_p$ is the spectrum which is the infinite delooping of $\mathbb{Z} \times \mathrm{BG}\mathbb{S}_p$. In [Bh], the first author uses Atiyah Hirzebruch spectral sequence to obtain a lower bound on k for which $1 + p^k\lambda$ extends to $\Sigma^{-1}\mathbb{CP}^n$. Moreover, the first author expected the classical image of J spectrum to extend p-adically to a spectrum, say $\mathcal{J}_p$, such that τ admits a lift

$$\tau : S^1 \to \Sigma\,\mathcal{J}_p \to \mathrm{bG}\mathbb{S}_p\,. \tag{1}$$

Therefore it was conjectured (see [Bh, Conjecture 2]) that the obstruction to A_n-structures on $M_p(k)$ lies in the image of J part of stable homotopy groups of $M_p(k)$. In this paper we prove that factorization similar to (1) is possible if we work in p-local settings instead of p-complete settings (see Remark 3).

Indeed, in this article, we will completely solve the problem (at odd primes) of the height of associate structures on the Moore spectra that are induced by spherical fibrations. Invoking the map J_q in the statement of p-local stable Adams conjecture (for a suitable choice of prime q) allows us to perform the constructions necessary to solve the problem.

Theorem. *Given an odd prime p, the spectrum $M_p(k)$ admits an A_n-structure induced by a spherical fibration if and only if $n < p^k$. For $p = 2$ and $k > 1$, $M_2(k+1)$ admits an A_{2^k-1}-structure induced by a spherical fibration but it cannot support any such $A_{2^{k+1}}$-structure.*

Here is a brief outline of the strategy employed to prove the above theorem. Recall that $\mathrm{G}\mathbb{S}_{(p)}$ denotes the group like E_∞-space of stable self homotopy equivalences of the p-local sphere. Also recall that:

$$\pi_0(\mathrm{G}\mathbb{S}_{(p)}) = \mathbb{Z}_{(p)}^\times.$$

One can also obtain $M_p(k)$ as a Thom spectrum for the virtual dimension zero fibration given by any pointed map:

$$\tau = 1 + p^k\lambda : S^1 \longrightarrow \mathbb{Z} \times \mathrm{BG}\mathbb{S}_{(p)}$$

where λ is prime to p. Recall that the space $\mathbb{Z} \times \mathrm{BG}\mathbb{S}_{(p)}$ deloops to a spectrum $\mathrm{bG}\mathbb{S}_{(p)}$. Invoking the p-local stable Adams conjecture, we may lift the map τ to the spectrum $\mathrm{hCE}(\psi_{\mathrm{q}})$, where $\mathrm{hCE}(\psi_{\mathrm{q}})$ denotes the homotopy co-equalizer of a suitable stable Adams operation acting on p-localized complex K-theory. This lift of τ to $\mathrm{hCE}(\psi_{\mathrm{q}})$ allows us to construct an explicit extension of τ through a complex projective space of the right dimension.

Since we will make extensive use of the methods introduced by Friedlander in [F], let us briefly review Friedlander's argument (see the next section for more details). The argument involves constructing the J homomorphism as a map between two Γ-spaces $\mathcal{B}\mathrm{Gl}(\mathbb{C})$ and $\mathcal{B}\mathrm{G}$ (see Definition 2.1 for Γ-spaces, and Examples 2.5, 2.6). The Γ-space $\mathcal{B}\mathrm{Gl}(\mathbb{C})$ is constructed from the permutative category of finite dimensional complex vector spaces. The Γ-space $\mathcal{B}\mathrm{G}$ is the classifying space for sectioned even dimensional spherical fibrations over Γ-spaces (for the classification theorem see 2). The J homomorphism $J : \mathcal{B}\mathrm{Gl}(\mathbb{C}) \longrightarrow \mathcal{B}\mathrm{G}$ is defined to be the map that classifies the spherical fibration over $\mathcal{B}\mathrm{Gl}(\mathbb{C})$ given by fiberwise compactifying the tautological vector bundles.

A fundamental result due to G. Segal shows that each Γ-space gives rise to a spectrum in a functorial fashion (see Theorem 2.3). As such, the map J described above represents the classical J-homomorphism from complex connective K-theory to the theory of stable spherical fibrations represented by the spectrum of units of the sphere spectrum.

Friedlander describes the p-completion of $\mathcal{B}\mathrm{G}$ as the space classifying even dimensional p-completed spherical fibrations as defined in [F]. The p-completion functor above has the property that it p-completes the infinite loop space underlying the Γ-space. By using algebraic models for Grassmannians, and invoking Etalé homotopy theory, Friedlander describes the Adams operation ψ_q as an automorphism of the p-completed spherical fibration over $\mathcal{B}\mathrm{Gl}(\mathbb{C})\hat{}_p$ for any prime $q \neq p$. By the uniqueness result for classifying maps, it follows that one has an equality (up to homotopy) of maps between Γ-spaces:

$$J \circ \psi_q = J : \mathcal{B}\mathrm{Gl}(\mathbb{C})\hat{}_p \longrightarrow \mathcal{B}\mathrm{G}\,\hat{}_p.$$

Since the p-completion functor does not commute with smash products, a p-completed fibrations is not a fibration in the usual sense (see Remark 3). For the applications we have in mind in this paper, it will be important to get around this problem. We will therefore need to work with p-localization instead. More precisely, we will construct a Γ-space $\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$ equivalent to the object wise localization of the Γ-space $\mathcal{B}\mathrm{Gl}(\mathbb{C})$, so that it supports a sectioned fibration with fibers equivalent to even dimensional p-localized spheres. In addition, the Adams operation ψ_q will lift to an automorphism of this spherical fibration over $\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$. We let $\mathcal{B}\mathrm{G}\mathbb{S}_{(p)}$ denote the classifying Γ-space for sectioned fibrations with fibers equivalent to even dimensional p-local spheres. By the results of Friedlander, we have a unique map $J : \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{G}\mathbb{S}_{(p)}$ up to homotopy that classifies this fibration. The uniqueness of J also implies that we have an equality up to homotopy:

$$J \circ \psi_q = J : \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{G}\mathbb{S}_{(p)} .$$

In particular, the map J factors through the homotopy co-equalizer $\mathrm{hCE}(\psi_\mathrm{q})$ of ψ_q acting on $\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$. The spectrum underlying $\mathcal{B}\mathrm{G}\mathbb{S}_{(p)}$ (as constructed by G. Segal) is none other than the spectrum $\mathrm{bG}\mathbb{S}_{(p)}$ introduced earlier, and represents the cohomology theory whose value on a compact space is the Grothendieck group of fibrations with fiber having the homotopy type of an even dimensional p-local sphere, endowed with a "base point" section.

Remark 3. *We point out to the reader that the Γ-space $\mathcal{B}\mathrm{G}\hat{\ }_p$ (as defined in [F]) does not classify sectioned fibrations with fibres being equivalent to even dimensional p-complete spheres. As was pointed out earlier, the reason for this fact is that the p-completion functor does not commute with smash products. In contrast, $\mathcal{B}\mathrm{G}\mathbb{S}_{(p)}$ does indeed classify sectioned fibrations with fibers being equivalent to even dimensional p-local spheres. In particular, the spaces $\mathcal{B}\mathrm{G}\mathbb{S}_{(p)}$ and $\mathcal{B}\mathrm{G}\hat{\ }_p$ are quite different inspite of what the notation suggests. For example, notice that the fundamental group of the infinite loop space underlying $\mathcal{B}\mathrm{G}\mathbb{S}_{(p)}$ is isomorphic to the p-local units, while that of the infinite loop space underlying $\mathcal{B}\mathrm{G}\hat{\ }_p$ is trivial at odd primes. This distinction will turn out to be crucial for our purposes.*

Before we begin, we would like to acknowledge our debt to the beautiful papers by Eric Friedlander [BF, F, F2, FM]. Our paper is built around his ideas and would not have been possible otherwise. We also thank Vigleik Angeltveit, Andrew Blumberg, David Gepner, Jesper Grodal, Mike Hopkins, Michael Mandell and Mona Merling for several helpful conversations pertaining to this article. And finally, the second author would like to thank the Simons Foundation and the Max Planck Institute for Mathematics, Bonn, for their hospitality during the period when this work was completed.

2. Review of Γ-spaces after Friedlander.

We now define the objects of interest beginning with the definition of Γ-spaces, and moving on to the theory of sectioned fibrations and their classifying spaces. Complete details can be found in [BF] and [F]. For convenience, we will use the notation from [F].

In staying faithful to the context used by Friedlander, we will work in the category of simplicial sets, which we refer to as "spaces". We reserve the term "topological spaces" for the category of usual spaces.

Definition 2.1. *A* Γ*-space is a functor:*

$$\mathcal{B} : \mathcal{F} \longrightarrow s\mathcal{S}et_*,$$

from the category $\mathcal{F}$ *of finite pointed sets, to the category* $s\mathcal{S}et_*$ *of pointed simplicial sets, such that* $\mathcal{B}$ *sends each singleton to the constant point simplicial set. We denote the category of* Γ*-spaces* $\mathcal{F}[s\mathcal{S}et_*]$ *to be the category of such functors. Note that it is sufficient to define a* Γ*-space on the full sub-category generated by the objects* $\mathbf{n} = \{0, 1, \ldots, n\}$ *(with the element* 0 *being the basepoint). Henceforth, we shall consider all* Γ*-spaced restricted to this sub-category.*

Example 2.2. Let the set of non-negative numbers $\mathbb{N}$ be seen as a discrete pointed simplicial set, with 0 being the basepoint. Define a Γ-space $\mathcal{N}$ by demanding that $\mathcal{N}(\mathbf{n}) = \mathbb{N}^{\times n}$ for $n > 0$ and given $\alpha : \mathbf{n} \longrightarrow \mathbf{k}$ in $\mathcal{F}$:

$$\alpha(i_1, \ldots, i_n) = (j_1, \ldots, j_k), \quad \text{where} \quad j_t = \sum_{s \in \alpha^{-1}(t)} i_s, \quad \text{if} \quad \alpha^{-1}(t) \neq \emptyset,$$

and the element j_t is defined to be 0 if $\alpha^{-1}(t)$ is empty.

All the Γ-spaces $\mathcal{B}$ we consider in this article will admit a canonical map to $\mathcal{N}$. In other words, the examples we consider are naturally objects in $\mathcal{F}[s\mathcal{S}et_*]/\mathcal{N}$, the category over $\mathcal{N}$. Given an n-tuple of non-negative integers $I = (i_1, \ldots, i_n) \in \mathbb{N}^{\times n}$, we may define $\mathcal{B}_I(\mathbf{n})$ to be the pre-image of I under the map $\mathcal{B}(\mathbf{n}) \longrightarrow \mathcal{N}(\mathbf{n})$. In particular, we have a decomposition:

$$\mathcal{B}(\mathbf{n}) = \coprod_{I \in \mathbb{N}^{\times n}} \mathcal{B}_I(\mathbf{n}).$$

Since $\mathbf{n}$ is implicit in the index I , we will henceforth denote $\mathcal{B}_I(\mathbf{n})$ by just $\mathcal{B}_I$.

The category $\mathcal{F}[s\mathcal{S}et_*]$ has the structure of a closed model category, with weak equivalences being defined object wise. Let Σ be the finite simplicial set representing the circle with two non-degenerate simplices, and Σ^n denotes its n-fold smash product. For a Γ-space $\mathcal{B}$, we define $\Phi(\mathcal{B})$ to be the spectrum given by spectrifying the pre-spectrum:

$$\{\mathcal{B}(\Sigma^n),\, \Sigma \wedge \mathcal{B}(\Sigma^n) \longrightarrow \mathcal{B}(\Sigma^{n+1}),\ n \geq 1\},$$

where $\mathcal{B}(\Sigma^n)$ is defined to be the diagonal of the bi-simplicial set $\mathcal{B}(\Sigma^n_\bullet)$. The suspension map $\Sigma \wedge \mathcal{B}(\Sigma^n) \longrightarrow \mathcal{B}(\Sigma^{n+1})$ is induced by applying $\mathcal{B}$ to the maps of finite sets parametrized by the simplices of Σ:

$$\{x\} \times \Sigma^n_k \longrightarrow \Sigma^{n+1}_k, \quad x \in \Sigma_k, \quad k \geq 0.$$

The following fundamental result is essentially due to Greame Segal [Se]:

Theorem 2.3. [BF] [Se] *The map* $\mathcal{B} \mapsto \Phi(\mathcal{B})$ *defines a functor*

$$\Phi : \mathrm{Ho}\,\mathcal{F}[s\mathcal{S}et_*] \longrightarrow \mathrm{Ho(Sp)}$$

from the homotopy category of Γ*-spaces, to the homotopy category of Spectra. Furthermore, the pre-spectrum* $\{\mathcal{B}(\Sigma^n),\, \Sigma \wedge \mathcal{B}(\Sigma^n) \longrightarrow \mathcal{B}(\Sigma^{n+1})\}$*, is an* Ω*-spectrum if* $\mathcal{B}$ *is "special" i.e. satisfies the property that the following map of simplicial sets is a weak equivalence:*

$$\prod_{i=1}^{n} \pi_i : \mathcal{B}(\mathbf{n}) \longrightarrow \prod_{i=1}^{n} \mathcal{B}(\mathbf{1}),$$

where $\pi_i : \mathbf{n} \longrightarrow \mathbf{1}$ *denote the various projections that send* $i \in \mathbf{n}$ *to* 1*, and* $j \neq i$ *to* 0*.*

Before we give important examples of special Γ-spaces, we will need to introduce a convenient simplicial model for the classifying space of a topological category:

Definition 2.4. *Let $\mathcal{C}$ be a topological category endowed with a distinguished base-point object $*$. Then the pointed simplicial set $\mathrm{B}\mathcal{C}$ is defined to be the diagonal of the bi-simplicial set obtained by taking level wise singular simplicies on the nerve of $\mathcal{C}$:*

$$\mathrm{B}\mathcal{C} := \Delta \circ \mathit{Sing}_\bullet\, \mathrm{N}_\bullet(\mathcal{C}).$$

Observe that the trivial category $*$ given by the identity morphism about $*$ maps to $\mathcal{C}$ endowing $\mathrm{B}\mathcal{C}$ with a base point. This construction is clearly natural with respect to functors of pairs $\{\mathcal{C}, *\}$. A particular example of this construction is the classifying space $\mathrm{B}\mathcal{M}$ of a topological monoid $\mathcal{M}$.

Remark 4. *The above definition is an example of a more general construction we will see throughout this article. We take this opportunity to set some conventions. If G is a (topological) group that acts on a simplicial (topological) space $X_\bullet$, then recall the standard the Bar construction $\mathrm{B}_\bullet(\mathrm{G}, X_\bullet)$ for this action gives rise to a bisimplicial (topological) space whose $(n, \bullet)$-simplicies are given by the product $\mathrm{G}^{\times n} \times X_\bullet$. In particular, one can apply the Singular functor to get a trisimplicial set $\mathit{Sing}_\bullet\, \mathrm{B}_\bullet(\mathrm{G}, X_\bullet)$ whose $(m, n, \bullet)$ simplicies are given by $\mathit{Sing}_m(\mathrm{G}^{\times n} \times X_\bullet)$. We may recover a regular simplicial set $\mathrm{B}(\mathrm{G}, X)$ from this construction by taking the diagonal*

$$\mathrm{B}(\mathrm{G}, X) := \Delta \circ \mathit{Sing}_\bullet\, \mathrm{B}_\bullet(\mathrm{G}, X_\bullet).$$

The homotopy type of $\mathrm{B}(\mathrm{G}, X)$ is equivalent to the Borel construction for the G-action on $|X|$ given by geometrically realizing the simplicial (topological) space: $\mathrm{G}^{\times \bullet} \times |X|$.

The above definition leads us to important examples of special Γ-spaces.

Example 2.5. Let X denote a pointed simplicial set, and let X^n denote its n-fold smash product. Let $\mathrm{G}(|X^n|)$ denote the topological monoid of self homotopy equivalences of the realization $|X^n|$. Consider the topological category $\mathcal{C}(X)$ with objects given by the collection $\{X^n,\, n \in \mathbb{N}\}$. We demand that X^0 is the distinguished object $*$. The topological space of morphisms between X^n and X^m is trivial unless $m = n$, in which case it is given by the monoid $\mathrm{G}(|X^n|)$. The category $\mathcal{C}(X)$ is a permutative category with monoidal structure given by the "block decomposition":

$$\Box : \mathcal{C}(X) \times \mathcal{C}(X) \longrightarrow \mathcal{C}(X), \qquad X^n \Box X^m = X^{n+m}.$$

The effect of $\Box$ on morphisms is induced via the canonical homomorphism:

$$\mathrm{G}(|X^n|) \times \mathrm{G}(|X^m|) \longrightarrow \mathrm{G}(|X^{n+m}|).$$

It is shown in [M], that there is a special Γ-space $\mathcal{B}\mathrm{GX}$ with:

$$\mathcal{B}\mathrm{GX}(\mathbf{1}) = \mathrm{B}\mathcal{C}(X) = \coprod_{n \geq 0} \mathrm{BG}(|X^n|), \quad \mathcal{B}\mathrm{GX}_I = \prod_{j=1}^{n} \mathrm{BG}(|X^{i_j}|), \quad \text{where} \quad I = (i_1, \cdots, i_n).$$

Taking X to be the two sphere S^2 in the previous example gives rise to the Γ-space $\mathcal{B}\mathrm{G}$ considered earlier.

Example 2.6. Consider the topological category $\mathcal{U}$ with objects given by the collection of standard complex Euclidean spaces: $\{\mathbb{C}^n,\, n \in \mathbb{N}\}$, with $\mathbb{C}^0 = *$ being the distinguished object. The topological space of morphism between $\mathbb{C}^n$ and $\mathbb{C}^m$ is trivial unless $n = m$, in which case it is given by the general linear group $\mathrm{Gl}_n(\mathbb{C})$. The permutative structure of $\mathcal{U}$ is again given by "block decomposition":

$$\Box : \mathcal{U} \times \mathcal{U} \longrightarrow \mathcal{U}, \qquad \mathbb{C}^n \Box \mathbb{C}^m = \mathbb{C}^{n+m}.$$

As before, there is a special Γ-space $\mathcal{B}\mathrm{Gl}(\mathbb{C})$ with the property:

$$\mathcal{B}\mathrm{Gl}(\mathbb{C})(\mathbf{1}) = \mathrm{B}\mathcal{U} = \coprod_{n \geq 0} \mathrm{BGl}_n(\mathbb{C}), \qquad \mathcal{B}\mathrm{Gl}_I(\mathbb{C}) = \prod_{j=1}^{n} \mathrm{BGl}_{i_j}(\mathbb{C}), \quad \text{where} \quad I = (i_1, \cdots, i_n).$$

The spectrum $\Phi(\mathcal{B}\mathrm{Gl}(\mathbb{C}))$ is precisely the spectrum representing connective complex K-theory as was mentioned earlier.

Remark 5. *Notice that the construction of $\mathcal{B}\mathrm{Gl}(\mathbb{C})$ is natural in the field $\mathbb{C}$. In particular, $\mathbb{C}$ can be replaced by an arbitrary topological ring $\mathbb{W}$ to give rise to a special Γ-space $\mathcal{B}\mathrm{Gl}(\mathbb{W})$:*

$$\mathcal{B}\mathrm{Gl}(\mathbb{W})(\mathbf{1}) = \coprod_{n \geq 0} \mathrm{BGl}_n(\mathbb{W}), \qquad \mathcal{B}\mathrm{Gl}_I(\mathbb{W}) = \prod_{j=1}^{n} \mathrm{BGl}_{i_j}(\mathbb{W}), \quad \textit{where} \quad I = (i_1, \cdots, i_n).$$

We now come to an important definition of a sectioned fibration of Γ-spaces. The first part of the following definition defines a arbitrary sectioned map, and the next part defines a sectioned fibration:

Definition 2.7. *A map $f : \mathcal{E} \longrightarrow \mathcal{B}$ of Γ-spaces over $\mathcal{N}$ is said to be sectioned if for an $n > 0$, and any pointed subset $S \subseteq \mathbf{n}$, there exists a sub-simplicial set $\mathcal{E}^S(\mathbf{n}) \subseteq \mathcal{E}(\mathbf{n})$ such that:*

$$\mathcal{E}^{\{0\}}(\mathbf{n}) \cong \mathcal{B}(\mathbf{n}) \quad \textit{via the map f, and} \qquad \mathcal{E}^T(\mathbf{n}) \subseteq \mathcal{E}^S(\mathbf{n}), \quad \textit{if} \quad T \subseteq S.$$

Given a morphism $d : \mathbf{n} \longrightarrow \mathbf{t}$ in $\mathcal{F}$, let $(d : S) \subseteq \mathbf{t}$ be defined by $0 \neq j \in (d : S)$ if and only if $d^{-1}(j) \subseteq S$. Then we require that d restricts to a map: $d : \mathcal{E}^S(\mathbf{n}) \longrightarrow \mathcal{E}^{(d:S)}(\mathbf{t})$.

Now let X be a pointed simplicial set with X^i denoting the i-fold smash product of X. A sectioned map $f : \mathcal{E} \longrightarrow \mathcal{B}$ is called an X-fibration if the for any n-tuple $I = (i_1, \ldots, i_n) \in \mathbb{N}^{\times n}$, the induced map: $f_I : \mathcal{E}_I \longrightarrow \mathcal{B}_I$, is a fibration with fiber being equivalent to $X^{i_1} \times \cdots \times X^{i_n}$. Furthermore, one requires that the following map is a fiber homotopy equivalence over $\mathcal{B}_I$:

$$\prod_{j=1}^{n} \pi_j : \mathcal{E}_I \longrightarrow \prod_{j=1}^{n} \mathcal{E}_{i_j} \times_{\prod \mathcal{B}_{i_j}} \mathcal{B}_I.$$

Finally, for any pointed subset $S \subseteq \mathbf{n}$, we require that the restriction of f_I is also a fibration: $f_I^S : \mathcal{E}_I^S \longrightarrow \mathcal{B}_I$ with the property that given any map $\alpha : \mathbf{n} \longrightarrow \mathbf{k}$ in $\mathcal{F}$ which restricts to an isomorphism on S, one has a fiber homotopy equivalence over $\mathcal{B}_I$:

$$(\alpha \times f_I) : \mathcal{E}_I^S \longrightarrow \mathcal{E}_{\alpha(I)} \times_{\mathcal{B}_{\alpha(I)}} \mathcal{B}_I.$$

The following theorem of Friedlander classifies sectioned X-fibrations:

Theorem 2.8. [F] *Given a pointed simplicial set X, the Γ-space $\mathcal{B}\mathrm{GX}$ constructed in Example 2.5 supports a universal sectioned X-fibration. In other words, equivalence classes of sectioned X-fibrations over an arbitrary Γ-space $\mathcal{B}$ are in bijective correspondence with maps from $\mathcal{B}$ to $\mathcal{B}\mathrm{GX}$ in the homotopy category of Γ-spaces over $\mathcal{N}$.*

3. A SPHERICAL FIBRATION OVER $\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$ AND THE p-LOCAL STABLE ADAMS CONJECTURE.

In this section, we extend Friedlander's constructions introduced earlier to build a sectioned $S^2_{(p)}$-fibration $\pi : \mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$ which supports an action of the Adams operation ψ_q for any prime $q \neq p$. By the classification results of Friedlander, this fibration would then be classified by a J-homomorphism invariant under ψ_q. The strategy we employ to construct the above fibration is to use an arithmetic fracture square using a suitable ring $\mathbb{W}$.

Fix a prime p and let $q \neq p$ be any other prime. Henceforth, we take $\mathbb{W}$ to be the ring of Witt vectors on the separable closure of the finite field $\mathbb{F}_q$ of q-elements. In particular, the Frobenius automorphism $\mathbf{Fr}_q$ lifts to an automorphism of $\mathbb{W}$. Let us also fix an embedding $\iota : \mathbb{W} \subset \mathbb{C}$. Notice that the units: $\mathrm{Gl}_1(\overline{\mathbb{F}}_q) = \overline{\mathbb{F}}_q^{\times}$ belong to $\mathbb{W}^{\times}$ and can be identified within the roots of unity in $\mathbb{C}$ under the above embedding. The central result in this context is the result of Friedlander and Mislin:

Theorem 3.1. [FM] *Consider the maps of topological rings:*

$$\mathbb{W} \longrightarrow \overline{\mathbb{F}}_q, \qquad \iota : \mathbb{W} \longrightarrow \mathbb{C},$$

with $\overline{\mathbb{F}}_q$ and $\mathbb{W}$ endowed with the discrete topology. Then, the above maps induce equivalences of classifying spaces when completed at the prime p:

$$\mathrm{BGl}_i(\mathbb{W})\hat{}_p \longrightarrow \mathrm{BGl}_i(\overline{\mathbb{F}}_q)\hat{}_p, \qquad \mathrm{BGl}_i(\mathbb{W})\hat{}_p \longrightarrow \mathrm{BGl}_i(\mathbb{C})\hat{}_p.$$

Furthermore, under these equivalences, the automorphism of $\mathrm{BGl}_i(\overline{\mathbb{F}}_q)\hat{}_p$ induced by the Frobenius automorphism $\mathbf{Fr}_q$ is equivalent to the unstable Adams operaton ψ_q on $\mathrm{BGl}_i(\mathbb{C})\hat{}_p$.

With the above setup in place, we begin by defining various Γ-spaces that we will use to construct an arithmetic square.

Definition 3.2. *For $i \geq 0$, we define the subgroups of the general linear group:*

$$\mathrm{N}_i(\mathbb{W}) := \Sigma_i \ltimes \mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times i} \subseteq \mathrm{Gl}_i(\mathbb{W}), \qquad \mathrm{N}_i(\mathbb{C}) := \Sigma_i \ltimes \mathrm{Gl}_1(\mathbb{C})^{\times i} \subseteq \mathrm{Gl}_i(\mathbb{C})$$

and their classifying spaces (as described in Definition 2.4):

$$\mathrm{BN}_i(\mathbb{W}) = \mathrm{B}(\Sigma_i \ltimes \mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times i}), \qquad \mathrm{BN}_i(\mathbb{C}) = \mathrm{B}(\Sigma_i \ltimes \mathrm{Gl}_1(\mathbb{C})^{\times i}).$$

Remark 6. *Notice that the spaces $\mathrm{BN}_i(\mathbb{W})$ and $\mathrm{BN}_i(\mathbb{C})$ fiber over $\mathrm{B}\Sigma_i$ with fibers $\mathrm{BGl}_1(\overline{\mathbb{F}}_q)^{\times i}$ and $\mathrm{BGl}_1(\mathbb{C})^{\times i}$ respectively. In particular, one may invoke Theorem 3.1 to deduce that the following canonical map is also an equivalence:*

$$\mathrm{B}_\iota : \mathrm{BN}_i(\mathbb{W})\hat{}_p \longrightarrow \mathrm{BN}_i(\mathbb{C})\hat{}_p.$$

Notice also that the automorphism of $\mathrm{BN}_i(\mathbb{W})\hat{}_p$ induced by the Frobenius $\mathbf{Fr}_q$ agrees (under the map B_ι) with the automorphism of $\mathrm{BN}_i(\mathbb{C})\hat{}_p$ induced by the q-power self map of $\mathrm{Gl}_1(\mathbb{C})^{\times i}$.

Definition 3.3. *Let $\eta : \mathcal{R}\mathrm{BN}_i(\mathbb{C}) \longrightarrow \mathrm{BN}_i(\mathbb{C})\hat{}_p$ denote a functorial fibrant replacement of the p-completion map for $\mathrm{BN}_i(\mathbb{C})$, where p-completion denotes the Bousfield localization [B] with respect to $\mathrm{H}\mathbb{Z}/\mathrm{p}$. Define the space BN_i as the pullback (this space is equivalent to $\mathrm{BN}_i(\mathbb{C})$ by Remark 6) :*

$$\begin{tikzcd} \mathrm{BN}_i \arrow[r, "\eta"] \arrow[d, "\varphi_i"] & \mathrm{BN}_i(\mathbb{W})\hat{}_p \arrow[d, "\mathrm{B}_\iota"] , \\ \mathcal{R}\mathrm{BN}_i(\mathbb{C}) \arrow[r, "\eta"] & \mathrm{BN}_i(\mathbb{C})\hat{}_p \end{tikzcd}$$

where we have used η to also denote the induced map on the pullback. Using the naturally of the above construction, we have three Γ-spaces $\mathcal{B}\mathrm{N}$, $\mathcal{R}\mathcal{B}\mathrm{N}(\mathbb{C})$ and $\mathcal{B}\mathrm{N}(\mathbb{W})$:

$$\mathcal{B}\mathrm{N}_I = \prod_{j=1}^{n} \mathrm{BN}_{i_j}, \qquad \mathcal{R}\mathcal{B}\mathrm{N}_I(\mathbb{C}) = \prod_{j=1}^{n} \mathcal{R}\mathrm{BN}_{i_j}(\mathbb{C}), \qquad \mathcal{B}\mathrm{N}_I(\mathbb{W}) = \prod_{j=1}^{n} \mathrm{BN}_{i_j}(\mathbb{W}).$$

where $I = (i_1, \cdots, i_n)$. Notice from the above construction that the Γ-space $\mathcal{B}\mathrm{N}$ admits maps:

$$\varphi : \mathcal{B}\mathrm{N} \longrightarrow \mathcal{R}\mathcal{B}\mathrm{N}(\mathbb{C}), \qquad \textit{and} \qquad \eta : \mathcal{B}\mathrm{N} \longrightarrow \mathcal{B}\mathrm{N}(\mathbb{W})\hat{}_p.$$

Remark 7. *Recall that Remark 6 shows that the map: $\varphi : \mathcal{B}\mathrm{N} \longrightarrow \mathcal{R}\mathcal{B}\mathrm{N}(\mathbb{C})$ is a homotopy equivalence. Furthermore, the automorphism of $\mathcal{B}\mathrm{N}(\mathbb{W})\hat{}_p$ induced by the Frobenius automorphism $\mathbf{Fr}_q$ of $\mathbb{W}$, and the automorphism of $\mathcal{R}\mathcal{B}\mathrm{N}(\mathbb{C})$ induced by the q-power map on each factor $\mathrm{Gl}_1(\mathbb{C})^{\times i_j}$ agree along $\mathcal{B}\mathrm{N}$. We denote this common automorphism of $\mathcal{B}\mathrm{N}$ by ψ_q.*

We are ready to describe an arithmetic fracture square:

Definition 3.4. *Let ρ_I denote the composition of η_I with the natural inclusion of $\mathcal{B}\mathrm{N}_I(\mathbb{W})$ into $\mathcal{B}\mathrm{Gl}_I(\mathbb{C})$ given by the sequence of natural maps:*

$$\rho_I : \mathcal{B}\mathrm{N}_I \longrightarrow \mathcal{B}\mathrm{N}_I(\mathbb{W})\hat{}_p \longrightarrow \mathcal{B}\mathrm{Gl}_I(\mathbb{W})\hat{}_p \longrightarrow \mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_p.$$

Now let $\mathcal{R}\rho_I$ denote the functorial fibrant replacement of the $\mathbb{Q}$-localization of ρ_I above:

$$\mathcal{R}\rho_I : \mathcal{R}(\mathcal{B}\mathrm{N}_I)_{\mathbb{Q}} \longrightarrow \mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_{\mathbb{Q}_p},$$

where $\mathbb{Q}$-localization of a space denotes the Bousfield localization with respect to rational homology. Similarly $\mathbb{Q}_p$ (or adelic) completion denotes p-completion followed by $\mathbb{Q}$-localization.

We define the ψ_q-equivariant Γ-space $\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$ by demanding $\mathcal{B}\mathrm{Gl}_I(\mathbb{C})_{(p)}$ to be the pullback:

$$\begin{tikzcd} \mathcal{B}\mathrm{Gl}_I(\mathbb{C})_{(p)} \arrow[r] \arrow[d] & \mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_p \arrow[d] . \\ \mathcal{R}(\mathcal{B}\mathrm{N}_I)_{\mathbb{Q}} \arrow[r, "\mathcal{R}\rho_I"] & \mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_{\mathbb{Q}_p} \end{tikzcd}$$

The next step, which is really the heart of our argument, is to construct a ψ_q-equivariant sectioned $S^2_{(p)}$-fibration $\mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$ over $\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$ represented by products of the fiberwise compactification of the universal i-dimensional complex vector bundle over $\mathrm{BGl}_i(\mathbb{C})$. For our purposes, we will need this spherical fibration to support an action of the Frobenius map that has fiberwise degree q^i. This spherical fibration will be constructed as a pullback diagram that lifts the pullback given in Definition 3.4 above.

In [F], §8, Friedlander defines the simplicial topological space S_c^{2i} defined as the simplicial cone on the topological space $\mathbb{C}^i - \{0\}$:

$$S_c^{2i} = * \cup \{(\mathbb{C}^i - \{0\}) \times \Delta[1]\} \cup *.$$

He then defines $\mathrm{BGl}_i(\mathbb{C}, S_c^{2i})$ by applying the diagonal to the tri-simplicial set given by singular simplices on the Bar construction of the canonical action of $\mathrm{Gl}_i(\mathbb{C})$ on the simplicial topological space S_c^{2i} (see Remark 4). Interpreting S_c^{2i} as the $\mathbb{C}$-points of a simplicial algebraic variety (defined over $\mathbb{Z}$) with an action of the algebraic group $\mathrm{Gl}_i(\mathbb{C})$, Friedlander used Etalé homotopy theory to express the natural (p-completed) projection map:

$$\mathrm{BGl}_i(\mathbb{C}, S_c^{2i})\hat{}_p \longrightarrow \mathrm{BGl}_i(\mathbb{C})\hat{}_p,$$

functorially in terms of the *algebraic* structure. In particular, the above projection map commutes with all (possibly discontinuous) field automorphisms of $\mathbb{C}$, (see [F] §9 and [F2] for details). Friedlander then assembles such maps into a *p-completed* S^2-fibration $\mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})\hat{}_p$ over the Γ-space $\mathcal{B}\mathrm{Gl}(\mathbb{C})\hat{}_p$:

$$\mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})\hat{}_p \longrightarrow \mathcal{B}\mathrm{Gl}(\mathbb{C})\hat{}_p, \tag{2}$$

and shows that this fibration admits an action of all field automorphisms of $\mathbb{C}$ that lift the Frobenius automorphism of $\mathbb{W}$ under the embedding $\iota : \mathbb{W} \subset \mathbb{C}$. Taking Friedlander's construction as motivation, we extend Definition 3.2:

Definition 3.5. *Define simplicial topological spaces as simplicial cones:*

$$S(\mathbb{W}) := * \cup \{\overline{\mathbb{F}}_q^\times \times \Delta[1]\} \cup *, \qquad S(\mathbb{C}) := * \cup \{(\mathbb{C} - \{0\}) \times \Delta[1]\} \cup *,$$

with the simplicial topological space $S(\mathbb{W})$ endowed with the discrete topology. Furthermore, one has canonical continuous maps of simplicial topological spaces:

$$S(\mathbb{W})^{\wedge i} \longrightarrow S(\mathbb{C})^{\wedge i} \longrightarrow S_c^{2i},$$

where the first map is induced by the inclusion $\mathbb{W} \subset \mathbb{C}$, and the second map is the standard homotopy equivalence as described in [F] *(§8). Notice that the inclusions of simplicial topological spaces above are compatible with the natural sequence of groups defined in Definition 3.2:*

$$\mathrm{N}_i(\mathbb{W}) \subset \mathrm{N}_i(\mathbb{C}) \subset \mathrm{Gl}_i(\mathbb{C}).$$

Consequently, the sequence of maps between the simplicial topological spaces defined above extends to the respective Bar constructions (see Remark 4):

$$\mathrm{BN}_i(S(\mathbb{W})^{\wedge i}) := \mathrm{B}(\mathrm{N}_i(\mathbb{W}), S(\mathbb{W})^{\wedge i}) \longrightarrow \mathrm{BN}_i(S(\mathbb{C})^{\wedge i}) := \mathrm{B}(\mathrm{N}_i(\mathbb{C}), S(\mathbb{C})^{\wedge i}) \longrightarrow \mathrm{BGl}_i(\mathbb{C}, S_c^{2i}).$$

Remark 8. *It is important to point out that the space $S(\mathbb{C})$ can be seen as the complex points of a simplicial algebraic variety defined over $\mathbb{Z}$, and that the map $\mathrm{BN}_i(S(\mathbb{C})^{\wedge i}) \longrightarrow \mathrm{BGl}_i(\mathbb{C}, S_c^{2i})$ itself can be lifted to a map of simplicial algebraic varieties defined over $\mathbb{Z}$. In particular, any lift of the Frobenius automorphism of $\mathbb{W}$ to the field $\mathbb{C}$ and acting on $\mathrm{BGl}_i(\mathbb{C}, S_c^{2i})\hat{}_p$ (as constructed by Friedlander in* [F]*§9), restricts to the Frobenius automorphism on $\mathbf{Fr}_q$ on $\mathrm{BN}_i(S(\mathbb{W})^{\wedge i})\hat{}_p$.*

Next, we prove a claim that extends Remark 6 to the level of spherical fibrations:

Claim 3.6. *The following canonical map is an equivalence:*

$$s\,\mathrm{B}_\iota : \mathrm{BN}_i(S(\mathbb{W})^{\wedge i})\hat{}_p \longrightarrow \mathrm{BN}_i(S(\mathbb{C})^{\wedge i})\hat{}_p.$$

Furthermore, the automorphism of $\mathrm{BN}_i(S(\mathbb{W})^{\wedge i})\hat{}_p$ induced by the Frobenius $\mathbf{Fr}_q$ agrees (under the map $s\,\mathrm{B}_\iota$) with the automorphism of $\mathrm{BN}_i(S(\mathbb{C})^{\wedge i})\hat{}_p$ induced by the q-power self maps of $\mathrm{Gl}_1(\mathbb{C})^{\times i}$ and extending the self map of $S(\mathbb{C})^{\wedge i}$ induced by the q-power self map of $\mathbb{C} - \{0\}$. In particular, the p-completed spherical fibration $\mathrm{BN}_i(S(\mathbb{W})^{\wedge i})\hat{}_p \longrightarrow \mathrm{BN}_i(\mathbb{W})\hat{}_p$ supports a rational model endowed with an automorphism ψ_q of fiberwise degree q^i which is compatible with the Frobenius automorphism on $\mathrm{BN}_i(S(\mathbb{W})^{\wedge i})\hat{}_{\mathbb{Q}_p}$.

Proof. First notice that both groups $\mathrm{N}_i(\mathbb{W})$ and $\mathrm{N}_i(\mathbb{C})$ map to Σ_i with kernels being $\mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times i}$ and $\mathrm{Gl}_1(\mathbb{C})^{\times i}$ respectively. Hence both spaces $\mathrm{BN}_i(S(\mathbb{W})^{\wedge i})$ and $\mathrm{BN}_i(S(\mathbb{C})^{\wedge i})$ fiber over $\mathrm{B}\Sigma_i$ with fibers being the spaces $\mathrm{B}(\mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times i}, S(\mathbb{W})^{\wedge i})$ and $\mathrm{B}(\mathrm{Gl}_1(\mathbb{C})^{\times i}, S(\mathbb{C})^{\wedge i})$ which are defined as the Bar constructions of the action of the groups $\mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times i}$ and $\mathrm{Gl}_1(\mathbb{C})^{\times i}$ on $S(\mathbb{W})^{\wedge i}$ and $S(\mathbb{C})^{\wedge i}$ respectively. It is therefore sufficient to demonstrate an equivalence between the spaces $\mathrm{B}(\mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times i}, S(\mathbb{W})^{\wedge i})\hat{}_p$ and $\mathrm{B}(\mathrm{Gl}_1(\mathbb{C})^{\times i}, S(\mathbb{C})^{\wedge i})\hat{}_p$. Towards this end, consider the equivariant cellular decomposition of the spaces $S(\mathbb{W})^{\wedge i}$ and $S(\mathbb{C})^{\wedge i}$ under the groups $\mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times i}$ and $\mathrm{Gl}_1(\mathbb{C})^{\times i}$ respectively. The (equivariant) cells of both spaces are in bijection, and the corresponding isotropy group of a cell in $S(\mathbb{W})^{\wedge i}$ includes into the isotropy group of the corresponding cell in $S(\mathbb{C})^{\wedge i}$ as an approximation of the form $\mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times j} \longrightarrow \mathrm{Gl}_1(\mathbb{C})^{\times j}$ for some $j \leq i$. Taking the Bar construction converts this to the inclusion on the level of classifying spaces which is known to be an isomorphism in homology with $\mathbb{Z}/p$-coefficients by Theorem 3.1. Since we have a finite number of equivariant cells, it follows easily from an induction on the skeleta that $\mathrm{B}(\mathrm{Gl}_1(\overline{\mathbb{F}}_q)^{\times i}, S(\mathbb{W})^{\wedge i})\hat{}_p$ is equivalent to $\mathrm{B}(\mathrm{Gl}_1(\mathbb{C})^{\times i}, S(\mathbb{C})^{\wedge i})\hat{}_p$ and consequently that $\mathrm{BN}_i(S(\mathbb{W})^{\wedge i})\hat{}_p$ is equivalent to $\mathrm{BN}_i(S(\mathbb{C})^{\wedge i})\hat{}_p$. The compatibility of the Frobenius with the degree q-self map is straightforward to verify since $\overline{\mathbb{F}}_q^{\times} \subset \mathbb{W} \subset \mathbb{C}$ includes as roots of unity. $\square$

We are now in a position to lift the diagram in Definition 3.4 to obtain a diagram of spherical fibrations.

Definition 3.7. *As in Definition 3.4, let ρ_I denote the composition:*

$$\rho_I : \mathcal{B}\mathrm{N}_I \longrightarrow \mathcal{B}\mathrm{N}_I(\mathbb{W})\hat{}_p \longrightarrow \mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_p,$$

and let $\mathbf{S}\mathcal{B}\mathrm{N}_I\hat{}_p \longrightarrow \mathcal{B}\mathrm{N}_I\hat{}_p$ denote the pullback of the fibration (2) along the p-completion of ρ_I. By Remark 8 and Claim 3.6, we know that the $\mathbb{Q}_p$-completion of this fibration supports a rational model $(\mathbf{S}\mathcal{B}\mathrm{N}_I)_{\mathbb{Q}} \longrightarrow (\mathcal{B}\mathrm{N}_I)_{\mathbb{Q}}$ admitting a fiberwise automorphism ψ_q of degree a power of q, that is compatible with the Frobenius automorphism acting on $\mathbf{S}\mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_{\mathbb{Q}_p}$. Let $\mathcal{R}s\rho_I$ denote the functorial fibrant replacement of the $\mathbb{Q}$-localization of the map $s\rho_I$ over ρ_I:

$$\mathcal{R}s\rho_I : \mathcal{R}(\mathbf{S}\mathcal{B}\mathrm{N}_I)_{\mathbb{Q}} \longrightarrow \mathbf{S}\mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_{\mathbb{Q}_p}.$$

We define the ψ_q-equivariant $S^2_{(p)}$-fibration of Γ-spaces $\mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$ by demanding $\mathbf{S}\mathcal{B}\mathrm{Gl}_I(\mathbb{C})_{(p)}$ to be the pullback that lifts the corresponding diagram in Definition 3.4:

$$\begin{tikzcd}
\mathbf{S}\mathcal{B}\mathrm{Gl}_I(\mathbb{C})_{(p)} \arrow[r] \arrow[d] & \mathbf{S}\mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_p \arrow[d] \\
\mathcal{R}(\mathbf{S}\mathcal{B}\mathrm{N}_I)_{\mathbb{Q}} \arrow[r, "\mathcal{R}s\rho_I"] & \mathbf{S}\mathcal{B}\mathrm{Gl}_I(\mathbb{C})\hat{}_{\mathbb{Q}_p}.
\end{tikzcd}$$

In other words, the previous definition shows:

Theorem 3.8. *There is a ψ_q equivariant sectioned $S^2_{(p)}$-fibration $\mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$, such that for every singleton: $I = \{i\}$, the map $\mathbf{S}\mathcal{B}\mathrm{Gl}_i(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{Gl}_i(\mathbb{C})_{(p)}$ is equivalent to the fiberwise compactification of the universal i-dimensional complex vector bundle ξ_i. Furthermore, the action of ψ_q on this bundle has fiberwise degree q^i.*

Proof. We apply the construction of the previous claim to $\mathcal{B} = \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$. The construction is natural in ψ_q and therefore the sectioned map π is clearly ψ_q-equivariant. Moreover, by construction the fiber of $\mathbf{S}\mathcal{B}\mathrm{Gl}_i(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{Gl}_i(\mathbb{C})_{(p)}$ is $S^{2i}_{(p)}$. It follows that the fiber of the map $\pi : \mathbf{S}\mathcal{B}\mathrm{Gl}_I(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{Gl}_I(\mathbb{C})_{(p)}$ is equivalent to $\prod S^{2i_s}_{(p)}$. The rest of the properties in the second part of Definition 2.7 are readily checked. ☐

Let $\mathcal{B}\mathrm{G}\mathbb{S}_{(p)} := \mathcal{B}\mathrm{G}(S^2_{(p)})$ denote the classifying space of sectioned $S^2_{(p)}$ fibrations, as in Theorem 2. We therefore obtain a J-homomorphism, which is the unique map of Γ-spaces classifying the fibration in the previous theorem:

$$J : \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{G}\mathbb{S}_{(p)} \,.$$

Furthermore, uniqueness also implies that this map is fixed under ψ_q, i.e. we have the equality: $J = J \circ \psi_q$. Let $\mathrm{hCE}(\psi_\mathrm{q})$ denote the homotopy co-equalizer of ψ_q and the identity map, on $\mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$. One therefore has an extension:

$$J_q : \mathrm{hCE}(\psi_\mathrm{q}) \longrightarrow \mathcal{B}\mathrm{G}\mathbb{S}_{(p)} \,.$$

Of course, such an extension J_q may not be unique. However, we may make a canonical choice for J_q as follows. Consider the sectioned $S^2_{(p)}$-fibration:

$$\pi_q : \Delta[1] \times_{\psi_q} \mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \longrightarrow \Delta[1] \times_{\psi_q} \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(q)} = \mathrm{hCE}(\psi_\mathrm{q}),$$

where $\Delta[1] \times_{\psi_q} \mathcal{B}\mathrm{Gl}(\mathbb{C})$ is defined as the mapping cylinder of ψ_q given by the push out :

$$\begin{tikzcd}
(\Delta[0]_0 \coprod \Delta[0]_1) \times \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \arrow[r] \arrow[d, "id \coprod \psi_q"] & \Delta[0] \times \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \arrow[d] \\
\Delta[1] \times \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \arrow[r] & \Delta[1] \times_{\psi_q} \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}.
\end{tikzcd}$$

We may perform the same construction for $\mathbf{S}\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$. The extension J_q we seek is the unique map that classifies the sectioned $S^2_{(p)}$-fibration π_q:

$$J_q : \mathrm{hCE}(\psi_\mathrm{q}) = \Delta[1] \times_{\psi_q} \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)} \longrightarrow \mathcal{B}\mathrm{G}\mathbb{S}_{(p)} \,.$$

Now let us apply Segal's functor Φ (2.3) to J_q to obtain a stable map also denoted by J_q:

$$J_q : \Phi\, \mathrm{hCE}(\psi_\mathrm{q}) = \mathrm{hCE}(\Phi\psi_\mathrm{q}) \longrightarrow \Phi\, \mathcal{B}\mathrm{G}\mathbb{S}_{(p)},$$

where we have used the fact that Φ preserves colimits. Now let $\mathrm{hE}(\Phi\psi_\mathrm{q})$ [1]denote the homotopy equalizer of $\Phi\psi_q$ and the identity map, on $\Phi\, \mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$. It is straightforward to see that $\mathrm{hCE}(\Phi\psi_\mathrm{q})$ is equivalent to $\Sigma\, \mathrm{hE}(\Phi\psi_\mathrm{q})$. In particular, we see that:

$$\pi_1(\Phi\, \mathrm{hCE}(\psi_\mathrm{q})) = \pi_0(\mathrm{hE}(\Phi\psi_\mathrm{q})) = \mathbb{Z}, \qquad \pi_1(\Phi\, \mathcal{B}\mathrm{G}\mathbb{S}_{(p)}) = \mathbb{Z}^\times_{(p)}.$$

[1] $\mathrm{hE}(\Phi\psi_\mathrm{q})$ is equivalent to the full image of J spectrum for the right choice of q, see Remark 10.

Claim 3.9. *The map $J_q : \Phi\,\mathrm{hCE}(\psi_\mathrm{q}) \longrightarrow \Phi\,\mathcal{B}\mathrm{G}\mathbb{S}_{(p)}$ sends a generator of $\pi_1(\Phi\,\mathrm{hCE}(\psi_\mathrm{q})) = \mathbb{Z}$ to the element $q \in \mathbb{Z}_{(p)}^{\times} = \pi_1(\Phi\,\mathcal{B}\mathrm{G}\mathbb{S}_{(p)})$.*

Proof. Notice that J_q identifies $\pi_0(\Phi\,\mathrm{hCE}(\psi_\mathrm{q}))$, with $\pi_0(\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}) = \mathbb{Z}$. Hence the components of $\Omega^\infty\Phi\,\mathrm{hCE}(\psi_\mathrm{q}))$ and $\Omega^\infty\,\mathcal{B}\mathrm{Gl}(\mathbb{C})_{(p)}$ can be identified with $\mathbb{Z}$. We will use the notation Ω_1^∞ to denote the component indexed by $1 \in \mathbb{Z}$. By the definition of the functor Φ, one notices that there is a canonical inclusion maps, which can be seen to be isomorphisms on the fundamental group:

$$\Delta[1] \times_{\psi_q} \mathrm{BGl}_1(\mathbb{C})_{(p)} \longrightarrow \Omega_1^\infty\Phi\,\mathrm{hCE}(\psi_\mathrm{q}), \qquad \mathrm{BG}(S^2_{(p)}) \longrightarrow \Omega_1^\infty\Phi\,\mathcal{B}\mathrm{G}\mathbb{S}_{(p)}\,.$$

Moreover, one has a commutative diagram:

$$\begin{tikzcd}
\Delta[1] \times_{\psi_q} \mathrm{BGl}_1(\mathbb{C})_{(p)} \arrow[r] \arrow[d, "J_q"] & \Omega_1^\infty\Phi\,\mathrm{hCE}(\psi_\mathrm{q}) \arrow[d, "J_q"] \\
\mathrm{BG}(S^2_{(p)}) \arrow[r] & \Omega_1^\infty\Phi\,\mathcal{B}\mathrm{G}\mathbb{S}_{(p)}\,.
\end{tikzcd}$$

Now the map ψ_q induces an automorphism of degree q on the fiber over the base point over $\mathrm{SBGl}_1(\mathbb{C})_{(p)} \longrightarrow \mathrm{BGl}_1(\mathbb{C})_{(p)}$ using Theorem 3.8. In particular, the left vertical map in the above commutative diagram sends the generator of $\pi_1(\Delta[1] \times_{\psi_q} \mathrm{BGl}_1(\mathbb{C})_{(p)}) = \mathbb{Z}$ to the element $q \in \mathbb{Z}_{(p)}^{\times} = \pi_1(\mathrm{BG}(S^2_{(p)}))$. The result follows by chasing the diagram. □

As an immediate consequence, we obtain a proof of the p-local stable Adams conjecture:

Theorem 3.10. *Given a prime p, let $q \neq p$ be any other prime. Then the J-homomorphism, seen as a stable map from $\mathrm{kU}_{(p)}$ to $\mathrm{bG}\mathbb{S}_{(p)}$, extends to a canonical map:*

$$J_q : \mathrm{hCE}(\psi_\mathrm{q}) \longrightarrow \mathrm{bG}\mathbb{S}_{(p)}\,.$$

Moreover, the map J_q has the property that it sends a generator of $\pi_1(\mathrm{hCE}(\psi_\mathrm{q})) = \mathbb{Z}$ to the element $q \in \mathbb{Z}_{(p)}^{\times} = \pi_1(\mathrm{bG}\mathbb{S}_{(p)})$.

4. A_n-STRUCTURES ON $M_p(k)$ INDUCED BY SPHERICAL FIBRATIONS.

Having established the p-local stable Adams conjecture in the previous section, let us turn our attention to the main application to the height of associative structures on the Moore spectra that are induced by spherical fibrations 1.1. More precisely, in this section we will establish the main application:

Theorem 4.1. *Given an odd prime p, the spectrum $M_p(k)$ admits an A_n-structure induced by a spherical fibration if and only if $n < p^k$. For $p = 2$ and $k > 1$, $M_2(k+1)$ admits an A_{2^k-1}-structure induced by a spherical fibration but it cannot support any such $A_{2^{k+1}}$-structure.*

If p is an odd prime then it is not hard (with the Dirichlet's theorem) to see that one may find another prime q such that q is a topological generator of $\mathbb{Z}_{(p)}^{\times}$. Then we may write $q^{(p-1)p^{k-1}} = 1 + p^k\lambda$, where λ is prime to p. If $p = 2$, then taking $q = 3$ and $k > 1$, we notice that $3^{2^{k-1}} = 1 + 2^{k+1}\lambda$ for some odd number λ.

Recall, the Thom spectrum for the virtual dimension zero fibration given by any pointed map

$$\tau : S^1 \longrightarrow \mathbb{Z} \times \mathrm{BG}\mathbb{S}_p = \Omega^\infty \,\mathrm{bG}\mathbb{S}_p$$

in the homotopy class of $1+p^k\lambda$, is the Moore spectrum $M_p(k)$. Recall also that by Stasheff [St, Theorem 8.4], τ is an A_n-map if and only if it extends to a stable map of the form

$$\tau : S^1 \longrightarrow \Sigma^{-1}\mathbb{CP}^n \longrightarrow \mathrm{bG}\mathbb{S}_p\,.$$

Now, by the p-local stable Adams conjecture 3.10, any map τ of degree $q^{(p-1)p^{k-1}}$ factors through the map J_q

$$\tau : S^1 \xrightarrow{\mu} \mathrm{hCE}(\psi_{\mathrm{q}}) \xrightarrow{J_q} \mathrm{bG}\mathbb{S}_{(p)},$$

where μ is any map in the homotopy class of $(p-1)p^{k-1}$ times the generator of $\pi_1(\mathrm{hCE}(\psi_{\mathrm{q}}))$. Hence to show that this map τ is an A_n-map, it is sufficient to establish a lift of the form:

(3)
$$\begin{tikzcd} S^1 \arrow[r, "\mu"] \arrow[d] & \mathrm{hCE}(\psi_{\mathrm{q}}) \\ \Sigma^{-1}\mathbb{CP}^n \arrow[ur, dashed] & \end{tikzcd}$$

To solve the extension problem (3) we need the following result.

Lemma 4.2. *Fix $m > 1$ and $q > 1$. If $f \in \mathbb{Q}[[x]]/(x^m)$ satisfies the relation*

(4)
$$f((1+x)^q - 1) \equiv q f(x) \mod x^m$$

then $f(x) \equiv c\ln(1+x) \mod x^m$ for some constant $c \in \mathbb{Q}$.

Proof. Putting $x = 0$, we get $f(0) = 0$. Now let $f(x) \equiv \sum_{i=1}^{m-1} a_i x^i \mod x^m$ and consider the formal difference

$$f((1+x)^q - 1) - q f(x) \equiv d_1 x + d_2 x^2 + \dots d_{m-1}x^{m-1} \mod x^m$$

It is easy to see that $d_1 = 0$, $d_2 = (q^2-q)a_2 - \binom{q}{2}a_1$ and in general, d_k, for $k < m$, is a linear combination of $a_1, \dots, a_k$ where coefficient of a_k is $q^k - q$. When $q > 1$ and f satisfies (4), the values of a_k for $k > 1$ are decided by the value of a_1. Hence $f(x)$ is uniquely determined by a_1, the coffecient of x. Since, $a_1\ln(1+x) \mod x^m$ satisfies (4) with a_1 as the coefficient of x, the result follows. ☐

Proposition 4.3. *The map μ in (3), in the homotopy class of $(p-1)p^{k-1}$ times the generator of $\pi_1(\mathrm{hCE}(\psi_{\mathrm{q}}))$ extends to a map from $\Sigma^{-1}\mathbb{CP}^{(p^k-1)}$ but does not extend to a map from $\Sigma^{-1}\mathbb{CP}^{p^k}$.*

Proof. Since target of μ is an infinite loop space the extension problem (3) is equivalent to the finding values of n such that we have a stable factorization of the form:

$$\mu : S^1 \longrightarrow \Sigma^{-1}\mathbb{CP}^n \longrightarrow \mathrm{hCE}(\psi_{\mathrm{q}})\,.$$

But recall that $\mathrm{hCE}(\psi_{\mathrm{q}}) = \Sigma\,\mathrm{hE}(\psi_{\mathrm{q}})$, where $\mathrm{hE}(\psi_{\mathrm{q}})$ denotes the homotopy equalizer of ψ_q acting on the p-local complex connective K-theory spectrum $\mathrm{kU}_{(p)}$. Hence we seek a stable factorization of p^{k-1}-times the unit. Namely, we ask for a map:

$$\mu = (p-1)p^{k-1}\mathbf{1} : S^0 \longrightarrow \Sigma^{-2}\mathbb{CP}^n \longrightarrow \mathrm{hE}(\psi_{\mathrm{q}})\,.$$

We may rephrase this question as follows. Consider the fibration of spectra:

$$\mathrm{hE}(\psi_{\mathrm{q}}) \longrightarrow \mathrm{kU}_{(p)} \longrightarrow \mathrm{kU}_{(p)},$$

with the second map being $\psi_q - 1$. Mapping $\Sigma^{-2}\mathbb{CP}^n$ into this fibration gives rise to a long exact sequence, reducing the question to finding a class $y \in \mathrm{kU}^2_{(p)}(\mathbb{CP}^n)$ such that y restricts to $(p-1)p^{k-1} \in \mathrm{kU}^2_{(p)}(\mathbb{CP}^1) = \mathbb{Z}_{(p)}$ and is fixed under ψ_q.

Now, recall that, $\mathrm{kU}^*_{(p)}\,\mathbb{CP}^n \cong \mathbb{Z}_{(p)}[[e]]/(e^{n+1})[\beta^{\pm}]$, where β is the Bott class in degree -2, and, $e = \gamma - 1$ is a class in degree 0, where γ is the cannonical line bundle over $\mathbb{CP}^n$. Note that $\psi_q(\beta) = q\beta$ and

$$\psi_q(e) = \psi_q(\gamma - 1) = \gamma^q - 1 = (1+e)^q - 1.$$

If $y = \beta^{-1}f(e) \in \mathrm{kU}^2_{(p)}\,\mathbb{CP}^n$, where $f(e) \in \mathbb{Z}_{(p)}[[e]]/(e^{n+1})$, is fixed by ψ_q, then $f(e)$ must satisfy

$$f((1+e)^q - 1) = qf(e).$$

By Lemma 4.2, we know that rationally f must be

$$f(e) = c\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} e^i \in \mathbb{Q}_{(p)}[[e]]/(e^{n+1}).$$

Moreover, $c = (p-1)p^{k-1}$ as y must restrict to $(p-1)p^{k-1} \in \mathrm{kU}^2_{(p)}(\mathbb{CP}^1)$. However, the polynomial $f \notin \mathbb{Z}_{(p)}[[e]]/(e^{n+1})$ unless $n < p^k$. Hence, the result follows. □

Remark 9. *The proof of the previous proposition shows that the result also holds if $\mathrm{kU}_{(p)}$ is replaced by the p-localization or p-completion of (periodic) complex K-theory without changing the conclusion of the proposition. We may further replace $(p-1)p^{k-1}$ by any number of the form $p^{k-1}\lambda$, with λ having trivial p-adic valuation.*

To complete our proof of Theorem 4.1, it remains to establish an upper bound for A_n-structures induced by spherical fibrations. We begin the argument by first considering odd primes. Let us assume the existence of a lift:

$$\begin{tikzcd} S^1 \arrow[r, "\mu"] \arrow[d] & \mathrm{bG}\mathbb{S}_p \\ \Sigma^{-1}\mathbb{CP}^{p^k} \arrow[ur, dashed] & \end{tikzcd}$$

where μ has degree $1 + p^k\lambda$ for some $\lambda \in \mathbb{Z}_p^\times$ and for an odd prime p. Let us compose the above map with the canonical map:

$$\mathrm{bG}\mathbb{S}_p \longrightarrow \mathrm{L}_{K(1)}\,\mathrm{bG}\mathbb{S}_p,$$

where $\mathrm{L}_{K(1)}\,\mathrm{bG}\mathbb{S}_p$ denotes the Bousfield localization of $\mathrm{bG}\mathbb{S}_p$ with respect to the first Morava K-theory $K(1)$ at the prime p. It is well known [Ra] that at odd primes p, this localization is equivalent to the homotopy co-equalizer of ψ_q and the identity map acting on the p-completed (periodic) complex K-theory spectrum: $\mathrm{KU}\hat{}_p$. More precisely, the $K(1)$-localization of the map J_q constructed in the previous section is an equivalence. In particular, for odd primes p, our assumption yeilds a lift of the form:

$$\begin{tikzcd} S^1 \arrow[r, "\mu"] \arrow[d] & \mathrm{L}_{K(1)}\,\mathrm{hCE}(\psi_{\mathrm{q}}) \\ \Sigma^{-1}\mathbb{CP}^{p^k} \arrow[ur, dashed] & \end{tikzcd}$$

As mentioned above, $\mathrm{L}_{K(1)}\,\mathrm{hCE}(\psi_{\mathrm{q}})$ is none other than the co-equalizer of ψ_q and the identity map acting on (periodic) p-complete complex K-theory. Furthermore, the degree of the map μ is of the form $p^{k-1}\lambda$ for some $\lambda \in \mathbb{Z}_p^\times$ (see [R] for an explicit formula). In light of Remark 9, such a lift is impossible.

The prime $p = 2$ behaves slightly differently. Assume a lift of the form:

$$\begin{tikzcd} S^1 \arrow[r, "\mu"] \arrow[d] & \mathrm{bG}\mathbb{S}_2 \\ \Sigma^{-1}\mathbb{CP}^{2^{k+1}} \arrow[ur, dashed] & \end{tikzcd}$$

where μ has degree of the form $1 + 2^{k+1}\lambda$ for some $\lambda \in \mathbb{Z}_2^\times$. To show a contradiction, we may proceed in the same fashion and consider the map

$$\mathrm{bG}\mathbb{S}_2 \longrightarrow \mathrm{L}_{K(1)}\,\mathrm{bG}\mathbb{S}_2\,.$$

However, $\mathrm{L}_{K(1)}\,\mathrm{bG}\mathbb{S}_2$ is known to be the co-equalizer of ψ_3 and the identity map acting on 2-complete (periodic) real K-theory: $\mathrm{KO}\hat{}_2$. In particular, we may compose further to the co-equalizer of ψ_3 and the identity map acting on $\mathrm{KU}\hat{}_2$ as before. We therefore have a map (which is not an equivalence):

$$\rho : \mathrm{L}_{K(1)}\,\mathrm{bG}\mathbb{S}_2 \longrightarrow \mathrm{L}_{K(1)}\,\mathrm{hCE}(\psi_{\mathrm{q}})\,.$$

Using explicit formulas in [R], we see that the image of the map ρ on π_1 has the property that it sends μ to a class that is of the form $2^k\lambda$ for some $\lambda \in \mathbb{Z}_2^\times$. The upshot of this observation is that we have a lift of the form:

$$\begin{tikzcd} S^1 \arrow[r, "\rho\circ\mu"] \arrow[d] & \mathrm{L}_{K(1)}\,\mathrm{hCE}(\psi_{\mathrm{q}}) \\ \Sigma^{-1}\mathbb{CP}^{2^{k+1}} \arrow[ur, dashed] & \end{tikzcd}$$

with $\rho \circ \mu$ having degree of the form $2^k\lambda$ for some $\lambda \in \mathbb{Z}_2^\times$. Now invoking Remark 9, we see that such a lift is impossible.

This completes the proof of Theorem 4.1.

Remark 10. *The reader will notice that the spectrum* $\mathrm{hE}(\psi_{\mathrm{q}})$ *is the* p*-local image of* J *spectrum for odd primes* p*. In particular, the obstruction to extending the above* A_{p^k-1}*-structure to an* A_{p^k}*-structure is an element in* $\pi_{2p^k-3}\,M_p(k)$ *represented by a generator of the same degree in the image of* J *(see* [Bh, Conjecture 2]*).*

Remark 11. *For odd primes p, our bound is optimal for for A_n-structures on $M_p(k)$ only when this structure is induced by a spherical fibration 1.1. One cannot rule out the possibility that $M_p(k)$ may admit "exotic" A_n-structures, which are not induced by spherical fibrations. Our argument does not address such structures. It would be very interesting to identify these exotic A_n-structures.*

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