THE LUBIN–TATE THEORY OF CONFIGURATION SPACES: I

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Abstract. We construct a spectral sequence converging to the Morava E-theory of unordered configuration spaces and identify its E^2-page as the homology of a Chevalley–Eilenberg-like complex for Hecke Lie algebras. Based on this, we compute the E-theory of the weight p summands of iterated loop spaces of spheres (parametrising the weight p operations on $\mathbb{E}_n$-algebras), as well as the E-theory of the configuration spaces of p points on a punctured surface. We read off the corresponding Morava K-theory groups, which appear in a conjecture by Ravenel. Finally, we compute the $\mathbb{F}_p$-homology of the space of unordered configurations of p particles on a punctured surface.

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1. Introduction

Configuration spaces govern several objects in mathematics and theoretical physics, ranging from $\mathbb{E}_n$-algebras in topology to Hurwitz spaces in geometry and phase spaces in mechanics.

The ordered configuration space of k non-colliding particles in a manifold M is given by

$$\mathrm{Conf}_k(M) = \left\{(x_1, \ldots, x_k) \in M^k \mid x_i \neq x_j \text{ for all } i \neq j\right\},$$

and the corresponding unordered configuration space is defined as $B_k(M) = \mathrm{Conf}_k(M)/\Sigma_k$. Here the symmetric group Σ_k acts by permuting the particles.

It is a classical challenge to compute the homology groups of configuration spaces for various manifolds M. When $M = \mathbb{R}^n$ is n-dimensional Euclidean space, these groups are of particular significance as they classify Dyer–Lashof operations acting on the homology of $\mathbb{E}_n$-algebras, which are a valuable tool in computations [KA56, Bro60, DL62, CLM76]. Through the Snaith splitting and its generalisations, this problem is closely connected to the equally classical study of the homology of iterated loop spaces and other mapping spaces (cf. [Sna74, CMT78]).

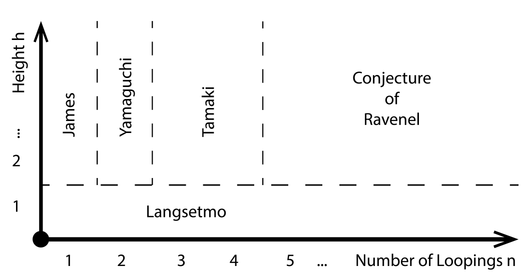
The rational homology groups of configuration spaces are well-understood in many cases of interest (cf. [BCT89, FT00, Tot96]). The mod p homology groups are very computable when either $p = 2$ or M is an odd-dimensional manifold (cf. [BCT89, ML88, BCM93]), but they remain mysterious in general, in particular when M is a surface.

Knowledge is also scarce for *generalised* homology theories. The simplest example of such a theory is given by complex K-theory, a classical invariant measuring the "twistedness" of a space through its complex vector bundles. Vector bundles on configuration spaces are of particular interest in theoretical physics, where they connect to the Knizhnik–Zamolodchikov equations in conformal field theory (cf. [EFK98]).

Chromatic homotopy theory provides, for every natural number h and every prime p, two distinct generalisations of complex K-theory. The first is known as *Morava K-theory* $K(h)$. Its value on a point is given by $K(h)_* \cong \mathbb{F}_p[u^{\pm 1}]$, and it behaves in several ways like a field interpolating between $\mathbb{Q}$ and $\mathbb{F}_p$. The second is the more refined *Morava E-theory* $E = E_h$, a highly structured analogue of Lubin–Tate space in number theory, which satisfies $E_* \cong W(\mathbb{F}_p)[[u_1, \ldots, u_{h-1}]][\beta^{\pm 1}]$ for $|\beta| = 2$. For $h = 1$, Morava E-theory recovers p-completed complex K-theory, whereas Morava K-theory is a variant of complex K-theory mod p.

The Morava K-theory of 2-fold, 3-fold, and even 4-fold loop spaces of spheres at arbitrary height h has been computed by Yamaguchi [Yam88] and Tamaki [Tam02].

At height $h = 1$, the Morava K-theory of n-fold loop spaces of spheres has been determined by Langsetmo [Lan93], who combined an equivalence of Mahowald–Thompson [MT92] with McClure's computation of the K-theory of spaces QX [BMMS86, Chapter 9].



For general n and h, there is a conjectural description due to Ravenel [Rav98, Conjecture 3].

As for E-theory, Langsetmo has essentially solved the height $h = 1$ case [Lan96a], i.e. computed the p-completed complex K-theory of n-fold loop spaces of spheres for all n. Ravenel has formulated a conjecture concerning the E-homology of $\Omega^2 S^3$ [Rav93, Conjecture 3.4] at general heights, which Goerss has linked to the theory of Dieudonné modules (cf. [Goe99]).

In this work, we will introduce a new method for calculating the Morava E-theory of configuration spaces, and apply it to perform several new computations. Setting the height h to 1, we obtain new results about the p-adic K-theory of configuration spaces. By taking the limit as h tends to ∞, we additionally deduce new results about the classical, mod p homology of configuration spaces of surfaces.

1.1. **Statement of Results.** The main tool used in this work is a convergent spectral sequence, together with an algebraic identification of its E^2-page as the homology of an explicit complex.

The spectral sequence arises from a connection between configuration spaces and Lie algebras explored by the third author in [Knu18]. Motivated by the work of Beilinson–Drinfeld on chiral algebras [BD04], as generalised and reinterpreted by Francis–Gaitsgory in [FG12], this connection takes the form of an adjunction between $\mathbb{E}_n$-algebras and spectral Lie algebras (in the sense of Salvatore [Sal98] and Ching [Chi05]). The existence of this adjunction allows one to interpret the configuration spaces of $\mathbb{R}^n$ as a kind of universal enveloping algebra. Combining a version of the Poincaré–Birkhoff–Witt theorem with the theory of factorisation homology [AF15], one obtains a formula expressing the stable homotopy types of configuration spaces of manifolds M in terms of the Lie algebra homology $C^{\mathcal{L}}$ of related spectral Lie algebras.

When M is a framed manifold and X is any spectrum, this equivalence takes the form

$$\bigoplus_k \Sigma^\infty_+ \mathrm{Conf}_k(M) \underset{\Sigma_k}{\otimes} X^{\otimes k} \xrightarrow{\ \simeq\ } C^{\mathcal{L}}(\mathrm{Free}_{\mathrm{Lie}}(\Sigma^{n-1}X)^{M^+}).$$

Here $\mathrm{Free}_{\mathrm{Lie}}(\Sigma^{n-1}X)^{M^+}$ denotes the spectral Lie algebra of maps from the one-point compactification of M to the free spectral Lie algebra $\mathrm{Free}_{\mathrm{Lie}}(\Sigma^{n-1}X)$ on $\Sigma^{n-1}X$ (cf. Theorem 5.1 below). As the functor $C^{\mathcal{L}}$ can be computed by a simplicial spectrum, we obtain a spectral sequence converging to the E-theory of configuration spaces. In good cases, we can identify its E^2-page with the derived abelianisation $H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g}(M;X))$ of the (unshifted) Hecke Lie algebra[1]

$$\mathfrak{g}(M;X) := E^\wedge_*(\mathrm{Free}_{\mathrm{Lie}}(\Sigma^{n-1}X)^{M^+}).$$

Hecke Lie algebras are purely algebraic objects, which were introduced by the first author in order to describe the natural operations acting on the E-theory of spectral Lie algebras (cf. [Bra17, Theorem 4.4.4]). Roughly speaking, Hecke Lie algebras are Lie algebras in E_*-modules, equipped with an additive action by the cohomology $\mathrm{Ext}^*_\Gamma(\overline{E}_0, \overline{E}_0)$ of Rezk's ring Γ, which is in turn closely related to the Hecke algebra of $\mathrm{GL}_n(\mathbb{Z}_p)$, cf. [Rez09], [Rez06, Section 14]. At the prime $p = 2$, there are additional non-additive operations witnessing certain congruences. To make this definition precise, one must keep careful track of the way in which operations compose, which is somewhat subtle as they lower homological degree. We refer to [Bra17, Definition 4.4.2] for a precise definition.

When working at an odd prime p, which we fix throughout this paper, the definition of Hecke Lie algebras simplifies significantly, and this simplification is recorded for the reader's convenience as Definition 4.11 below. In this case, we construct an analogue $\mathrm{CE}^{\mathcal{H}_u}$ of the classical Chevalley–Eilenberg complex (cf. Definition 4.16) by first killing the additive operations in a

[1]We will deviate in our grading conventions from [Bra17] and consider an unshifted variant of Hecke Lie algebras. To remind the reader of this minor difference, we will use the letter $\mathcal{H}_u$ instead of $\mathcal{H}$ throughout.

derived fashion, and then taking the derived abelianisation of the resulting Lie algebra. In good cases, this complex computes Hecke Lie algebra homology—Theorem 4.18 below.

Combining these observations, we arrive at the following result:

Theorem 5.5. (Hecke spectral sequence) *Let M be a framed n-manifold and X a spectrum, and suppose that the Hecke Lie algebra $\mathfrak{g}(M;X) := E^{\wedge}_{*}(\mathrm{Free}_{\mathrm{Lie}}(\Sigma^{n-1}X)^{M^+})$ is a finite and free E_*-module in each weight. There is a convergent weighted spectral sequence*

$$\mathrm{E}^2_{s,t} \cong H_{s+1}(\mathrm{CE}_{\mathcal{H}_u}(\mathfrak{g}(M;X)))_{t-1} \implies \bigoplus_{k\geq 0} E^{\wedge}_{s+t}(B_k(M;X)).$$

Remark 1.1. We will construct this spectral sequence for any form of Morava E-theory, meaning any Morava E-theory associated to a formal group over a perfect field of characteristic $p>2$.

In Section 6, we apply this result to compute the completed E-homology of the p^{th} Snaith summand of $\Omega^n S^r$ for all n, r at arbitrary chromatic height h, and establish the following result:

Theorem 6.10. (E-theory, Euclidean case). Let E be a Morava E-theory at an odd prime p. For any positive integer n and integer k, the E_*-module $E^{\wedge}_{*}\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{h\Sigma_p} (S^k)^{\otimes p}\right)$ is given by one of the following E_*-modules:

$$E^{\wedge}_{*}(\mathbb{E}_n(S^k))(p) \cong \begin{cases} \Sigma^{kp}E_* \oplus \Sigma^{pk+n-1}E_* \oplus \Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n}{2}-1}) & \text{for } n \text{ even, } k \text{ even} \\ \Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n}{2}}) & \text{for } n \text{ even, } k \text{ odd} \\ \Sigma^{kp}E_* \oplus \Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n-1}{2}}) & \text{for } n \text{ odd, } k \text{ even} \\ \Sigma^{k+(2k+n-1)(\frac{p-1}{2})}E_* \oplus \Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n-1}{2}}) & \text{for } n \text{ odd, } k \text{ odd} \end{cases}$$

Here (tr) denotes the transfer ideal associated to the inclusion of the trivial group, whereas $e \in E^0(B\Sigma_p)$ is the Euler class of the reduced complex standard representation.

The differentials in our spectral sequence exhibit intriguing behaviour familiar from other spectral sequences (cf. [Hun96]): there is a divided power class $\gamma_p(x)$ which affords a nontrivial differential d^{p-1} landing on a p-torsion element; hence x^p survives, while $\gamma_p(x)$ does not.

Remark 1.2. We can interpret Theorem 6.10 as a description of the weight p power operations acting on the (completed) E-homology of $\mathbb{E}_n$-algebras. The torsion-free classes are related to expressions x^p, $x^{p-2}\cdot[x,x]$, and $x\cdot[x,x]^{\frac{p-1}{2}}$ coming from the Poisson structure; the torsion classes are a new chromatic phenomenon.

Remark 1.3. At height 1, our result is particularly simple and stated as Theorem 6.1 below. In this case, it can (with some care) also be read off from the work of Langsetmo (cf. [Lan93, Lan96b]), who obtains this computation by entirely different means.

Remark 1.4. Combining Theorem 6.10 with the work of Zhu [Zhu14], we can give very concrete formulae at height 2. For example, at $p=3$, we have

$$E^{\wedge}_{*}(B_3(\mathbb{R}^{11})) \quad\cong\quad E_* \oplus \Sigma^{-1}E_*[\alpha]/(\alpha^4 - 6\alpha^2 + (h-9)\alpha - 3, \alpha^5)$$

We refer to Section 6.4 for more detailed computations.

Remark 1.5. The analogue of Theorem 6.10 at $p=2$ is much easier, since the configuration spaces can be expressed in terms of real projective spaces (compare Remark 6.11 below).

Remark 1.6. It is not difficult to read off the Morava K-theory groups $K(h)_*(\mathbb{E}_n(S^k))(p)$ from Theorem 6.10, and we will record the resulting groups in Theorem 6.19 below. In Section 6.5, we will also outline the connection between our results and the classical work of Langsetmo, Yamaguchi, and Tamaki.

The E-cohomology of the configuration space of p points in a general punctured surface can be computed along similar lines, and we obtain the following result (again at arbitrary height):

Theorem 7.1. (E-theory, surface case). Let E denote a Morava E-theory at an odd prime p.

(1) The E-cohomology of the space $B_p(\dot{T})$ of p unordered points in the punctured torus satisfies

$$E^*(B_p(\dot{T}))) \cong \left(\bigoplus_{0 \leq i < p} \Sigma^i E_*^{\oplus \lfloor \frac{3i+2}{2} \rfloor} \right) \oplus \Sigma^p E_*^{\oplus(p+1)}.$$

(2) The E-cohomology of the unordered configuration space of p points in a punctured orientable genus g surface $\mathcal{S}_{g,1}$ is given by

$$E^*(B_p(\mathcal{S}_{g,1})) \cong \bigoplus_{0 \leq i \leq p} \Sigma^i E_*^{\oplus \beta_i},$$

where $\beta_0, \ldots, \beta_p$ are integers specified on p.55 of the main text.

Remark 1.7. At height 1, Theorem 7.1 is a statement about the p-adic K-theory of $B_p(\mathcal{S}_{g,1})$, i.e. about vector bundles on configuration spaces of punctured surfaces.

Remark 1.8. The above computation makes use of the corresponding computation over the rationals, which is originally due to Bödigheimer–Cohen [BC88], and has been revisited by the second author and Drummand-Cole in [Knu17] and [DCK17].

Remark 1.9. The absence of torsion in Theorem 7.1 may be thought of as a reflection of the fact that $E_*^{\wedge}(B_p(\mathbb{R}^2))$ is a free E_*-module by Theorem 6.10.

Theorem 6.10 and Theorem 7.1 illustrate how the general method introduced in Theorem 5.5 can be used to generate new chromatic information about specific labelled configuration spaces.

Our final result is somewhat more surprising, as it gives new information about the p-primary part of the ordinary homology of configuration spaces (in the hard case where p is odd and the manifold M is even-dimensional):

Theorem 1.10. *For any odd prime p and any genus g, the integral (co)homology of $B_p(\mathcal{S}_{g,1})$ has no p-power torsion.*

Corollary 1.11. *The $\mathbb{F}_p$-Betti numbers of $B_p(\mathcal{S}_{g,1})$ coincide with the rational Betti numbers. Hence $\dim_{\mathbb{F}_p}(H_i(B_p(\mathcal{S}_{g,1}); \mathbb{F}_p) = \beta_i$, where the numbers β_i are specified in Theorem 7.1.*

1.2. **Future directions.** Our work points to further questions, to which we hope to return.

(1) *Higher weights.* We have restricted attention to computations in Snaith weight p. A more organised approach will lead to similar computations in higher weights, and thereby perhaps even give a proof of a form of Ravenel's conjecture [Rav98, Conjecture 3] at all weights. The coherently cocommutative coalgebra structure on stabilised configuration spaces (defined geometrically by splitting configurations) will be a helpful tool.
(2) *Manifolds.* After incorporating certain actions of tangential structure groups, our methods extend to configuration spaces of non-parallelisable manifolds. One could therefore emulate the rational computation of [DCK17] and attempt to treat all surfaces.

(3) *Vector bundles.* We hope that our techniques will yield information about the complex K-theory of configuration spaces which is helpful for the classification of vector bundles.
(4) *Coefficients.* We see this paper as a model, in the case of Lubin–Tate theory, for a program that is valid for any homology theory. We have seen that our methods give new information for the mod p homology of configuration spaces of manifolds M, about which little is known unless $\dim M$ is odd or $p = 2$ ([BCT89], $M = \mathbb{R}^n$ [CLM76, III], or $M = S^2$ [Sch]).

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2. Preliminaries

Let R be a graded-commutative ring, where the grading is parametrised by the integers. In this section, we will briefly recall the basic homological algebra of graded R-modules in the context of interest to us, and fix some notation for the remainder of this paper.

2.1. **Weighted graded R-modules.** We start with the abelian category Mod_R of graded R-modules and grading-preserving maps between them. It carries a symmetric monoidal structure given by the graded relative tensor product $\otimes$. The symmetry isomorphism of $\otimes$ incorporates the usual Koszul sign rule, which means that the isomorphism $M \otimes N \cong N \otimes M$ sends an element $m \otimes n$ to $(-1)^{|m||n|} n \otimes m$.

The free graded R-algebra on a graded R-module M takes the form $\mathrm{Sym}_R(M) = \bigoplus_w M^{\otimes w}_{\Sigma_w}$ and hence naturally splits as an infinite direct sum of "weighted pieces" indexed by the naturals. In our later spectral sequences, it will be important to effectively access weighted pieces of this kind. We therefore introduce an additional grading:

Definition 2.1 (Weighted graded modules)**.** The category $\mathrm{Mod}_R^{\mathbb{N}}$ of *weighted graded R-modules* is given by the category of functors from the discrete category $\mathbb{N}$ of nonnegative integers to Mod_R. Day convolution equips $\mathrm{Mod}_R^{\mathbb{N}}$ with a symmetric monoidal structure, which we will denote by $\otimes$.

Concretely, an object $M \in \mathrm{Mod}_R^{\mathbb{N}}$ is simply an $\mathbb{N}$-indexed collection of $\mathbb{Z}$-graded R-modules

$$M(0)\ ,\ M(1)\ ,\ M(2)\ ,\ M(3)\ ,\ \ldots .$$

Given $M, N \in \mathrm{Mod}_R^{\mathbb{N}}$, the weight w component of $M \otimes N$ is given by $\bigoplus_{u+v=w} M(u) \otimes N(v)$.

We shall call the $\mathbb{Z}$-grading the *internal grading* and the $\mathbb{N}$-grading the *weight grading*. This means that elements in internal degree i and weight w belong to the group $M(w)_i$. Note that given $M, N \in \mathrm{Mod}_R^{\mathbb{N}}$, the symmetry isomorphism $M \otimes N \cong N \otimes M$ only implements the Koszul sign rule with respect to the internal grading: this means that if $m \in M(u)_i$ and $n \in N(v)_j$, then $m \otimes n \in M \otimes N$ is sent to $(-1)^{i+j} n \otimes m$ in $N \otimes M$.

The category $\mathrm{Mod}_R^{\mathbb{N}}$ of weighted graded R-modules admits a canonical endofunctor:

Definition 2.2 (Suspension)**.** Given $n \in \mathbb{Z}$, the n^{th} *suspension* of $M \in \mathrm{Mod}_R^{\mathbb{N}}$ is the unique weighted graded R-module $\Sigma^n M$ with $(\Sigma^n M)_i = M_{i-n}$ and with R-action inherited from M.

Notation 2.3. Given a nonnegative integer $w \in \mathbb{N}$ and a graded R-module $M \in \mathrm{Mod}_R$, we write $M\{w\} \in \mathrm{Mod}_R^{\mathbb{N}}$ for the weighted graded R-module which is M concentrated in weight w.

Definition 2.4. A weighted graded R-module $M \in \mathrm{Mod}_R^{\mathbb{N}}$ is said to be

1. *finitely generated* if the underlying ordinary R-module has this property.
2. *free* if it is a direct sum of modules $\Sigma^r R\{w\}$ with $r \in \mathbb{Z}, w \in \mathbb{N}$.
3. *projective* if it is projective as an object in the abelian category $\mathrm{Mod}_R^{\mathbb{N}}$; by the usual argument, this is equivalent to being a summand of a free module.
4. *flat* if the functor $M \otimes -$ preserves short exact sequences.

Write $\mathrm{Mod}_{R,\mathrm{f}}^{\mathbb{N}}$ ($\mathrm{Mod}_{R,\mathrm{ff}}^{\mathbb{N}}$) for the full subcategory spanned by all (finite) free modules.

The following classical result will be indispensable (cf. [Laz69], and [FF74] for the graded case):

Theorem 2.5 (Lazard)**.** *A module $M \in \mathrm{Mod}_R^{\mathbb{N}}$ is flat if and only if it is a filtered colimit of finite free modules.*

2.2. **The derived category of $\mathrm{Mod}_R^{\mathbb{N}}$.** In order to compute derived functors, we will need to work in the (nonnegative) derived ∞-category $\mathcal{D}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ of $\mathrm{Mod}_R^{\mathbb{N}}$ (cf. [Lur03, Section 1.3.2]). As an ∞-category, this can be constructed by freely adding geometric realisations to $\mathrm{Mod}_{R,f}^{\mathbb{N}}$. For us, it will however be more helpful to model $\mathcal{D}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ by two concrete model categories.

Chain complexes. The first such model is given by the category $\mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ of (homologically) nonnegatively graded chain complexes of weighted graded R-modules

$$\ldots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \to 0 \to \ldots .$$

Chain maps, homology modules, and quasi-isomorphisms are defined in the usual way.

Notation 2.6. A chain complex $M \in \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ is equipped with three different gradings. The *homological grading* measures in which piece of the chain complexes we are in; it is indexed by $\mathbb{N}$. The second and third grading use that all pieces M_n are weighted graded R-modules; as before, we will call them the *internal grading* and *weight grading*, respectively. Hence elements of homological degree n, internal degree i, and weight w belong to the R_0-module $M_n(w)_i$.

The category of chain complexes carries a second natural endofunctor:

Definition 2.7 (Shift)**.** Given an integer $n \in \mathbb{Z}$, the n^{th} *(homological) shift* of a chain complex $M = (M_i, d_i)$ is given by the chain complex with $(M[n])_i := M_{i-n}$ with differential $(-1)^n d_i$. Here we use the convention that $M_i = 0$ for all $i < 0$.

Warning. *The shift operator $[n]$ should not be confused with the suspension operator Σ^n (obtained by applying Definition 2.2 in each component).*

Definition 2.8. A chain complex in $\mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ is *levelwise free* if it is a free weighted graded R-module in each homological degree (in the sense of Definition 2.4). The terms *levelwise finitely generated*, *levelwise projective*, and *levelwise flat* are defined in a similar way.

The category $\mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ is equipped with the (cofibrantly generated) projective model structure. Its weak equivalences are the quasi-isomorphisms, its fibrations are the levelwise surjective chain maps, and its cofibrations are the levelwise injective chain maps with levelwise projective cokernel.

Moreover, $\mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ carries a natural symmetric monoidal structures given by the usual tensor product of chain complexes. Given chain complexes $M, N \in \mathrm{Mod}_R^{\mathbb{N}}$, the symmetry isomorphism $M \otimes N \cong N \otimes M$ implements the Koszul sign rule with respect to the internal and homological gradings, but not the weight grading: this means that if $m \in M_a(u)_i$ and $n \in N_b(v)_j$, then $m \otimes n \in M \otimes N$ is sent to $(-1)^{i+j+a+b} n \otimes m$ in $N \otimes M$.

Simplicial modules. The second model for the (nonnegative) derived category is $\mathrm{sMod}_R^{\mathbb{N}}$, the category of simplicial objects in $\mathrm{Mod}_R^{\mathbb{N}}$. These are contravariant functors from the category Δ of nonempty finite linearly ordered sets to $\mathrm{Mod}_R^{\mathbb{N}}$. Once more, its objects carry three different gradings, which we will again refer to as the *internal*, *homological*, and *weight* gradings.

Definition 2.9. An object in $\mathrm{sMod}_R^{\mathbb{N}}$ is said to be *levelwise free* if it is a free weighted graded R-module in each homological (i.e. simplicial) degree (cf. Definition 2.4). The terms *levelwise finitely generated*, *levelwise projective*, and *levelwise flat* are defined analogously.

The category $\mathrm{sMod}_R^{\mathbb{N}}$ carries a (cofibrantly generated) model structure as well: its weak equivalences and fibrations are precisely those maps which induce weak equivalences or fibrations on underlying simplicial sets. For a well-known, more explicit description of fibrations and cofibrations, we refer to [GS07, Lemma 4.3]. The levelwise tensor product gives $\mathrm{sMod}_R^{\mathbb{N}}$ a symmetric monoidal structure.

One pleasant feature of $\mathrm{sMod}_R^{\mathbb{N}}$ is that filtered colimits are automatically homotopy colimits:

Lemma 2.10. *If $M : I \to \mathrm{sMod}_R^{\mathbb{N}}$ is a functor with I filtered, then $\underset{i\in I}{\mathrm{hocolim}}\, M_i \to \underset{i\in I}{\mathrm{colim}}\, M_i$ is a weak equivalence.*

Proof. Quasi-isomorphisms of chain complexes, and hence weak equivalences in $\mathrm{sMod}_R^{\mathbb{N}}$, are detected by compact objects. This implies that filtered colimits of weak equivalences are again weak equivalences. □

The Dold–Kan correspondence. The two model categories $\mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ and $\mathrm{sMod}_R^{\mathbb{N}}$ have the same underlying ∞-category. This fact is witnessed by a well-known Quillen equivalence arising as a composite of adjunctions

$$\mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}}) \rightleftarrows (\mathrm{Mod}_R^{\mathbb{N}})^{\Delta^{op}_{\mathrm{inj}}} \underset{\iota^*}{\overset{\iota_!}{\rightleftarrows}} (\mathrm{Mod}_R^{\mathbb{N}})^{\Delta^{op}} = \mathrm{sMod}_R^{\mathbb{N}}\,.$$

Here $\iota : \Delta^{op}_{\mathrm{inj}} \to \Delta^{op}$ denotes the inclusion of the wide subcategory on order-preserving injections. The upper arrow on the left identifies chain complexes with semisimplicial objects $M_\bullet$ for which d_i vanishes on M_n whenever $i < n$. Its right adjoint sends a semisimplicial object $M_\bullet$ to the chain complex whose n^{th} term is given by $\bigcap_{i=0}^{n-1} \ker(d_i : M_n \to M_{n-1})$, with differential $(-1)^n d_n$. We write Γ and N for the composite left and right adjoint, respectively, and observe:

Lemma 2.11. *The functors N and Γ each preserve levelwise projectivity, levelwise flatness, and levelwise finite generation. The functor Γ also preserves levelwise freeness.*

Proof. The claims concerning the right adjoint N all follow from the fact that the inclusion $N(M)_n \subseteq M_n$ is split with complement the span of the images of the degeneracies with target M_n. The assertions about the functor Γ follow from the well-known formula $\Gamma(V)_n = \bigoplus_{\pi:[n]\twoheadrightarrow[k]} V_k$, where the sum is indexed over the set of surjective maps in Δ with domain $[n]$. □

The adjunction $(\Gamma \dashv N)$ can be used to identify the two models for the derived category:

Theorem 2.12 (Dold–Kan correspondence)**.** *The pair $\Gamma : \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}}) \leftrightarrows \mathrm{sMod}_R^{\mathbb{N}} : N$ forms an adjoint equivalence of categories, and in fact a Quillen equivalence of model categories.*

While the Dold–Kan correspondence is an equivalence, it does *not* respect the symmetric monoidal structure. The categories of commutative algebra objects in $\mathrm{Ch}_{\geq 0}(\mathrm{grMod}_R)$ and $\mathrm{sMod}_R^{\mathbb{N}}$ are therefore not equivalent. In fact, only the latter category has a well-behaved homotopy theory for general rings R.

This relies on the symmetric power functor preserving weak equivalences between cofibrant (i.e. levelwise projective) simplicial modules. In fact, we will show in Corollary 2.15 that the symmetric power functor also preserves equivalences between the slightly larger class of levelwise flat simplicial modules. This may be thought of as a non-additive variant of the flat resolution lemma in homological algebra.

Our argument will rely on the following simplicial variant of Lazard's theorem:

Lemma 2.13 (Simplicial Lazard Theorem)**.** *A simplicial module $M \in \mathrm{sMod}_R^{\mathbb{N}}$ is levelwise flat if and only if it is a filtered colimit of simplicial modules which are levelwise finite free.*

Proof. The "if" direction is immediate from Theorem 2.5 above. For the converse, it suffices to show that any levelwise flat chain complex $V \in \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ is a filtered colimit of levelwise finite free chain complexes, since N and Γ preserve colimits, N preserves flatness, and Γ preserves freeness by Lemma 2.11. We may further assume that V_n vanishes above some given degree N, since an arbitrary chain complex is a filtered colimit of its truncations.

We will show by downward induction on r that for every $r \geq 0$, the complex V is a filtered colimit of complexes $\{V^i\}_{i\in I}$ satisfying the following two properties:

(1) V_n^i is finite free for $r \leq n \leq N$ and all $i \in I$.
(2) the map $V_n^i \to V_n$ is an isomorphism for all $0 \leq n < r$ and all $i \in I$.

The case $r = 0$ is the statement we intend to prove. The base case $r = N + 1$ is trivially true. So assume that the claim holds for a given $r \geq 1$, and that this is verified by a filtered diagram $\{V^i\}$ of chain complexes over V. By Theorem 2.5, we can write $\mathrm{colim}_{j\in J} W^j \cong V_{r-1}$ with J filtered and each W^j finite free.

Let K denote the category of triples $(i \in I, j \in J, \delta : V_r^i \to W^j)$ such that the diagram

$$\begin{tikzcd} V_r^i \arrow[r] \arrow[d, "\delta"'] & V_r \arrow[d, "d"] \\ W^j \arrow[r] & V_{r-1} \end{tikzcd}$$

commutes and the composite $V_{r+1}^i \xrightarrow{d} V_r^i \xrightarrow{\delta} W^j$ vanishes. A morphism from (i, j, δ) to (i', j', δ') is a pair of morphisms in I and J compatible with δ and δ'.

By the conditions on objects of K, replacing $V_{r-1} \cong V_{r-1}^i$ with W^j in V^i defines a chain complex, and we obtain in this way a diagram $\varphi : K \to \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R^{\mathbb{N}})$ over V.

Thus, it will suffice to show, first, that the colimit of φ is V; and, second, that K is filtered.

It is enough to verify the first claim in each degree n, ignoring the differential. There is an obvious forgetful functor $\pi : K \to I \times J$, and we have a commuting diagram

$$\begin{tikzcd} K \arrow[r, "\pi"] \arrow[d, "\varphi"'] & I \times J \arrow[d, "\psi"] \\ \mathrm{Ch}_R \arrow[r, "(-)_n"] & \mathrm{Mod}_R, \end{tikzcd}$$

where

$$\psi(i,j) = \begin{cases} V_n^i & n \geq r \\ W^j & n = r-1 \\ V_n & n < r-1. \end{cases}$$

By the assumptions on $\{V^i\}_{i\in I}$, it suffices to check that π is a cofinal functor, i.e. that for all $(i, j) \in I \times J$, the comma category $K_{(i,j)/}$ is nonempty and connected. This follows easily using that each V_r^i is a compact object and that I and J are filtered.

The verification that K is filtered proceeds in a similar manner, using compactness and the fact I and J are filtered. □

2.3. **Symmetric powers.** The k^{th} symmetric power of a module $M \in \mathrm{Mod}_R^{\mathbb{N}}$ is defined as

$$\mathrm{Sym}_R^k(M) = (M^{\otimes k})_{\Sigma_k}.$$

This functor preserves filtered colimits. The total symmetric power is $\mathrm{Sym}_R(M) = \bigoplus_k \mathrm{Sym}_R^k(M)$.

In this section, we will study the homotopical behaviour of the functor $\mathrm{Sym}_R^k : \mathrm{sMod}_R^{\mathbb{N}} \to \mathrm{sMod}_R^{\mathbb{N}}$ obtained by applying Sym_R^k in each simplicial degree. This will lead to a technical tool for our subsequent study of Lie algebra homology.

It is well-known [DP58] that symmetric powers preserve weak equivalences between levelwise projective modules. In fact, the following slightly stronger claim holds:

Lemma 2.14. *If M is levelwise flat, then $\mathbb{L}\,\mathrm{Sym}_R^k(M) \to \mathrm{Sym}_R^k(M)$ is a weak equivalence.*

Proof. By Lemma 2.13, we can write $M \cong \mathrm{colim}_{i\in I} M_i$ with I filtered and each M_i levelwise finite free. Consider the following commutative square:

$$\begin{tikzcd}
\mathbb{L}\,\mathrm{Sym}_R^k\,(\operatorname*{colim}_{i\in I} M_i) \arrow[r] & \mathrm{Sym}_R^k(\operatorname*{colim}_{i\in I} M_i) \\
\operatorname*{colim}_{i\in I} \mathbb{L}\,\mathrm{Sym}_R^k\,(M_i) \arrow[u] \arrow[r, "\sim"] & \operatorname*{colim}_{i\in I} \mathrm{Sym}_R^k\,(M_i). \arrow[u]
\end{tikzcd}$$

The right vertical arrow is an equivalence since Sym_R^k preserves filtered colimits. This implies that the left derived functor $\mathbb{L}\,\mathrm{Sym}^k$ preserves filtered homotopy colimits, which, by 2.10, shows that the left vertical arrow is an equivalence. This lower horizontal arrow is an equivalence since levelwise finite free modules are projective. □

Corollary 2.15 (Invariance of symmetric powers for flat modules)**.** *If $f : M \to M'$ is a weak equivalence of levelwise flat simplicial R-modules, then $\mathrm{Sym}_R^k(f)$ is a weak equivalence for all $k \geq 0$.*

We will also need the following fact in our later computations:

Lemma 2.16. *Assume that 2 is invertible in R. If $M \in \mathrm{sMod}_R^{\mathbb{N}}$ is levelwise free (respectively levelwise projective or levelwise flat), then $\mathrm{Sym}_R^k(M)$ has the same property.*

Proof. Since Sym_R^k satisfies a binomial formula on direct sums, it suffices to treat the case $M = \Sigma^n R$ to verify that Sym_R^k preserves freeness. Indeed, observe that for $k > 1$, we have

$$\mathrm{Sym}_R^k\,(\Sigma^n R) \cong \begin{cases} \Sigma^{nk} R & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

If M is projective, then M is a summand of a free R-module F, so $\mathrm{Sym}_R^k(M)$ is a summand of $\mathrm{Sym}_R^k(F)$, which is free by the previous case. If M is flat, then M is a filtered colimit of levelwise finite free R-modules by Lemma 2.13. Since $\mathrm{Sym}_R^k(M)$ preserves filtered colimits, a second application of Lemma 2.13 shows that $\mathrm{Sym}_R^k(M)$ is levelwise flat. □

2.4. **Divided Powers.** The n^{th} divided power of a module $M \in \mathrm{Mod}_R^{\mathbb{N}}$ is given by

$$\Gamma_R^n(M) = (M^{\otimes n})^{\Sigma_n}.$$

The functor Γ_R preserves filtered colimits. The total divided power is $\Gamma_R(M) = \bigoplus_k \Gamma_R^k(M)$.

Proposition 2.17. *Divided powers satisfy the following well-known properties:*

(1) There is a natural isomorphism $\Gamma_R(M_1 \oplus M_2) \cong \Gamma_R(M_1) \otimes \Gamma_R(M_2)$ for all M_1, M_2.
(2) If $M \cong R\langle x\rangle$ is free on one generator, then there is an isomorphism of R-modules

$$\Gamma_R(M) \cong \begin{cases} R\langle \gamma_i(x) : 0 \leq i < \infty,\ |\gamma_i(x)| = i|x|\rangle & |x| \text{ even} \\ R \oplus R\langle x\rangle & |x| \text{ odd.} \end{cases}$$

Proof. The first fact is standard (cf. [Rob63, Theorem III.4]). The second and third claim follow from [Nei10, Lemma 5.1.2] after tensoring up from $\mathbb{Z}$ to R. □

Divided powers satisfy many of the same desirable properties of symmetric powers. The following results are classical and proven along the same lines as in the preceeding section; we will therefore be brief:

Lemma 2.18 (Invariance of divided powers for flat modules)**.** *If $f : M \to M'$ is a weak equivalence of levelwise flat simplicial R-modules, then $\Gamma_R^k(f)$ is a weak equivalence for all $k \geq 0$.*

Lemma 2.19. *Assume that 2 is invertible in R. If $M \in \mathrm{sMod}_R^{\mathbb{N}}$ is levelwise free (respectively levelwise projective or levelwise flat), then $\Gamma_R^k(M)$ has the same property.*

3. Lie algebras and their homology

Given a Lie algebra over a graded ring R, we can divide out the ideal spanned by all brackets and hence obtain its abelianisation. Carrying out this construction in a suitably derived fashion leads to a definition of the Lie algebra homology $H^{\mathrm{Lie}}(\mathfrak{g})$ of $\mathfrak{g}$.

In this section, we show that if $\mathfrak{g}$ is flat over R, then $H^{\mathrm{Lie}}(\mathfrak{g})$ can be computed by the classical Chevalley–Eilenberg complex $\mathrm{CE}(\mathfrak{g})$ of $\mathfrak{g}$.

3.1. **Simplicial Lie algebras over a graded ring.**

We fix a graded-commutative ring R with $2 \in R^{\times}$ and work in the category $\mathrm{sMod}_R^{\mathbb{N}}$ of weighted graded R-modules.

Definition 3.1 (Weighted graded Lie algebra)**.** A (weighted, graded) *Lie algebra* in $\mathrm{Mod}_R^{\mathbb{N}}$ consists of a weighted graded R-module $\mathfrak{g} \in \mathrm{Mod}_R^{\mathbb{N}}$ together with a map

$$[-,-] : \mathfrak{g} \otimes_R \mathfrak{g} \to \mathfrak{g}$$

satisfying the following identities for all homogeneous elements $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$, $c \in \mathfrak{g}_k$:

(1) $[a,b] + (-1)^{ij}[b,a] = 0$
(2) $(-1)^{ik}[a,[b,c]] + (-1)^{ji}[b,[c,a]] + (-1)^{kj}[c,[a,b]] = 0$
(3) $[a,[a,a]] = 0$.

A map of Lie algebras is an R-module map intertwining the respective brackets. We write $\mathrm{Lie}_R^{\mathbb{N}}$ for the category of (weighted, graded) Lie algebras over R.

Due to our standing assumption that $2 \in R^{\times}$, we have $[a,a] = 0$ for a in even internal degree.

Remark 3.2. There is disagreement in the literature over the definition of Lie algebras over general rings. An operadic definition would only enforce the first two axioms (and hence $3 \cdot [a,[a,a]] = 0$ for any a), but the resulting "operadic Lie algebras" in general fail to inject into their universal enveloping algebras, since the third axiom is satisfied by any Lie algebra obtained from an associative algebra. Since the relationship with the universal enveloping algebra is crucial in what we do, we have chosen the above set of axioms.

The forgetful functor $\mathrm{U}_{\mathrm{Mod}_R}^{\mathrm{Lie}_R} : \mathrm{Lie}_R^{\mathbb{N}} \to \mathrm{Mod}_R^{\mathbb{N}}$ admits a left adjoint $\mathrm{Free}_{\mathrm{Mod}_R}^{\mathrm{Lie}_R}$, the free Lie algebra functor. The category $\mathrm{Lie}_R^{\mathbb{N}}$ is equivalent to algebras for the corresponding monad $\mathbf{L}$. Carrying out these constructions degreewise, we obtain the category $\mathrm{sLie}_R^{\mathbb{N}}$ of simplicial Lie algebras. It is linked to $\mathrm{sMod}_R^{\mathbb{N}}$ by a (monadic) free-forgetful adjunction. Abusing notation, we will write $\mathrm{Free}_{\mathrm{Mod}_R}^{\mathrm{Lie}_R}$ and $\mathrm{U}^{\mathrm{Lie}}$ for its constituent functors, and denote the resulting monad by $\mathbf{L}$.

Lemma 3.3. *If $M \in \mathrm{sMod}_R^{\mathbb{N}}$ is levelwise free (respectively levelwise projective or levelwise flat), then $\mathbf{L}(M)$ has the same property.*

Proof. The levelwise free case reduces via extension of scalars to the case $R = \mathbb{Z}[\frac{1}{2}]$ of [Nei10, Prop. 8.5.1] (the standing assumption that $2 \in R^{\times}$ is used to guarantee that our definition of Lie algebra coincides with [Nei10, Def. 8.1.1]). If M is levelwise projective, then M is a summand of a levelwise free simplicial module, so the same is true of $\mathbf{L}(M)$ by the levelwise free case. If M is levelwise flat, then M is a filtered colimit of levelwise finite free modules by Lemma 2.13, so the same is true of $\mathbf{L}(M)$ by the free case since $\mathrm{U}_{\mathrm{Mod}}^{\mathrm{Lie}}$ preserves filtered colimits. ☐

The category $\mathrm{sLie}_R^{\mathbb{N}}$ of simplicial Lie algebras carries a standard model structure:

Proposition 3.4 (Model structure on simplicial Lie algebras). *The right transferred model structure along the following adjunction exists:*

$$\mathrm{sMod}_R^{\mathbb{N}} \underset{\mathrm{U}^{\mathrm{Lie}_R}_{\mathrm{Mod}_R}}{\overset{\mathrm{Free}^{\mathrm{Lie}_R}_{\mathrm{Mod}_R}}{\rightleftarrows}} \mathrm{sLie}_R^{\mathbb{N}}$$

Its weak equivalences and fibrations are defined on underlying simplicial modules.

Proof. Since every object in $\mathrm{sMod}_R^{\mathbb{N}}$ is fibrant and the path object of $\mathrm{sMod}_R^{\mathbb{N}}$ lifts to $\mathrm{sLie}_R^{\mathbb{N}}$, the claim follows from well-known existence criteria—see e.g. [JN14, Thm. 3.2, Rmk. 3.3]. □

Notation 3.5 (Bar construction). Given a monad $\mathbf{T}$ acting on a category C, a right $\mathbf{T}$-functor $F : \mathrm{C} \to \mathrm{D}$, and X a simplicial $\mathbf{T}$-algebra, we write $\mathrm{Bar}_\bullet(F, \mathbf{T}, X)$ for the simplicial object of D given by taking the diagonal of the bisimplicial object obtained by applying the two-sided monadic bar construction on F and $\mathbf{T}$ levelwise to X.

Using this, we can construct an explicit cofibrant replacement functor :

Lemma 3.6. *Let $\mathfrak{g} \in \mathrm{sLie}_R^{\mathbb{N}}$ be a simplicial (weighted and graded) Lie algebra. The natural map $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}}_{\mathrm{Mod}}, \mathbf{L}, \mathfrak{g}) \to \mathfrak{g}$ is a weak equivalence. Moreover, if $\mathfrak{g}$ is levelwise projective, then $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}}_{\mathrm{Mod}}, \mathbf{L}, \mathfrak{g})$ is a cofibrant object in $\mathrm{sLie}_R^{\mathbb{N}}$.*

Proof. That the map in question is a weak equivalence follows from [JN14, Prop. 3.13], and cofibrancy follows from [JN14, Prop. 3.17, 3.22]. □

3.2. **The universal enveloping algebra.** If $A \in \mathrm{Alg}_R^{\mathbb{N}}$ is an associative algebra object in $\mathrm{Mod}_R^{\mathbb{N}}$, then A determines a Lie algebra in the sense of Definition 3.1 with bracket given by $[a, b] = ab - (-1)^{|a||b|}ba$. This resulting functor $\mathrm{Alg}_R^{\mathbb{N}} \to \mathrm{Lie}_R^{\mathbb{N}}$ admits a left adjoint, the *universal enveloping algebra* functor, which may be constructed explicitly as the quotient

$$U(\mathfrak{g}) = \frac{T_R(\mathfrak{g})}{[a, b] = a \otimes b - (-1)^{|a||b|} b \otimes a}.$$

Here T_R denotes the tensor algebra over R. Note that, since the Lie algebra 0 is terminal and $U(0) = R$, the functor U factors canonically through the category of augmented algebras.

The algebra $U(\mathfrak{g})$ is naturally filtered by word length. After passing to the associated graded algebra, the defining relation becomes that of the free graded-commutative R-algebra $\mathrm{Sym}_R(\mathfrak{g})$. The following classical theorem summarises this observation:

Theorem 3.7 (Poincaré–Birkhoff–Witt). *If $\mathfrak{g} \in \mathrm{Lie}_R^{\mathbb{N}}$ is a Lie algebra whose underlying module is flat, then the natural map*

$$\mathrm{Sym}_R(\mathfrak{g}) \to \mathrm{gr}\, U(\mathfrak{g})$$

of augmented bigraded R-algebras is an isomorphism.

Remark 3.8. A reference for the flat but ungraded case is [Hig69], and the same argument applies in the graded context after adding appropriate Koszul signs. We do not spell out this straightforward adaptation, noting only that the assumption that $2 \in R^\times$ seems to be necessary.

Proposition 3.9 (Invariance of the universal enveloping algebra). *The functor U preserves weak equivalences between levelwise flat simplicial Lie algebras.*

Proof. The claim follows from the five lemma, Theorem 3.7, and repeated use of Lemma 2.15. □

3.3. **Lie algebra homology.** A module $M \in \mathrm{Mod}_{\mathbb{R}}^{\mathbb{N}}$ determines a Lie algebra $\mathrm{T}_{\mathrm{Lie}}^{\mathrm{Mod}}(M)$ in $\mathrm{Lie}_R^{\mathbb{N}}$ with underlying module M and vanishing Lie bracket. This construction gives a functor $\mathrm{T}_{\mathrm{Lie}}^{\mathrm{Mod}} : \mathrm{Mod}_R^{\mathbb{N}} \to \mathrm{Lie}_R^{\mathbb{N}}$, which admits a left adjoint $\mathrm{Q}_{\mathrm{Lie}}^{\mathrm{Mod}}$. Explicitly, $\mathrm{Q}_{\mathrm{Lie}}^{\mathrm{Mod}}(\mathfrak{g})$ is obtained by forming the quotient of the underlying module of $\mathfrak{g}$ by the submodule generated by all iterated brackets of elements of $\mathfrak{g}$. Applying these functors levelwise, we obtain an adjunction at the level of categories of simplicial objects. Since fibrations and weak equivalences of simplicial Lie algebras are defined on underlying modules, it follows that this is in fact a Quillen adjunction.

Definition 3.10 (Lie algebra homology)**.** The *Lie algebra homology* of a Lie algebra $\mathfrak{g} \in \mathrm{sLie}_R^{\mathbb{N}}$ is defined as

$$H^{\mathrm{Lie}}(\mathfrak{g}) = H\left(N\left(\mathbb{L}\mathrm{Q}_{\mathrm{Lie}}^{\mathrm{Mod}}(\mathfrak{g})\right)[1] \oplus R\right).$$

Remark 3.11. Since $H^{\mathrm{Lie}}(\mathfrak{g})$ is the homology of a chain complex of objects in $\mathrm{Mod}_R^{\mathbb{N}}$, it is a graded object in $\mathrm{Mod}_R^{\mathbb{N}}$, i.e., a bigraded weighted R-module.

There is a small complex that is often available for computing this Lie algebra homology. To state the following construction (essentially due to [CE48] and [May66]), we will need the divided power functors from Section 2.4:

Definition 3.12 (Chevalley–Eilenberg, May)**.** The *Chevalley–Eilenberg complex* of a Lie algebra $\mathfrak{g} \in \mathrm{Lie}_R^{\mathbb{N}}$ is the chain complex

$$\mathrm{CE}(\mathfrak{g}) = (\Gamma_R(\mathfrak{g}[1]), d),$$

where d is defined as follows; if $a_1, \ldots, a_m \in \mathfrak{g}^{\mathrm{odd}}$ have odd total degree and $b_1, \ldots, b_n \in \mathfrak{g}^{\mathrm{even}}$ have even total degree, then the value of d on a generic element

$$\gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma b_1, \cdots, \sigma b_n \rangle$$

of $\Gamma_R(\mathfrak{g}[1])$ is given by the following formula:

$$\begin{aligned}
&\sum_{1 \le i < j \le m} \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_i - 1}(\sigma a_i) \cdots \gamma_{r_j - 1}(\sigma a_j) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma[a_i, a_j], \sigma b_1, \cdots, \sigma b_n \rangle \\
&+ \sum_{1 \le i < j \le n} (-1)^{i+j-1} \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma[b_i, b_j], \sigma b_1, \cdots \widehat{\sigma b_i}, \cdots, \widehat{\sigma b_j}, \cdots, \sigma b_n \rangle \\
&+ \frac{1}{2} \sum_{i=1}^{m} \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_i - 2}(\sigma a_i) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma[a_i, a_i], \sigma b_1, \cdots, \sigma b_n \rangle \\
&+ \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{j-1} \gamma_1(\sigma[a_i, b_j])\, \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_i - 1}(\sigma a_i) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma b_1, \cdots, \widehat{\sigma b_j}, \cdots, \sigma b_n \rangle
\end{aligned}$$

If $\mathfrak{g} \in \mathrm{sLie}_R^{\mathbb{N}}$ is a simplicial Lie algebra, we define $\mathrm{CE}(\mathfrak{g})$ by applying the previous construction in each simplicial degree, thereby obtaining a simplicial chain complex in $\mathrm{Mod}_R^{\mathbb{N}}$. Its homotopy groups are isomorphic to the homology of the corresponding total complex (obtained by using the Dold–Kan correspondence).

We have the following theorem about this homology, which in some form goes back at least as far as [CE48]—see also [May66, Pri70] for settings closer to ours:

Theorem 3.13. *If $\mathfrak{g} \in \mathrm{sLie}_R^{\mathbb{N}}$ is levelwise flat, then there is a natural isomorphism*

$$H(\mathrm{CE}(\mathfrak{g})) \cong H^{\mathrm{Lie}}(\mathfrak{g}).$$

Remark 3.14. Away from characteristic zero, it is important to remember that $\mathrm{CE}(\mathfrak{g})$ is a simplicial chain complex with simplicial structure induced by that of $\mathfrak{g}$. In particular, it is not in general the result of an operation applied to any differential graded Lie algebra.

Remark 3.15. Our standing assumption is that 2 is invertible, but Definition 3.12 and Theorem 3.13 extend to $p = 2$ if one adopts the small changes descriped in [May66, Section 5].

Although this result is classical, we do not know of a statement in the literature covering exactly the required level of generality. In order to be self-contained, we offer a complete proof. The first step is to reinterpret Lie algebra homology as Tor groups over the universal enveloping algebra from Section 3.2. A proof of the following well-known result is contained in the following subsection:

Proposition 3.16. *Given any Lie algebra $\mathfrak{g} \in \mathrm{Lie}_R^{\mathbb{N}}$ whose underlying module is flat, there is a natural isomorphism*
$$H^{\mathrm{Lie}}(\mathfrak{g}) \cong \mathrm{Tor}^{U(\mathfrak{g})}(R, R).$$

Assuming this result for now, the second step is to connect the Chevalley–Eilenberg complex from Definition 3.12 to a suitable $U(\mathfrak{g})$-resolution of the ground ring R.

Definition 3.17. The *extended Chevalley–Eilenberg complex* of $\mathfrak{g}$ is the bigraded R-module
$$\overline{\mathrm{CE}}(\mathfrak{g}) = U(\mathfrak{g}) \otimes_R \Gamma_R(\mathfrak{g}[1])$$
equipped with the differential d sending
$$(-1)^{|a_0|} d\left(a_0 \otimes \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_m}(\sigma a_m) \ \otimes\ \langle \sigma b_1, \cdots, \sigma b_n \rangle\right)$$
to the following expression:
$$\begin{aligned}
& \sum_{i=1}^{m} a_0 a_i \otimes \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_i - 1}(\sigma a_i) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma b_1, \cdots, \sigma b_n \rangle \\
+ \quad & \sum_{j=1}^{n} (-1)^{j-1} a_0 b_j \otimes \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma b_1, \cdots, \widehat{\sigma b_j}, \cdots, \sigma b_n \rangle \\
+ \quad & \sum_{1 \le i < j \le m} a_0 \otimes \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_i - 1}(\sigma a_i) \cdots \gamma_{r_j - 1}(\sigma a_j) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma[a_i, a_j], \sigma b_1, \cdots, \sigma b_n \rangle \\
+ \quad & \sum_{1 \le i < j \le n} (-1)^{i+j-1} a_0 \otimes \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma[b_i, b_j], \sigma b_1, \cdots \widehat{\sigma b_i}, \cdots, \widehat{\sigma b_j}, \cdots, \sigma b_n \rangle \\
+ \quad & \frac{1}{2} \sum_{i=1}^{m} a_0 \otimes \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_i - 2}(\sigma a_i) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma[a_i, a_i], \sigma b_1, \cdots, \sigma b_n \rangle \\
+ \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{j-1} a_0 \otimes \gamma_1\left(\sigma[a_i, b_j]\right) \gamma_{r_1}(\sigma a_1) \cdots \gamma_{r_i - 1}(\sigma a_i) \cdots \gamma_{r_m}(\sigma a_m)\ \langle \sigma b_1, \cdots, \widehat{\sigma b_j}, \cdots, \sigma b_n \rangle
\end{aligned}$$

Lemma 3.18. *If $\mathfrak{g} \in \mathrm{Lie}_R^{\mathbb{N}}$ is R-flat, then the augmentation $\overline{\mathrm{CE}}(\mathfrak{g}) \to R$ is a quasi-isomorphism. Therefore, if $\mathfrak{g}$ is R-projective, then $\overline{\mathrm{CE}}(\mathfrak{g})$ is a $U(\mathfrak{g})$-projective resolution of R.*

Proof. It suffices to prove the claim after passing to the associated graded for the diagonal filtration induced by the natural filtrations of $U(\mathfrak{g})$ (cf. Section 3.2) and $\Gamma_R(\mathfrak{g}[1])$. Invoking the Poincaré–Birkhoff–Witt theorem, this complex is isomorphic to the Koszul complex
$$(\mathrm{Sym}_R(\mathfrak{g}) \otimes_R \Gamma_R(\mathfrak{g}[1]),\, \partial) \to R$$

for the graded R-module underlying $\mathfrak{g}$, which is acyclic for flat modules. Indeed, the flat case reduces to the case of a suspension of R via filtered colimits and direct sums, and we may set $R = \mathbb{Z}$ without loss of generality, in which case the claim is classical (cf. [McC00, Prop. 7.1]). $\square$

Proof of Theorem 3.13. In light of the isomorphism $\mathrm{CE}(\mathfrak{g}) \cong R \otimes_{\otimes_{U(\mathfrak{g})}} \overline{\mathrm{CE}}(\mathfrak{g})$, the R-projective case follows from Proposition 3.16 and Lemma 3.18. In the general case, it suffices to show that levelwise application of $\mathrm{CE}(-)$ preserves weak equivalences between R-flat simplicial Lie algebras, which follows by induction along the filtration of $\mathrm{CE}(\mathfrak{g})$ from Lemma 2.18. $\square$

3.4. **Derivations and the proof of Proposition 3.16.** We begin by observing an immediate consequence of Lemma 3.6.

Corollary 3.19. *There is a natural weak equivalence* $\mathbb{L}\mathrm{Q}^{\mathrm{Mod}}_{\mathrm{Lie}}(\mathfrak{g}) \simeq \mathrm{Bar}_\bullet(\mathrm{id}, \mathbf{L}, \mathfrak{g})$ *for R-projective Lie algebras* $\mathfrak{g}$.

In order to compare this monadic bar construction to the derived tensor product in question, we will make use of some classical ideas (cf. [Qui70, Bar96]) relating algebraic homology theories to modules and derivations. We begin with several definitions.

Definition 3.20. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-*module* is a module N equipped with a linear map

$$\mathfrak{g} \otimes_R N \to N,$$

written $a \otimes x \mapsto ax$, such that $[a, b]x = a(bx) - (-1)^{|a||b|}b(ax)$ for all homogeneous $a, b \in \mathfrak{g}$ and $x \in N$. A map of $\mathfrak{g}$-modules is an R-linear map intertwining the action maps.

Write $\mathrm{Mod}_{\mathfrak{g}}$ for the category of $\mathfrak{g}$-modules, and note that this category is naturally isomorphic to the category of (left) $U(\mathfrak{g})$-modules.

Construction 3.21. Let N be a $\mathfrak{g}$-module. Define a bracket on $\mathfrak{g} \ltimes N := \mathfrak{g} \oplus N$ by the formula

$$[(a, x), (b, y)] = \Big([a, b], ay + (-1)^{|b||x|}by\Big)$$

on homogeneous elements. One checks from the definition of a $\mathfrak{g}$-module that this bracket satisfies the axioms of a graded Lie algebra in the sense of Definition 3.1.

We refer to $\mathfrak{g} \ltimes N$ as the *split square-zero extension* of $\mathfrak{g}$ by N. This construction extends in the obvious way to a limit-preserving functor $\mathrm{Mod}_{\mathfrak{g}} \to \mathrm{Lie}_{R/\mathfrak{g}}$ with left adjoint $\Omega_{\mathfrak{g}}$. This object has a further functoriality for base change in that, given maps of Lie algebras $\mathfrak{g}' \to \mathfrak{g}_1 \to \mathfrak{g}_2$, there is a canonical map $\Omega_{\mathfrak{g}_1}(\mathfrak{g}') \to \Omega_{\mathfrak{g}_2}(\mathfrak{g}')$ of $\mathfrak{g}_1$-modules, where the target is viewed as a $\mathfrak{g}_1$-module by restriction along the second map. This map arises as the adjoint of the dashed filler in the following commuting diagram of Lie algebras:

$$\begin{tikzcd}
 & \mathfrak{g}_1 \ltimes \Omega_{\mathfrak{g}_2}(\mathfrak{g}') \arrow[r] \arrow[d] & \mathfrak{g}_2 \ltimes \Omega_{\mathfrak{g}_2}(\mathfrak{g}') \arrow[d] \\
\mathfrak{g}' \arrow[r] \arrow[ur, dashed] & \mathfrak{g}_1 \arrow[r] & \mathfrak{g}_2.
\end{tikzcd}$$

Lemma 3.22. *Let $\mathfrak{g}' \to \mathfrak{g}$ be a map of Lie algebras, and regard* $\mathrm{Free}^{\mathrm{Lie}}_{\mathrm{Mod}} \circ \mathbf{L}^{\circ n}(\mathfrak{g}')$ *as an object of* $\mathrm{Lie}_{R/\mathfrak{g}}$ *via the structure map to $\mathfrak{g}'$. There is an isomorphism of $\mathfrak{g}$-modules (natural in $\mathfrak{g}' \to \mathfrak{g}$):*

$$\Omega_{\mathfrak{g}}(\mathrm{Free}^{\mathrm{Lie}}_{\mathrm{Mod}} \circ \mathbf{L}^{\circ n}(\mathfrak{g}')) \cong U(\mathfrak{g}) \otimes_R \mathbf{L}^{\circ n}(\mathfrak{g}')$$

.

Proof. By adjunction, for any $\mathfrak{g}$-module N, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}_{\mathfrak{g}}}(\Omega_{\mathfrak{g}}(\mathrm{Free}^{\mathrm{Lie}}_{\mathrm{Mod}} \circ \mathbf{L}^{\circ n}(\mathfrak{g}')), N) &\cong \mathrm{Hom}_{\mathrm{Lie}_{R/\mathfrak{g}}}(\mathrm{Free}^{\mathrm{Lie}}_{\mathrm{Mod}} \circ \mathbf{L}^{\circ n}(\mathfrak{g}'), \mathfrak{g} \ltimes N) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_{R/U^{\mathrm{Lie}}_{\mathrm{Mod}}(\mathfrak{g})}}(\mathbf{L}^{\circ n}(\mathfrak{g}'), \mathfrak{g} \oplus N) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_R}(\mathbf{L}^{\circ n}(\mathfrak{g}'), N) \\ &\cong \mathrm{Hom}_{\mathrm{Mod}_{\mathfrak{g}}}(U(\mathfrak{g}) \otimes_R \mathbf{L}^{\circ n}(\mathfrak{g}'), N).\end{aligned}$$

□

Split square-zero extensions are closely related to the theory of derivations.

Definition 3.23. Let N be a $\mathfrak{g}$-module, A *derivation* of $\mathfrak{g}$ into N is an R-module map $f : \mathfrak{g} \to N$ such that

$$f([a, b]) = af(b) + (-1)^{|a||b|} bf(a)$$

for all homogeneous $a, b \in \mathfrak{g}$. We write $\mathrm{Der}(\mathfrak{g}, N)$ for set of derivations of $\mathfrak{g}$ into N.

Lemma 3.24. *There is a bijection* $\mathrm{Hom}_{\mathrm{Lie}_{R/\mathfrak{g}}}(\mathfrak{g}, \mathfrak{g} \ltimes N) \cong \mathrm{Der}(\mathfrak{g}, N)$ *naturally in* N *and* $\mathfrak{g}$.

Proof. Both sets inject into the set of R-module maps $f : \mathfrak{g} \to N$, so it suffices to show that the condition of being a derivation is the same as the condition that $(\mathrm{id}_{\mathfrak{g}}, f)$ be a map of Lie algebras. This comparison is implied by the following simple computation in $\mathfrak{g} \ltimes N$:

$$[a + f(a), b + f(b)] = [a, b] + af(b) + (-1)^{|a||b|} bf(a)$$

□

Using this universal property, we now connect $\Omega_{\mathfrak{g}}$ to the universal enveloping algebra. As a matter of notation, we write $I(A)$ for the augmentation ideal of an augmented algebra.

Lemma 3.25. *There is a natural isomorphism of* $\mathfrak{g}$*-modules* $\Omega_{\mathfrak{g}}(\mathfrak{g}) \cong I(U(\mathfrak{g}))$.

Proof. By Lemma 3.24, it suffices to show that $I(U(\mathfrak{g}))$ corepresents the functor $\mathrm{Der}(\mathfrak{g}, -)$.

First, if $f : \mathfrak{g} \to N$ is a derivation, then we obtain a map $\overline{f} : I(T_R(\mathfrak{g})) \to N$ by setting

$$\overline{f}(a_1 \otimes \cdots \otimes a_n) = a_1 \cdots a_{n-1} f(a_n).$$

Since $\mathfrak{g}$ is a $\mathfrak{g}$-module, we observe that for all homogeneous $t_1, t_2 \in T_R(\mathfrak{g})$ with $t_2 \neq 1$, we have:

$$\overline{f}\left(t_1 \left(a \otimes b - (-1)^{|a||b|} b \otimes a - [a, b]\right) t_2\right) = 0.$$

In the other case, we can use that f is a derivation to deduce

$$\overline{f}\left(t_1 \left(a \otimes b - (-1)^{|a||b|} b \otimes a - [a, b]\right)\right) = 0.$$

Thus, $\overline{f}$ descends to the quotient $I(U(\mathfrak{g}))$, and the resulting map is a map of $\mathfrak{g}$-modules by construction.

Conversely, given any map $f : I(U(\mathfrak{g})) \to N$ of $\mathfrak{g}$-modules, the composite $\mathfrak{g} \to U(\mathfrak{g}) \to N$ is a derivation, since

$$0 = f(ab) - (-1)^{|a||b|} f(ba) - f([a, b]) = af(b) - (-1)^{|a||b|} bf(a) - f([a, b]).$$

These constructions define inverse bijections $\mathrm{Hom}_{\mathrm{Mod}_{\mathfrak{g}}}(I(U(\mathfrak{g})), N) \cong \mathrm{Der}(\mathfrak{g}, N)$, and naturality is obvious. □

Corollary 3.26. *Levelwise application of $\Omega_{\mathfrak{g}}$ preserves weak equivalences of levelwise flat simplicial Lie algebras over $\mathfrak{g}$.*

In particular, $\Omega_{\mathfrak{g}}$ preserves weak equivalences between cofibrant objects and so admits a total left derived functor.

Lemma 3.27. *If $\mathfrak{g}$ is an R-projective Lie algebra, then the augmentation*

$$N(\Omega_{\mathfrak{g}}(\mathrm{Bar}_{\bullet}(\mathrm{Free}^{\mathrm{Lie}}_{\mathrm{Mod}}, \mathbf{L}, \mathfrak{g}))) \to \Omega_{\mathfrak{g}}(\mathfrak{g}) \cong I(U(\mathfrak{g}))$$

is a cofibrant replacement in the category of chain complexes of $U(\mathfrak{g})$-modules.

Proof. In each chain degree, the chain complex in question is $U(\mathfrak{g})$-free on a projective R-module by Lemmas 3.3 and 3.22. To see that the map is a quasi-isomorphism, we note that the lefthand side computes $\mathbb{L}\Omega_{\mathfrak{g}}(\mathfrak{g})$ by Lemma 3.6, while the righthand side computes $\mathbb{L}\Omega_{\mathfrak{g}}(\mathfrak{g})$ by the projectivity of $\mathfrak{g}$, Lemma 3.25, and Corollary 3.26. □

Corollary 3.28. *For R-projective simplicial Lie algebras $\mathfrak{g}$, there is a natural weak equivalence*

$$\mathbb{L}\mathrm{Q}^{\mathrm{Mod}}_{\mathrm{Lie}}(\mathfrak{g}) \simeq R \otimes^{\mathbb{L}}_{U(\mathfrak{g})} I(U(\mathfrak{g})).$$

Proof of Proposition 3.16. Assume first that $\mathfrak{g}$ is R-projective. The short exact sequence

$$I(U(\mathfrak{g})) \to U(\mathfrak{g}) \to R$$

of $\mathfrak{g}$-modules gives rise to a cofiber sequence

$$\mathbb{L}\mathrm{Q}^{\mathrm{Mod}}_{\mathrm{Lie}}(\mathfrak{g}) \to R \to R \otimes^{\mathbb{L}}_{U(\mathfrak{g})} R.$$

The augmentation $U(\mathfrak{g}) \to R$ gives rise to a retraction of the righthand map, and the claim follows. In the general case, it suffices to verify that $R \otimes^{\mathbb{L}}_{U(\mathfrak{g})} R$ preserves levelwise weak equivalences between levelwise flat simplicial Lie algebras, which follows from the fact that $U(\mathfrak{g})$ is flat for R-flat $\mathfrak{g}$ by Theorem 3.7 and the functor U preserves weak equivalences between R-flat simplicial Lie algebras by Proposition 3.9. □

4. A Chevalley–Eilenberg complex for Hecke Lie algebras

In this section, we recall the definition of Hecke Lie algebras and develop their homotopy theory. Hecke Lie algebras were introduced in the first author's thesis [Bra17] in order to describe the operations acting on the E-theory of $K(h)$-local Lie algebras. In this work, we restrict attention to the case of an odd prime p, where Hecke Lie algebras are particularly simple.

We fix a (smooth, 1-dimensional, commutative) formal group G_0 of height $0 < h < \infty$ over a perfect field k of characteristic p. We write E for the corresponding Lubin–Tate spectrum constructed by Goerss, Hopkins, and Miller [GH04] [Rez98]. The ring spectrum E is complex orientable, and we fix a complex orientation $x^E \in \widetilde{E}^0(\mathbb{C}P^\infty)$. Given a (virtual) complex vector bundle $V \to X$, there is a Thom class $\tau_V \in \widetilde{E}^0(X^V)$ such that multiplication by τ_V defines an isomorphism $E^*(X) = \widetilde{E}^*(X_+) \xrightarrow{\cong} \widetilde{E}^*(X^V)$. In the case of the trivial complex line bundle over a point, the Thom isomorphism gives rise to an isomorphism $E^* \cong E^{*-2}$ and hence to a periodicity generator $u \in \pi_2(E) = E_2$.

4.1. **Power operations on $\mathbb{E}_\infty$-rings.** We begin with some recollections concerning power operations on $K(h)$-local $\mathbb{E}_\infty$-E-algebras, as developed in [Wil82], [Hop14], and [Rez09]).

Given an integer $i \in \mathbb{Z}$, we write Γ^{-i} for Rezk's (uncompleted) ring of additive degree i power operations, which acts naturally on the i^{th} homotopy group $\pi_i(R)$ for any $K(h)$-local $\mathbb{E}_\infty$-E-algebra R (cf. [Rez09, Section 6.2.]). The ring Γ^{-i} is endowed with a canonical weight grading $\Gamma^{-i} \cong \bigoplus_w \Gamma^{-i}(w)$, where

$$\Gamma^{-i}(w) \cong \ker \left(\pi_i\left((\Sigma^{-i}E)^{\otimes w}_{h\Sigma_w}\right) \to \bigoplus_{0<j<w} \pi_i\left((\Sigma^{-i}E)^{\otimes w}_{h(\Sigma_j \times \Sigma_{w-j})}\right)\right). \tag{1}$$

The map shown is induced by the transfer. We have[2] $\Gamma^{-i}(w) = 0$ unless w is a power of p.

Warning. *The rings Γ^i are not related to the free divided power functor Γ^* appearing in the Chevalley–Eilenberg complex; the meaning of the symbol Γ will be clear from the context.*

We can also consider *weighted* $K(h)$-local $\mathbb{E}_\infty$-E-algebras, i.e., commutative algebra objects in the functor ∞-category $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Mod}_E^\wedge)$ with its Day convolution symmetric monoidal structure (cf. [Gla16],[Lur03, Section 2.2.6]). In this context, Rezk's rings act in a weighted manner, in the sense that there are action maps

$$\Gamma^{-i}(w) \times \pi_i(R(n)) \longrightarrow \pi_i(R(w \cdot n))$$

refining the usual action on the underlying $K(h)$-local $\mathbb{E}_\infty$-E-algebra $L_{K(h)}\left(\bigoplus_n R(n)\right)$.

The rings Γ^{-i} are linked by *twisting* homomorphisms $E_k \otimes_{E_0} \Gamma^{-i} \otimes_{E_0} E_{-k} \to \Gamma^{-i-k}$ sending a tensor $\lambda \otimes \alpha \otimes \mu$ to the operation $x \mapsto (\lambda \cdot \alpha(\mu \cdot x))$. This twisting map respects weight and is an isomorphism whenever k is even.

More interesting are the *suspension* homomorphisms

$$\ldots \xrightarrow{\cong} \Gamma^2 \hookrightarrow \Gamma^1 \xrightarrow{\cong} \Gamma^0 \hookrightarrow \Gamma^{-1} \xrightarrow{\cong} \Gamma^{-2} \hookrightarrow \ldots.$$

These morphisms, which we denote generically by Susp, also respect weight. They are defined as follows. First, note that for all w and i, there is an identification of E_0-modules

$$\pi_i\left((\Sigma^i E)^{\otimes w}_{h\Sigma_w}\right) \cong \widetilde{E}_i^\wedge\left((S^i)^{\otimes w}_{h\Sigma_w}\right).$$

[2] We differ from Rezk's "logarithmic" grading convention.

Smashing the diagonal $S^1 \to (S^1)^{\otimes w}$ with $(S^i)^{\otimes w}$ and applying $(-)_{\Sigma_w}$ produces a map

$$\widetilde{E}^{\wedge}_i\left((S^i)^{\otimes w}_{h\Sigma_w}\right) \cong \widetilde{E}^{\wedge}_{i+1}\left(S^1 \otimes_{h\Sigma_w} (S^i)^{\otimes w}\right) \to \widetilde{E}^{\wedge}_{i+1}\left((S^{i+1})^{\otimes w}_{h\Sigma_w}\right),$$

which restricts to the desired map $\Gamma^{-i}(w) \to \Gamma^{-i-1}(w)$ (cf. e.g. [Rez09, Remark 7.5]).

The following result provides a useful alternative description of the suspension.

Proposition 4.1 (Suspensions via Euler class)**.** *For each $w \geq 0$, there is a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \Gamma^0(w) \arrow[r, hook, "(e\cdot -)^\vee"] \arrow[d, "\cong"'] & \Gamma^0(w) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Gamma^0(w) \arrow[r, hook, "(e\cdot -)^\vee"] \arrow[d, "\cong"'] & \Gamma^0(w) \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & \Gamma^2(w) \arrow[r, hook, "\mathrm{Susp}"] & \Gamma^1(w) \arrow[r, "\mathrm{Susp}"] & \Gamma^0(w) \arrow[r, hook, "\mathrm{Susp}"] & \Gamma^{-1}(w) \arrow[r] & \cdots,
\end{tikzcd}$$

where $e \in \widetilde{E}^0(B\Sigma_{w+})$ is the Euler class of the reduced complex standard representation of Σ_w.

Proof. When $i = 2j$, we can identify $(S^{2j})^{\otimes w}_{h\Sigma_w}$ with the Thom spectrum $B\Sigma_w^{j\cdot V_w}$, where V_w denotes the (complexified) standard representation of Σ_w. Under this identification, the map $\Sigma^2(S^{2j})^{\otimes w}_{h\Sigma_w} \to (S^{2(j+1)})^{\otimes w}_{h\Sigma_w}$ induced by the diagonal, as above, corresponds to the "Thomification" of the map of Σ_m-representations $j \cdot V_m \oplus \mathbb{C} \to (j+1)\cdot V_m$ induced by the diagonal embedding of the trivial representation $\mathbb{C}$ into the standard representation V_m.

The E_0-linear dual of the Thom isomorphism provides the identification $\widetilde{E}^{\wedge}_{2j}(B\Sigma_w^{j\cdot V_w}) \cong \widetilde{E}^{\wedge}_0(B\Sigma_{w+})$, and, since the Thom isomorphism respects transfers, this identification restricts to an isomorphism of left E_0-modules $\Gamma^{-2j}(w) \cong \Gamma^0(w)$ defined in (1). We have used repeatedly that the completed E-homology of symmetric groups and the E_0-modules $\Gamma^i(w)$ are finitely generated and free [Str98, Theorem 1.1.]. The composite

$$\widetilde{E}^0(B\Sigma_{w+}) \cong \widetilde{E}^{2(j+1)}(B\Sigma_{w+}^{(j+1)\cdot V_m}) \to \widetilde{E}^{2(j+1)}(B\Sigma_{w+}^{j\cdot V_m \oplus \mathbb{C}}) \cong \widetilde{E}^0(B\Sigma_{w+})$$

induced by the diagonal $\mathbb{C} \to V_m$ is well known to be multiplication by $e \in \widetilde{E}^0(B\Sigma_{w+})$ – see [Str98, p.14], for example. Passing to E_0-linear duals and to the respective submodules of additive operations, and then inserting the odd degree rings "by hand" (using that every second suspension is an isomorphism), we obtain the result. □

At weight $w = p$, the module $\Gamma^0(p)^\vee$ is given by $E^0(B\Sigma_p)/(\mathrm{tr})$, there tr denotes the image of the transfer map from the trivial group (cf. [Rez09, Section 10.3]). The Euler class of the reduced standard representation of $B\Sigma_p$ may be viewed as an element $e \in E^0(B\Sigma_p)$, and by abuse of notation, we shall also use e to denote its image in $\Gamma^0(p)^\vee$.

Example 4.2 (The K-theory of $\mathbb{E}_\infty$-rings)**.** The E-theory corresponding to the commutative formal group law $\widehat{\mathbb{G}}_m$ of height 1 over $\mathbb{F}_p$ is given by p-complete complex K-theory. We recall from [Hop14] that there is a $K(1)$-local equivalence $S^0_{h\Sigma_p} \simeq B\Sigma_{p+} \xrightarrow{(\epsilon,\mathrm{tr})} S^0 \oplus S^0$, where ϵ denotes the collapse map and tr is the transfer map. The homotopy unique map $\psi : S^0 \to B\Sigma_{p+}$ with $\epsilon\circ\psi \simeq 1$ and $\mathrm{tr}\circ\psi \simeq 0$ defines a class $\Psi_0 \in K^{\wedge}_0(B\Sigma_{p+}) = \pi_0(L_{K(1)}K^{\otimes p}_{h\Sigma_p})$. As the transfer vanishes, this class lies in $\Gamma^0(p)$. Using periodicity and the suspensions $\Gamma^{2n+1}(p) \xrightarrow{\cong} \Gamma^{2n}(p)$, we obtain classes $\Psi_i \in \Gamma^i$ for all i. At height 1, the Euler class appearing in Proposition 4.1 is just p, and the suspension homomorphism $\Gamma^i \to \Gamma^{i-1}$ is therefore determined by the assignment

$$\Psi_i \quad \mapsto \quad \begin{cases} \Psi_{i-1} & \text{for } i \text{ odd} \\ p\cdot\Psi_{i-1} & \text{for } i \text{ even.} \end{cases}$$

The class Ψ_i generates Γ^i freely and there is an equivalence $\Gamma^i = \mathbb{Z}_p[\Psi_i]$.

In fact, we can explicitly describe the ring $\Gamma^0(p)^\vee$ and the Euler class e at all heights:

Proposition 4.3 (Euler class basis)**.** *If the coordinate on E-theory is chosen so that the resulting formal group law is p-typical, then there are isomorphisms*

$$\Gamma^0(p)^\vee \cong E^0(B\Sigma_p)/(\mathrm{tr}) \cong E_0[e]/f(e),$$

where e is the Euler class of the reduced standard representation and $f(e) = e^{\frac{p^h-1}{p-1}} + \cdots + p$ is the unique monic degree $\frac{p^h-1}{p-1}$ polynomial over E_0 for which $f(-x^{p-1}) = \frac{[p](x)}{x}$ in $E_0[[x]]/[p](x)$.

Proof. A standard transfer argument shows that $E^*(B\Sigma_p)$ is the subring of $(\mathbb{Z}/p)^\times = \mathrm{Aut}(C_p)$-fixed points inside $E^*(BC_p)$. The Gysin sequence associated to the fibration $S^1 \longrightarrow BC_p \longrightarrow BS^1 \simeq \mathbb{CP}^\infty$ yields the formula $E^*(BC_p) \cong E_*[[x]]/[p](x)$. The Euler class $e \in E^*(B\Sigma_p)$ sits inside of $E^*(BC_p)$ as $\prod_{k=1}^{p-1}[k](x) = -x^{p-1}$, where the last equality is a consequence of the p-typicality of the formal group law. Finally, the transfer ideal (tr) is generated as an E_*-module by $\frac{[p](x)}{x}$. For details, we refer to [HKR00] and [Mar10, Section 4.3.6]. □

4.2. **Hecke modules and Hecke Lie algebras.** We now review the theory of Hecke modules and Hecke Lie algebras [Bra17, Section 4.3-4]. As we will see in Section 4.5, these definitions exactly capture the structure of the operations acting on the completed E-homology of $K(h)$-local Lie algebras.

Remark 4.4 (Grading convention)**.** Our conventions differ from those of [Bra17] by a shift. Specifically, if $\mathfrak{g}$ is a shifted Lie algebra with shifted bracket $[-,-]'$, then $\Sigma^{-1}\mathfrak{g}$ becomes a graded Lie algebra with bracket defined by the formula $[u,v] := (-1)^{\deg(u)}\sigma^{-1}([\sigma(u),\sigma(v)]')$, where $\sigma : \Sigma^{-1}\mathfrak{g} \to \mathfrak{g}$ denotes the evident shifting bijection and σ^{-1} its inverse.

Our definitions of the Hecke power ring, Hecke modules, and Hecke Lie algebras will therefore all differ from the corresponding notions in [Bra17] by a shift. To make this small difference clear throughout, we have added the subscript "u" in various places, thereby stressing that we are working *u*nshifted Hecke Lie algebras.

Definition 4.5 (Power rings)**.** A *power ring* is a collection $P = \{P_i^j(w)\}_{(i,j,w)\in\mathbb{Z}^2\times\mathbb{N}}$ of Abelian groups with elements $\iota_{ii} \in P_i^i(1)$ for all i, together with associative and unital composition maps

$$P_i^j(v) \otimes P_j^k(w) \longrightarrow P_i^k(vw).$$

Example 4.6. The power ring P^{Comm} of additive operations on $K(h)$-local $\mathbb{E}_\infty$-rings is given by

$$(P^{\mathrm{Comm}})_i^j(w) = E_{j-i} \underset{E_0}{\otimes} \Gamma^{-i}(w),$$

with composition defined using the twisting maps and multiplication in the Γ^i.

Definition 4.7 (Modules over power rings)**.** A (weighted) *module* over the power ring P is a weighted graded Abelian group $M \in \mathrm{Mod}_{\mathbb{Z}}^{\mathbb{N}}$ equipped with multiplication maps $P_i^j(w)\otimes M_i \to M_j$ compatible with composition in P. A map of P-modules is a map of weighted graded Abelian groups intertwining with multiplication.

We can now define the power ring of primary interest:

Definition 4.8 (Hecke operations on Lie algebras)**.** The power ring $\mathcal{H}_u^{\mathrm{Lie}}$ of additive operations on (unshifted) $K(h)$-local spectral Lie algebras is given by

$$(\mathcal{H}_u^{\mathrm{Lie}})_i^j(w) = \begin{cases} \mathrm{Ext}^a_{\Gamma^i}(E_0, E_{-i+j+a}) & \text{if } w = p^a \\ 0 & \text{if } w \text{ is not a power of } p, \end{cases}$$

where we regard E_0 and E_{-i+j+a} as trivial Γ^i-modules, with composition defined as the counterclockwise composite in the commutative diagram

$$\begin{tikzcd}
(\mathcal{H}_u^{\mathrm{Lie}})_i^j(p^a) \otimes (\mathcal{H}_u^{\mathrm{Lie}})_j^k(p^b) \arrow[r, dashed] \arrow[d, equal] & (\mathcal{H}_u^{\mathrm{Lie}})_i^k(p^{a+b}) \arrow[ddd, equal] \\
(\mathrm{Ext}^a_{\Gamma^i}(E_0, E_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Gamma^j}(E_0, E_{-j+k+b})) \arrow[d] & \\
(\mathrm{Ext}^a_{\Gamma^i}(E_0, E_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Gamma^{j+a}}(E_0, E_{-j+k+b})) \arrow[d] & \\
(\mathrm{Ext}^a_{\Gamma^i}(E_0, E_{-i+j+a})) \otimes (\mathrm{Ext}^b_{\Gamma^i}(E_{-i+j+a}, E_{-i+k+a+b})) \arrow[r] & (\mathrm{Ext}^{a+b}_{\Gamma^i}(E_0, E_{-i+k+a+b})),
\end{tikzcd}$$

The first map is suspension, the second twisting, and the third is the Yoneda product.

Example 4.9 (Hecke operations at height one)**.** For p-adic K-theory defined over $\mathbb{Z}_p$, we have

$$(\mathcal{H}_u^{\mathrm{Lie}})_i^j(w) = \begin{cases} E_{j-i} \cdot \iota_i & \text{if } w = 1 \\ E_{j-i+1} \cdot \alpha_i & \text{if } w = p \\ 0 & \text{else}, \end{cases}$$

Here ι_i is the identity operation in degree i, and the weight of α_i is p. Composition is defined as

$$\begin{aligned}
(\lambda_{j-i} \cdot \iota_i) \otimes (\lambda_{k-j} \cdot \iota_j) &\mapsto (\lambda_{k-j}\lambda_{j-i} \cdot \iota_i) \\
(\lambda_{j-i} \cdot \alpha_i) \otimes (\lambda_{k-j+1} \cdot \iota_{j-1}) &\mapsto (\lambda_{k-j+1}\lambda_{j-i} \cdot \alpha_i) \\
(\lambda_{j-i} \cdot \iota_i) \otimes (\lambda_{k-j+1} \cdot \alpha_j) &\mapsto (\lambda_{k-j+1}\lambda_{j-i} \cdot \alpha_i) \\
(\lambda_{j-i} \cdot \alpha_i) \otimes (\lambda_{k-j+1} \cdot \alpha_{j-1}) &\mapsto 0.
\end{aligned}$$

Here we have used that the Frobenius on $\mathbb{Z}_p = W(\mathbb{F}_p)$ is trivial at height $h = 1$.

Definition 4.10 (Hecke modules)**.** A *Hecke module* is a weighted module over the power ring $\mathcal{H}_u^{\mathrm{Lie}}$. We write $\mathrm{Mod}^{\mathbb{N}}_{\mathcal{H}_u}$ for the category of Hecke modules.

Note that a Hecke module is in particular an E_*-module.

The category $\mathrm{Mod}^{\mathbb{N}}_{\mathcal{H}_u}$ is the category of modules for an additive monad $\mathbf{A}^{\mathcal{H}_u}$ on $\mathrm{Mod}^{\mathbb{N}}_{E_*}$, where $\mathbf{A}^{\mathcal{H}_u}(M)_j$ is the quotient of the free Abelian monoid on symbols $\{\ [\alpha|x] \mid \alpha \in (\mathcal{H}_u^{\mathrm{Lie}})_i^j, x_i \in M_i\}$ by all relations $[\alpha_1, x] + [\alpha_2, x] = [\alpha_1 + \alpha_2, x], [\alpha, x+y] = [\alpha, x] + [\alpha, y]$ with $\alpha \in (\mathcal{H}_u^{\mathrm{Lie}})_i^j, x_i \in M_i$ and all relations $[\lambda|x] = [1|\lambda x]$ with $\lambda \in (\mathcal{H}_u^{\mathrm{Lie}})_i^j(1)$ a scalar and $x_i \in M_i$. Here we implicitly endow symbols in $\mathbf{A}^{\mathcal{H}_u}(M)$ with their natural weight.

We write $\mathrm{Free}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}$ and $\mathrm{U}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}$ the corresponding free and forgetful functors.

Definition 4.11 (Hecke Lie algebras)**.** A *Hecke Lie algebra* consists of an (unshifted, weighted) Hecke module $\mathfrak{g} \in \mathrm{Mod}^{\mathbb{N}}_{\mathcal{H}_u}$ equipped with the structure of a Lie algebra on its underlying weighted E_*-module, subject to the relation $[x, \alpha(y)] = 0$ for all $x \in \mathfrak{g}_k, y \in \mathfrak{g}_i$, and $\alpha \in (\mathcal{H}_u^{\mathrm{Lie}})_i^j(w)$ with $w > 1$. A map of Hecke Lie algebras is a map of Hecke modules intertwining the brackets.

We write $\mathrm{Lie}^{\mathbb{N}}_{\mathcal{H}_u}$ for the resulting category of Hecke Lie algebras. This category is the category of modules for a monad $\mathbf{L}^{\mathcal{H}_u}$ on $\mathrm{Mod}^{\mathbb{N}}_{E_*}$, and we denote the corresponding free and forgetful functors by $\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}$ and $\mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}$, respectively. These adjunctions factor through free and forgetful adjunctions to $\mathrm{Lie}^{\mathbb{N}}_{E_*}$ and $\mathrm{Mod}^{\mathbb{N}}_{\mathcal{H}_u}$, which we write as $(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Lie}_{E_*}}, \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Lie}_{E_*}})$ and $(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}}, \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}})$, respectively. We now place these notions within a homotopical setting. The next result follows from the same considerations as Proposition 3.4 above.

Proposition 4.12 (Transferred model structures)**.** *Right transferred model structures exist along the following adjunctions:*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{sLie}^{\mathbb{N}}_{E_*} \arrow[r, shift left=2, "\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Lie}_{E_*}}"] \arrow[r, phantom, "\perp"] \arrow[d, shift left=2, "\mathrm{U}^{\mathrm{Lie}_{E_*}}_{\mathrm{Mod}_{E_*}}"] \arrow[d, phantom, "\dashv"] & \mathrm{sLie}^{\mathbb{N}}_{\mathcal{H}_u} \arrow[l, shift left=2, "\mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Lie}_{E_*}}"] \arrow[d, shift right=2, "\mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}}"'] \arrow[d, phantom, "\vdash"] \\
\mathrm{sMod}^{\mathbb{N}}_{E_*} \arrow[u, shift left=2, "\mathrm{Free}^{\mathrm{Lie}_{E_*}}_{\mathrm{Mod}_{E_*}}"] \arrow[r, shift right=2, "\mathrm{Free}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}"'] \arrow[r, phantom, "\top"] & \mathrm{sMod}^{\mathbb{N}}_{\mathcal{H}_u}, \arrow[l, shift right=2, "\mathrm{U}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}"'] \arrow[u, shift right=2, "\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}}"']
\end{tikzcd}$$

Bar constructions again provide convenient cofibrant replacements in these model categories:

Lemma 4.13 (Cofibrant replacement)**.** *Let $\mathfrak{g}$ be a simplicial Hecke Lie algebra and M a simplicial Hecke module.*

(1) The natural maps $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g}) \to \mathfrak{g}$ *and* $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{A}^{\mathcal{H}_u}, M) \to M$ *are weak equivalences.*

(2) If $\mathfrak{g}$ is levelwise projective, then $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g})$, $\mathrm{U}^{\mathrm{Lie}^{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}}(\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g}))$ *are both cofibrant.*

(3) If M is levelwise projective, then $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{A}^{\mathcal{H}_u}, M)$ *is cofibrant.*

Proof. The first and third claim follow in the same way as Lemma 3.6. For the second, one needs the further observation that $\mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}} \circ \mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}$ takes values in free Hecke modules. □

4.3. **Indecomposables.** In this section, we will extend the adjunction considered in Section 3.3 to a commuting diagram of adjunctions of the form

$$\begin{tikzcd}
\mathrm{Lie}^{\mathbb{N}}_{\mathcal{H}_u} \arrow[rr, shift left=2, "\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}"] \arrow[rr, phantom, "\perp"] \arrow[d, shift right=2, "\mathrm{Q}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Lie}_{\mathcal{H}_u}}"'] \arrow[d, phantom, "\dashv"] & & \mathrm{Lie}^{\mathbb{N}}_{E_*} \arrow[ll, shift left=2, "\mathrm{T}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}"] \arrow[d, shift left=2, "\mathrm{Q}^{\mathrm{Mod}}_{\mathrm{Lie}}"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Mod}^{\mathcal{H}_u}_{E_*} \arrow[u, shift right=2, "\mathrm{T}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Lie}_{\mathcal{H}_u}}"'] \arrow[rr, shift right=2, "\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}}"'] \arrow[rr, phantom, "\top"] & & \mathrm{Mod}^{\mathbb{N}}_{E_*} \arrow[ll, shift right=2, "\mathrm{T}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}}"'] \arrow[u, shift left=2, "\mathrm{T}^{\mathrm{Mod}}_{\mathrm{Lie}}"]
\end{tikzcd}$$

The right adjoints take an algebraic structure and produce a richer one by defining certain operations to be identically zero. The left adjoints are functors of indecomposables, which take an algebraic structure and produce a simpler one by forming the quotient by the image of certain operations.

More formally, we make the following definitions at the level of objects for an E_*-module M, a Hecke module N, a Lie algebra $\mathfrak{g}$, and Hecke Lie algebra $\mathfrak{h}$.

(1) The underlying Hecke module of $\mathrm{T}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Lie}_{\mathcal{H}_u}}(N)$ is N, and the Lie bracket is the zero map. The underlying E_*-module of $\mathrm{Q}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Lie}_{\mathcal{H}_u}}(\mathfrak{h})$ is the quotient of $\mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}(\mathfrak{h})$ by the submodule generated by the elements of the form $\alpha([x_1,[x_2,[\ldots,[x_{n-1},x_n]\ldots]]])$ with $\alpha\in\mathcal{H}_u^{\mathrm{Lie}}$, $n>1$, and $x_1,\ldots,x_n\in\mathfrak{g}$, and the Hecke operations descend to the quotient.

(2) The underlying graded Abelian group of $\mathrm{T}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}}(M)$ is that of M, and the Hecke module structure is defined by setting

$$\alpha(x)=\begin{cases}\alpha\cdot x & \text{if } \alpha\in(\mathcal{H}_u^{\mathrm{Lie}})_i^j[1]\cong E_{j-i}\\ 0 & \text{if } w>1\end{cases}$$

for $x\in M_i$ and $\alpha\in(\mathcal{H}_u^{\mathrm{Lie}})_i^j(w)$. The E_*-module $\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}}(N)$ is the quotient of $\mathrm{U}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}(N)$ by the submodule generated by the elements of the form $\alpha(x)$ with $\alpha\in(\mathcal{H}_u^{\mathrm{Lie}})_i^j(w)$ with $w>1$ and $x\in N_i$.

(3) The underlying Hecke module of $\mathrm{T}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}(\mathfrak{g})$ is $\mathrm{T}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}}(\mathrm{U}^{\mathrm{Lie}}_{\mathrm{Mod}}(\mathfrak{g}))$, the underlying Lie algebra is $\mathfrak{g}$, and the two define a Hecke Lie algebra structure. The underlying E_*-module of $\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}(\mathfrak{h})$ is $\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}}(\mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}}(\mathfrak{h}))$, and the Lie bracket descends to the quotient.

These definitions extend to arrows in obvious ways, and it is easily checked that the claimed adjunctions hold and that the respective diagrams of left and right adjoints commute up to unique natural isomorphism.

We obtain a simplicial variant of our diagram of adjunctions by applying $\mathrm{Fun}(\Delta^{op},-)$. These adjunctions are all Quillen, since the right adjoints each preserve fibrations and trivial fibrations by definition.

In particular, we obtain a Quillen adjunction

$$\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}} : \mathrm{sLie}^{\mathbb{N}}_{\mathcal{H}_u} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{sMod}^{\mathbb{N}}_{E_*} : \mathrm{T}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}},$$

where $\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}} = \mathrm{Q}^{\mathrm{Mod}}_{\mathrm{Lie}} \circ \mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}} \cong \mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}} \circ \mathrm{Q}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Lie}_{\mathcal{H}_u}}$.

Definition 4.14 (Hecke Lie algebra homology)**.** The *Hecke Lie algebra homology* of a simplicial Hecke Lie algebra $\mathfrak{g}$ is the bigraded E_*-module

$$H^{\mathrm{Lie}_{\mathcal{H}_u}}(\mathfrak{g}) = H\left(N\left(\mathbb{L}\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}(\mathfrak{g})\right)[1] \oplus E_*\right).$$

4.4. **The Hecke–Chevalley–Eilenberg complex.** In this section, we introduce a small complex for computing Hecke Lie algebra homology. Using the identity

$$\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}} = \mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{E_*}} \circ \mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}},$$

our strategy will be to pair an understanding of the left derived functor of $\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}$ with our earlier exploration of Lie algebra homology.

Construction 4.15 (The additive resolution)**.** Given a Hecke Lie algebra $\mathfrak{g}$, we endow

$$\mathrm{Bar}_\bullet(\mathrm{id}, \mathbf{A}^{\mathcal{H}_u}, \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}}(\mathfrak{g}))$$

with the structure of a simplicial Lie algebra by defining the Lie bracket by the equation

$$\Big[\, [\alpha_1|\cdots|\alpha_n|x]\ ,\ [\beta_1|\cdots|\beta_n|y]\,\Big] = \begin{cases} \alpha_1\cdots\alpha_n\beta_1\cdots\beta_n[x,y] & \text{if all } \alpha_i,\beta_j \text{ have weight 1} \\ 0 & \text{otherwise.}\end{cases}$$

One checks that this operation is well-defined, satisfies the identities of a Lie algebra, and respects the simplicial structure maps. The same formula defines the structure of a simplicial Hecke Lie algebra on the simplicial Hecke module $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{A}^{\mathcal{H}_u}, \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}}(\mathfrak{g}))$.

We denote the algebraically enhanced bar constructions of Construction 4.15 by $\mathrm{AR}(\mathfrak{g})$ and $\overline{\mathrm{AR}}(\mathfrak{g})$, respectively, and refer to the former as the *additive resolution* of $\mathfrak{g}$.

Definition 4.16 (Hecke Chevalley–Eilenberg complex)**.** The *Hecke Chevalley–Eilenberg complex* of a Hecke Lie algebra $\mathfrak{g}$ is the simplicial chain complex of weighted E_*-modules

$$\mathrm{CE}_{\mathcal{H}_u}(\mathfrak{g}) = \mathrm{CE}\left(\mathrm{AR}(\mathfrak{g})\right).$$

If $\mathfrak{g}$ is a simplicial Hecke Lie algebra, then $\mathrm{CE}_{\mathcal{H}_u}(\mathfrak{g})$ is defined as the bisimplicial chain complex obtained by applying the previous construction levelwise.

Notation 4.17 (Gradings in $\mathrm{CE}_{\mathcal{H}_u}(\mathfrak{g})$)**.** Given a (weighted) Hecke Lie algebra $\mathfrak{g}$, the Hecke Chevalley–Eilenberg complex $\mathrm{CE}_{\mathcal{H}_u}(\mathfrak{g})$ is a simplicial chain complex in the abelian category $\mathrm{Mod}^{\mathbb{N}}_{E_*}$ of weighted graded E_*-modules. It is therefore equipped with four different gradings, which we will list as a tuple (i, j, r, w) in the following order:

1. i is the *internal degree* (coming from the abelian category of graded E_*-modules).
2. j is the *homological degree* (corresponding to the divided power degree in the Chevalley–Eilenberg complex in Definition 3.12, i.e. the chain complex direction)
3. r is the *simplicial degree* (indicating the position in the additive resolution $\mathrm{AR}(\mathfrak{g})$)
4. w is the weight (coming from the ambient weight-grading on the category $\mathrm{Mod}^{\mathbb{N}}_{E_*}$).

We can now state the main theorem of this section.

Theorem 4.18. *There is a natural isomorphism $H^{\mathrm{Lie}_{\mathcal{H}_u}}(\mathfrak{g}) \cong H(\mathrm{CE}_{\mathcal{H}_u}(\mathfrak{g}))$ for E_*-projective simplicial Hecke Lie algebras $\mathfrak{g}$.*

The theorem follows by combining Theorem 3.13 with the following result.

Proposition 4.19. *There is a natural weak equivalence of simplicial Lie algebras*

$$\mathbb{L}\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}(\mathfrak{g}) \simeq \mathrm{AR}(\mathfrak{g})$$

for E_*-projective simplicial Hecke Lie algebras $\mathfrak{g}$.

Proof. Because brackets of Hecke power operations vanish, the Hecke Lie structure map of $\mathfrak{g}$ factors canonically as the composite $\mathbf{L}^{\mathcal{H}_u}(\mathfrak{g}) \to \mathbf{A}^{\mathcal{H}_u}(\mathfrak{g}) \to \mathfrak{g}$, and these maps extend to maps $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g}) \to \overline{\mathrm{AR}}(\mathfrak{g}) \to \mathfrak{g}$ of simplicial Hecke Lie algebras, from which we obtain the commutative diagram

$$\begin{array}{ccccc}
\mathbb{L}\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\left(\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g})\right) & \longrightarrow & \mathbb{L}\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\left(\overline{\mathrm{AR}}(\mathfrak{g})\right) & \longrightarrow & \mathbb{L}\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}(\mathfrak{g}) \\
\downarrow & & \downarrow & & \\
\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\left(\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g})\right) & \longrightarrow & \mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\left(\overline{\mathrm{AR}}(\mathfrak{g})\right). & &
\end{array}$$

We will argue that all of the arrows in this diagram are weak equivalences. With this claim established, the result will follow, since the map $\overline{\mathrm{AR}}(\mathfrak{g}) \to \mathrm{AR}(\mathfrak{g})$ induced by the augmentation of $\mathbf{A}^{\mathcal{H}_u}$ induces an isomorphism

$$\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\left(\overline{\mathrm{AR}}(\mathfrak{g})\right) \xrightarrow{\simeq} \mathrm{AR}(\mathfrak{g})$$

of simplicial Lie algebras.

The top row of horizontal maps and the lefthand vertical map are weak equivalences by Lemma 4.13, so it remains to verify that the weak equivalence $\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g}) \xrightarrow{\sim} \overline{\mathrm{AR}}(\mathfrak{g})$ is preserved by $\mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}$, which, after applying $\mathrm{U}^{\mathrm{Lie}_{E_*}}_{\mathrm{Mod}_{E_*}}$ and using the isomorphism

$$U^{\mathrm{Lie}_{E_*}}_{\mathrm{Mod}_{E_*}} \circ \mathrm{Q}^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}} \cong \mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}} \circ \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}},$$

is equivalent to showing that $\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}}$ preserves the weak equivalence

$$\mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}}\left(\mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g})\right) \xrightarrow{\sim} \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}}\left(\overline{\mathrm{AR}}(\mathfrak{g})\right) \cong \mathrm{Bar}_\bullet\left(\mathrm{Free}^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}, \mathbf{A}^{\mathcal{H}_u}, \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}}(\mathfrak{g})\right).$$

Since $\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}}$ is left Quillen, it suffices to observe that the source and target of this weak equivalence are both cofibrant by Lemma 4.13 $\square$

4.5. **Spectral Lie algebras and Hecke Lie algebras.** The structure of a Hecke Lie algebra is the shadow in E-homology cast by a much richer structure, which lifts the Lie algebra axioms from classical algebra to the world of stable homotopy. These *spectral Lie algebras* are spectra equipped with an action of the spectral Lie operad, the operadic Koszul dual of the commutative operad in spectra. This operad was first studied in [Sal98] and [Chi05], and its algebras have been the subject of much recent study in a variety of contexts [AC15, Kja16, BR17, Heu18].

In the context of Lubin–Tate theory, we have the following result, which was established in the first author's thesis [Bra17, Theorem 4.4.4.]. We write $\mathrm{Free}^{\mathscr{L}_E}$ for the left adjoint to the forgetful functor on the ∞-category of spectral Lie algebras in $K(h)$-local E-module spectra.

Theorem 4.20 (Brantner)**.**

(1) The homotopy groups of a (weighted) spectral Lie algebra in $K(h)$-local E-module spectra canonically form a (weighted) Hecke Lie algebra.

(2) For a flat (weighted) E-module spectrum M, the canonical map

$$\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}(\pi_*(M)) \to \pi_*\left(\mathrm{Free}^{\mathscr{L}_E}(L_{K(h)}(M))\right)$$

induces an isomorphism on completions.

We pause to explain the meaning of the weighted variant of the theorem, which follows immediately from the argument of [Bra17, Theorem 4.4.4.] by formally recording weights. The $K(h)$-local Lie operad acts on weighted $K(h)$-local E-module spectra (i.e. elements of the functor category $\mathrm{Fun}(\mathbb{N}, \mathrm{Mod}_E^\wedge)$ with Day convolution) by placing the entire operad in weight 0.

At the level of homotopy groups, the resulting *weighted $K(h)$-local Lie algebras* are naturally equipped with the following asserted in Theorem 4.20 (1):

$$[-,-] : \pi_i(\mathfrak{g}(a)) \times \pi_j(\mathfrak{g}(b)) \to \pi_{i+j}(\mathfrak{g}(a+b))$$
$$(\mathcal{H}_u^{\mathrm{Lie}})_i^j(b) \times \pi_i(\mathfrak{g}(a)) \to \pi_j(\mathfrak{g}(b \cdot a))$$

This *weighted Hecke Lie algebra* structure is controlled by an extension of the monad $\mathbf{L}^{\mathcal{H}_u}$ to weighted E_*-modules. For this, we endow the free Hecke Lie algebra $\mathbf{L}^{\mathcal{H}_u}(M)$ on a weighted graded E_*-module M with a weight-grading. It has the property that if $\{m_s \in M_{i_s}(a_s)\}_{s=1}^k$ are homogeneous, w is a Lyndon word in k letters involving the i^{th} letter n_i times, and $\alpha \in (\mathcal{H}_u^{\mathrm{Lie}})^j_{\sum_s i_s}(b)$ is a Hecke operation of weight b, then $\alpha(w(m_1, \ldots, m_k))$ has weight $b \cdot (\sum_s m_s a_s)$.

The free $K(h)$-local Lie algebra on a finite and free E-module spectrum is usually neither finite nor free due to the presence of completed infinite direct sums. Instead, it is only *completed-free*, i.e. the $K(h)$-localisation of a free E-module spectrum. This is reflected by the algebraic completion on homotopy groups appearing in part (2) of Theorem 4.20.

The weighted context allows us to bypass this complication by restricting attention to the full subcategory $\mathrm{Fun}(\mathbb{N}, \mathrm{Mod}_E^\wedge)^{\mathrm{pff},>0}$ of graded E-module spectra which are pointwise finite free in each weight and *concentrated in positive weights*. Using the explicit description of free Hecke Lie algebras provided in [Bra17, Section 4.4.2], we see that the free Hecke Lie algebra on some $M \in \mathrm{Fun}(\mathbb{N}, \mathrm{Mod}_E^\wedge)^{\mathrm{pff},>0}$ is again finite and free in each weight. Hence, we can deduce the following simplification of Theorem 4.20:

Corollary 4.21. *The monad $\mathrm{Free}_E^{\mathscr{L}}$ preserves $\mathrm{Fun}(\mathbb{N}, \mathrm{Mod}_E^\wedge)^{\mathrm{pff},>0}$, and the canonical map*

$$\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}(\pi_*(M)) \to \pi_*\left(\mathrm{Free}^{\mathscr{L}_E}(L_{K(h)}(M))\right)$$

is an isomorphism for any $M \in \mathrm{Fun}(\mathbb{N}, \mathrm{Mod}_E^\wedge)^{\mathrm{pff},>0}$.

4.6. **Looping.** The ∞-category of weighted $K(h)$-local Lie algebras admits small limits and the forgetful functor creates them; in particular, we can form the loop object $\Omega\mathfrak{g} \simeq 0 \times_{\mathfrak{g}} 0$ on such a Lie algebra $\mathfrak{g}$, the underlying spectrum of which is simply the desuspension of the underlying spectrum of $\mathfrak{g}$.

Our next result describes the Hecke Lie structure on $\pi_*(\Omega^n\mathfrak{g})$ in terms of that on $\pi_*(\mathfrak{g})$.

Proposition 4.22 (Hecke operations on looped Lie algebras)**.** *For $\mathfrak{g}$ a weighted $K(h)$-local Lie algebra and $n > 0$, the Hecke Lie algebra $\pi_*(\Omega^n\mathfrak{g})$ has vanishing Lie bracket, and the Hecke module structure is given by the following commutative diagram:*

$$\begin{tikzcd}
(\mathcal{H}_u^{\mathrm{Lie}})_i^j(w) \times \pi_i(\Omega^n\mathfrak{g}(a)) \arrow[r] \arrow[d, "\mathrm{Susp}^n \times (\cong)"'] & \pi_j(\Omega^n\mathfrak{g}(w\cdot a)) \arrow[d, "\cong"] \\
(\mathcal{H}_u^{\mathrm{Lie}})_{i+n}^{j+n}(w) \times \pi_{i+n}(\mathfrak{g}(a)) \arrow[r] & \pi_{j+n}(\mathfrak{g}(w\cdot a)).
\end{tikzcd}$$

For $w = p^s$, the morphism $(\mathcal{H}_u^{\mathrm{Lie}})_i^j(p^s) \xrightarrow{\mathrm{Susp}^n} (\mathcal{H}_u^{\mathrm{Lie}})_{i+n}^{j+n}(p^s)$ is given by the morphism

$$\mathrm{Ext}^s_{\Gamma^i}(E_0, E_{-i+j+s}) \to \mathrm{Ext}^s_{\Gamma^{i+n}}(E_0, E_{-i+j+s})$$

induced by the suspension morphism $\Gamma^{i+n} \to \Gamma^i$ linking Rezk's Γ-rings.

Proof. Let $\alpha \in \pi_j(\mathrm{Free}^{\mathscr{L}_E}(\Sigma^i E))$ be a universal operation from degree i to degree j and suppose that $x \in \pi_i(\Omega^n\mathfrak{g})$ is represented by a map $\Sigma^i E \to \Omega^n\mathfrak{g}$.

The class $\alpha(x)$ is then represented by the following diagram:

$$\begin{tikzcd}
\Sigma^j E \arrow[r, "\alpha"] & \mathrm{Free}^{\mathscr{L}_E}(\Sigma^i E) \arrow[rr, "\overline{x}"] \arrow[d] & & \Omega^n(\mathfrak{g}) \\
& \mathrm{Free}^{\mathscr{L}_E}(\Omega^n(\mathfrak{g})) \arrow[rru] \arrow[rr, dashed] & & \Omega^n(\mathrm{Free}^{\mathscr{L}_E}(\mathfrak{g})) \arrow[u]
\end{tikzcd}$$

Here we have factored the structure map of the Lie algebra $\Omega^n(\mathfrak{g})$ through the canonical map $\mathrm{Free}^{\mathscr{L}_E}(\Omega^n(\mathfrak{g})) \to \Omega^n(\mathrm{Free}^{\mathscr{L}_E}(\mathfrak{g}))$ and the structure map of $\mathfrak{g}$; a more detailed construction of this map is given in the proof of Proposition 5.9.

The map $\alpha(x) : \Sigma^j E \to \Omega^n(\mathfrak{g})$ can therefore be represented by shifting the composite from top left to bottom right in the following diagram down by n:

$$\begin{tikzcd}
\Sigma^{j+n} E \arrow[r, "\Sigma^n(\alpha)"] \arrow[dr, dashed] & \Sigma^n \mathrm{Free}^{\mathscr{L}_E}(\Sigma^i E) \arrow[r] \arrow[d] & \Sigma^n \mathrm{Free}^{\mathscr{L}_E}(\Omega^n(\mathfrak{g})) \arrow[d] & \\
& \mathrm{Free}^{\mathscr{L}_E}(\Sigma^{i+n} E) \arrow[r] & \mathrm{Free}^{\mathscr{L}_E}(\mathfrak{g}) \arrow[r] & \mathfrak{g}
\end{tikzcd}$$

The first claim of the proposition now follows by observing that the dotted diagonal arrow picks out $\mathrm{Susp}^n(\alpha)$. The second claim is established in Theorem 4.2.19. of [Bra17]. $\square$

Motivated by Proposition 4.22, we introduce the following definition:

Definition 4.23 (Looping Hecke Modules)**.** Given a Hecke module M, we write ΩM for the Hecke module with $\mathrm{U}_{\mathrm{Mod}}^{\mathrm{Mod}^{\mathcal{H}_u}}(\Omega M) = \Sigma^{-1} M$ and Hecke operations defined by the formula of Proposition 4.22. This construction extends to an endofunctor of Hecke modules.

In order to understand this endofunctor, we will use that Hecke operations of weight p have a particularly simple description. By the Koszulness of the ring Γ^i [Rez12], we have

$$(\mathcal{H}_u^{\mathrm{Lie}})_i^{i-1}(p) = \mathrm{Ext}^1_{\Gamma^i}(E_0, E_0) \cong \Gamma^i(p)^\vee.$$

Combining this isomorphism with the E_0-linear dual of Proposition 4.1 and invoking Proposition 4.22, we deduce the following:

Proposition 4.24 (Suspending via the Euler class)**.** *There is a commutative diagram*

$$\begin{tikzcd}
\dots \arrow[r, "\cong"] & \Gamma^0(p)^\vee \arrow[r, hook, "(e\cdot -)"] \arrow[d, "\cong"'] & \Gamma^0(p)^\vee \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Gamma^0(p)^\vee \arrow[r, hook, "(e\cdot -)"] \arrow[d, "\cong"'] & \Gamma^0(p)^\vee \arrow[r, "\cong"] \arrow[d, "\cong"'] & \dots \\
\dots \arrow[r, "\cong"] & \Gamma^{-1}(p)^\vee \arrow[r, hook] \arrow[d, "\cong"'] & \Gamma^0(p)^\vee \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Gamma^1(p)^\vee \arrow[r, hook] \arrow[d, "\cong"'] & \Gamma^2(p)^\vee \arrow[r, "\cong"] \arrow[d, "\cong"'] & \dots \\
\dots \arrow[r, "\cong"] & (\mathcal{H}_u^{\mathrm{Lie}})_{-1}^{-2}(p) \arrow[r, hook] & (\mathcal{H}_u^{\mathrm{Lie}})_0^{-1}(p) \arrow[r, "\cong"] & (\mathcal{H}_u^{\mathrm{Lie}})_1^{0}(p) \arrow[r, hook] & (\mathcal{H}_u^{\mathrm{Lie}})_2^{1}(p) \arrow[r, "\cong"] & \dots
\end{tikzcd}$$

Corollary 4.25. *For $\mathfrak{g}$ a weighted $K(h)$-local Lie algebra and $n > 0$, the following diagram commutes:*

$$\begin{tikzcd}
 & \Gamma^0(p)^\vee \times \pi_0(\Omega^i(\mathfrak{g}(a))) \arrow[r] \arrow[d] \arrow[dl, "\left(e^{\lfloor \frac{i}{2} \rfloor}\cdot -\right)"'] & (\mathcal{H}_u^{\mathrm{Lie}})_0^{-1}(p) \times \pi_0(\Omega^i(\mathfrak{g}(a))) \arrow[r] \arrow[d, "\mathrm{Susp}\otimes(\cong)"'] & \pi_{-1}(\Omega^i(\mathfrak{g}(p\cdot a))) \arrow[d, "\cong"] \\
\Gamma^0(p)^\vee \times \pi_i(\mathfrak{g}(a)) \arrow[r, "(\mathrm{Thom})^\vee \times \mathrm{id}"'] & \Gamma^i(p)^\vee \times \pi_i(\mathfrak{g}(a)) \arrow[r] & (\mathcal{H}_u^{\mathrm{Lie}})_i^{i-1}(p) \times \pi_i(R(a)) \arrow[r] & \pi_{i-1}(\mathfrak{g}(p\cdot a))
\end{tikzcd}$$

4.7. **Delooping.** In good circumstances, maps between such "looped" Hecke modules can be "delooped":

Proposition 4.26 (Delooping maps of Hecke modules)**.** *Let $f : \Omega M_1 \to \Omega M_2$ be a map of Hecke modules. If M_1 and M_2 are both torsion-free, then $f = \Omega g$ for a unique map of Hecke modules $g : M_1 \to M_2$.*

In order to prove this result, we shall need to bound the torsion of the cokernel of suspensions. We begin by introducing the following notation:

Definition 4.27 (Torsion cokernel)**.** Let N be an integer. We say that a map of abelian groups $f : A \to B$ has *torsion cokernel of exponent dividing* N if, for any $b \in B$, the element $N \cdot b$ lies in the image of f.

Proposition 4.3 allows us to deduce the following (known) result:

Proposition 4.28 (Torsion cokernel for operations on $\mathbb{E}_\infty$-rings)**.** *The k-fold suspension homomorphism $\Gamma^i(p^a) \hookrightarrow \Gamma^{i-k}(p^a)$ has torsion cokernel of exponent dividing p^{ak}.*

Proof. If $w = p$, then Proposition 4.3 (or in fact also [Rez09, Proposition 10.6]) shows that the above suspension morphism has torsion cokernel of exponent dividing p^k. If $w = p^a$, we use that Γ^{i-k} is Koszul with respect to the "p-logarithmic" weight-grading (cf. [Rez12]) to write α as a composite of a weight p operations. The exponent therefore divides $(p^k)^a$. If w is not a power of p, there is nothing to prove. $\square$

Remark 4.29. The above exponent is not optimal. One can find a sharper bound by taking the parity of i into account.

On the Lie algebra side, we can deduce the following statement:

Proposition 4.30 (Torsion cokernel for operations on Lie algebras)**.** *The k-fold suspension homomorphism $(\mathcal{H}_u^{\mathrm{Lie}})_i^j(p^a) \to (\mathcal{H}_u^{\mathrm{Lie}})_{i+k}^{j+k}(p^a)$ has torsion cokernel of exponent dividing p^{a^2k}.*

Proof. The morphism in question given by the map $\mathrm{Ext}^a_{\Gamma^i}(E_0, E_{-i+j+a}) \to \mathrm{Ext}^a_{\Gamma^{i+k}}(E_0, E_{-i+j+a})$ induced by the suspension homomorphism $\Gamma^{i+k} \to \Gamma^i$. By Koszulness of the ring Γ^{i+k}, we can represent every class in $\mathrm{Ext}^a_{\Gamma^{i+k}}(E_0, E_{-i+j+a})$ by an E_0-linear map $f : \Gamma^{i+k}(p)^{\otimes a} \to E_{-i+j+a}$.

For any $x \in \Gamma^i(p)^{\otimes a}$, Proposition 4.28 implies that $(p^{ak})^a x$ lies in $\Gamma^{i+k}(p)^{\otimes a}$, where we have identified $\Gamma^{i+k}(p)^{\otimes a}$ with its image under the injection $(\Gamma^{i+k}(p))^{\otimes a} \hookrightarrow (\Gamma^i(p))^{\otimes a}$. We can therefore define a map $g : \Gamma^i(p)^{\otimes a} \to E_{-i+j+a}$ by $g(x) = f(p^{a^2k}x)$ The composite $\Gamma^{i+k}(p))^{\otimes a} \to \Gamma^i(p))^{\otimes a} \xrightarrow{g} E_{-i+j+a}$ agrees with $p^{a^2k}f$, and this clearly implies that the class $p^{a^2k}[f] \in \mathrm{Ext}^a_{\Gamma^{i+k}}(E_0, E_{-i+j+a})$ lies in the image of the map $(\mathcal{H}_u^{\mathrm{Lie}})_i^j(p^a) \to (\mathcal{H}_u^{\mathrm{Lie}})_{i+k}^{j+k}(p^a)$. □

We can finally establish the delooping claim made in the beginning of this section.

Proof of Proposition 4.26. For $i = 1, 2$, the E_*-modules ΩM_1 and ΩM_2 are simply given by applying the shift functor Σ^{-1} to M_1 and M_2, respectively. As a map of E_*-modules, define

$$g(x) := \Sigma(f(\Sigma^{-1}x)).$$

It remains to check that g is indeed a map of Hecke modules. For this, assume $\alpha \in (\mathcal{H}_u^{\mathrm{Lie}})_i^j(w)$ is a given Hecke operation and that $x \in (M_1)_i$ is an element. By Proposition 4.30, we can then choose an integer N such that $N\alpha = \mathrm{Susp}(\beta)$ for some $\beta \in (\mathcal{H}_u^{\mathrm{Lie}})_{i-1}^{j-1}(w)$.

Since f is a map of Hecke Lie algebras, we have

$$\begin{aligned} g((N\alpha)(x)) &= g(\mathrm{Susp}(\beta)(x)) = \Sigma(f(\Sigma^{-1}\,\mathrm{Susp}(\beta)(x))) = \Sigma(f(\beta(\Sigma^{-1}(x)))) \\ &= \Sigma(\beta(f(\Sigma^{-1}(x)))) = \Sigma(\beta(\Sigma^{-1}g(x))) = \mathrm{Susp}(\beta)g(x) = (N\alpha)(g(x)) \end{aligned}$$

Hence $N(g(\alpha(x)) - \alpha(g(x))) = 0$, which implies $g(\alpha(x)) = \alpha(g(x))$ by torsion-freeness. □

5. The E-theory of labeled configuration spaces

In this section, we will construct spectral sequences converging to the Morava E-theory of (possibly labelled) configuration spaces of manifolds and examine some of their basic properties. Throughout the remainder of the paper, all manifolds are assumed to be of finite type.

5.1. Stable formulas for configuration spaces.

The homotopy types of configuration spaces are subtle invariants of the background manifold. For example, according to a theorem of Longoni–Salvatore [LS05], these homotopy types distinguish certain pairs of lens spaces which are homotopy equivalent but not homeomorphic. In particular, configuration spaces are not homotopy invariants even of compact manifolds of equal dimension.

This subtlety is an unstable phenomenon. In fact, there is a relatively simple formula expressing the stable homotopy types of the configuration spaces of M in terms of the pointed homotopy type of the one-point compactification M^+. This formula is valid for configuration spaces labeled by a spectrum X, which are defined as

$$B_k(M;X) := \Sigma^\infty_+ \operatorname{Conf}_k(M) \otimes_{\Sigma_k} X^{\otimes k}.$$

We naturally consider $B_k(M;X)$ as a weighted spectrum placed in weight k.

Writing $\mathscr{L}$ for the free spectral lie algebra monad acting on spectra or weighted spectra and $\mathrm{Free}^{\mathscr{L}}$ for the corresponding free functor, we recall the following equivalence established by the third author (cf. [Knu18, Section 3.4.]:

Theorem 5.1 (Knudsen)**.** *Let M be a framed n-manifold and X a spectrum. There is a natural equivalence of weighted spectra*

$$\bigoplus_{k\geq 1} B_k(M;X) \simeq \Sigma \left|\mathrm{Bar}\left(\mathrm{id}, \mathscr{L}, \mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M^+}\right)\right|.$$

Here the lefthand side is weighted by the index k and the righthand side by operadic arity. The superscript indicates the cotensor in the ∞-category of spectral Lie algebras.

Example 5.2. The wedge of labeled configuration spaces appearing in Theorem 5.1 has several familiar interpretations.

(1) In the case $M = \mathbb{R}^n$, this wedge forms the free $\mathbb{E}_n$-algebra on the spectrum X. For general M, it is the factorisation homology (alias topological chiral homology) of M with coefficients in this algebra [Sal01, Lur03, AF15].
(2) In the case $X = \Sigma^\infty Y$ for Y a connected pointed space, this same wedge is equivalent to the suspension spectrum of the space $\mathrm{Map}_c(M, \Sigma^n Y)$ of compactly supported maps [McD75, Böd87]. In particular, setting $M = \mathbb{R}^n$ yields a formula for $\Sigma^\infty\Omega^n\Sigma^n Y$.

Remark 5.3. The above discussion can be made valid for non-parallelisable and non-smooth manifolds M at the cost of keeping track of actions of tangential structure groups—see [Knu18]. We restrict ourselves to the framed setting in the present work for the sake of simplicity.

Remark 5.4. The proof of Theorem 5.1 given in [Knu18] uses factorization homology and Koszul duality. The authors have subsequently been informed by Arone that an alternative proof using Goodwillie calculus is available in the case of a suspension spectrum.

Via the canonical filtration of the bar construction, Theorem 5.1 supplies spectral sequences converging to $E_*B_k(M;X)$ for each k, M, X, and homology theory E. In characteristic zero, these spectral sequences all collapse, and the E^2 page may be identified with the classical Lie algebra homology of a certain graded Lie algebra, which is eminently computable [Knu17, DCK17]. We show that the computational utility of Theorem 5.1 is much wider in scope, and in particular applies to Morava E-theory.

5.2. **The E^2-page of the bar spectral sequence.** For the remainder of this paper, we take E to be a Lubin–Tate theory of height h as in Section 4. We now state our first main result.

Theorem 5.5 (Hecke spectral sequence)**.** *Let M be a framed n-manifold and X a spectrum, and suppose that the weighted Hecke Lie algebra*

$$\mathfrak{g}(M;X) := E_*^\wedge\left(\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M^+}\right)$$

is a finite and free E_*-module in each weight. There is a convergent weighted spectral sequence

$$\mathrm{E}^2_{s,t} \cong H_{s+1}(\mathrm{CE}_{\mathcal{H}_u}(\mathfrak{g}(M;X)))_{t-1} \implies \bigoplus_{k\geq 0} E^\wedge_{s+t}(B_k(M;X)).$$

Notation 5.6. We denote Lubin-Tate theory by an italic E, whereas the n^{th} page of a (homological) spectral sequence is denoted by E^n.

Proof of Theorem 5.5. We apply the functor $L_{K(h)}(E\otimes -)$ weightwise to the equivalence of weighted spectra asserted in Theorem 5.1 and obtain an equivalence of $K(h)$-local E-module spectra

$$\left|\Sigma\mathrm{Bar}_\bullet\left(\mathrm{id},\,\mathscr{L}_E,\,L_{K(h)}\left(E\otimes\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M^+}\right)\right)(k)\right| \simeq L_{K(h)}(E\otimes\Sigma^\infty_+B_k(M;X)).$$

for every $k\geq 1$. Taking the skeletal filtration, we obtain a spectral sequence

$$\mathrm{E}^2_{s,t}(k) = H_s\left(\pi_t\left(\Sigma\mathrm{Bar}_\bullet\left(\mathrm{id},\,\mathscr{L}_E,\,L_{K(h)}\left(E\otimes\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M^+}\right)\right)(k)\right)\right) \Rightarrow E^\wedge_{s+t}(B_k(M;X)),$$

which converges, as does the homotopy spectral sequence for any bounded below filtered spectrum (cf. e.g. [Lur03, Prop. 1.2.2.14] for this classical fact).

Collecting weights, adding a copy of E_* in weight 0 and bidegree $(s,t)=(-1,1)$, and applying Proposition 4.21 repeatedly yields the spectral sequence of the theorem. We conclude by identifying the E^2 page as

$$\begin{aligned}
\mathrm{E}^2_{s,t} &= H_{s+1}\left(\mathrm{Bar}_\bullet\left(\mathrm{id},\,\mathbf{L}^{\mathcal{H}_u},\,\mathfrak{g}(M;X)\right)[1]\oplus E_*\right)_{t-1}\\
&\cong H_{s+1}\left(\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\,\mathrm{Bar}_\bullet\left(\mathbf{L}^{\mathcal{H}_u},\ ,\mathbf{L}^{\mathcal{H}_u},\,\mathfrak{g}(M;X)\right)[1]\oplus E_*\right)_{t-1}\\
&\cong H_{s+1}\left(\mathbb{L}\,\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\left(\mathfrak{g}(M;X)\right)[1]\oplus E_*\right)_{t-1}\\
&\cong H^{\mathrm{Lie}^{\mathcal{H}_u}}_{s+1}\left(\mathfrak{g}(M;X)\right)_{t-1}\\
&\cong H_{s+1}(\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(M;X)\right))_{t-1},
\end{aligned}$$

where the second isomorphism uses the isomorphism $\mathrm{Q}^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\,\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}\cong\mathrm{id}$, the third uses Lemma 4.13, the fourth is Definition 4.14, and the last uses Theorem 4.18. $\square$

We can also set up a cohomological version of this spectral sequence:

Theorem 5.7 (Cohomological Hecke spectral sequence)**.** *Let M be a framed n-manifold of finite type and X a spectrum, and suppose that the weighted Hecke Lie algebra*

$$\mathfrak{g}(M;X) := E_*^\wedge\left(\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M^+}\right)$$

is a finite and free E_*-module in each weight. There is a weighted spectral sequence

$$\mathrm{E}_2^{s,t} \cong H^{s+1}\left(\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(M;X)\right)^\vee\right)_{t+1} \implies \bigoplus_k E^{t-s}(B_k(M;X)).$$

which converges completely (cf. e.g. [BK72] *or* [GJ09, Definition 6.18.]*)*

Proof. We apply the functor $E^{(-)} = \mathrm{Map}_{\mathrm{Sp}}(-,E)$ weightwise to the equivalence asserted in Theorem 5.1 and obtain an equivalence of $K(h)$-local E-module spectra

$$\mathrm{Tot}\left(E^{\Sigma\mathrm{Bar}_\bullet\left(\mathrm{id},\mathscr{L}_E,L_{K(h)}\left(E\otimes\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M^+}\right)\right)(k)}\right) \simeq E^{B_k(M;X))_+}.$$

for every $k \geq 0$. The coskeletal tower of this cosimplicial spectrum gives rise to a spectral sequence

$$\mathrm{E}_2^{s,t} = H^s\left(\pi_t\left(\Sigma\mathrm{Bar}_\bullet\left(\mathrm{id},\,\mathscr{L}_E,\,L_{K(h)}\left(E\otimes\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M^+}\right)\right)(k)\right)^\vee\right) \Rightarrow E^{t-s}(B_k(M;X))$$

By repeated application of Proposition 4.21, we can rewrite the E^2-page as

$$\mathrm{E}_2^{s,t}(k) = H^s\left(\mathrm{Bar}_\bullet\left(\mathrm{id},\,\mathbf{L}^{\mathcal{H}_u},\,\mathfrak{g}(M;X)\right)^\vee\right)_{t+1}(k).$$

Since $\mathfrak{g}(M;X)$ is concentrated in weights ≥ 1, we know that for $n > k$, every element in $(\mathbf{L}^{\mathcal{H}_u})^{\circ n}(\mathfrak{g}(M;X))(k)$ is the degeneracy of an element in $(\mathbf{L}^{\mathcal{H}_u})^{\circ k}(\mathfrak{g}(M;X))(k)$. The cosimplicial E_*-module $\mathrm{Bar}_\bullet\left(\mathrm{id},\,\mathbf{L}^{\mathcal{H}_u},\,\mathfrak{g}(M;X)\right)^\vee(k)$ is therefore k-coskeletal, and hence $E_s^{s,t}(k)$ vanishes for $s > k$. By [GJ09, Corollary 6.20.], this implies that the spectral sequence converges completely.

The description of the E_2 page then follows as in Theorem 5.5. □

Remark 5.8. We can compute $H^*\left(\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(M;X)\right)^\vee\right)$ from $H_*\left(\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(M;X)\right)\right)$ by a universal coefficient spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Ext}^p\left(H_*\left(\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(M;X)\right)\right),E_*\right)_q \Rightarrow H^{q-p}\left(\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(M;X)\right)^\vee\right).$$

5.3. **The Hecke Lie algebras of interest.** In the remainder of the paper, we will use Theorem 5.5 as a practical tool for explicit calculations. For this, it is necessary to understand $\mathfrak{g}(M;X)$ as a Hecke Lie algebra. There are three parts to this problem:

(1) compute $\mathfrak{g}(M;X)$ as an E_*-module;
(2) calculate the Lie bracket;
(3) understand the Hecke operations.

The answers to (1) and (2) can be obtained from knowledge of $\widetilde{E}^*(M^+)$ as a nonunital E^*-algebra (with the cup product) and $E_*(X)$ as an E_*-module.

Proposition 5.9 (Lie bracket from cup product)**.** *If $\widetilde{E}^*(M^+)$ and $E_*(X)$ are both finite free E_*-modules, then*

$$\mathrm{U}_{\mathrm{Lie}_{E_*}}^{\mathrm{Lie}_{\mathcal{H}_u}}\left(\mathfrak{g}(M;X)\right) \cong \widetilde{E}^*(M^+)\otimes_{E_*}\mathbf{L}^{\mathcal{H}_u}(E_*(\Sigma^{n-1}X))$$

with Lie bracket given by the formula $[\alpha\otimes x,\beta\otimes y] = (-1)^{|x||\beta|}\alpha\beta\otimes[x,y]$.

Proof. We begin by briefly recalling several well-known facts about algebras over monads in our context; a detailed treatment can be found in [Lur03, Section 3.4.3]. Given a spectral Lie algebra $\mathfrak{g}$ and a space Z, the Z-shaped diagram with constant value $\mathfrak{g}$ in spectral Lie algebras admits a limit. Moreover, the underlying spectrum of this limit is given by the mapping spectrum $\mathfrak{g}^{Z_+} = \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ Z, \mathfrak{g})$. Since the limit (in Lie algebras) maps to all its factors via Lie algebra maps, we obtain a $Z^{\triangleleft}$-shaped diagram of arrows informally given by

$$\begin{tikzcd}
\mathscr{L}(\mathfrak{g}^{Z_+}) \arrow[r] \arrow[d, dashed] & \mathfrak{g}^{Z_+} \arrow[d, dashed] \\
\mathscr{L}(\mathfrak{g}) \arrow[r] & \mathfrak{g}
\end{tikzcd}$$

Passing to the limit therefore gives rise to a factorisation of the structure map of $\mathfrak{g}^{Z_+}$ as

$$\mathscr{L}(\mathfrak{g}^{Z_+}) \to (\mathscr{L}(\mathfrak{g}))^{Z_+} \to \mathfrak{g}^{Z_+}, \tag{2}$$

where the first map is the natural one defined for every limit, and the second map is obtained from the structure map of $\mathfrak{g}$. If instead we consider a pointed space $* \to Z$, then we define $\mathfrak{g}^Z := \mathfrak{g}^{Z_+} \times_{\mathfrak{g}^{S^0}} \mathfrak{g}^* = \mathrm{fib}(\mathfrak{g}^{Z_+} \to \mathfrak{g})$. We can then use the previous observation to see that the structure map of $\mathfrak{g}^Z$ is given by $\mathscr{L}(\mathfrak{g}^Z) \to (\mathscr{L}(\mathfrak{g}))^Z \to \mathfrak{g}^Z$, where the first map is the natural one defined for every pointed limit, and the second again comes from the structure map of $\mathfrak{g}$.

We now take $\mathfrak{g} = \mathscr{L}(\Sigma^{n-1}X)$ and $Z = M^+$, the one-point-compactification of M. The first claim follows from Proposition 4.21 in light of our assumptions, since limits in spectral Lie algebras are computed in spectra and $\mathscr{L}(\Sigma^{n-1}X)^{M^+} \simeq \mathbb{D}(M^+) \wedge \mathscr{L}(\Sigma^{n-1}X)$ because $\Sigma^\infty M^+$ is dualisable. For the second claim, we need to examine the structure map of $\mathscr{L}(\Sigma^{n-1}X)^{M^+}$ in weight 2. Observe that if Z is a finite pointed space and $\mathfrak{g}$ is a spectrum, then the canonical map of naive Σ_2-spectra $(\mathfrak{g}^Z)^{\wedge 2} \to (\mathfrak{g}^{\wedge 2})^Z$ is given by $(\mathfrak{g}^Z)^{\wedge 2} = (\mathfrak{g}\wedge\mathfrak{g})^{Z\wedge Z} \xrightarrow{\mathfrak{g}^{\delta^*}} (\mathfrak{g}\wedge\mathfrak{g})^Z = (\mathfrak{g}^{\wedge 2})^Z$, where δ denotes the diagonal map on Z. If Z is finite, then $\Sigma^{-1}(-)_{h\Sigma_2} : \mathrm{Sp}^{\Sigma_2} \to \mathrm{Sp}$ is exact, and we deduce that the canonical map $\Sigma^{-1}(\mathfrak{g}^Z)^{\wedge 2}_{h\Sigma_2} \to (\Sigma^{-1}\mathfrak{g}^{\wedge 2}_{h\Sigma_2})^Z$ is given by the composite

$$\Sigma^{-1}(\mathfrak{g}^Z)^{\wedge 2}_{h\Sigma_2} = \Sigma^{-1}\left((\mathfrak{g}\wedge\mathfrak{g})^{Z\wedge Z}\right)_{h\Sigma_2} \xrightarrow{\Sigma^{-1}\mathfrak{g}^{\delta^*}_{h\Sigma_2}} \Sigma^{-1}\left((\mathfrak{g}\wedge\mathfrak{g})^{Z}\right)_{h\Sigma_2} = (\Sigma^{-1}\mathfrak{g}^{\wedge 2}_{h\Sigma_2})^Z.$$

For $\mathfrak{g} = \mathscr{L}(\Sigma^{n-1}X)$ and $Z = M^+$, this is the left part of structure map (2) of $\mathscr{L}(\Sigma^{n-1}X)^{M^+}$ in weight 2. Using again that M^+ is dualisable, we can therefore rewrite the structure map of $\mathscr{L}(\Sigma^{n-1}X)^{M^+}$ in weight 2 as

$$\Sigma^{-1}\mathbb{D}(M^+)^{\wedge 2}_{h\Sigma_2} \wedge \mathscr{L}(\Sigma^{n-1}X)^{\wedge 2}_{h\Sigma_2} \xrightarrow{\Sigma^{-1}\mathbb{D}(\delta^*)\wedge\mu} \Sigma^{-1}\mathbb{D}(M^+) \wedge \mathscr{L}(\Sigma^{n-1}X),$$

where μ_2 denotes the structure map of $\mathscr{L}(\Sigma^{n-1}X)$ in weight 2. The second claim now follows by a straightforward diagram chase (keeping careful track of signs). □

We can therefore read off the following statement from Corollary 4.21 above:

Corollary 5.10. *If $\widetilde{E}^*(M^+)$ and $E_*(X)$ are both E_*-free and finite, then so is $\mathfrak{g}(M;X)$.*

In particular, Theorem 5.5 applies under the assumptions made in Proposition 5.9.

In the examples of greatest interest, X is the suspension spectrum of a sphere, so the second condition is satisfied, and it is often the case that the first is as well. In Sections 6 and 7 below, we study two such examples, namely $M = \mathbb{R}^n$ and M a punctured orientable surface $\mathcal{S}_{g,1}$. In the former example, $\widetilde{E}^*((\mathbb{R}^n)^+) \cong \widetilde{E}^*(S^n)$ is free of rank 1, and, in the latter example, the standard CW structure splits after suspensions, so that $\widetilde{E}^*\left(\mathcal{S}_{g,1}^+\right) = \widetilde{E}^*\left(\mathcal{S}_{g,1}\right)$ is free of rank $2g+2$.

The computation of the Hecke operations on $\mathfrak{g}$ is a more delicate task, but may also be accomplished in many situations of interest.

Lemma 5.11 (Stably split manifolds)**.** *Let M, X be as in Theorem 5.5. Assume that M^+ admits the structure of a CW complex whose skeletal filtration splits stably. There is an isomorphism*

$$\mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}}\left(\mathfrak{g}(M;X)\right) \cong \bigoplus_{r\geq 0}\bigoplus_{A_r} \Omega^r\, \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}}\, \mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}\left(\Sigma^{n-1}E_*(X)\right),$$

where A_r denotes the set of r-cells of M^+.

Proof. We proceed inductively along the skeletal filtration of M^+, using Proposition 4.22 for the base case. For the inductive step, our assumptions and Proposition 4.22 imply that the sequence of Hecke Lie modules

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{A_r} \Omega^r\, \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}}\, \mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}}\left(E_*(X)\right) \arrow[d] & & \\ & \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}}\left(E_*\left(\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M_r^+}\right)\right) \arrow[r] & \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{\mathcal{H}_u}}\left(E_*\left(\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}X)^{M_{r-1}^+}\right)\right) \arrow[r] & 0 \end{tikzcd}$$

is exact. Since M is of finite type, and since the skeletal filtration of M^+ splits stably, the collapse map $M_r^+ \to \bigvee_{A_r} S^r$ splits after a finite suspension. Thus, the lefthand map in the above short exact sequence splits after applying a finite iteration of Ω and hence splits outright by Proposition 4.26. Applying the inductive hypothesis completes the proof. $\square$

5.4. **Inverting the implicit prime.** We can rationalise the E-theory of a given a space Y in two ways: first, we can tensor the E-cohomology groups with the rationals to obtain $p^{-1}E^*(Y)$; second, we can consider its cohomology $(p^{-1}E)^*(Y)$ with respect to the rationalisation of E. In general, these procedures will lead to different results. For example, the p-adic K-theory of $B\Sigma_p$ is free on two generators, whereas $H^*(B\Sigma_p, \mathbb{Q}_p[\beta^{\pm 1}]) \cong \mathbb{Q}_p[\beta^{\pm 1}]$ is generated by one element. However, they agree, for example, when Y is a finite CW complex (as $p^{-1}E_*$ is flat over E_*).

In this section, we will examine the effect of these two procedures on the cohomological Hecke spectral sequence from Theorem 5.7. Working with cohomology instead of homology is simpler in this context, as mapping spectra to E are automatically $K(h)$-local, whereas smash products with E are not. To avoid any confusion concerning the letter "E", recall Notation 5.6.

Assume that M is a stably split, framed n-manifold such that $\widetilde{E}^*(M^+)$ is a finite free E_*-module, and let $X = S^a$. The spectral sequence $\{\mathrm{E}_r(M;S^a)\}$ in Theorem 5.7 has signature

$$\mathrm{E}_2^{s,t} = H^s\left(\mathrm{Bar}_\bullet\left(\mathrm{id}, \mathbf{L}^{\mathcal{H}_u}, \mathfrak{g}(M;S^a)\right)^\vee\right)_{t+1} \Longrightarrow \bigoplus_k E^{t-s}(B_k(M;S^a)) \tag{3}$$

Let $\{p^{-1}\mathrm{E}_r(M;S^a)\}$ be the spectral sequence obtained from $\{\mathrm{E}_r(M;S^a)\}$ by inverting the prime p. The main purpose of this section is to establish the following helpful computational tool, which will play a role in Section 7:

Theorem 5.12. *Under the above assumptions, $\{p^{-1}\mathrm{E}_r(M;S^a)\}$ collapses at the E_2-page. In other words, every non-trivial differential d_r in $\{\mathrm{E}_r(M;S^a)\}$ with $r\geq 2$ has p-power torsion target.*

To prove this theorem, we will consider a variant of (3) for the cohomology theory $(p^{-1}E)^*(-)$. Indeed, write $\mathbf{L}^{p^{-1}E_*}$ for the monad on $\mathrm{Mod}^{\mathbb{N}}_{p^{-1}E_*}$ parametrising Lie algebras in the sense of Definition 3.1. Since $p^{-1}E \simeq p^{-1}(\mathrm{W}(k)[[u_1,\dots,u_{n-1}]])[u_n^{\pm 1}]$ is a form of rational cohomology, a much easier variant of [Bra17, Theorem 4.4.4] asserts the existence of a natural isomorphism

$$\mathbf{L}^{p^{-1}E_*}(\pi_*(Y)) \to \pi_*\left(\mathrm{Free}^{\mathscr{L}_{p^{-1}E}}\, Y\right)$$

for any $p^{-1}E$-module spectrum Y. The same argument as in Theorem 5.7 then shows that there is a completely convergent weighted spectral sequence $\widehat{\mathrm{E}} = \{\widehat{\mathrm{E}}_r(M;S^a)\}$ with signature

$$\widehat{\mathrm{E}}_2^{s,t} = H^s\left(\mathrm{Bar}_\bullet(\mathrm{id},\, \mathbf{L}^{p^{-1}E_*},\, \mathfrak{h}(M;S^a)_{p^{-1}})^\vee\right)_{t+1} \Longrightarrow \bigoplus_k (p^{-1}E)^{t-s}(B_k(M;S^a))$$

Here $\mathfrak{h}(M;S^a)_{p^{-1}}$ denotes the Lie algebra $\left(p^{-1}E\right)_*(\mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}S^a)^{M^+})$. A rational variant of the argument for Proposition 5.9 allows us to identify this Lie algebra as

$$\mathfrak{h}(M;S^a)_{p^{-1}} \cong \widetilde{(p^{-1}E)}^*(M^+) \underset{p^{-1}E_*}{\otimes} \mathbf{L}^{p^{-1}E_*}\left((p^{-1}E)_*(S^a)\right),$$

where the Lie bracket on the right hand side is again defined in terms of the cup product. Since M^+ is a finite CW complex, the canonical map $p^{-1}(\widetilde{E}^*(M^+)\otimes_{E_*}\mathbf{L}^{E_*}(E_*(S^a))) \to \mathfrak{h}(M;S^a)_{p^{-1}}$ is an isomorphism. Hence, we have identified an integral form $\mathfrak{h}(M;S^a) \in \mathrm{Lie}_{E_*}$ of $\mathfrak{h}(M;S^a)_{p^{-1}}$.

Sending the simplicial weighted spectrum $\mathrm{Bar}_\bullet(\mathrm{id}, \mathscr{L}, \mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}S^a)^{M^+}))$ into the map $E \to p^{-1}E$, we obtain a canonical map of spectral sequences

$$\phi_r : p^{-1}\mathrm{E}_r(M;S^a) \to \widehat{\mathrm{E}}_r(M;S^a).$$

A key step in the proof of Theorem 5.12 will be to show that these two rationalisations agree:

Proposition 5.13. *Under the above assumptions, the map ϕ_r is an isomorphism for all $r \geq 2$.*

Proof. In this proof, we will often omit forgetful functors from the notation to increase readability. We first need to explicitly identify the map ϕ_1 on E_1-pages. To this end, consider the following composite of maps of cosimplicial modules:

$$\begin{aligned} p^{-1}\mathrm{Bar}_\bullet\left(\mathrm{id},\, \mathbf{L}^{\mathcal{H}_u},\, \mathfrak{g}(M;S^a)\right)^\vee &\xrightarrow{\phi_1} (\mathrm{Bar}_\bullet(\mathrm{id},\, \mathbf{L}^{p^{-1}E_*},\, \mathfrak{h}(M;S^a)_{p^{-1}}))^\vee \\ &\xrightarrow{\cong} \left(p^{-1}\mathrm{Bar}_\bullet(\mathrm{id},\, \mathbf{L}^{E_*},\, \mathfrak{h}(M;S^a))\right)^\vee. \end{aligned}$$

The second map identifies the Bar construction of $\mathfrak{h}(M;S^a)_{p^{-1}}$ with the rationalised Bar construction of its integral form $\mathfrak{h}(M;S^a)$.

Unravelling the computation of the E-cohomology of free spectral Lie algebras, we see that the map ϕ_1 kills all Hecke operations of higher weight. We deduce that the above composite is obtained by applying the functor $(p^{-1}(-))^\vee$ to the following natural inclusion:

$$\mathrm{Bar}_\bullet(\mathrm{id},\, \mathbf{L}^{E_*},\, \mathfrak{h}(M;S^a)) \to \mathrm{Bar}_\bullet\left(\mathrm{id},\, \mathbf{L}^{\mathcal{H}_u},\, \mathfrak{g}(M;S^a)\right) \tag{4}$$

To prove Proposition 5.13, it suffices to show that (4) is a weak equivalence after inverting p. The map (4) is obtained by applying the functor $Q^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{E_*}}$ from Section 4.3 to the natural inclusion (5)

$$\mathfrak{h}(M;S^a) \xrightarrow{\simeq} \mathrm{Bar}_\bullet(\mathrm{Free}^{\mathrm{Lie}_{E_*}}_{\mathrm{Mod}_{E_*}},\, \mathbf{L}^{E_*},\, \mathfrak{h}(M;S^a)) \to Q^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}}\mathrm{Bar}_\bullet\left(\mathrm{Free}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}_{E_*}},\, \mathbf{L}^{\mathcal{H}_u},\, \mathfrak{g}(M;S^a)\right)$$

Since $p^{-1}(Q^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{E_*}}(\mathfrak{g})) \cong Q^{\mathrm{Mod}_{p^{-1}E_*}}_{\mathrm{Lie}_{p^{-1}E_*}}(p^{-1}\mathfrak{g})$ (as the corresponding square of right adjoints evidently commutes), it suffices to check that (5) is a weak equivalence after inverting p. It suffices to verify this on modules, and since $U^{\mathrm{Mod}_{E_*}}_{\mathrm{Lie}_{E_*}} \circ Q^{\mathrm{Lie}_{E_*}}_{\mathrm{Lie}_{\mathcal{H}_u}} \cong Q^{\mathrm{Mod}_{E_*}}_{\mathrm{Mod}_{\mathcal{H}_u}} \circ U^{\mathrm{Mod}_{\mathcal{H}_u}}_{\mathrm{Lie}_{\mathcal{H}_u}}$, proceeding as in the proof of Proposition 4.19 allows us to further reduce to the claim that the following map is a weak equivalence after inverting p:

$$\mathfrak{h}(M; S^a) \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathbf{A}^{\mathcal{H}_u}, \mathrm{U}^{\mathrm{Lie}_{\mathcal{H}_u}}_{\mathrm{Mod}^{\mathcal{H}_u}}(\mathfrak{g}(M; S^a))). \tag{6}$$

Since M is stably split, we may by Lemma 5.11 assume without restriction that $M = \mathbb{R}^n$. In this case, the above map of modules is given by

$$\Omega^n \mathbf{L}^{E_*}(x_{n+a-1}) \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathbf{A}^{\mathcal{H}_u}, \Omega^n \mathbf{L}^{\mathcal{H}_u}(x_{n+a-1})), \tag{7}$$

On the left, $\Omega^n \mathbf{L}^{E_*}(x_{n+a-1})$ is simply the $(-n)^{th}$ shift of $\mathbf{L}^{E_*}(x_{n+a-1})$. On the right, we have used the looping functor Ω^n on Hecke modules from Definition 4.23. We now observe a commuting diagram

$$\begin{tikzcd}
\Omega^n \mathbf{L}^{E_*}(x_{n+a-1}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Bar}_\bullet(\mathrm{id}, \mathbf{A}^{\mathcal{H}_u}, \Omega^n \mathbf{L}^{\mathcal{H}_u}(x_{n+a-1})) \arrow[r] \arrow[d, "\mathrm{Bar}(\mathrm{id}, \mathrm{Susp}^a, -)"'] & \Omega^n \mathbf{L}^{\mathcal{H}_u}(x_{n+a-1}) \arrow[d, "\cong"'] \\
\mathbf{L}^{E_*}(x_{n+a-1})[-n] \arrow[r] & \mathrm{Bar}_\bullet(\mathrm{id}, \mathbf{A}^{\mathcal{H}_u}, \mathbf{L}^{\mathcal{H}_u}(x_{n+a-1}))[-n] \arrow[r] & \mathbf{L}^{\mathcal{H}_u}(x_{n+a-1})[-n]
\end{tikzcd}$$

The middle map sends a typical element $[\alpha_1|\ldots|\alpha_r|\ y\ ]$ to $[\mathrm{Susp}^a(\alpha_1)|\ldots|\mathrm{Susp}^a(\alpha_r)|\ y\ ]$, and the lower composite is the identity.

Since the suspension morphisms $(\mathcal{H}_u^{\mathrm{Lie}})_i^j(p^s) \xrightarrow{\mathrm{Susp}^a} (\mathcal{H}_u^{\mathrm{Lie}})_{i+a}^{j+a}(p^s)$ are rational isomorphisms for all i, j, it suffices to prove that the map

$$\mathbf{L}^{E_*}(x_{n+a-1}) \to \mathrm{Bar}_\bullet(\mathrm{id}, \mathbf{A}^{\mathcal{H}_u}, \mathbf{L}^{\mathcal{H}_u}(x_{n+a-1}))$$

is a rational equivalence, which holds as $\mathbf{L}^{\mathcal{H}_u}(x_{n+a-1}) \cong \mathbf{A}^{\mathcal{H}_u} \circ \mathbf{L}^{E_*}(x_{n+a-1})$. □

In order to conclude Theorem 5.12, we require one further result.

Lemma 5.14. *Under the above assumptions, $\{\widehat{\mathrm{E}}_r(M; S^a)\}$ degenerates at the $\widehat{\mathrm{E}}_2$-page.*

Proof. Write $\{\overline{\mathrm{E}}_r(M; S^a)\}$ for the spectral sequence obtained by mapping the weighted simplicial spectrum $\mathrm{Bar}_\bullet(\mathrm{id}, \mathscr{L}, \mathrm{Free}^{\mathscr{L}}(\Sigma^{n-1}S^a)^{M^+}))$ into the rational Eilenberg–MacLane spectrum of $\mathbb{Q}$. There is an evident map of spectral sequences

$$(p^{-1}E_*) \otimes_{\mathbb{Q}} \overline{\mathrm{E}}_r(M; S^a) \to \widehat{\mathrm{E}}_r(M; S^a),$$

and this map is an isomorphism as $p^{-1}E$ is a rational spectrum.

It the refore suffices to prove that the $\mathbb{Q}$-based spectral sequence $\overline{\mathrm{E}}_r(M; S^a)$ degenerates. This follows from the main result of [Knu17], which shows that the rational cohomology of the weighted spectrum $\bigoplus_k B_k(M; S^a)$ agrees with the $\overline{\mathrm{E}}_2$-page of this spectral sequence. Thus, no further differentials can occur. □

Remark 5.15. For general X, higher differentials do occur. These differentials are "tensored up" from differentials that occur rationally, which are related to Massey products in $\widetilde{H}^*(M^+; \mathbb{Q})$. In particular, the conclusion of Lemma 5.14 holds for general X under the assumption that M^+ is rationally formal.

6. Configurations of p points in $\mathbb{R}^n$

Given a Euclidean space $\mathbb{R}^n$ and an integer k, we consider the following spectrum

$$\Sigma^\infty_+ \mathrm{Conf}_p(\mathbb{R}^n) \otimes_{h\Sigma_p} (S^k)^{\otimes p},$$

which is the weight p component in the free $\mathbb{E}_n$-algebra $\mathbb{E}_n(S^k)$ on a single generator in degree k. If k is nonnegative, then Snaith's theorem shows that $\mathbb{E}_n(S^k)$ is equivalent to $\Sigma^\infty_+\Omega^n S^{n+k}$. In this section, we will use the methods introduced above to compute the E-theory of these spaces for all heights h and all dimensions n.

6.1. **Warmup: The K-theory of p points in $\mathbb{R}^n$.** We shall begin at height 1, where calculations are notationally simpler, geometrically most significant, and an illustrative template for the general height calculations in Section 6.3 below.

We fix a specific height one Morava E-theory K with $E_* = K_* = \mathbb{Z}_p[u^{\pm}]$, namely p-completed complex K-theory. In this section, we will prove:

Theorem 6.1 (K-theory of configurations in Euclidean space)**.** *Given $n \geq 1$ and $k \in \mathbb{Z}$, there are isomorphisms of K_*-modules*

$$K^\wedge_*\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{h\Sigma_p} (S^k)^{\otimes p}\right) \;\cong\; \begin{cases} \Sigma^{kp}K_* \oplus \Sigma^{pk+n-1}K_* \oplus \Sigma^{k-1}K_*/p^{\frac{n}{2}-1} & \textit{for } n \textit{ even, } k \textit{ even} \\ \Sigma^{k-1}K_*/p^{\frac{n}{2}} & \textit{for } n \textit{ even, } k \textit{ odd} \\ \Sigma^{kp}K_* \oplus \Sigma^{k-1}K_*/p^{\frac{n-1}{2}} & \textit{for } n \textit{ odd, } k \textit{ even} \\ \Sigma^{k+(2k+n-1)(\frac{p-1}{2})}K_* \oplus \Sigma^{k-1}K_*/p^{\frac{n-1}{2}} & \textit{for } n \textit{ odd, } k \textit{ odd} \end{cases}$$

Remark 6.2. As a sanity check, observe that the torsion-free components take the expected form. Indeed, $K_*\left(\mathbb{E}_n(S^k)\right)$ forms a Poisson algebra with commutative product $\cdot$ and Lie bracket $[-,-]$ of degree $n-1$ satisfying $[x,x] = (-1)^{|x|+n}[x,x]$.

Heuristically, we therefore expect to see the following torsion-free classes:

$k\backslash n$	even	odd
even	$[x_k, x_k]\cdot x_k^{p-2}$, x_k^p	x_k^p
odd		$[x_k,x_k]^{\frac{p-1}{2}}\cdot x_k$

To prove Theorem 6.1, we introduce the following very simple Hecke Lie algebras:

Definition 6.3 (Atomic Hecke Lie algebras at height 1)**.** Given positive integers n, w and an integer a, the atomic Hecke Lie algebra $\mathfrak{g}^{(w)}_{a,n}$ is defined as follows:

(1) The underlying K_*-module of $\mathfrak{g}$ is free on two generators:
  - a generator x in internal degree a and weight w
  - a generator y_{a-1} in internal degree $a-1$ and weight pw.

(2) The Hecke operations are determined by (cf. Example 4.9):

$$\alpha x = \alpha_a x = \begin{cases} p^{\lfloor \frac{n}{2} \rfloor} y & \text{if } a \text{ is even} \\ p^{\lceil \frac{n}{2} \rceil} y & \text{if } a \text{ is odd} \end{cases}$$

(3) All Lie brackets vanish.

In the following sections, we will use small subscripts $(-)_{(i,j,r,w)}$ to indicate the quadruple grading[3] on the Hecke Chevalley–Eilenberg complex, which was defined in Notation 4.17 above. Our computation will rely on knowing the homology of the atomic algebras from Definition 6.3:

Lemma 6.4 (Homology of atomic Hecke Lie algebras)**.** *In weights $k \leq pw$, we have isomorphisms:*

$$\text{For } a \text{ even: } H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g}_{a,n}^{(w)})(k) \cong (\Lambda_{K_*}[x_{(a,1,0,w)}] \oplus K_*/p^{\lfloor \frac{n}{2} \rfloor}[y_{(a-1,1,0,pw)}])(k).$$

$$\text{For } a \text{ odd: } H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g}_{a,n}^{(w)})(k) \cong (\Gamma_{K_*}[x_{(a,1,0,w)}] \oplus K_*/p^{\lceil \frac{n}{2} \rceil}[y_{(a-1,1,0,pw)}])(k).$$

Here Λ_{K_*} constructs the free exterior and Γ_{K_*} the free divided K_*-algebra.

Proof of Lemma 6.4. We depict the additive resolution $\mathrm{AR}(\mathfrak{g}_{a,n}^{(w)})$ in the following diagram:

$r\backslash$*weight*	$1w$	pw	p^2w	$\cdots$
0	$[x]_{(a,1,0,1w)}$	$[y]_{(a-1,1,0,pw)}$	0	$\cdots$
1	$[1\vert x]_{(a,1,1,1w)}$	$[\alpha\vert x]_{(a-1,1,1,pw)}$ $[1\vert y]_{(a-1,1,1,pw)}$	$[\alpha\vert y]_{(a-2,1,1,p^2w)}$	$\cdots$
2	$[1\vert 1\vert x]_{(a,1,2,1w)}$	$[\alpha\vert 1\vert x]_{(a-1,1,2,pw)}$ $[1\vert \alpha\vert x]_{(a-1,1,2,pw)}$ $[1\vert 1\vert y]_{(a-1,1,2,pw)}$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The second index records the homological degree in the simplicial chain complex $\mathrm{CE}(\mathrm{AR}(\mathfrak{g}_{a,n}^{(w)}))$. The shift by $+1$ in the Chevalley–Eilenberg complex is reflected by the second entries being 1 rather than 0. By Theorem 4.18, $H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g}_{a,n}^{(w)})$ is the homology of the complex

$$\mathrm{CE}(\mathrm{AR}(\mathfrak{g}_{a,n}^{(w)})) = \Gamma_{K_*}(\mathrm{AR}(\mathfrak{g}_{a,n}^{(w)})[1]).$$

For a even, $[x]$ has odd total degree in the chain complex $\mathrm{AR}_0(\mathfrak{g}_{a,n}^{(w)})[1]$. Since the tensor product of chain complexes of K_*-modules incorporates the Koszul sign rule for both internal and chain degree, this means that $[x]^i$ vanishes in $\mathrm{CE}(\mathrm{AR}(\mathfrak{g}_{a,n}^{(w)}))$ when $i > 1$. The normalised chain complex of $\mathrm{CE}(\mathrm{AR}(\mathfrak{g}_{a,n}^{(w)}))$ is therefore given by

$$\begin{aligned} \ldots \to 0 \to 0 \to 0 \to 0 \to [1] \quad & \text{in weight } 0,\\ \ldots \to 0 \to 0 \to 0 \to [x] \to 0 \quad & \text{in weight } w,\\ \ldots \to 0 \to [\alpha|x] \xrightarrow{p^{\lfloor \frac{n}{2} \rfloor}} [y] \to 0 \quad & \text{in weight } pw, \end{aligned}$$

and vanishes for all other smaller weights.

For a odd, a similar argument shows that the normalised chain complex of $\mathrm{CE}(\mathrm{AR}(\mathfrak{g}_{a,n}^{(w)}))$ is

$$\begin{aligned} \ldots \to 0 \to 0 \to 0 \to 0 \to 0 \to \ [x]^i\ \to 0\ \to \ldots\ \to 0 \quad & \text{in weight } iw \text{ for } i = 0, 1, \ldots, (p-1),\\ \ldots \to \gamma_p([x]) \to 0 \to \ldots \to 0 \to [\alpha|x] \xrightarrow{p^{\lceil \frac{n}{2} \rceil}} [y] \to 0 \quad & \text{in weight } pw, \end{aligned}$$

and vanishes in all other weights below pw. Here $\gamma_p([x])$ denotes the p^{th} divided power of $[x]$. $\square$

[3] We grade the Hecke Chevalley–Eilenberg complex by (internal degree, homological degree, simplicial degree, weight).

We recall the explicit structure of some free Hecke Lie algebras from [Bra17, Section 4.4.2]:

Corollary 6.5 (Free Hecke Lie algebras on one generator at height 1)**.** *If x_a is a generator in even degree a, then $\mathbf{L}^{\mathcal{H}_u}(x_a)$ is generated as a K_*-module by $x \in \mathbf{L}^{\mathcal{H}_u}(x_a)_a$ and $y \in \mathbf{L}^{\mathcal{H}_u}(x_a)_{i-a}$. The Lie bracket vanishes, and the Hecke operations are determined by $\alpha_a(x) = y$ and $\alpha(y) = 0$.*

If x_a is a generator in odd degree a, then $\mathbf{L}^{\mathcal{H}_u}(x_a)$ is generated as K_*-module by four classes

$$x \in \mathbf{L}^{\mathcal{H}_u}(x_a)_a \ , \quad y \in \mathbf{L}^{\mathcal{H}_u}(x_a)_{a-1} \ , \quad \widetilde{x} \in \mathbf{L}^{\mathcal{H}_u}(x_i)_{2a} \ , \quad \widetilde{y} \in \mathbf{L}^{\mathcal{H}_u}(x_i)_{2a-1}$$

The Lie bracket satisfies $[x, x] = \widetilde{x}$ and vanishes otherwise; the Hecke operations are determined by

$$\alpha_a(x) = y \ , \quad \alpha_{2a}(\widetilde{x}) = \widetilde{y} \ , \quad \alpha_{a-1}(y) = \alpha_{2a-1}(\widetilde{y}) = 0.$$

With the homology of atomic Lie algebras at hand, we can carry out the desired computation. Our proof will proceed in three steps:

(1) Identify the Hecke Lie algebra $\mathfrak{g}(\mathbb{R}^n, S^k) := K_*^{\wedge}(\Omega^n \operatorname{Free}_{\mathrm{Lie}}(\Sigma^{n+k-1}))$.
(2) Compute the Hecke Lie algebra homology of $\mathfrak{g}(\mathbb{R}^n, S^k) := K_*^{\wedge}(\Omega^n \operatorname{Free}_{\mathrm{Lie}}(\Sigma^{n+k-1}))$.
(3) Compute the E^{∞}-page of the spectral sequence and solve extension problems.

In order to apply Theorem 5.5, we start with the following observation:

Proposition 6.6. *The Hecke Lie algebra $\mathfrak{g}(\mathbb{R}^n, S^k)$ is determined as follows in terms of the atomic Hecke Lie algebras from Definition 6.3:*

$k\backslash n$	*even*	*odd*
even	$\mathfrak{g}^{(1)}_{k-1,n} \oplus \mathfrak{g}^{(2)}_{n+2k-2,\lfloor \frac{n}{2} \rfloor}$	$\mathfrak{g}^{(1)}_{k-1,n}$
odd	$\mathfrak{g}^{(1)}_{k-1,n}$	$\mathfrak{g}^{(1)}_{k-1,n} \oplus \mathfrak{g}^{(2)}_{n+2k-2,n}$

Proof. By Proposition 4.22, $\mathfrak{g}(\mathbb{R}^n, S^k)$ is obtained by first taking the free Hecke Lie algebra $\mathbf{L}^{\mathcal{H}_u}(x_{n+k-1})$ on a class in degree $n + k - 1$, then desuspending n times in internal degree (which kills the Lie bracket), and then acting through n-fold suspended Hecke operations. We will freely use Corollary 6.5 in our arguments below.

If $n + k - 1$ is even, then $\mathbf{L}^{\mathcal{H}_u}(x_{n+k-1})$ is generated (as a K_*-module) by a class u in internal degree $n + k - 1$ and a class $v = \alpha_{n+k-1}(u)$ in internal degree $n + k - 2$. Hence $\mathfrak{g}(\mathbb{R}^n, S^k) = \Sigma^{-n} \mathbf{L}^{\mathcal{H}_u}(x_{n+k-1})$ is generated by classes $x = \Sigma^{-n}u$ and $y = \Sigma^{-n}v$ in internal degree $k - 1$ and $k - 2$, respectively. To act with α_{k-1} on x, we can (by Proposition 4.22) instead act by $\Sigma^n(\alpha_{k-1})$ on u and then apply Σ^{-n} the resulting class.

If $k-1$ is even, then Proposition 4.24 and the identification of the Euler class e with p implies:

$$\alpha_{k-1}(x) = \Sigma^{-n}(\operatorname{Susp}^n(\alpha_{k-1})(u)) = \Sigma^{-n}(p^{\lfloor \frac{n}{2} \rfloor}\alpha_{n+k-1}(u)) = p^{\lfloor \frac{n}{2} \rfloor} y$$

As the action on y vanishes, $\mathfrak{g}(\mathbb{R}^n, S^k)$ is equal to the atomic Lie algebra $\mathfrak{g}^{(1)}_{k-1,n}$.

If $k - 1$ is odd, a similar argument shows that

$$\alpha_{k-1}(x) = \Sigma^{-n}(\operatorname{Susp}^n(\alpha_{k-1})(u)) = \Sigma^{-n}(p^{\lceil \frac{n}{2} \rceil}\alpha_{n+k-1}(u)) = p^{\lceil \frac{n}{2} \rceil} y$$

which implies that $\mathfrak{g}(\mathbb{R}^n, S^k)$ is equal to the atomic Lie algebra $\mathfrak{g}^{(1)}_{k-1,n}$.

If $n+k-1$ is odd, then $\mathbf{L}^{\mathcal{H}_u}(x_{n+k-1})$ is generated (as a K_*-module) by four classes:

- u in internal degree $n+k-1$ and weight 1;
- $v = \alpha_{n+k-1}(u)$ in internal degree $n+k-2$ and weight p;
- $u' = [u,u]$ in internal degree $2n+2k-2$ and weight 2;
- $v' = \alpha_{2n+2k-2}[u,u]$ in internal degree $2n+2k-3$ and weight $2p$.

Hence $\mathfrak{g}(\mathbb{R}^n, S^k)$ is generated by the four classes $x = \Sigma^{-n}u$, $y = \Sigma^{-n}v$, $x' = \Sigma^{-n}u'$, and $y' = \Sigma^{-n}v'$ in internal degrees $k-1$, $k-2$, $n+2k-2$, and $n+2k-3$, respectively.

If $k-1$ is even, then $n+2k-2$ is odd, and so a similar argument as before shows that we have $\alpha_{k-1}(x) = p^{\lfloor \frac{n}{2} \rfloor} y$ and $\alpha_{n+2k-2}(x) = p^{\lceil \frac{n}{2} \rceil} y'$. Hence $\mathfrak{g}(\mathbb{R}^n, S^k) = \mathfrak{g}^{(1)}_{k-1,n} \oplus \mathfrak{g}^{(2)}_{n+2k-2,n}$.

If $k-1$ is odd, $\alpha_{k-1}(x) = p^{\lceil \frac{n}{2} \rceil} y$ and $\alpha_{n+2k-2}(x) = p^{\lfloor \frac{n}{2} \rfloor} y'$, so $\mathfrak{g}(\mathbb{R}^n, S^k) = \mathfrak{g}^{(1)}_{k-1,n} \oplus \mathfrak{g}^{(2)}_{n+2k-2,n}$. $\square$

Combining this with our previous work, we can immediately deduce:

Proposition 6.7. *The homology of $\mathfrak{g}(\mathbb{R}^n, S^k)$ in weight p is generated by the following elements:*

$k\backslash n$	*even*	*odd*
even	$\gamma_p([x_{(k-1,1,0,1)}])$ $[y_{(k-2,1,0,p)}]/p^{\lceil \frac{n}{2} \rceil}$ $[x_{(k-1,1,0,1)}]^{p-2} \cdot [x'_{(n+2k-2,1,0,2)}]$	$\gamma_p([x_{(k-1,1,0,1)}])$ $[y_{(k-2,1,0,p)}]/p^{\lceil \frac{n}{2} \rceil}$
odd	$[y_{(k-2,1,0,p)}]/p^{\lfloor \frac{n}{2} \rfloor}$	$[x_{(k-1,1,0,1)}] \cdot [x'_{(n+2k-2,1,0,2)}]^{\frac{p-1}{2}}$ $[y_{(k-2,1,0,p)}]/p^{\lfloor \frac{n}{2} \rfloor}$

Proof. This follows from Proposition 6.6, Lemma 6.4 since $H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g} \oplus \mathfrak{g}')(w)$ is isomorphic to $(H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g}) \otimes H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g}'))(w)$ for all involved Hecke Lie algebras $\mathfrak{g}, \mathfrak{g}'$ and all weights $w \leq p$. $\square$

Remark 6.8. Before proceeding to the proof Theorem 6.1, we note that elements $[z_{(i,j,r,w)}]$ in internal degree i, homological degree j, simplicial degree r, and weight w (cf. Notation 4.17) contribute to the $\mathrm{E}^2_{j+r-1,i+1} = H^{\mathrm{Lie}^{\mathcal{H}_u}}_{j+r}(\mathfrak{g}(\mathbb{R}^n, S^k))_i$ term of the weight w component of our spectral sequence in Theorem 5.5.

Proof of Theorem 6.1. By Proposition 6.7, the E^2-term $\mathrm{E}^2_{s,t} = H^{\mathrm{Lie}^{\mathcal{H}_u}}_{s+1}(\mathfrak{g}(\mathbb{R}^n, S^k))_{t-1}$ of the spectral sequence Theorem 5.5 at weight p is given by:

$$\mathrm{E}^2_{0,*}(p) \quad = \quad H^{\mathrm{Lie}^{\mathcal{H}_u}}_1(\mathfrak{g}(\mathbb{R}^n, S^k))(p))_{*-1} \quad = \quad \begin{cases} \Sigma^{k-1} K_*/p^{\lceil \frac{n}{2} \rceil} & \text{for } n \text{ even, } k \text{ even} \\ \Sigma^{k-1} K_*/p^{\lfloor \frac{n}{2} \rfloor} & \text{for } n \text{ even, } k \text{ odd} \\ \Sigma^{k-1} K_*/p^{\lceil \frac{n}{2} \rceil} & \text{for } n \text{ odd, } k \text{ even} \\ \Sigma^{k-1} K_*/p^{\lfloor \frac{n}{2} \rfloor} & \text{for } n \text{ odd, } k \text{ odd} \end{cases}$$

$$\mathrm{E}^2_{\frac{p-1}{2},*-\frac{p-1}{2}}(p) = H^{\mathrm{Lie}^{\mathcal{H}_u}}_{\frac{p+1}{2}}(\mathfrak{g}(\mathbb{R}^n,S^k))(p))_{*-\frac{p+1}{2}} = \begin{cases} 0 & \text{for } n \text{ even, } k \text{ even} \\ 0 & \text{for } n \text{ even, } k \text{ odd} \\ 0 & \text{for } n \text{ odd, } k \text{ even} \\ \Sigma^{(2k+n-1)(\frac{p-1}{2})+k}K_* & \text{for } n \text{ odd, } k \text{ odd} \end{cases}$$

$$\mathrm{E}^2_{p-2,*-p+2}(p) = H^{\mathrm{Lie}^{\mathcal{H}_u}}_{p-1}(\mathfrak{g}(\mathbb{R}^n,S^k))(p))_{*-p+1} = \begin{cases} \Sigma^{kp+n-1}K_* & \text{for } n \text{ even, } k \text{ even} \\ 0 & \text{for } n \text{ even, } k \text{ odd} \\ 0 & \text{for } n \text{ odd, } k \text{ even} \\ 0 & \text{for } n \text{ odd, } k \text{ odd} \end{cases}$$

$$\mathrm{E}^2_{p-1,*-p+1}(p) = H^{\mathrm{Lie}^{\mathcal{H}_u}}_{p}(\mathfrak{g}(\mathbb{R}^n,S^k))(p))_{*-p} = \begin{cases} \Sigma^{kp}K_* & \text{for } n \text{ even, } k \text{ even} \\ 0 & \text{for } n \text{ even, } k \text{ odd} \\ \Sigma^{kp}K_* & \text{for } n \text{ odd, } k \text{ even} \\ 0 & \text{for } n \text{ odd, } k \text{ odd} \end{cases}$$

The modules $\mathrm{E}^2_{s,t}(p)$ in our spectral sequence vanish for all other values of s and t.

Step 3). We will now compute the differentials $d_r : \mathrm{E}^r_{s,t}(p) \to \mathrm{E}^r_{s-r,t+r-1}(p)$.

Case 1: n even, k odd. In this simplest case, the spectral sequence is concentrated on the $(s=0)$-line, which allows us to read off the result.

Case 2: n odd, $k>0$ even. The spectral sequence is concentrated on the lines $(s=0)$ and $(s=p-1)$. Hence $E^2_{s,t}(p) \cong \ldots \cong E^{p-1}_{s,t}(p)$. The only possible non-vanishing differential is

$$d_{p-1} : \mathrm{E}^{p-1}_{p-1,t}(p) \to \mathrm{E}^{p-1}_{0,t+p-2}(p).$$

Since $\mathrm{E}^{p-1}_{0,*}(p)$ is concentrated in odd and $\mathrm{E}^{p-1}_{p-1,*}(p)$ in even degrees, we cannot conclude that d_{r-1} is zero. Indeed, a second thought reveals that it in fact should not vanish.

Already for $n=1$, the James splitting shows that $K^\wedge_*(\Omega S^{k+1})$ is torsion-free, and hence the p-torsion generator $[y_1] \in \mathrm{E}^2_{0,*}(p,1,n) = \Sigma^{k-1}K_*/p$ on the zero-line must be hit under d_{p-1} (up to a unit) by the class $\gamma_p([x])$. Hence $\gamma_p(x)$ dies, whereas x^p survives.

This observation propagates to higher loop spaces, and we can use it to compute d_{p-1} for all higher odd n. For this, write $\mathrm{E}^{p-1}_{s,t}(p,n,k)$ for the spectral sequence converging to $K^\wedge_*(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{h\Sigma_p} (S^k)^{\otimes p})$. The map $\Sigma^\infty_+\Omega S^{1+k} \to \Sigma^\infty_+\Omega^n S^{n+k}$ corresponds to the canonical map of spectral Lie algebras

$$\Omega\,\mathrm{Free}^{\mathscr{L}}(S^k) \to \Omega^n\,\mathrm{Free}^{\mathscr{L}}(S^{n+k-1}).$$

We obtain a map of spectral sequences $\phi : \mathrm{E}^r_{s,t}(p,1,k) \to \mathrm{E}^r_{s,t}(p,n,k)$. On E^1, it is obtained by applying the Hecke–Chevalley–Eilenberg complex to the induced map of Hecke Lie algebras

$$\mathfrak{g}^{(1)}_{k-1,1} = \mathfrak{g}(\mathbb{R}^1,S^k) \xrightarrow{\ \phi\ } \mathfrak{g}(\mathbb{R}^n,S^k) = \mathfrak{g}^{(1)}_{k-1,n}.$$

We can describe this map entirely explicitly. To this end, denote the natural generators by

$$x_1 \in \mathfrak{g}(\mathbb{R}^1, S^k)_{k-1}\ ,\ y_1 \in \mathfrak{g}(\mathbb{R}^1, S^k)_{k-2}, \quad \text{and} \quad x_n \in \mathfrak{g}(\mathbb{R}^n, S^k)_{k-1}\ ,\ y_n \in \mathfrak{g}(\mathbb{R}^n, S^k)_{k-2}.$$

As $\Omega\,\mathrm{Free}^{\mathscr{L}}(S^k) \to \Omega^n\,\mathrm{Free}^{\mathscr{L}}(S^{n+k-1})$ is obtained by looping a free map, we find

$$\phi(x_1) = x_n\ , \quad \phi(y_1) = p^{\lfloor \frac{n}{2} \rfloor} y_n.$$

Hence the induced map on the $(s = 0)$–line of E_2-pages is given by

$$\mathrm{E}^2_{0,*}(p,1,k) = \Sigma^{k-1}K^*/p \xrightarrow{p^{\frac{n}{2}}} \Sigma^{k-1}K_*/p^{\frac{n+1}{2}} = \mathrm{E}^2_{0,*}(p,n,k)$$

Since $\mathrm{E}^{p-1}_{0,*}(p,1,k) \cong \mathrm{E}^2_{0,*}(p,1,k)$ is killed by d^{p-1} and $\mathrm{E}^{p-1}_{0,*}(p,1,k) \to \mathrm{E}^{p-1}_{0,*}(p,n,k)$ respects the differentials, we deduce $\mathrm{E}^p_{0,*}(p,n,k) = \Sigma^{k-1}K_*/p^{\frac{n-1}{2}}$ and $\mathrm{E}^p_{p-1,*-p-1}(p,n,k) = \langle p\gamma_p([x])\rangle = \langle [x]^p\rangle$.

As no further differentials can occur, we have computed the E^∞-page. But $\mathrm{E}^\infty_{p-1,*-p-1}$ is a free K_*-module, and the only other nontrivial line is at $(s = 0)$, so there are no extension problems. This finishes the proof for n odd, $k > 0$ even.

If n and k have the same parity and $k > 0$, we can avoid further hard work by remembering that $p > 2$. In particular, setting $2m = n + k$, we may apply Serre's p-local equivalence [Ser53]:

$$\Omega S^{2m}_{(p)} \simeq S^{2m-1}_{(p)} \times \Omega S^{4m-1}_{(p)}.$$

Taking $n - 1$ additional loops yields the p-local equivalence

$$\Omega^n S^{n+k}_{(p)} \simeq \Omega^{n-1} S^{n+k-1}_{(p)} \times \Omega^n S^{2n+2k-1}_{(p)},$$

which is given [Grb06, §2.1] by the product of the suspension map $E : \Omega^{n-1}S^{n+k-1} \to \Omega^n S^{n+k}$ with the n-fold loops of the Whitehead square $[\iota,\iota] : S^{2n+2k-1} \to S^{n+k}$. After stabilising, the map $\Sigma^\infty\Omega^n[\iota,\iota]$ has domain a free $\mathbb{E}_n$-algebra with generator mapping to a weight 2 class. Adding basepoints, stabilising, and passing to weight p, we obtain a p-local decomposition

$$\Sigma^\infty_+\Omega^n S^{n+k}(p) \simeq_{(p)} \bigoplus_{i+2j=p} (\Sigma^\infty_+\Omega^{n-1}S^{n+k-1})(i) \otimes (\Sigma^\infty_+\Omega^n S^{2n+2k-1})(j).$$

As the reduced p-adic K-theory of symmetric groups $B\Sigma_i$ vanishes whenever $i < p$, the summand $K^\wedge_*(\mathbb{E}_n(S^a)(w))$ is isomorphic to the weight w component in the free Poisson algebra in K_*-modules on a class in degree a, for all $w < p$. In particular, they are finite free K_*-modules in this range. This can also be proven directly from our spectral sequence. We obtain an isomorphism

$$K^\wedge_*(\Sigma^\infty_+\Omega^n S^{n+k}(p)) \cong \bigoplus_{i+2j=p} K^\wedge_*(\mathbb{E}_{n-1}(S^k)(i)) \otimes_{K_*} K^\wedge_*(\mathbb{E}_n(S^{n+2k-1})(j)).$$

If k and n are even, then $n+2k-1$ is odd, which implies that the right hand side vanishes for degree reasons unless $j = 0$ (which gives a copy of $\Sigma^{kp}K_* \oplus \Sigma^{k-1}K_*/p^{\lfloor \frac{n-1}{2} \rfloor}$) or $j = 1$ (which gives a copy of $\Sigma^{pk+n-1}K_*$).

If k and n are odd, then the right hand side vanishes unless $i = 1$ (in which case $j = \frac{p-1}{2}$ and we obtain a copy of K_*), or $i = p$ (which contributes a copy of $\Sigma^{k-1}K_*/p^{\lfloor \frac{n-1}{2} \rfloor}$. This finishes the proof whenever $k > 0$.

Finally, the computation extends to general k by using that over a complex oriented homology theory, we have an equivariant equivalence of (naïve) Σ_n-spectra $K \otimes (S^2)^{\otimes n} \simeq \Sigma^{2n}K$.

$\square$

6.2. **Interpretation in terms of power operations.** Recall that the free $K(1)$-local $\mathbb{E}_\infty$-$K_p^\wedge$-algebra on a class x in degree 2 is given by

$$\left(K \wedge \Sigma^\infty_+ \Omega^\infty S^2\right)^\wedge_p .$$

By a classical computation (cf. e.g. [Wil82] [Hop14]), its homotopy groups are isomorphic to the completion of the algebra $\mathbb{Z}_p[u^\pm][x, \theta x, \theta\theta x, \cdots]$. Here, the class x is in weight 1, while the class θx is in weight p, the class $\theta\theta x$ is in weight p^2, and so on. One sees that the entire algebra is generated by the class x together with a single Dyer-Lashof operation θ. The weight p component is given by a free K_*-module on x^p and θx.

The above calculations allow us to understand this component not only for free $\mathbb{E}_\infty$-algebras, but also for free $\mathbb{E}_n$-algebras with $n < \infty$. Indeed, the free $K(1)$-local $\mathbb{E}_n$-$K_p^\wedge$-algebra on a class x in degree 2 is given by

$$\left(K \wedge \Sigma^\infty_+ \Omega^n S^{n+2}\right)^\wedge_p .$$

By examining the weight p component of the above algebra, we can determine what vestige of the θ operation remains present in $\mathbb{E}_n$-algebras. According to the above calculations, the nature of this weight p component varies depending upon the parity of n.

When n is odd, we see from Theorem 6.1 that the weight p-component of the free $K(1)$-local $\mathbb{E}_n$-$K_p^\wedge$-algebra on a degree 2 class is equivalent to

$$\Sigma^{2p} K_p^\wedge \oplus \Sigma K/p^{\frac{n-1}{2}} .$$

The copy of $\Sigma^{2p} K_p^\wedge$ corresponds to the element x^p in the free $\mathbb{E}_\infty$-algebra, while the torsion module $\Sigma K/p^{\frac{n-1}{2}}$ is related to θx. Indeed, there is a diagram

$$\begin{tikzcd}
\left(K \wedge \Sigma^\infty_+ \Omega^n S^{n+2}\right)^\wedge_p \arrow[r] & \left(K \wedge \Sigma^\infty_+ \Omega^\infty S^2\right)^\wedge_p \\
\Sigma K/p^{\frac{n-1}{2}} \arrow[u] \arrow[r, "\text{Bockstein}"] & \Sigma^2 K_p^\wedge . \arrow[u, "\theta x"']
\end{tikzcd}$$

The sequence of suspension maps $\Omega S^3 \longrightarrow \Omega^3 S^5 \longrightarrow \Omega^5 S^7 \longrightarrow \cdots \longrightarrow \Omega^\infty S^2$ induces a sequence of $K_p^\wedge$-module Bocksteins

$$0 \longrightarrow \Sigma K/p \longrightarrow \Sigma K/p^2 \longrightarrow \Sigma K/p^3 \longrightarrow \cdots \longrightarrow \Sigma^2 K_p^\wedge ,$$

exhibiting the relation $\mathrm{hocolim}_n(\Sigma K/p^n) \simeq \Sigma^2 K_p^\wedge$ in $K(1)$-local K-modules.

When n is even, we read off from Theorem 6.1 that the weight p-component of the free $\mathbb{E}_n$-$K_p^\wedge$-algebra on a degree 2 class is equivalent to

$$\Sigma^{2p} K_p^\wedge \oplus \Sigma^{2p+n-1} K_p^\wedge \oplus \Sigma K/p^{\frac{n}{2}} .$$

The copy of $\Sigma^{2p} K_p^\wedge$ is related to x^p in the free $\mathbb{E}_\infty$-algebra, and $\Sigma K/p^{\frac{n}{2}}$ is a shadow of θx.

The copy of $\Sigma^{2p+n-1} K_p^\wedge$ represents a phenomenon specific to $\mathbb{E}_n$-algebras: their homotopy groups form Poisson algebras in K_*-modules, with an $(n-1)$-shifted Lie bracket $[-,-]$. The class in degree $(2p+n-1)$ then corresponds to the Poisson word $[x,x]x^{p-2}$.

Remark 6.9. Throughout the above discussion, we have focused on the situation of a free $K(1)$-local $\mathbb{E}_n$-$K_p^\wedge$-algebra generated by a degree 2 class x. We could similarly use Theorem 6.1 to understand the free $\mathbb{E}_n$-algebra on a class in any other degree. Four cases will arise, depending on what the parities of n and k imply about the vanishing of x^2, $[x,x]$ based on Koszul sign rules.

6.3. **The E-theory of p points in $\mathbb{R}^n$.** Fix a form of E-theory at height h and prime $p > 2$. We will carry out the following computation, which, to the best of our knowledge, is new:

Theorem 6.10 (E-theory, Euclidean case)**.** *For any positive integer n and integer k, the E_*-module $E_*^\wedge\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{h\Sigma_p} (S^k)^{\otimes p}\right)$ is given by*

$$E_*^\wedge(\mathbb{E}_n(S^k))(p) = \begin{cases} \Sigma^{kp}E_* \oplus \Sigma^{pk+n-1}E_* \oplus \Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n}{2}-1}) & \text{for } n \text{ even, } k \text{ even} \\ \Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n}{2}}) & \text{for } n \text{ even, } k \text{ odd} \\ \Sigma^{kp}E_* \oplus \Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n-1}{2}}) & \text{for } n \text{ odd, } k \text{ even} \\ \Sigma^{k+(2k+n-1)(\frac{p-1}{2})}E_* \oplus \Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n-1}{2}}) & \text{for } n \text{ odd, } k \text{ odd} \end{cases}$$

Here (tr) *denotes the transfer ideal associated to the inclusion of the trivial group, whereas $e \in E^0(B\Sigma_p)$ is the Euler class of the reduced complex standard representation.*

Remark 6.11. The $p = 2$ analogue of the above theorem is much more elementary. Here, one may rely on the equivalence

$$\mathrm{Conf}_2(\mathbb{R}^n) \otimes_{\Sigma_2} (S^k)^{\otimes 2} \simeq \Sigma^k \mathbb{RP}_k^{n+k-1},$$

where $\mathbb{RP}_k^{n+k-1}$ denotes the cofiber of the inclusion $\mathbb{RP}^{k-1} \to \mathbb{RP}^{n+k-1}$. This reduces the question to the calculation of the E-theory of real projective spaces, which is classically accomplished via the Gysin sequences associated to the fibrations $S^1 \to \mathbb{RP}^{2n+1} \to \mathbb{CP}^n$.

While our computation will be notationally more cumbersome, it relies on the same strategy as the preceding section 6.1. First, we need a higher height variant of the atomic Lie algebras from Definition 6.3, which will make use of the Hecke power ring introduced in Definition 4.8:

Definition 6.12 (Atomic Hecke Lie algebras at height h)**.** Given positive integers n, w and an integer a, we define the atomic Hecke Lie algebra $\mathfrak{h} = \mathfrak{h}_{a,s}^{(w)}$ as follows:

(1) The underlying graded abelian group of $\mathfrak{h}$ is given by

$$\mathfrak{h}_* = \bigoplus_{0 \le \ell \le h} (\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+*}(p^\ell).$$

We shall denote the element corresponding to some $\phi \in (\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+i}(p^\ell)$ by the symbol x_ϕ; it will live in homological degree i and weight $p^\ell w$.

(2) Hecke operations in $(\mathcal{H}_u^{\mathrm{Lie}})_i^j(w)$ act on $(\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+i}(p^k) \subset \mathfrak{h}_i$ via

$$(\mathcal{H}_u^{\mathrm{Lie}})_i^j[w] \times (\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+i}[p^k] \xrightarrow{\mathrm{Susp}^b \times \mathrm{id}} (\mathcal{H}_u^{\mathrm{Lie}})_{b+i}^{b+j}[w] \times (\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+i}[p^k] \to (\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+j}[p^k] \subset \mathfrak{h}_j$$

(3) All Lie brackets vanish.

We generalise Lemma 6.4 to higher heights:

Lemma 6.13 (Homology of atomic Hecke Lie algebras at height h)**.** *In weights $k \leq pw$, we have:*

$$\begin{aligned}
\text{For } a \text{ even: } H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{h}_{a,n}^{(w)})(k) &\cong (\Lambda_{E_*}[x_{(a,1,0,w)}] \oplus (\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+*}(p)/_{\mathrm{Susp}^n})(k)\\
&\cong (\Lambda_{E_*}[x_{(a,1,0,w)}] \oplus \Sigma^{a-1}E^{-*}(B\Sigma_p)/(\mathrm{tr}, e^{\lfloor \frac{n}{2} \rfloor}))(k)\\
\text{For } a \text{ odd: } H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{h}_{a,n}^{(w)})(k) &\cong (\Gamma_{E_*}[x_{(a,1,0,w)}] \oplus (\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+*}(p)/_{\mathrm{Susp}^n})(k)\\
&\cong (\Gamma_{E_*}[x_{(a,1,0,w)}] \oplus \Sigma^{a-1}E^{-*}(B\Sigma_p)/(\mathrm{tr}, e^{\lceil \frac{n}{2} \rceil}))(k).
\end{aligned}$$

As above, (tr) *denotes the transfer ideal associated to the trivial subgroup, and $e \in E^0(B\Sigma_p)$ is the Euler class of the reduced complex standard representation.*

Proof. The symbols x_ϕ introduced in Definition 6.12 are E_*-linear, i.e. satisfy $\lambda \cdot x_\phi = x_{\lambda\cdot\phi}$ for all $\lambda \in E_*$. Write $x_1 = x$ for the canonical generator in degree a and weight 1. We draw $\mathrm{AR}(\mathfrak{h}_{a,n}^{(w)})$:

$r\backslash$*weight*	$1w$	pw	p^2w	$\cdots$
0	$[x]_{(a,1,0,1w)}$	$[x_\alpha]_{(a-1,1,0,pw)}$	$[x_\beta]_{(a-2,1,0,p^2w)}$	$\cdots$
1	$[1\vert x]_{(a,1,1,1w)}$	$[\alpha'\vert x]_{(a-1,1,1,pw)}$ $[1\vert x_\alpha]_{(a-1,1,1,pw)}$	$[1\vert x_\beta]_{(a-2,1,1,p^2w)}$ $[\alpha'\vert x_\alpha]_{(a-2,1,1,p^2w)}$ $[\beta'\vert x]_{(a-2,1,1,p^2w)}$	
2	$[1\vert 1\vert x]_{(a,1,2,1w)}$	$[\alpha'\vert 1\vert x]_{(a-1,1,2,pw)}$ $[1\vert \alpha'\vert x]_{(a-1,1,2,pw)}$ $[1\vert 1\vert x_\alpha]_{(a-1,1,2,pw)}$	$[1\vert 1\vert x_\beta]_{(a-2,1,2,p^2w)}$ $[1\vert \beta'\vert x]_{(a-2,1,2,p^2w)}$ $[\beta'\vert 1\vert x]_{(a-2,1,2,p^2w)}$ $[1\vert \alpha''\vert x_\alpha]_{(a-2,1,2,p^2w)}$ $[\alpha''\vert 1\vert x_\alpha]_{(a-2,1,2,p^2w)}$ $[\alpha''\vert \alpha'\vert x]_{(a-2,1,2,p^2w)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Here α runs over some fixed E_0-module basis for $(\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+a-1}(p)$, the element α' runs over some fixed basis for $(\mathcal{H}_u^{\mathrm{Lie}})_a^{a-1}(p)$, the element α'' runs over some fixed basis for $(\mathcal{H}_u^{\mathrm{Lie}})_{a-1}^{a-2}(p)$, etc. Similarly, β runs over some fixed basis for $(\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+a-2}(p^2)$, β' runs over some fixed basis for $(\mathcal{H}_u^{\mathrm{Lie}})_a^{a-2}(p^2)$, β'' runs over some fixed basis for $(\mathcal{H}_u^{\mathrm{Lie}})_a^{a-2}(p^2)$, etc. The subscripts again reflect the quadruple grading from Notation 4.17. We will again use Theorem 4.18 to compute $H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{h}_{a,n}^{(w)})$ as the homology of the complex $\mathrm{CE}(\mathrm{AR}(\mathfrak{h}_{a,n}^{(w)})) = \Gamma_{K_*}(\mathrm{AR}(\mathfrak{h}_{a,n}^{(w)})[1])$.

For a even, $[x]$ has odd total degree in the chain complex of graded E_*-modules $\mathrm{AR}_0(\mathfrak{h}_{a,n}^{(w)})[1]$ which implies that $[x]^i$ vanishes when $i > 1$. Using the structure maps in Definition 6.12, we see that the normalised chain complex of $\mathrm{CE}(\mathrm{AR}(\mathfrak{h}_{a,n}^{(w)}))$ is therefore given by the complexes

$$\ldots \to 0 \to 0 \to 0 \to 0 \to [1] \text{ in weight } 0,$$
$$\ldots \to 0 \to 0 \to 0 \to [x] \to 0 \text{ in weight } w,$$
$$\ldots \to 0 \to (\mathcal{H}_u^{\mathrm{Lie}})_a^*(p) \xrightarrow{\mathrm{Susp}^n} (\mathcal{H}_u^{\mathrm{Lie}})_{n+a}^{n+*}(p) \to 0 \text{ in weight } pw,$$

and vanishes for all other smaller weights.

For a odd, $[x]$ has even total degree in the chain complex of graded E_*-modules $\mathrm{AR}_0(\mathfrak{h}^{(w)}_{a,n})[1]$. We deduce that the normalised chain complex of $\mathrm{CE}(\mathrm{AR}(\mathfrak{h}^{(w)}_{a,n}))$ is given by

$$\ldots \to 0 \to 0 \to 0 \to 0 \to 0 \to [x]^i \to 0 \to \ldots \to 0 \qquad \text{in weight } iw \text{ for } i = 0, 1, \ldots, (p-1),$$

$$\ldots \to \gamma_p([x]) \to 0 \to \ldots \to 0 \to (\mathcal{H}^{\mathrm{Lie}}_u)^*_a(p) \xrightarrow{\mathrm{Susp}^n} (\mathcal{H}^{\mathrm{Lie}}_u)^{n+*}_{n+a}(p) \to 0 \quad \text{in weight } pw,$$

and vanishes in all other weights below pw. This proves the first part of both cases; the remaining claims follow from Proposition 4.24. □

With this computation at hand, we can prove Theorem 6.10 along the same lines as Theorem 6.1. We recall the structure of free Hecke Lie algebras on a single class from [Bra17, Section 4.4.2]:

Corollary 6.14 (Free Hecke Lie algebras on one generator at height h)**.**

If x_a is a generator in even degree a, then

$$\mathbf{L}^{\mathcal{H}_u}(x_a)_* = \bigoplus_{0 \le \ell \le h} (\mathcal{H}^{\mathrm{Lie}}_u)^*_a(p^\ell).$$

Write elements corresponding to $\phi \in (\mathcal{H}^{\mathrm{Lie}}_u)^*_a(p^\ell)$ as x_ϕ. The Lie bracket vanishes, and the Hecke operations are determined by $\alpha(x_\phi) = x_{\alpha\phi}$.

If x_a is a generator in odd degree a, then $\mathbf{L}^{\mathcal{H}_u}(x_a)$ is given by

$$\mathbf{L}^{\mathcal{H}_u}(x_a)_* = \bigoplus_{0 \le \ell \le h} (\mathcal{H}^{\mathrm{Lie}}_u)^*_a(p^\ell) \oplus \bigoplus_{0 \le \ell \le h} (\mathcal{H}^{\mathrm{Lie}}_u)^*_{2a}(p^\ell).$$

Write elements corresponding to $\phi \in (\mathcal{H}^{\mathrm{Lie}}_u)^*_a(p^\ell)$ on the left as x_ϕ and to $\phi \in (\mathcal{H}^{\mathrm{Lie}}_u)^*_{2a}(p^\ell)$ on the right as $\widetilde{x}_\phi$. The Lie bracket satisfies $[x_\lambda, x_\mu] = \widetilde{x}_{\lambda\mu}$ for all scalars $\lambda, \mu \in E_* \cong (\mathcal{H}^{\mathrm{Lie}}_u)^*_a(1)$, and vanishes otherwise. The Hecke operations are determined by $\alpha(x_\phi) = x_{\alpha\phi}$ and $\alpha(\widetilde{x}_\phi) = \widetilde{x}_{\alpha\phi}$.

Proof of Theorem 6.10. We will follow the same strategy as in the proof of Theorem 6.1.

Proposition 4.22 shows that $\mathfrak{g}(\mathbb{R}^n, S^k)$ is obtained by desuspending $\mathbf{L}^{\mathcal{H}_u}(x_{n+k-1})$ exactly n times in internal degree, and then acting through n-fold suspended Hecke operations. As in the proof of Theorem 6.1, Corollary 6.14 then implies that $\mathfrak{g}(\mathbb{R}^n, S^k)$ is given as indicated below:

$k \backslash n$	even	odd
even	$\mathfrak{h}^{(1)}_{k-1,n} \oplus \mathfrak{h}^{(2)}_{n+2k-2,\lfloor \frac{n}{2} \rfloor}$	$\mathfrak{h}^{(1)}_{k-1,n}$
odd	$\mathfrak{h}^{(1)}_{k-1,n}$	$\mathfrak{h}^{(1)}_{k-1,n} \oplus \mathfrak{g}^{(2)}_{n+2k-2,n}$

We read off $H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g}(\mathbb{R}^n, S^k))(p)$ from Lemma 6.4 and the isomorphism $H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{h} \oplus \mathfrak{h}')(w) \cong (H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{h}) \otimes H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{h}'))(w)$, which holds for all involved Hecke Lie algebras $\mathfrak{h}, \mathfrak{h}'$ in weights $w \le p$.

$k\backslash n$	even	odd
even	$\Sigma^{p(k-1)}E_*$ $\Sigma^{k-2}E^{-*}(B\Sigma_p)/(\mathrm{tr},e^{\lceil\frac{n}{2}\rceil})$ $\Sigma^{p(k-1)+n}E_*$	$\Sigma^{p(k-1)}E_*$ $\Sigma^{k-2}E^{-*}(B\Sigma_p)/(\mathrm{tr},e^{\lceil\frac{n}{2}\rceil})$
odd	$\Sigma^{k-2}E^{-*}(B\Sigma_p)/(\mathrm{tr},e^{\lfloor\frac{n}{2}\rfloor})$	$\Sigma^{(k-1)+\frac{p-1}{2}(n+2k-2)}E_*$ $\Sigma^{k-2}E^{-*}(B\Sigma_p)/(\mathrm{tr},e^{\lfloor\frac{n}{2}\rfloor})$

FIGURE 1. The Hecke Lie algebra homology $H^{\mathrm{Lie}^{\mathcal{H}_u}}(\mathfrak{g}(\mathbb{R}^n,S^k))(p)$.

Hence, the E_2-term $\mathrm{E}^2_{s,t}=H^{\mathrm{Lie}^{\mathcal{H}_u}}_{s+1}(\mathfrak{g}(\mathbb{R}^n,S^k))_{t-1}$ of the spectral sequence at weight p is given by:

$$\mathrm{E}^2_{0,*}(p) = H^{\mathrm{Lie}^{\mathcal{H}_u}}_1(\mathfrak{g}(\mathbb{R}^n,S^k))(p))_{*-1} = \begin{cases}\Sigma^{k-1}E^{-*}(B\Sigma_p)/(\mathrm{tr},e^{\lceil\frac{n}{2}\rceil}) & \text{for } n \text{ even, } k \text{ even}\\ \Sigma^{k-1}E^{-*}(B\Sigma_p)/(\mathrm{tr},e^{\lfloor\frac{n}{2}\rfloor}) & \text{for } n \text{ even, } k \text{ odd}\\ \Sigma^{k-1}E^{-*}(B\Sigma_p)/(\mathrm{tr},e^{\lceil\frac{n}{2}\rceil}) & \text{for } n \text{ odd, } k \text{ even}\\ \Sigma^{k-1}E^{-*}(B\Sigma_p)/(\mathrm{tr},e^{\lfloor\frac{n}{2}\rfloor}) & \text{for } n \text{ odd, } k \text{ odd}\end{cases}$$

$$\mathrm{E}^2_{\frac{p-1}{2},*-\frac{p-1}{2}}(p) = H^{\mathrm{Lie}^{\mathcal{H}_u}}_{\frac{p+1}{2}}(\mathfrak{g}(\mathbb{R}^n,S^k))(p))_{*-\frac{p+1}{2}} = \begin{cases}0 & \text{for } n \text{ even, } k \text{ even}\\ 0 & \text{for } n \text{ even, } k \text{ odd}\\ 0 & \text{for } n \text{ odd, } k \text{ even}\\ \Sigma^{(2k+n-1)(\frac{p-1}{2})+k}E_* & \text{for } n \text{ odd, } k \text{ odd}\end{cases}$$

$$\mathrm{E}^2_{p-2,*-p+2}(p) = H^{\mathrm{Lie}^{\mathcal{H}_u}}_{p-1}(\mathfrak{g}(\mathbb{R}^n,S^k))(p))_{*-p+1} = \begin{cases}\Sigma^{kp+n-1}E_* & \text{for } n \text{ even, } k \text{ even}\\ 0 & \text{for } n \text{ even, } k \text{ odd}\\ 0 & \text{for } n \text{ odd, } k \text{ even}\\ 0 & \text{for } n \text{ odd, } k \text{ odd}\end{cases}$$

$$\mathrm{E}^2_{p-1,*-p+1}(p) = H^{\mathrm{Lie}^{\mathcal{H}_u}}_{p}(\mathfrak{g}(\mathbb{R}^n,S^k))(p))_{*-p} = \begin{cases}\Sigma^{kp}E_* & \text{for } n \text{ even, } k \text{ even}\\ 0 & \text{for } n \text{ even, } k \text{ odd}\\ \Sigma^{kp}E_* & \text{for } n \text{ odd, } k \text{ even}\\ 0 & \text{for } n \text{ odd, } k \text{ odd}\end{cases}$$

The modules $\mathrm{E}^2_{s,t}(p)$ in our spectral sequence vanish for all other values of s and t.

We will now compute the E^∞-page of the spectral sequence and solve extension problems.

Case 1: n even, k odd. The spectral sequence is concentrated on the $(s=0)$-line.

Case 2: n odd, $k>0$ even. The spectral sequence is concentrated at $(s=0)$ and $(s=p-1)$, so $E^2_{s,t}(p)\cong E^3_{s,t}(p)\cong\ldots\cong E^{p-1}_{s,t}(p)$. As before, the next differential d_{p-1} is again nonzero. Indeed, let $\mathrm{E}^r_{s,t}(p,n,k)$ denote the spectral sequence associated to associated to the pair (n,k).

For $n = 1$, the James splitting shows that $E_*^\wedge(\Omega S^{k+1})$ is torsion-free. Hence the generator $[y_1]$ of $\mathrm{E}^2_{0,*}(p,1,n) = \Sigma^{k-1}E^{-*}(B\Sigma_p)/(\mathrm{tr}, e) = \Sigma^{k-1}E_*/p$ must lie in the image of d^{p-1}.

For higher n, we obtain a map of spectral sequences $\phi : \mathrm{E}^r_{s,t}(p,1,k) \to \mathrm{E}^r_{s,t}(p,n,k)$ from the weight p sequence attached to $(1,k)$ to the sequence attached to (n,k). For E^1, this map is determined by applying the Hecke Chevalley–Eilenberg complex to the map

$$\mathfrak{h}^{(1)}_{k-1,1} = \mathfrak{h}(\mathbb{R}^1, S^k) \xrightarrow{\ \phi\ } \mathfrak{g}(\mathbb{R}^n, S^k) = \mathfrak{h}^{(1)}_{k-1,n}.$$

We can describe this map explicitly. Writing $x_1 \in \mathfrak{g}(\mathbb{R}^1, S^k)_{k-1}$ and $x_n \in \mathfrak{g}(\mathbb{R}^n, S^k)_{k-1}$ for the canonical generators, we note that $\phi(x_1) = x_2$. More generally, if $x_{1,\alpha} \in \mathfrak{g}(\mathbb{R}^1, S^k)_i = (\mathfrak{h}^{(1)}_{k-1,n})_i$ corresponds to some class $\alpha \in (\mathcal{H}^{\mathrm{Lie}}_u)^{1+i}_{1+(k-1)}(p)$, then $\phi(x_{1,\alpha}) = x_{2,\mathrm{Susp}^{n-1}(\alpha)}$ corresponds to $\mathrm{Susp}^{n-1}(\alpha) \in (\mathcal{H}^{\mathrm{Lie}}_u)^{n+i}_{n+(k-1)}(p)$. This shows that the induced map on the $(s = 0)$–line of E^2-pages is given by the natural map

$$\mathrm{E}^2_{0,*}(p,1,k) = \Sigma^{k-1}E^{-*}(B\Sigma_p)/(\mathrm{tr}, e^1) \xrightarrow{\ e^{\frac{n}{2}}\ } \Sigma^{k-1}E^{-*}(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n+1}{2}}) = \mathrm{E}^2_{0,*}(p,n,k)$$

Since $\mathrm{E}^{p-1}_{0,*}(p,1,k) \cong \mathrm{E}^2_{0,*}(p,1,k)$ is killed by d^{p-1} and the map $\mathrm{E}^{p-1}_{0,*}(p,1,k) \to \mathrm{E}^{p-1}_{0,*}(p,n,k)$ respects the differentials, we deduce that $\mathrm{E}^p_{0,*}(p,n,k)$ is given by $\Sigma^{k-1}E^{-*}(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n-1}{2}})$. The kernel of d^{p-1} on the summand $\langle \gamma_p([x]) \rangle$ is spanned by $[x]^p$. As there is no space for further differentials and all extension problems are trivial, this finishes the proof for n odd, $k > 0$ even. We then extend to all n and k using the Serre fibration [Ser51] and complex orientability of E-theory following the same strategy as in the proof of Theorem 6.1.

□

6.4. **A height 2 example.** We end by emphasising that the above formulas are explicit and workable in examples beyond p-adic K-theory. To this end, consider the height 2 Morava E-theory at $p = 3$ investigated by Yifei Zhu [Zhu14] arising from the moduli of elliptic curves with a choice of a point of exact order 4.

This cohomology theory E has coefficient ring $E_* = \mathbb{Z}_9[[h]][u^\pm]$, where $\mathbb{Z}_9 = W(\mathbb{F}_9)$, h is a class in degree 0, and u is a class in degree 2. Zhu calculates [Zhu14, Corollary 2.6] the ring $E^0(B\Sigma_3)/\mathrm{tr}$ to be given by

$$\mathbb{Z}_9[[h]][\alpha]/(\alpha^4 - 6\alpha^2 + (h-9)\alpha - 3).$$

The element α is closely related to the Euler class e. Rather than spelling out this relationship in detail, we simply note that there is an isomorphism of E_0-modules

$$E_0[e]/(f(e), e^n) \cong E_0[\alpha]/(\alpha^4 - 6\alpha^2 + (h-9)\alpha - 3, \alpha^n),$$

which can be constructed by observing that both sides compute the cokernel of the dual of the $2n$-fold suspension map

$$\Gamma^0(p) \longrightarrow \Gamma^{-2n}(p).$$

Indeed, Propositions 4.1 and 4.3 imply that $E_0[e]/(f(e), e^n)$ computes this cokernel, whereas one can read off the formula for the double suspension in terms of α from Zhu's Cartan formula [Zhu14, Proposition 3.6(v)] applied to [Zhu14, Example 3.7].

We suspect it will often be the case, when working with explicit E-theories of algebro-geometric origin, that the double suspension operation is presented more easily in a basis for

$E^0(B\Sigma_p)/(\mathrm{tr})$ other than that given by the Euler class e. In any case, it is the cokernel of the double suspension operation that we are really after. For example, we have the following calculations.

- Consider the free $\mathbb{E}_1$-E-algebra on a generator x in degree 0. The weight 3 component of this algebra is a free E_*-module, generated by x^3. This is reflected in the calculation
$$E_*^\wedge(B_3(\mathbb{R}^1)) \quad\cong\quad E_* \oplus \Sigma^{-1}E_*[\alpha]/(\alpha^4 - 6\alpha^2 + (h-9)\alpha - 3, \alpha^0) \cong E_*$$
which is of course clear since $B_3(\mathbb{R}^1)$ is contractible.
- On the other hand, we calculate
$$E_*^\wedge(B_3(\mathbb{R}^3)) \quad\cong\quad E_* \oplus \Sigma^{-1}E_*[\alpha]/(\alpha^4 - 6\alpha^2 + (h-9)\alpha - 3, \alpha^1) \cong E_* \oplus \Sigma^{-1}E_*/3$$
We can thus see that, in the free $\mathbb{E}_3$-E-algebra on a generator x in degree 0, there is more than the E_*-multiples of x^p in weight 3. There is additionally a copy of $\Sigma^{-1}E_*/3$ that constitutes a kind of Dyer-Lashof operation on x.
- We calculate
$$\begin{aligned} E_*^\wedge(B_3(\mathbb{R}^5)) \quad&\cong\quad E_* \oplus \Sigma^{-1}E_*[\alpha]/(\alpha^4 - 6\alpha^2 + (h-9)\alpha - 3, \alpha^2) \\ &\cong\quad E_* \oplus \Sigma^{-1}E_*\langle 1, \alpha\rangle/\langle 3 - (h-9)\alpha, 3\alpha\rangle \end{aligned}$$
In an $\mathbb{E}_5$-algebra, the weight 3 Dyer-Lashof operations on x now consist of an E_*-module on two generators with two relations.
- We can easily continue with such calculations; for example
$$E_*^\wedge(B_3(\mathbb{R}^{11})) \quad\cong\quad E_* \oplus \Sigma^{-1}E_*[\alpha]/(\alpha^4 - 6\alpha^2 + (h-9)\alpha - 3, \alpha^5)$$
We obtain the weight 3 Dyer-Lashof operations in the free $\mathbb{E}_{11}$-E-algebra on a degree 0 class.

6.5. **Morava K-theory.** One of the main advantages of our approach to iterated loop spaces of spheres, in contrast with the approaches of Yamaguchi [Yam88] (for 2-fold loop spaces) and Tamaki [Tam02] (for 3-fold and 4-fold loop spaces), is that we do not require a Künneth formula, and that our computation does not increase in difficulty with the number n of loops. Thus, we are able to compute Morava E-theory for all n simultaneously, rather than computing Morava K-theory by induction on n. Nevertheless, since E-theory is a refinement of K-theory, it is to be expected that our E-theory calculations yield Morava K-theory calculations as byproducts.

In this section we will explain how to conclude facts about Morava K-theory from our work, which may be of interest as it settles the computational challenge raised by Ravenel's conjecture [Rav98, Conjecture 3] at weight p. As usual, the height one situation provides valuable intuition:

Example 6.15. For $K = K_p^\wedge$ the p-adic complex K-theory spectrum, we have computed that, for $n > 0$ even and k odd, there is an isomorphism
$$K_*^\wedge\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right) \quad\cong\quad \Sigma^{k-1}K_*/p^{\frac{n}{2}}.$$
Since $\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}$ is a finite spectrum, we in fact did not have to $K(1)$-localise; hence
$$K_*\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right) \quad\cong\quad \Sigma^{k-1}K_*/p^{\frac{n}{2}}.$$
This implies that there exists a cofibre sequence of K-module spectra
$$\Sigma^{k-1}K \xrightarrow{p^{\frac{n}{2}}} \Sigma^{k-1}K \longrightarrow K \otimes \Sigma^\infty\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right).$$

Notation 6.16. The Morava K-theory spectrum $K(1)$ is defined as the mod p K-theory spectrum K/p . Note in particular our convention that $K(1)$ is a 2-periodic theory, rather than the $(2p-2)$-periodic Adams summand.

The map $p^{\frac{n}{2}}$ becomes 0 after tensoring down to K/p, and so we obtain a cofibre sequence

$$\Sigma^{k-1}K(1) \xrightarrow{0} \Sigma^{k-1}K(1) \longrightarrow K(1)\otimes\left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right).$$

We have therefore established the following special case of [Lan93, Theorem 4a]:

Corollary 6.17. *For n even and p odd, there are equivalences*

$$K(1)\otimes\left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right) \simeq \Sigma^{k-1}K(1)\vee\Sigma^k K(1) \simeq K(1)\vee \Sigma K(1).$$

Remark 6.18. The $K(1)$-homology contains far less information that the p-complete K-theory. In particular, it is unable to distinguish the spectra $\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}$ for different even n.

We now turn to general heights h, as well as general parities of n and k:

Theorem 6.19. *Let $K(h)$ be the 2-periodic Morava K-theory associated to a Morava E-theory of height h over a perfect field k of characteristic $p>2$.*

The $K(h)$-module $K(h)\otimes\left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)$ is equivalent to

$$\begin{cases} \left(\Sigma^{k-1}K(h)\oplus\Sigma^k K(h)\right)^{\oplus \min\left(\frac{p^h-1}{p-1},\frac{n}{2}-1\right)}\oplus\Sigma^{kp}K(h)\oplus\Sigma^{pk+n-1}K(h) & \text{for } n \text{ even, } k \text{ even}\\ \left(\Sigma^{k-1}K(h)\oplus\Sigma^k K(h)\right)^{\oplus \min\left(\frac{p^h-1}{p-1},\frac{n}{2}\right)} & \text{for } n \text{ even, } k \text{ odd}\\ \left(\Sigma^{k-1}K(h)\oplus\Sigma^k K(h)\right)^{\oplus \min\left(\frac{p^h-1}{p-1},\frac{n-1}{2}\right)}\oplus\Sigma^{kp}K(h) & \text{for } n \text{ odd, } k \text{ even}\\ \left(\Sigma^{k-1}K(h)\oplus\Sigma^k K(h)\right)^{\oplus \min\left(\frac{p^h-1}{p-1},\frac{n-1}{2}\right)}\oplus\Sigma^{k+(2k+n-1)(\frac{p-1}{2})}K(h) & \text{for } n \text{ odd, } k \text{ odd}\end{cases}$$

Proof. Let E denote the form of Morava E-theory in question. Our strategy will be to first understand the E-module spectrum $E\otimes\left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)$, and then tensor down to $K(h)$.

Choosing a p-typical coordinate, we may, as in Proposition 4.3, write $E^*(B\Sigma_p)/(\mathrm{tr}) \cong E_*[e]/f(e)$ for a monic polynomial $f(e)=e^{\frac{p^h-1}{p-1}}+\cdots+p$ over E_0 with $f(-e^{p-1})=\frac{[p](e)}{e}$ in $E^*(B\Sigma_p)$.

Given $m\geq 0$, let A_m be the free E-module of rank m with homotopy groups $\pi_*(A_m)\cong E_*[e]/e^m$. Write $\phi_m : A_m\to A_m$ for the unique E-module map such that $\pi_*(\phi_m): E_*[e]/e^m \to E_*[e]/e^m$ is given by multiplication by $f(e)$. Note that $\pi_*(\phi_m)$ is injective, as $f(e)$ having constant term p implies that the corresponding matrix has non-vanishing determinant. Letting C_m denote the cofibre of ϕ_m, it follows that $\pi_*(C_m)\cong E_*[e]/(f(e),e^m)$.

By Theorem 6.10, the E-module spectrum $E\otimes\left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)$ is given by

$$\begin{cases} \Sigma^{kp}E\oplus\Sigma^{pk+n-1}E\oplus\Sigma^{k-1}C_{\frac{n}{2}-1} & \text{for } n \text{ even, } k \text{ even}\\ \Sigma^{k-1}C_{\frac{n}{2}} & \text{for } n \text{ even, } k \text{ odd}\\ \Sigma^{kp}E\oplus\Sigma^{k-1}C_{\frac{n-1}{2}} & \text{for } n \text{ odd, } k \text{ even}\\ \Sigma^{k+(2k+n-1)(\frac{p-1}{2})}E\oplus\Sigma^{k-1}C_{\frac{n-1}{2}} & \text{for } n \text{ odd, } k \text{ odd}\end{cases}$$

Indeed, consider for example the case where n and k are both even. We have calculated that

$$E^\wedge_*\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)\cong \Sigma^{kp}E_*\oplus\Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr},e^{\frac{n}{2}-1})\oplus\Sigma^{pk+n-1}E_*.$$

Since $\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)$ is a finite CW-complex, the left hand side of the above is just

$$\pi_*\left(E\otimes\left(\mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)\right).$$

There is a quotient map of E_*-modules

$$\Sigma^{kp}E_*\oplus\Sigma^{k-1}E^*(B\Sigma_p)/(e^{\frac{n}{2}-1})\oplus\Sigma^{pk+n-1}E_* \to \Sigma^{kp}E_*\oplus\Sigma^{k-1}E^*(B\Sigma_p)/(\mathrm{tr}, e^{\frac{n}{2}-1})\oplus\Sigma^{pk+n-1}E_*,$$

Since the domain is a free E_*-module this map can be refined to a map of E-module spectra

$$\Sigma^{kp}E \oplus \Sigma^{k-1}A_{\frac{n}{2}-1} \oplus \Sigma^{pk+n-1}E \to E \otimes \left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right).$$

The following composition is zero on the level of homotopy groups:

$$\Sigma^{k-1}A_{\frac{n}{2}-1} \xrightarrow{\Sigma^{k-1}\phi_{\frac{n}{2}-1}} \Sigma^{k-1}A_{\frac{n}{2}-1} \longrightarrow E \otimes \left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)$$

Hence, it vanishes as a map of E-modules since the domain is free. We see that the map $\Sigma^{k-1}A_{\frac{n}{2}-1} \to E \otimes \left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)$ extends over $\Sigma^{k-1}C_{\frac{n}{2}-1}$. The resulting map

$$\Sigma^{kp}E \oplus \Sigma^{k-1}C_{\frac{n}{2}-1} \oplus \Sigma^{pk+n-1}E \longrightarrow E \otimes \left(\Sigma^\infty \mathrm{Conf}_p(\mathbb{R}^n)_+ \otimes_{\Sigma_p} (S^k)^{\otimes p}\right)$$

induces an isomorphism on homotopy groups.

To finish the proof, it remains to compute $C_m \otimes_E K(h)$ for a general m, which can be done using the defining cofibre sequence $A_m \xrightarrow{\phi_m} A_m \longrightarrow C_m$. Indeed, note that by Proposition 4.3

$$f(-e^{p-1}) = u\frac{[p](e)}{e} \equiv u'e^{p^h-1} \bmod \mathfrak{m},$$

where u, u' are units and $\mathfrak{m} = (p, u_1, \ldots, u_{m-1})$ is the maximal ideal in E_0. Modulo $\mathfrak{m}$, $f(e)$ is a unit times $e^{\frac{p^h-1}{p-1}}$, and we obtain an equivalence $C_m \otimes_E K(h) \simeq \bigoplus_{\min(\frac{p^h-1}{p-1}, m)} K(h) \vee \Sigma K(h)$. □

As $h \to \infty$, we recover a classical result of Cohen:

Corollary 6.20. *The $\mathbb{F}_p$-vector space $H_*\left(\mathrm{Conf}_p(\mathbb{R}^{2m})_+ \otimes_{\Sigma_p} (S^1)^{\otimes p}; \mathbb{F}_p\right)$ has dimension $2m$.*

Proof. Let $K(h)$ be a Morava K-theory with $K(h)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$. For $h \gg 0$ very large, the Atiyah–Hirzebruch spectral sequence of the finite space $(\mathrm{Conf}_p(\mathbb{R}^{2m})_+ \otimes_{\Sigma_p} (S^1)^{\otimes p}$ degenerates, which leads to an isomorphism of $K(h)$-modules

$$K(h)_*\left(\mathrm{Conf}_p(\mathbb{R}^{2m})_+ \otimes_{\Sigma_p} (S^1)^{\otimes p}\right) \cong H_*\left(\mathrm{Conf}_p(\mathbb{R}^{2m})_+ \otimes_{\Sigma_p} (S^1)^{\otimes p}; \mathbb{F}_p\right)[\beta^{\pm 1}].$$

□

Remark 6.21. This result agrees with the work of Cohen [CLM76, III], which implies that $H_*\left(\mathrm{Conf}_p(\mathbb{R}^{2m})_+ \otimes_{\Sigma_p} (S^1)^{\otimes p}; \mathbb{F}_p\right)$ has a basis given by $\{\beta^\epsilon Q^s \mid 1 \le s \le m, \epsilon \in \{0,1\}\}$. Cohen's result is stronger, as it also gives the dimensions of individual homology groups, while our method only sees the even and odd degree, respectively. We believe that this deficiency can be removed by studying $2(p^h-1)$-periodic versions of $K(h)$ using the prime-to-p subgroup of the Morava stabiliser group, which acts on our spectral sequence. We will not pursue this approach here.

We briefly compare this with several previously known results mentioned in the introduction.

Height one. Langsetmo computed the $K(1)$-homology of iterated loop spaces [Lan93, Lan96b], utilising the work of Mahowald-Thompson [MT92] and McClure [BMMS86, Chapter 9].

Write $T^a(x_1, x_2, \ldots)$ for the truncated polynomial algebra $\mathbb{F}_p[x_1, x_2, \ldots]/(x_1^a, x_2^a, \ldots)$. Then:

Theorem 6.22 (Langsetmo)**.** *For $h = 1$ and p odd, there are isomorphisms*

$$K(1)_*(\ \Omega^{2m}\ S^{2m+1}) \ \cong K(1)_* \otimes \Lambda(u_i \mid i \le m) \otimes T^{p^m}(x_i \mid i \ge 1)$$

The u_i have odd and the x_i even degrees, respectively. Elements with index a live in weight p^a.

For all integers $m \geq 1$, our Theorem 6.19 gives an isomorphism between the $K(1)$-homology of the p^{th} Snaith summand of $\Omega^{2m}S^{2m+1}$ and $K(1)_* \oplus \Sigma K(1)_*$ These two copies correspond precisely to the classes u_1 and x_1 in Langsetmo's result.

Double loop spaces. The Morava K-theory of double loop spaces was computed at all heights by Yamaguchi (cf. [Yam88]), using the Atiyah–Hirzebruch spectral sequence:

Theorem 6.23 (Yamaguchi)**.** *Let p be an odd prime. For all heights $h \geq 0$, the Atiyah-Hirzebruch spectral sequence collapses and gives rise to an isomorphism*

$$K(h)_*(\Omega^2 S^{2m+1}) \cong K(h)_* \otimes \Lambda(u_a \mid a \leq h) \otimes T^{p^h}(x_i \mid i \geq 1).$$

The u_i have odd and the x_i even degrees, respectively. Elements with index e live in weight p^e.

Theorem 6.19 asserts an isomorphism between the $K(h)$-homology of the p^{th} Snaith summand of $\Omega^2 S^{2r+1}$ and $K(h)_* \oplus \Sigma K(h)_*$; these two copies of $K(h)_*$ correspond to y_1 and x_0, respectively.

Triple loop spaces. The case of triple loop spaces is more delicate, and the problem has an interesting history in this case. For some time, it was believed that the Atiyah–Hirzebruch spectral sequence would degenerate (cf. [May88]), which led to a faulty computation and a surpring, yet incorrect, counterexample (cf. [MT89]) to a conjecture of Miller–Snaith [MS79] and Mahowald–Ravenel [MR87] (which later became a theorem of Bousfield in [Bou94, Theorem 14.8]).

This state of affairs was resolved by Tamaki, who used a version of the Eilenberg–Moore spectral sequence [Tam94] to climb from double to triple loop spaces (cf. [Tam02, Proposition 6.1]):

Theorem 6.24 (Tamaki)**.** *If p is an odd prime, then the Eilenberg–Moore spectral sequence degenerates and gives rise to an isomorphism*

$$K(h)_*(\Omega^3 S^{2m+1}) \cong K(h)_* \otimes P(u_a \mid a \leq h) \otimes \Lambda(x_{i,j} \mid i+j \leq h, i > 0) \otimes \bigotimes_{1 \leq j \leq h-1} T^{p^{h-j}}(y_{i,j} \mid i > 0)$$

where u_i as Snaith weight p^i and the elements $x_{i,j}$ and $y_{i,j}$ have weight p^{i+j}.

Our Theorem 6.19 shows that the $K(h)$-homology of the p^{th} Snaith summand in $\Omega^3 S^{2m+1}$ $K(h)_* \oplus \Sigma K(h)_* \oplus \Sigma K(h)$; this corresponds to the classes u_0^p, u_1, and $x_{1,0}$ in Snaith weight p.

A remark on a conjecture of Ravenel. We will briefly comment on Ravenel's [Rav98, Conjecture 3], which concerns the $K(h)$-homology of iterated loop spaces of spheres. Ravenels conjecture suggests that for a fixed number of loops n, the weight p part of $K(h)_*(\Omega^n S^{n+1})$ does not change in size as we vary the chromatic height h.

This is true whenever $n \leq 3$, as in this case, the multiplicities $\min(\frac{p^h-1}{p-1}, \frac{n}{2})$ (for n even) and $\min(\frac{p^h-1}{p-1}, \frac{n-1}{2})$ (for n odd) appearing in Theorem 6.19 are both equal to 1 for all heights $h \geq 1$.

However, this pattern does not persist for general n, where Ravenel's conjecture does not seem to be true. Indeed, Theorem 6.19 shows that for $n \geq 4$, the size of $K(h)_*(\Omega^n S^{n+1})(p)$ depends on h. This can in fact already be observed from Langsetmo's Theorem 6.22.

7. Configurations of p points in surfaces

In this section, we examine the E-theory and $\mathbb{F}_p$-homology of unordered configuration spaces of points on the once-punctured orientable surface $\mathcal{S}_{g,1}$ of genus $g \geq 0$, a framed manifold.

7.1. **The E-theory of $B_p(\mathcal{S}_{g,1})$.** Let E be an E-theory of height h over a perfect field k of characteristic $p > 2$. We compute the E-theory of $B_p(\mathcal{S}_{g,1}) = \mathrm{Conf}_p(\mathcal{S}_{g,1})/\Sigma_p$ at all heights:

Theorem 7.1 (E-theory, surface case)**.**

(1) The E-cohomology of the unordered configuration space of p points in the punctured torus is

$$E^*(B_p(\dot{T}))) \cong \left(\bigoplus_{0 \leq i < p} \Sigma^i E_*^{\oplus \lfloor \frac{3i+2}{2} \rfloor} \right) \oplus \Sigma^p E_*^{\oplus (p+1)}.$$

(2) More generally, the E-cohomology of the unordered configuration space of p points in the punctured orientable genus g surface $\mathcal{S}_{g,1}$ is given by

$$E^*(B_p(\mathcal{S}_{g,1})) \cong \bigoplus_{0 \leq i \leq p} \Sigma^i E_*^{\oplus \beta_i}.$$

For $0 < p$, we have

$$\beta_i = \sum_{\substack{0 \leq j \leq g \\ j \equiv i \bmod 2}} \left(\binom{2g}{j} - \binom{2g}{j-2} \right) \binom{2g + \frac{i-j}{2} - 1}{2g-1} + \sum_{\substack{g+1 \leq j \leq 2g+1 \\ i \equiv j \bmod 2}} \left(\binom{2g}{j-1} - \binom{2g}{j+1} \right) \binom{2g + \frac{i-j}{2} - 1}{2g-1}$$

For $i = p$, we have

$$\beta_p = \sum_{\substack{0 \leq j \leq g \\ j \equiv i \bmod 2}} \left(\binom{2g}{j} - \binom{2g}{j-2} \right) \binom{2g + \frac{i-j}{2} - 1}{2g-1}.$$

To apply Theorem 5.7, we determine the Hecke Lie algebra $\mathfrak{g}(\mathcal{S}_{g,1}, S^0) = E_*^{\wedge}(\mathrm{Free}_{\mathrm{Lie}}(S^{-1})^{\mathcal{S}_{g,1}})$. In the height one case, this Hecke Lie algebra is especially simple:

Lemma 7.2. *At height 1, the underlying K_*-module of $\mathfrak{g}(\mathcal{S}_{g,1}; S^0)$ is free on $8g+4$ generators:*

$i \backslash weight$	1	2	p	$2p$
-2	0	0	cy	0
-1	cx	0	$a_1y, \dots, a_gy$ $b_1y, \dots, b_gy$	$c\widetilde{y}$
0	$a_1x, \dots, a_gx$ $b_1x, \dots, b_gx$	$c\widetilde{x}$	0	$a_1\widetilde{y}, \dots, a_g\widetilde{y}$ $b_1\widetilde{y}, \dots, b_g\widetilde{y}$
1	0	$a_1\widetilde{x}, \dots, a_g\widetilde{x}$ $b_1\widetilde{x}, \dots, b_g\widetilde{x}$	0	0

Here i denotes the internal degree. The Hecke module structure is determined by the equations

$$\alpha(ex) = \begin{cases} ey & \text{for } e = a_1, \dots a_g, b_1, \dots, b_g \\ pey & \text{for } e = c \end{cases}$$
$$\alpha(e\widetilde{x}) = pe\widetilde{y}, \qquad \text{for } e = a_1, \dots, a_g, b_1, \dots, b_g, c.$$

All other Hecke operations vanish. The only nonzero components of the Lie bracket are given by $[a_ix, b_ix] = -c\widetilde{x}$ for $i = 1, \dots, g$.

Proof. We recall three well-known facts about $\mathcal{S}_{g,1}$, the closed oriented surface of genus g:

(1) The cohomology $\widetilde{E}^{-*}(\mathcal{S}_{g,1})$ is free on generators $a_1,\dots,a_g,\ b_1,\dots,b_g$ (in homological degree -1) and c (in homological degree -2).
(2) We have $a_ib_i=-b_ia_i=c$. The remaining products of the generators vanish.
(3) The standard cell decomposition of $\mathcal{S}_{g,1}$ is stably split.

By Corollary 6.5, the free Hecke Lie algebra $\mathbf{L}^{\mathcal{H}_u}(K_*(S^1))$ on a class in degree 1 is generated as K_*-module by four classes

$$x\in\mathbf{L}^{\mathcal{H}_u}(K_*(S^1))_1\ ,\quad y\in\mathbf{L}^{\mathcal{H}_u}(K_*(S^1))_0\ ,\quad \widetilde{x}\in\mathbf{L}^{\mathcal{H}_u}(K_*(S^1))_2\ ,\quad \widetilde{y}\in\mathbf{L}^{\mathcal{H}_u}(K_*(S^1))_1$$

The Lie bracket satisfies $[x,x]=\widetilde{x}$ and vanishes otherwise; the Hecke operations satisfy

$$\alpha(x)=y\ ,\quad \alpha(\widetilde{x})=\widetilde{y}\ ,\quad \alpha(\widetilde{x})=\alpha\widetilde{y}=0.$$

Combining facts (1) and (2) above with Proposition 5.9, we see that the underlying E_*-module of $\mathfrak{g}(\dot{T};S^0)$ is generated by the $8g+4$ indicated elements, and that the Lie bracket behaves as claimed. The third fact allows us to apply Proposition 4.24 and Lemma 5.11 to determine the structure of $\mathfrak{g}(\dot{T};S^0)$ as a Hecke module. □

We immediately generalise this claim to higher heights:

Lemma 7.3. *The underlying E_*-module of $\mathfrak{g}(\mathcal{S}_{g,1};S^0)$ is given by*

$$E_*\langle a_1,\dots,a_g,b_1,\dots,b_g,c\rangle\otimes_{E_*}\left(\bigoplus_{0\le\ell\le h}(\mathcal{H}_u^{\mathrm{Lie}})^*_1(p^\ell)\oplus\bigoplus_{0\le\ell\le h}(\mathcal{H}_u^{\mathrm{Lie}})^*_2(p^\ell)\right)$$

Write elements corresponding to $\phi\in(\mathcal{H}_u^{\mathrm{Lie}})^*_1(p^\ell)$ or $\phi\in(\mathcal{H}_u^{\mathrm{Lie}})^*_2(p^\ell)$ as x_ϕ or $\widetilde{x}_\phi$, respectively. Elements $e\otimes x_\phi$ with $e\in\{a_1,\dots,a_g,b_1,\dots b_g,c\}$ and $\phi\in(\mathcal{H}_u^{\mathrm{Lie}})^*_1(p^\ell)$ have weight p^ℓ. Elements $e\otimes\widetilde{x}_\phi$ with $e\in\{a_1,\dots,a_g,b_1,\dots b_g,c\}$ and $\phi\in(\mathcal{H}_u^{\mathrm{Lie}})^*_2(p^\ell)$ have weight $2p^\ell$.

If $\phi\in(\mathcal{H}_u^{\mathrm{Lie}})^*_1$, then

$$\alpha(e\otimes x_\phi)=\begin{cases}e\otimes x_{\mathrm{Susp}(\alpha)\cdot\phi} & \text{for } e=a_1,\dots a_g,b_1,\dots,b_g \text{ and } \alpha\in(\mathcal{H}_u^{\mathrm{Lie}})^j_{*-1}\\ e\otimes x_{\mathrm{Susp}^2(\alpha)\cdot\phi} & \text{for } e=c \text{ and } \alpha\in(\mathcal{H}_u^{\mathrm{Lie}})^j_{*-2}\end{cases}$$

If $\phi\in(\mathcal{H}_u^{\mathrm{Lie}})^*_2$, then

$$\alpha(e\otimes \widetilde{x}_\phi)=\begin{cases}e\otimes \widetilde{x}_{\mathrm{Susp}(\alpha)\cdot\phi} & \text{for } e=a_1,\dots a_g,b_1,\dots,b_g \text{ and } \alpha\in(\mathcal{H}_u^{\mathrm{Lie}})^j_{*-1}\\ e\otimes \widetilde{x}_{\mathrm{Susp}^2(\alpha)\cdot\phi} & \text{for } e=c \text{ and } \alpha\in(\mathcal{H}_u^{\mathrm{Lie}})^j_{*-2}\end{cases}$$

The Lie bracket satisfies $[a_i\otimes x_\lambda,b_i\otimes x_\mu]=-c\otimes\widetilde{x}_{\lambda\mu}$ for all scalars $\lambda,\mu\in E_*\cong(\mathcal{H}_u^{\mathrm{Lie}})^*_1(1)$ and all $i=1,\dots g$. It vanishes otherwise.

Proof. We proceed as in the proof of Lemma 7.2: the structure of $\mathfrak{g}(\mathcal{S}_{g,1};S^0)$ as an E_*-Lie algebra is implied by Corollary 6.14 together with the well-known ring structure of $E^*(\mathcal{S}_{g,1})$. Moreover, Lemma 5.11 determines the structure of $\mathfrak{g}(\mathcal{S}_{g,1};S^0)$ as a Hecke module. □

With this description of $\mathfrak{g}(\mathcal{S}_{g,1};S^0)$ at hand, we can now prove the main result of this section.

Proof of 7.1. Recall that Theorem 5.7 gives a cohomological spectral sequence with signature

$$\mathrm{E}_2^{s,t} \cong H^{s+1}\left(\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(\mathcal{S}_{g,1};S^0)\right)^\vee\right)_{t+1} \implies \bigoplus_k E^{t-s}(B_k(\mathcal{S}_{g,1};S^0)).$$

In the interest of readability, we present the argument at height $h=1$; we will then indicate how it generalises to higher heights in a second step.

Recall the description of $\mathfrak{g}(\mathcal{S}_{g,1};S^0)$ given in Lemma 7.2. We depict the additive resolution $\mathrm{AR}(\mathfrak{g}(\mathcal{S}_{g,1};S^0))$ of the relevant Hecke Lie algebra up to weight p:

$r\backslash weight$	1	2	p	$\cdots$
0	$[cx]_{(-1,1,0,1)}$ $[a_ix]_{(0,1,0,1)}$ $[b_ix]_{(0,1,0,1)}$	$[c\widetilde{x}]_{(0,1,0,2)}$ $[a_i\widetilde{x}]_{(1,1,0,2)}$ $[b_i\widetilde{x}]_{(1,1,0,2)}$	$[cy]_{(-2,1,0,p)}$ $[a_iy]_{(-1,1,0,p)}$ $[b_iy]_{(-1,1,0,p)}$	$\cdots$
1	$[1\vert cx]_{(-1,1,1,1)}$ $[1\vert a_ix]_{(0,1,1,1)}$ $[1\vert b_ix]_{(0,1,1,1)}$	$[1\vert c\widetilde{x}]_{(0,1,1,2)}$ $[1\vert a_i\widetilde{x}]_{(1,1,1,2)}$ $[1\vert b_i\widetilde{x}]_{(1,1,1,2)}$	$[1\vert cy]_{(-2,1,1,p)}$ $[1\vert a_iy]_{(-1,1,1,p)}$ $[1\vert b_iy]_{(-1,1,1,p)}$ $[\alpha\vert cx]_{(-2,1,1,p)}$ $[\alpha\vert a_ix]_{(-1,1,1,p)}$ $[\alpha\vert b_ix]_{(-1,1,1,p)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Thus, basis elements of weight p in $\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(\mathcal{S}_{g,1};S^0)\right)$ fall into two different classes:

(1) weight p elements in the additive resolution (these belong to $\Gamma^1_{K_*}\left(\mathrm{AR}(\mathfrak{g}(\mathcal{S}_{g,1};S^0))\right)$)
(2) products or divided powers of elements in weight 1 or 2 in the additive resolution (these belong to $\Gamma^{>1}_{K_*}\left(\mathrm{AR}(\mathfrak{g}(\mathcal{S}_{g,1};S^0))\right)$).

The simplicial structure maps and the differential send elements in each class to linear combinations of elements in the same class. This gives a decomposition of simplicial chain complexes

$$\mathrm{CE}_{\mathcal{H}_u}\left(\mathfrak{g}(\mathcal{S}_{g,1};S^0)\right)(p) = \mathrm{CE}_1(p) \oplus \mathrm{CE}_2(p).$$

The explicit description of the Hecke action on $\mathfrak{g}(\mathcal{S}_{g,1};S^0)$ in Lemma 7.2 shows that the normalised chain complex of $\mathrm{CE}_1(p)$ is equivalent to $\ldots \to 0 \to [\alpha|cx] \xrightarrow{p} [cy] \to 0$. Hence, we have

$$H^{s+1}(\mathrm{CE}_1(p)^\vee)_{*+1} = \begin{cases} \Sigma K_*/p & \text{if } s=1 \\ 0 & \text{else} \end{cases}.$$

The second summand $\mathrm{CE}_2(p)$ can be understood by following the rational computations in [BC88], [Knu17, Section 6.2], and [DCK17, Section 3.2]. More precisely, consider the ordinary Lie algebra $\mathfrak{h}$ in $\mathrm{Mod}^{\mathbb{N}}_{K_*}$ with underlying module generated by $2g+1$ classes $a_1x,\ldots,a_gx,b_1x,\ldots,b_gx$ (in degree 0 and weight 1) and an additional class $c\widetilde{x}$ (in degree 0 and weight 2). The only non-vanishing Lie brackets of generators are given by $[a_ix,b_ix]=-c\widetilde{x}$ for $1\leq i\leq g$ Observe that $\mathrm{CE}_2(p)$ is isomorphic to the weight p component of the Chevalley–Eilenberg complex of

$$\mathfrak{h} \oplus \mathrm{T}(K_*\langle a_i\widetilde{x}, b_i\widetilde{x}, cx \mid 1\leq i\leq g\rangle),$$

where the right summand denotes a trivial Lie algebra on $2g+1$ classes $a_1\widetilde{x},\ldots,a_g\widetilde{x},b_1\widetilde{x},\ldots,b_g\widetilde{x}$ (in degree 1 and weight 2) and an additional class cx (in degree -1 and weight 1).

The Lie algebra *co*homology of $\mathrm{T}(K_*\langle a_i\widetilde{x}, b_i\widetilde{x}, cx \mid 1 \leq i \leq g\rangle)$ is therefore given by a polynomial algebra $P_{K_*}(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_g, \gamma)$, where the classes $\widetilde{\alpha}_i, \widetilde{\beta}_i$ sit in degree -2 and weight 2, and γ lives in degree 0 and weight 1. Here we have used that dualising divided powers of free K_*-modules gives symmetric powers.

Since $\mathfrak{h}$ sits in even degree, the Chevalley–Eilenberg complex does not contain any divided powers, and the cohomology $H^*(\mathrm{CE}(\mathfrak{h}^\vee)$ agrees with the homology of the complex

$$(\Lambda[\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \widetilde{\gamma}], d),$$

with differential satisfying $d(\widetilde{\gamma}) = -2(\alpha_1\beta_1 + \ldots + \alpha_g\beta_g)$ and vanishing otherwise. Here α_i, β_i sit in degree -1 and weight 1, and $\widetilde{\gamma}$ sits in degree -1 and weight 2. The cohomology of this complex is computed in [BC88, Theorem D], We can therefore describe the E_2-term in weight p:

$$\mathrm{E}_2^{s,*}(p) \cong \Big(H_{-s-1}(\Lambda[\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \widetilde{\gamma}], d) \otimes P_{K_*}(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_g, \gamma)\Big)(p) \oplus\ \Sigma^1 K_*[1].$$

Picking an embedding $\mathbb{R}^2 \cong D^2 \to \mathcal{S}_{g,1}$ from an open disc into the punctured surface of genus g, we obtain a map of configuration spaces $\mathrm{Conf}_p(\mathbb{R}^2) \to \mathrm{Conf}_p(\mathcal{S}_{g,1})$. Stably, this map corresponds to the map of spectral Lie algebras $\mathrm{Free}^{\mathscr{L}}(S^{-1})^{S^2} \to \mathrm{Free}^{\mathscr{L}}(S^{-1})^{\mathcal{S}_{g,1}}$ induced by the collapse map $\mathcal{S}_{g,1} \to S^2$.

We obtain maps of cohomological spectral sequences $\mathrm{E}_r^{s,t}(\mathcal{S}_{g,1}) \to \mathrm{E}_r^{s,t}(\mathbb{R}^2)$ from the spectral sequence computing $E^*(B_p(\mathcal{S}_{g,1}))$ to the corresponding sequence for $B_p(\mathbb{R}_2)$, which we have studied in great detail in the preceding section.

On the E_1-page, the map $\mathrm{E}_1^{s,t}(\mathcal{S}_{g,1}) \to \mathrm{E}_1^{s,t}(\mathbb{R}^2)$ is obtained by applying the cohomological Hecke Chevalley–Eilenberg complex to the map of Hecke Lie algebras

$$\mathfrak{g}(\mathbb{R}^2, S^0) \cong \widetilde{E}^*(S^2) \otimes_{\mathbb{E}_2} \mathbf{L}^{\mathcal{H}_u}(x_1) \longrightarrow \widetilde{E}^*(\mathcal{S}_{g,1}) \otimes_{\mathbb{E}_2} \mathbf{L}^{\mathcal{H}_u}(x_1) \cong \mathfrak{g}(\mathcal{S}_{g,1}, S^0).$$

Under this map, the classes $[x]$ and $[y]$ on the left (cf. Theorem 6.1 and its proof) are sent to the classes $[cx]$ and $[cy]$ on the right. As in the proof of Theorem 6.1, the class dual to $[x]$ kills the torsion class dual to $[y]$ in the spectral sequence $\mathrm{E}_r^{s,t}(\mathbb{R}^2)$, which implies that the class dual to $[cy]$ must have died on the p^{th} page of the spectral sequence for $\mathcal{S}_{g,1}$.

By Theorem 5.12, all nontrivial d_r-differentials with $r \geq 2$ must have torsion targets. This implies that the pages $\mathrm{E}_2^{s,t}(\mathcal{S}_{g,1}), \ldots, \mathrm{E}_{p-1}^{s,t}(\mathcal{S}_{g,1})$ have at most one torsion class $[cy]$, and that $\mathrm{E}_r^{s,t}(\mathcal{S}_{g,1})$ is torsion-free for all $r \geq p$. This implies that the spectral sequence degenerates on the E_p-page, with $\mathrm{E}_p^{s,t}(\mathcal{S}_{g,1}) \cong \mathrm{E}_\infty^{s,t}(\mathcal{S}_{g,1})$ given by

$$\Big(H_{-s-1}(\Lambda[\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \widetilde{\gamma}], d) \otimes P_{K_*}(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_g, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_g, \gamma)\Big)(p). \tag{8}$$

Since this is a free K_*-module, there are no extension problems. The claim for K-theory therefore follows by computing the dimensions in (8) explicitly, which is done in [DCK17, Section 3.2].

The argument generalises to E-cohomology by starting with Lemma 7.3 instead of Lemma 7.2. The important facts needed in the proof are:

- Up to scaling, Hecke operations of weight p shift degree down by 1 (cf. Definition 4.8)
- The suspension $(\mathcal{H}_u^{\mathrm{Lie}})_i^{i-1}(p) \to (\mathcal{H}_u^{\mathrm{Lie}})_{i-1}^{i-2}(p)$ is an isomorphism for all i even (cf. Proposition 4.24), and has cokernel E_*/p for i odd; this gives a description of $\mathrm{CE}_1(p)$ as E_*/p.
- The E-cohomology of $B_p(\mathbb{R}^2)$ is torsion-free (cf. Theorem 6.10).

□

7.2. **The $\mathbb{F}_p$-homology of $B_p(\mathcal{S}_{g,1})$.** We close this paper with an application in classical topology which, to the best of our knowledge, is new:

Proposition 7.4 ($\mathbb{F}_p$-homology, surface case)**.** *Let p be an odd prime. The $\mathbb{F}_p$-homology groups of the unordered configuration space of p points in the punctured torus satisfy*

$$H_{even}(B_p(\dot{T}));\mathbb{F}_p) = \bigoplus_i H_{2i}(B_p(\dot{T});\mathbb{F}_p) \cong \bigoplus_{0\leq i<p \ \ even} \mathbb{F}_p^{\oplus \frac{3i+2}{2}}.$$

$$H_{odd}(B_p(\dot{T}));\mathbb{F}_p) = \bigoplus_i H_{2i+1}(B_p(\dot{T});\mathbb{F}_p) \cong \left(\bigoplus_{0\leq i<p \ \ odd} \mathbb{F}_p^{\oplus \frac{3i+1}{2}}\right) \oplus \mathbb{F}_p^{\oplus(p+1)}.$$

More generally, the $\mathbb{F}_p$-homology of the unordered configuration space of p points in a punctured orientable genus g surface $\mathcal{S}_{g,1}$ satisfies:

$$H_{even}(B_p(\mathcal{S}_{g,1}));\mathbb{F}_p) := \bigoplus_i H_{2i}(B_p(\mathcal{S}_{g,1}));\mathbb{F}_p) \cong \bigoplus_{0\leq i<p \ \ even} \mathbb{F}_p^{\oplus\beta_i}.$$

$$H_{odd}(B_p(\mathcal{S}_{g,1}));\mathbb{F}_p) := \bigoplus_i H_{2i+1}(B_p(\mathcal{S}_{g,1}));\mathbb{F}_p) \cong \bigoplus_{0\leq i\leq p \ \ odd} \mathbb{F}_p^{\oplus\beta_i},$$

where the numbers β_i are defined as in Theorem 7.1.

Proof. Since $B_p(\mathcal{S}_{g,1})$ is a finite complex, we can pick $h \gg 0$ sufficiently large that the $K(h)$-based Atiyah-Hirzebruch spectral sequence degenerates; this implies

$$K(h)_*(B_p(\mathcal{S}_{g,1})) \cong H_*(B_p(\mathcal{S}_{g,1});\mathbb{F}_p)[\beta^{\pm 1}]. \tag{9}$$

Here $K(h)$ is the 2-periodic Morava K-theory attached to the E-theory E_h corresponding to the height h Honda formal group law over the field $\mathbb{F}_p$.

Writing $\mathbb{D}$ for the Spanier-Whitehead duality functor in spectra, we can also use the finiteness of $B_p(\mathcal{S}_{g,1})$ to obtain an equivalence

$$K(h)^{B_p(\mathcal{S}_{g,1})_+} \simeq K(h)\otimes \mathbb{D}(B_p(\mathcal{S}_{g,1})_+) \simeq K(h)\otimes_{E_h} (E_h \otimes \mathbb{D}(B_p(\mathcal{S}_{g,1})_+) \tag{10}$$

By Theorem 7.1, we know that the E-module $E_h \otimes \mathbb{D}(B_p(\mathcal{S}_{g,1})_+) \simeq E_h^{B_p(\mathcal{S}_{g,1})_+}$ is a direct sum of $\beta_{\text{ even }} := \bigoplus_{i<p \text{ even}} \beta_i$ many copies of E and $\beta_{\text{ odd }} := \bigoplus_{i\leq p \text{ odd}} \beta_i$ many copies of ΣE. Expression (10) then implies that $K(h)^*(B_p(\mathcal{S}_{g,1}))$ consists of $\beta_{\text{ even }}$ many copies of $K(h)_*$ and $\beta_{\text{ odd }}$ many copies of $\Sigma K(h)_*$. Since $K(h)$ is a generalised field, the corresponding claim holds for $K(h)$-based homology, and (9) implies the result. □

Proof of Theorem 1.10. By the universal coefficient theorem, the $\mathbb{F}_p$-Betti number in degree i is greater than or equal to the rational Betti number in degree i:

$$\dim_{\mathbb{Q}}(H_i(B_p(\mathcal{S}_{g,1});\mathbb{Q}) \leq \dim_{\mathbb{F}_p}(H_i(B_p(\mathcal{S}_{g,1});\mathbb{F}_p).$$

On the other hand, Proposition 7.4 and the known rational calculation show that the sum of the $\mathbb{F}_p$-Betti numbers coincides with the sum of the rational Betti numbers. Since these numbers are all non-negative, it follows that corresponding Betti numbers are in fact equal, and the claim follows by a second invocation of the universal coefficient theorem. □

References

[AC15] O. Antolín Camarena. *The mod 2 homology of free spectral Lie algebras*. PhD thesis, Harvard University, 2015.

[AF15] D. Ayala and J. Francis. Factorization homology of topological manifolds. *J. Topol.*, 8:1045–1084, 2015.

[Bar96] M. Barr. Cartan–Eilenberg cohomology and triples. *J. Pure Appl. Algebra*, 112:219–238, 1996.

[BC88] Carl-Friedrich Bödigheimer and Fred Cohen. Rational cohomology of configuration spaces of surfaces. In *Algebraic Topology and Transformation Groups*, volume 1361 of *Lecture Notes in Math.*, pages 7–13. Springer, 1988.

[BCM93] C.-F. Bödigheimer, F. R. Cohen, and R. J. Milgram. Truncated symmetric products and configuration spaces. *Math. Z.*, 214(2):179–216, 1993.

[BCT89] C.-F. Bödigheimer, F. R. Cohen, and L. R. Taylor. On the homology of configuration spaces. *Topology*, 28:111–123, 1989.

[BD04] A. Beilinson and V. Drinfeld. *Chiral Algebras*. American Mathematical Society, 2004.

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. *H_∞ ring spectra and their applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[Böd87] C.-F. Bödigheimer. Stable splittings of mapping spaces. In *Algebraic Topology*, volume 1286 of *Lecture Notes in Math.* Springer, 1987.

[Bou94] A. K. Bousfield. Localization and periodicity in unstable homotopy theory. *J. Amer. Math. Soc.*, 7(4):831–873, 1994.

[BR17] M. Behrens and C. Rezk. Spectral algebra models of unstable v_n-periodic homotopy theory. *arXiv preprint arXiv:1703.02186*, 2017.

[Bra17] D. L. B. Brantner. *The Lubin-Tate Theory of Spectral Lie Algebras*. PhD thesis, Harvard University, 2017. Edited Version available at https://people.maths.ox.ac.uk/brantner/brantnerthesis.pdf.

[Bro60] W. Browder. Homology operations and loop spaces. *Illinois journal of mathematics*, 4(3):347–357, 1960.

[CE48] C. Chevalley and S. Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.*, 1948.

[Chi05] M. Ching. Bar constructions for topological operads and the Goodwillie derivatives of the identity. *Geom. Topol.*, 9:833–933, 2005.

[CLM76] F. R. Cohen, T. J. Lada, and J. P. May. *Homology of Iterated Loop Spaces*. Number 533 in Lecture Notes in Math. Springer, 1976.

[CMT78] F. R. Cohen, J. P. May, and L. R. Taylor. Splitting of certain spaces CX. *Math. Proc. Camb. Phil. Soc.*, 84, 1978.

[DCK17] G. C. Drummond-Cole and B. Knudsen. Betti numbers of configuration spaces of surfaces. *J. London Math. Soc.*, 2017.

[DL62] E. Dyer and R. K. Lashof. Homology of iterated loop spaces. *Amer. J. Math.*, 84:35–88, 1962.

[DP58] Albrecht Dold and Dieter Puppe. Non-additive functors, their derived functors, and the suspension homomorphism. *Proc. Nat. Acad. Sci. U.S.A.*, 44:1065–1068, 1958.

[EFK98] P. I. Etingof, I. Frenkel, and A. A. Kirillov. *Lectures on representation theory and Knizhnik–Zamolodchikov equations*. American Mathematical Soc., 1998.

[FF74] R. Fossum and H.-B. Foxby. The category of graded modules. *Math. Scand.*, 1974.

[FG12] J. Francis and D. Gaitsgory. Chiral Koszul duality. *Selecta Math.*, 18(1):27–87, 2012.

[FT00] Yves Félix and Jean-Claude Thomas. Rational Betti numbers of configuration spaces. *Topology Appl.*, 102:139–149, 2000.

[GH04] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[GJ09] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*. Birkhäuser, 2009.

[Gla16] S. Glasman. Day convolution for ∞-categories. *Mathematical Research Letters*, 23(5):1369–1385, 2016.

[Goe99] Paul G. Goerss. Hopf rings, Dieudonné modules, and $E_*\Omega^2S^3$. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 115–174. Amer. Math. Soc., Providence, RI, 1999.

[Grb06] Jelena Grbić. Universal homotopy associative, homotopy commutative H-spaces and the EHP spectral sequence. *Math. Proc. Cambridge Philos. Soc.*, 140(3):377–400, 2006.

[GS07] Paul Goerss and Kristen Schemmerhorn. Model categories and simplicial methods. In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 3–49. Amer. Math. Soc., Providence, RI, 2007.

[Heu18] Gijs Heuts. Lie algebras and v_n-periodic spaces. *arXiv preprint arXiv:1803.06325*, 2018.

[Hig69] P. J. Higgins. Baer invariants and the Birkhoff-Witt theorem. *Journal of Algebra*, 11:469–482, 1969.

[HKR00] M. Hopkins, N. Kuhn, and D. C. Ravenel. Generalized group characters and complex oriented cohomology theories. *Journal of the American Mathematical Society*, 13(3):553–594, 2000.

[Hop14] M. Hopkins. $K(1)$-local E_∞-ring spectra. In *Topological modular forms*, volume 201 of *Math. Surveys Monogr.*, pages 287–302. Amer. Math. Soc., Providence, RI, 2014.

[Hun96] Thomas Hunter. On the homology spectral sequence for topological hochschild homology. *Transactions of the American Mathematical Society*, 348(10):3941–3953, 1996.

[JN14] N. Johnson and J. Noel. Lifting homotopy t-algebra maps to strict maps. *Advances in Mathematics*, 264:593–645, 2014.

[KA56] T. Kudo and S. Araki. Topology of H_n-spaces and H-squaring operations. *Mem. Fac. Sci. Kyūsyū Univ. Ser. A.*, 10:85–120, 1956.

[Kja16] J. J. Kjaer. On the odd primary homology of free algebras over the spectral Lie operad. *arXiv preprint arXiv:1608.06605*, 2016.

[Knu17] B. Knudsen. Betti numbers and stability for configuration spaces via factorization homology. *Alg. Geom. Topol.*, 17(5):3137–3187, 2017.

[Knu18] B. Knudsen. Higher enveloping algebras. *Geom. Topol.*, 22(7):4013–4066, 2018.

[Lan93] L. Langsetmo. The K-theory localization of loops on an odd sphere and applications. *Topology*, 32(3):577–585, 1993.

[Lan96a] L. Langsetmo. Further structure in $K(1)_*\Omega^k S^{2n+1}$. In *Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)*, volume 136 of *Progr. Math.*, pages 235–240. Birkhäuser, Basel, 1996.

[Lan96b] Lisa Langsetmo. Further structure in $K(1)_*\Omega^k S^{2n+1}$. In *Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)*, volume 136 of *Progr. Math.*, pages 235–240. Birkhäuser, Basel, 1996.

[Laz69] Daniel Lazard. Ass de modules plats. *C. R. Acad. Sci. Paris Sér. A-B*, 269:A65–A67, 1969.

[LS05] R. Longoni and P. Salvatore. Configuration spaces are not homotopy invariant. *Topology*, 44:375–380, 2005.

[Lur03] J. Lurie. Higher algebra. Preprint, `http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf`, 2003.

[Mar10] S. Marsh. The Morava E-theories of finite general linear groups. *arXiv preprint arXiv:1001.1949*, 2010.

[May66] J Peter May. The cohomology of restricted lie algebras and of hopf algebras. *Journal of Algebra*, 3(2):123–146, 1966.

[May88] J. G. Mayorquin. The mod 2 K-homology of $\Omega^3 S^3 X$. *Canad. J. Math.*, 40(1):142–196, 1988.

[McC00] J. McCleary. *A User's Guide to Spectral Sequences*, volume 58 of *Cambridge Studies in Advances Mathematics*. Cambridge University Press, 2000.

[McD75] D. McDuff. Configuration spaces of positive and negative particles. *Topology*, 14:91–107, 1975.

[ML88] R. James Milgram and Peter Löffler. The structure of deleted symmetric products. In *Braids (Santa Cruz, CA, 1986)*, volume 78 of *Contemp. Math.*, pages 415–424. Amer. Math. Soc., Providence, RI, 1988.

[MR87] Mark E. Mahowald and Douglas C. Ravenel. Toward a global understanding of the homotopy groups of spheres. In *The Lefschetz centennial conference, Part II (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 57–74. Amer. Math. Soc., Providence, RI, 1987.

[MS79] Haynes Miller and Victor Snaith. On the K-theory of the Kahn-Priddy map. *J. London Math. Soc. (2)*, 20(2):339–342, 1979.

[MT89] Mark Mahowald and Robert Thompson. K-theory and unstable homotopy groups. In *Algebraic topology (Evanston, IL, 1988)*, volume 96 of *Contemp. Math.*, pages 273–279. Amer. Math. Soc., Providence, RI, 1989.

[MT92] Mark Mahowald and Robert D. Thompson. The K-theory localization of an unstable sphere. *Topology*, 31(1):133–141, 1992.

[Nei10] J. Neisendorfer. *Algebraic Methods in Unstable Homotopy Theory*. Cambridge University Press, 2010.
[Pri70] Stewart B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970.
[Qui70] D. Quillen. On the (co-) homology of commutative rings. In *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, pages 65–87. Amer. Math. Soc., Providence, R.I., 1970.
[Rav93] D. C. Ravenel. The homology and Morava K-theory of $\Omega^2\mathrm{SU}(n)$. *Forum Math.*, 5(1):1–21, 1993.
[Rav98] D. C. Ravenel. What we still don't know about loop spaces of spheres. *Contemporary Mathematics*, 220:275–292, 1998.
[Rez98] C. Rezk. Notes on the Hopkins-Miller theorem. In *Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997)*, volume 220 of *Contemp. Math.*, pages 313–366. Amer. Math. Soc., Providence, RI, 1998.
[Rez06] Charles Rezk. The units of a ring spectrum and a logarithmic cohomology operation. *J. Amer. Math. Soc.*, 19(4):969–1014, 2006.
[Rez09] C. Rezk. The congruence criterion for power operations in Morava E-theory. *Homology Homotopy Appl.*, 11(2):327–379, 2009.
[Rez12] C. Rezk. Rings of power operations for Morava E-theories are Koszul. *arXiv preprint arXiv:1204.4831*, 2012.
[Rob63] Norbert Roby. Lois polynomes et lois formelles en théorie des modules. In *Annales scientifiques de l'École Normale Supérieure*, volume 80, pages 213–348, 1963.
[Sal98] Paolo Salvatore. *Configuration operads, minimal models and rational curves.* PhD thesis, University of Oxford, 1998.
[Sal01] P. Salvatore. Configuration spaces with summable labels. In *Cohomological Methods in Homotopy Theory*, volume 196 of *Progr. Math.* Birkhäuser, 2001.
[Sch] C. Schiessl. Integral cohomology of configuration spaces of the sphere. arXiv:801.04273.
[Ser51] J.-P. Serre. Homologie singulière des espaces fibrés. *Ann. of Math. (2)*, 54(3):425–505, 1951.
[Ser53] Jean-Pierre Serre. Groupes d'homotopie et classes de groupes abéliens. *Ann. of Math*, 58(2):258–294, 1953.
[Sna74] V. P. Snaith. A stable decomposition of $\Omega^n S^n X$. *J. London Math. Soc. (2)*, 7:577–583, 1974.
[Str98] N. P. Strickland. Morava E-theory of symmetric groups. *Topology*, 37(4):757–779, 1998.
[Tam94] Dai Tamaki. A dual Rothenberg-Steenrod spectral sequence. *Topology*, 33(4):631–662, 1994.
[Tam02] Dai Tamaki. The fiber of iterated Freudenthal suspension and Morava K-theory of $\Omega^k S^{2l+1}$. In *Recent progress in homotopy theory (Baltimore, MD, 2000)*, volume 293 of *Contemp. Math.*, pages 299–329. Amer. Math. Soc., Providence, RI, 2002.
[Tot96] Burt Totaro. Configuration spaces of algebraic varieties. *Topology*, 35(4):1057–1067, 1996.
[Wil82] Clarence Wilkerson. Lambda-rings, binomial domains, and vector bundles over $\mathbf{C}P(\infty)$. *Comm. Algebra*, 10(3):311–328, 1982.
[Yam88] A. Yamaguchi. Morava K-theory of double loop spaces of spheres. *Mathematische Zeitschrift*, 199(4):511–523, 1988.
[Zhu14] Yifei Zhu. The power operation structure on Morava E-theory of height 2 at the prime 3. *Algebr. Geom. Topol.*, 14(2):953–977, 2014.