DETECTING EXOTIC SPHERES IN LOW DIMENSIONS USING COKER J

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Abstract. Building off of the work of Kervaire and Milnor, and Hill, Hopkins, and Ravenel, Xu and Wang showed that the only odd dimensions n for which S^n has a unique differentiable structure are 1, 3, 5, and 61. We show that the only even dimensions below 140 for which S^n has a unique differentiable structure are 2, 6, 12, and 56, and perhaps 4.

Contents

1. Introduction

Let Θ_n denote the group of homotopy spheres. For $n \neq 4$, $\Theta_n = 0$ if and only if S^n has a unique differentiable structure. We wish to consider the following question:

For which n is $\Theta_n = 0$?

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The general belief is that there should be finitely many such n, and these n should be concentrated in relatively low dimensions.

Kervaire and Milnor [KM63], [Mil11] showed that there is an isomorphism

$$\Theta_{4k} \cong \operatorname{coker} J_{4k}$$

and there are exact sequences

$$0 \to \Theta^{bp}_{2k+1} \to \Theta_{2k+1} \to \operatorname{coker} J_{2k+1} \to 0,$$
$$0 \to \Theta_{4k+2} \to \operatorname{coker} J_{4k+2} \xrightarrow{\Phi_K} \mathbb{Z}/2 \to \Theta_{4k+1} \to 0.$$

Here Θ^{bp}_n is the subgroup of those homotopy spheres which bound a parallelizable manifold, coker J is the cokernel of the J homomorphism

$$J : \pi_n SO \to \pi^s_n,$$

and Φ_K is the Kervaire invariant.

Kervaire and Milnor showed that Θ^{bp}_{4k+3} is non-trivial for $k \geq 1$. The work of Hill-Hopkins-Ravenel [HHR16] implies that Φ_K is non-trivial only in dimensions 2, 6, 14, 30, 62, and perhaps 126. This implies that Θ^{bp}_{4k+1} is non-trivial, except possibly in dimensions 1, 5, 13, 29, 61, and 125. In dimensions 13 and 29, well established computations of π^s_* show coker J is non-trivial. With extraordinary effort, Wang and Xu recently showed $\pi^s_{61} = 0$ [WX17]. They also observe that coker $J_{125} \neq 0$ by producing an explicit element whose non-triviality is detected by tmf, the spectrum of topological modular forms. This concludes the analysis of odd dimensions.

The case of even dimensions, by the Kervaire-Milnor exact sequence, boils down to the question: for which k does coker J_{2k} have a non-trivial element of Kervaire invariant 0?

Over the years we have amassed a fairly detailed knowledge of the stable stems in low degrees using the Adams and Adams-Novikov spectral sequences. We refer the reader to [Rav86] for a good summary of the state of knowledge at odd primes, and [Isa14], [Isa16] for a detailed account of the current state of affairs at the prime 2. In particular, we have a complete understanding of π^s_n in a range extending somewhat beyond $n = 60$.[1]

However, we do not need to completely compute π^s_n to simply deduce coker J_n has a non-zero element of Kervaire invariant 0 — it suffices to produce such a single non-trivial element. There are two mechanisms which we explore to do this:

(1) Take a product or Toda bracket of existing elements in π^s_* to produce a new element in π^s_*, and show the resulting element is non-trivial.
(2) Use chromatic homotopy theory to produce infinite families of non-trivial elements in π^s_*.

[1]A recent breakthrough of Isaksen-Wang-Xu, using motivic homotopy theory and machine computations, is now bringing the range beyond $n = 90$.

The purpose of this paper is to simply enunciate what these techniques buy us in even, low dimensions. The main result is

Theorem 1.1. The only dimensions less than 140 for which S^n has a unique differentiable structure are 1,2,3,5, 6, 12, 56, 61, and perhaps 4.

This theorem will be established in large part by using the complete computation of $(\pi_*^s)_{(p)}$ for $* < 60$ for $p = 2$, $(\pi_*^s)_{(3)}$ for $* < 104$, and a small part of the vast knowledge of $(\pi_*^s)_{(5)}$ (computed in [Rav86] for $* < 1000$), though the 5 torsion is very sparse, and contributes very little to the discussion. Contributions from primes greater than 5 offer nothing to this range ($(\operatorname{coker} J_{82})_{(7)}$ is non-trivial, but we handle this dimension through other means).

Additional work will be required to produce non-trivial classes in $(\operatorname{coker} J_*)_{(2)}$ for $* \geq 60$. We will detect the non-triviality of products of known elements of $(\pi_*^s)_{(2)}$ using the Adams spectral sequence, and the Hurewicz homomorphism to the coefficient ring of topological modular forms (tmf) [DFHH14]. The greatest amount of work will involve an analysis of the tmf-based Adams spectral sequence for the generalized Moore spectrum $M(8, v_1^8)$.

Most of the classes we construct will by v_2-periodic - in fact 192-periodic for $p = 2$, 144-periodic for $p = 3$, and 48-periodic for $p = 5$. These periodic classes actually imply the existence of exotic spheres in infinitely many dimensions, and limit the remaining dimensions to certain congruence classes.

Actually, for some time we did not know how to construct a non-trivial class in $\operatorname{coker} J$ of Kervaire invariant 0 in dimension 126, but Dan Isaksen and Zhouli Xu came up with a clever argument which handles that case (Theorem 3.6).

We do not know if there are any non-trivial classes in $\operatorname{coker} J_{140}$. This is why we stop there. But we do end the paper with some remarks which explain that there are actually only a handful of dimensions in the range $140-200$ where we are unable to produce non-trivial classes in $\operatorname{coker} J$. Some of these classes were communicated to the authors by Zhouli Xu.

Conventions. Throughout this paper we will let $\{x\} \subset \pi_* X$ denote a coset of elements detected by an element x in the E_2-term of an Adams spectral sequence (ASS) or Adams-Novikov spectral sequence (ANSS). Conversely, for a class α in homotopy, we let $\underline{\alpha}$ denote an element in the ASS which detects it. We let A_* denote the dual Steenrod algebra, $A//A(2)_*$ denote the dual of the Hopf algebra quotient $A//A(2)$, and for an A_*-comodule M (or more generally an object of the derived category of A_*-comodules) we let

$$\operatorname{Ext}_{A_*}^{s,t}(M)$$

denote the group $\operatorname{Ext}_{A_*}^{s,t}(\mathbb{F}_2, M)$.

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2. The image of J and Browder's theorem

The only non-trivial even dimensional homotopy groups of SO are

$$\pi_{8k} SO = \mathbb{Z}/2,$$

and the J homomorphism is non-trivial in these degrees. The images of these elements have ASS names

$$\operatorname{im} J_{8k} = \begin{cases} \mathbb{Z}/2\{h_3 h_1\}, & k = 1, \\ \mathbb{Z}/2\{P^k c_0\}, & k > 1. \end{cases}$$

We are interested only in elements of coker J which have Kervaire invariant 0. The following theorem of Browder is essential [Bro69].

Theorem 2.1 (Browder)**.** *$x \in \pi_*^s$ has $\Phi_K(x) = 1$ if and only if it is detected in the ASS by h_j^2.*

For primes $p > 2$, the v_1-periodic stable homotopy groups of spheres coincide with the image of J, and thus do not contribute to coker J. However, at $p = 2$ the v_1-periodic localization of coker J is generated by the following elements [Mah81]:

$$v_1^{-1} \operatorname{coker} J_{8k+1} = \mathbb{Z}/2\{P^k h_1\},$$
$$v_1^{-1} \operatorname{coker} J_{8k+2} = \mathbb{Z}/2\{P^k h_1^2\}.$$

These elements generate the Hurewicz image of KO. Note that $\Phi_K(\{h_1^2\}) = 1$. We deduce

Proposition 2.2. *$\Theta_{8k+2} \neq 0$ for $k > 1$.*

Thus we are left to consider dimensions congruent to -2 mod 8, and dimensions congruent to 0 mod 4. The only such classes which can contribute to coker J in these dimensions must be v_n-periodic for $n \geq 2$.

3. Tabulation of some non-trivial elements in Coker J

Tables 1 (respectively 2) list non-zero elements in $(\operatorname{coker} J_{4k})_{(p)}$ (respectively non-zero elements in $(\operatorname{coker} J_{8k-2})_{(p)}$ with Kervaire invariant 0). A 0 entry indicates the group is known to be zero.

As the discussion in the last section indicates, a primary source of these non-zero elements are the v_2-periodic elements. In fact, all of the 3 and 5-primary elements in Tables 1 and 2 are v_2-periodic (in fact, 144-periodic for $p = 3$ [BP04] and 48-periodic for $p = 5$ [Smi70]). For $p = 2$, the classes in boldface are known (or tentatively known) to be 192-periodic, and (with the exception of $\kappa^2\bar{\kappa}$ and $\bar{\kappa}^6$) are detected by tmf via its Hurewicz homomorphism. (For a discussion of some of these cases, in various degrees of rigor, the reader is referred to the manuscript "From elliptic curves to homotopy theory" by the last two authors, in [DFHH14, Part II] — the authors hope to return to this subject in a later paper.)

Thus, while we are emphasizing the low dimensional aspects of the subject, in fact the v_2-periodic classes are giving exotic spheres in infinitely many dimensions, of

$4k$	$p=2$	$p=3$	$p=5$
4	0	0	0
8	$\boldsymbol{\epsilon}$	0	0
12	0	0	0
16	η_4	0	0
20	$\boldsymbol{\bar{\kappa}}$	β_1^2	0
24	$\eta_4\epsilon$	0	0
28	$\boldsymbol{\epsilon\bar{\kappa}}$	0	0
32	$\boldsymbol{\{q\}}$	0	0
36	$\{t\}$	$\beta_2\beta_1$	0
40	$\boldsymbol{\bar{\kappa}^2}$	β_1^4	0
44	$\{g_2\}$	0	0
48	$\boldsymbol{\kappa^2\bar{\kappa}}$	0	0
52	$\boldsymbol{\bar{\kappa}\{q\}}$	β_2^2	0
56	0	0	0
60	$\boldsymbol{\bar{\kappa}^3}$	0	0
64	η_6	0	0
68		$\langle\alpha_1,\beta_{3/2},\beta_2\rangle$	0
72		$\beta_2^2\beta_1^2$	0
76		0	β_1^2
80	$\boldsymbol{\bar{\kappa}^4}$	0	0
84		$\beta_5\beta_1$	0
88	$\{g_2^2\}$	0	0
92		$\beta_{6/3}\beta_1$	0
96	$\eta_6\{d_1\}$	0	0
100	$\boldsymbol{\bar{\kappa}^5}$	$\beta_2\beta_5$	0
104	$\boldsymbol{v_2^{16}\epsilon}$		0
108	$\eta_6\{g_2\}$		0
112		$\beta_{6/3}\beta_1^3$	0
116	$\boldsymbol{(v_2^{16}\kappa)\nu^2}$		0
120	$\boldsymbol{\bar{\kappa}^6}$		0
124			$\beta_2\beta_1$
128	η_7		0
132	$\{h_2h_6^2\}\nu$		0
136	$\eta_7\epsilon$		0

TABLE 1. Some non-trivial elements of $(\operatorname{coker} J_{4k})_{(p)}$ in low degrees.

certain congruence classes mod 192, 144, and 48. All in all, in our range, over half of the candidates are coming from v_2-periodic classes.

In dimensions less than 60 for $p=2$, less than 104 for $p=3$, and in all dimensions depicted for $p=5$, these non-trivial elements can be found in [Isa14], [Isa14] and [Rav86]. Note that Kochman-Mahowald [KM95] has errors, and in particular incorrectly states that $\operatorname{coker} J_{56} \neq 0$. The first author believes the computations of $(\pi_*^s)_{(3)}$ in [Rav86] has errors in the range 104-108. The computation $(\pi_*^s)_{(3)}$ in the range up to 103 was first independently done by Nakamura [Nak75] and Tangora [Tan75], and the answers agree.

$8k-2$	$p=2$	$p=3$	$p=5$
6	0	0	0
14	$\boldsymbol{\kappa}$	0	0
22	$\boldsymbol{\epsilon\kappa}$	0	0
30	0	β_1^3	0
38	$\{h_1 x\}$	$\beta_{3/2}$	β_1
46	$\boldsymbol{\{w\}\eta}$	$\beta_2\beta_1^2$	0
54	$\{h_5 h_0 i\}$	0	0
62		$\beta_2^2\beta_1$	0
70	$\boldsymbol{\langle\bar{\kappa}\{w\},\nu,\eta\rangle}$	0	0
78		β_2^3	0
86		$\beta_{6/2}$	β_2
94		β_5	0
102	$\boldsymbol{v_2^{16}\nu^2}$	$\beta_{6/3}\beta_1^2$	0
110	$\boldsymbol{v_2^{16}\kappa}$		0
118	$\boldsymbol{v_2^{16}\eta^2\bar{\kappa}}$		0
126	$\langle\theta_5, 2, \{X_2 + C'\}\rangle$ or $\theta_5\eta_6$		0
134			β_3

TABLE 2. Some non-trivial elements of $(\operatorname{coker} J_{8k-2})_{(p)}$ with Kervaire invariant 0 in low degrees.

Note that for any prime $p > 2$, the first non-trivial element of $(\operatorname{coker} J_*)_{(p)}$ is β_1 (see, for instance, [Rav86]), which lies in dimension $2(p^2-1)-2(p-1)-2$. The only prime greater than 5 for which this dimension is less than 200 is $p = 7$, for which $|\beta_1| = 82$, and this is the only nontrivial 7-torsion in coker J in our range. This adds nothing to the discussion as $82 \equiv 2 \mod 8$.

The 2-primary elements with names $v_2^{16}x$ for various x will be constructed later in this paper using the tmf-resolution — indeed that is the goal of all of the subsequent sections of this paper. These classes are all non-trivial because they are detected in the tmf-Hurewicz homomorphism. In all fairness, in each of these dimensions except for dimension 118, we could have manually constructed non-trivial classes using certain products and Toda brackets involving θ_5. However, as we do not know another means of producing a class in dimension 118, we would still have to use the tmf-resolution technique to handle that dimension. We also have a preference for the classes we construct, as we believe/know they are 192-periodic, and hence they actually account for an entire congruence class mod 192 of dimensions.

Below, we explain the origin and non-triviality of the remaining 2-primary elements in the tables. Note that in our range the E_2-term of the 2-primary ASS can be computed using software developed by Bruner [Bru93], [Bru].

dim 60: $\bar{\kappa}^3$**:** This class is non-zero in $\pi_*\mathrm{tmf}$, hence non-zero.

dim 64: η_6**:** The existence and non-triviality of this class was established by the fourth author [Mah77].

dim 70: $\langle\bar{\kappa}\{w\},\nu,\eta\rangle$: The element $\{w\}\in\pi^s_{45}$ supports a hidden ν-extension in the ASS for the sphere, so $\nu\{w\}$ is detected by $d_0e_0^2$ [Isa14, Lem.4.2.71]. Thus $\nu\bar{\kappa}\{w\}$ is detected in Adams filtration greater than or equal to 16. However, there are no non-trivial classes in π^s_{68} with Adams filtration greater than or equal to 16 [Isa16]. This means the proposed Toda bracket exists. The image of this Toda bracket in tmf does not contain zero, hence the Toda bracket in the sphere does not contain zero.

dim 80: $\bar{\kappa}^4$: This class is non-zero in $\pi_*\mathrm{tmf}$, hence non-zero.

dim 88: $\{g_2^2\}$: The element g_2 is a permanent cycle in the ASS. Computer computations of Ext_{A_*} in this range reveal there are no possible sources of a differential to kill g_2^2.

dim 96: $\eta_6\{d_1\}$: The element d_1 is a permanent cycle in the ASS. Its product with η_6 is detected in the ASS by the element $h_6h_1d_1$. There are no possible sources of a differential to kill this element.

dim 100: $\bar{\kappa}^5$: This class is non-zero in $\pi_*\mathrm{tmf}$, hence non-zero.

dim 108: $\eta_6\{g_2\}$: This element is detected in the ASS by the element $h_6h_1g_2$. There are no possible sources of a differential to kill this element.

dim 120: $\bar{\kappa}^6$: See below.

dim 126: See below.

dim 128: η_7: The existence and non-triviality of this class was established by the fourth author [Mah77].

dim 132: $\{h_2h_6^2\}\nu$: Bruner showed that the classes $h_j^2h_2$ are permanent cycles in the ASS. There are no classes in Ext which can support a nontrivial Adams differential killing $h_2^2h_6^2$.

dim 136: $\eta_7\epsilon$: This class is detected by $\{c_0h_7h_1\}$ in the ASS. There are no classes in Ext which can support a non-trivial Adams differential killing this class.

Showing $\bar{\kappa}^6$ is non-zero is slightly tricky, since its image in $\pi_*\mathrm{tmf}$ is actually zero. However, let $F(3)$ be the fiber

$$F(3)\to\mathrm{TMF}\xrightarrow{q-f}\mathrm{TMF}_0(3)$$

for the maps f and q of [MR09, Sec. 5]. Since q and f are E_∞-ring maps, $F(3)$ is an E_∞ ring spectrum, and in particular has a unit.

Proposition 3.1. The image of $\bar{\kappa}^6$ under the map

$$\pi_*S\to\pi_*F(3)$$

is non-zero. In particular, $\bar{\kappa}^6$ is non-zero in π^s_*.

Proof. Intuitively, this is true because in the ANSS for tmf, there is a differential

$$d_{23}(\Delta^5\eta) = \bar{\kappa}^6$$

but we have, by [MR09, Prop. 8.1],

$$(q-f)^*\Delta^5\eta = a_3^{18}a_1^6\eta \mod (2, a_1^7).$$

Thus $\bar{\kappa}^6$ gets "identified" with the image of $a_3^{18}a_1^6\eta + \cdots$ in $F(3)$.

To make this idea rigorous, we note that the element $\bar{\kappa} \in \pi_{20}^s$ factors over the 18-cell of the generalized Moore spectrum $M(8, v_1^8)$, hence so does $\bar{\kappa}^5$ - we let

$$\bar{\kappa}^5[18] \in \pi_{118}M(8, v_1^8)$$

denote such a lift. We let

$$\bar{\kappa}^5[18]_{F(3)} \in F(3)_{118}M(8, v_1^8)$$
$$\bar{\kappa}^5[18]_{\mathrm{TMF}} \in \mathrm{TMF}_{118}M(8, v_1^8)$$

denote the images of this lift in $F(3)$ and TMF-homology, respectively. We can use the computations of $\mathrm{tmf}_*M(8, v_1^8)$ in Section 6 to identify the term in the MASS which detects $\bar{\kappa}^5[18]_{\mathrm{TMF}}$, and we find that the product

$$\bar{\kappa} \cdot (\bar{\kappa}^5[18]_{\mathrm{TMF}}) \in \mathrm{tmf}_{138}M(8, v_1^8) \tag{3.2}$$

is non-trivial. By manually computing the Atiyah-Hirzebruch spectral sequence for $\mathrm{tmf}_*(M(3,8))$, one finds that the element (3.2) lifts to the element

$$c_4^2\Delta^5\eta[1]_{\mathrm{TMF}} \in \mathrm{TMF}_{138}M(8). \tag{3.3}$$

Applying the geometric boundary theorem [Beh12, Lem. A.4.1] to the fiber sequence of resolutions of $F(3)$

$$\begin{tikzcd}
F(3) \wedge \Sigma^{16}M(8) \arrow[r, "v_1^8"] \arrow[d] & F(3) \wedge M(8) \arrow[r] \arrow[d] & F(3) \wedge M(8, v_1^8) \arrow[r, "\partial"] \arrow[d] & F(3) \wedge \Sigma^{17}M(8) \arrow[d] \\
\mathrm{TMF} \wedge \Sigma^{16}M(8) \arrow[r, "v_1^8"] \arrow[d, "q-f"'] & \mathrm{TMF} \wedge M(8) \arrow[r] \arrow[d, "q-f"'] & \mathrm{TMF} \wedge M(8, v_1^8) \arrow[r] \arrow[d, "q-f"'] & \mathrm{TMF} \wedge \Sigma^{17}M(8) \arrow[d, "q-f"'] \\
\mathrm{TMF}_0(3) \wedge \Sigma^{16}M(8) \arrow[r, "v_1^8"] & \mathrm{TMF}_0(3) \wedge M(8) \arrow[r] & \mathrm{TMF}_0(3) \wedge M(8, v_1^8) \arrow[r] & \mathrm{TMF}_0(3) \wedge \Sigma^{17}M(8)
\end{tikzcd}$$

we see that $\partial(\bar{\kappa} \cdot \bar{\kappa}^5[18]_{F(3)})$ is computed by the usual formula for a connecting homomorphism: applying $q-f$ to the lift (3.3), divide the result by v_1^8, and then take the image in $F(3)$. Applying $q-f$ to (3.3) gives

$$(q-f)^*c_4^2\Delta^5\eta[1] = (a_3^{18}a_1^{14}\eta + \cdots)[1].$$

Dividing by v_1^8, we get

$$(a_3^{18}a_1^6\eta + \cdots)[1] \in \mathrm{TMF}_0(3)_{122}. \tag{3.4}$$

We deduce that $\partial(\bar{\kappa} \cdot \bar{\kappa}^5[18]_{F(3)})$ is detected by (3.4). Projecting to the top cell of $M(8)$, we deduce that the image of $\bar{\kappa}^6$ in $\pi_*F(3)$ is detected by

$$a_3^{18}a_1^6\eta + \cdots \in \mathrm{TMF}_0(3)_{121}.$$

This class can be seen to be non-zero in $\pi_*F(3)$, by again appealing to [MR09, Prop. 8.1]. $\square$

Remark 3.5. Since the map $S \to F(3)$ factors through the spectrum $Q(3)$ of [MR09], [BO16], the element $\bar{\kappa}^6$ is also detected in $Q(3)$.

Dimension 126 is handled by the following theorem, communicated to us by Isaksen and Xu.

Theorem 3.6 (Isaksen-Xu)**.** There exists a non-trivial element of coker J in dimension 126 of Kervaire invariant 0.

Proof. Suppose that $\theta_5\eta_6 \in \pi^s_{126}$ is non-trivial. Then we are done. Suppose however that $\theta_5\eta_6 = 0$. Consider the class $X_2 + C'$ in $\mathrm{Ext}^{7,63+7}_{A_*}(\mathbb{F}_2, \mathbb{F}_2)$. A detailed analysis of the 2-primary stable stems of Isaksen-Wang-Xu reveals that this class is a permanent cycle, and detects an element of order 2 in π^s_{63}. (We caution the reader that this analysis is not written down carefully at this time.) Thus the Toda bracket

$$\langle \theta_5, 2, \{X_2 + C'\}\rangle \in \pi^s_{126}$$

exists, and is detected by $h_6(X_2 + C')$ in the ASS. We would be done if we could show that this class is not the target of an Adams differential. There are only two classes which could support such a differential:

$$x_{6,99} \in \mathrm{Ext}^{6,127+6}_{A_*}(\mathbb{F}_2, \mathbb{F}_2),$$
$$h_6^2 h_1 \in \mathrm{Ext}^{3,127+3}_{A_*}(\mathbb{F}_2, \mathbb{F}_2).$$

Since $h_2 x_{6,99} = 0$ and $h_2(X_2 + C') \neq 0$ in Ext, there cannot be a differential

$$d_2(x_{6,99}) = X_2 + C'.$$

If our assumption on $\theta_5\eta_6$ holds, the class $h_6^2 h_1$ would detect the Toda bracket

$$\langle 2, \theta_5, \eta_6\rangle$$

and hence be a permanent cycle. □

4. The strategy for the remaining elements

For the remainder of this paper we will be working 2-locally. Let $M(2^i, v_1^j)$ denote the cofiber of a v_1^j-self map

$$v_1^j : \Sigma^{2j} M(2^i) \to M(2^i)$$

on the mod 2^i Moore spectrum (provided such a self map exists). In [BHHM08], the authors used the tmf-resolution to show the element

$$v_2^{32} \in \mathrm{tmf}_{192} M(2, v_1^4)$$

lifts to a v_2^{32}-self map of $M(2, v_1^4)$.

Our strategy will be to apply this kind of technique to lift elements from $\pi_*\mathrm{tmf}$ to π^s_*. Namely, given a v_1-torsion element $x \in \mathrm{tmf}_*$, we will lift it to

$$\widetilde{x} \in \mathrm{tmf}_* M(2^i, v_1^j)$$

so that the projection to the top cell maps $\widetilde{x}$ to x. We will then show, using the tmf-resolution, that $\widetilde{x}$ lifts to an element

$$\widetilde{y} \in \pi_* M(2^i, v_1^j).$$

Then the image

$$y \in \pi_*^s$$

given by projecting $\widetilde{y}$ to the top cell is an element whose image under the tmf-Hurewicz homomorphism is x.

The first major observation is the following.

Proposition 4.1. Every v_1-torsion element $x \in \pi_*\mathrm{tmf}$ is 8-torsion and v_1^8-torsion.

Proof. This is equivalently stated as asserting that every c_4-torsion class in $\pi_*\mathrm{tmf}$ is c_4^2-torsion and 8-torsion. Using the fact that c_4^2 has Adams filtration 8, this is easily checked from the Adams E_∞ page for tmf (see, for instance, [DFHH14, pp. 196-7]). $\square$

We therefore apply the strategy above to lift v_1-torsion elements in $\pi_*\mathrm{tmf}$ to the top cell of $M(8, v_1^8)$, and endeavor to lift these to homotopy classes using the tmf-resolution. We therefore proceed to analyze the tmf-resolution of $M(8, v_1^8)$.

The modified Adams spectral sequence. What we actually will do is analyze the modified Adams spectral sequence (MASS) for $M(8, v_1^8)$ (see [BHHM08, Sec. 3]). Define $H(8)$ to be the cofiber of the map

$$\Sigma^3\mathbb{F}_2[-3] \xrightarrow{h_0^3} \mathbb{F}_2$$

in the derived category of A_*-comodules, and define $H(8, v_1^8)$ to be the cofiber of the map

$$\Sigma^{24}H(8)[-8] \xrightarrow{v_1^8} H(8, v_1^8).$$

The MASS then takes the form

$$E_2^{s,t}(M(8, v_1^8)) = \mathrm{Ext}_{A_*}^{s,t}(H(8, v_1^8)) \Rightarrow \pi_{t-s}M(8, v_1^8).$$

There is also a MASS

$$E_2^{s,t}(\mathrm{tmf} \wedge M(8, v_1^8)) = \mathrm{Ext}_{A(2)_*}^{s,t}(H(8, v_1^8)) \Rightarrow \mathrm{tmf}_{t-s}M(8, v_1^8).$$

The algebraic tmf-resolution. The E_2-page of the MASS for $M(8, v_1^8)$ will be analyzed using an algebraic tmf-resolution (as in [BHHM08, Sec. 5]). Namely, for any object M of the derived category of A_*-comodules, we apply $\mathrm{Ext}_{A_*}(-)$ to the following diagram in the derived category:

$$\begin{tikzcd}
M \arrow[d] & \overline{A//A(2)}_*[-1] \otimes M \arrow[l] \arrow[d] & \overline{A//A(2)}_*^{\otimes 2}[-2] \otimes M \arrow[l] \arrow[d] & \cdots \arrow[l] \\
A//A(2)_* \otimes M & A//A(2)_* \otimes \overline{A//A(2)}_*[-1] \otimes M & A//A(2)_* \otimes \overline{A//A(2)}_*^{\otimes 2}[-2] \otimes M &
\end{tikzcd}$$

where $\overline{A//A(2)}_*$ is the cokernel of the unit

$$0 \to \mathbb{F}_2 \to A//A(2)_* \to \overline{A//A(2)}_* \to 0.$$

This results in the *algebraic* tmf*-resolution*

$$E_1^{s,t,n} = \mathrm{Ext}_{A(2)_*}^{s,t}(\overline{A//A(2)}_*^{\otimes n} \otimes M) \Rightarrow \mathrm{Ext}_{A_*}^{s+n,t}(M).$$

Useful facts about $M(8, v_1^8)$. The following facts about $M(8, v_1^8)$ will prove useful.

Proposition 4.2. $M(8, v_1^8)$ is a (possibly non-associative) ring spectrum.

Proof. We duplicate the argument given in [Mah78] in the case of $M(8, v_1^8)$. A similar argument in fact shows that $M(8, v_1^{4t})$ is a ring spectrum for all $t > 0$. It suffices to show there exists an extension

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & M(8, v_1^8) \\ M(8, v_1^8) \wedge M(8, v_1^8) \arrow[ur, dotted] & \end{tikzcd} \tag{4.3}$$

We will use the MASS

$$\mathrm{Ext}_{A_*}^{s,t}(H(8, v_1^8)^{\otimes 2}, H(8, v_1^8)) \Rightarrow [\Sigma^{t-s} M(8, v_1^8)^{\wedge 2}, M(8, v_1^8)]. \tag{4.4}$$

We shall say a cell $\Sigma^t \mathbb{F}_2[-s]$ in the derived category of A_*-comodules has bidgeree $(t-s, s)$. In the derived category of A_*-comodules, $H(8, v_1^8)$ has cells in the bidegrees

$$(0, 0),\ (1, 2),\ (2, 4),\ (17, 7),\ (18, 9),\ (19, 11),\ (34, 14),\ (35, 16),\ (36, 18). \tag{4.5}$$

We may therefore compute the E_2-term of the MASS (4.4) by the "algebraic Atiyah-Hirzebruch spectral sequence" obtained from the cellular filtration of $H(8, v_1^8)^{\otimes 2}$. The E_1-term of this algebraic Atiyah-Hirzebruch spectral sequence consists of terms

$$\mathrm{Ext}_{A_*}^{s-s_0, t-t_0}(H(8, v_1^8))$$

where $(t_0 - s_0, s_0)$ is in the list (4.5). The extension problem (4.3) is equivalent to showing that the class

$$1[0, 0] \in \mathrm{Ext}_{A_*}^{0,0}(H(8, v_1^8))$$

supported on the bottom cell of $H(8, v_1^8)^{\otimes 2}$ is a permanent cycle in both the algebraic Atiyah-Hirzebruch spectral sequence, and the MASS. The possible targets of a differential (in either spectral sequence) supported by this class are detected in the algebraic Atiyah-Hirzebruch spectral sequence by classes in

$$\mathrm{Ext}_{A_*}^{s_0+r, (k_0-1)+(s_0+r)}(H(8, v_1^8))$$

for $r \geq 1$ and (k_0, s_0) in the list (4.5). One can check these groups are all zero. This check can be performed by hand, or alternatively, in this range the groups are isomorphic to

$$\mathrm{Ext}_{A(2)_*}^{s_0+r, (k_0-1)+(s_0+r)}(H(8, v_1^8))$$

and these latter groups are displayed in Figure 6.4. □

Corollary 4.6. Both the algebraic tmf-resolution and the MASS for $M(8, v_1^8)$ are spectral sequences of algebras.

Lemma 4.7. The element

$$v_2^8 \in \mathrm{Ext}_{A(2)_*}^{8, 48+8}(H(8, v_1^8))$$

is a permanent cycle in the algebraic tmf-resolution, and gives rise to an element

$$v_2^8 \in \mathrm{Ext}_{A_*}^{8, 48+8}(H(8, v_1^8)).$$

Proof. In the May spectral sequence for $\mathrm{Ext}_{A_*}(\mathbb{F}_2)$ there is a differential

$$d_8(b_{3,0}^4) = b_{2,0}^4 h_5.$$

This implies that v_2^8 can be constructed as a lift of

$$h_5[0] \in \mathrm{Ext}_{A_*}^{1,31+1}(H(8))$$

in the long exact sequence

$$\mathrm{Ext}_{A_*}^{8,48+8}(H(8,v_1^8)) \xrightarrow{\partial} \mathrm{Ext}_{A_*}^{1,31+1}(H(8)) \xrightarrow{v_1^8} \mathrm{Ext}_{A_*}^{9,47+9}(H(8)).$$

Here and throughout this proof, we use $x[0]$ (respectively $x[1]$) to denote a class in $\mathrm{Ext}_{A_*}(H(8))$ corresponding to $x \in \mathrm{Ext}_{A_*}(\mathbb{F}_2)$ supported on the 0-cell (respectively 1-cell) of $H(8)$. We must show that

$$v_1^8 \cdot h_5[0] = 0 \in \mathrm{Ext}^{9,47+9}(H(8)).$$

The only other possibility is that $v_1^8 \cdot h_5[0] = B_1[1]$. If that were the case, then the image of h_5 under the composite

$$\mathrm{Ext}_{A_*}^{1,31+1}(\mathbb{F}_2) \to \mathrm{Ext}_{A_*}^{1,31+1}(H(8)) \xrightarrow{v_1^8} \mathrm{Ext}_{A_*}^{9,47+9}(H(8)) \to \mathrm{Ext}_{A_*}^{7,46+7}(\mathbb{F}_2)$$

would be B_1. Since $h_4 h_5 = 0$, this would imply that $h_4 B_1 = 0$. However, according to Bruner's computer calculations, $h_4 B_1 \neq 0$ [Bru]. We deduce that $v_1^8 \cdot h_5[0]$ is indeed zero. $\square$

5. bo-Brown-Gitler comodules

The analysis of the E_1-page of the algebraic tmf-resolution is simplified via the decomposition of $A(2)_*$-comodules

$$\overline{A//A(2)}_* \cong \bigoplus_{i>0} \Sigma^{8i} \underline{\mathrm{bo}}_i$$

of [BHHM08, Cor. 5.5]. Here $\underline{\mathrm{bo}}_i$ is the ith bo-Brown-Gitler comodule (denoted $N_1(i)$ in [BHHM08]) — it is the homology of the ith bo-Brown-Gitler spectrum.[2] We therefore have a decomposition of the E_1-page of the algebraic tmf-resolution for M given by

$$E_1^{s,t,n} \cong \bigoplus_{i_1,\ldots,i_n>0} \mathrm{Ext}_{A(2)_*}^{s,t}(\Sigma^{8(i_1+\cdots+i_n)} \underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \otimes M). \tag{5.1}$$

For any M, the computation of

$$\mathrm{Ext}_{A(2)_*}^{s,t}(\Sigma^{8(i_1+\cdots+i_n)} \underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \otimes M)$$

can be inductively determined from $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k} \otimes M)$ by means of a set of exact sequences of $A(2)_*$-comodules which relate the $\underline{\mathrm{bo}}_i$'s [BHHM08, Sec. 7] (see also [BOSS15]):

$$0 \to \Sigma^{8j} \underline{\mathrm{bo}}_j \to \underline{\mathrm{bo}}_{2j} \to A(2)//A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \to \Sigma^{8j+9} \underline{\mathrm{bo}}_{j-1} \to 0, \tag{5.2}$$

$$0 \to \Sigma^{8j} \underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1 \to \underline{\mathrm{bo}}_{2j+1} \to A(2)//A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \to 0 \tag{5.3}$$

Here, $\underline{\mathrm{tmf}}_j$ is the jth tmf-Brown-Gitler comodule.

[2]For a more detailed discussion of bo-Brown-Gitler spectra, and their relationship to the (topological) tmf-resolution, we refer the reader to [BOSS15].

Remark 5.4. Technically speaking, as is addressed in [BHHM08], the comodules $A(2)//A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1}$ in the above exact sequences have to be given a slightly different $A(2)_*$-comodule structure from the standard one arising from the tensor product. However, this different comodule structure ends up being Ext-isomorphic to the standard one. As we are only interested in Ext groups, the reader can safely ignore this subtlety.

Thus there are spectral sequences

$$\left\{ \begin{array}{c} \mathrm{Ext}^{s,t}_{A(1)_*}(\underline{\mathrm{tmf}}_{j-1} \otimes M) \oplus \\ \mathrm{Ext}^{s,t}_{A(2)_*}(\Sigma^{8j}\underline{\mathrm{bo}}_j \otimes M) \oplus \\ \mathrm{Ext}^{s,t}_{A(2)_*}(\Sigma^{8j+9}\underline{\mathrm{bo}}_{j-1} \otimes M[-1]) \end{array} \right\} \Rightarrow \mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_{2j} \otimes M),$$

$$\left\{ \begin{array}{c} \mathrm{Ext}^{s,t}_{A(1)_*}(\underline{\mathrm{tmf}}_{j-1} \otimes M) \oplus \\ \mathrm{Ext}^{s,t}_{A(2)_*}(\Sigma^{8j}\underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1 \otimes M) \end{array} \right\} \Rightarrow \mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_{2j+1} \otimes M).$$

These spectral sequences have been observed to collapse in low degrees (see [BOSS15]) but in general it seems possible they might not collapse. They inductively build $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_i \otimes M)$ out of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k} \otimes M)$ and $\mathrm{Ext}_{A(1)_*}(\underline{\mathrm{tmf}}_j \otimes M)$.

Define the ith bo-Brown-Gitler polynomial $f_i(s,t,x)$ inductively by the formulas (inspired from the exact sequences by ignoring the $A(2)//A(1)_*$ terms):

$$\begin{aligned} f_0 &= 1 \\ f_1 &= x \\ f_{2j} &= t^j f_j + s t^{j+1} f_{j-1} \\ f_{2j+1} &= t^j x f_j \end{aligned}$$

For a multi-index $I = (i_1, \dots, i_n)$, define

$$\underline{\mathrm{bo}}_I := \underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n}$$

and

$$f_I(s,t,x) = f_{i_1} \cdots f_{i_n}.$$

Write

$$f_I = \sum_{k,l,m} a(I)_{k,l,m} s^k t^l x^m.$$

Then inductively the exact sequences give rise to a sequence of spectral sequences

(5.5)
$$\left\{ \begin{array}{c} \bigoplus_{k,l,m} \mathrm{Ext}^{s,t}_{A(2)_*}(\Sigma^{8l+k}\underline{\mathrm{bo}}_1^{\otimes m}[-k] \otimes M)^{\oplus a(I)_{k,l,m}} \\ \oplus \mathrm{Ext}^{s,t}_{A(1)_*}(\text{something}) \end{array} \right\} \Rightarrow \mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_I \otimes M).$$

The following facts about the polynomials $f_i(s,t,x)$ can be easily established by induction.

Lemma 5.6.

(1) $a(i)_{k,l,m} = 0$ unless $i = l + m$.

(2) $f_i(s,t,x) \equiv t^{i-m}x^m \mod (s)$, where m is the number of 1's in the dyadic expansion of i.
(3) The highest power of x appearing in $f_i(s,t,x)$ is less than or equal to the number of digits in the dyadic expansion of i.
(4) The highest power of s appearing in f_i is the number of 1's to the left of the rightmost 0 in the dyadic expansion of i.

Finally, the following lemma explains that, in our $H(8, v_1^8)$ computations, we may disregard terms coming from $\mathrm{Ext}_{A(1)_*}$ in the sequence of spectral sequences (5.5).

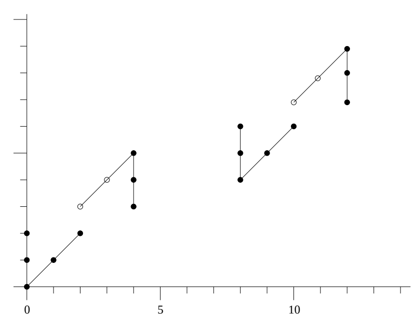
Lemma 5.7. In the algebraic tmf-resolution for $M = H(8, v_1^8)$, the terms

$$\mathrm{Ext}_{A(1)_*}(\text{something})$$

in (5.5) do not contribute to $\mathrm{Ext}^{s,t}_{A_*}(H(8, v_1^8))$ if

$$s > \frac{1}{7}(t-s) + \frac{51}{7}.$$

Proof. The connectivity of the n-line of the tmf-resolution for $H(8, v_1^8)$ is $8n-1$, meaning that the bottom cell of the n-line contributes to $\mathrm{Ext}^{n,8n}_{A_*}$. In particular, the contribution of the bottom cells rises on a line of slope $1/7$. The groups $\mathrm{Ext}^{s,t}_{A(1)_*}(H(8, v_1^8))$ are displayed below.



The lowest line of slope $1/7$ which bounds the pattern above is

$$s = \frac{1}{7}(t-s) + \frac{51}{7}.$$

The result follows. □

6. The MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$

We now turn our attention to the analysis of the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$. The groups $\mathrm{Ext}_{A_*}(H(8))$ are easily computed using the computation of $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$ (see, for example, the chart on p. 194 of [DFHH14]) using the long exact sequence on Ext induced by the cofiber sequence

$$\Sigma^3\mathbb{F}_2[-3] \xrightarrow{h_0^3} \mathbb{F}_2 \to H(8) \to \Sigma^3\mathbb{F}_2[-2].$$

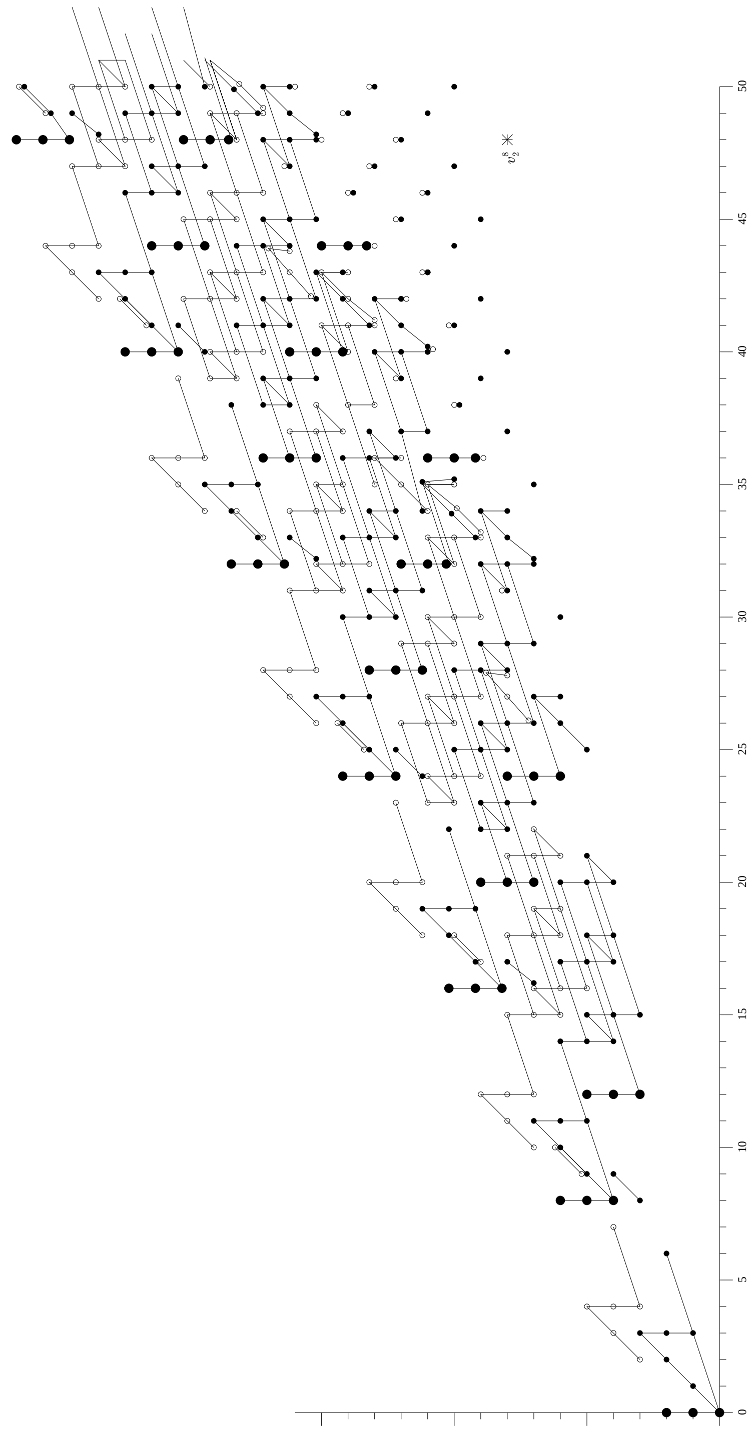


FIGURE 6.1. The groups $\mathrm{Ext}_{A(2)_*}(H(8))$.

The result is displayed in Figure 6.1. The computation is simplified by the fact that all h_0-torsion in $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$ is h_0^3-torsion. In this figure, solid dots correspond to classes carried by the "0-cell" of $H(8)$, and open circles correspond to classes carried by the "1-cell" of $H(8)$. The large solid circles correspond to h_0-torsion free classes of $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$ on the 0-cell of $H(8)$.

The computation of $\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$ is similarly accomplished by the long exact sequence on Ext given by the cofiber sequence

$$\Sigma^{24} H(8)[-8] \xrightarrow{v_1^8} H(8) \to H(8, v_1^8) \to \Sigma^{24} H(8)[-7].$$

Note that every class in $\mathrm{Ext}_{A(2)_*}(H(8))$ is v_1^4-periodic, so this computation is not difficult. The result is depicted in Figure 6.2. Here we retain the notation from Figure 6.1 with regard to solid dots, large solid dots, and open circles. The classes with solid boxes around them support $h_{2,1}$ towers, where $h_{2,1}$ corresponds to the class $[\xi_2^2]$ in the cobar complex for $A(2)_*$. Everything is v_2^8-periodic.

Figure 6.3 depicts the differentials in the MASS for $\mathrm{tmf} \wedge M(8)$ through a range. The complete computation of this MASS can be similarly accomplished, but it is not necessary for our purposes. For the most part the differentials are deduced from the maps of spectral sequences induced by the maps

$$\mathrm{tmf} \to \mathrm{tmf} \wedge M(8),$$
$$\mathrm{tmf} \wedge M(8) \to \Sigma \mathrm{tmf}.$$

The abutment of the spectral sequence, $\mathrm{tmf}_* M(8)$ is already easily computed from the long exact sequence associated to the cofiber sequence

$$S^0 \xrightarrow{8} S^0 \to M(8),$$

and this information can be used to deduce the remaining differentials. For the most part, these remaining differentials are related to hidden $\cdot 8$ extensions in the ASS for tmf via the geometric boundary theorem [Beh12, Lem. A.4.1].

Example 6.1. The names we use for elements are those indicated in the chart on p.195 of [DFHH14]. We use $x[k]$, for x an element of the ASS for tmf, to denote "x on the k-cell". Via the geometric boundary theorem, the differential

$$d_2((\underline{c_4\Delta + q})[1]) = \underline{8\Delta} \cdot \underline{(c_4 + \epsilon)}[0]$$

arises from the hidden extension

$$\underline{8 \cdot (c_4 \Delta + q)} = \underline{8\Delta} \cdot \underline{(c_4 + \epsilon)}.$$

Example 6.2. The most subtle differential in this range is

$$d_3(v_1^2 g^2[1]) = h_0^2 \underline{2c_4c_6\Delta}[0]. \tag{6.3}$$

This differential does *not* come from a hidden extension. In fact, in the ASS for tmf there is a differential

$$d_4(v_1^2 g^2) = v_1^8 \cdot \underline{2\nu\Delta}. \tag{6.4}$$

Naively, one might expect that differential (6.4) lifts to a differential

$$d_4(v_1^2 g^2[1]) = v_1^8 \cdot \underline{2\nu\Delta}[1],$$

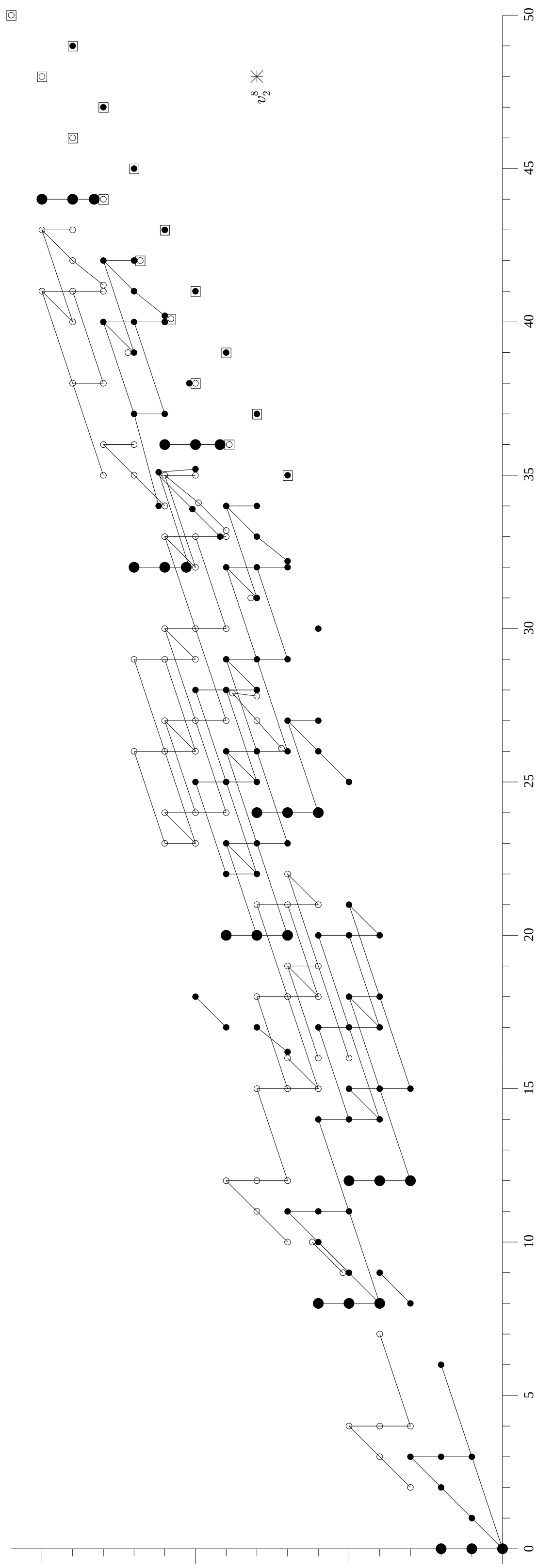


FIGURE 6.2. The groups $\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$.

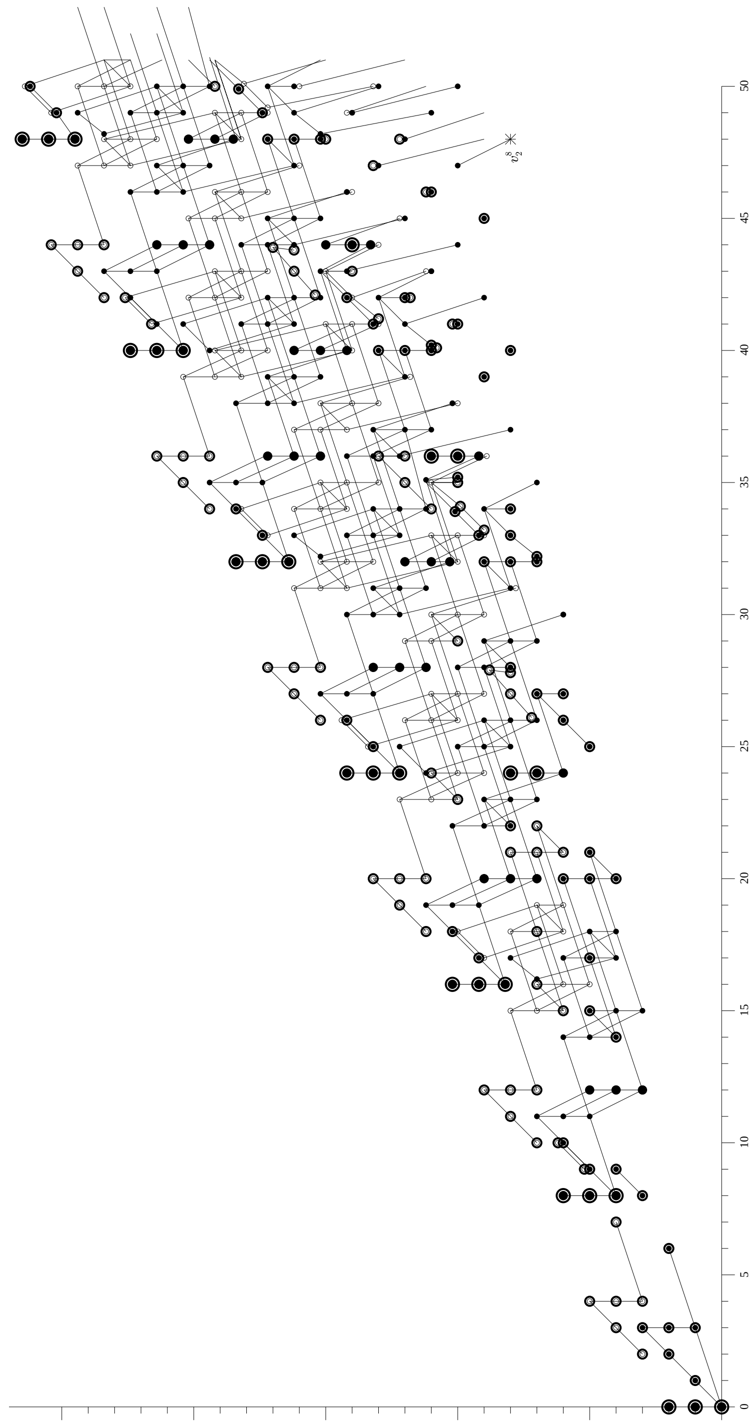


FIGURE 6.3. The MASS for tmf $\wedge M(8)$.

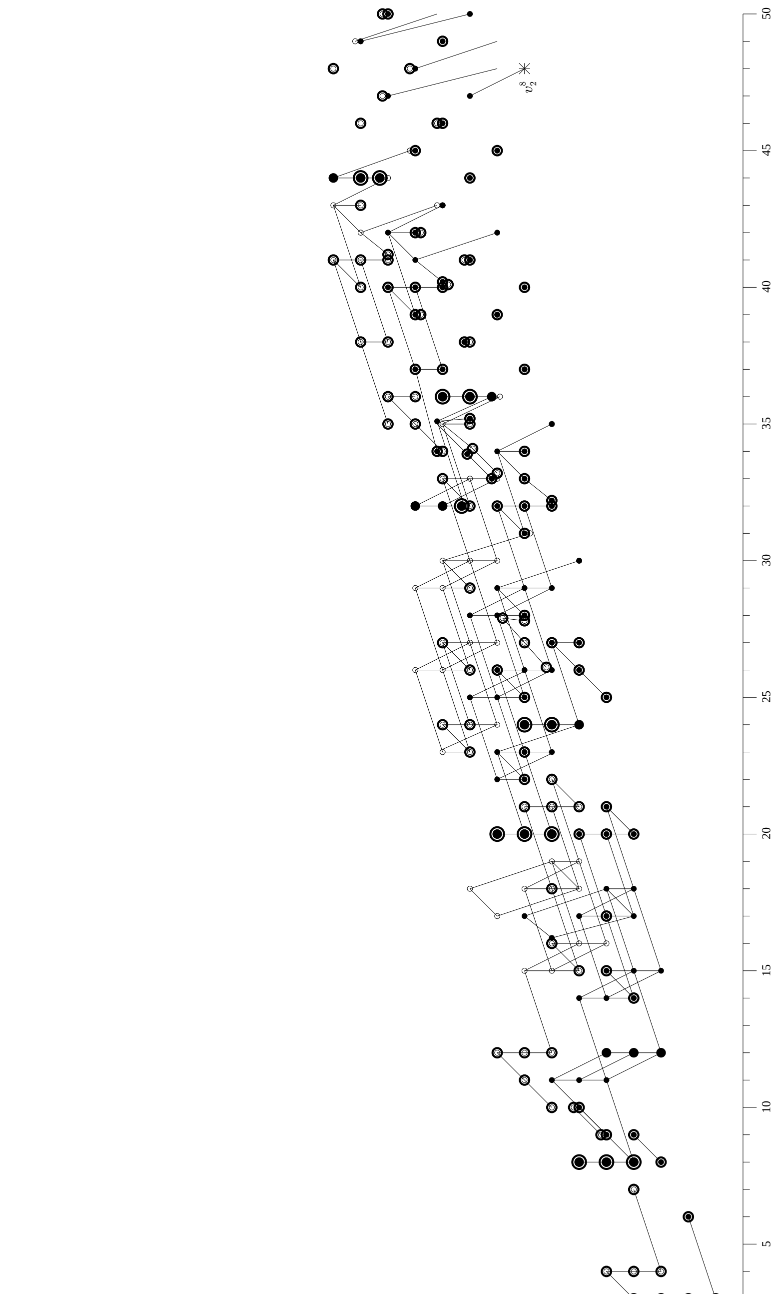


FIGURE 6.4. The MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$.

but that differential is preceded by (6.3). Differential (6.3) is forced when we analyze the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$, and compare it to the answer predicted by the Atiyah-Hirzebruch spectral sequence for $\mathrm{tmf}_* M(8, v_1^8)$.

Figure 6.4 depicts the differentials in the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$ through the same range. Again, the complete computation of this MASS can be similarly accomplished. Remarkably, since the E_2-term of the MASS for $\mathrm{tmf} \wedge M(8)$ is entirely $c_4^2 = v_1^8$-periodic, there end up being no hidden c_4^2-extensions, and all the differentials in the MASS for $M(8, v_1^8)$ are simply given by the images of the differentials in the MASS for $M(8)$ under the map of spectral sequences

$$\{E_r^{*,*}(M(8))\} \to \{E_r^{*,*}(M(8, v_1^8))\}.$$

Figure 6.5 displays $\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$ in the range 96-145. In this figure, classes coming from $h_{2,1}$-towers starting in dimensions below 96 are labeled with $\times$'s. We have circled the classes we are interested in, which detect lifts of certain homotopy elements to the top cell of $M(8, v_1^8)$. These are determined using the differentials in the MASS for $M(8)$ and $M(8, v_1^8)$ using the geometric boundary theorem [Beh12, Lem. A.4.1.]. We explain how this is done in one example.

Example 6.5. Consider the class $v_2^{16}\epsilon = \Delta^4 \epsilon$ in $\pi_{104}\mathrm{tmf}$. It lifts to an element

$$v_2^{16}\epsilon[1] \in \mathrm{tmf}_{105} M(8)$$

which is detected by the class

$$v_2^{16} c_0[1] \in \mathrm{Ext}_{A(2)_*}^{105+21,21}(H(8))$$

in the MASS for $\mathrm{tmf} \wedge M(8)$. Now lift $v_2^{16}\epsilon[1]$ to an element

$$v_2^{16}\epsilon[18] \in \mathrm{tmf}_{122} M(8, v_1^8).$$

We wish to determine an element of

$$\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$$

which detects one of these lifts in the MASS. In the MASS for $\mathrm{tmf} \wedge M(8)$, there is a differential

$$d_3(\Delta^4 v_1^4 e_0[1]) = \Delta^4 v_1^8 c_0[1].$$

It follows from the geometric boundary theorem that $v_2^{16}\epsilon[18]$ is detected by $\Delta^4 v_1^4 e_0[1]$ in the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$.

7. The algebraic tmf-resolution for $M(8, v_1^8)$

For $n > 0$, and $i_1, \ldots, i_n > 0$, the terms

$$\mathrm{Ext}_{A(2)_*}^{s,t}(\underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \otimes H(8, v_1^8))$$

that comprise the terms in the algebraic tmf-resolution for $H(8, v_1^8)$ are in some sense less complicated than $\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$.

Most of the features of these computations can already be seen in the computation of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$, which is displayed in Figure 7.1. This computation was performed by taking the computation of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1)$ (see, for example,

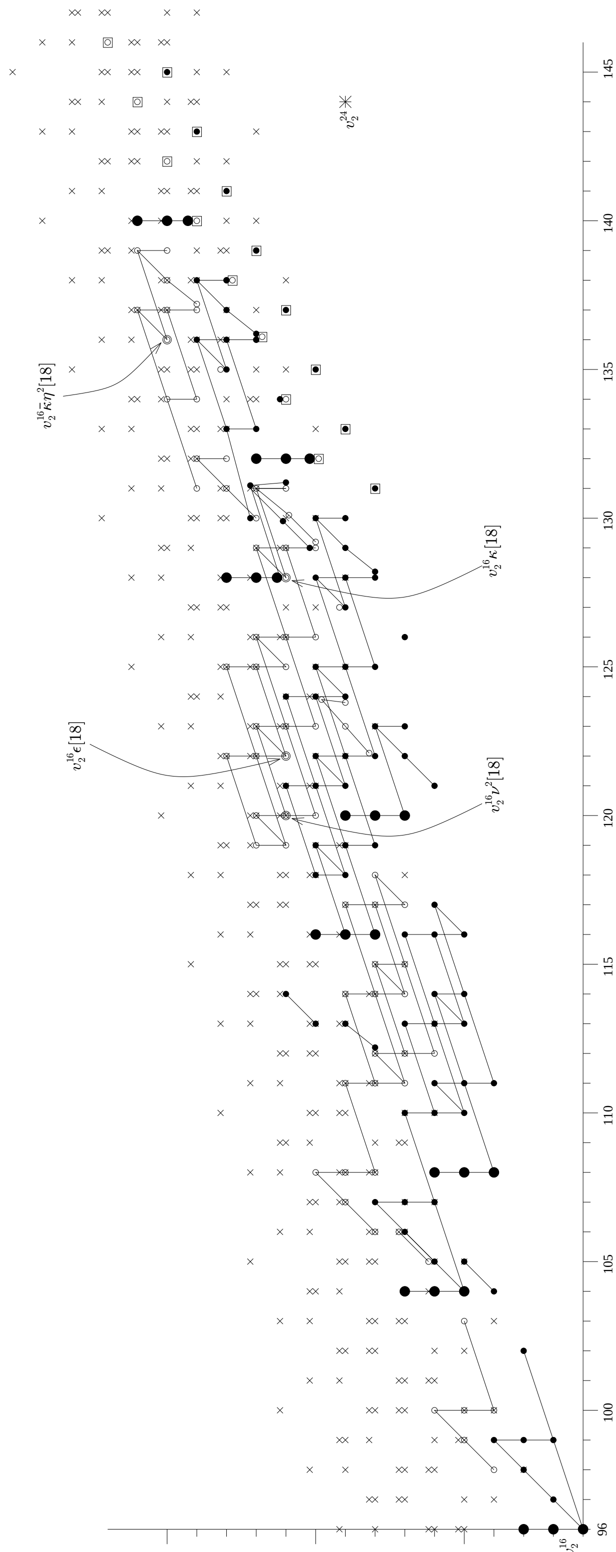


FIGURE 6.5. $\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$ in the range 96-145, with classes which detect certain lifts of homotopy elements to the top cell of $M(8, v_1^8)$.

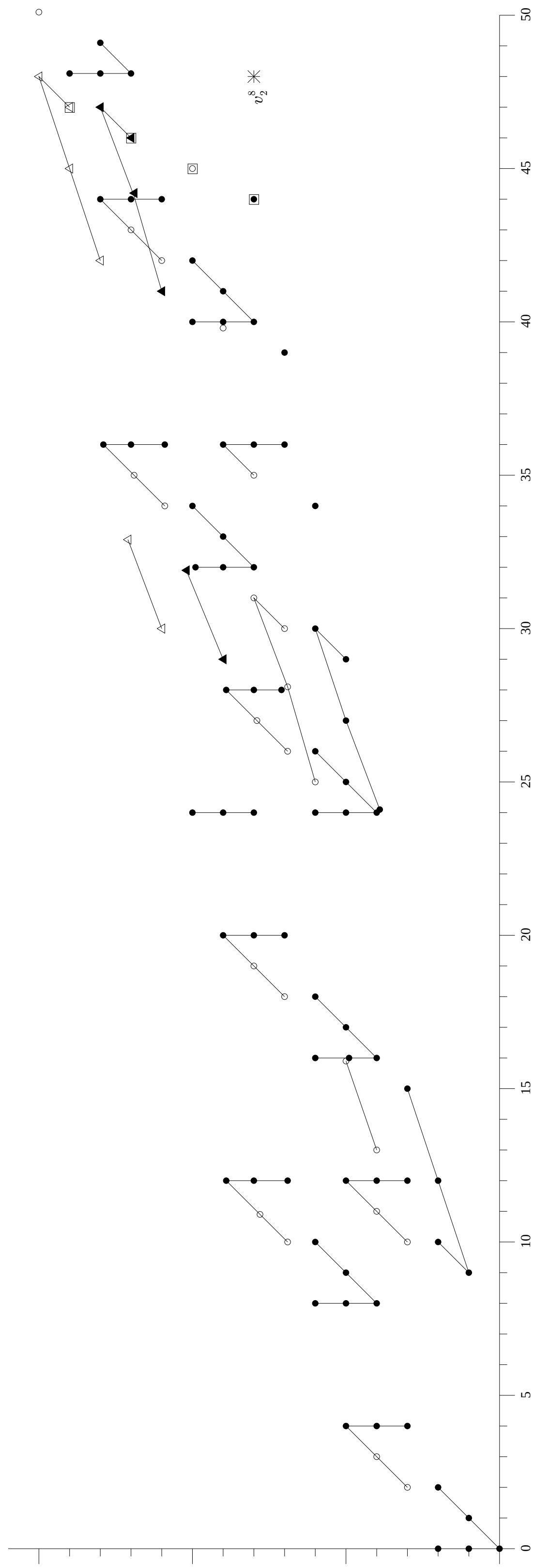


FIGURE 7.1. $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$.

[BHHM08]) and running the long exact sequences in Ext associated to the cofiber sequences

$$\Sigma^3\underline{\mathrm{bo}}_1[-3] \xrightarrow{h_0^3} \underline{\mathrm{bo}}_1 \to \underline{\mathrm{bo}}_1 \otimes H(8),$$

$$\Sigma^{24}\underline{\mathrm{bo}}_1 \otimes H(8)[-8] \xrightarrow{v_1^8} \underline{\mathrm{bo}}_1 \otimes H(8) \to \underline{\mathrm{bo}}_1 \otimes H(8, v_1^8).$$

In Figure 7.1, as before, solid dots represent generators carried by the 0-cell of $H(8, v_1^8)$ and open circles are carried by the 1-cell. Unlike the case of $\mathrm{Ext}_{A(2)_*}(H(8))$, there is v_1^8-torsion in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8))$. This results in classes in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$ carried by the 17-cell and the 18-cell of $H(8, v_1^8)$, which are represented by solid triangles and open triangles, respectively. A box around a generator indicates that that generator actually carries a copy of $\mathbb{F}_2[h_{2,1}]$. As before, everything is v_2^8-periodic.

One can similarly compute

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k} \otimes H(8, v_1^8))$$

for larger values of k by applying the same method to the corresponding computations of

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$$

in [BHHM08]. We do not bother to record the complete results of these computations for small values of k, but will freely use them in what follows. The sequences of spectral sequences 5.5 imply these computations control $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_I)$.

In this section and the next, if

$$\mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \otimes H(8, v_1^8))$$

has a unique non-zero element, we shall refer to it as

$$x_{t-s,s}(i_1, \ldots, i_n).$$

We will now prove the following.

Proposition 7.1. The elements

$$\underline{v_2^{16}\nu^2}[18],\ \underline{v_2^{16}\epsilon}[18],\ \underline{v_2^{16}\kappa}[18],\ \underline{v_2^{16}\bar{\kappa}\eta^2}[18] \in \mathrm{Ext}^{s,t}_{A(2)}(H(8, v_1^8))$$

of Figure 6.5 are permanent cycles in the algebraic tmf-resolution for $H(8, v_1^8)$.

Corollary 7.2. The elements

$$\underline{v_2^{16}\nu^2}[18], \underline{v_2^{16}\epsilon}[18], \underline{v_2^{16}\kappa}[18], \underline{v_2^{16}\bar{\kappa}\eta^2}[18] \in \mathrm{Ext}^{s,t}_{A(2)}(H(8, v_1^8))$$

lift to elements of $\mathrm{Ext}_{A_*}(H(8, v_1^8))$.

Proof of Proposition 7.1. We begin by enumerating the targets of possible differentials supported by these classes in the algebraic tmf-resolution

$$\mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \otimes H(8, v_1^8)) \Rightarrow \mathrm{Ext}^{s+n, t+8(i_1+\cdots+i_n)}_{A(2)_*}.$$

Here, a d_r differential raises n by r. Since the classes in question lie on the $n = 0$ line, the possible targets of differentials will all lie in terms with $n > 0$. We begin with the "edge" case where $i_1 = i_2 = \cdots = i_n = 1$.

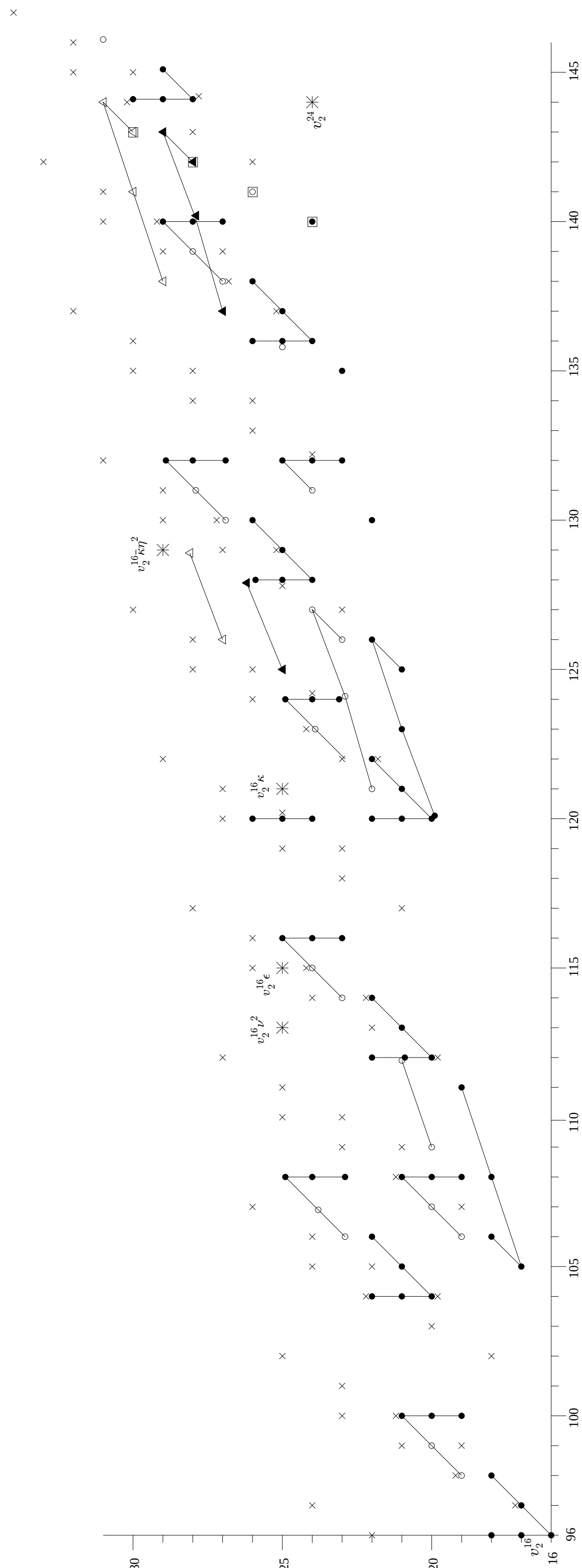


FIGURE 7.2. $\mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$ in the region $96 \leq t - s \leq 144$.

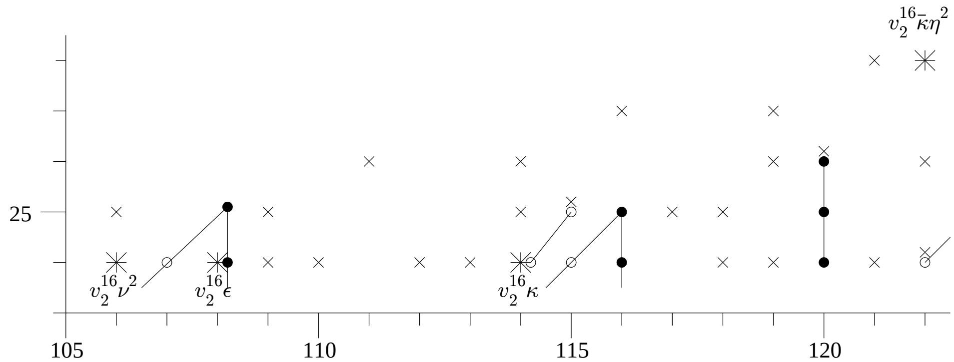


FIGURE 7.3. Potential targets of differentials in $\mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2} \otimes H(8, v_1^8))$

The first term to consider is $\mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$, displayed in Figure 7.2. These Ext groups can be easily determined from Figure 7.1 by multiplying everything by v_2^{16}, and propagating the $h_{2,1}$-towers, and their v_2^8-multiples. The classes belonging to $h_{2,1}$-towers are labeled with $\times$'s. In this figure, we have indicated the relative position of the classes from the $n = 0$-term of the algebraic tmf-resolution we wish to analyze with $\ast$'s. We have set things up so that targets of differentials in the algebraic tmf-resolution look like Adams d_1-differentials. For example, we have

$$d_1(\underline{v_2^{16}\nu^2}[18]) \in \mathrm{Ext}^{26,112+26}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$$

and there are no non-trivial targets for such a differential. Actually, the only possibility for a non-trivial d_1 on the classes in question hitting a class in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$ is

$$d_1(\underline{v_2^{16}}\kappa[18]) = x_{120,26}(1) \in \mathrm{Ext}^{26,120+26}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8).$$

However, since $\underline{\kappa} = d_0 \in \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$ lifts to a class

$$d_0[18] \in \mathrm{Ext}_{A(2)_*}(H(8, v_1^8)),$$

using Lemma 4.7 we have

$$d_1(\underline{v_2^{16}\kappa}[18]) = v_2^{16} d_1(d_0[18]).$$

Since $x_{120,26}(1)$ is not v_2^8 divisible, we deduce that it cannot be the target of such a differential.

Figures 7.3 and 7.4 display the corresponding targets of potential d_1-differentials coming from the terms

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2} \otimes H(8, v_1^8)), \quad \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 3} \otimes H(8, v_1^8))$$

in the algebraic tmf-resolution. As the figures indicate, there are no possible non-zero targets of such d_1 differentials. One finds that there are no contributions from

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k} \otimes H(8, v_1^8))$$

for $k \geq 4$ since the classes in question lie past the vanishing edge of these Ext groups.

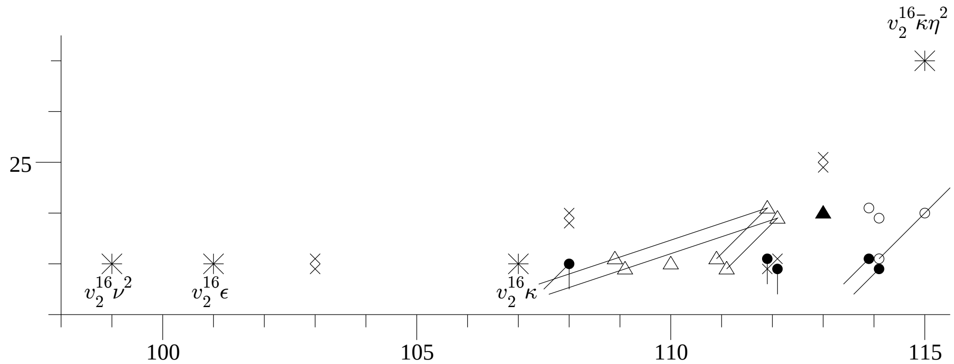


FIGURE 7.4. Potential targets of differentials in $\mathrm{Ext}^{s,t}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 3} \otimes H(8, v_1^8))$

An elementary analysis, using the technology of Section 5, shows that the classes lie beyond the vanishing edge of all of the other terms in the algebraic tmf-resolution. □

Remark 7.3. One could also attempt to prove Proposition 7.1 by showing the classes ν^2, ϵ, κ, and $\bar{\kappa}\eta^2$ lift to the top cell of $M(8, v_1^8)$, and then use Lemma 4.7. However, the obstruction that showed up in the case of $v_2^{16}\kappa$ also shows up in the case of κ, but is harder to eliminate because one no longer can appeal to v_2^8 divisibility.

8. The MASS for $M(8, v_1^8)$

We are now in a position to proof our main result.

Theorem 8.1. The elements

$$\underline{v_2^{16}\nu^2}[18],\ \underline{v_2^{16}\epsilon}[18],\ \underline{v_2^{16}\kappa}[18],\ \underline{v_2^{16}\bar{\kappa}\eta^2}[18] \in \mathrm{Ext}^{s,t}_{A_*}(H(8, v_1^8))$$

of Figure 6.5 are permanent cycles in the MASS for $M(8, v_1^8)$.

Proof. We analyze the potential targets of Adams differentials supported by these elements x using the algebraic tmf-resolution. We can rule out any possible contributions from

$$\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$$

because if $d_r(x)$ is detected by $\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$, the map of MASS's induced by the map

$$M(8, v_1^8) \to \mathrm{tmf} \wedge M(8, v_1^8)$$

would result in a corresponding non-trivial differential in the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$. However, we already know that these classes x exist in $\mathrm{tmf}_* M(8, v_1^8)$. We are left with showing that $d_r(x)$ cannot be detected by

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \otimes H(8, v_1^8))$$

in the algebraic tmf-resolution. Consulting the analysis of the proof of Proposition 7.1, we see that the only possible targets of $d_r(x)$ could be detected in

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8)).$$

By Figure 7.2, these possibilities are:

$$d_2(\underline{v_2^{16}\nu^2}[18]) = x_{112,27}(1),$$
$$d_2(\underline{v_2^{16}}\kappa[18]) = x_{120,27}(1).$$

In each of these two cases, the source is $g = h_{2,1}^4$-torsion, while the target is g-torsion free. The source is checked to be g-torsion by checking there are no terms in higher filtration in the algebraic tmf-resolution which could detect a g-tower supported by the source. The target is checked to be g-torsion free, as the only possible way for a g-tower on the target to be truncated in the algebraic tmf-resolution would be for it to be killed by a g-tower in $\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$. The only possible g-towers which could do such a killing originate in stems smaller than the stems of the targets in question, and would therefore kill the targets themselves. □

9. Tentative results in the range 141 − 200

To edify the reader's curiosity, we include tables with some tentative results on exotic spheres beyond dimension 140. The classes in dimensions 158, 160, 168, and 192 where pointed out by Zhouli Xu. A key resource are the computations of Christian Nassau [Nas].

Some non-trivial elements of $(\mathrm{coker}\, J_{4k})_{(p)}$**.**

$4k$	$p = 2$	$p = 3$	$p = 5$
140			0
144	$\boldsymbol{((v_2^{16}\eta w)\eta/2)\eta}$		0
148	$\boldsymbol{v_2^{16}\epsilon\bar{\kappa}}$		0
152			β_1^4
156	$\boldsymbol{\langle v_2^{24}\nu^2, 2\nu, \eta^2\rangle}$		0
160	$\eta_5\eta_7$		0
164	$\boldsymbol{v_2^{24}\kappa\nu^2}$	$\beta_{10}\beta_1$	0
168	$\eta_7\{f_1\}$		0
172			$\beta_3\beta_1$
176			0
180		$\beta_{11}\beta_1$	0
184		$\beta_{10}\beta_1^3$	0
188			0
192	η_6^3		0
196		$\beta_{11}\beta_2$	0
200	$\boldsymbol{v_2^{32}\epsilon}$		$\beta_2\beta_1^3$

Some non-trivial elements of $(\mathrm{coker}\, J_{8k-2})_{(p)}$ **with Kervaire invariant** 0**.**

$8k-2$	$p=2$	$p=3$	$p=5$
142	$\boldsymbol{v_2^{16}\eta w}$		0
150	$\boldsymbol{(v_2^{16}\epsilon\bar{\kappa})\eta^2}$		0
158	$\eta_7\theta_4$		0
166			0
174	$\boldsymbol{\beta_{32/8}}$	$\beta_{10}\beta_1^2$	0
182	$\boldsymbol{\beta_{32/4}}$	$\beta_{12/2}$	β_4
190		$\beta_{11}\beta_1^2$	β_1^5
198	$\boldsymbol{v_2^{32}\nu^2}$		0

The only dimensions in this range where we do not know if exotic spheres exist are 140, 166, 176, and 188.

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