CHROMATIC SPLITTING FOR THE $K(2)$-LOCAL SPHERE AT $p = 2$

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Abstract. We calculate the homotopy type of $L_1L_{K(2)}S^0$ and $L_{K(1)}L_{K(2)}S^0$ at the prime 2, where $L_{K(n)}$ is localization with respect to Morava K-theory and L_1 localization with respect to 2-local K theory. In $L_1L_{K(2)}S^0$ we find all the summands predicted by the Chromatic Splitting Conjecture, but we find some extra summands as well. An essential ingredient in our approach is the analysis of the continuous group cohomology $H^*(\mathbb{G}_2, E_0)$ where $\mathbb{G}_2$ is the Morava stabilizer group and $E_0 = \mathbb{W}[[u_1]]$ is the ring of functions on the height 2 Lubin-Tate space. We show that the inclusion of the constants $\mathbb{W} \to E_0$ induces an isomorphism on group cohomology, a radical simplification.

Contents

1. Introduction

The problem of understanding the homotopy groups of spheres has always been central to algebraic topology. A period of calculation beginning with Serre's computation of the cohomology of Eilenberg- MacLane spaces and Toda's work with the EHP sequence culminated, in the late 1970s, with the work of Miller, Ravenel, and Wilson on periodic phenomena in the homotopy groups of spheres and Ravenel's nilpotence conjectures. The solutions to most of these conjectures in the middle

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1980s established the primacy of the chromatic point of view, which uses the algebraic geometry of smooth 1-parameter formal groups to organize the search for large scale phenomena in stable homotopy theory.

This has been remarkably successful. Much of what we know about stable homotopy theory can be motivated and conjectured by analyzing the moduli stack of formal groups and its quasi-coherent sheaves. See, for example, the table in Section 2 of [HG94]. In particular, this stack has a stratification by height and working along this stratification highlights two distinct lines of research. First, we'd like to discover all that can be learned by working at a single height; or, put another way, we make calculations in $K(n)$-local homotopy theory. Second, we need to assemble the information from different heights. This is the chromatic assembly problem.

In this paper, we give an analysis of the Chromatic Splitting Conjecture of Hopkins at $p = 2$ and $n = 2$. At first glance, this is a chromatic assembly question, but as we proceed here we need extensive information from height 2 calculations. Thus, the questions of single-height calculations and chromatic assembly remain closely related.

The Chromatic Splitting Conjecture predicts a splitting of $L_1 L_{K(2)} S^0$. We do get a splitting, and it contains the expected summands, but it contains other summands as well. That there has to be more was already proved in [Bea17a], which in turn builds on the papers [Bea15] and [Bea17b], all by the first author. There are hints at extra summands in the work of Shimomura and Wang [SW02] as well.

Fix a prime p and let $K(n)$ be the nth Morava K-theory spectrum; by convention, $K(0) = H\mathbb{Q}$, the rational homology spectrum. Let L_n be localization with respect to the homology theory represented by $K(0) \vee K(1) \vee \cdots \vee K(n)$. This is the same as localization with respect to the Johnson-Wilson theory $E(n)$. Then for all spectra X there is a natural map $L_n X \to L_{n-1} X$. The Hopkins-Ravenel Chromatic Convergence Theorem of §8.6 of [Rav92] then says that if X is a finite CW spectrum the induced map

$$X \longrightarrow \operatorname{holim} L_n X$$

is localization at the homology theory $H_*(-, \mathbb{Z}_{(p)})$.

Next, let $L_{K(n)} X$ be localization with respect to $K(n)$. Then there is map $L_n X \to L_{K(n)} X$ and, for any spectrum X, the map $L_n X \to L_{n-1} X$ can be recovered from the *chromatic fracture square*

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X. \end{tikzcd} \tag{1.1.1}$$

That this square is a homotopy pull-back can be found in Theorem 6.19 of [HS99], but is implicit in [Rav84] as well.

The chromatic fracture square and chromatic convergence together imply that we can recover the homotopy type of a finite CW spectrum from $L_{K(n)} X$ for all n and all p, provided we can complete the assembly process of (1.1.1).

Calculations of $L_{K(n)} X$ usually come down to analysis of the $K(n)$-local Adams-Novikov Spectral Sequence. The cohomology theory $K(n)^*$ is complex orientable and the associated formal group Γ_n is of height n. Let $\mathbb{G}_n$ be the automorphisms of the pair $(\mathbb{F}_{p^n}, \Gamma_n)$ and let E_n be the Morava E-theory associated to the pair $(\mathbb{F}_{p^n}, \Gamma_n)$. By the Hopkins-Miller Theorem the profinite group $\mathbb{G}_n$ acts on E_n, and hence on $(E_n)_*X := \pi_* L_{K(n)}(E_n \wedge X)$. If X is a finite CW spectrum, we then have a spectral sequence

$$H^s(\mathbb{G}_n, (E_n)_t X) \Longrightarrow \pi_{t-s} L_{K(n)} X \tag{1.1.2}$$

where cohomology is continuous group cohomology. Much of $K(n)$-local homotopy theory comes down to the analysis of the group $\mathbb{G}_n$ and the action of $\mathbb{G}_n$ on $(E_n)_*$. We give some more details in Section 2 and add references to the large literature on the subject there.

Even assuming we can master the $K(n)$-local calculations, there remains the assembly question. Let $M_n X$ be the fiber of $L_n X \to L_{n-1} X$; then the key result needed to establish the chromatic fracture square of (1.1.1) is that $M_n X \to M_n L_{K(n)} X$ is an equivalence. Crucial to assembly is the other fiber; that is, the fiber of $L_{n-1} X \to L_{n-1} L_{K(n)} X$. This is the main subject of this paper. In fact, we investigate the homotopy type of the map $L_{n-1} X \to L_{n-1} L_{K(n)} X$. The Chromatic Splitting Conjecture, due to Hopkins, is a very specific conjecture about this map, based on hard calculations. We will get into this below; before that, however, we will state our main results.

Let $p = 2$ and write $E = E_2$ for our choice of Morava E-theory at height 2. Let $\mathbb{W}$ be the Witt vectors on $\mathbb{F}_4$. Then there is an isomorphism $E_* \cong \mathbb{W}[[u_1]][u^{\pm 1}]$ where the power series ring is in degree 0 and u has degree -2. Our first result is a $K(2)$-local result. See Theorem 5.4.1 and Theorem 5.4.4

Theorem 1. The inclusion of constants into the power series ring induce isomorphisms in group cohomology

$$H^*(\mathbb{G}_2, \mathbb{F}_4) \to H^*(\mathbb{G}_2, E_0/2)$$

and

$$H^*(\mathbb{G}_2, \mathbb{W}) \to H^*(\mathbb{G}_2, E_0).$$

This is a remarkable simplification. We conjecture that the analogous result is true at all heights and all primes. It is true wherever it has been checked; that is, for $n \leq 2$ and all primes. If $n = 1$ this is a tautology. For $n = 2$ and $p > 3$, it can be deduced from [SY95] (see also Corollaire 4.5 of [Lad13]). This basic case was also proved later in [Koh13] using different techniques. For $n = 2$ and $p = 3$ it can be deduced from [HKM13]; [GHM14]. The primes 2 and 3 are harder, as the group $\mathbb{G}_2$ contains p-torsion subgroups.

The next step is to calculate differentials in the Adams-Novikov spectral sequence (1.1.2) for $X = S^0$. By an old result of Morava and Lazard we know that for all n and p the cohomology ring $H^*(\mathbb{G}_n, \mathbb{W}) \otimes \mathbb{Q}$ is an exterior algebra on n generators of degree $2i - 1$ for $1 \leq i \leq n$; then Theorem 1 implies $\pi_i L_{K(2)} S^0$ has a torsion-free generator when $i = 0$, -1, -3, or -4. To get further, we need to get some control on the torsion.

For all n and p, the group $\mathbb{G}_n$ comes equipped with a determinant map

$$\det\colon \mathbb{G}_n \to \mathbb{Z}_p^\times$$

to the units in the p-adic integers. If $p = 2$, there is an isomorphism $\mathbb{Z}_2^\times \cong \mathbb{Z}_2 \times C_2$, where $C_2 = \{\pm 1\}$ is the cyclic group of order 2. We thus get a map

$$E(\zeta_2) \otimes \mathbb{F}_2[\chi] \cong H^*(\mathbb{Z}_2 \times C_2, \mathbb{F}_2) \to H^*(\mathbb{G}_2, \mathbb{F}_2).$$

Here $E(-)$ denotes the exterior algebra over $\mathbb{F}_2$. These cohomology classes will be discussed in Remark 5.1.1. We will show in Proposition 5.3.1 (but see also Theorem 6.3.24 of [Rav86]), that this map induces an injection

$$E(\zeta_2) \otimes \mathbb{F}_2[\chi]/(\chi^3) \xrightarrow{\subseteq} H^*(\mathbb{G}_2, \mathbb{F}_2).$$

Here is one place when the prime 2 is fundamentally different. At odd primes, $\mathbb{Z}_p^\times \cong \mathbb{Z}_p \times C_{p-1}$, where C_{p-1} is a cyclic group of order $p-1$. Thus $H^*(\mathbb{Z}_p^\times, \mathbb{F}_p) \cong E(\zeta_2)$. The class χ only appears at $p = 2$.

The class ζ_2 is the reduction of a class of infinite order in $H^1(\mathbb{G}_2, \mathbb{Z}_2)$; that this class is a permanent cycle in the Adams-Novikov Spectral Sequence is well understood. In a standard abuse of notation we also write $\zeta_2 \in \pi_{-1} L_{K(2)} S^0$ for a particular homotopy class detected by the cohomology class ζ_2. See [DH04] and Proposition 2.2.1 for details. We will show that the Bockstein on the class χ is also a permanent cycle in $H^2(\mathbb{G}_2, \mathbb{Z}_2)$ and detects a class $x \in \pi_{-2} L_{K(2)} S^0$ of order 2. We will then show that the Bockstein on $\zeta_2 \chi$ detects the class $\zeta_2 x \in \pi_{-3} L_{K(2)} S^0$, also of order 2.

Let $V(0)$ be the mod 2 Moore spectrum and let $\iota : S^0 \to L_{K(2)} S^0$ be the unit. The classes ι, ζ_2, x and $\zeta_2 x$ together with choices for the torsion free generators of $\pi_i L_{K(2)} S^0$ for $i = -3$ and $i = -4$ can be used to define a map

$$f : S^0 \vee S^{-1} \vee S^{-3} \vee S^{-4} \vee \Sigma^{-2} V(0) \vee \Sigma^{-3} V(0) \longrightarrow L_{K(2)} S^0 \tag{1.1.3}$$

Our main result then describes chromatic splitting at $n = p = 2$. Let S_2^n denote the 2-completed n-sphere.

Theorem 2. The map f defines a weak equivalence

$$L_1(S_2^0 \vee S_2^{-1}) \vee L_0(S_2^{-3} \vee S_2^{-4}) \vee L_1(\Sigma^{-2} V(0) \vee \Sigma^{-3} V(0)) \simeq L_1 L_{K(2)} S^0.$$

In particular, $L_1 S_2^0 \to L_1 L_{K(2)} S^0$ is a split inclusion.

The spherical summands are predicted by the Chromatic Splitting Conjecture, but the Moore spectrum summands are the new phenomenon.

The strategy for proving Theorem 2 is to use the chromatic fracture square (1.1.1) to deduce the homotopy type $L_1 L_{K(2)} S^0$. More precisely, we prove the following rational result. See Theorem 5.5.2 and Theorem 7.5.2.

Theorem 3. Let S_2^n be the 2-complete n-sphere. Then the map f of (1.1.3) induces a weak equivalence

$$L_0(S_2^0 \vee S_2^{-1} \vee S_2^{-3} \vee S_2^{-4}) \simeq L_0 L_{K(2)} S^0.$$

Furthermore, we prove the following $K(1)$-local result. See Theorem 7.4.1.

Theorem 4. The restriction of the map f of (1.1.3) to $S^0 \vee S^{-1} \vee \Sigma^{-2} V(0) \vee \Sigma^{-3} V(0)$ induces a weak equivalence

$$L_{K(1)}\big(S^0 \vee S^{-1} \vee \Sigma^{-2} V(0) \vee \Sigma^{-3} V(0)\big) \simeq L_{K(1)} L_{K(2)} S^0.$$

Much of the work in this paper goes into this last theorem. Theorem 2 then follows exactly as in the prime 3 case; see [GHM14] and Theorem 7.5.2.

We conclude this introduction by revisiting the Chromatic Splitting Conjecture. This is due to Hopkins and can be found in the literature in [Hov95]. As we mentioned above, there is a result of Morava and Lazard that the cohomology ring $H^*(\mathbb{G}_n, \mathbb{W}) \otimes \mathbb{Q}$ is an exterior algebra on n generators of degree $2i-1$ for $1 \leq i \leq n$. Part of the Chromatic Splitting Conjecture is that, for some choice of generators $x_i \in H^{2i-1}(\mathbb{G}_n, \mathbb{W})$, the exterior algebra $E_{\mathbb{Z}_p}(x_1, \ldots, x_n)$ over $\mathbb{Z}_p$ maps non-trivially to permanent cycles in $H^*(\mathbb{G}_n, (E_n)_*)$. This would give a map out of a wedge of spheres to $L_{K(n)}S^0$ indexed on the monomial basis of the exterior algebra. The following would completely describe the gluing data (1.1.1).

Conjecture 5 (Strong Chromatic Splitting Conjecture)**.** This map out of the wedge of spheres induces an equivalence

$$L_{n-1}S_p^0 \vee \bigvee_{1 \leq i_1 < \ldots < i_j \leq n} L_{n-i_j} S_p^{-2(\sum i_k)+j} \simeq L_{n-1}L_{K(n)}S^0 \ .$$

The conjecture is known to hold for $n = 1$ if $p \geq 2$ and for $n = 2$ if $p \geq 3$ ([Beh12], [GHM14], [SY95]). However, the results of [Bea17a] already implies that Conjecture 5 does not hold when $n = p = 2$ and Theorem 2 makes this completely precise. We note that although Conjecture 5 does not hold at $p = 2$, there is no evidence that it should fail at odd primes, or at least when p is large with respect to n

We can also write down a weaker form of the conjecture, which holds for all p and $n \leq 2$; that is, for all cases where we've been able to check.

Conjecture 6 (Weak Chromatic Splitting Conjecture)**.** If X is the p-completion of a finite spectrum, the map $L_{n-1}X \to L_{n-1}L_{K(n)}X$ is a split inclusion.

This second conjecture would imply that there are maps $L_{K(n)}S^0 \to L_{K(n-1)}S^0$ such that $S^0 \simeq \operatorname{holim}_n L_{K(n)}S^0$; that is, S^0 can be recovered from its Morava K-theory localizations.

Organization of the paper. In Section 2, we review some of the background from chromatic homotopy theory, including some of the more technical techniques at the prime $p = 2$. In Section 3, we recall some classical results from homotopy theory; most of this can be summarized in the remark that extra care is needed because the order of the identity of the mod 2 Moore spectrum is not equal to two. Then in Section 4 we give a detailed review of $K(1)$–local computations at the prime $p = 2$ which are used in later sections. In Section 5, we begin our analysis of the case $n = p = 2$. We describe the cohomology of various subgroups G of $\mathbb{G}_2$. The main result in this section is the proof of Theorem 1 in Theorem 5.4.4. Section 6 is dedicated to one of the key technical results of the paper: the class χ is a d_3–cycle in the $K(2)$-local Adams-Novikov Spectral Sequence for the Moore spectrum. Section 7 contains the proof of Theorem 4. This theorem is shown by studying the v_1–localized Adams-Novikov Spectral Sequences computing $L_{K(1)}L_{K(2)}V(0)$ and especially $L_{K(1)}L_{K(2)}Y$ where $Y = V(0) \wedge C(\eta)$. The spectrum Y was used in Mahowald's proof of the Telescope Conjecture at $p = 2$ and $n = 1$. See [Mah82]. We also deduce Theorem 2 and Theorem 3 at the end of Section 7.

We have been slightly disingenuous in our presentation above, as we actually prove Theorem 4 before constructing the map f of (1.1.3); that is, we only construct the $K(1)$-localization of the map f in Section 7. In Section 8 we complete the difficult task of constructing f.

Acknowledgements. This project had its genesis in conversations with Mark Mahowald, who was trying to come to terms with calculations of Shimomura and Wang [SW02]. Specifically, Mark thought that those authors had identified v_1-torsion-free summands in the E_2-term of the Adams-Novikov Spectral Sequence for $\pi_*L_{K(2)}V(0)$ which could not be explained by the Chromatic Splitting Conjecture. In some sense this entire paper, as well as [Bea17a] and [Bea17b], are an attempt to ratify and explain this insight.

Careful readers of Section 6 below will recognize that the techniques and ideas there are completely different from the rest of the paper. This lateral move arises from an insight of Mike Hopkins: namely, that the isomorphism of Theorem 1 could be extended to a homomorphism of homotopy fixed point spectral sequences. See Remark 6.1.2 for more details. We don't completely prove that, but we do get enough information from this idea to prove our key Theorem 6.1.1. We extend heartfelt thanks to Hopkins for sharing this idea.

Finally, this work has taken place over a number years and at a number of places. We thank the Hausdorff Institute of Mathematics, the Université de Strasbourg, and the University of Colorado all for providing such wonderful places to work.

2. Preliminaries

We begin by introducing the $K(n)$-local category, Morava E-theory, the Morava stabilizer group, and general convergence results for the $K(n)$-local Adams-Novikov Spectral Sequence. We then get specific at $n = 2$ and $p = 2$, discussing the role of formal groups from supersingular elliptic curves. We close the section with some background on algebraic and topological duality resolutions.

2.1. **The $K(n)$-local category.** Fix a prime p. Let Γ_n be a formal group of height n over the finite field $\mathbb{F}_p$ of p elements and let $\mathrm{End}(\Gamma_n/\mathbb{F}_p)$ be the endomorphism ring of Γ_n over $\mathbb{F}_p$. The unique map rings $\mathbb{Z}_p \to \mathrm{End}(\Gamma_n/\mathbb{F}_p)$ is an inclusion into the center. Because Γ_n is defined over $\mathbb{F}_p$ the Frobenius map $\xi(x) = x^p$ also defines an endomorphism of Γ_n. We will assume the endomorphism $\xi^n(x) = x^{p^n}$ satisfies an equation

$$\xi^n = ap \in \mathrm{End}_{\mathbb{F}_{p^n}}(\Gamma_n/\mathbb{F}_p) \tag{2.1.1}$$

where $a \in \mathbb{Z}_p^\times$ is a unit. The Honda formal group of height n satisfies these criteria: this has a formal group law which is p-typical and with p-series $[p](x) = x^{p^n} = \xi^n(x)$. However, if $n = 2$ and $p = 2$, then the formal group of a supersingular elliptic curve defined over $\mathbb{F}_2$ will also do, and this will be our preferred choice if $p = 2$; see Section 2.4.

Let $i\colon \mathbb{F}_p \hookrightarrow k$ be any extension of $\mathbb{F}_p$ and let $\mathrm{Aut}(\Gamma_n/k)$ be the group of automorphisms of $i^*\Gamma_n$ over k. Our assumption (2.1.1) implies that for any extension $\mathbb{F}_{p^n} \subseteq k$ there is an isomorphism

$$\mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \xrightarrow{\cong} \mathrm{Aut}(\Gamma_n/k).$$

To shorten notation we define

$$\mathbb{S}_n = \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}). \tag{2.1.2}$$

If we choose a coordinate for Γ_n, then every element of $\mathbb{S}_n$ can be expressed as a power series $\phi(x) \in x\mathbb{F}_{p^n}[[x]]$ invertible under composition. The map $\phi(x) \mapsto \phi'(0)$ defines a surjective map

$$\mathbb{S}_n \longrightarrow \mathbb{F}_{p^n}^{\times}.$$

We define S_n to be the kernel of this map; this is the p-Sylow subgroup of the profinite group $\mathbb{S}_n$. There is an isomorphism $S_n \rtimes \mathbb{F}_{p^n}^{\times} \cong \mathbb{S}_n$.

Define the extended Morava stabilizer $\mathbb{G}_n$ as the automorphism group of the pair $(\mathbb{F}_{p^n}, \Gamma_n)$. Elements of $\mathbb{G}_n$ are pairs (f, ϕ) where $f \in \mathrm{Aut}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ and $\phi\colon \Gamma_n \to f^*\Gamma_n$ is an isomorphism of formal groups. Since Γ_n is defined over $\mathbb{F}_p$, there is an isomorphism

$$\mathbb{G}_n \cong \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p). \tag{2.1.3}$$

We next define Morava K-theory; there are many variants, all of which have the same Bousfield class and define the same localization. To be specific, let $K(n) = K(\mathbb{F}_{p^n}, \Gamma_n)$ be the 2-periodic ring spectrum with homotopy groups

$$K(n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]$$

and with associated formal group Γ_n. Here the class u is in degree -2. This slightly unclassical choice of $K(n)$ has the property that it receives a map $E_n \to K(n)$ from Morava E-theory defined below in (2.1.4).

We will spend a great deal of time working in the $K(n)$-local category and, when doing so, all our spectra will implicitly be localized. In particular, we emphasize that we will write $X \wedge Y$ for $L_{K(n)}(X \wedge Y)$, as this is the smash product internal to the $K(n)$-local category.

We now define the Lubin-Tate spectrum $E = E_n = E(\mathbb{F}_{p^n}, \Gamma_n)$. This is the complex oriented, Landweber exact, 2-periodic, E_∞-ring spectrum with

$$E_* = (E_n)_* \cong \mathbb{W}[[u_1, \cdots, u_{n-1}]][u^{\pm 1}] \tag{2.1.4}$$

with u_i in degree 0 and u in degree -2. Here $\mathbb{W} = W(\mathbb{F}_{p^n})$ is the ring of Witt vectors on $\mathbb{F}_{p^n}$. Note that E_0 is a complete local ring with residue field $\mathbb{F}_{p^n}$; the formal group over E_0 is a choice of universal deformation of the formal group Γ_n over $\mathbb{F}_{p^n}$. (We will be specific about this choice at $n = p = 2$ below in Section 2.4.) By the Hopkins-Miller theorem, the group $\mathbb{G}_n = \mathrm{Aut}(\Gamma_n) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on E.

Remark 2.1.1 (Morava modules). If X is a spectrum we define

$$E_*X = \pi_* L_{K(n)}(E \wedge X).$$

Despite the notation, the functor $E_*(-)$ is not a homology theory, as it does not take wedges to sums in general. Nonetheless, it is our most important algebraic invariant intrinsic to the $K(n)$-local category.

The E_*-module E_*X is equipped with the $\mathfrak{m}$-adic topology where $\mathfrak{m}$ is the maximal ideal in E_0. This topology is always complete, but need not be separated in general – although, in fact, all the E_*-modules we consider in this paper will be complete and separated. See [GHMR05] §2 for some precise assumptions which guarantee that E_*X is complete and separated.

The action of $\mathbb{G}_n$ on E_n determines a continuous action of $\mathbb{G}_n$ on $(E_n)_*X$. If $n = 1$, we can choose $E_* = K_*$, p-complete K-theory, and $\mathbb{G}_1 = \mathbb{Z}_p^\times$, the units in the p-adics. The action is then through the Adams operations and in that case we might write ψ^ℓ for the action of $\ell \in \mathbb{Z}_p^\times$; for example, as in Remark 4.1.1. Not withstanding this, as general rule we will simply write g_* for the action of $g \in \mathbb{G}_n$ on E_*X.

This action is twisted in the sense that if $g \in \mathbb{G}_n$, $a \in E_*$ and $x \in E_*X$, then $g_*(ax) = g_*(a)g_*(x)$. We will call such modules either Morava modules or *twisted E_*-$\mathbb{G}_n$-modules.* When we consider closed subgroups H of $\mathbb{G}_n$ and E_*-modules with an action of H satisfying the analogous formula for $h \in H$ then we call such modules *twisted E_*-H-modules.*

Remark 2.1.2. The E_*-algebra E_*E has a $\mathbb{G}_n$-action on both the left and right factor. The action of the left factor defines the Morava module structure. Using the action on the right factor we get a composite map

$$E_*E \times \mathbb{G}_n \longrightarrow E_*E \xrightarrow{m} E_*$$

where m is induced by the multiplication $E \wedge E \to E$. The adjoint to this map is an isomorphism

$$E_*E \cong \mathrm{map}_{cts}(\mathbb{G}_n, E_*) \tag{2.1.5}$$

of Morava modules. Here $\mathrm{map}_{cts}(-,-)$ denotes the set of continuous maps. On the right hand side of this equation, E_* acts on the target and the $\mathbb{G}_n$-action is the diagonal action given by $(g_*\phi))(x) = g_*\phi(g_*^{-1}x)$.

Caution is needed here, as the isomorphism (2.1.5) need not hold for the Lubin-Tate homology theory $E(k, \Gamma)$ for an arbitrary height n formal group Γ over a field $k \subseteq \overline{\mathbb{F}}_p$. In the literature (2.1.5) is proved for the Honda formal group; see, for example, Theorem 12 of [Str00]. In examining the proof there we see that what is needed is our assumption from (2.1.1). The details needed to then produce the isomorphism of (2.1.5), and more information as well, can be found in §5 of [Hen18].

Now suppose X is a finite p-local spectrum. From (2.1.5) it follows (again see [Str00]) that the $K(n)$-local E-based Adams Spectral Sequence for X has the form

$$H^s(\mathbb{G}_n, E_tX) \Longrightarrow \pi_{t-s}L_{K(n)}X. \tag{2.1.6}$$

Group cohomology here is *continuous* group cohomology. There is extensive discussion of this spectral sequence in [DH04].

Complex orientations define maps of ring spectra

$$BP \longleftarrow MU \longrightarrow E\ .$$

If we localize at a prime and if X is a finite p-local spectrum, we get a diagram of spectral sequences where the upward arrows are isomorphisms

$$\begin{array}{ccc} \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*X) & \Longrightarrow & \pi_*X \\ \cong\uparrow & & \uparrow\cong \\ \mathrm{Ext}^{*,*}_{MU_*MU}(MU_*, MU_*X) & \Longrightarrow & \pi_*X \\ \downarrow & & \downarrow \\ H^*(\mathbb{G}_n, E_*X) & \Longrightarrow & \pi_*L_{K(n)}X\ . \end{array} \tag{2.1.7}$$

Remark 2.1.3. For example, let $F \subseteq \mathbb{G}_n$ be a closed subgroup. In [DH04] Devinatz and Hopkins defined and studied a homotopy fixed point spectrum E^{hF} with the property that the isomorphism of (2.1.5) descends to an isomorphism of Morava modules

$$E_*E^{hF} \cong \mathrm{map}_{cts}(\mathbb{G}_n/F, E_*) \tag{2.1.8}$$

and a spectral sequence for X a finite CW spectrum

$$H^s(F, E_tX) \Longrightarrow \pi_{t-s}(E^{hF} \wedge X)\ . \tag{2.1.9}$$

If $F = \mathbb{G}_n$ itself this spectral sequence is the $K(n)$-local Adams-Novikov Spectral Sequence for $L_{K(n)}S^0$ and, if F is finite, this spectral sequence is the homotopy fixed point spectral sequence for the action of F on E. The Devinatz-Hopkins construction is natural in F and, in particular, if $F_1 \subseteq F_2$ are two closed subgroups, we have a commutative diagram of spectral sequences

$$\begin{tikzcd} H^s(F_2, E_tX) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(E^{hF_2} \wedge X) \arrow[d] \\ H^s(F_1, E_tX) \arrow[r, Rightarrow] & \pi_{t-s}(E^{hF_1} \wedge X) \end{tikzcd} \tag{2.1.10}$$

where the vertical arrows are the natural maps induced by the inclusion of F_1 into F_2.

Again, some caution is needed here, as the Devinatz-Hopkins paper works only with the Lubin-Tate theory defined for the Honda formal group. However, a close reading of [DH04] shows that they need not have been so specific. They use only the Hopkins-Miller theorem, which states that the space of E_∞-self maps of $E(k, \Gamma_n)$ is homotopically discrete, as well as the isomorphism of (2.1.5). The Hopkins-Miller theorem holds for any Lubin-Tate spectrum $E(k, \Gamma)$ where $k \subseteq \overline{\mathbb{F}}_p$ and Γ is of finite height. In addition, (2.1.5) holds for the formal group of Section 2.4. This caution will come up again below, but we won't repeat this comment.

2.2. **Subgroups of $\mathbb{G}_n$.** Various subgroups of $\mathbb{G}_n$ will play a role in this paper; we discuss some here. Some finite subgroups will appear later, in Section 2.4, after we have introduced our choice of formal group.

The right action of $\mathbb{S}_n = \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n})$ on $\mathrm{End}(\Gamma_n/\mathbb{F}_{p^n})$ defines a determinant map $\det\colon \mathbb{S}_n \to \mathbb{Z}_p^\times$ which extends to a determinant map

$$\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{\det \times 1} \mathbb{Z}_p^\times \times \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{p_1} \mathbb{Z}_p^\times. \tag{2.2.1}$$

Define the *reduced determinant* N to be the composition

$$\begin{tikzcd} \mathbb{G}_n \arrow[r, "\det"] \arrow[rr, bend right, "N"'] & \mathbb{Z}_p^\times \arrow[r] & \mathbb{Z}_p^\times/C \cong \mathbb{Z}_p \end{tikzcd} \tag{2.2.2}$$

where $C \subseteq \mathbb{Z}_p^\times$ is the maximal finite subgroup. For example, $C = \{\pm 1\}$ if $p = 2$. There are isomorphisms $\mathbb{Z}_p^\times/C \cong \mathbb{Z}_p$. We choose one for now, although we will be much more specific in Remark 4.1.1 below. Write $\mathbb{G}_n^1$ for the kernel of N, $\mathbb{S}_n^1 = \mathbb{S}_n \cap \mathbb{G}_n^1$, and $S_n^1 = S_n \cap \mathbb{G}_n^1$. The map $N\colon S_n \to \mathbb{Z}_p$ is split surjective and we have semi-direct product decompositions for all of $\mathbb{G}_n$, $\mathbb{S}_n$, and S_n; for example, there is an isomorphism

$$\mathbb{S}_n^1 \rtimes \mathbb{Z}_p \cong \mathbb{S}_n.$$

If n is prime to p, we can choose a central splitting and the semi-direct product is actually a product, but that is not the case of interest here.

The surjective homomorphism $N\colon \mathbb{G}_n \to \mathbb{Z}_p$ defines a non-zero cohomology class $\zeta_n \in H^1(\mathbb{G}_n, \mathbb{Z}_p)$. Also write

$$\zeta_n \in H^1(\mathbb{G}_n, E_0) \tag{2.2.3}$$

for the image of this class under the map induced by the unique continuous homomorphism of rings $\mathbb{Z}_p \to E_0$.

The class ζ_n detects a homotopy class in $\pi_{-1}L_{K(n)}S^0$. Let $\pi \in \mathbb{G}_n$ be any element so that $N(\pi) \in \mathbb{Z}_p$ is a topological generator. Then we have a fibration sequence

$$L_{K(n)}S^0 \simeq E^{h\mathbb{G}_n} \longrightarrow E^{h\mathbb{G}_n^1} \xrightarrow{\pi-1} E^{h\mathbb{G}_n^1} \tag{2.2.4}$$

The following result can be found in Proposition 8.2 of [DH04].

Proposition 2.2.1. *The class ζ_n is a non-zero permanent cycle in the Adams-Novikov Spectral Sequence*

$$H^s(\mathbb{G}_n, E_t) \Longrightarrow \pi_{t-s}L_{K(n)}S^0$$

detecting the image of the unit in $\pi_0 E^{h\mathbb{G}_2^1}$ under the boundary map

$$\pi_0 E^{h\mathbb{G}_n^1} \longrightarrow \pi_{-1}L_{K(n)}S^0.$$

In an abuse of notation we will call this homotopy class ζ_n as well.

For many purposes the action of the Galois group $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \subseteq \mathbb{G}_n$ is harmless. This can be made precise in the following two results, which are from Section 1 of [BG18]. The key observation is that the extension of rings $\mathbb{Z}_p \to \mathbb{W} = W(\mathbb{F}_{p^n})$ is Galois with Galois group Gal.

Lemma 2.2.2. *Let $F \subseteq \mathbb{G}_n$ be a closed subgroup and let $F_0 = F \cap \mathbb{S}_n$. Suppose the canonical map*

$$F/F_0 \longrightarrow \mathbb{G}_n/\mathbb{S}_n \cong \mathrm{Gal}$$

is an isomorphism. Then for any twisted $\mathbb{G}_n$-module M we have isomorphisms

$$H^*(F, M) \cong H^*(F_0, M)^{\mathrm{Gal}}$$
$$H^*(F_0, M) \cong \mathbb{W} \otimes_{\mathbb{Z}_p} H^*(F, M)\ .$$

This has the following topological analog. If X space, let X_+ denote X with disjoint basepoint.

Lemma 2.2.3. *Let $F \subseteq \mathbb{G}_n$ be a closed subgroup and let $F_0 = F \cap \mathbb{S}_n$. Suppose the canonical map*

$$F/F_0 \longrightarrow \mathbb{G}_n/\mathbb{S}_n \cong \mathrm{Gal}$$

is an isomorphism. Then there is a Gal*-equivariant equivalence*

$$\mathrm{Gal}_+ \wedge E^{hF} \to E^{hF_0}.$$

Remark 2.2.4. For a finite spectrum X and closed subgroup F of $\mathbb{G}_n$ the natural map $E^{hF} \wedge X \to (E \wedge X)^{hF}$ is an equivalence. Combining Lemma 2.2.2 and Lemma 2.2.3 thus yields an isomorphism of spectral sequences

$$\begin{tikzcd}
\mathbb{W} \otimes H^*(F, E_*X) \arrow[r, Rightarrow] \arrow[d, "\cong"'] & \mathbb{W} \otimes \pi_*(E \wedge X)^{hF} \arrow[d, "\cong"] \\
H^*(F_0, E_*X) \arrow[r, Rightarrow] & \pi_*(E \wedge X)^{hF_0}
\end{tikzcd}$$

where the differentials on the top line are the $\mathbb{W}$-linear differentials extended from the spectral sequence for F.

2.3. **Strong vanishing lines.** The $K(n)$-local Adams-Novikov Spectral Sequence satisfies a very strong horizontal vanishing line condition; see Definition 2.3.1, Lemma 2.3.2, and Theorem 2.3.5 below. We will need this throughout Section 7. These results are in the literature, but a bit hard to pull together in complete detail, so we provide a summary here. The main references are in the proof of Corollary 15 of [Str00] and Theorem 8.9 of [HS99].

Let us write

$$\begin{tikzcd}
X \arrow[dr] & & Z \arrow[ll, dashed] \\
& Y \arrow[ur] &
\end{tikzcd}$$

for a cofiber sequence (i.e., a triangle) $X \to Y \to Z \to \Sigma X$. Suppose we have a diagram, where each of the triangles is a cofiber sequence.

(2.3.1)
$$\begin{tikzcd}
X \arrow[dr] & & X_1 \arrow[ll, dashed] \arrow[dr] & & X_2 \arrow[ll, dashed] \arrow[dr] & & X_3 \arrow[ll, dashed] \arrow[dr] & & \cdots \arrow[ll, dashed] \\
& F_0 \arrow[ur] & & F_1 \arrow[ur] & & F_2 \arrow[ur] & & F_3 \arrow[ur] &
\end{tikzcd}$$

From (2.3.1) we obtain a spectral sequence (with $s \geq 0$)

$$E_1^{s,t} = \pi_t F_s \Longrightarrow \pi_{t-s} X. \tag{2.3.2}$$

Recall that the spectral sequence (2.3.2) is strongly convergent if

$$\lim_s \, \pi_{t+s} X_s = 0 = \lim^1_s \, \pi_{t+s} X_s \tag{2.3.3}$$

for all t.

Definition 2.3.1. The spectral sequence of (2.3.2) has a *strong horizontal vanishing line* at N if for all s and t the map $\pi_t X_s \to \pi_{t-N} X_{s-N}$ is zero.

The following result justifies this nomenclature. The proof is a diagram chase.

Lemma 2.3.2. *Suppose the spectral sequence of* (2.3.2) *has a strong horizontal vanishing line at N. Then*

(1) the spectral sequence is strongly convergent;
(2) for all s and t, $E_N^{s,t} \cong E_\infty^{s,t}$; and,
(3) for all $s \geq N$, $E_\infty^{s,t} = 0$.

Recall that we write $Y \wedge Z$ for $L_{K(n)}(Y \wedge Z)$ when we are in the $K(n)$-local category and we write $E = E_n$ for our chosen Morava E-theory at n. Then the $K(n)$-local Adams-Novikov Spectral Sequence is obtained from the standard exact couple diagram constructed from the E-based cobar complex in the $K(n)$-local category:

(2.3.4)
$$\begin{tikzcd}[column sep=small]
L_{K(n)}S^0 \arrow[dr] & & \overline{E} \arrow[ll, dashed] \arrow[dr] & & \overline{E}\wedge\overline{E} \arrow[ll, dashed] \arrow[dr] & & \cdots \arrow[ll, dashed] \\
& E \arrow[ur] & & E\wedge\overline{E} \arrow[ur] & & E\wedge\overline{E}\wedge\overline{E} \arrow[ur] &
\end{tikzcd}$$

Given a spectrum X, the $K(n)$-local Adams-Novikov Spectral Sequence has E_2-term isomorphic to

$$E_2^{s,t} \cong \pi^s\pi_t L_{K(n)}(E^{\wedge\bullet} \wedge X).$$

If Z is a finite spectrum and $G \subseteq \mathbb{G}_n$ is closed, then we can set $X = E^{hG} \wedge Z$. Then by Proposition 6.6 of [DH04] the E_2-term becomes

$$E_2^{s,t} \cong H^s(G, E_t Z).$$

Following Strickland (see Corollary 15 of [Str00]), we write $i_1 = i\colon \overline{E} \to \Sigma L_{K(n)}S^0$ for the boundary map in the first triangle. Then the later boundary maps can be written as

$$i_s = i \wedge \overline{E}^{\wedge(s-1)}\colon \overline{E}^{\wedge s} \to \Sigma\overline{E}^{\wedge(s-1)}.$$

Furthermore we have identifications

$$i_1 \circ \cdots \circ i_s = i^{\wedge s}\colon \overline{E}^{\wedge s} \to \Sigma^s L_{K(n)}S^0$$

and, more generally,

$$i_{r+1} \circ \cdots \circ i_s = \overline{E}^{\wedge r} \wedge i^{\wedge(s-r)}\colon \overline{E}^{\wedge s} \to \Sigma^{s-r}\overline{E}^{\wedge r}. \tag{2.3.5}$$

As in the proof of Corollary 15 of [Str00] we now have the following result.

Lemma 2.3.3. *Let Z be in the thick subcategory of $K(n)$-local spectra generated by E. Then there exists an integer N depending only on Z so that*

$$i^{\wedge s} \wedge Z : \overline{E}^{\wedge s} \wedge Z \to \Sigma^s Z$$

is null-homotopic for $s \geq N$.

Proof. If $Z = E$ itself, then $E \to E \wedge E$ has a retraction given the multiplication $E \wedge E \to E$. In that case, we can take $N = 1$. Furthermore, if we have a cofiber sequence $Z_1 \to Z_2 \to Z_3$ and the result holds for Z_1 and Z_3 with integers N_1 and N_3 respectively, then the result holds for Z_2 with integer $N_1 + N_3$, using (2.3.5). Finally, if Z_0 is a retract of Z and the result holds for Z with integer N, it also holds for Z_0 with integer N. □

By Theorem 8.9 of [HS99], the $K(n)$-local sphere is in the thick subcategory of the $K(n)$-local category generated by E. Thus Lemma 2.3.3 gives the following result.

Corollary 2.3.4. *There is an integer $N = N_{S^0}$ so that*

$$i^N : \overline{E}^{\wedge N} \longrightarrow \Sigma^N L_{K(n)}S^0$$

is null-homotopic.

This result immediately implies the following.

Theorem 2.3.5. *For all spectra X, the $K(n)$-local Adams-Novikov Spectral Sequence has a strong horizontal vanishing line at an integer $N \leq N_{S^0}$.*

Corollary 2.3.6. *For all $K(n)$-local finite spectra Z and and all closed subgroups $G \subseteq \mathbb{G}_n$, the $K(n)$-local Adams-Novikov Spectral Sequence*

$$H^s(G, E_t Z) \Longrightarrow \pi_{t-s}(E^{hG} \wedge Z)$$

has a strong horizontal vanishing line at an integer $N \leq N_{S^0}$.

Corollary 2.3.7. *Let Z be a finite complex with a self map $f : \Sigma^k Z \to Z$. Then for all closed subgroups $G \subseteq \mathbb{G}_n$, the localized $K(n)$-local Adams-Novikov Spectral Sequence*

$$f_*^{-1} H^s(G, E_t Z) \Longrightarrow f_*^{-1} \pi_{t-s}(E^{hG} \wedge Z)$$

has a strong horizontal vanishing line at an integer $N \leq N_{S^0}$.

2.4. **Elliptic curves and subgroups of $\mathbb{G}_2$ at $p = 2$.** Here we spell out what we need from the theory of elliptic curves at $p = 2$; this will give us a preferred formal group and a preferred universal deformation. Choose Γ_2 to be the formal group obtained from the elliptic curve C_0 over $\mathbb{F}_2$ defined by the Weierstrass equation

$$y^2 + y = x^3. \tag{2.4.1}$$

This is a standard representative for the unique isomorphism class of supersingular curves over $\overline{\mathbb{F}}_2$; see [Sil86], Appendix A. As a result Γ_2 has height 2, as the notation indicates. In addition, if ξ is the Frobeniues, we have $\xi^2 = -2$ in the endomorphism ring of Γ_2 over $\mathbb{F}_2$. Thus this formal group satisfied our assumption from (2.1.1). See Lemma 3.1.1 of [Bea17b].

Following Strickland (see [Bea17b, Section 2]), let C be the elliptic curve over $\mathbb{W}[[u_1]]$ defined by the Weierstrass equation

$$y^2 + 3u_1 xy + (u_1^3 - 1)y = x^3. \tag{2.4.2}$$

Remark 2.4.1. The curve C reduces to C_0 modulo $\mathfrak{m} = (2, u_1)$. In the equation (2.4.2) note that the coefficient of xy is congruent to u_1 modulo 2; hence $v_1 = u^{-1}u_1$ for the formal group of C over $\mathbb{F}_4[[u_1]]$. See Proposition 6.1 of [Bea17b]. From this it follows that the formal group C over $\mathbb{W}[[u_1]]$ is a choice of the universal deformation of Γ_2.

In [Bea17b], C_0 was called $\mathcal{C}$ and C was called $\mathcal{C}_U$. The current notation is closer to what we find in the number theory literature.

Again turning to [Sil86], Appendix A we have

$$\mathrm{Aut}(C_0/\mathbb{F}_4) \cong Q_8 \rtimes \mathbb{F}_4^\times \tag{2.4.3}$$

where the action of $\mathbb{F}_4^\times \cong C_3$ on Q_8 is determined by a cyclic permutation of generators i, j and their product $k = ij$. We will denote this group by G_{24}. Define

$$G_{48} = \mathrm{Aut}(\mathbb{F}_4, C_0) \cong \mathrm{Aut}(C_0/\mathbb{F}_4) \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2). \tag{2.4.4}$$

Since any automorphism of the pair $(\mathbb{F}_4, C_0)$ induces an automorphism of the pair $(\mathbb{F}_4, \Gamma_2)$ we get a map $G_{48} \to \mathbb{G}_2$. This map is an injection and we identify G_{48} with its image.

Remark 2.4.2. Let $C_0[3]$ be the subgroup scheme of C_0 consisting of points of order 3; over $\mathbb{F}_4$, this becomes abstractly isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$. The group G_{48} acts linearly on $C_0[3]$ and choosing a basis for the $\mathbb{F}_4$-points of $C_0[3]$ determines an isomorphism $G_{48} \cong GL_2(\mathbb{Z}/3)$.

We will be interested in various subgroups of G_{48}. The ones of interest are as follows:

Remark 2.4.3. The following subgroups will play an important role in this paper.

(1) $C_2 = \{\pm 1\} \subseteq Q_8$;
(2) $C_6 = C_2 \times \mathbb{F}_4^\times$;
(3) G_{24} and G_{48} themselves.

Remark 2.4.4. For the formal group Γ_2 of C_0, the group $\mathbb{S}_2 = \mathrm{Aut}(\Gamma_2/\mathbb{F}_2)$ has a very concrete description. Let $\mathbb{W} = W(\mathbb{F}_4)$ be the Witt vectors on $\mathbb{F}_4$. The endomorphism ring of Γ_2 is the non-commutative extension of $\mathbb{W}$

$$\mathbb{W}\langle T\rangle/(T^2 = -2, aT = Ta^\sigma)$$

where $a \in \mathbb{W}$ and $(-)^\sigma$ is the action of the Frobenius on $\mathbb{W}$. Then $\mathbb{S}_2$ is the group of units in this ring. See §3 of [Bea17b].

Remark 2.4.5. Up to conjugacy in $\mathbb{S}_2$, the subgroup $Q_8 \subseteq \mathbb{S}_2$ is the unique maximal finite 2-primary subgroup. Similarly, up to conjugacy in $\mathbb{S}_2$, G_{24} is the unique maximal finite subgroup. See [Buj12]. However, for $\mathbb{S}_2^1$, neither of these statements remains true.

More precisely, any finite subgroup of $\mathbb{S}_2$ is necessarily in $\mathbb{S}_2^1$, but if $\pi \in \mathbb{S}_2$ is any element such that $N(\pi)$ is a topological generator of $\mathbb{Z}_2$, then $\pi G_{24}\pi^{-1}$ is not conjugate to G_{24} in $\mathbb{S}_2^1$. If $\omega \in \mathbb{W}$ is a primitive cube root of untiy, then we can choose $\pi = 1 + 2\omega$, as in (2.3.3) of [Bea15] and we define

$$G_{24}' := \pi G_{24}\pi^{-1}$$

to be a representative of this other conjugacy class.

The element $\pi = 1+2\omega \in \mathbb{W}^\times$ satisfies $\det(\pi) = 3$. There is another key element that arises at various points. From Section 2.3 of [Bea15] (see also Remark 7.2.9 below) we know there is an element $\alpha \equiv 1-2\omega$ modulo (4) in $\mathbb{W}$ so that $\det(\alpha) = -1$.

Remark 2.4.6. We have been discussing G_{24} as a subgroup of $\mathbb{S}_2$, but it can also be thought of as a quotient as well. Inside of $\mathbb{S}_2$ there is a normal torsion-free pro-2-subgroup K which has the property that the composition

$$G_{24} \longrightarrow \mathbb{S}_2 \longrightarrow \mathbb{S}_2/K$$

is an isomorphism. Thus we have a decomposition $K \rtimes G_{24} \cong \mathbb{S}_2$. The group K is a Poincaré duality group of dimension 4. We also define $K^1 = \mathbb{S}_2^1 \cap K$; then, K^1 is a Poincaré duality group of dimension 3 and we have a decomposition $K^1 \rtimes G_{24} \cong \mathbb{S}_2^1$. The composition

$$G_{24}' \longrightarrow \mathbb{S}_2 \longrightarrow \mathbb{S}_2/K$$

remains an isomorphism and defines an alternate splitting. The subgroups K and K^1 play a key technical role in this paper, and we will supply more information when we need it below. See Remark 5.1.5, Lemma 5.1.6, and Remark 5.2.5. The main reference is [Bea15].

Remark 2.4.7. In order to calculate the algebraic duality spectral sequences or topological duality spectral sequences defined below in Section 2.5 we will need some information on $H^*(F, E_*)$ and $\pi_* E^{hF}$ for the finite subgroups $F = C_6$ and $F = G_{24}$. There is a detailed summary of literature in §2 of [BG18] and here we make only a few remarks. The spectral sequence

$$H^s(C_6, E_t) \Longrightarrow \pi_{t-s} E^{hC_6}$$

is relatively simple and can be deduced from [MR09]. The spectral sequence

$$H^s(G_{24}, E_t) \Longrightarrow \pi_{t-s} E^{hG_{24}}$$

is much harder, but extensively studied; see [Bau08] or [DFHH14]. For now we record that there are classes c_4, c_6, Δ, and j in $H^0(G_{24}, E_*)$, of degrees 8, 12, 24, and 0 respectively. These are the modular forms of the curve (2.4.2), hence necessarily invariant under the automorphisms of the curve. Then there is an isomorphism

$$\mathbb{W}[[j]][c_4, c_6, \Delta^{\pm 1}]/I \cong H^0(G_{24}, E_*)$$

where I is the ideal generated by

$$c_4^3 - c_6^2 = (12)^3 \Delta \qquad \text{and} \qquad j\Delta = c_4^3.$$

The classes c_4, j, and Δ^8 are permanent cycles, but c_6 and Δ^i, $i \not\equiv 0$ modulo 8 are not. We will give more information in Remark 8.1.1 below.

2.5. **The duality resolutions.** A key computational tool in this paper is the algebraic duality resolution of [Bea15] and its topological analog from [BG18]. We begin with the algebraic version.

If $X = \lim X_\alpha$ is a profinite set, define $\mathbb{Z}_2[[X]] = \lim_{n,\alpha} \mathbb{Z}/2^n[X_\alpha]$, where $\mathbb{Z}/2^n[Y]$ denotes the free $\mathbb{Z}/2^n$-module on the set Y. If G is a profinite group let $IG \subseteq \mathbb{Z}_2[[G]]$ be the augmentation ideal; that is, the kernel of the augmentation $\epsilon\colon \mathbb{Z}_2[[G]] \to \mathbb{Z}_2$.

The following can be found in Theorems 1.2.1 and 1.2.6 of [Bea15]. The subgroups G_{24}, C_6, and G'_{24} have been defined in Remarks 2.4.3 and 2.4.5, and the subgroup K^1 was discussed in Remark 2.4.6.

Theorem 2.5.1 (**Algebraic Duality Resolution**). *Let $F_0 = G_{24}$, $F_1 = F_2 = C_6$ and $F_3 = G'_{24}$.*

(1) There is an exact sequence of continuous $\mathbb{Z}_2[[\mathbb{S}_2^1]]$–modules

$$0 \longrightarrow \mathbb{Z}_2[[\mathbb{S}_2^1/F_3]] \xrightarrow{\partial_3} \mathbb{Z}_2[[\mathbb{S}_2^1/F_2]] \xrightarrow{\partial_2} \mathbb{Z}_2[[\mathbb{S}_2^1/F_1]] \xrightarrow{\partial_1} \mathbb{Z}_2[[\mathbb{S}_2^1/F_0]] \xrightarrow{\epsilon} \mathbb{Z}_2 \longrightarrow 0.$$

(2) The maps ∂_1 and ∂_3 are trivial modulo IK^1. Furthermore, ∂_2 is multiplication by 2 modulo $(8, IS_2^1)$.

Remark 2.5.2. Note that part (2) of Theorem 2.5.1 implies that modulo $I\mathbb{S}_2^1$ we have $\partial_1 = 0 = \partial_3$ and ∂_2 is multiplication by 2 modulo $(8, I\mathbb{S}_2^1)$. For many arguments, this is all we will need.

Remark 2.5.3 (**Algebraic Duality Spectral Sequence**)**.** If M is a profinite $\mathbb{Z}_2[[\mathbb{S}_2^1]]$-module, we have a natural isomorphisms

$$\mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathbb{Z}_2[[\mathbb{S}_2^1/F_p]], M) \cong H^q(F_p, M) \tag{2.5.1}$$

and part (1) of Theorem 2.5.1 immediately gives a spectral sequence

$$E_1^{p,q} = H^q(F_p, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M)$$

with differentials $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. We will call this the *algebraic duality spectral sequence*, which we may abbreviate as ADSS.

We can induce the exact sequence of Theorem 2.5.1 up to an exact sequence of complete $\mathbb{Z}_2[[\mathbb{G}_2^1]]$–modules

$$\begin{aligned} 0 \to \mathbb{Z}_2[[\mathbb{G}_2^1/F_3]] &\xrightarrow{\partial_3} \mathbb{Z}_2[[\mathbb{S}_G^1/F_2]] \\ &\xrightarrow{\partial_2} \mathbb{Z}_2[[\mathbb{G}_2^1/F_1]] \xrightarrow{\partial_1} \mathbb{Z}_2[[\mathbb{G}_2^1/F_0]] \to \mathbb{Z}_2[[\mathbb{G}_2/\mathbb{S}_2^1]] \to 0 \ . \end{aligned}$$

For any closed subgroup of F of $\mathbb{G}_2$, the isomorphism of (2.1.8) gives us an isomorphism of twisted $\mathbb{G}_2$-modules

$$E_*E^{hF} \cong \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2[[\mathbb{G}_2/F]], E_*).$$

From these observations we get the following result. Note that since G_{24} and G'_{24} are conjugate in $\mathbb{G}_2$, we have $E^{hG'_{24}} \simeq E^{hG_{24}}$.

Corollary 2.5.4. *There is an exact sequence of twisted* $\mathbb{G}_2$*-modules*

$$0 \to E_*E^{h\mathbb{S}_2^1} \to E_*E^{hG_{24}} \to E_*E^{hC_6} \to E_*E^{hC_6} \to E_*E^{hG_{24}} \to 0.$$

The following is the main theorem of [BG18]. This is the *topological duality resolution.* Note that it follows from Remark 2.4.7 that multiplication by Δ^k induces an isomorphism of Morava modules $E_*\Sigma^{24k}E^{hG_{24}} \cong E_*E^{hG_{24}}$ for all k, but this extends to an equivalence $\Sigma^{24k}E^{hG_{24}} \simeq E^{hG_{24}}$ if and only if $k \equiv 0$ modulo 8, as Δ^k is a permanent cycle in the homotopy fixed point spectral sequence only for those k. See [DFHH14] for history and details. This all means that the suspension factor on the last spectrum in the following topological resolution is significant.

Theorem 2.5.5. *The algebraic resolution of Corollary 2.5.4 can be realized by a sequence of spectra*

$$E^{h\mathbb{S}_2^1} \longrightarrow E^{hG_{24}} \to E^{hC_6} \to E^{hC_6} \to \Sigma^{48}E^{hG_{24}}.$$

In this sequence all compositions and all Toda brackets are zero modulo indeterminacy. The sequence can be refined to a tower of fibrations

(2.5.2)
$$\begin{tikzcd} E^{h\mathbb{S}_2^1} \arrow[r] & Z_2 \arrow[r] & Z_1 \arrow[r] & E^{hG_{24}} \\ \Sigma^{45}E^{hG_{24}} \arrow[u] & \Sigma^{-2}E^{hC_6} \arrow[u] & \Sigma^{-1}E^{hC_6} \arrow[u] & \end{tikzcd}$$

Remark 2.5.6 (Topological Duality Spectral Sequence). We will write $\mathscr{E}_p$ for the pth spectrum in the topological duality resolution of Theorem 2.5.5. Thus

$$\mathscr{E}_p = \begin{cases} E^{hG_{24}}, & p = 0; \\ E^{hC_6}, & p = 1, 2; \\ \Sigma^{48}E^{hG_{24}}, & p = 3. \end{cases}$$

Using the notation of Remark 2.5.3, we get spectral sequences

$$H^s(F_p, E_t) \Longrightarrow \pi_{t-s}\mathscr{E}_p.$$

As a complement to the spectral sequence of Remark 2.5.3, the tower of (2.5.2) gives a spectral sequence

$$E_1^{p,q} = \pi_q \mathscr{E}_p \Longrightarrow \pi_{q-p} E^{h\mathbb{S}_2^1}.$$

We refer to this spectral sequences as the *topological duality spectral sequence*. We may abbreviate this as the TDSS. There are variants we use; for example, we could smash the tower of (2.5.2) with a spectrum Y to get a spectral sequence converging to $\pi_* L_{K(2)}(E^{h\mathbb{S}_2^1} \wedge Y)$.

Remark 2.5.7. It may be helpful to note that the resolution of Theorem 2.5.5 can be refined to a tower over $E^{h\mathbb{S}_2^1}$

(2.5.3)
$$\begin{tikzcd}
E^{h\mathbb{S}_2^1} \arrow[dr] & & D_1 \arrow[ll, dashed] \arrow[dr] & & D_2 \arrow[ll, dashed] \arrow[dr] & & D_3 \arrow[ll, dashed] \arrow[dr, "\simeq"] & \\
& E^{hG_{24}} \arrow[ur] & & E^{hC_6} \arrow[ur] & & E^{hC_6} \arrow[ur] & & \Sigma^{48} E^{hG_{24}}
\end{tikzcd}$$

As before we write

$$\begin{tikzcd}
X \arrow[dr] & & Z \arrow[ll, dashed] \\
& Y \arrow[ur] &
\end{tikzcd}$$

for a cofiber sequence $X \to Y \to Z \to \Sigma X$. The dotted arrows in (2.5.3) have Adams-Novikov filtration 1. The resulting spectral sequence is isomorphic to the topological duality spectral sequence. For more details see Section 3 of [BG18].

3. Recollections from homotopy theory

In this section we gather together some of the material we need from the homotopy theory literature. First, we discuss some qualitative aspects of the homotopy groups of $V(0)$, making explicit some interesting behavior which can be traced back to the fact that the order of the identity of $V(0)$ is not 2. Then we give some basic background on how the Hopf map $\sigma \in \pi_7 S^0$ appears in the Adams-Novikov Spectral Sequence. Both of these topics arise repeatedly in the following sections.

In this section, and all subsequent sections, we will be working in the 2-local stable category.

3.1. **Some basic homotopy theory of $V(0)$.** Let $\iota\colon S^0 \to V(0)$ be the inclusion of the bottom cell and $2\colon V(0) \to S^1$ the collapse map onto the top cell so that

(3.1.1)
$$S^0 \xrightarrow{\times 2} S^0 \xrightarrow{\iota} V(0) \xrightarrow{p} S^1$$

is the standard cofiber sequence for $V(0)$. For a spectrum X, let $j = X \wedge \iota\colon X \simeq X \wedge S^0 \to X \wedge V(0)$ and $q = X \wedge p\colon X \wedge V(0) \to \Sigma X$. We then have a cofiber sequence

$$X \xrightarrow{\times 2} X \xrightarrow{j} X \wedge V(0) \xrightarrow{q} \Sigma X.$$

Remark 3.1.1. We let $\beta\colon \pi_n(X \wedge V(0)) \to \pi_{n-1}(X \wedge V(0))$ be the homotopy Bockstein homomorphism induced by the composite

$$X \wedge V(0) \xrightarrow{q} \Sigma X \xrightarrow{j} \Sigma X \wedge V(0).$$

Note that any element in the image of β has order 2.

Remark 3.1.2. One way to see that the order of the identity of $V(0)$ is not 2 is to note that the induced short exact sequence in mod 2 cohomology

$$0 \longleftarrow H^*V(0) \xleftarrow{j^*} H^*(V(0) \wedge V(0)) \xleftarrow{q^*} H^*\Sigma V(0) \longleftarrow 0$$

is not split as a module over the Steenrod algebra. This implies immediately that the exact sequence

$$0 \to \mathbb{Z}/2 \cong \pi_1 V(0) \xrightarrow{j_*} \pi_1(V(0) \wedge V(0)) \xrightarrow{q_*} \pi_0 V(0) \cong \mathbb{Z}/2 \to 0$$

cannot be split and, hence,

$$\pi_1(V(0) \wedge V(0)) \cong \mathbb{Z}/4\ . \tag{3.1.2}$$

Specifically, let $\iota \in \pi_0 V(0)$ is the inclusion of the bottom cell. Define $i_0 = j_*(\iota) \in \pi_0(V(0) \wedge V(0))$ and $i_1 \in \pi_1(V(0) \wedge V(0))$ to be any class with $q_*(i_1) = \iota$, then

$$2i_1 = i_0\eta \ = j_*(\iota)\eta \tag{3.1.3}$$

where $\eta \in \pi_1 S^0$ is the generator.[1] This is the universal example of the following result.

Lemma 3.1.3. *Let $x \in \pi_n X$ have order 2. Then there is a class $y \in \pi_{n+1}(X \wedge V(0))$ so that*

$$2y = j_*(x)\eta \in \pi_{n+1}(X \wedge V(0)).$$

Furthermore, $q_*(y) = x$.

Proof. Extend x to a map $\overline{x}\colon \Sigma^n V(0) \to X$ and then contemplate the diagram

$$\begin{tikzcd}
\Sigma^n V(0) \arrow[r] \arrow[d, "\overline{x}"'] & \Sigma^n V(0) \wedge V(0) \arrow[r] \arrow[d, "\overline{x} \wedge V(0)"] & \Sigma^{n+1} V(0) \arrow[d, "\Sigma\overline{x}"] \\
X \arrow[r, "j"'] & X \wedge V(0) \arrow[r, "q"'] & \Sigma X\ .
\end{tikzcd}$$

The class y is then the image of a generator of $\pi_1(V(0) \wedge V(0))$ under the map $(\overline{x} \wedge V(0))_*$. $\square$

Remark 3.1.4. We can refine the Lemma 3.1.3 in the case when the class of order 2 is an η-multiple. Let $x \in \pi_n X$ be a class of order 2. Then the Toda bracket $\langle x, 2, \eta \rangle \subseteq \pi_{n+2} X$ is defined with indeterminacy[2]

$$x\pi_2 S^0 + \pi_{n+1} X\eta = \pi_{n+1} X\eta.$$

It follows that $\langle x, 2, \eta \rangle\, 2$ is well-defined and the shuffle

$$\langle x, 2, \eta \rangle\, 2 = x\, \langle 2, \eta, 2 \rangle = x\eta^2 \tag{3.1.4}$$

shows that if $x\eta^2 \neq 0$, then any element of $\langle x, 2, \eta \rangle$ has order 4. We also have that $x\eta^2$ must be divisible by 2.

---

[1] We write η as the right hand factor in this product as this fits better with our Toda bracket conventions. See Remark 3.1.4.

[2] Note that we are writing Toda brackets in composition order.

Remark 3.1.5. Here are a few simple applications of the Lemma 3.1.3 and Remark 3.1.4. We will only use (3.1.5) in the sequel, but (3.1.6) and the charts of Figure 1 make for a more complete story.

Let $v_1 \in \pi_2 V(0)$ be either of the two classes which map to $\eta \in \pi_1 V(0)$ under the boundary map $\pi_2 V(0) \to \pi_1 S^0$. Since $v_1 \in \langle \iota, 2, \eta \rangle$, (3.1.4) gives that

$$2v_1 = \eta^2 \neq 0. \tag{3.1.5}$$

Let us also write v_1 for $j_*(v_1) \in \pi_2(V(0) \wedge V(0))$. The image of j_* has order two in $\pi_2(V(0) \wedge V(0))$, and hence $2v_1 = 0$. We can then choose an element

$$v_1^2 \in \langle v_1, 2, \eta \rangle \subseteq \pi_4(V(0) \wedge V(0)),$$

so named because of its Hurewicz image in $BP_*(V(0) \wedge V(0))$. Then Remark 3.1.2 and (3.1.4) yield relations in $\pi_*(V(0) \wedge V(0))$:

$$\begin{aligned} 2i_1 &= i_0 \eta \\ 2v_1^2 &= v_1 \eta^2 \ . \end{aligned} \tag{3.1.6}$$

Remark 3.1.6. The basic relations of (3.1.6) come as exotic extensions in any Adams-Novikov Spectral Sequence. Using the standard calculations of [MRW77] or even Table 2 of [Rav78], it is an exercise to compute the BP-based Adams-Novikov Spectral Sequence for $V(0)$ and $V(0) \wedge V(0)$ in a small range. The following two charts display the E_∞-pages for $V(0)$ and $V(0) \wedge V(0)$ respectively. The charts display additive extensions, η-multiplications, and ν-multiplications. A vertical line denotes multiplication by 2. Notice that, in particular, we must have $d_3(v_1^2) = \eta^3$.

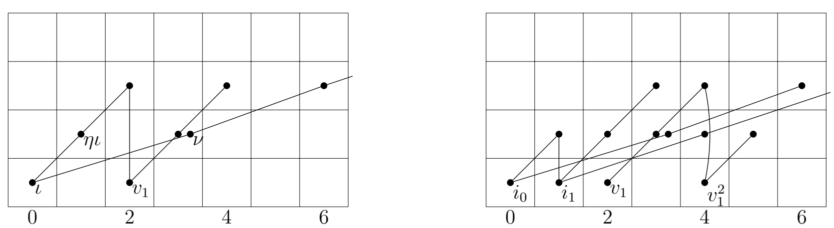

FIGURE 1. The Adams-Novikov E_∞-pages for $\pi_* V(0)$ (left) and $\pi_*(V(0) \wedge V(0))$ (right).

We close this section with a lemma about Spanier-Whitehead duality. If X is a finite spectrum, let DX denote its Spanier-Whitehead dual. For ι and p as in (3.1.1), then

$$DS^0 \xleftarrow{D\iota} DV(0) \xleftarrow{Dp} DS^1$$

is equivalent to

$$S^0 \xleftarrow{p} \Sigma^{-1} V(0) \xleftarrow{\iota} S^{-1}.$$

Lemma 3.1.7. *Let $y\colon S^n \to X \wedge V(0)$ and let $x\colon \Sigma^{n-1}V(0) \simeq \Sigma^n DV(0) \to X$ be Spanier-Whitehead dual to y. Then we have a commutative diagram*

$$\begin{tikzcd} S^n \arrow[r, "\iota"] \arrow[d, "y"'] & \Sigma^n V(0) \arrow[d, "x"] \\ X \wedge V(0) \arrow[r, "q"'] & \Sigma X \end{tikzcd} .$$

If $\beta\colon \pi_n(X \wedge V(0)) \to \pi_{n-1}(X \wedge V(0))$ is the homotopy Bockstein homomorphism of Remark 3.1.1, then

$$\beta(y) = j_* x_*(\iota).$$

Proof. We have a commutative diagram

$$\begin{tikzcd} S^n \wedge S^{-1} \arrow[r, "S^n \wedge Dp"] \arrow[d, "y \wedge S^{-1}"'] & S^n \wedge DV(0) \arrow[d, "y \wedge DV(0)"] \arrow[dd, bend left=60, "x"] \\ X \wedge V(0) \wedge S^{-1} \arrow[r, "X \wedge V(0) \wedge Dp"'] \arrow[dr, "q"'] & X \wedge V(0) \wedge DV(0) \arrow[d, "X \wedge m"] \\ & X \wedge S^0 \end{tikzcd}$$

where $m\colon V(0) \wedge DV(0) \to S^0$ is the duality pairing. The right vertical composite defines x. By Spanier-Whitehead duality the composite

$$V(0) \wedge S^{-1} \xrightarrow{V(0)\wedge Dp} V(0) \wedge DV(0) \xrightarrow{m} S^0$$

is the desuspension of p. Now suspend.

The equation for $\beta(y)$ is immediate from the commutativity of this diagram. □

3.2. **Finding σ in the $K(n)$-local sphere.** A crucial actor in our proof of the decomposition of $L_{K(1)}L_{K(2)}S^0$ is the Hopf class $\sigma \in \pi_7 S^0 \cong \mathbb{Z}/16$. This class remains non-zero in $\pi_* L_{K(2)} S^0$ and also in $\pi_* E^{h\mathbb{S}_2^1}$, and here we discuss how it is detected. We begin with a preliminary remark.

Remark 3.2.1 (Geometric Boundary Theorems). At various points in this paper we use a Geometric Boundary Theorem to name elements in an Adams-Novikov Spectral Sequence. Specifically, we will have a cofiber sequence $X \longrightarrow Y \longrightarrow Z$ which induces a short exact sequence after applying a homology theory $E_*(-)$

$$0 \longrightarrow E_*X \longrightarrow E_*Y \longrightarrow E_*Z \longrightarrow 0\ .$$

Here E might be different than E_n. This gives a long exact sequence on the E_2-term of the Adams-Novikov Spectral Sequence. Suppose $x \in \pi_{t-s}Z$ is detected by a class $a \in E_2^{s,t}Z$. We would like to assert the image of X in $\pi_{t-s-1}X$ is detected by the image of a under the connecting homomorphism

$$\delta\colon E_2^{s,t}Z \to E_2^{s+1,t}X.$$

Finding a proof of this fact in the literature in complete generality is a challenge. We will use this result in two cases. In the first case, E will be a connected ring spectrum such as MU or BP, and then we can appeal to Theorem 2.3.4 of [Rav86]. In the second case $E = E_n$ for some n and $E_*X = \pi_* L_{K(n)}(E_n \wedge X)$, and then we can appeal to Proposition A.10 of [DH04]. In either case, we will call this the Geometric Boundary Theorem.

We must be precise about what we mean by σ. It will be the generator of $\pi_7 S^0$ detected in $\mathrm{Ext}^{1,8}_{BP_*BP}(BP_*, BP_*)$ by the Greek letter construction of Miller, Ravenel, and Wilson. Recall from Theorem 4.3.2 of [Rav86] that there is an isomorphism

$$\mathbb{F}_2[v_1] \cong \mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*/2)$$

and that the α-family in $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*)$ is defined by taking the image of the powers of v_1 under various Bockstein homomorphisms. Specifically, from Corollary 4.23 of [MRW77] we know that the class

$$v := v_1^4 - 8v_1v_2 \in BP_8 \tag{3.2.1}$$

becomes a comodule primitive in $BP_8/16$ and we define $\alpha_{4/4}$ to be the image of this class under the connecting Bockstein homomorphism associated to the short exact sequence

$$0 \longrightarrow BP_* \xrightarrow{\times 16} BP_* \longrightarrow BP_*/16 \longrightarrow 0\ . \tag{3.2.2}$$

Then $\pi_7 S^0 \to \mathrm{Ext}^{1,8}_{BP_*BP}(BP_*, BP_*)$ is an isomorphism and we define σ to be the unique homotopy class which maps to $\alpha_{4/4}$ under this map. We also write σ for this class (that is, for $\alpha_{4/4}$) in $\mathrm{Ext}^{1,8}_{BP_*BP}(BP_*, BP_*)$. This is an abuse of notation, but a standard one: we typically do it for η and ν as well.

Using the equation of (3.2.1) we have that $v \equiv v_1^4$ modulo 2. Using the Geometric Boundary Theorem of Remark 3.2.1 it follows that v itself detects a class $x \in \pi_8(S^0/16)$ which maps to σ under the map $\partial\colon \pi_8 S^0/16 \to \pi_7 S^0$. Here, $S^0/16$ is the cofiber of the map $16\colon S^0 \to S^0$. The class x is not unique, as the kernel of ∂ is $\mathbb{Z}/2\times\mathbb{Z}/2$, but the elements of the kernel all have higher Adams-Novikov filtration, so all choices of x are detected by v. The Adams-Novikov chart for $S^0/16$ in this range is displayed in Table 3 of [Rav78]. The class v of (3.2.1) is clearly displayed there, which is why we have used that notation.

Let $E = E_n$ be a choice of a Lubin-Tate theory at height n. We now write down a detection result for $\sigma \in \pi_7 E^{hH}$, where $H \subseteq \mathbb{G}_n$ is a closed subgroup. We note that the image of $v_1 \in \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*/(2))$ in $H^*(H, E_*/(2))$ – see diagram (2.1.7) – is the element v_1 of Remark 2.4.1. Consider the Bockstein homomorphism in cohomology

$$\delta^{(4)} = \delta^{(4)}_H\colon H^0(H, E_8/16) \to H^1(H, E_8)$$

determined by the short exact sequence

$$0 \longrightarrow E_8 \xrightarrow{\times 16} E_8 \longrightarrow E_8/16 \longrightarrow 0.$$

Proposition 3.2.2. *Let $H \subseteq \mathbb{G}_n$ be a closed subgroup and let $R = H^0(H, E_0)$. Let $c \in E_8/16$ be a class so that*

(1) $c \equiv v_1^4$ modulo 2,
(2) c is invariant under the action of H, and
(3) $H^0(H, E_8/2)$ is a cyclic R-module generated by v_1^4.

Then, up to multiplication by a unit in R, the image of $\sigma \in \pi_7 E^{hH}$ is detected in the spectral sequence

$$H^s(H, E_t) \Longrightarrow \pi_{t-s} E^{hH}$$

by the class $\delta^{(4)}(c) \in H^1(H, E_8)$.

Proof. As above, let $x \in \pi_8(S^0/16)$ be any class which maps to σ under the boundary map $\pi_8(S^0/16) \to \pi_7 S$. We will follow this element through the following diagram of spectral sequences

$$\begin{tikzcd}
\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*/16) \arrow[r, Rightarrow] & \pi_*(S^0/16) \\
\mathrm{Ext}^{*,*}_{MU_*MU}(MU_*, MU_*/16) \arrow[r, Rightarrow] \arrow[u, "\cong"] \arrow[d] & \pi_*(S^0/16) \arrow[u, "\cong"'] \arrow[d] \\
H^*(H, E_*/16) \arrow[r, Rightarrow] & \pi_*(E^{hH} \wedge S^0/16)\ .
\end{tikzcd}$$

To obtain this diagram we combine the diagram of spectral sequences in (2.1.7) with the diagram of spectral sequences in (2.1.10) with $F_1 = H \subseteq \mathbb{G}_2 = F_2$.

The class x is detected by v in $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*/16)$. Since $v \equiv v_1^4$ in $BP_*/2$, we have that v maps to an element in $H^0(H, E_8/16)$ congruent to v_1^4 modulo 2. Here we use the identification of v_1 from Remark 2.4.1. It now follows that x is non-zero in $\pi_8(E^{hH} \wedge S^0/16)$. The assumptions imply that c generates $H^0(H, E_8/16)$; therefore, c detects ax for some unit $a \in R/16$. The result then follows from the Geometric Boundary Theorem of Remark 3.2.1. □

Note that we are not asserting that $\sigma \neq 0 \in \pi_7 E^{hH}$. We will take this up at various points later. See Section 4 and Section 7 and, in particular, Proposition 4.1.7 and Theorem 7.2.4.

4. A review of calculations in $K(1)$-local homotopy theory at $p = 2$

In our main arguments, we will identify wedge summands of $L_{K(1)}L_{K(2)}S^0$ equivalent to $L_{K(1)}S^0$ and $L_{K(1)}V(0)$. As background for this, we record here known results from $K(1)$-local homotopy theory at the prime 2. For example, the class $\zeta_1 \in \pi_{-1}L_{K(2)}S^0$ is closely related to the Hopf map $\sigma \in \pi_7 S^0$ and it is important for us to make this relationship explicit. We will also discuss $L_{K(1)}V(0)$ and $L_{K(1)}Y$ where $Y = V(0) \wedge C(\eta)$ is the spectrum studied by Mahowald in his proof of the telescope conjecture at $n = 1$ and $p = 2$. See Section 2 of [Mah82].

None of the material in this section is new; it can be put together from [Mah82], [MRW77], and [Rav86] among many sources.

We begin with some basic calculations in the $K(1)$-local Adams-Novikov Spectral Sequence. As noted earlier, we can choose 2-completed K-theory for our version of Lubin-Tate theory at height 1, so we will write K for E_1.

The group $\mathbb{G}_1 = \mathbb{Z}_2^\times$ acts on $K_*X = \pi_* L_{K(1)}(K \wedge X)$. We write the action of $k \in \mathbb{Z}_2^\times$ using the Adams operations notation; that is, if $x \in K_*X$, we write $\psi^k(x)$ for the action of k on x.

We are interested in the spectral sequence

$$H^s(\mathbb{Z}_2^\times, K_t X) \Longrightarrow \pi_{t-s} L_{K(1)} X.$$

4.1. **Calculating $\pi_* L_{K(1)} S^0$ and $\pi_* L_{K(1)} V(0)$.** We begin with these, the most basic spectra. We take up the important auxiliary spectrum $Y = V(0) \wedge C(\eta)$ in Section 4.2.

Remark 4.1.1. Recall that $K_{2n} = \mathbb{Z}_2 u^{-n}$ where $u \in K_{-2}$. The operations ψ^k are $\mathbb{Z}_2$-linear and $\psi^k(u^n) = k^n u^n$. Here are a few crossed homomorphisms defining elements in the first cohomology groups $H^1(\mathbb{Z}_2^\times, K_*)$.[3]

(1) $\chi_1 \colon \mathbb{Z}_2^\times \to \mathbb{Z}/2 = K_0/2$ is given by
$$\chi_1(1 + 2k_0) = \overline{k}_0.$$
where $\overline{k}_0$ is the mod 2 reduction of k_0.

(2) $\zeta_1 \colon \mathbb{Z}_2^\times \to \mathbb{Z}_2 = K_0$ is defined as the composite
$$\mathbb{Z}_2^\times \longrightarrow \mathbb{Z}_2^\times/\{\pm 1\} \xrightarrow{\cong} (1 + 4\mathbb{Z}_2) \xrightarrow[\cong]{\frac{1}{4}\log(-)} \mathbb{Z}_2 \ .$$
Here the first map is the projection, the second map is the inverse of the isomorphism defined by the composition
$$1 + 4\mathbb{Z}_2 \subseteq \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\} \ ,$$
and $\log(1 + x) = \sum_{n \geq 1} (-1)^{n+1} x^n/n$.

(3) If n is odd, $\alpha_n \colon \mathbb{Z}_2^\times \to K_{2n}$ is given by
$$\alpha_n(k) = \frac{1}{2}(k^{-n} - 1)u^{-n}.$$

(4) If $n = 2^i(2t + 1)$ with $i > 0$, then $\alpha_{n/i+2} \colon \mathbb{Z}_2^\times \to K_{2n}$ is given by
$$\alpha_{n/i+2}(k) = \frac{1}{2^{i+2}}(k^{-n} - 1)u^{-n}.$$

We had a class $\alpha_{4/4}$ above; see after (3.2.1). Lemma 4.1.3 below implies that this new class $\alpha_{4/4}$ will be a unit multiple of the image of the older class, and we won't need to distinguish between the two as we try to find σ in the homotopy groups of the localizations of in $V(0)$, Y and $V(0) \wedge Y$. The class ζ_1 was already discussed in §2. See (2.2.3), and Proposition 2.2.1.

Since we are at height one, we have $v_1 = u^{-1} \in K_2/2$.

Lemma 4.1.2. *The crossed homomorphisms of Remark 4.1.1 satisfy the following formulas:*

(1) if n is odd, then $v_1^n \chi_1 \equiv \alpha_n$ modulo 2;

(2) if n is even, then $v_1^n \zeta_1 \equiv \alpha_{n/i+2}$ modulo 2.

Proof. As a topological abelian group $\mathbb{Z}_2^\times$ is generated by -1 and 5; thus we need only show that the identities hold when evaluating at these elements. The formula (1) is a simple calculation. For formula (2) we use
$$\frac{1}{4}\log(1 + 4k) \equiv k \qquad \text{modulo } 2.$$
□

[3]Our notation differs from that of Ravenel in Lemma 2.1 of [Rav77].

We also have the following result, which is immediate from Remark 4.1.1. Let $\delta^{(i)}$ be the ith Bockstein

$$\delta^{(i)}\colon H^s(\mathbb{Z}_2^\times, K_*/2^i) \to H^{s+1}(\mathbb{Z}_2^\times, K_*) \ .$$

Write $\delta = \delta^{(1)}$.

Lemma 4.1.3. *We have formulas for the Bockstein homomorphisms:*

(1) *if n is odd, $\delta(v_1^n) = \alpha_n$;*

(2) *if $n = 2^i(2t+1)$ with $i > 0$, then $\delta^{(i+2)}u^{-n} = \alpha_{n/i+2}$ and u^{-n} reduces to v_1^n modulo 2.*

The next proposition gives some basic detection results.

Proposition 4.1.4. *(1) The class $\eta \in \pi_1 S^0$ is non-zero in $\pi_* L_{K(1)} S^0$ detected by α_1.*

(2) The class $\sigma \in \pi_7 S^0$ is non-zero in $\pi_* L_{K(1)} S^0$ detected by a unit multiple of $\alpha_{4/4}$.

Proof. Part (1) follows from Lemma 4.1.3 and the fact that η is always detected by the Bockstein of v_1. Part (2) follows from Proposition 3.2.2, Lemma 4.1.3, and the isomorphisms

$$\mathbb{F}_2[v_1^{\pm 1}] \cong K_*/2 \cong H^0(\mathbb{Z}_2^\times, K_*/2) \ . \qquad \square$$

Remark 4.1.5. Differentials in the $K(1)$-local Adams-Novikov Spectral Sequence at $p = 2$ are largely determined by a standard d_3. We expand on this observation. In [Rav78, p.430], there is a generator

$$\alpha_3 \in \mathrm{Ext}^{1,6}_{BP_*BP}(BP_*, BP_*) \cong \mathbb{Z}/2$$

which is the Bockstein on v_1^3. There, it is shown that $d_3(\alpha_3) = \eta^4$. Furthermore, α_3 reduces to ηv_1^2 in $\mathrm{Ext}^{1,6}_{BP_*BP}(BP_*, BP_*V(0))$, so that $\eta d_3(v_1^2) = \eta^4$. Since there is no η torsion on the E_3-term of the Adams-Novikov Spectral Sequence for $V(0)$ in bidegree $(3,6)$, this forces the differential $d_3(v_1^2) = \eta^3$.

Since α_3 is the Bockstein on v_1^3, it maps to the class we named α_3 in Remark 4.1.1. See Lemma 4.1.3.

In general, for a 2-local MU-algebra spectrum E, the E-based Adams-Novikov Spectral Sequence for a spectrum X is a module over the BP-based Adams-Novikov Spectral Sequence for the sphere. There is a universal d_3–differential

$$d_3(\alpha_3 z) = \eta^4 z + \alpha_3 d_3(z).$$

Further, if 2 annihilates $E_*(X)$, this gives a universal differential

$$d_3(\eta v_1^2 z) = \eta^4 z + \eta v_1^2 d_3(z).$$

If there is no η-torsion on the E_3–term, this implies that $d_3(v_1^2 z) = \eta^3 z + v_1^2 d_3(z)$.

Warning 4.1.6. The spectrum $V(0)$ is not a ring spectrum. However, since $BP_*V(0) \cong BP_*/(2)$ is a graded commutative comodule algebra, the E_2-term of an Adams-Novikov Spectral Sequence for $V(0)$ is often a bigraded commutative ring. Typically the spectral sequence looses the ring structure at E_3, where we have $d_3(v_1) = 0$ and $d_3(v_1^2) = \eta^3$. See Remark 4.1.5.

Nonetheless, it is often convenient to write down the E_2-term as a ring and we will do so. These issues are classical, and we hope this doesn't cause confusion.

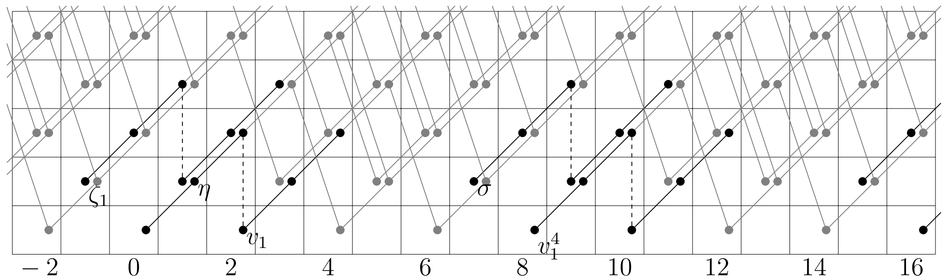

FIGURE 2. The spectral sequence computing the homotopy groups of $L_{K(1)}V(0)$. A $\bullet$ denotes a copy of $\mathbb{Z}/2$. Dashed vertical lines denote exotic multiplication by 2.

Remark 4.1.5 implies the following well-known result. See Figure 2. We use that $V(0)$ has a v_1^4-self map. The fact that ζ_1 is a permanent cycle was covered in Proposition 2.2.1 and the additive extension is from (3.1.5). In this result, $E(-)$ denotes the exterior algebra over $\mathbb{F}_2$.

Proposition 4.1.7. *We have an isomorphism*

$$\mathbb{F}_2[v_1^{\pm 1}, \eta] \otimes E(\zeta_1) \cong \mathbb{F}_2[v_1^{\pm 1}, \eta] \otimes E(\sigma) \cong H^*(\mathbb{Z}_2^\times, K_*/2)$$

with $v_1^4\zeta_1 = \sigma$. All non-zero differentials in the spectral sequence

$$H^*(\mathbb{Z}_2^\times, K_*/2) \Longrightarrow \pi_* L_{K(1)}V(0)$$

are determined by v_1^4–linearity, the facts that v_1, η, σ and ζ_1 are permanent cycles and

$$d_3(v_1^2 x) = \eta^3 x + v_1^2 d_3(x).$$

The spectral sequence collapses at E_4 and the only additive extensions are implied by v_1^4-linearity, multiplication by ζ_1, and

$$2v_1 = \eta^2 \ .$$

Remark 4.1.8. Using Proposition 4.1.7 and naturality it is possible to work out the spectral sequence for the homotopy of $L_{K(1)}S^0$. See Figure 3. Here are the results in brief. We have the following non-trivial differentials

$$\begin{aligned} d_3(\alpha_{4t+3}) &= \eta^3\alpha_{4t+1} \\ d_3(\alpha_{2(2^i t+1)/3}) &= \eta^3\alpha_{2^{i+1}t/i+3}, \quad t \not\equiv 0 \mod 2 \\ d_3(\alpha_{2/3}) &= \eta^3\zeta_1 \ . \end{aligned}$$

The last formula can be thought of as a case of the second formula with $i = \infty$. There are additive extensions as well. In fact, by (3.1.4) we see that $\eta^2\alpha_{4t+1}$ must be divisible by 2. This implies that for any $t \in \mathbb{Z}$

$$4(2\alpha_{4t+2/3}) = \eta^2\alpha_{4t+1}.$$

Proposition 4.1.7 has the following consequence. Recall that we are writing σ for both the element in homotopy and the class in the E_2-term of the Adams-Novikov Spectral Sequence which detects it.

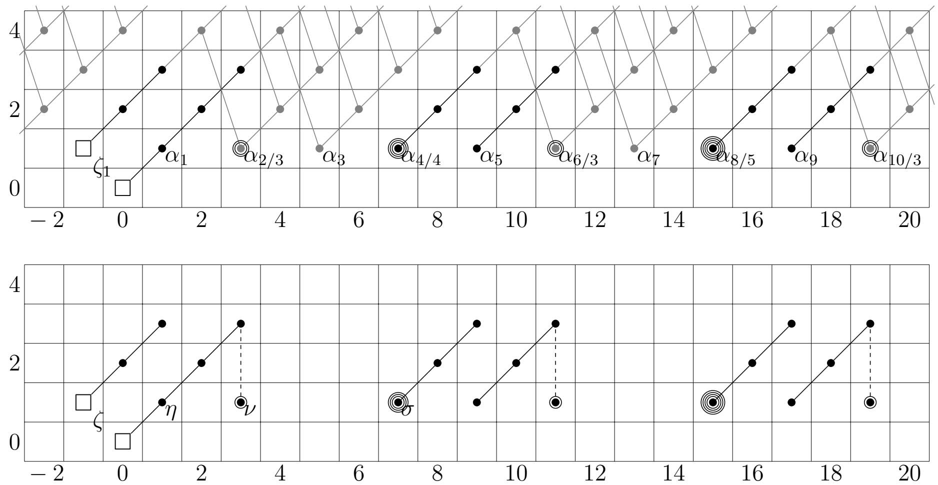

FIGURE 3. The E_2 (top) and E_∞ (bottom) pages of the spectral sequence for the homotopy of $L_{K(1)}S^0$. Here, a $\square$ denotes a copy of $\mathbb{Z}_2$, a $\bullet$ denotes a copy of $\mathbb{Z}/2$, a $\odot$ a copy of $\mathbb{Z}/4$ and so on. Dashed lines denote exotic multiplications by 2.

Corollary 4.1.9. *(1) In $H^*(\mathbb{Z}_2^\times, K_*)$ we have*

$$\alpha_{4/4}^2 = 0 = \sigma^2.$$

(2) In $\pi_* L_{K(1)}S^0$ we have $\sigma^2 = 0$.

Proof. For part (1), it is sufficient to prove $\alpha_{4/4}^2 = 0$ as σ is a unit multiple of $\alpha_{4/4}$. We use that

$$H^s(\mathbb{Z}_2^\times, K_*) \to H^s(\mathbb{Z}_2, K_*/2)$$

is an injection if $s > 1$. By part (2) of Lemma 4.1.2 we have $\alpha_{4/4} \equiv v_1^4\zeta_1$ modulo 2. Since $\zeta_1^2 = 0$, the result follows.

For part (2), we deduce from Proposition 4.1.7 that $\pi_{14}L_{K(1)}V(0) = 0$. From the long exact sequence on homotopy groups, it follows that $\pi_{14}L_{K(1)}S^0 = 0$. Alternatively we could read part (2) off of Figure 3. $\square$

Proposition 4.1.7 is explicit about the Hopf maps η and σ. The other Hopf map ν plays a more subtle role. See Figure 2 and Figure 3.

Proposition 4.1.10. *(1) Let $\nu \in \pi_3 S^0 \cong \mathbb{Z}/8$ be a generator. Then ν is non-zero in $\pi_* L_{K(1)}S^0$ and detected by a unit multiple of $2\alpha_{2/3}$.*

(2) The class ν is non-zero in $\pi_* L_{K(1)}V(0)$ detected by $v_1\zeta_1\eta^2$.

(3) In $\pi_* L_{K(1)}V(0)$, ν is a multiple of η.

Proof. From the discussion in Remark 4.1.8 we know that $\pi_3(L_{K(1)}S^0)$ sits in a short exact sequence with kernel given by $E_4^{3,6} \cong \mathbb{Z}/2$ generated by η^3 and quotient given by $E_4^{1,4} \cong \mathbb{Z}/4$ generated by $2\alpha_{2/3}$. The result follows because of $4\nu = \eta^3 \in \pi_3(S^0)$.

For the second statement, we have that $\nu \neq 0 \in \pi_* L_{K(1)} V(0)$ and that $2\alpha_{2/3} = 0 \in H^*(\mathbb{Z}_2^\times, K_*/2)$. It follows that ν must be detected by a class in filtration $s = 2$ or higher and, by Proposition 4.1.7, the only class available at E_∞ is $v_1\zeta_1\eta^2$.

For the third statement, we have that ν is detected by a class which is a multiple of η in the E_∞-page for $\pi_* L_{K(1)} V(0)$. Since there are no non-zero elements of higher filtration in the $t - s = 3$ stem, the claim follows. □

4.2. **The $K(1)$-local homotopy of the spectrum Y.** Let $C(\eta)$ be the cone on $\eta \in \pi_1 S^0$ and let $Y = V(0) \wedge C(\eta)$. The spectrum Y is a type 1 complex with a v_1-self map $v_1 \colon \Sigma^2 Y \to Y$. The map v_1 is not unique, but the induced map $\pi_* \Sigma^2 L_{K(1)} Y \to \pi_* L_{K(1)} Y$ is independent of the choice. Indeed, for any k, $\pi_k L_{K(1)} Y$ is one-dimensional over $\mathbb{Z}/2$. See Proposition 4.2.2.

Remark 4.2.1. By the construction of Y, there is a cofiber sequence

$$\Sigma V(0) \xrightarrow{V(0)\wedge\eta} V(0) \xrightarrow{\jmath} Y \xrightarrow{h} \Sigma^2 V(0) \tag{4.2.1}$$

which gives rise to a short exact sequence of BP_*BP-comodules

$$0 \to BP_*V(0) \to BP_*Y \to BP_*\Sigma^2 V(0) \to 0 \ .$$

This is not split, but is the non-zero element

$$\eta \in \mathrm{Ext}^1_{BP_*BP}(\Sigma^2 BP_*/2, BP_*/2) \cong \mathbb{Z}/2.$$

Proposition 4.2.2. *We have an isomorphism of $\mathbb{F}_2[v_1^{\pm 1}]$-modules*

$$\mathbb{F}_2[v_1^{\pm 1}] \otimes E(\zeta_1) \cong \mathbb{F}_2[v_1^{\pm 1}] \otimes E(\sigma) \cong H^*(\mathbb{Z}_2^\times, K_*Y)$$

with $v_1^4\zeta_1 = \sigma$. The spectral sequence

$$H^*(\mathbb{Z}_2^\times, K_*Y) \Longrightarrow \pi_* L_{K(1)} Y$$

collapses. If $\iota \colon S^0 \to Y$ is the inclusion of the bottom cell, then $\pi_* L_{K(1)} Y$ is a free module over $\mathbb{F}_2[v_1^{\pm 1}]$ on generators ι and $\iota\sigma$ of degrees 0 and 7 respectively.

Proof. By Landweber exactness and Remark 4.2.1 we have a short exact sequence of $\mathbb{Z}_2^\times$ modules

$$0 \to K_*V(0) \to K_*Y \to K_*\Sigma^2 V(0) \to 0 \ .$$

Furthermore, in the long exact sequence in group cohomology, the boundary map is given be multiplication by η. The claim about $H^*(\mathbb{Z}_2^\times, K_*Y)$ now follows from Proposition 4.1.7. The spectral sequence for $\pi_* L_{K(1)} Y$ must collapse for degree reasons. □

Let $\iota \colon S^0 \to Y$ and $p \colon Y \to S^3$ be the inclusion of the bottom cell and the collapse to the top cell of Y respectively, and similarly for $\iota \colon S^0 \to V(0)$ and $p \colon V(0) \to S^1$.

Proposition 4.2.3. *Let $2 \colon Y \to Y$ be the degree 2 map. Then there is a factoring*

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[rr, bend right, "2"'] & S^3 \arrow[r, "\nu"] & Y \end{tikzcd}$$

where ν is the composite $S^3 \xrightarrow{\nu} S^0 \xrightarrow{\iota} Y$. *After localization, the degree* 2 *map*

$$2\colon L_{K(1)}Y \to L_{K(1)}Y$$

is null-homotopic.

Proof. There is factoring of $2\colon V(0) \to V(0)$ as

$$V(0) \xrightarrow{p} S^1 \xrightarrow{\eta} V(0)\ .$$

Since $\eta = 0$ in π_*Y, this gives a factoring of $2\colon Y \to Y$ as a map

$$Y \xrightarrow{h} \Sigma^2 V(0) \xrightarrow{f} Y.$$

Since $K_*h\colon K_*Y \to K_*\Sigma^2V(0)$ is onto and $2K_*Y = 0$, we have that $K_*f = 0$. Since $\pi_2 Y \to K_2 Y$ an injection onto the summand generated by v_1, this implies f factors as a composition

$$\Sigma^2 V(0) \xrightarrow{\Sigma^2 p} S^3 \xrightarrow{g} Y.$$

The mod 2 cohomology of Y is cyclic over the Steenrod algebra and, hence, there can be no splitting

$$H^*(Y \wedge V(0)) \cong H^*Y \oplus H^*\Sigma Y$$

as modules over the Steenrod algebra. Thus the order of the identity on Y is not 2 and we see $g \neq 0$. Finally, since $\pi_3 Y \cong \mathbb{Z}/2$ generated by ν, the first statement follows. The second statement follows as $\nu = 0$ in $\pi_*L_{K(1)}Y$; indeed, we showed in Proposition 4.1.10 that ν is divisible by η in $\pi_*L_{K(1)}V(0)$. □

Recall that classes i_0 and i_1 in $\pi_*V(0) \wedge V(0)$ of degree 0 and 1 respectively were defined in Remark 3.1.2. We abuse notation and let $i_0 = (\jmath \wedge V(0))_* i_0$ and $i_1 = (\jmath \wedge V(0))_* i_1$ be the corresponding classes in $\pi_*(Y \wedge V(0))$.

Corollary 4.2.4. *There is an equivalence*

$$L_{K(1)}(Y \wedge V(0)) \simeq L_{K(1)}Y \vee \Sigma L_{K(1)}Y\ .$$

Furthermore, $\pi_*L_{K(1)}(Y \wedge V(0))$ *is a free module over* $\mathbb{F}_2[v_1^{\pm 1}] \otimes E(\sigma)$ *on generators* i_0 *and* i_1 *in degrees* 0 *and* 1.

Proof. This is an immediate consequence of Proposition 4.2.3. □

5. Height 2 cohomology calculations

In this section we collect together some calculations of the group cohomology of $\mathbb{G}_2$ and many of its closed subgroups. Much of the material here is either background for, or a consequence of, the following result. Let G be a closed subgroup of $\mathbb{G}_2$ containing $\mathbb{S}_2^1$ as a normal subgroup. Then the inclusion of $\mathbb{W} \to E_0 \cong \mathbb{W}[[u_1]]$ of continuous G-modules yields an isomorphism

$$H^*(G, \mathbb{W}) \xrightarrow{\cong} H^*(G, E_0).$$

See Theorem 5.4.4 below. This is a remarkable simplification and at the heart of much of what we can prove in $K(2)$-local homotopy theory.

5.1. **Preliminaries and recollections.** A key to many of our calculations is the behavior and properties of the classes in $H^1(\mathbb{G}_2, \mathbb{F}_2)$; these play a central role in the story we are telling here. This is also a point where the prime 2 has extra phenomena not seen at odd primes. The first part of this section is devoted to analyzing these cohomology classes. We will also include some material on the cohomology of some Poincaré duality subgroups of $\mathbb{S}_2$ collected from [Bea15].

Remark 5.1.1. In (2.2.1) we defined the extended determinant map

$$\det\colon \mathbb{G}_2 \longrightarrow \mathbb{Z}_2^\times \ .$$

We also have projections

$$p_1\colon \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2 \qquad \text{and} \qquad p_2\colon \mathbb{Z}_2^\times \to \mathbb{Z}/4^\times \cong \{\pm 1\} \ .$$

We gave an explicit isomorphism $\mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2$ in Remark 4.1.1. We then get surjective homomorphisms [4]

$$\zeta_2 := p_1 \circ \det\colon \mathbb{G}_2 \longrightarrow \mathbb{Z}_2 \qquad \text{and} \qquad \chi_2 := p_2 \circ \det\colon \mathbb{G}_2 \longrightarrow \mathbb{Z}/2 \tag{5.1.1}$$

and hence cohomology classes we label, by abuse of notation, by the same names; thus, we have classes $\zeta_2 \in H^1(G, \mathbb{Z}_2)$ and $\chi_2 \in H^1(G, \mathbb{F}_2)$ for any subgroup $G \subseteq \mathbb{G}_2$. We also write $\zeta = \zeta_2$ and $\chi = \chi_2$ when there can be no confusion.

The cohomology class ζ_2 is the reduced determinant discussed earlier in (2.2.2) and (2.2.3); see also Proposition 2.2.1. The class χ_2 is the analog of the class χ_1 defined in Remark 4.1.1 for $n = 1$.

As a final bit of notation, we will also write $\zeta_2 \in H^1(\mathbb{G}_2, \mathbb{F}_2)$ to be the reduction of $\zeta_2 \in H^1(\mathbb{G}_2, \mathbb{Z}_2)$. We also define

$$\widetilde{\chi} \in H^2(\mathbb{G}_2, \mathbb{Z}_2)$$

to be the image of χ_2 under the Bockstein $H^1(\mathbb{G}_2, \mathbb{F}_2) \to H^2(\mathbb{G}_2, \mathbb{Z}_2)$.

Remark 5.1.2. Looking back at Remark 4.1.1 we see that the projections p_1 and p_2 of Remark 5.1.1 have arisen before. In fact $p_1 = \zeta_1$ and $p_2 = \chi_1$, up to the isomorphism $\mathbb{Z}/4^\times \cong \mathbb{Z}/2$. We can write

$$\zeta_2 = \zeta_1 \circ \det \qquad \text{and} \qquad \chi_2 = \chi_1 \circ \det .$$

We now begin our calculations.

Lemma 5.1.3. *(1) The cohomology group $H^1(\mathbb{S}_2, \mathbb{F}_2)$ is of dimension 2 over $\mathbb{F}_2$ with basis χ_2 and ζ_2.*

(2) The cohomology group $H^1(\mathbb{S}_2, \mathbb{Z}_2)$ is free of rank one over $\mathbb{Z}_2$ generated by ζ_2.

Proof. The first statement follows from Theorem 6.3.12 of Ravenel [Rav77] or, alternatively, Proposition 5.3 of [Hen17] by taking covinvariants with respect to the residual action of $\mathbb{F}_4^\times$. We will see below in Proposition 5.2.12, that $0 \neq \chi_2^2 \in H^2(\mathbb{S}_2^1, \mathbb{F}_2)$, and hence in $H^2(\mathbb{S}_2^1, \mathbb{F}_2)$. The second statement follows. ☐

[4]**Warning!** In Lemma 2.1 of Ravenel [Rav77] the mod 2 reductions of these classes have the names $\zeta_2 + \rho_2$ and ζ_2 respectively.

Remark 5.1.4. Here are some more refined details about the behavior of ζ_2 and χ_2. From Remark 2.4.6 we have decompositions $K \rtimes G_{24} \cong \mathbb{S}_2$ and $K^1 \rtimes G_{24} \cong \mathbb{S}_2^1$.

Recall that $\mathbb{W}^\times \subseteq \mathbb{S}_2$, with $\mathbb{W}$ the Witt vectors on $\mathbb{F}_4$. Let $\omega \in \mathbb{W}$ be a primitive third root of unity. The element $\pi = 1 + 2\omega \in \mathbb{W}^\times$ satisfies $\det(\pi) = 3$. From Section 2.3 of [Bea15] (see also Remark 7.2.9 below) we know there is an element $\alpha \equiv 1 - 2\omega$ modulo (4) in $\mathbb{W}$ so that $\det(\alpha) = -1$. Furthermore, α and π are elements of $K \subseteq \mathbb{S}_2$. The elements π and α were discussed before in Remark 2.4.5.

(1) Since $\det(\alpha) = -1$, it follows that $\chi(\alpha) = -1$ and $\zeta(\alpha) = 0$. Thus $\alpha \in K^1$ and $\chi\colon \mathbb{S}_2^1 \to \mathbb{F}_2$ restricts to a non-zero class of the same name in $H^1(\mathbb{S}_2^1, \mathbb{F}_2)$ and even in $H^1(K^1, \mathbb{F}_2)$. By definition $\mathbb{S}_2^1$ is in the kernel of the reduced determinant; hence, ζ restricts to zero in $H^1(\mathbb{S}_2^1, \mathbb{Z}_2)$.

(2) Note also that $\zeta_2^2 = 0$ in cohomology, wherever it appears, as it is the image of a generator of $H^1(\mathbb{Z}_2, \mathbb{Z}_2)$. The exponent of χ is an important point addressed below in Proposition 5.3.1.

Remark 5.1.5. We now add some further recollections on the cohomology of the subgroups K and K^1 of $\mathbb{S}_2$. Since $\pi \in K$ and $\det(\pi) = 3$ maps to a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$, the composition

$$K \xrightarrow{\subseteq} \mathbb{S}_2 \xrightarrow{\zeta_2} \mathbb{Z}_2$$

remains surjective and defines a non-zero cohomology class $\zeta_2 \in H^1(K, \mathbb{Z}_2)$. Any choice of splitting of this surjection gives isomorphisms $\mathbb{S}_2^1 \rtimes \mathbb{Z}_2 \cong \mathbb{S}_2$ and $K^1 \rtimes \mathbb{Z}_2 \cong K$. A choice of splitting is given by sending a generator of $\mathbb{Z}_2$ to π.

The following can be found in Corollary 2.5.12 and Theorem 2.5.13 of [Bea15].[5]

Lemma 5.1.6. *(1) The subgroups K^1 and K are oriented Poincaré duality groups of dimensions 3 and 4 respectively.*

(2) There is an isomorphism

$$H^*(K, \mathbb{F}_2) \cong \mathbb{F}_2[\zeta, \chi, x_1, x_2]/(\zeta^2, \chi^2, x_1^2 + \chi x_1, x_2^2 + \chi x_2)$$

where ζ, χ, x_1 and x_2 are in degree 1.

(3) There is an isomorphism

$$H^*(K^1, \mathbb{F}_2) \cong \mathbb{F}_2[\chi, x_1, x_2]/(\chi^2, x_1^2 + \chi x_1, x_2^2 + \chi x_2)$$

where χ, x_1 and x_2 are in degree 1.

Remark 5.1.7. We also write down the cohomology of $Q_8 \rtimes \mathbb{F}_4^\times \cong G_{24}$ and C_6. Recall that Q_8 has periodic cohomology with a periodicity class $k \in H^4(Q_8, \mathbb{Z}_2)$ of order 8. We will also write $k \in H^4(Q_8, \mathbb{F}_2)$ for the reduction of k.

We have an isomorphism

$$\mathbb{F}_2[x, y, k]/(x^2 + xy + y^2, x^2y + xy^2) \cong H^*(Q_8, \mathbb{F}_2)$$

[5] In Section 2.5 of [Bea15], the restriction of χ to $H^1(K^1, \mathbb{F}_2)$ is denoted by x_0 and the restriction of ζ to $H^1(K, \mathbb{F}_2)$ by x_4. We use the same name for the restrictions as for the original classes.

where x and y are in degree 1. For a choice $\omega \in \mathbb{F}_4$ of a primitive cube root of unity we have $\omega_* x = y$, $\omega_* y = x + y$, and $\omega_* k = k$; from this it follows that there is an isomorphism

$$\mathbb{F}_2[z,k]/(z^2) \cong H^*(Q_8, \mathbb{F}_2)^{\mathbb{F}_4^\times} \cong H^*(G_{24}, \mathbb{F}_2)$$

where $z = x^2 y = xy^2$. Finally, since k has order 8 in $H^4(Q_8, \mathbb{Z}_2)$, the Universal Coefficient Theorem gives an isomorphism

$$\mathbb{Z}_2[k]/(8k) \cong H^*(G_{24}, \mathbb{Z}_2) \ .$$

Recall further that

$$H^*(C_6, \mathbb{F}_2) \cong \mathbb{F}_2[h]$$

for a class h in degree 1 and that

$$H^*(C_6, \mathbb{Z}_2) \cong \mathbb{Z}_2[g]/(2g)$$

for a class g in degree 2 which is the image of h under the connecting homomorphism for $\mathbb{Z}_2 \xrightarrow{2} \mathbb{Z}_2 \to \mathbb{F}_2$. The inclusion $C_6 = \{\pm 1\} \times \mathbb{F}_4^\times \subseteq G_{24}$ yields a map on cohomology

$$H^*(G_{24}, \mathbb{Z}_2) \cong \mathbb{Z}_2[k]/(8k) \to \mathbb{Z}_2[g]/(2g) \cong H^*(C_6, \mathbb{Z}_2)$$

sending k to g^2. With $\mathbb{F}_2$ coefficients the map

$$H^*(G_{24}, \mathbb{F}_2) \cong \mathbb{F}_2[z,k]/(z^2) \to \mathbb{F}_2[h] \cong H^*(C_6, \mathbb{F}_2)$$

sends z to 0 and k to h^4.

5.2. **The cohomology of $\mathbb{S}_2^1$.** In this section we use the algebraic duality spectral sequence of Section 2.5 to calculate the integral and mod 2 cohomology of $\mathbb{S}_2^1$ as graded modules over $H^*(G_{24}, \mathbb{Z}_2)$. From Remark 2.5.3 we have

$$E_1^{p,q} = H^q(F_p, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M) \ . \tag{5.2.1}$$

We are particularly interested in the cases $M = \mathbb{Z}_2$ and $M = \mathbb{F}_2$.

This spectral sequence has a split edge homomorphism. The augmentation map $\mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \to \mathbb{Z}_2$ induces, through the isomorphism of (2.5.1), an edge homomorphism

$$H^*(\mathbb{S}_2^1, \mathbb{Z}_2) \to H^*(G_{24}, \mathbb{Z}_2)$$

of the spectral sequence (5.2.1). This is induced by the inclusion of $G_{24} \subseteq \mathbb{S}_2^1$. By Remark 2.4.6 there is a projection $\mathbb{S}_2^1 \to \mathbb{S}_2^1/K^1 \cong G_{24}$ which splits this inclusion, so we immediately have the following result.

Lemma 5.2.1. *Let $M = \mathbb{Z}_2$ or $\mathbb{F}_2$. The map of algebras*

$$H^*(\mathbb{S}_2^1, M) \to H^*(G_{24}, M)$$

induced by the inclusion of the subgroup G_{24} has an algebra splitting.

Remark 5.2.2. From Remark 5.1.7 and Lemma 5.2.1 we get an injective map

$$\mathbb{Z}_2[k]/(8k) \cong H^*(G_{24}, \mathbb{Z}_2) \to H^*(\mathbb{S}_2^1, \mathbb{Z}_2).$$

We confuse k with its image in $H^4(\mathbb{S}_2^1, \mathbb{Z}_2)$ and in the cohomology of most of its various subgroups. To be specific, the composition

$$H^*(G_{24}, \mathbb{Z}_2) \to H^*(\mathbb{S}_2^1, \mathbb{Z}_2) \to H^*(G_{24}', \mathbb{Z}_2)$$

is an isomorphism; this follows from the fact that the inclusion $G'_{24} \subseteq \mathbb{S}_2^1$ also splits the projection $\mathbb{S}_2^1 \to G_{24}$. Also, as in Remark 5.1.7, the map

$$H^*(G_{24}, \mathbb{Z}_2) \to H^*(\mathbb{S}_2^1, \mathbb{Z}_2) \to H^*(C_6, \mathbb{Z}_2) \cong \mathbb{Z}_2[g]/(2g)$$

sends k to g^2.

Remark 5.2.3. It will be useful to compare the algebraic duality spectral sequence for $\mathbb{S}_2^1$ to a spectral sequence for a quotient group. Let $C_2 = \{\pm 1\} \subseteq \mathbb{S}_2$ be the central subgroup of order 2. For any group $G \subseteq \mathbb{S}_2$ which contains C_2 define $PG = G/C_2$. Note that C_2 is a subgroup of all of the groups $\mathbb{S}_2^1$, C_6, G_{24}, and G'_{24}. Whenever $C_2 \subseteq F$ we have isomorphisms of $\mathbb{S}_2^1$-coset spaces $\mathbb{S}_2^1/F \cong P\mathbb{S}_2^1/PF$ and, hence isomorphisms of continuous $\mathbb{S}_2^1$-modules

$$\mathbb{Z}_2[[\mathbb{S}_2^1/F]] \cong \mathbb{Z}_2[[P\mathbb{S}_2^1/PF]]\ .$$

The resolution of Theorem 2.5.1 is in fact constructed as a resolution of continuous $P\mathbb{S}_2^1$-modules

$$\begin{aligned} 0 \to \mathbb{Z}_2[[P\mathbb{S}_2^1/PF_3]] &\xrightarrow{\partial_3} \mathbb{Z}_2[[P\mathbb{S}_2^1/PF_2]] \\ &\xrightarrow{\partial_2} \mathbb{Z}_2[[P\mathbb{S}_2^1/PF_1]] \xrightarrow{\partial_1} \mathbb{Z}_2[[P\mathbb{S}_2^1/PF_0]] \to \mathbb{Z}_2 \to 0\ . \end{aligned} \tag{5.2.2}$$

We have

$$\begin{aligned} PF_0 &\cong A_4 := (C_2 \times C_2) \rtimes C_3 \\ PF_1 &\cong PF_2 \cong C_3 \\ PF_3 &\cong A'_4 \end{aligned}$$

with $A'_4 = \pi A_4 \pi^{-1}$, where $\pi = 1 + 2\omega \in \mathbb{W}^\times$. Note that, in particular, that if $p = 1$ or 2, then $\mathbb{Z}_2[[P\mathbb{S}_2^1/PF_p]] \cong \mathbb{Z}_2[[P\mathbb{S}_2^1/C_3]]$ is a projective $\mathbb{Z}_2[[P\mathbb{S}_2^1]]$-module. This often makes arguments with this resolution simpler. For example, if M is a profinite $P\mathbb{S}_2^1$ module we have an analog of the algebraic duality spectral sequence

$$H^q(PF_p, M) \Longrightarrow H^{p+q}(P\mathbb{S}_2^1, M)$$

with $E_1^{p,q} = H^q(C_3, M) = 0$ if $p = 1$, 2 and $q > 0$.

Finally, these considerations give a diagram of spectral sequences for any profinite $P\mathbb{S}_2^1$-module M

$$\begin{tikzcd} H^q(PF_p, M) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(P\mathbb{S}_2^1, M) \arrow[d] \\ H^q(F_p, M) \arrow[r, Rightarrow] & H^{p+q}(\mathbb{S}_2^1, M) \end{tikzcd}$$

with the vertical maps induced by the evident quotient homomorphisms.

Lemma 5.2.4. *Let $M = \mathbb{Z}_2$ or $\mathbb{F}_2$. The spectral sequence*

$$H^q(PF_p, M) \Longrightarrow H^{p+q}(P\mathbb{S}_2^1, M)$$

collapses at the E_2-page. Furthermore, if $M = \mathbb{F}_2$, it collapses at the E_1-page.

Proof. Since $H^q(PF_p, M) = 0$ for $p = 1, 2$ and $q > 0$, and since $H^*(PF_0, M)$ is a retract of $H^*(P\mathbb{S}_2^1, M)$, the spectral sequence collapses at E_2. The d_1-differential is induced by the maps ∂_i of Theorem 2.5.1. By part (2) of Theorem 2.5.1 we have that $d_1 \equiv 0$ modulo 2, so the spectral sequence collapses at E_1 for $M = \mathbb{F}_2$. $\square$

Remark 5.2.5. It is also useful to compare these spectral sequences to yet another one defined for a subgroup of $\mathbb{S}_2^1$. Recall there is a semidirect product decomposition $\mathbb{S}_2^1 \cong K^1 \rtimes G_{24}$; this implies a decomposition $P\mathbb{S}_2^1 \cong K^1 \rtimes A_4$. Thus, the resolution (5.2.2) is a resolution of projective $\mathbb{Z}_2[[K^1]]$-modules. For any profinite $P\mathbb{S}_2^1$-module M, we get a diagram of spectral sequences

$$\begin{tikzcd} E_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(\mathbb{Z}_2[[P\mathbb{S}_2^1/PF_p]], M) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(P\mathbb{S}_2^1, M) \arrow[d] \\ E_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[[K^1]]}(\mathbb{Z}_2[[P\mathbb{S}_2^1/PF_p]], M) \arrow[r, Rightarrow] & H^{p+q}(K^1, M) \end{tikzcd} \tag{5.2.3}$$

and in the bottom row the Ext groups vanish if $q > 0$. This allows us to prove the following lemma. Note that part (1) of Lemma 5.1.6 implies $H^3(K^1, M) \cong M$ for M either $\mathbb{Z}_2$ or $\mathbb{F}_2$.

Lemma 5.2.6. *Let $M = \mathbb{Z}_2$ or $\mathbb{F}_2$. The sequence*

$$0 \to H^3(A_4, M) \longrightarrow H^3(P\mathbb{S}_2^1, M) \longrightarrow H^3(K^1, M) \cong M \to 0$$

is split short exact, where the maps are induced by the projection to A_4 and the inclusion of K^1 in $P\mathbb{S}_2^1 \cong K^1 \rtimes A_4$. Furthermore the restriction homomorphism

$$H^3(\mathbb{S}_2^1, M) \longrightarrow H^3(K^1, M)$$

is split surjective.

Proof. The second statement follows from the first since the map $H^3(P\mathbb{S}_2^1, M) \to H^3(K^1, M)$ factors through $H^3(\mathbb{S}_2^1, M) \to H^3(K^1, M)$.

For the first statement, we use the map of spectral sequences (5.2.3). Since $\mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']] \cong \mathbb{Z}_2[[K^1]]$ as a K^1-module, $\mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(\mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']], M) \cong M$ and, at $E_1^{3,0}$, the map (5.2.3) is the inclusion of the invariants

$$M^{A_4'} \cong \mathrm{Hom}_{\mathbb{Z}_2[[P\mathbb{S}_2^1]]}(\mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']], M) \to \mathrm{Hom}_{\mathbb{Z}_2[[K^1]]}(\mathbb{Z}_2[[P\mathbb{S}_2^1/A_4']], M) \cong M.$$

The action of A_4' on M is trivial, so this is an isomorphism.

By part (2) of Theorem 2.5.1, $E_2^{3,0} \cong E_1^{3,0}$ for both K^1 and $P\mathbb{S}_2^1$ as the differential $d_1 \colon E_1^{2,0} \to E_1^{3,0}$ is induced by the map ∂_3, which is zero modulo IK^1. Both spectral sequences collapse at the E_2-term. For K^1 this follows for degree reasons and for $P\mathbb{S}_2^1$, this is Lemma 5.2.4. So the map in (5.2.3) at $E_\infty^{3,0}$ is an isomorphism. Finally, for $P\mathbb{S}_2^1$ we have $E_\infty^{2,1} = E_\infty^{1,2} = 0$ by Remark 5.2.3. Since $E_\infty^{0,3} \cong H^3(PF_0, M)$ is a retract of $H^3(P\mathbb{S}_2^1, M)$, the extension is split. $\square$

Remark 5.2.7. In results to follow we will use extra structure on the algebraic duality spectral sequence. For all profinite $\mathbb{S}_2^1$-modules M, there is a natural action of

$$\mathrm{Ext}^*_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathbb{Z}_2, \mathbb{Z}_2) \cong H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$$

on $\mathrm{Ext}^*_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(M, \mathbb{Z}_2)$ so, in particular, the algebraic duality spectral sequence

$$H^q(F_p, \mathbb{Z}_2) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, \mathbb{Z}_2)$$

is a spectral sequence of $H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$-modules. Furthermore, the action of $H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$ on $E_1^{p,*} \cong H^*(F_q, \mathbb{Z}_2)$ is through the restriction homomorphism induced by the inclusion $F_p \subseteq \mathbb{S}_2^1$.

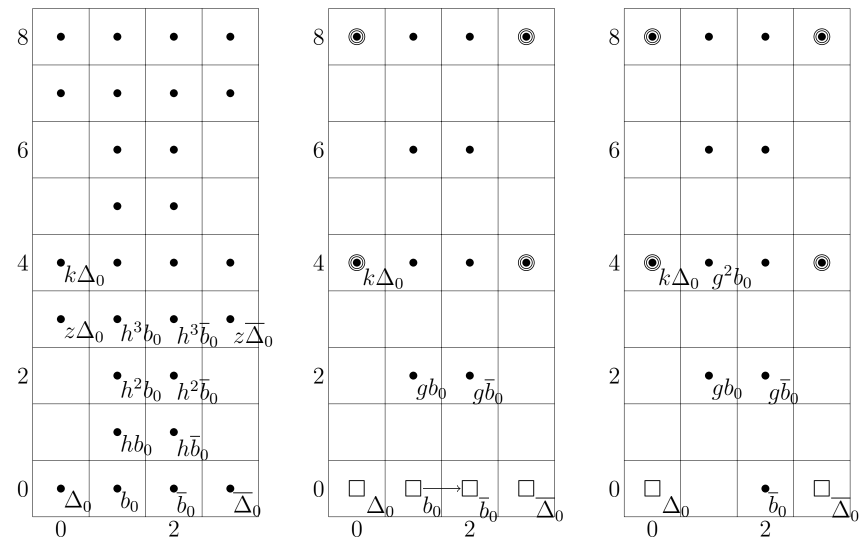

FIGURE 4. The $E_1 = E_\infty$-term (left) of the ADSS for $H^*(\mathbb{S}_2^1, \mathbb{F}_2)$ and the E_1-term (center) and $E_2 = E_\infty$-term (right) of the ADSS for $H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$.

We go into more detail on this algebra action in Section 7.1; see the material after Lemma 7.1.1.

Remark 5.2.8. We are now ready to use the algebraic duality spectral sequence to compute the cohomology of $\mathbb{S}_2^1$ with $\mathbb{Z}_2$ and $\mathbb{F}_2$ coefficients. In arguments below we may use the notation $E_r^{*,*}(\mathbb{F}_2)$ or $E_r^{*,*}(\mathbb{Z}_2)$ for the algebraic duality spectral sequences converging to $H^*(\mathbb{S}_2^1, \mathbb{F}_2)$ or $H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$ respectively.

Figure 4 displays the E_1-term with $\mathbb{F}_2$-coefficients in the left column, the E_1-term with $\mathbb{Z}_2$-coefficients in the middle column, and the E_∞-term with $\mathbb{Z}_2$-coefficients in the right column. We will have $E_1 = E_\infty$ for $\mathbb{F}_2$-coefficients. See Corollary 5.2.11.

We explain the notation in this figure; let $M = \mathbb{F}_2$ or $\mathbb{Z}_2$. Then a vertical subcolumn in an E_1-term displays a copy of $H^*(F_p, M)$ where $F_p = G_{24}$ if $p = 0$, C_6 if $p = 1$ or 2, or G'_{24} if $p = 3$. The cohomology rings of G_{24} and C_6 were discussed in Remark 5.1.7. The symbol $\circledcirc$ denotes a copy of $\mathbb{Z}/8$, while a $\bullet$ denotes a copy of $\mathbb{Z}/2$. A $\square$ denotes a copy of $\mathbb{Z}_2$.

We write $\Delta_0 \in H^0(F_0, M)$, $b_0 \in H^0(F_1, M)$, $\overline{b}_0 \in H^0(F_2, M)$, $\overline{\Delta}_0 \in H^0(F_3, M)$ for the obvious generators. This notation is used to facilitate references to [Bea17b]. Thus, for example, $E_1^{1,*} \cong H^*(C_6, M)b_0$ as a module over $H^*(\mathbb{S}_2^1, M)$.

The generators of Δ_0, b_0, $\overline{b}_0$, and $\overline{\Delta}_0$ are, strictly speaking, each an element of $H^0(F_p, M)$ for some p. In the next few results we will conflate them with their images under the edge homomorphism

$$H^0(F_p, M) \longrightarrow E_\infty^{p,0} \subseteq H^p(\mathbb{S}_2^1, M)$$

of the algebraic duality spectral sequence.

Before getting to our main theorems, we give some preliminary results about the classes b_0, $\bar{b}_0$, and $\overline{\Delta}_0$ in $E_1^{*,0}$ for both $\mathbb{F}_2$ and $\mathbb{Z}_2$. Recall from Remark 5.1.4 that $\widetilde{\chi} \in H^2(\mathbb{G}_2, \mathbb{Z}_2)$ is the Bockstein on $\chi = \chi_2 \in H^1(\mathbb{G}_2, \mathbb{F}_2)$.

Lemma 5.2.9. *(1) The class $b_0 \in E_1^{1,0}(\mathbb{F}_2)$ detects $\chi \in H^1(\mathbb{S}_2^1, \mathbb{F}_2)$.*

(2) The class $\bar{b}_0 \in E_1^{2,0}(\mathbb{F}_2)$ detects $\chi^2 \in H^2(\mathbb{S}_2^1, \mathbb{F}_2)$, and the class $\bar{b}_0 \in E_1^{2,0}(\mathbb{Z}_2)$ detects the class $\widetilde{\chi} \in H^2(\mathbb{S}_2^1, \mathbb{Z}_2)$.

(3) There is a torsion-free class $e \in H^3(\mathbb{S}_2^1, \mathbb{Z}_2)$ detected by $\overline{\Delta}_0 \in E_2^{3,0}(\mathbb{Z}_2)$ which restricts to a generator of

$$H^3(K^1, \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Proof. For part (1) we see that in the spectral sequence for $H^*(\mathbb{S}_2^1, \mathbb{F}_2)$ (the left-hand column in Figure 4) we have $E_\infty^{1,0}(\mathbb{F}_2) = \mathbb{F}_2$ generated by b_0 and $E_\infty^{0,1} = 0$; hence $H^1(\mathbb{S}_2^1, \mathbb{F}_2) \cong \mathbb{F}_2$. Since $\chi \in H^1(\mathbb{S}_2^1, \mathbb{F}_2)$ is not zero, it must be detected by b_0.

For part (2) we recall from part (2) of Theorem 2.5.1 that in the integral spectral sequence (the central column in Figure 4)

$$d_1 \colon E_1^{1,0}(\mathbb{Z}_2) \to E_1^{2,0}(\mathbb{Z}_2)$$

is multiplication by 2. Thus $H^2(\mathbb{S}_2^1, \mathbb{Z}_2) \cong \mathbb{Z}/2$ generated by a class detected by $\bar{b}_0$. This implies that the connecting homomorphism $H^1(\mathbb{S}_2^1, \mathbb{F}_2) \to H^2(\mathbb{S}_2^1, \mathbb{Z}_2)$ must be non-zero. Since $H^1(\mathbb{S}_2^1, \mathbb{F}_2) \cong \mathbb{F}_2$ generated by χ, we must have that the generator of $H^2(\mathbb{S}_2^1, \mathbb{Z}_2)$ is $\widetilde{\chi}$. From this it follows that $\bar{b}_0$ detects $\chi^2 \in H^2(\mathbb{S}_2^1, \mathbb{F}_2)$.

This leaves part (3). The integral algebraic duality spectral sequence has an edge homomorphism

$$\mathbb{Z}_2 \cong \mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}']], \mathbb{Z}_2) = E_1^{3,0} \to H^3(\mathbb{S}_2^1, \mathbb{Z}_2).$$

This map is injective. To see this, note that $\bar{b}_0$ is a permanent cycle; thus $d_1 \colon E_1^{2,0} \to E_1^{3,0}$ is zero. All other differentials with target $E_r^{3,0}$ have zero source.

Let e be the image of $\overline{\Delta}_0$. That e restricts to a generator of $H^3(K^1, \mathbb{Z}_2)$ follows from Lemma 5.2.6. □

We can now give the calculation of $H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$ as a module over $H^*(G_{24}, \mathbb{Z}_2)$. We give the integral calculation first as there are fewer possible differentials. Some of the generators in this result are written as products; this is meant only to be evocative of their antecedents in the spectral sequence. Thus, for example, $g\chi$ is not a product in the cohomology ring $H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$, but a class detected by gb_0.

Theorem 5.2.10. *As an $H^*(G_{24}, \mathbb{Z}_2) \cong \mathbb{Z}_2[k]/(8k)$-module $H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$ is generated by elements*

$$1,\ g\chi,\ g^2\chi,\ \widetilde{\chi},\ g\widetilde{\chi},\ e$$

of degrees 0, 3, 5, 2, 4, *and* 3 *respectively and subject only to the relations*

$$2g\chi = 2g^2\chi = 2\widetilde{\chi} = 2g\widetilde{\chi} = 0.$$

Proof. As a reminder, we are using the algebraic duality spectral sequence of Remark 2.5.3 with $M = \mathbb{Z}_2$. The E_1-page is determined as an $H^*(G_{24}, \mathbb{Z}_2)$-module by Remark 5.1.7 and the result is displayed in the center column of Figure 4. Then

Theorem 2.5.1 part (2) implies that $0 = d_1 \colon E_1^{p,*} \to E_1^{p+1,*}$ if $p = 0$ or $p = 2$ and the same result implies

$$d_1 \colon E_1^{1,q} \to E_1^{2,q}$$

is multiplication by 2 and, hence, non-zero only if $q = 0$. From E_2 onwards, the spectral sequence is too sparse for differentials. The result will now follow if we can show that there are no extensions.

Because we have a spectral sequence of modules over $H^*(\mathbb{S}_2^1, \mathbb{Z}_2)$, as in the Remark 5.2.7, we have a spectral sequence of $H^*(G_{24}, \mathbb{Z}_2)$-modules. By periodicity with respect to k, we need only check that the extensions

$$0 \to \mathbb{Z}_2 \cong E_\infty^{3,0} \to H^3(\mathbb{S}_2^1, \mathbb{Z}_2) \to E_\infty^{1,2} \cong \mathbb{F}_2 \to 0$$

and

$$0 \to \mathbb{Z}/2 \cong E_\infty^{2,2} \to H^4(\mathbb{S}_2^1, \mathbb{Z}_2) \to E_\infty^{0,4} \cong \mathbb{Z}/8 \to 0$$

are split. That the first is split follows from the second statement of Lemma 5.2.9 with $M = \mathbb{Z}_2$. That the second is split follows from Lemma 5.2.1. □

We now turn to the calculation of $H^*(\mathbb{S}_2^1, \mathbb{F}_2)$, again using the algebraic duality spectral sequence of Remark 2.5.3 with $M = \mathbb{F}_2$. The result is displayed in the left column of Figure 4.

As is Remark 5.1.7, write $H^*(G_{24}, \mathbb{F}_2) \cong \mathbb{F}_2[z, k]/(z^2)$ and $H^*(C_6, \mathbb{F}_2) \cong \mathbb{F}_2[h]$. The class $k \in H^4(G_{24}, \mathbb{Z}_2)$ reduces to the class of the same name in $H^4(G_{24}, \mathbb{F}_2)$ and the class $g \in H^2(C_6, \mathbb{Z}_2)$ reduces to $h^2 \in H^2(C_6, \mathbb{F}_2)$.

Corollary 5.2.11. *The algebraic duality spectral sequence for $H^*(\mathbb{S}_2^1, \mathbb{F}_2)$ collapses at E_1. As a module over $\mathbb{F}_2[k]$, $H^*(\mathbb{S}_2^1, \mathbb{F}_2)$ is freely generated by classes*

$$1,\ z,\ e,\ ze,\ h^i\chi,\ h^i\chi^2$$

with $0 \le i \le 3$. These classes are of degrees 0, 3, 3, 6, $1+i$, $2+i$ respectively.

Proof. The E_1-page of the spectral sequence is displayed in the left column of Figure 4. That this spectral sequence collapses follows from Lemma 5.2.1, the Universal Coefficient Theorem and Theorem 5.2.10. □

A key input for our main result on the homotopy type of $L_{K(1)}L_{K(2)}S^0$ is the exact nilpotence order of the class $\chi = \chi_2 \in H^1(\mathbb{G}_2, \mathbb{F}_2)$. The following is a preliminary step. The final result is below in Proposition 5.3.1.

Proposition 5.2.12. *Let $\chi \in H^1(\mathbb{S}_2^1, \mathbb{F}_2)$ be the restriction of $\chi \in H^1(\mathbb{G}_2, \mathbb{F}_2)$. Then $\chi^2 \neq 0$ and $\chi^3 = 0$ in $H^*(\mathbb{S}_2^1, \mathbb{F}_2)$.*

Proof. We already have $\chi^2 \neq 0$, by Corollary 5.2.11, so we need to show $\chi^3 = 0$. The homomorphism $\chi \colon \mathbb{S}_2^1 \to \mathbb{F}_2$ is trivial on the central element $-1 \in \mathbb{S}_2$, hence it comes from a unique class $b \in H^1(P\mathbb{S}_2^1, \mathbb{F}_2)$. This class restricts to zero in $H^1(A_4, \mathbb{F}_2)$. We will show that $b^3 = 0$.

To see this, recall the decomposition $P\mathbb{S}^1 = K^1 \rtimes A_4$. Lemma 5.2.6 implies that the map

$$H^3(P\mathbb{S}_2^1, \mathbb{F}_2) \to H^3(K^1, \mathbb{F}_2) \times H^3(A_4, \mathbb{F}_2)$$

defined by the two restriction maps is an isomorphism. Then, from part (2) of Lemma 5.1.6, and the fact that χ (and hence b) restricts to the class with the same name in $H^1(K^1, \mathbb{F}_2)$, we have that b^3 restricts to 0 in $H^3(K^1, \mathbb{F}_2)$. Since b restricts to 0 in $H^1(A_4, \mathbb{F}_2)$, the result follows. □

5.3. **The exponent of χ_2 in the cohomology of $\mathbb{S}_2$.** We next turn to the analysis of the cohomology of $\mathbb{S}_2$ itself. It is natural to ask for a complete calculation of $H^*(\mathbb{S}_2, \mathbb{F}_2)$; however, at this point, we have no succinct story to tell, and we won't need this calculation in later sections, so we won't pick up that story here. This is partly because we do not have direct access to an algebraic duality spectral sequence in this case. For this reason, to prove the next result we will examine the Lyndon-Hochschild-Serre Spectral Sequence (LHSSS) for the split group extension $K \to \mathbb{S}_2 \to G_{24}$ of Remark 2.4.6.

Proposition 5.3.1. *In $H^*(\mathbb{S}_2, \mathbb{F}_2)$, the class $\chi = \chi_2 \in H^1(\mathbb{S}_2, \mathbb{F}_2)$ has nilpotence order exactly 3; that is, $\chi^2 \neq 0$ and $\chi^3 = 0$.*

Proof. We write $H^*(G) = H^*(G, \mathbb{F}_2)$ in this argument. Recall from Proposition 5.2.12 that χ maps to the like-named class $\chi \in H^1(\mathbb{S}_2^1)$ and there we have $\chi^2 \neq 0$ and $\chi^3 = 0$. It follows immediately that $\chi^2 \neq 0$ in $H^2(\mathbb{S}_2)$. It remains to show $\chi^3 = 0$ in $H^*(\mathbb{S}_2, \mathbb{F}_2)$.

We will use that χ^3 restricts to zero in $H^*(\mathbb{S}_2^1)$ and a comparison of Lyndon-Hochschild-Serre Spectral Sequences induced by the inclusion $i \colon \mathbb{S}_2^1 \to \mathbb{S}_2$:

$$\begin{tikzcd} H^p(G_{24}, H^q(K)) \arrow[r, Rightarrow] \arrow[d, "i^*"'] & H^{p+q}(\mathbb{S}_2) \arrow[d, "i^*"] \\ H^p(G_{24}, H^q(K^1)) \arrow[r, Rightarrow] & H^{p+q}(\mathbb{S}_2^1) \ . \end{tikzcd} \tag{5.3.1}$$

From Lemma 5.1.6 we have a decomposition $H^*(K) \cong H^*(K^1) \otimes E(\zeta)$. Since ζ lifts to $H^*(\mathbb{G}_2)$, it is necessarily G_{24}-invariant; hence, this is an isomorphism of G_{24}-algebras and we have an isomorphism

$$H^*(G_{24}, H^*(K)) \cong H^*(G_{24}, H^*(K^1)) \otimes E(\zeta) \ .$$

Note that $\zeta \in E_2^{0,1} \cong H^0(G_{24}, H^1(K, \mathbb{F}_2))$; therefore, we can deduce an isomorphism

$$E_2^{p,q} \cong H^p(G_{24}, H^q(K)) \cong H^p(G_{24}, H^q(K^1)) \oplus H^p(G_{24}, H^{q-1}(K^1))\zeta \ . \tag{5.3.2}$$

The map i^* of E_2 terms in (5.3.1) is the algebra map sending ζ to zero. The class χ is detected in $E_2^{0,1} \cong H^0(G_{24}, H^1(K, \mathbb{F}_2))$.

As with any spectral sequence, the LHSSS gives a filtration

$$F^{3,0} \subseteq F^{2,1} \subseteq F^{1,2} \subseteq F^{0,3} = H^3(\mathbb{S}_2)$$

with

$$E_\infty^{p,q} = F^{p,q}/F^{p+1,q-1}$$

a subquotient of $E_2^{p,q} \cong H^p(G_{24}, H^q(K))$. There is a similar filtration for $H^2(\mathbb{S}_2)$.

We will show that $\chi^3 \in F^{3,0}$ and hence it is in the image of the map $H^*(G_{24}) \to H^*(\mathbb{S}_2)$ induced by the projection $\mathbb{S}_2 \to G_{24}$. Since this map has a section by Lemma 5.2.1 and χ restricts to 0 in $H^1(G_{24}) = 0$, we will have $\chi^3 = 0$ in $H^*(\mathbb{S}_2)$.

Since $\chi^2 = 0$ in $H^2(K, \mathbb{F}_2)$ we have $\chi^2 \in F^{1,1}$. Thus χ^2 is detected by a permanent cycle

$$\gamma \in H^1(G_{24}, H^1(K)) \ .$$

It follows that the class χ^3 is detected at $E_\infty^{1,2}$ by the class $\chi\gamma$.

By Remark 5.1.7 we know $H^1(G_{24},\mathbb{F}_2) = H^2(G_{24},\mathbb{F}_2) = 0$. Then (5.3.2) forces the map

$$i^*\colon H^p(G_{24}, H^1(K)) \to H^p(G_{24}, H^1(K^1))$$

at the $E_2^{p,1}$-term to be an isomorphism for $p = 1, 2$; in particular, if $\chi\gamma \neq 0$ in $H^1(G_{24}, H^2(K))$ then $i^*(\chi\gamma) = \chi i^*(\gamma) \neq 0$ in $H^1(G_{24}, H^2(K^1))$. But $\chi^3 = 0$ in $H^*(\mathbb{S}_2^1)$, so we have $\chi\gamma = 0$ and we can conclude $\chi^3 \in F^{2,1}$.

Let $\chi^3 \in H^*(\mathbb{S}_2)$ be detected by a permanent cycle ε in $H^2(G_{24}, H^1(K))$. Since $\chi^3 = 0$ in $H^*(\mathbb{S}_2^1)$ and i^* is an isomorphism at $E_2^{2,1}$, we have an equation $d_2(\omega) = i^*(\varepsilon)$ in the LHSSS for $H^*(\mathbb{S}_2^1)$. Since the map i^* of E_2-terms in (5.3.1) is a surjection, this implies ε itself is in the image of d_2. Thus $\chi^3 \in F^{3,0}$ as needed. □

5.4. **The cohomology of $H^*(\mathbb{G}_2, E_0)$.** We now come to one of the key results of the paper, Theorem 5.4.4.

We begin with the following basic case, which gives a tight control over the algebraic duality spectral sequence of Remark 2.5.3 for $M = E_0/2$. As in Remark 5.2.8 and Lemma 5.2.9 we write Δ_0, b_0, $\overline{b}_0$, and $\overline{\Delta}_0$ for the obvious generator of $H^0(F_p, \mathbb{F}_2)$, for $p = 0$ through 3 respectively. This gives generators in $H^0(F_p, E_0/2)$ as well.

Theorem 5.4.1. *Let $i\colon \mathbb{F}_4 \to E_0/2 \cong \mathbb{F}_4[[u_1]]$ be the inclusion of the constants. Then the induced map of algebraic duality spectral sequences*

(5.4.1)
$$\begin{tikzcd} H^q(F_p,\mathbb{F}_4) \arrow[r, Rightarrow] \arrow[d, "i_*"'] & H^{p+q}(\mathbb{S}_2^1,\mathbb{F}_4) \arrow[d, "i_*"] \\ H^q(F_p,E_0/2) \arrow[r, Rightarrow] & H^{p+q}(\mathbb{S}_2^1,E_0/2) \end{tikzcd}$$

becomes an isomorphism at E_2 and yields an isomorphism

$$i_*\colon H^*(\mathbb{S}_2^1,\mathbb{F}_4) \cong H^*(\mathbb{S}_2^1, E_0/2).$$

Proof. We first show that the map is one-to-one. Recall from Remark 2.4.1 that $v_1 = u^{-1}u_1$ for our formal group. This is a $\mathbb{G}_2$-invariant element modulo 2; hence the quotient map

$$E_*/2 \longrightarrow E_*/(2, v_1) \cong \mathbb{F}_4[u^{\pm 1}]$$

is $\mathbb{G}_2$-equivariant. The composite map

$$H^*(\mathbb{S}_2^1,\mathbb{F}_4) \to H^*(\mathbb{S}_2^1, E_0/2) \to H^*(\mathbb{S}_2^1, E_0/(2,u_1))$$

is then an isomorphism because it is induced by the isomorphism $\mathbb{F}_4 \cong E_0/(2,u_1)$ on coefficients. For similar reasons the map on E_1-terms of (5.4.1) must also be an injection.

The hard part is to show that the map induced by i is surjective on the E_2-term. Here we need Theorem 1.2.2 of [Bea17b]. From there we read that the $E_2 = E_\infty$-term of the algebraic duality spectral sequence for $E_0/2$ is free over $\mathbb{F}_4[k]$

on elements

$$\begin{aligned} \Delta_0 &\in H^0(F_0, E_0/2) \\ \nu^2 y \Delta_{-1} &\in H^3(F_0, E_0/2) \\ h^i b_0 &\in H^i(F_1, E_0/2), \qquad 0 \le i \le 3, \\ h^i \overline{b}_0 &\in H^i(F_2, E_0/2), \qquad 0 \le i \le 3, \\ \overline{\Delta}_0 &\in H^0(F_3, E_0/2) \\ \nu^2 y \overline{\Delta}_{-1} &\in H^3(F_3, E_0/2)\ . \end{aligned}$$

This, the left column of Figure 4, and Corollary 5.2.11 imply that on the E_2-pages of spectral sequences (5.4.1) i_* induces an injective homomorphism of free graded $\mathbb{F}_4[k]$-modules with the same number of generators in each bidegree. Thus our map must be onto. □

We can now give an integral statement. By Lemma 2.2.2 here is an isomorphism

$$\mathbb{W} \otimes H^*(\mathbb{S}_2^1, \mathbb{Z}_2) \cong H^*(\mathbb{S}_2^1, \mathbb{W}). \tag{5.4.2}$$

Hence Theorem 5.2.10 gives an explicit computation of $H^*(\mathbb{S}_2^1, \mathbb{W})$ using the algebraic duality spectral sequence.

Corollary 5.4.2. *Let $i\colon \mathbb{W} \to E_0 \cong \mathbb{W}[[u_1]]$ be the inclusion of the constants. Then i induces an isomorphism*

$$H^*(\mathbb{S}_2^1, \mathbb{W}) \cong H^*(\mathbb{S}_2^1, E_0).$$

Proof. That i induces an isomorphism

$$H^*(\mathbb{S}_2^1, \mathbb{W}/2^n) \cong H^*(\mathbb{S}_2^1, E_0/2^n)$$

follows by induction on n. The base case, $n = 1$, is Theorem 5.4.1. For the inductive step, we use the five lemma. The integral result is shown by taking inverse limits and using the $\lim^1$-exact sequence. □

We record the following companion result to Corollary 5.4.2 for use in the proof of Theorem 8.1.5. If M is a profinite $\mathbb{S}_2^1$-module, let $E_r^{p,q}(M)$ be the rth page of the algebraic duality spectral sequence of M. See Remark 2.5.3. Recall that $\widetilde{\chi} \in H^2(\mathbb{G}_2, \mathbb{Z}_2)$ is the Bockstein on $\chi \in H^1(\mathbb{G}_2, \mathbb{F}_2)$.

Lemma 5.4.3. *(1) The unit map $i\colon \mathbb{W} \to E_0$ induces an isomorphism*

$$E_2^{*,0}(\mathbb{W}) \to E_2^{*,0}(E_0)$$

where $E_2^{*,*}(M)$ denotes the E_2-term of the algebraic duality spectral sequence for M.

(2) For $0 \le p \le 3$ we have isomorphisms

$$E_2^{p,0}(E_0) \cong \begin{cases} \mathbb{W} & p = 0 \\ 0 & p = 1 \\ \mathbb{F}_4 & p = 2 \\ \mathbb{W} & p = 3\ . \end{cases}$$

Each of the non-zero groups $E_2^{p,0}(E_0)$ is generated by the cohomology class of the unit in $E_1^{p,0} = H^0(F_p, E_0)$.

Proof. For part (1) we use that $E_2^{*,0}(M)$ is the cohomology of the torsion-free cochain complexes $E_1^{*,0} = H^0(F_*, M)$. Now, let $M = \mathbb{W}$ or E_0. For both choices of M and any p, $H^1(F_p, M) = 0$. For $M = \mathbb{W}$ this follows from Remark 5.1.7 and for $M = E_0$ this follows from [Bau08] and [MR09], but see also Section 2 of [BG18]. From this it follows that for all $n > 1$ there is a short exact sequences of complexes

$$0 \longrightarrow H^0(F_*, M) \xrightarrow{2^n} H^0(F_*, M) \longrightarrow H^0(F_*, M/2^n) \longrightarrow 0.$$

Thus we deduce that we have a short exact sequence of chain complexes

$$0 \longrightarrow H^0(F_*, M/2^n) \xrightarrow{2} H^0(F_*, M/2^{n+1}) \longrightarrow H^0(F_*, M/2) \longrightarrow 0.$$

The map of cochain complexes

$$H^0(F_*, \mathbb{W}/2) \longrightarrow H^0(F_*, E_0/2)$$

induces an isomorphism in cohomology by Corollary 5.4.2. Then, using the five lemma, we have that

$$H^0(F_*, \mathbb{W}/2^n) \longrightarrow H^0(F_*, E_0/2^n)$$

is an isomorphism. The integral result then follows by taking inverse limits and using the $\lim^1$-exact sequence.

Part (2) is proved by combining (5.4.2), Lemma 5.2.9, and Theorem 5.2.10. $\square$

We can now extend Corollary 5.4.2 to a larger class of groups which includes $\mathbb{S}_2$, $\mathbb{G}_2$, and $\mathbb{G}_2^1$.

Theorem 5.4.4. *Let $G \subseteq \mathbb{G}_2$ be any closed subgroup which contains $\mathbb{S}_2^1$ as a normal subgroup. Then the inclusion of $\mathbb{Z}_2[[G]]$-modules $i\colon \mathbb{W} \to E_0$ induces an isomorphism*

$$i_* : H^*(G, \mathbb{W}) \cong H^*(G, E_0).$$

Proof. This follows from Corollary 5.4.2 and the following diagram of Lyndon-Hochschild-Serre Spectral Sequences

$$\begin{tikzcd}
H^p(G/\mathbb{S}_2^1, H^q(\mathbb{S}_2^1, \mathbb{W})) \arrow[r, Rightarrow] \arrow[d, "\cong"', "i_*"] & H^{p+q}(G, \mathbb{W}) \arrow[d] \\
H^p(G/\mathbb{S}_2^1, H^q(\mathbb{S}_2^1, E_0)) \arrow[r, Rightarrow] & H^{p+q}(G, E_0)
\end{tikzcd}$$

$\square$

Remark 5.4.5. Theorem 5.4.4 extends to an isomorphism

$$i_* : H^*(G, \mathbb{W}/2^n) \cong H^*(G, E_0/2^n)$$

for all $n \geq 1$.

Remark 5.4.6. One point that might be worth clarifying is the effect of the extension of scalars $\mathbb{Z}_2 \to \mathbb{W}$. Using Lemma 2.2.2 and Theorem 5.4.4 we can conclude that

$$H^*(\mathbb{S}_2, \mathbb{Z}_2) \cong H^*(\mathbb{G}_2, E_0).$$

Similar remarks apply to the inclusion $\mathbb{S}_2^1 \subseteq \mathbb{G}_2^1$.

Remark 5.4.7. Lemma 5.1.3 and Theorem 5.4.4 imply

$$H^1(\mathbb{G}_2, E_0) \cong \mathbb{Z}_2$$

generated by ζ_2 and Lemma 5.1.3 and Remark 5.4.5 imply

$$H^1(\mathbb{G}_2, E_0/2) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

generated by ζ_2 and χ_2.

5.5. **The rational cohomology of $\mathbb{S}_2$ and $\mathbb{G}_2$.** In this subsection we make the rational calculations needed to deduce the homotopy type of $L_1 L_{K(2)} S^0$ from the homotopy type of $L_{K(1)} L_{K(2)} S^0$. The main result is Proposition 5.5.1, which completely calculates $H^*(\mathbb{G}_2, E_*) \otimes \mathbb{Q}$.

By Lemma 5.2.9, there exists a class

$$e \in H^3(\mathbb{S}_2^1, \mathbb{Z}_2)$$

which maps to a generator of $H^3(K^1, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Let $E_{\mathbb{Q}_2}(-)$ denote exterior algebras over $\mathbb{Q}_2 \cong \mathbb{Z}_2 \otimes \mathbb{Q}$.

Proposition 5.5.1. *(1) There is an isomorphism of graded rings*

$$E_{\mathbb{Q}_2}(e) \cong H^*(\mathbb{S}_2^1, \mathbb{Z}_2) \otimes \mathbb{Q}.$$

(2) There is an isomorphism of graded rings

$$E_{\mathbb{Q}_2}(\zeta, e) \cong H^*(\mathbb{S}_2, \mathbb{Z}_2) \otimes \mathbb{Q}.$$

Proof. The first part follows from Theorem 5.2.10. For the second part consider the diagram of Lyndon-Hochschild-Serre Spectral Sequences induced by the inclusion $K \subseteq \mathbb{S}_2$:

$$\begin{tikzcd}
H^p(\mathbb{Z}_2, H^q(\mathbb{S}_2^1, \mathbb{Z}_2)) \arrow[r] \arrow[d] & H^{p+q}(\mathbb{S}_2, \mathbb{Z}_2) \arrow[d] \\
H^p(\mathbb{Z}_2, H^q(K^1, \mathbb{Z}_2)) \arrow[r] & H^{p+q}(K, \mathbb{Z}_2).
\end{tikzcd}$$

Since $K \subseteq \mathbb{S}_2$ and $K^1 \subseteq \mathbb{S}_2^1$ are finite index subgroups, the vertical maps are rational monomorphisms. By part (1) of Lemma 5.1.6, we have $H^4(K, \mathbb{Z}_2) \otimes \mathbb{Q} \cong \mathbb{Q}_2$. Thus the action of $\mathbb{Z}_2$ on $H^3(K^1, \mathbb{Z}_2) \otimes \mathbb{Q} \cong \mathbb{Q}_2$ must be trivial, as needed. $\square$

This extends immediately to a much larger calculation.

Theorem 5.5.2. *Let $i \colon \mathbb{W} \to E_0 \cong \mathbb{W}[[u_1]]$ be the inclusion of the constants. This map induces an isomorphism*

$$E_{\mathbb{Q}_2}(\zeta, e) \cong H^*(\mathbb{G}_2, \mathbb{W}) \otimes \mathbb{Q} \cong H^*(\mathbb{G}_2, E_*) \otimes \mathbb{Q}.$$

Proof. From Theorem 5.4.4, Remark 5.4.6, and Proposition 5.5.1 we get isomorphisms

$$E_{\mathbb{Q}_2}(\zeta, e) \cong H^*(\mathbb{G}_2, \mathbb{W}) \otimes \mathbb{Q} \cong H^*(\mathbb{G}_2, E_0) \otimes \mathbb{Q}.$$

To complete the proof, we must show $H^*(\mathbb{G}_2, E_t)$ is torsion for $t \neq 0$. This is very standard, but we give the proof as it is short.

Define $\mathbb{Z}_2^\times \to \mathbb{S}_2 \subseteq \mathbb{G}_2$ by sending ℓ to the ℓ-series $[\ell](x)$ of our chosen formal group law. This identifies $\mathbb{Z}_2^\times$ with the center of $\mathbb{G}_2$. Let $\mathbb{Z}_2 \subset \mathbb{Z}_2^\times$ be the subgroup topologically generated by an integer ℓ with $\ell \equiv 1$ modulo 4. Now consider the Lyndon-Hochschild-Serre Spectral Sequence for the extension

$$0 \to \mathbb{Z}_2 \to \mathbb{G}_2 \to \mathbb{G}_2/\mathbb{Z}_2 \to 0.$$

If $t \neq 0$, then ℓ acts on E_{2t} by multiplication by ℓ^{-t}, hence $H^0(\mathbb{Z}_2, E_{2t}) = 0$ and $H^1(\mathbb{Z}_2, (E_2)_{2t})$ is torsion. Since $E_{2t+1} = 0$, the result follows. $\square$

Remark 5.5.3. By Proposition 5.5.1 there must be a class in $\pi_{-3}L_{K(2)}S^0$ which maps to a multiple of $e \in \mathbb{Q} \otimes \pi_{-3}L_{K(2)}S^0$. We can be very specific about this. We will see below in Remark 8.1.7 that

$$\pi_{-3}L_{K(2)}S^0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2 \subseteq H^3(\mathbb{G}_2, E_0)$$

generated by the classes $4e$ and $\zeta_2\widetilde{\chi}$.

A similar phenomenon happened at $p = 3$ and $n = 2$; there was a class e so that $3e$ detects a copy of $\mathbb{Z}_3$. See [GHM14].

6. The class χ_2 is a d_3-cycle

As should be clear by now, the class $\chi = \chi_2 \in H^1(\mathbb{G}_2, E_0/2)$ is one key to the extra subtleties we encounter at $p = 2$ in $K(2)$-local homotopy theory. Over the next few sections we will work on some of the specific implications of the existence of this class. We show that χ is a permanent cycle in the $K(2)$-local Adams-Novikov Spectral Sequence in Theorem 8.2.1; see also Theorem 7.3.3 for a $K(1)$-local application. However, it turns out that evaluating d_3 requires an entirely different set of techniques, which we isolate in this section. The central idea of this section is due to Mike Hopkins.

The goal is to prove the following result.

Theorem 6.1.1. *In the spectral sequence*

$$E_2^{s,t} = H^s(\mathbb{G}_2, E_tV(0)) \Longrightarrow \pi_{t-s}L_{K(2)}V(0)$$

the classes χ and χ^2 are d_3–cycles.

Remark 6.1.2. Before proving this, we give some background to explain our methods.

We proved above in Theorem 5.4.4 and Remark 5.4.6 that the inclusion of the constant power series $\mathbb{Z}_2 \to E_0$ induced an isomorphism

$$H^*(\mathbb{S}_2, \mathbb{Z}_2) \cong H^*(\mathbb{G}_2, \mathbb{W}) \cong H^*(\mathbb{G}_2, E_0).$$

We would like to exploit a homotopy fixed point spectral sequence for the trivial action of $\mathbb{S}_2$ on the 2-completed sphere S^0_2; the issue is that it would take considerable effort to define such a spectral sequence. Fortunately, the class $\chi^i \in H^*(\mathbb{S}_2, \mathbb{F}_2)$ is the restriction of the non-zero class $h^i \in H^*(C_2, \mathbb{F}_2)$ under a quotient map $\chi\colon \mathbb{G}_2 \to C_2$. See Remark 5.1.1. We can then examine the map on cohomology

$$H^*(C_2, \pi_*S^0_2) \to H^*(\mathbb{G}_2, \pi_*S^0_2) \to H^*(\mathbb{G}_2, E_*).$$

We will show below in Lemma 6.1.4 that this composite map does extend to a map of homotopy fixed point spectral sequences.

For the trivial action of C_2 on the 2-completed sphere spectrum S^0_2, we have

$$(S^0_2)^{hC_2} \simeq F(BC_{2+}, S^0_2) \simeq F(\mathbb{R}P^\infty_+, S^0_2).$$

Here $F(X, Y)$ is the function spectrum. The homotopy fixed point spectral sequence

$$H^s(C_2, \pi_t S_2^0) \Longrightarrow \pi_{t-s} F(BC_{2+}, S_2^0)$$

is the Atiyah-Hirzebruch Spectral Sequence for the cohomotopy of $\mathbb{R}P^\infty$. It follows from Lin's Theorem [LDMA80] that $(S_2^0)^{hC_2} \simeq S_2^0 \vee \mathbb{R}P_+^\infty$; hence, the homotopy fixed point spectral sequence is an impractical approach to Lin's theorem. Low-dimensional calculations can still be informative, however, and we use them to give us the information we need.

We now set up the map of spectral sequences we will use.

Remark 6.1.3. We recall some results from Devinatz [Dev05], especially sections 2 and 3. This paper by Devinatz undertakes a thorough analysis iterating the homotopy fixed point constructions. Let H and K be closed subgroups of $\mathbb{G}_n$ and suppose H is normal in K. This implies that there is a K/H-equivariant map $E^{hK} \to E^{hH}$ of E_∞-ring spectra. Let X be an E^{hK}-module. Then, in the $K(n)$-local category of E^{hK}-modules, we can form an E^{hH}-based Adams-Novikov resolution of E^{hK} and apply the mapping space functor $F_{E^{hK}}(X, -)$ to obtain a spectral sequence

$$H^s(K/H, \pi_t F_{E^{hK}}(X, E^{hH})) \Longrightarrow \pi_{t-s} F_{E^{hK}}(X, E^{hK}).$$

The subtle part is to identify the E_2-term. This is Theorem 3.1 of [Dev05]. If Y is a finite CW-spectrum and DY its Spanier-Whitehead dual, we can set $X = E^{hK} \wedge DY$ to obtain a spectral sequence

$$H^s(K/H, \pi_t (E \wedge Y)^{hH}) \Longrightarrow \pi_{t-s} (E \wedge Y)^{hK}. \tag{6.1.1}$$

Furthermore, in the Appendix of [Dev05], Devinatz shows that if K/H is finite, then this is the usual homotopy fixed point spectral sequence.

This is natural in the pair (K, H) in the sense that if $K_0 \subseteq K$ and $H_0 \subseteq H$ then we get a diagram of spectral sequences

$$\begin{tikzcd} H^s(K/H, \pi_t (E \wedge Y)^{hH}) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s} (E \wedge Y)^{hK} \arrow[d] \\ H^s(K_0/H_0, \pi_t (E \wedge Y)^{hH_0}) \arrow[r, Rightarrow] & \pi_{t-s} (E \wedge Y)^{hK_0}. \end{tikzcd} \tag{6.1.2}$$

Now let $H_0 = \{e\}$, $K_0 = K = \mathbb{G}_2$, and H the kernel of the map

$$\chi \colon \mathbb{G}_2 \to (\mathbb{Z}/4)^\times \cong C_2$$

of Remark 5.1.1. Then $K/H \cong C_2$ and $K_0/H_0 = \mathbb{G}_2$. If we set $Y = V(0)$, then (6.1.2) gives a diagram of spectral sequences

$$\begin{tikzcd} H^s(C_2, \pi_t (E \wedge V(0))^{hH}) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s} L_{K(2)} V(0) \arrow[d] \\ H^s(\mathbb{G}_2, E_t V(0)) \arrow[r, Rightarrow] & \pi_{t-s} L_{K(2)} V(0). \end{tikzcd} \tag{6.1.3}$$

Now consider that map

$$V(0) \simeq S_2^0 \wedge V(0) \to E^{hH} \wedge V(0) \simeq (E \wedge V(0))^{hH}$$

of C_2-equivariant spectra, with trivial action on $V(0)$. Since the top row of (6.1.3) is the usual homotopy spectral sequence we get a map of spectral sequences

$$\begin{tikzcd} H^s(C_2,\pi_tV(0)) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}V(0)^{hC_2} \arrow[d] \\ H^s(C_2,\pi_t(E\wedge V(0))^{hH}) \arrow[r, Rightarrow] & \pi_{t-s}L_{K(2)}V(0). \end{tikzcd} \tag{6.1.4}$$

The top spectral sequence here is in the usual (unlocalized) stable category. Many variants are possible here; for example, we could use the sphere as well.

We now have the following result.

Lemma 6.1.4. *There is a map of spectral sequences*

$$\begin{tikzcd} H^s(C_2,\pi_tV(0)) \arrow[r, Rightarrow] \arrow[d, "\varphi"'] & \pi_{t-s}V(0)^{hC_2} \arrow[d] \\ H^s(\mathbb{G}_2,E_tV(0)) \arrow[r, Rightarrow] & \pi_{t-s}L_{K(2)}V(0). \end{tikzcd} \tag{6.1.5}$$

from the homotopy fixed point spectral sequence for $V(0)$ to the Adams-Novikov Spectral Sequence of $L_{K(2)}V(0)$. The map φ at the E_2-page is the map on cohomology induced by the quotient map $\chi\colon \mathbb{G}_2\to(\mathbb{Z}/4)^\times\cong C_2$ and the Hurewicz map $\pi_*V(0)\to E_*V(0)$.

Proof. Combine the maps of spectral sequences (6.1.3) with (6.1.4). Note the top spectral sequence of (6.1.4) is the homotopy fixed point spectral sequence for $V(0)$ with a trivial C_2 action. □

Since $E\wedge V(0)$ doesn't have a ring structure, neither of the spectral sequences of (6.1.5) is a spectral sequence of rings. Nonetheless, the E_2-terms are graded commutative rings and the map φ is a ring map. Let h be the non-zero element in $H^1(C_2,\pi_0V(0))\cong\mathbb{F}_2$ and write h^i for its powers. Then, by definition, $\varphi(h^i)=\chi^i$.

Finally, since the homotopy fixed point spectral sequence for $V(0)$ is the Atiyah-Hirzebruch Spectral Sequence for $V(0)^*(\mathbb{R}P^\infty_+)$ we have the following observation. The inclusion of the skeleton $\mathbb{R}P^n\to\mathbb{R}P^\infty$ induces a map of spectral sequences

$$\begin{tikzcd} H^s(C_2,\pi_tV(0)) \arrow[r, Rightarrow] \arrow[d, "Skel_n"] & \pi_{t-s}V(0)^{hC_2} \arrow[d] \\ H^s(\mathbb{R}P^n_+,\pi_tV(0)) \arrow[r, Rightarrow] & \pi_{t-s}F(\mathbb{R}P^n_+,V(0))\ . \end{tikzcd}$$

where the lower spectral sequence is the Atiyah-Hirzebruch Spectral Sequence for $V(0)^*(\mathbb{R}P^n_+)$.

Proof of Theorem 6.1.1. To prove that χ is a d_3–cycle, we prove that $d_2(h)=0$ in the fixed point spectral sequence and that the only target for $d_3(h)$ maps to zero under the map φ of (6.1.5). We will use our result Proposition 5.3.1 on the nilpotence order of χ.

We begin by analyzing the d_2–differential. First, note that

$$Skel_3\colon H^s(C_2,\pi_tV(0))\to H^s(\mathbb{R}P^3_+,\pi_tV(0))$$

is an isomorphism if $0 \leq s \leq 2$ and injective if $s = 3$. Hence, to prove that h is a d_2–cycle, it suffices to prove that this holds for $Skel_3(h)$.

First, note that $\mathbb{R}P^3_+ \simeq S^0 \vee \Sigma V(0) \vee S^3$. Hence,

$$D\mathbb{R}P^3_+ \wedge V(0) \simeq V(0) \vee \Sigma^{-3} V(0) \vee (\Sigma^{-2} V(0) \wedge V(0)).$$

Therefore,

$$\pi_{-1}(D\mathbb{R}P^3_+ \wedge V(0)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4.$$

The needed calculations are elementary, but see also Section 3.1.

We now turn to the analysis of the relevant Atiyah-Hirzebruch spectral sequences. In the spectral sequence

$$E_2^{s,t} = H^s(\mathbb{R}P^3_+, \pi_t V(0)) \Longrightarrow \pi_{t-s}(D\mathbb{R}P^3_+ \wedge V(0)),$$

the only non-zero contributions to $\pi_{-1}(D\mathbb{R}P^3_+ \wedge V(0))$ in $E_2^{s,t}$ are:

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/4\{v_1 h^3\} & (s,t) = (3,2) \\ \mathbb{Z}/2\{\eta h^2\} & (s,t) = (2,1) \\ \mathbb{Z}/2\{h\} & (s,t) = (1,0). \end{cases}$$

Thus $d_2(h)$ must be zero, or the E_∞-term would be too small to account for the size of $\pi_{-1}(D\mathbb{R}P^3_+ \wedge V(0))$.

We turn to the d_3–differential. For this we do not need to restrict to any skeleton, but can directly use the diagram of spectral sequences (6.1.5). The targets for $d_3(h)$ are in

$$E_2^{4,2} = H^4(C_2, \pi_2 V(0)) \cong \mathbb{Z}/2\{v_1 h^4\}.$$

If $d_3(h) = \lambda v_1 h^4$ for $\lambda \in \mathbb{Z}/2$, then, by naturality,

$$d_3(\chi) = \lambda v_1 \varphi(h^4) = \lambda v_1 \chi^4.$$

However, by Proposition 5.3.1, $\chi^4 = 0$. This finishes the proof that χ is a d_3–cycle. By the Geometric Boundary Theorem (see Remark 3.2.1) and the fact that χ^2 is the Bockstein of χ, we have that χ^2 is also a d_3–cycle. $\square$

7. The decomposition of $L_{K(1)} L_{K(2)} S^0$

In this section, we prove one of our main results; that is, we show that

$$L_{K(1)} L_{K(2)} S^0 \simeq L_{K(1)} \big(S^0 \vee S^{-1} \vee \Sigma^{-2} V(0) \vee \Sigma^{-3} V(0) \big) \, .$$

See Theorem 7.4.1 below. In Section 7.5 we will use this result to calculate $L_1 L_{K(2)} S^0$.

The subtle point to the argument is to make an analysis of the action of the element $\zeta_1 \in \pi_* L_{K(1)} S^0$ on $\pi_* L_{K(1)} L_{K(2)} S^0$. In Proposition 4.1.7 we saw that $v_1^4 \zeta_1 = \sigma \in \pi_* L_{K(1)} V(0)$, where $\sigma \in \pi_7 S^0$ is the Hopf map. We gave some further background on the role of σ in $K(n)$-local homotopy theory in Section 3.2, but now we get specific and analyze the action of σ in $\pi_* L_{K(2)} S^0$.

7.1. **A Bockstein Lemma and detecting products with σ.** Fix a prime p and let $C = (C^\bullet, \partial_C)$ be a torsion-free cochain complex over $\mathbb{Z}_{(p)}$. We assume all cochain complexes are zero in negative degrees. Write

$$\delta = \delta_C \colon H^s(C/p^n) \longrightarrow H^{s+1}(C)$$

for the connecting homomorphism in the long exact sequence in cohomology induced by the short exact sequence of cochain complexes

$$0 \longrightarrow C \xrightarrow{p^n} C \longrightarrow C/p^n \longrightarrow 0.$$

This is, of course, the nth Bockstein homomorphism. We may write $\delta_C^{(n)}$ if we need to emphasize the n and we may abbreviate the Bockstein to δ_G if C is a complex for computing the group cohomology of a group G with coefficients in some module.

In working with the Bockstein we write $\overline{a}$ for the image of $a \in C$ under the reduction to C/p^n and likewise we write $\overline{y}$ for the the image of $y \in H^s(C)$ under the reduction to $H^s(C/p^n)$. By construction, if $x \in H^s(C/p^n)$ is any cohomology class, then $\delta_C(x)$ is represented by $\partial_C(a)/p^n$, where $a \in C^s$ is such that $\overline{a}$ is a representative for x. The next result then follows from the Leibniz rule.

Lemma 7.1.1. *Let A be a torsion-free differential graded algebra over $\mathbb{Z}_{(p)}$ and let $a \in H^*(A/p^n)$ be a cohomology class. Let M be a torsion-free differential graded module over A and $y \in H^*(M)$. Then*

$$\delta_M(a\overline{y}) = \delta_A(a)y.$$

Let $A = (A^\bullet, \partial_A)$ be a torsion-free differential graded algebra and

$$M_0 \xrightarrow{d_1} M_1 \xrightarrow{d_1} \cdots \xrightarrow{d_1} M_{n-1} \xrightarrow{d_1} M_n \longrightarrow \dots \tag{7.1.1}$$

a cochain complex of differential graded A-modules. This is a bicomplex, but the internal differentials $\partial_M \colon M_p^q \to M_p^{q+1}$ and the external differentials d_1 have different behavior with respect to the A-module structure:

$$\partial_M(ax) = \partial_A(a)x + (-1)^{\deg(a)} a\partial_M(x) \qquad \text{and} \qquad d_1(ax) = ad_1(x). \tag{7.1.2}$$

Nonetheless, we get a total complex $T(M)$ and, by filtering by degree in p, a spectral sequence

$$E_1^{p,q} = H^q M_p \Longrightarrow H^{p+q}T(M)$$

with $d_1 = H^*(d_1)$ and $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. It follows from (7.1.2) that this is a spectral sequence of $H^*(A)$ modules. Finally, there is an edge homomorphism

$$E_2^{p,0} = H^p(H^0(M_\bullet), d_1) \longrightarrow H^p(T(M)).$$

The following result has generalizations, but will suffice for our purposes. The proof is a diagram chase.

Lemma 7.1.2. *Let $x \in H^p(T(M))$ be the image of a d_1-cocycle $y \in H^0(M_p)$ under the edge homomorphism. Let $a \in H^0(A/p^n)$ be a class so that there is an element $c \in A$ satisfying*

(1) the class $\overline{c}$ is a cocycle representing a, and
(2) the class cy is a cocycle for the coboundary operator $\partial \colon M_p^0 \to M_p^1$.

Then $\delta^{(n)}_{T(M)}(a\overline{x}) \in H^{p+1}(T(M))$ *is the image under the edge homomorphism of*

$$\Big[\frac{d_1(cy)}{p^n}\Big] \in H^0(M_{p+1})$$

where $[z]$ *denotes the cohomology class of* z.

We apply Lemma 7.1.2 to the algebraic duality spectral sequence of Section 2.5. Applying the functor $\mathrm{Hom}_{cts}(-,E_*)$ to the algebraic duality resolution of Theorem 2.5.1 gives us an exact sequence of twisted E_*-$\mathbb{S}_2^1$-modules

$$\begin{aligned} 0 \to E_* \to & \mathrm{Hom}_{cts}(\mathbb{Z}_2[[\mathbb{S}_2^1/F_0]],E_*) \to \mathrm{Hom}_{cts}(\mathbb{Z}_2[[\mathbb{S}_2^1/F_1]],E_*) \\ & \to \mathrm{Hom}_{cts}(\mathbb{Z}_2[[\mathbb{S}_2^1/F_2]],E_*) \to \mathrm{Hom}_{cts}(\mathbb{Z}_2[[\mathbb{S}_2^1/F_3]],E_*) \to 0. \end{aligned}$$

We now let $A = C^\bullet(\mathbb{S}_2^1,E_*)$ be the differential graded algebra of continuous cochains with

$$C^n(\mathbb{S}_2^1,E_*) = \mathrm{map}_{cts}(\mathbb{S}_2^1\times\ldots\times\mathbb{S}_2^1,E_*) \cong \mathrm{Hom}_{cts}(\mathbb{Z}_2[[\mathbb{S}_2^1\times\ldots\times\mathbb{S}_2^1]],E_*)$$

with n factors of $\mathbb{S}_2^1$. Furthermore let M_p for $0\le p\le 3$ be the differential graded A module with

$$M_p^n = \mathrm{Hom}_{cts}(\mathbb{Z}_2[[\mathbb{S}_2^1\times\ldots\times\mathbb{S}_2^1]]\otimes\mathbb{Z}_p[[\mathbb{S}_2^1/F_p]],E_*)\ ,$$

again with n factors of $\mathbb{S}_2^1$. The tensor product has to be taken in the appropriate category of profinite $\mathbb{Z}_2[[\mathbb{S}_2^1]]$-modules, so in fact it is a completed tensor product. Then we get a complex of differential graded A-modules

$$0\to M_0\to M_1\to M_2\to M_3\to 0$$

to which we will apply Lemma 7.1.2. Note that the cohomology of the differential graded algebra A is equal to $H^*(\mathbb{S}_2^1,E_*)$ while the cohomology of M_p is given by $\mathrm{Ext}^*_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathbb{Z}_2[[\mathbb{S}_2^1/F_p]],E_*)\cong H^*(F_p,E_*)$. The spectral sequence of the total complex $T(M)$ is isomorphic to the algebraic duality spectral sequence of Remark 2.5.3 with E_1-term given as

$$E_1^{p,q} = H^q(F_p,E_*) \Longrightarrow H^{p+q}(\mathbb{S}_2^1,E_*).$$

We now get specific about the class c needed for Lemma 7.1.2. Recall from Remark 2.4.1 that $v_1 = u_1u^{-1}\in H^0(\mathbb{G}_2,E_2/2)$. We have, from Lemma 5.2.2 of [Bea17b], that the class

$$c_4 = 9(u_1^4u^{-4}+8u_1u^{-4}) \tag{7.1.3}$$

is invariant for the action of G_{24} and invariant modulo 16 with respect to the action of $\mathbb{G}_2$. Reducing modulo 16 we have class

$$\overline{c}_4\in H^0(\mathbb{G}_2,E_8/16)$$

which reduces to v_1^4 modulo 2. Let $H\subseteq\mathbb{G}_2$ be a closed subgroup and consider the Bockstein homomorphism $\delta^{(4)}\colon H^0(H,E_8/16)\to H^1(H,E_8)$.

Proposition 7.1.3. *Let* $H\subseteq\mathbb{G}_2$ *be any closed subgroup which contains* $\mathbb{S}_2^1$. *Up to multiplication by a unit in* $\mathbb{Z}_2$, *the image of* σ *in* π_*E^{hH} *is detected in the spectral sequence*

$$H^s(H,E_t)\Longrightarrow \pi_{t-s}E^{hH} \tag{7.1.4}$$

by the class $\delta^{(4)}(\overline{c}_4)\in H^1(H,E_8)$.

Proof. From [Bea17b], Theorem 1.8, we have an isomorphism $\mathbb{F}_4[v_1] \cong H^*(\mathbb{S}_2^1, E_*/2)$. Since v_1 is $\mathbb{G}_2$ invariant modulo 2, it immediately follows that we have an isomorphism

$$\mathbb{F}_2[v_1] \cong H^*(\mathbb{G}_2, E_*/2)\ .$$

We can now apply Proposition 3.2.2 to prove the claim for $H = \mathbb{G}_2$. The result follows for general H by restriction in group cohomology. □

We have not yet shown that $\sigma \neq 0$ in $\pi_* E^{hH}$. This will follow from Theorem 7.2.4 below. For now we have the following detection results.

Lemma 7.1.4. *Let $0 \le p < 3$. Let $x \in H^p(\mathbb{S}_2^1, E_*)$ be the image of a d_1-cocycle $y \in H^0(F_p, E_*)$ under the edge homomorphism*

$$H^p(H^0(F_\bullet, E_*), d_1) \to H^p(\mathbb{S}_2^1, E_*).$$

Then $\delta^{(4)}(c_4\overline{x}) \in H^{p+1}(\mathbb{S}_2^1, E_*)$ is the image under the edge homomorphism of

$$\Big[\frac{d_1(c_4 y)}{16}\Big] \in H^{p+1}(H^0(F_\bullet, E_*), d_1)\ .$$

Proof. This is an immediately rephrasing of Lemma 7.1.2 in this context. □

Lemma 7.1.1, Proposition 7.1.3 and Lemma 7.1.4 can be combined to prove the following result.

Proposition 7.1.5. *Let $0 \le s < 3$. Let $x \in \pi_{t-s} E^{h\mathbb{S}_2^1}$ be detected in the spectral sequence*

$$H^s(\mathbb{S}_2^1, E_t) \Longrightarrow \pi_{t-s} E^{h\mathbb{S}_2^1}$$

by the image of a class $y \in H^s(H^0(F_\bullet, E_t))$ under the edge homomorphism

$$H^s(H^0(F_\bullet, E_t)) \longrightarrow H^s(\mathbb{S}_2^1, E_t)$$

of the algebraic duality spectral sequence. Then σx is detected in $H^{s+1}(\mathbb{S}_2^1, E_{t+8})$ by the image of the class

$$\Big[\frac{d_1(c_4 y)}{16}\Big] \in H^{s+1}(H^0(F_\bullet, E_{t+8}))$$

under the edge homomorphism.

7.2. **The group cohomology of $v_1^{-1}E_*V(0)$.** In the next few results, we identify products with σ in the algebraic duality spectral sequence and then use this to give a description of $v_1^{-1}H^*(\mathbb{G}_2, E_*V(0))$. See Theorem 7.2.6 below. We begin with a connection to homotopy theory.

Remark 7.2.1. If Z is a type 1 complex with a v_1-self map $f\colon \Sigma^d Z \to Z$ and if X is any spectrum, let

$$v_1^{-1}(X \wedge Z) = (1 \wedge f)^{-1}(X \wedge Z).$$

This is independent of the choice of f and functorial by the essential uniqueness of v_n-self maps. See [HS98]. For the same reason, this construction preserves cofiber sequences.

There is a weak equivalence

$$v_1^{-1}(X \wedge Z) \simeq L_{K(1)}(X \wedge Z)$$

and, more generally, a weak equivalence

$$L_{K(1)}X \simeq \operatorname{holim} v_1^{-1}(X \wedge S/2^n)$$

where $S/2^n$ is the mod p^n-Moore spectrum.

From this we have that for any closed subgroup of $G \subseteq \mathbb{G}_2$ there is a localized spectral sequence

$$v_1^{-1}H^*(G, E_*V(0)) \Longrightarrow \pi_*L_{K(1)}(E^{hG} \wedge V(0))\ .$$

This spectral sequence has a strong horizontal vanishing line and converges strongly. See (2.3.3), Definition 2.3.1, Lemma 2.3.2, and Corollary 2.3.7.

Recall from Warning 4.1.6 that even though $V(0)$ is not a ring spectrum, we often describe the E_2-term of a spectral sequences for the homotopy of a localization of $V(0)$ as a ring.

The following result from [Bea17b] is the key to all computations. The cohomology classes $\chi = \chi_2$ and e appeared in Corollary 5.2.11.

Theorem 7.2.2. *The localized cohomology group $v_1^{-1}H^*(\mathbb{S}_2^1, E_*V(0))$ is free on six generators over the subring $\mathbb{F}_4[v_1^{\pm 1}, \eta]$. Specifically, in the algebraic duality resolution spectral sequence*

$$H^q(F_p, E_tV(0)) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, E_tV(0))$$

there are permanent cycles

(1) Δ_0 of degree $(p, q, t) = (0, 0, 0)$ detecting the unit;
(2) b_0 of degree $(p, q, t) = (1, 0, 0)$ detecting χ;
(3) b_1 of degree $(p, q, t) = (1, 0, 6)$;
(4) $\overline{b}_0$ of degree $(p, q, t) = (2, 0, 0)$ detecting χ^2;
(5) $\overline{b}_1$ of degree $(p, q, t) = (2, 0, 6)$;
(6) $\overline{\Delta}_0$ of degree $(p, q, t) = (3, 0, 0)$ detecting e.

After inverting v_1, there is an isomorphism of $\mathbb{F}_4[v_1^{\pm 1}, \eta]$-modules

$$v_1^{-1}H^*(\mathbb{S}_2^1, (E_2)_*V(0)) \cong \mathbb{F}_4[v_1^{\pm 1}, \eta]\{\Delta_0, b_0, b_1, \overline{b}_0, \overline{b}_1, \overline{\Delta}_0\} \tag{7.2.1}$$

Proof. The only point to add to Corollary 1.2.3 of [Bea17b] is the explanation of why b_0 detects χ, $\overline{b}_0$ detects χ^2, and $\overline{\Delta}_0$ detects e. But this follows by combining Lemma 5.2.9 and Theorem 5.4.1. □

Remark 7.2.3. The generators of Δ_0, b_0, b_1, $\overline{b}_0$, $\overline{b}_1$ and $\overline{\Delta}_0$ of (7.2.1) are, strictly speaking, each an element of $H^0(F_p, E_*V(0))$ for some p. We have conflated them with their images under the edge homomorphism

$$H^0(F_p, E_*V(0)) \longrightarrow E_\infty^{p,0} \subseteq H^p(\mathbb{S}_2^1, E_*V(0))$$

of the algebraic duality spectral sequence. There should be no ambiguity. This issue was also discussed in Remark 5.2.8.

Our main new computation is the following. It will allow us to rewrite the classes b_1, $\overline{b}_1$, and $\overline{\Delta}_0 = e$ in terms of σ, v_1 and the other generators of Theorem 7.2.2. Note that this result is a statement about cohomology classes *before* v_1-localization.

Theorem 7.2.4. *Multiplication by σ gives the following identities in the cohomology ring $H^*(\mathbb{S}_2^1, (E_2)_*V(0))$:*

(1) $\sigma = \sigma\Delta_0 = v_1 b_1$;
(2) $\sigma\bar{b}_0 = v_1^4\overline{\Delta}_0$ *or, equivalently,* $\sigma\chi^2 = v_1^4\overline{\Delta}_0$; *and*
(3) $\sigma b_0 \equiv \epsilon_0 v_1 \bar{b}_1 + \epsilon_1 v_1^4 \bar{b}_0$ *for some* $\epsilon_0 \in \mathbb{F}_4^\times$ *and* $\epsilon_1 \in \mathbb{F}_4$.

This is proved below in Proposition 7.2.8, Proposition 7.2.10, and Proposition 7.2.11. For now we will draw some consequences. In the following two results, $E(-)$ denotes the exterior algebra over $\mathbb{F}_2$.

Theorem 7.2.5. *There is an isomorphism of graded algebras*

$$\mathbb{F}_4[v_1^{\pm 1}, \eta] \otimes E(\sigma) \otimes \mathbb{F}_2[\chi]/(\chi^3) \cong v_1^{-1}H^*(\mathbb{S}_2^1, (E_2)_*V(0)) \ .$$

Proof. By Corollary 4.1.9 and Proposition 5.3.1 we have $\sigma^2 = 0$ and $\chi^3 = 0$ in $v_1^{-1}H^*(\mathbb{S}_2^1, (E_2)_*V(0))$. The result now follows from (7.2.1), and Theorem 7.2.4. □

Theorem 7.2.6. *There are isomorphisms of graded algebras*

$$\mathbb{F}_2[v_1^{\pm 1}, \eta] \otimes E(\sigma) \otimes \mathbb{F}_2[\chi]/(\chi^3) \cong v_1^{-1}H^*(\mathbb{G}_2^1, (E_2)_*V(0))$$

and

$$\mathbb{F}_2[v_1^{\pm 1}, \eta] \otimes E(\sigma, \zeta_2) \otimes \mathbb{F}_2[\chi]/(\chi^3) \cong v_1^{-1}H^*(\mathbb{G}_2, (E_2)_*V(0)) \ .$$

Proof. The first statement follows immediately from Theorem 7.2.5 by taking Galois fixed points since v_1, η, σ and χ are Galois invariant. See Lemma 2.2.3.

To prove the second statement, note that the elements v_1, η, σ and χ are all in the image of the restriction map

$$H^*(\mathbb{G}_2, (E_2)_*V(0)) \to H^*(\mathbb{G}_2^1, (E_2)_*V(0)) \ .$$

Thus the Lyndon-Hochschild-Serre Spectral Sequence for the split extension $\mathbb{G}_2^1 \to \mathbb{G}_2 \to \mathbb{Z}_2$ reads

$$v_1^{-1}H^*(\mathbb{G}_2^1, (E_2)_*V(0)) \otimes E(\zeta_2) \Longrightarrow v_1^{-1}H^*(\mathbb{G}_2, (E_2)_*V(0)).$$

This must collapse and, since $\zeta_2^2 = 0$ in $H^*(\mathbb{G}_2, E_0)$, there can be no multiplicative extensions. □

We now turn to the proof of Theorem 7.2.4. Our basic computational tool is Proposition 7.1.5. The proofs involve studying σ in the integral algebraic duality spectral sequence

$$E_1^{p,q,*} \cong H^q(F_p, E_*) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, E_*) \tag{7.2.2}$$

and then reducing modulo 2. We need the following preliminary result.

Proposition 7.2.7. *Let X be a finite spectrum. In the algebraic duality spectral sequence*

$$E_1^{p,q} = H^q(F_p, E_*X) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, E_*X)$$

multiplication by σ raises filtration in p.

Proof. We saw in Remark 5.2.7 that this is a spectral sequence of modules over $H^*(\mathbb{S}_2^1, E_*)$. The module structure on the E_1-term is given via the restriction homomorphisms $H^*(\mathbb{S}_2^1, E_*) \to H^*(F_p, E_*)$, hence it suffices to show that the restriction of σ considered as a class in $H^1(\mathbb{S}_2^1, E_8)$ restricts trivially to $H^1(F_p, E_8)$ for all p. Because $F_1 = F_2 \subset F_0$ it suffices to show this for $F_0 = G_{24}$ and $F_3 = G_{24}'$. In fact, because σ comes from $H^1(\mathbb{S}_2, E_8)$ — and even from $H^1(\mathbb{G}_2, E_8)$ — and because F_0 and F_3 are conjugate in $\mathbb{S}_2$, it suffices to consider the case of G_{24}.

In Section 3.2 the class σ was defined to be the Greek letter element $\alpha_{4/4} = \delta^{(4)}(v)$. Furthermore, by Proposition 3.2.2 the image of v in $H^0(\mathbb{G}_2, E_8/(16))$ is equal to the reduction $\overline{c}_4$ of the integral class c_4, up to a 2-adic unit. The class c_4 was defined in (7.1.3). By Lemma 5.2.2 of [Bea17b] $\overline{c}_4 \in H^0(G_{24}, E_8/16)$ lifts to $H^0(G_{24}, E_8)$, so we have

$$\delta^{(4)}_{G_{24}}(\overline{c}_4) = 0 \in H^1(G_{24}, E_8)$$

as needed. □

We now come to part (1) of Theorem 7.2.4.

Proposition 7.2.8. *The mod-2 reduction of $\sigma \in H^1(\mathbb{S}^1_2, E_8)$ to $H^1(\mathbb{S}^1_2, E_8V(0))$ satisfies the equation*

$$\sigma\Delta_0 = v_1 b_1.$$

Proof. By Theorem 7.2.2, the class

$$\Delta_0 \in H^0(G_{24}, E_0) = H^0(F_0, E_0)$$

detects the unit in $H^*(\mathbb{S}^1_2, E_*V(0))$. By Proposition 7.1.5, $d_1(c_4\Delta_0)/16$ detects $\sigma\Delta_0$ in the algebraic duality spectral sequence. It follows from Definition 5.2.3 and Proposition 5.2.4 of [Bea17b] that

$$\frac{d_1(c_4\Delta_0)}{16} \equiv v_1 b_1 \qquad \text{mod } 2 .$$

Therefore,

$$\sigma\Delta_0 = v_1 b_1 \in E_\infty^{1,0,8} \subseteq H^1(\mathbb{S}^1_2, E_8V(0)).$$ □

Remark 7.2.9. The proof of the next result uses heavily the techniques and language of Section 5 of [Bea17b] and in particular the finer structure of the group $\mathbb{S}_2$. We recall some of the details.

In Remark 2.4.4 we recorded that

$$\mathbb{S}_2 \cong (\mathbb{W}\langle T\rangle/(T^2 = -2, aT = Ta^\sigma))^\times$$

and in Remark 2.4.5 we defined the element $\pi = 1+2\omega \in \mathbb{S}_2$, where ω is a primitive third root of unity. Another important element of $\mathbb{S}_2$ is

$$\alpha := \frac{1-2\omega}{\sqrt{-7}}$$

where $\sqrt{-7}$ is chosen to be congruent to 1 modulo 4. Then

$$\alpha \equiv 1 + \omega T^2 + T^4 \qquad \text{modulo } (T^6).$$

Note that α and π are both in $\mathbb{W}^\times$ and hence they commute in $\mathbb{S}_2$.

In Remark 2.4.1 we discussed the subgroup $Q_8 \subseteq \mathbb{S}_2$ and its elements i, j, and k.

We can now prove part (2) of Theorem 7.2.4.

Proposition 7.2.10. *In $H^3(\mathbb{S}^1_2, (E_2)_8V(0))$ there is an equation*

$$\sigma\chi^2 = v_1^4\overline{\Delta}_0.$$

Proof. In this proof we will write

$$E_1^{p,q,t} = H^p(F_q, E_t) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, E_t)$$

for the algebraic duality spectral sequence computing $H^*(\mathbb{S}_2^1, E_*)$. As in Figure 4 and Remark 5.2.8, we also write

$$\bar{b}_0 \in H^0(C_6, E_0) = H^0(F_2, E_0) \cong E_1^{2,0,0}$$

for the integral lift of the class $\bar{b}_0$ of Theorem 7.2.2. This integral lift is the unit in $H^0(C_6, E_0)$ and detects $\widetilde{\chi}$ by Lemma 5.2.9. Proposition 7.1.5 implies that $\sigma\widetilde{\chi}$ is detected by $d_1(c_4\bar{b}_0)/16$. By Theorem 1.1.1 of [Bea17b], $d_1\colon E_1^{2,0,t} \to E_1^{3,0,t}$ is given by the action of

$$\pi(e+i+j+k)(e-\alpha^{-1})\pi^{-1} = \pi(e+i+j+k)\pi^{-1}(e-\alpha^{-1}).$$

Lemma 5.2.2 of [Bea17b] applied to $\alpha^{-1} \equiv 1+\omega T^2$ modulo T^3 together with the fact that π_*^{-1} is an isomorphism congruent to the identity modulo $(2, u_1)$ implies that

$$\pi_*^{-1}(e-\alpha^{-1})_*(c_4) \equiv 16u_1u^{-4} = 16v_1v_2 \mod (32, 16u_1^2)$$

where $v_1 = u_1u^{-1}$ and $v_2 = u^{-3}$. Since π and α commute with the elements of C_6, $\pi_*^{-1}(e-\alpha^{-1})_*(c_4)$ is an element of $E_8^{C_6}$, and we can improve the congruence to

$$\pi_*^{-1}(e-\alpha^{-1})_*(c_4) \equiv 16v_1v_2 \mod (32, 16u_1^4).$$

We compute the action of $e+i+j+k$ on the right hand side. Since we are reducing modulo 32, it suffices to study the action modulo 2.

The elements i, j and k fix $v_1 \equiv u_1u^{-1}$ modulo 2. Furthermore, modulo 2 the formulas in Section 2.4 of [Bea17b] give

$$i_*(v_2) = v_2(u_1+1)^3, \qquad j_*(v_2) = v_2(\omega u_1+1)^3, \qquad k_*(v_2) = v_2(\omega^2u_1+1)^3,$$

where $\omega \in \mathbb{W}$ is a third root of unity. A direct computation using the fact that $1+\omega+\omega^2 = 0$ implies that

$$(e+i+j+k)_*(v_1v_2) = v_1^4 \mod (2).$$

Since $e+i+j+k$ sends the ideal $(2, u_1^4)$ to $(2, u_1^5)$ and π is an isomorphism congruent to the identity modulo $(2, u_1)$, we conclude that

$$(\pi(e+i+j+k)\pi^{-1}(e-\alpha^{-1}))_*(c_4) = 16v_1^4 \mod (32, 16u_1^5).$$

Therefore $16^{-1}d_1(c_4\bar{b}_0) \equiv v_1^4\overline{\Delta}_0 \mod (2, u_1^5)$, so that

$$\sigma\bar{b}_0 \equiv v_1^4\overline{\Delta}_0 \mod (2, u_1^5)$$

in $E_2^{3,0,8}$.

By functoriality, this identity also holds in the Algebraic Duality Spectral Sequence with coefficients $E_*V(0)$. By Theorem 1.2.2 of [Bea17b] we have $\mathbb{F}_4\{v_1^4\overline{\Delta}_0\} \cong E_\infty^{3,0,8} \subseteq H^3(\mathbb{S}_2^1, E_8V(0))$ in that spectral sequence and this proves the claim. $\square$

We now come to the proof of the final part of Theorem 7.2.4.

Proposition 7.2.11. *There exists $\epsilon_0 \in \mathbb{F}_4^\times$ and $\epsilon_1 \in \mathbb{F}_4$ such that*

$$\sigma b_0 = \epsilon_0 v_1\bar{b}_1 + \epsilon_1 v_1^4\bar{b}_0$$

in $H^2(\mathbb{S}_2^1, (E_2)_8V(0))$.

Proof. Since $b_0 \in E_\infty^{1,0,0}$ and, by Proposition 7.2.7, multiplication by σ raises filtration in the algebraic duality spectral sequence,

$$\sigma b_0 \in E_\infty^{2,0,8} \cong \mathbb{F}_4\{v_1\bar{b}_1, v_1^4\bar{b}_0\}.$$

The calculation of $E_\infty^{2,0,8}$ is from Theorem 1.2.2 of [Bea17b]. Because of this, we have an equation

$$\sigma b_0 = \epsilon_0 v_1 \bar{b}_1 + \epsilon_1 v_1^4 \bar{b}_0$$

with $\epsilon_i \in \mathbb{F}_4$. Let β be the Bockstein homomorphism for $H^*(\mathbb{S}_2^1, E_*V(0))$ associated to the short exact sequence

$$0 \to (E_2)_*/2 \to (E_2)_*/4 \to (E_2)_*/2 \to 0\ .$$

Since since b_0 detects χ and $\bar{b}_0$ detects χ^2, we have $\beta(b_0) = \bar{b}_0$. Then, because σ is an integral class, we have

$$\beta(\sigma b_0) = \sigma \bar{b}_0 = v_1^4 \overline{\Delta}_0 \neq 0.$$

The second equality is Proposition 7.2.10. However, v_1^4 is invariant modulo 4, so $\beta(v_1^4\bar{b}_0) = v_1^4\beta(\bar{b}_0) = 0$. This implies that $\epsilon_0 \neq 0$. □

7.3. **The localized Adams-Novikov Spectral Sequence for $L_{K(1)}L_{K(2)}V(0)$.** In Theorem 7.2.6 we calculated the E_2-term of the localized spectral sequence to be

$$\mathbb{F}_2[v_1^{\pm 1}, \eta] \otimes E(\sigma, \zeta_2) \otimes \mathbb{F}_2[\chi]/(\chi^3) \cong v_1^{-1}H^*(\mathbb{G}_2, (E_2)_*V(0))\ .$$

In this section, we compute the differentials and extensions. The classes 1, η, σ and ζ_2 are permanent cycles. We will show that the classes χ and χ^2 are permanent cycles as well, which is the key step in the computation. We will see below in Theorem 8.2.1 that, in fact, they are permanent cycles before v_1-localization, but that requires considerably more work.

We break up the computation into a number of results. We will use the universal differential of Remark 4.1.5. There is no η-torsion at E_3 so this becomes becomes

$$d_3(v_1^2 z) = v_1^2 d_3(z) + \eta^3 z\ . \tag{7.3.1}$$

Our first result is a list of basic structural properties of our spectral sequences.

Lemma 7.3.1. *The localized Adams-Novikov Spectral Sequences*

$$v_1^{-1}H^*(\mathbb{G}_2^1, (E_2)_*V(0)) \Longrightarrow \pi_* L_{K(1)}(E^{\mathbb{G}_2^1} \wedge V(0))$$

$$v_1^{-1}H^*(\mathbb{G}_2, (E_2)_*V(0)) \Longrightarrow \pi_* L_{K(1)}L_{K(2)}V(0)$$

have the following properties:

(1) Both spectral sequences are modules over $\mathbb{F}_2[v_1^{\pm 4}, \eta] \otimes E(\sigma, \zeta_2)$.
(2) In the first spectral sequence ζ_2 acts by zero.
(3) In both spectral sequences $d_{2r} = 0$ for all $r \geq 1$.
(4) In both spectral sequences $\eta^3 E_4^{*,*} = 0$.
(5) For the first spectral sequence $E_4^{s,t} = 0$ for $s \geq 6$ and for the second spectral sequence $E_4^{s,t} = 0$ for $s \geq 7$.
(6) In both spectral sequences the classes 1 and v_1 are permanent cycles.

Proof. For the first statement we use that η, σ, and ζ_2 are permanent cycles for the sphere and that $V(0)$ has a v_1^4 self-map. For the second statement, we use that $\zeta_2 = 0$ in $\pi_* E^{h\mathbb{G}_2^1}$. See Proposition 2.2.1. Since $E_*V(0)$ is concentrated in even degrees, the third statement is the standard sparseness result for the Adams-Novikov Spectral Sequence.

In particular the first differential is d_3. There is no η-torsion at E_3, so if $d_3(z) = 0$, (7.3.1) gives $d_3(v_1^2 z) = \eta^3 z$. Part (4) follows.

To get the vanishing lines, note that every element at the E_2-term of the spectral sequence of $\pi_* L_{K(1)} L_{K(2)} V(0)$ is an η-multiple of the classes

$$v_1^j \chi^i \sigma^{\epsilon_1} \zeta_2^{\epsilon_2}$$

with $j \in \mathbb{Z}$, $0 \leq i \leq 2$ and $\epsilon_i = 0$ or 1. These classes have s-filtration at most 4, so the result follows from part (4). In the spectral sequence for $\pi_* L_{K(1)}(E^{h\mathbb{G}_2^1} \wedge V(0))$, we have $\epsilon_2 = 0$, and we get the better vanishing line.

The final statement is true for the BP-based Adams-Novikov Spectral Sequence for the sphere, so is true here as well. Compare Figure 1; see also (2.1.7). □

In the next proof we will use the homotopy Bockstein homomorphism

$$\beta \colon \pi_n L_{K(1)} L_{K(2)} V(0) \to \pi_{n-1} L_{K(1)} L_{K(2)} V(0).$$

See Remark 3.1.1. Any element in the image of β is of order 2.

Our next result is a preliminary calculation with $E^{h\mathbb{G}_2^1}$. Figure 5 below gives a partial illustration.

Lemma 7.3.2. *For the spectral sequence*

$$v_1^{-1} H^*(\mathbb{G}_2^1, (E_2)_* V(0)) \Longrightarrow \pi_* L_{K(1)}(E^{h\mathbb{G}_2^1} \wedge V(0)) \qquad (7.3.2)$$

we have the following:

1. *The classes χ, χ^2, and $v_1\chi^2$, are permanent cycles. If y is any class detected by χ, then $\beta(y)$ is detected by χ^2, and $2y = \beta(y)\eta$.*
2. *The spectral sequence collapses at E_4. The non-trivial differentials are determined by the differential*
$$d_3(v_1\chi) = \eta^2\chi^2$$
and the fact that the spectral sequence is a module over $\mathbb{F}_2[v_1^{\pm 4}, \eta] \otimes E(\sigma)$.
3. *The class $\eta v_1 \chi + v_1^2\chi^2$ is a permanent cycle. If x is any class detected by $\eta v_1\chi + v_1^2\chi^2$, then $\beta(x)$ is detected by $\eta^2\chi + \eta v_1 \chi^2$, and $2x = \beta(x)\eta$.*

Proof. Recall we have

$$\mathbb{F}_2[v_1^{\pm 1}, \eta] \otimes E(\sigma) \otimes \mathbb{F}_2[\chi]/(\chi^3) \cong v_1^{-1} H^*(\mathbb{G}_2^1, (E_2)_* V(0)) \ .$$

We begin part (1). The class χ is a d_3–cycle by Theorem 6.1.1, hence it must be a permanent cycle by part (5) of Lemma 7.3.1. Let y be a class detected by χ in the spectral sequence (7.3.2). Since χ^2 is the algebraic Bockstein of χ, the Geometric Boundary Theorem (see Remark 3.2.1) implies χ^2 is a permanent cycle for this spectral sequence and detects the Bockstein $\beta(y)$. By Lemma 3.1.3 there must be an additive extension $2y = \beta(y)\eta$ in the spectral sequence.

To see $v_1\chi^2$ is a permanent cycle, note that Theorem 6.1.1 and the Geometric Boundary Theorem imply that $\widetilde{\chi} \in H^2(\mathbb{G}_2, E_0)$ is a d_3-cycle in the Adams-Novikov

Spectral Sequence for $L_{K(2)}S^0$. Since χ^2 is the reduction of $\widetilde{\chi}$ and v_1 is a d_3-cycles, we have $\chi^2 v_1$ is a d_3-cycle. It is then a permanent cycle by the vanishing line of part (5) of Lemma 7.3.1.

We now take on part (2). Since the class $\eta\chi^2$ detects a class divisible by 2, we must have $\eta^2\chi^2 = 0$ at E_∞. It follows that there must be a non-trivial differential originating at $E_3^{1,2}$. This vector space has a basis given by $v_1^{-3}\sigma$ and $v_1\chi$. The class $v_1^{-3}\sigma$ is a permanent cycle, since v_1 is a permanent cycle and the spectral sequence is a module over $\mathbb{F}_2[v_1^{\pm 4}, \sigma]$; hence, we must have the claimed differential

$$d_3(v_1\chi) = \eta^2\chi^2.$$

We now apply Lemma 7.3.1 parts (1), (2), and (5) to get the collapse at E_4 and complete this step in the argument.

We now turn to part (3). From this, the differential from part (2), and (7.3.1), we can deduce the following differentials:

$$d_3(v_1^2\chi) = \eta^3\chi \qquad d_3(v_1^2\chi^2) = \eta^3\chi^2$$
$$d_3(v_1^3\chi^2) = \eta^3 v_1\chi^2 \qquad d_3(v_1^3\chi) = \eta^3 v_1\chi + \eta^2 v_1^2\chi^2.$$

Then $\eta v_1\chi + v_1^2\chi^2$ is a d_3-cycle and hence a permanent cycle, again by the the vanishing line from Lemma 7.3.1.

Now let x be a class detected by $\eta v_1\chi + v_1^2\chi^2$. We have an equation for the algebraic Bockstein

$$\beta(\eta v_1\chi + v_1^2\chi^2) = \beta(\eta v_1\chi) = \eta^2\chi + \eta v_1\chi^2.$$

The Geometric Boundary Theorem implies $\eta^2\chi + \eta v_1\chi^2$ is a permanent cycle and detects the Bockstein $\beta(x)$. By Lemma 3.1.3 there must be an additive extension $2x = \beta(x)\eta$ in the spectral sequence. □

We now come to one the main calculations of this paper, Theorem 7.3.3. This result shows that the localized Adams-Novikov Spectral Sequence (7.3.3) for the spectrum $L_{K(1)}L_{K(2)}V(0)$ can be obtained from the spectral sequence of (7.3.2) for $L_{K(1)}(E^{h\mathbb{G}_2^1} \wedge V(0))$ by tensoring with the exterior algebra $E(\zeta_2)$ and extending the differentials by the Leibniz rule. This is by no means formal. It requires a statement and argument which, on the surface, looks like a repetition of Lemma 7.3.2, but the logic requires us to prove part (2) of Lemma 7.3.2 before proving part (1) of Theorem 7.3.3.

Theorem 7.3.3. *The spectral sequence*

$$v_1^{-1}H^*(\mathbb{G}_2, (E_2)_*V(0)) \Longrightarrow \pi_* L_{K(1)}L_{K(2)}V(0). \tag{7.3.3}$$

collapses at E_4. We have

1. *The classes χ, χ^2, and $v_1\chi^2$, are permanent cycles. If y is any class detected by χ, then $\beta(y)$ is detected by χ^2, and $2y = \beta(y)\eta$.*
2. *The spectral sequence collapses at E_4. The non-trivial differentials are determined by the differential*
$$d_3(v_1\chi) = \eta^2\chi^2$$
and the fact that the spectral sequence is a module over $\mathbb{F}_2[v_1^{\pm 4}, \eta] \otimes E(\sigma, \zeta_2)$.
3. *The class $\eta v_1\chi + v_1^2\chi^2$ is a permanent cycle. If x is any class detected by $\eta v_1\chi + v_1^2\chi^2$, then $\beta(x)$ is detected by $\eta^2\chi + \eta v_1\chi^2$, and $2x = \beta(x)\eta$.*

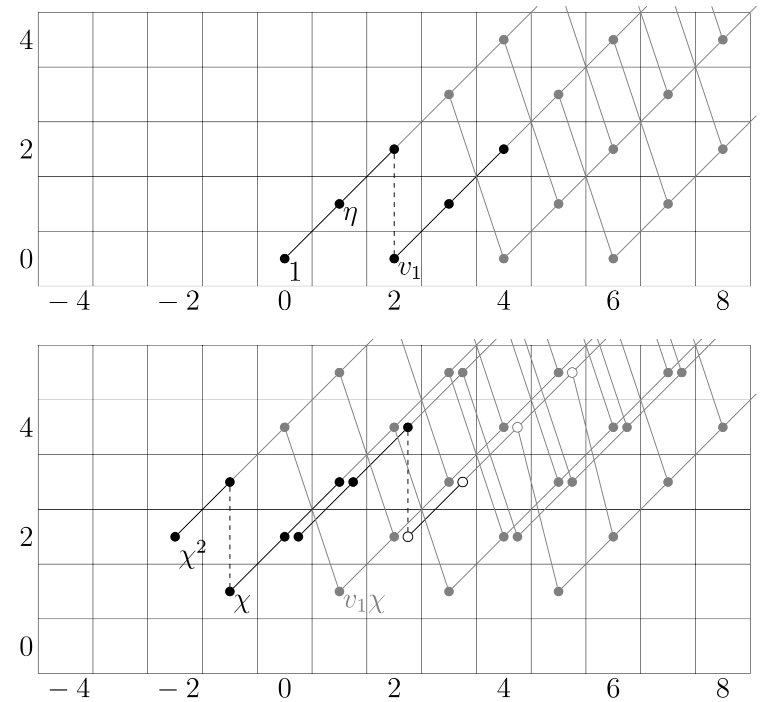

FIGURE 5. Patterns for the differentials on certain classes in the spectral sequences (7.3.2) and (7.3.3). The top figure illustrates the differential pattern on the unit 1 and the bottom figure the differential pattern on χ and χ^2. There, a $\circ$ denotes a copy of $\mathbb{F}_2$ generated by an η-multiple of the class $\eta v_1\chi + v_1^2\chi^2$. The dashed lines denote the extension by 2 at E_∞.

Proof. We have

$$\mathbb{F}_2[v_1^{\pm 1}, \eta] \otimes E(\sigma, \zeta_2) \otimes \mathbb{F}_2[\chi]/(\chi^3) \cong v_1^{-1}H^*(\mathbb{G}_2, (E_2)_*V(0)) \ .$$

We begin with part (1). The class χ is a d_3–cycle by Theorem 6.1.1. Thus, by parts (3) and (5) of Lemma 7.3.1, χ will be a permanent cycle if it is a d_5-cycle. Using part (4) of Lemma 7.3.1, we see that the only possible non-zero differential is

$$d_5(\chi) = v_1^{-4}\zeta_2\sigma\eta^2\chi^2.$$

However, by naturality with respect to the map $\mathbb{G}_2^1 \to \mathbb{G}_2$ and by part (2) of Lemma 7.3.2, we have $d_3(v_1\chi) = \eta^2\chi^2 + \zeta_2 x$ for some class $x \in E_2^{4,4}$. Since $\zeta_2^2 = 0$ in $H^*(\mathbb{G}_2, E_*)$, we can calculate

$$d_3(v_1^{-4}\zeta_2\sigma v_1\chi) = v_1^{-4}\zeta_2\sigma d_3(v_1\chi) = v_1^{-4}\zeta_2\sigma\eta^2\chi^2 \ .$$

Hence $v_1^{-4}\zeta_2\sigma\eta^2\chi^2$ is zero on the E_4-term. We conclude that χ is a d_5-cycle. The remainder of the proof of part (1) goes through exactly as for part (1) of Lemma 7.3.2.

We now turn to part (2). As in part (2) of Lemma 7.3.2, there must be a class in $a \in E_3^{1,2}$ with $d_3(a) = \eta^2\chi^2$. The vector space $E_3^{1,2}$ has basis $v_1\chi$, $v_1\zeta_2$, $v_1^{-3}\sigma$.

The class $v_1\zeta_2$ is a permanent cycle by (1) and (3) of Lemma 7.3.1. The class $v_1^{-3}\sigma$ is a permanent cycle exactly as in the proof of (2) in Lemma 7.3.2 . Thus we have the indicated differential.

To get the collapse at E_4 we use Lemma 7.3.1 parts (1), (2), (3), and (5). Part (3) is proved exactly as in part (3) of Lemma 7.3.2. □

Remark 7.3.4. We have not listed all the additive extensions in the spectral sequences (7.3.2) and (7.3.3). In fact we have missed only those exotic extensions by 2 implied by the extension from v_1 to η^2 already visible in Figure 1 of Section 3. Indeed, it will follow from Theorem 7.4.1 below that all additive extensions are determined by v_1^4 and ζ_2 multiplications and the exotic extensions by 2 from v_1 to η^2, from χ to $\chi^2\eta$, and from $\eta v_1\chi + v_1^2\chi^2$ to

$$\eta^3\chi + \eta^2 v_1\chi^2 \equiv \eta^2 v_1\chi^2.$$

The last equivalence follows from the fact that $\eta^3 = 0$ at E_4. These extension are all shown in Figure 5. The upper figure shows a pattern from $L_{K(1)}V(0)$, and the lower figure shows a pattern from $L_{K(1)}(V(0) \wedge V(0))$. Compare Figure 1 and Figure 2.

7.4. **The homotopy type of $L_{K(1)}L_{K(2)}S^0$.** We are now ready to prove the decomposition of the $K(1)$-localization of $L_{K(2)}S^0$. Choose a class

$$y \in \pi_{-1}L_{K(1)}L_{K(2)}V(0) \tag{7.4.1}$$

detected by $\chi \in v_1^{-1}H^1(\mathbb{G}_2, E_*V(0))$; this is possible by Theorem 7.3.3. The class y is not unique, but any two choices differ by an element of higher Adams-Novikov filtration. The proof of Theorem 7.4.1 comes down to an Adams-Novikov Spectral Sequence argument, and we will see that any choice will do. Theorem 8.2.1 below will allow us to refine our choice.

If $D(-)$ denotes the Spanier-Whitehead duality functor, we then obtain a map from y

$$x\colon \Sigma^{-2}V(0) = \Sigma^{-1}DV(0) \longrightarrow L_{K(1)}L_{K(2)}S^0 . \tag{7.4.2}$$

This gives us maps

$$\begin{aligned} \iota\colon S^0 &\longrightarrow L_{K(2)}S^0 \\ \zeta_2\colon S^{-1} &\longrightarrow L_{K(2)}S^0 \\ x\colon \Sigma^{-2}V(0) &\longrightarrow L_{K(1)}L_{K(2)}S^0 \\ x\zeta_2\colon \Sigma^{-3}V(0) &\longrightarrow L_{K(1)}L_{K(2)}S^0 \end{aligned}$$

The first of these maps is the unit; the second is the class of Proposition 2.2.1. These maps assemble into a map

$$f = \iota \vee \zeta_2 \vee x \vee x\zeta_2\colon S^0 \vee S^{-1} \vee \Sigma^{-2}V(0) \vee \Sigma^{-3}V(0) \longrightarrow L_{K(1)}L_{K(2)}S^0. \tag{7.4.3}$$

Here is one of our main results.

Theorem 7.4.1. *The map f induces a $K(1)$-local equivalence*

$$L_{K(1)}\big(S^0 \vee S^{-1} \vee \Sigma^{-2}V(0) \vee \Sigma^{-3}V(0)\big) \simeq L_{K(1)}L_{K(2)}S^0 .$$

This is a consequence of Theorem 7.4.6, but to get there we need some preliminaries.

Lemma 7.4.2. *Let $f\colon X_1 \to X_2$ be a map of spectra. Suppose there is a type 1 complex Z so that the induced map*

$$L_{K(1)}(X_1 \wedge Z) \to L_{K(1)}(X_2 \wedge Z)$$

is an equivalence. Then $L_{K(1)}X_1 \to L_{K(1)}X_2$ is an equivalence.

Proof. We will use that

$$L_{K(1)}X \simeq \operatorname{holim} v_1^{-1}(X \wedge S/2^n)$$

where $S/2^n$ is the mod 2^n Moore spectrum. See Remark 7.2.1. From this we see that it is sufficient to show that

$$v_1^{-1}(X_1 \wedge Y) \to v_1^{-1}(X_2 \wedge Y)$$

is an equivalence for all type 1 spectra Y. The class of type 1 spectra Y for which we have such an equivalence is a thick subcategory of 2-local finite spectra. Since there is such an equivalence for one type 1 spectrum, there is for all. □

From this last result we see that it is enough to make a judicious choice of type 1 complex. Recall that $Y = V(0) \wedge C(\eta)$. See Section 4.2.

Proposition 7.4.3. *There are isomorphisms of $\mathbb{F}_2[v_1^{\pm 1}] \otimes E(\sigma)$–modules*

$$\mathbb{F}_2[v_1^{\pm 1}] \otimes E_{\mathbb{F}_2}(\sigma) \otimes \mathbb{F}_2[\chi]/(\chi^3) \cong v_1^{-1}H^*(\mathbb{G}_2^1, E_*Y)$$

$$\mathbb{F}_2[v_1^{\pm 1}] \otimes E_{\mathbb{F}_2}(\sigma, \zeta_2) \otimes \mathbb{F}_2[\chi]/(\chi^3) \cong v_1^{-1}H^*(\mathbb{G}_2, E_*Y)\ .$$

Proof. By Landweber exactness and Remark 4.2.1 we have a short exact sequence of $\mathbb{G}_2$-modules

$$0 \to E_*V(0) \to E_*Y \to E_*\Sigma^2 V(0) \to 0\ .$$

Furthermore, in the long exact sequence in group cohomology, the boundary map is given be multiplication by η. We then get a long exact sequence for the localized cohomology groups $v_1^{-1}H^*(\mathbb{G}_2, -)$ and the result follows from Theorem 7.2.6. □

Let $\iota\colon S^0 \to Y$ be inclusion of the bottom cell and let ζ_2 and σ be the images of the same named class under the induced map

$$\iota_*\colon \pi_*L_{K(1)}L_{K(2)}S^0 \to \pi_*L_{K(1)}L_{K(2)}Y\ .$$

Similarly, let $\jmath\colon V(0) \to Y$ be the inclusion of the bottom two cells; see(4.2.1). Let $y \in \pi_{-1}L_{K(1)}L_{K(2)}V(0)$ be as in (7.4.1) and call the images of y and $\beta(y)$ under the induced map

$$\jmath_*\colon \pi_*L_{K(1)}L_{K(2)}V(0) \to \pi_*L_{K(1)}L_{K(2)}Y$$

by the same names.

Proposition 7.4.4. *The localized spectral sequence*

$$v_1^{-1}H^s(\mathbb{G}_2, (E_2)_tY) \Longrightarrow \pi_{t-s}L_{K(1)}L_{K(2)}Y$$

strongly converges, collapses at the E_2-term, and has no additive extensions.

Proof. The E_2-term was calculated in Proposition 7.4.3. Convergence follows from Lemma 2.3.2 and Corollary 2.3.7. Since Y has a v_1-self map, this is a spectral sequence of modules over $\mathbb{F}_2[v_1^{\pm 1}] \otimes E(\sigma, \zeta_2)$. The class ζ_2 is a permanent cycle by Proposition 2.2.1. The E_2–term has a horizontal vanishing at $s = 5$ and, by sparseness, $E_2 = E_3$. By Theorem 6.1.1, the class χ is a d_3–cycle, and this forces the spectral sequence to collapse. It follows from Theorem 7.3.3 that y is detected by χ and $\beta(y)$ is detected by χ^2.

We turn to extensions; to do this, we must show 2 annihilates the homotopy of $\pi_* L_{K(1)} L_{K(2)} Y$. In fact, we have

$$0 = 2\colon L_{K(1)} L_{K(2)} Y \longrightarrow L_{K(1)} L_{K(2)} Y.$$

To see this, recall that in Proposition 4.2.3 we showed there is a factoring

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[rr, bend right, "2"'] & S^3 \arrow[r, "\nu"] & Y \end{tikzcd}$$

where $\nu \in \pi_3 S^0$ is the Hopf map and p is collapse onto the top cell. In Proposition 4.2.3, we also showed that $0 = \nu \in \pi_* L_{K(1)} Y$; hence, $0 = \nu \in \pi_* L_{K(1)} L_{K(2)} Y$. $\square$

We have one more preliminary result. We gave a short analysis of $\pi_*(V(0) \wedge V(0))$ in Remark 3.1.5 and found classes i_0 and i_1 of degree 0 and 1 respectively with the property that $\beta(i_1) = i_0$, where β is the Bockstein in homotopy. The class x in the following result first appeared in (7.4.2).

Lemma 7.4.5. *Let $x\colon \Sigma^{-2} V(0) \longrightarrow L_{K(1)} L_{K(2)} S^0$ be the Spanier-Whitehead dual of a class $y \in \pi_{-1} L_{K(1)} L_{K(2)} V(0)$ detected by χ. Then under the map*

$$x \wedge V(0)\colon \Sigma^{-2} V(0) \wedge V(0) \longrightarrow L_{K(1)} L_{K(2)} V(0)$$

the class $(x \wedge V(0))_*(i_0)$ is detected by χ^2 and $(x \wedge V(0))_*(i_1)$ is detected by χ.

Proof. Lemma 3.1.7 applied to the diagram

$$\begin{tikzcd} S^{-1} \arrow[r, "\iota"] \arrow[d, "y"] \arrow[rr, bend left, "i_0"] & \Sigma^{-1} V(0) \arrow[r, "j"] \arrow[d, "x"] & \Sigma^{-1} V(0) \wedge V(0) \arrow[d, "x \wedge V(0)"] \\ L_{K(1)} L_{K(2)} V(0) \arrow[r, "p"] \arrow[rr, bend right, "\beta"'] & \Sigma L_{K(1)} L_{K(2)} S^0 \arrow[r, "\iota"] & \Sigma L_{K(1)} L_{K(2)} V(0) \end{tikzcd}$$

shows that $(x \wedge V(0))_*(i_0)$ is given by $\beta(y)$. Then the Geometric Boundary Theorem (see Remark 3.2.1) implies that $(x \wedge V(0))_*(i_0)$ is detected by χ^2 and the homotopy Bockstein $\beta(i_1) = i_0$ implies that $(x \wedge V(0))_*(i_1)$ is detected by some class whose algebraic Bockstein is χ^2. By Theorem 7.2.5 the only such class is χ. $\square$

Theorem 7.4.1 now follows from Lemma 7.4.2 and the following result.

Theorem 7.4.6. *The map f of* (7.4.3) *induces a $K(1)$-local equivalence*

$$L_{K(1)}\big(S^0 \vee S^{-1} \vee \Sigma^{-2}V(0) \vee \Sigma^{-3}V(0)\big) \wedge Y \simeq L_{K(1)}L_{K(2)}Y \ .$$

Proof. As a module over $\mathbb{F}_2[v_1^{\pm 1}] \otimes E(\sigma)$, $\pi_* L_{K(1)}Y$ is free of rank 1 on a class ι of degree 0 and $\pi_* L_{K(1)}(Y \wedge V(0))$ is free of rank 2 on the classes i_0 and i_1 of degrees 0 and 1 respectively. See Proposition 4.2.2, Corollary 4.2.4, and the remarks before Corollary 4.2.4. By Proposition 7.4.3 and Proposition 7.4.4 we have that $\pi_* L_{K(1)}L_{K(2)}Y$ is free of rank 6 over $\mathbb{F}_2[v_1^{\pm 1}] \otimes E(\sigma)$ on generators ι, ζ_2, y, $\beta(y)$, $\zeta_2 y$ and $\zeta_2\beta(y)$ of degrees 0, -1, -1, -2, -2 and -3 respectively. See the remarks before Proposition 7.4.4 as well.

We show that the maps

$$\begin{aligned}
\iota \wedge Y\colon\ & L_{K(1)}Y \longrightarrow L_{K(1)}L_{K(2)}Y \\
\zeta_2 \wedge Y\colon\ & \Sigma^{-1}L_{K(1)}Y \longrightarrow L_{K(1)}L_{K(2)}Y \\
x \wedge Y\colon\ & \Sigma^{-2}L_{K(1)}(V(0) \wedge Y) \longrightarrow L_{K(1)}L_{K(2)}Y \\
x\zeta_2 \wedge Y\colon\ & \Sigma^{-3}L_{K(1)}(V(0) \wedge Y) \longrightarrow L_{K(1)}L_{K(2)}Y
\end{aligned}$$

are all injective on homotopy and exhaust the various summands.

The map

$$\iota \wedge Y\colon L_{K(1)}(S^0 \wedge Y) \to L_{K(1)}L_{K(2)}(S^0 \wedge Y)$$

is injective on homotopy and maps onto the $\mathbb{F}_2[v_1^{\pm 1}] \otimes E(\sigma)$ summand generated by ι. Similarly $\zeta_2 \wedge Y$ maps injectively onto the summand generated by ζ_2.

These leaves the other four summands. Using the inclusion $\jmath\colon V(0) \to Y$ and Lemma 7.4.5 we have that

$$x \wedge Y\colon \Sigma^{-2}L_{K(1)}(V(0) \wedge Y) \longrightarrow L_{K(1)}L_{K(2)}Y$$

sends the generators i_0 and i_1 in degrees -2 and -1 to non-zero classes detected by χ^2 and χ respectively. By multiplying with ζ_2 we can conclude that

$$x\zeta_2 \wedge Y\colon \Sigma^{-3}L_{K(1)}(V(0) \wedge Y) \longrightarrow L_{K(1)}L_{K(2)}Y$$

sends the generators i_0 and i_1 in degrees -3 and -2 to non-zero classes detected by $\chi^2\zeta_2$ and $\chi\zeta_2$ respectively. Since y is detected by χ and $\beta(y)$ by χ^2, the result follows from Proposition 7.4.3. $\square$

7.5. **The homotopy types of $L_0L_{K(2)}S^0$ and $L_1L_{K(2)}S^0$.** We end this section by recording one of the main consequences of Theorem 7.4.1, which was stated in the introduction as Theorem 2. Recall that X_p denote the p-completion of X.

Theorem 7.5.1 (**Chromatic Splitting**)**.** *There is an equivalence*

$$L_1L_{K(2)}S^0 \simeq L_1(S^0_2 \vee S^{-1}_2) \vee L_0(S^{-3}_2 \vee S^{-4}_2) \vee L_1(\Sigma^{-2}V(0) \vee \Sigma^{-3}V(0)) \ .$$

Theorem 7.5.1 is proved exactly as in Theorem 5.11 of [GHM14], i.e. by examining the homotopy pull-back

$$\begin{tikzcd}
L_1L_{K(2)}S^0 \arrow[r] \arrow[d] & L_{K(1)}L_{K(2)}S^0 \arrow[d] \\
L_0L_{K(2)}S^0 \arrow[r] & L_0L_{K(1)}L_{K(2)}S^0 \ .
\end{tikzcd}$$

For this, we need the following result which describes $L_0 L_{K(2)} S^0$.

Theorem 7.5.2. *There is an equivalence*

$$L_0 L_{K(2)} S^0 \simeq L_0(S_2^0 \vee S_2^{-1} \vee S_2^{-3} \vee S_2^{-4}).$$

Proof. By Theorem 5.5.2, the spectral sequence

$$E_2^{s,t} = H^s(\mathbb{G}_2, E_t) \otimes \mathbb{Q} \Longrightarrow \pi_{t-s} L_0 L_{K(2)} S_2^0.$$

has E_2–term $E_{\mathbb{Q}_2}(\zeta_2, e)$. It collapses with no extensions for degree reasons, and it converges by Lemma 2.3.2 and Corollary 2.3.7.

The homotopy classes detected by 1, ζ_2, e, $e\zeta_2$ combine to give a map

$$S_2^0 \vee S_2^{-1} \vee S_2^{-3} \vee S_2^{-4} \to L_0 L_{K(2)} S^0$$

which is a rational equivalence. □

8. Lifting to $L_{K(2)} S^0$

In Section 7 we constructed a map

$$x \vee x\zeta_2 : \Sigma^{-2} V(0) \vee \Sigma^{-3} V(0) \longrightarrow L_{K(1)} L_{K(2)} S^0$$

which became part of our $K(1)$-equivalence. The map x was not unique, but any choice yielded the equivalence. In this section we show that there is a choice for this map that factors through the localization map

$$L_{K(2)} S^0 \to L_{K(1)} L_{K(2)} S^0.$$

This is important for the chromatic assembly process.

The map x is defined to be Spanier-Whitehead dual to the class

$$y \in \pi_{-1} L_{K(1)} L_{K(2)} V(0)$$

detected by $\chi = \chi_2 \in v_1^{-1} H^1(\mathbb{G}_2, E_* V(0))$. The key result of this section is Theorem 8.2.1, which shows that there is a choice of y in the image of the map from $\pi_{-1} L_{K(2)} V(0)$.

There are two main steps: we first show $\pi_{-1}(E^{h\mathbb{G}_2^1} \wedge V(0)) \cong \mathbb{Z}/4$ generated by a class detected by $\chi \in H^1(\mathbb{G}_2^1, E_0/2)$; this uses the topological duality spectral sequence of Theorem 2.5.5. We then show this generator actually comes from $\pi_{-1} L_{K(2)} V(0)$ using entirely different information, ultimately going back to Theorem 6.1.1.

8.1. Computing $\pi_i E^{h\mathbb{G}_2^1}$, $-3 \leq i \leq 0$.

For this calculation we use the topological duality spectral sequence of Remark 2.5.6:

$$\pi_q \mathscr{E}_p \Longrightarrow \pi_{q-p} E^{h\mathbb{S}_2^1}.$$

We will use Lemmas 2.2.2 and 2.2.3 to deduce information about $\pi_* E^{h\mathbb{G}_2^1}$. The result is recorded in Theorem 8.1.5. The homotopy of $E^{h\mathbb{G}_2^1}$ in the indicated range is given in Corollary 8.1.6.

The first step is to record what we need about the homotopy groups of the fibers $\mathscr{E}_p$. Since we will be trying to compute $\pi_i E^{h\mathbb{S}_2^1}$ for i near 0, we will only need information in a small range of dimensions.

Remark 8.1.1. If $p = 0$, then $\mathscr{E}_p = E^{hG_{24}}$; if $p = 3$, then $\mathscr{E}_p = \Sigma^{48} E^{hG_{24}}$. Write $G_{48} = G_{24} \rtimes \mathrm{Gal}$. Then the computation of $\pi_* E^{hG_{48}}$ can be mined from [Bau08] or [DFHH14]; it is very elaborate, but the part we need is fairly simple. We then have $\pi_* E^{hG_{24}} \cong \mathbb{W} \otimes \pi_* E^{hG_{48}}$. See Lemma 2.2.3.

Recall from Remark 2.4.7 that there are classes c_4, c_6, Δ, and j in $H^0(G_{48}, E_*)$, of degrees 8, 12, 24, and 0 respectively, and an isomorphism

$$\mathbb{Z}_2[[j]][c_4, c_6, \Delta^{\pm 1}]/I \cong H^0(G_{48}, E_*)$$

where I is the ideal generated by

$$c_4^3 - c_6^2 = (12)^3 \Delta \qquad \text{and} \qquad j\Delta = c_4^3.$$

The class j is a permanent cycles. The class Δ^{-2} is not a permanent cycle, but $4\Delta^{-2}$ and $j\Delta^{-2}$ are. This will explain the complicated form of $\pi_{-48} E^{hG_{24}}$ in (8.1.1) and the factor of 4 in part (1) of Corollary 8.1.6. See also Remark 5.5.3. Let η and ν be the Hopf classes in the homotopy groups of spheres.

We pause to explain some notation. All homotopy groups $\pi_* E^{hG_{24}}$ are modules over $\mathbb{W}[[j]] \cong \pi_0 E^{hG_{24}}$. For a quotient ring R of $\mathbb{W}[[j]]$ we write $R\{x\}$ for the free R-module generated by x. If $J \subseteq R$ is an ideal, we write $J\{x\} \subseteq R\{x\}$ for the sub-R-module generated by J. The name of a generator, if given, is the label given to the class in the cohomology $H^*(G_{24}, E_*) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} H^*(G_{48}, E_*)$ which detects the element in question. In all the degrees we list, $\pi_i E^{hG_{24}}$ is a sub-$\mathbb{W}[[j]]$-module of a single cohomology group $E_2^{s,t} \cong H^s(G_{24}, E_t)$, and we indicate that cohomology group.

With this in mind we now record the homotopy groups we need.

$$\pi_i E^{hG_{24}} \cong \begin{cases} \mathbb{W}[[j]] \cong E_2^{0,0} & i = 0 \\ \mathbb{F}_4[[j]]\{\eta^i\} \cong E_2^{i,2i} & i = 1, 2 \\ \mathbb{W}/8\{\nu\} \cong E_2^{1,4} & i = 3 \\ 0 & i = -3, -2, -1 \\ (4, j)\{\Delta^{-2}\} \subseteq \mathbb{W}[[j]]\{\Delta^{-2}\} \cong E_2^{0,-48} & i = -48 \\ (j)\{\eta\Delta^{-2}\} \subseteq \mathbb{F}_4[[j]]\{\eta\Delta^{-2}\} \cong E_2^{1,-46} & i = -47 \\ (j)\{\eta^2\Delta^{-2}\} \subseteq \mathbb{F}_4[[j]]\{\eta^2\Delta^{-2}\} \cong E_2^{2,-44} & i = -46 \\ \mathbb{W}/8\{\nu\Delta^{-2}\} \cong E_2^{1,-44} & i = -45 \end{cases} \tag{8.1.1}$$

The edge homomorphism $h\colon \pi_i E^{hG_{24}} \to E_i^{G_{24}}$ is an isomorphism when $i = 0$ and injective when $i = -48$.

If $p = 1$ or 2, then $F_p = E^{hC_6}$. The following computation is relatively simple and can be read off of Section 4 of [MR09] or, more explicitly, from Section 2 of [BG18]. The notation is analogous to that in (8.1.1), except that $\pi_3 E^{hC_6} \cong E_\infty^{3,6}$ is only a quotient of $E_2^{3,6}$.

$$\pi_i E^{hC_6} \cong \begin{cases} 0 & i = -1, -2 \\ \mathbb{W}[[u_1^3]] \cong E_2^{0,0} & i = 0 \\ \mathbb{F}_4[[u_1^3]]\{\eta^i\} \cong E_2^{i,2i} & i = 1, 2 \\ \mathbb{F}_4\{\nu\} & i = 3 \end{cases} \tag{8.1.2}$$

The edge homomorphism $h\colon \pi_0 E^{hC_6} \to E_0^{C_6}$ of the homotopy fixed point spectral sequence is an isomorphism.

Finally, we will need the following result in our calculations below.

Lemma 8.1.2. *Let $G = G_{24}$ and $G = C_6$. Multiplication by η induces isomorphisms*

$$\eta\colon H^0(G, E_0/2) \xrightarrow{\cong} H^1(G, E_2/2)$$
$$\eta^2\colon H^0(G, E_0/2) \xrightarrow{\cong} H^2(G, E_4/2).$$

Furthermore, for $i = 1, 2$ reduction modulo 2 induces an isomorphism

$$H^i(G, E_{2i}) \xrightarrow{\cong} H^i(G, E_{2i}/2).$$

Proof. For these statements we will need some information about $H^s(G, E_t)$ for low values of s and t. There are many references for this; see [MR09] for C_6, [DFHH14] for G_{24}, or §2 of [BG18] for more references and a convenient chart for $G = G_{24}$.

From these references we see that there are isomorphisms

$$H^0(G, E_0)/2 \underset{\cong}{\xrightarrow{\eta}} H^1(G, E_2) \underset{\cong}{\xrightarrow{\eta}} H^2(G, E_4).$$

We also have that $H^s(G, E_t) = 0$ for $(s, t) = (1, 0)$, $(2, 2)$, and $(3, 4)$. The result follows from the long exact sequence in group cohomology induced by the short exact sequence

$$0 \longrightarrow E_* \xrightarrow{\times 2} E_* \longrightarrow E_*/2 \longrightarrow 0. \qquad \square$$

We now begin our calculation of the topological duality spectral sequence for $E^{h\mathbb{S}_2^1}$. The result is illustrated in Figure 6. The main calculations needed to produce the E_2-term can be found in Lemma 5.4.3 and Theorem 8.1.5. There is an explanatory remark for these charts immediately following.

FIGURE 6. The E_1 (left) and E_2 (right) terms of the TDSS for $\pi_*(E^{h\mathbb{S}_2^1})$.

Remark 8.1.3. The two charts are respectively the E_1-term and E_2-term of the topological duality spectral sequence of Remark 2.5.6 for $\pi_*(E^{h\mathbb{S}_2^1})$. We use the Adams grading with $E_1^{p,q}$ in Remark 2.5.6 in the $(q-p, p)$ box.

A $\square$ denotes a copy of $\mathbb{W}[[j]]$ if $s = 0, 3$ and $\mathbb{W}[[u_1^3]]$ if $s = 1, 2$. Similarly, a $\circ$ denotes a copy of $\mathbb{F}_4[[j]]$ or $\mathbb{F}_4[[u_1^3]]$. A $\bullet$ is a copy of $\mathbb{F}_4$. A $\blacksquare$ is a copy of $\mathbb{W}$. The symbol $\circledcirc$ denotes a copy of $\mathbb{W}/8$.

Horizontal and curved lines indicate multiplication by η respectively ν. The dashed vertical line on the E_2-term is there to indicate the additive extension on the E_∞-term. See Theorem 8.1.5. In the top row of the left chart, we have depicted the ideal $(4,j)\{\Delta^{-2}\} \subseteq \pi_{-48}E^{hG_{24}}$ using the $\mathbb{W}$-module structure as $\mathbb{W}\{4\}\oplus\mathbb{W}[[j]]\{j\} \subseteq \mathbb{W}[[j]]$. The $\blacksquare$ represents $\mathbb{W}\{4\}$ and the $\square$ represents $\mathbb{W}[[j]]\{j\}$.

The upper right corners of both charts contribute to $\pi_1 E^{h\mathbb{S}_2^1}$. The elements in these spots do not support differentials in the topological duality spectral sequence, so don't come into the calculations of π_0. As they don't affect our answers, we don't include them in the discussion and leave these boxes marked with a question mark.

We now come to the calculation of $\pi_* E^{h\mathbb{S}_2^1}$ in the range we need. We compute using the topological duality spectral sequence of Remark 2.5.6:

$$E_1^{p,q} = \pi_q \mathscr{E}_p \Longrightarrow \pi_{q-p}(E_2^{h\mathbb{S}_2^1}) \ .$$

We will use information from Remark 8.1.1. The E_1-term is displayed in Figure 6. The notation Δ_0, b_0, $\bar{b}_0$, and $\overline{\Delta}_0$ was explained in Remark 5.2.8.

Theorem 8.1.4. *There are isomorphisms*

(1) $\pi_{-3}(E_2^{h\mathbb{S}_2^1}) \cong \mathbb{W}$ *with generator detected by* $4\overline{\Delta}_0 \in E_1^{3,0} = \pi_0\Sigma^{48}E^{hG_{24}}$.
(2) $\pi_{-2}(E_2^{h\mathbb{S}_2^1}) \cong \mathbb{F}_4$ *detected by* $\bar{b}_0 \in E_1^{2,0} = \pi_0 E^{hC_6}$.
(3) $\pi_{-1}(E_2^{h\mathbb{S}_2^1}) \cong \mathbb{F}_4$ *with generator detected by* $\eta\bar{b}_0$ *in* $E_1^{2,1} = \pi_1 E^{hC_6}$.
(4) $\pi_0(E^{h\mathbb{S}_2^1}) \cong \mathbb{W} \oplus \mathbb{W}/4 \oplus \mathbb{W}/8$.

Furthermore, for $\pi_0 E^{h\mathbb{S}_2^1}$,

(i) a generator of the summand $\mathbb{W}$ *is detected by the unit in* $E_1^{0,0} = \pi_0 E^{hG_{24}}$*;*
(ii) a generator of the summand $\mathbb{W}/4$ *is detected by* $\eta b_0 \in E_1^{1,1} = \pi_1 E^{hC_6}$*;*
(iii) a generator of the summand $\mathbb{W}/8$ *is detected by* $\nu\overline{\Delta}_0 \in E_1^{3,3} = \pi_3\Sigma^{48}E^{hG_{24}}$.

If a class in $\pi_0 E^{h\mathbb{S}_2^1}$ *is detected by* $\eta b_0 \in E_1^{1,1}$ *then twice that class is detected by* $\eta^2\bar{b}_0 \in E_1^{2,2} \cong \pi_2 E^{hC_6}$.

Proof. This requires a multi-part argument and we break it into steps.

Calculating d_1 with $q = 0$. The fundamental calculation is with the sequence

$$E_1^{0,0} \xrightarrow{d_1} E_1^{1,0} \xrightarrow{d_1} E_1^{2,0} \xrightarrow{d_1} E_1^{3,0} \ .$$

Using the calculations of Remark 8.1.1, this can be fit into a diagram of chain complexes

$$\begin{tikzcd}
\pi_0 E^{hG_{24}} \arrow[r, "d_1"] \arrow[d, "\cong"'] & \pi_0 E^{hC_6} \arrow[r, "d_1"] \arrow[d, "\cong"'] & \pi_0 E^{hC_6} \arrow[r, "d_1"] \arrow[d, "\cong"'] & \pi_{-48}E^{hG_{24}} \arrow[d, "\subseteq"'] \\
H^0(G_{24}, E_0) \arrow[r, "d_1"] & H^0(C_6, E_0) \arrow[r, "d_1"] & H^0(C_6, E_0) \arrow[r, "d_1"] & H^0(G_{24}, E_0) \\
H^0(G_{24}, \mathbb{W}) \arrow[r, "d_1"] \arrow[u] & H^0(C_6, \mathbb{W}) \arrow[r, "d_1"] \arrow[u] & H^0(C_6, \mathbb{W}) \arrow[r, "d_1"] \arrow[u] & H^0(G_{24}, \mathbb{W}) \arrow[u]
\end{tikzcd} \ .$$

The vertical maps down from the top row are edge homomorphisms. All but the last of these maps are isomorphisms. The last map is injective with image the ideal $(4,j) \subseteq \mathbb{W}[[j]]$. The map from the bottom to the middle row is induced by the unit map $i\colon \mathbb{W} \to E_0$; thus, by Lemma 5.4.3, the bottom two rows have the same cohomology. It then follows that in the topological duality spectral sequence we have

$$E_2^{0,0} \cong \mathbb{W} \qquad E_2^{1,0} = 0 \qquad E_2^{2,0} \cong \mathbb{F}_4 \qquad E_2^{3,0} \cong \mathbb{W}.$$

The generator for $E_2^{2,0}$ is $\bar{b}_0$ and the generator for $E_2^{3,0}$ is $4\overline{\Delta}_0$.

Calculating d_1 with $q = 1$. The next calculation is with the sequence

$$E_1^{0,1} \xrightarrow{d_1} E_1^{1,1} \xrightarrow{d_1} E_1^{2,1} \xrightarrow{d_1} E_1^{3,1} .$$

This can be fit into a diagram

$$\begin{tikzcd}
\pi_1 E^{hG_{24}} \arrow[r, "d_1"] \arrow[d, "\cong"'] & \pi_1 E^{hG_6} \arrow[r, "d_1"] \arrow[d, "\cong"'] & \pi_1 E^{hG_6} \arrow[r, "d_1"] \arrow[d, "\cong"'] & \pi_{-47} E^{hG_{24}} \arrow[d, "\subseteq"] \\
H^1(G_{24}, E_2) \arrow[r, "d_1"] \arrow[d, "\cong"'] & H^1(C_6, E_2) \arrow[r, "d_1"] \arrow[d, "\cong"'] & H^1(C_6, E_2) \arrow[r, "d_1"] \arrow[d, "\cong"'] & H^1(G_{24}, E_2) \arrow[d, "\cong"'] \\
H^1(G_{24}, E_2/2) \arrow[r, "d_1"] & H^1(C_6, E_2/2) \arrow[r, "d_1"] & H^1(C_6, E_2/2) \arrow[r, "d_1"] & H^1(G_{24}, E_2/2) \\
H^0(G_{24}, E_2/2) \arrow[r, "d_1"] \arrow[u, "\eta", "\cong"'] & H^0(C_6, E_2/2) \arrow[r, "d_1"] \arrow[u, "\eta", "\cong"'] & H^0(C_6, E_2/2) \arrow[r, "d_1"] \arrow[u, "\eta", "\cong"'] & H^0(G_{24}, E_2/2) \arrow[u, "\eta", "\cong"']
\end{tikzcd}$$

Using the calculations of Remark 8.1.1, we have all vertical maps from the top row are isomorphisms except the edge homomorphism on the right hand side. This is an inclusion onto the submodule $\mathbb{F}_4[[j]]\{j\eta\} \subseteq \mathbb{F}_4[[j]]\{\eta\}$. The maps from the second row to the third row are reduction modulo 2 and the maps from the bottom row to the third row are multiplication by η. All are isomorphisms by Lemma 8.1.2. It follows that in the topological duality spectral sequence we have

$$E_2^{0,1} \cong \mathbb{F}_4 \qquad E_2^{1,1} = \mathbb{F}_4 \qquad E_2^{2,1} \cong \mathbb{F}_4 \qquad E_2^{3,1} \cong 0.$$

The generator for $E_2^{0,1}$ is $\eta\Delta_0$, that for $E_2^{1,1}$ is ηb_0 and the generator for $E_2^{2,1}$ is $\eta\bar{b}_0$.

Calculating d_1 with $q = 2$. The exact same argument, with η replaced by η^2, shows that

$$E_2^{0,2} \cong \mathbb{F}_4 \qquad E_2^{1,2} = \mathbb{F}_4 \qquad E_2^{2,2} \cong \mathbb{F}_4 \qquad E_2^{3,2} \cong 0$$

and that the generator for $E_2^{0,2}$ is $\eta^2\Delta_0$, that for $E_2^{1,2}$ is $\eta^2 b_0$ and the generator for $E_2^{2,2}$ is $\eta^2\bar{b}_0$.

Calculating d_1 with $q = 3$. Finally, we calculate with the sequence

$$E_1^{0,3} \xrightarrow{d_1} E_1^{1,3} \xrightarrow{d_1} E_1^{2,3} \xrightarrow{d_1} E_1^{3,3} .$$

In this case we need a different style of argument. By Remark 8.1.1 we have

$$E_1^{0,3} \cong \mathbb{W}/8 \qquad E_1^{1,3} = \mathbb{F}_4 \qquad E_1^{2,3} \cong \mathbb{F}_4 \qquad E_1^{3,3} \cong \mathbb{W}/8$$

generated by $\nu\Delta_0$, νb_0, $\nu\bar{b}_0$, and $\nu\overline{\Delta}_0$ respectively. Then $d_1 = 0$ by the calculation for $q = 0$ and the ν-linearity of d_1.

Calculating d_2 and d_3. We first turn to d_2. By looking at Figure 6, we see that in the range $0 \leq q - p \leq 1$, the only classes which could support a non-zero d_2 are the unit, detected by Δ_0 and the classes detected by $\eta\Delta_0$ and $\eta^2 b_0$ in $E_2^{0,1}$ and

$E_2^{1,2}$. The unit and $\eta\Delta_0$ are evidently permanent cycles and $\eta^2 b_0$ is an η-multiple of ηb_0, which is permanent cycle. Thus $d_2 = 0$.

This leaves only d_3. But in the range $0 \leq q - p \leq 1$, the only class which could support a non-zero d_3 is η times the unit, so $d_3 = 0$ as well and we have shown

$$E_2^{p,q} \cong E_\infty^{p,q}, \qquad -3 \leq q - p \leq 0.$$

Settling extensions. At this point we have a filtration of the $\mathbb{W}$-module $\pi_0 E^{hG_{24}}$

$$\begin{tikzcd} 0 \arrow[r, "\subseteq"] & F_3 \arrow[r, "\subseteq"] \arrow[d, "="'] & F_2 \arrow[r, "\subseteq"] \arrow[d] & F_1 \arrow[r, "\subseteq"] \arrow[d] & \pi_0 E^{hG_{24}} \arrow[d] \\ & E_\infty^{3,3} \cong \mathbb{W}/8 & E_\infty^{2,2} \cong \mathbb{W}/2 & E_\infty^{1,1} \cong \mathbb{W}/2 & E_\infty^{0,0} \cong \mathbb{W} \end{tikzcd}$$

Note that the Galois equivariant extension of the unit of $E^{h\mathbb{S}_2^1}$ splits off the torsion-free summand. Thus we need only show that a generator of $E_\infty^{1,1}$ detects an element of order 4. This follows from Remark 3.1.4 and, in particular, from (3.1.4). Specifically, $\overline{b}_0 \in E_\infty^{2,0}$ detects a unique Galois invariant homotopy class z of order 2 in $\pi_{-2} E^{h\mathbb{S}_2^1}$. Since a generator $E_\infty^{2,2}$ is detected by $\eta^2 \overline{b}_0$, we have $2\langle z, 2, \eta\rangle = \eta^2 z \neq 0$ and, in particular that $\langle z, 2, \eta\rangle$ does not contain zero. Since $\langle z, 2, \eta\rangle$ is Galois invariant, it can only be detected by ηb_0. $\square$

The next result describes how the generators of $\pi_* E^{h\mathbb{S}_2^1}$ in our range are detected in the homotopy fixed point spectral sequence. In reading the statement, a bit of care is needed. The class e is not a permanent cycle, so the class νe of part (4) is not a product; it is named by the class that detects it in the spectral sequence.

Theorem 8.1.5. *In the homotopy fixed point spectral sequence*

$$H^s(\mathbb{S}_2^1, E_t) \Longrightarrow \pi_{t-s} E^{h\mathbb{S}_2^1}$$

(1) a generator of $\pi_{-3}(E_2^{h\mathbb{S}_2^1}) \cong \mathbb{W}$ *is detected by* $4e$ *in* $H^3(\mathbb{S}_2^1, E_0)$*;*
(2) a generator of $\pi_{-2}(E_2^{h\mathbb{S}_2^1}) \cong \mathbb{F}_4$ *is detected by* $\widetilde{\chi}$ *in* $H^2(\mathbb{S}_2^1, E_0)$*;*
(3) a generator of $\pi_{-1}(E_2^{h\mathbb{S}_2^1}) \cong \mathbb{F}_4$ *is detected by* $\eta\widetilde{\chi}$ *in* $H^3(\mathbb{S}_2^1, E_2)$*;*
(4) $\pi_0(E^{h\mathbb{S}_2^1}) \cong \mathbb{W} \oplus \mathbb{W}/4 \oplus \mathbb{W}/8$.

Furthermore, for $\pi_0 E^{h\mathbb{S}_2^1}$*,*

(i) a generator of the summand $\mathbb{W}$ *is detected by the unit in* $H^0(\mathbb{S}_2^1, E_0)$*;*
(ii) a generator of the summand $\mathbb{W}/4$ *is detected by an element in the Massey product*

$$\langle \widetilde{\chi}, 2, \eta\rangle \in H^1(\mathbb{S}_2^1, E_1);$$

(iii) a generator of the summand $\mathbb{W}/8$ *is detected by* $\nu e \in H^3(\mathbb{S}_2^1, E_3)$.

If a class in $\pi_0 E^{h\mathbb{S}_2^1}$ *is detected by* $\langle \widetilde{\chi}, 2, \eta\rangle$ *then twice that class is detected by*

$$\eta^2 \widetilde{\chi} \in H^4(\mathbb{S}_2^1, E_4).$$

Proof. This follows Theorem 8.1.4 and Lemma 5.2.9. For part (1) we know from Lemma 5.2.9 that $4\overline{\Delta}_0$ detects $4e$ in the algebraic duality spectral sequence. For part (2), the same result shows that $\overline{b}_0$ detects $\widetilde{\chi}$. Part (3) then follows. It remains to discuss the generators of $\pi_0 E^{h\mathbb{S}_2^1}$.

At the very end of the proof of Theorem 8.1.4 we argued that if $z \in \pi_{-2}E^{h\mathbb{S}_2^1}$ and is the unique non-zero Galois invariant class, then the Toda bracket $\langle z, 2, \eta\rangle$ does not contain zero and is detected in the topological duality spectral sequence by ηb_0. This forces the Massey product $\langle \widetilde{\chi}, 2, \eta\rangle$ to be non-zero. Since z is detected by $\widetilde{\chi}$ and $\eta^2 z \neq 0$, the exotic extension follows from the standard juggling formula $2\langle z, 2, \eta\rangle = \eta^2 z$ of (3.1.4). □

We next pass to $\mathbb{G}_2^1$. The result is a corollary of Theorem 8.1.5 obtained by taking Galois fixed points.

Corollary 8.1.6. *There are isomorphisms*

(1) $\pi_{-3}E^{h\mathbb{G}_2^1} \cong \mathbb{Z}_2$ *generated by a class detected by* $4e \in H^3(\mathbb{G}_2^1, E_0)$*;*
(2) $\pi_{-2}E^{h\mathbb{G}_2^1} \cong \mathbb{Z}/2$ *generated by a class detected by* $\widetilde{\chi} \in H^2(\mathbb{G}_2^1, E_0)$*;*
(3) $\pi_{-1}E^{h\mathbb{G}_2^1} \cong \mathbb{Z}/2$ *generated by a class detected by* $\eta\widetilde{\chi} \in H^3(\mathbb{G}_2^1, E_2)$*;*
(4) $\pi_0 E^{h\mathbb{G}_2^1} \cong \mathbb{Z}_2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8$.

Furthermore, the summands of $\pi_0 E^{h\mathbb{G}_2^1}$ *are generated by the unit and classes detected by* $\langle \widetilde{\chi}, 2, \eta\rangle \in H^2(\mathbb{G}_2^1, E_2)$ *and* $\nu e \in H^4(\mathbb{G}_2^1, E_4)$*. If a class is detected by* $\langle \widetilde{\chi}, 2, \eta\rangle$*, then twice that class is detected by* $\eta^2\widetilde{\chi}$*.*

Remark 8.1.7. Using the fiber sequence of (2.2.4)

$$L_{K(2)}S^0 \longrightarrow E^{h\mathbb{G}_2^1} \xrightarrow{\pi - 1} E^{h\mathbb{G}_2^1}$$

we can make analogous calculations for the sphere itself. For example

$$\pi_{-3}L_{K(2)}S^0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}/2 \subseteq H^3(\mathbb{G}_2, E_0)$$

generated by the classes $4e$ and $\zeta_2\widetilde{\chi}$.

We now pass to an analysis of $\pi_*(E^{h\mathbb{G}_2^1} \wedge V(0))$. Recall from Section 3.1 that q and j are defined by the cofiber sequence

$$E^{h\mathbb{G}_2^1} \xrightarrow{\times 2} E^{h\mathbb{G}_2^1} \xrightarrow{j} E^{h\mathbb{G}_2^1} \wedge V(0) \xrightarrow{q} \Sigma E^{h\mathbb{G}_2^1}.$$

From Remark 3.1.1 we have $\beta = jq$. Let $z\colon S^{-2} \to E^{h\mathbb{G}_2^1}$ be the class detected by $\widetilde{\chi}$. Since this class has order two, it factors through

$$q\colon \Sigma^{-1}E^{h\mathbb{G}_2^1} \wedge V(0) \longrightarrow E^{h\mathbb{G}_2^1}$$

and we get a map

$$y_1\colon S^{-1} \longrightarrow E^{h\mathbb{G}_2^1} \wedge V(0). \tag{8.1.3}$$

such that $q_*(y_1) = z$. We now have the following result.

Proposition 8.1.8. *There are isomorphisms*

(a) $\pi_{-1}(E^{h\mathbb{G}_2^1} \wedge V(0)) \cong \mathbb{Z}/4$ *generated by the class* y_1*. This class is detected by* $\chi \in H^1(\mathbb{G}_2^1, E_0/2)$*.*
(b) $\pi_{-2}(E^{h\mathbb{G}_2^1} \wedge V(0)) \cong \mathbb{Z}/2$*. The generator is detected by* $\chi^2 \in H^2(\mathbb{G}_2^1, E_0/2)$*.*

Furthermore, the class $2y_1$ *is detected by* $\eta\chi^2 \in H^3(\mathbb{G}_2^1, E_2/2)$*.*

Proof. From Corollary 8.1.6 it follows that

$$\pi_{-2}(E^{h\mathbb{G}_2^1} \wedge V(0)) \cong \mathbb{Z}/2$$

generated by $j_*(z)$. This class is detected by the image of $\widetilde{\chi}$ in $H^2(\mathbb{G}_2^1, E_0/2)$, which is exactly χ^2. From the same result we have a short exact sequence

$$0 \to \mathbb{Z}/2 \to \pi_{-1}(E^{h\mathbb{G}_2^1} \wedge V(0)) \to \mathbb{Z}/2 \to 0.$$

Since $j_*(z)$ is detected by the Bockstein on χ and $j_*(y_1) = j_*q_*(z) = \beta(z)$, the Geometric Boundary Theorem (see Remark 3.2.1) implies that the class y_1 is detected by χ. The generator of the kernel is $\eta j_*(z)$, detected by $\eta\chi^2$. It remains to show y_1 has order 4; this follows from Lemma 3.1.3. $\square$

8.2. **Producing the lifting.** We now show that the class $y_1 \in \pi_{-1}(E^{h\mathbb{G}_2^1} \wedge V(0))$ of order 4 constructed above in Proposition 8.1.8 is in the image of the unit map $\pi_* L_{K(2)}V(0) \to \pi_*(E^{h\mathbb{G}_2^1} \wedge V(0))$. Recall from Theorem 7.3.3 that the class $\chi \in v_1^{-1}H^1(\mathbb{G}_2, E_*V(0))$ is a non-zero permanent cycle in the localized Adams-Novikov Spectral Sequence computing $\pi_* L_{K(1)}L_{K(2)}V(0)$.

Theorem 8.2.1. *There is a class $y_0 \in \pi_{-1}L_{K(2)}V(0)$ detected by $\chi \in H^1(\mathbb{G}_2, E_0/2)$. Under the map*

$$\pi_{-1}L_{K(2)}V(0) \longrightarrow \pi_{-1}(E^{h\mathbb{G}_2^1} \wedge V(0))$$

the class y_0 maps to y_1 and under the map

$$\pi_{-1}L_{K(2)}V(0) \longrightarrow \pi_{-1}L_{K(1)}L_{K(2)}V(0)$$

the class y_0 is non-zero and detected by the class $\chi \in v_1^{-1}H^*(\mathbb{G}_2, E_*V(0))$.

Proof. There is a fiber sequence

$$L_{K(2)}V(0) \longrightarrow E^{h\mathbb{G}_2^1} \wedge V(0) \xrightarrow{\pi-1} E^{h\mathbb{G}_2^1} \wedge V(0) \tag{8.2.1}$$

where $\pi \in \mathbb{G}_2$ is an element that generates $\mathbb{G}_2/\mathbb{G}_2^1 \cong \mathbb{Z}_2$. Since $\pi_{-1}(E^{h\mathbb{G}_2^1} \wedge V(0))$ is isomorphic to $\mathbb{Z}/4$, we have[6] that $\pi_* y_1 = \pm y_1$ so either $(\pi - 1)_* y_1 = 0$ or $(\pi - 1)_* y_1 = 2y_1$.

If the first case applies, then we can choose a class $y_2 \in \pi_{-1}L_{K(2)}V(0)$ which maps to y_1. By Remark 5.4.7 we have an isomorphism

$$H^1(\mathbb{G}_2, E_0V(0)) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

with generators χ and ζ_2. The class ζ_2 maps to zero in $H^1(\mathbb{G}_2^1, E_0V(0))$; hence, we have that y_2 is detected by $\chi + \epsilon\zeta_2$ where $\epsilon = 0$ or 1. Since ζ_2 is a permanent cycle by Proposition 2.2.1 detecting a homotopy class also called ζ_2, we set $y_0 = y_2 + \epsilon\zeta_2$. Then y_0 is detected by χ. Finally, by Theorem 7.2.6, the map

$$H^1(\mathbb{G}_2, E_0V(0)) \to v_1^{-1}H^*(\mathbb{G}_2, E_*V(0))$$

is an injection in degree 1, so the final statement follows.

If the second case applies we would have that y_1 maps to a class of order 2 in $\pi_{-2}L_{K(2)}V(0)$ under the boundary map in the long exact sequence in homotopy obtained from the cofiber sequence of (8.2.1). To rule this out, we use the following result, Theorem 8.2.2. $\square$

[6]We use π_* for the action of a group element $\pi \in \mathbb{G}_n$ on the homotopy groups of $E^{hG_2^1} \wedge V(0)$. Our apologies. Here at the prime 2 there is a classical choice of $\pi \in \mathbb{G}_2$. See Remark 2.4.5.

Theorem 8.2.2. *Under the boundary homomorphism*

$$\partial \colon \pi_{-1}(E^{h\mathbb{G}_2^1} \wedge V(0)) \longrightarrow \pi_{-2} L_{K(2)} V(0)$$

the class $x_1 = \partial(y_1)$ is a class of exact order 4 detected by $\zeta\chi \in H^2(\mathbb{G}_2, E_0/2)$. The class $2x_1$ is detected by $\eta\zeta\chi^2 \in H^4(\mathbb{G}_2, E_2/2)$.

This is an application of the Geometric Boundary Theorem; see Remark 3.2.1. The proof will be below, after we have given some background.

Since $E_* E^{\mathbb{G}_2^1} \cong \mathrm{map}_{cts}(\mathbb{G}_2/\mathbb{G}_2^1, E_*)$ we can apply E_* to the cofiber sequence (8.2.1) and obtain a short exact sequence of Morava modules

$$0 \to E_* V(0) \longrightarrow E_*(E^{h\mathbb{G}_2^1} \wedge V(0)) \xrightarrow{(\pi-1)_*} E_*(E^{h\mathbb{G}_2^1} \wedge V(0)) \to 0 \tag{8.2.2}$$

and hence a diagram of spectral sequences

$$\begin{array}{ccc} H^s(\mathbb{G}_2^1, E_t/2) & \Longrightarrow & \pi_{t-s}(E^{h\mathbb{G}_2^1} \wedge V(0)) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \partial} \\ H^{s+1}(\mathbb{G}_2, E_t/2) & \Longrightarrow & \pi_{t-s-1} L_{K(2)} V(0) \end{array} \tag{8.2.3}$$

where δ is the algebraic connecting map

$$H^s(\mathbb{G}_2^1, E_t/2) \cong H^s(\mathbb{G}_2, E_t(E^{h\mathbb{G}_2^1} \wedge V(0))) \xrightarrow{\ \delta\ } H^{s+1}(\mathbb{G}_2, E_t/2)$$

in the long exact sequence induced by the short exact sequence (8.2.2).

Lemma 8.2.3. *Suppose the class $a \in H^s(\mathbb{G}_2^1, E_*/2)$ is the image of an element $b \in H^s(\mathbb{G}_2, E_*/2)$ under the restriction*

$$H^s(\mathbb{G}_2, E_*/2) \to H^s(\mathbb{G}_2^1, E_*/2)\ .$$

Then $\delta(a) = \zeta_2 b$.

Proof. The connecting homomorphism δ is a homomorphism of $H^*(\mathbb{G}_2, E_*/2)$-modules. By the definition of ζ_2 (see Remark 5.1.1), we have $\delta(1) = \zeta_2$, and the result follows. ∎

Proof of Theorem 8.2.2. Using Lemma 8.2.3 and the diagram of spectral sequences (8.2.3) we have that $\partial(y_1) \in \pi_{-2} L_{K(2)} V(0)$ is detected by $\zeta_2\chi \in H^2(\mathbb{G}_2, E_0/2)$. Since $2y_1$ is detected by $\eta\chi^2$, we can again use Lemma 8.2.3 and the diagram of spectral sequences (8.2.3) to conclude that $\partial(2y_1) \in \pi_{-2} L_{K(2)} V(0)$ is detected by

$$\zeta_2 \eta \chi^2 \in H^4(\mathbb{G}_2, E_2/2).$$

This class is non-zero by Theorem 7.2.6. Thus we need to check that this class cannot be a boundary in the spectral sequence. Since $E_*/2 = 0$ in odd degrees, the only possible differential is

$$d_3 \colon H^1(\mathbb{G}_2, E_0/2) \to H^4(\mathbb{G}_2, E_2/2).$$

By Remark 5.4.7 we have an isomorphism $H^1(\mathbb{G}_2, E_0/2) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ with generators ζ_2 and χ. We know by Proposition 2.2.1 that ζ_2 is a permanent cycle and we know by Theorem 6.1.1 that χ is a d_3-cycle. Thus $d_3 = 0$ on $H^1(\mathbb{G}_2, E_0/2)$ and the result follows. ∎

References

[Bau08] Tilman Bauer. Computation of the homotopy of the spectrum `tmf`. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 11–40. Geom. Topol. Publ., Coventry, 2008. URL: `https://doi.org/10.2140/gtm.2008.13.11`.

[Bea15] Agnès Beaudry. The algebraic duality resolution at $p = 2$. *Algebr. Geom. Topol.*, 15(6):3653–3705, 2015. URL: `https://doi.org/10.2140/agt.2015.15.3653`.

[Bea17a] Agnès Beaudry. The chromatic splitting conjecture at $n = p = 2$. *Geom. Topol.*, 21(6):3213–3230, 2017. URL: `https://doi.org/10.2140/gt.2017.21.3213`.

[Bea17b] Agnès Beaudry. Towards the homotopy of the $K(2)$-local Moore spectrum at $p = 2$. *Adv. Math.*, 306:722–788, 2017. URL: `https://doi.org/10.1016/j.aim.2016.10.020`.

[Beh12] Mark Behrens. The homotopy groups of $S_{E(2)}$ at $p \geq 5$ revisited. *Adv. Math.*, 230(2):458–492, 2012. URL: `https://doi.org/10.1016/j.aim.2012.02.023`.

[BG18] Irina Bobkova and Paul G. Goerss. Topological resolutions in $K(2)$-local homotopy theory at the prime 2. *Journal of Topology*, 11(4):917–956, 2018. `doi:10.1112/topo.12076`.

[Buj12] Cédric Bujard. Finite subgroups of extended Morava stabilizer groups. *ArXiv e-prints*, June 2012. `arXiv:1206.1951`.

[Dev05] Ethan S. Devinatz. A Lyndon-Hochschild-Serre spectral sequence for certain homotopy fixed point spectra. *Trans. Amer. Math. Soc.*, 357(1):129–150, 2005. URL: `https://doi.org/10.1090/S0002-9947-04-03394-X`.

[DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill, editors. *Topological modular forms*, volume 201 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.

[DH04] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004. URL: `https://doi.org/10.1016/S0040-9383(03)00029-6`.

[GHM14] Paul G. Goerss, Hans-Werner Henn, and Mark E. Mahowald. The rational homotopy of the $K(2)$-local sphere and the chromatic splitting conjecture for the prime 3 and level 2. *Doc. Math.*, 19:1271–1290, 2014.

[GHMR05] Paul G. Goerss, Hans-Werner Henn, Mark E. Mahowald, and Charles Rezk. A resolution of the $K(2)$-local sphere at the prime 3. *Ann. of Math. (2)*, 162(2):777–822, 2005. URL: `https://doi.org/10.4007/annals.2005.162.777`.

[Hen17] Hans-Werner Henn. A Mini-Course on Morava Stabilizer Groups and Their Cohomology. *ArXiv e-prints*, February 2017. `arXiv:1702.05033`.

[Hen18] Hans-Werner Henn. The centralizer resolution of the $K(2)$-local spheres at the prime 2. *HAL e-prints*, hal-01697478, 2018.

[HG94] Michael J. Hopkins and Benedict H. Gross. The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):76–86, 1994. URL: `https://doi.org/10.1090/S0273-0979-1994-00438-0`.

[HKM13] Hans-Werner Henn, Nasko Karamanov, and Mark E. Mahowald. The homotopy of the $K(2)$-local Moore spectrum at the prime 3 revisited. *Math. Z.*, 275(3-4):953–1004, 2013. URL: `https://doi.org/10.1007/s00209-013-1167-4`.

[Hov95] Mark Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *The Čech centennial (Boston, MA, 1993)*, volume 181 of *Contemp. Math.*, pages 225–250. Amer. Math. Soc., Providence, RI, 1995. URL: `https://doi.org/10.1090/conm/181/02036`.

[HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998. URL: `https://doi.org/10.2307/120991`.

[HS99] Mark Hovey and Neil P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999. URL: `https://doi.org/10.1090/memo/0666`.

[Koh13] Jan Kohlhaase. On the Iwasawa theory of the Lubin-Tate moduli space. *Compos. Math.*, 149(5):793–839, 2013. URL: `https://doi.org/10.1112/S0010437X12000723`.

[Lad13] Olivier Lader. *Une résolution projective pour le second groupe de Morava pour $p \geq 5$ et applications.* Theses, Université de Strasbourg, October 2013. URL: `https://tel.archives-ouvertes.fr/tel-00875761`.

[LDMA80] Wên Hsiung Lin, Donald M. Davis, Mark E. Mahowald, and John. F. F. Adams. Calculation of Lin's Ext groups. *Math. Proc. Cambridge Philos. Soc.*, 87(3):459–469, 1980. URL: `https://doi.org/10.1017/S0305004100056899`.

[Mah82] Mark E. Mahowald. The image of J in the EHP sequence. *Ann. of Math. (2)*, 116(1):65–112, 1982. URL: `https://doi.org/10.2307/2007048`.

[MR09] Mark E. Mahowald and Charles Rezk. Topological modular forms of level 3. *Pure Appl. Math. Q.*, 5(2, Special Issue: In honor of Friedrich Hirzebruch. Part 1):853–872, 2009. URL: `https://doi.org/10.4310/PAMQ.2009.v5.n2.a9`.

[MRW77] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. of Math. (2)*, 106(3):469–516, 1977. URL: `https://doi.org/10.2307/1971064`.

[Rav77] Douglas C. Ravenel. The cohomology of the Morava stabilizer algebras. *Math. Z.*, 152(3):287–297, 1977. URL: `https://doi.org/10.1007/BF01488970`.

[Rav78] Douglas C. Ravenel. A novice's guide to the Adams-Novikov spectral sequence. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 404–475. Springer, Berlin, 1978.

[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984. URL: `https://doi.org/10.2307/2374308`.

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[Sil86] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986. URL: `https://doi.org/10.1007/978-1-4757-1920-8`.

[Str00] Neil P. Strickland. Gross-Hopkins duality. *Topology*, 39(5):1021–1033, 2000. URL: `https://doi.org/10.1016/S0040-9383(99)00049-X`.

[SW02] Katsumi Shimomura and Xiangjun Wang. The Adams-Novikov E_2-term for $\pi_*(L_2S^0)$ at the prime 2. *Math. Z.*, 241(2):271–311, 2002. URL: `https://doi.org/10.1007/s002090200415`.

[SY95] Katsumi Shimomura and Atsuko Yabe. The homotopy groups $\pi_*(L_2S^0)$. *Topology*, 34(2):261–289, 1995. URL: `https://doi.org/10.1016/0040-9383(94)00032-G`.

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