

BENRENS: Real K-theory + TAF

Note Title

10/24/2009

Joint with MJH

Background

" n -periodic Chern theories

" n -periodicity + J-hom

$\mathbb{Z} \cong \mathbb{Z} \cong \mathbb{Z} \cong 0 \cong \mathbb{Z} \cong 0 \cong 0 \cong \mathbb{Z} \dots$

Global

Local

$n=1$ { good KU
better $KO = KU^{h\mathbb{C}_2}$

$$KU_+ = E_1 \cong \bigvee_{i=0}^{k-2} \Sigma^{2i} E(1)_p$$
$$KO_p \cong E_1^{h\mathbb{C}_2}$$

$n=2$ { good Ell_C parallel curve
better TMF
 $p=2,3$

$$E(2)_+, E_2 = \bigvee_{K(2)} E(2)_p \otimes \mathbb{Z}/p^2$$
$$EO_2 = E_2^{h\mathbb{C}_2} \cong TMF_{K(2)}$$

$S_n \supset G_n$ finite

general n { good "antonomorphic"
better TAF

$$E(n)_+, E_n = \bigvee_{K(n)} \Sigma^{2i} E(n)_p \otimes \mathbb{Z}/p^n$$
$$EO_n = E_n^{hG_n}$$

$\mathbb{Z}/p^n = W(\mathbb{F}_{p^n})$

Q What is relation between EO_n and $TAF_{K(n)}$?

More background

$$\begin{array}{ccc}
 \mathbb{Z}_p^n [u_1, \dots, u_{n+1}] = \Pi_0 E_n & \tilde{H}_n = \text{Lubin-Tate universal deformation} & \mathcal{G}_n = \tilde{H}_n / E_n \\
 \downarrow \Pi_p^n & & \\
 \mathbb{F}_p^n & H_n = \text{Honda height } n \text{ FGL} &
 \end{array}$$

$\mathcal{S}_n = \text{Aut}(H_n) = \text{Möbius stabilizers}$

$$\begin{array}{ccc}
 \mathcal{S}_n & \begin{array}{c} \downarrow \\ \Pi_0 E_n \end{array} & \mathcal{G}_n = \mathcal{S}_n \rtimes \text{Gal} \\
 & \begin{array}{c} \uparrow \\ \text{Gal}(\mathbb{F}_p^n / \mathbb{F}_p) \end{array} &
 \end{array}$$

Hopkins-Miller: Factorization $\rightarrow E_n$ for $G \subset S_n$ maximal finite

$$EO_n := E_n^{<G>} \quad \text{e.g. } EO_3 = \begin{cases} KO_3 & \text{for } p=2 \\ E(1)_p & p \text{ odd} \end{cases}$$

For $(p-1) \mid n$ and $n > 1$, the maximal finite subgroup is not unique + must be chosen

Thm (T. Hewett) Let $n = (p-1)p^{m-1}$ with $p \neq 2$
Then $\exists!$ $G_{n,\alpha} \subset E_n$ for $0 \leq \alpha \leq m$ (or $1 \leq \alpha \leq m$ for $p=2$)

s.t. $G_{n,\alpha}$ has elts with maximal exponent p^α

TAF?

<u>n</u>	<u>alg group</u>	<u>Moduli space</u>	<u>spectrum</u>
1	GL_1	one pt	KO
2	GL_2	M_{ell}	TMF
n	$U(1, n-1)$	sh "Shimura Variety"	TAF (Can do this for) $\sim 1/2$ of primes

What is the moduli space sh?

Data F $n \bar{n}$ $F = \text{quadratic extension of } \mathbb{Q}$
 $2 \mid$ 1 $p = \text{prime which splits}$
 \mathbb{Q} p

$V = n \text{ dim vector space } / F$

$\langle -, - \rangle = \text{all Hermitian form on } V \text{ with}$
 $\text{sign } (1, n-1)$

$Sh = \{ (A, i, \tau) \}$ *Polarization is an iso with dual of A*

$(A, \tau) = \text{polarized abelian variety of dim } n$

$i: \mathcal{O}_F \rightarrow \text{End}(A)$ ring-map (complex
multiplications)

$\hat{A} = \text{formal completion of } A$
 $= \hat{A}_n \oplus \hat{A}_{\bar{n}}$ where $\dim \hat{A}_n = 1$

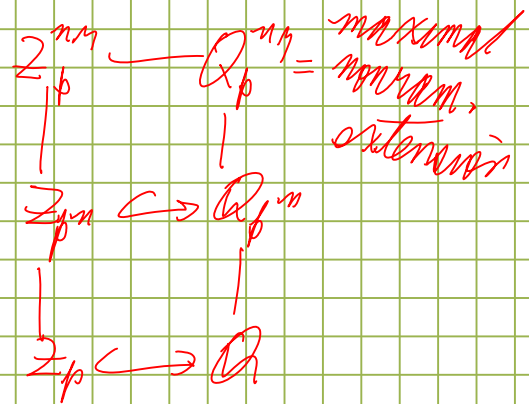
Thm (B-Lawson)

$$TAF \cong \left(\bigvee_{(A, i, \lambda) \in \text{Sh}^{[m]}(\mathbb{F}_p)} E_m^{h\text{Cent}(A, i, \lambda)} \right)^{h\text{Gal}_{\mathbb{F}_p}}$$

↑
ht of \hat{A}_m is n

$$\text{Cent}(A, i, \lambda) \triangleleft S_m$$

finite



The indexing set is like a class number.

When is $\text{Cent}(A, i, \lambda)$ a maximal finite subgroup?

TAF depends on choice of F and $\langle -, - \rangle$

New Question

Given $G_{m,\alpha} \subset \mathbb{F}_m$, \exists ? choice of $(F, \langle \cdot, \cdot \rangle)$

s.t. $\exists (A, i, \lambda) \in \text{Sh}^{\text{[n]}}(\mathbb{F}_p)$ s.t.

$$\text{Aut}(A, i, \lambda) = G_{m,\alpha} \quad ?$$

Answer Thom (B-Hopkins) Yes if $p \equiv 1$ and
 $n = (p-1)p^{m-1}$ and $\alpha = 1$

No if p odd, n not as above.

Unknown if $p=2$, n not as above

Conjecture prize: for all $n = (p-1)p^{m-1}$ we can
find an (A, i, λ) for which $\text{Aut}(\cdot)$ has an elt
of order p^n . (Possibly true for all n)

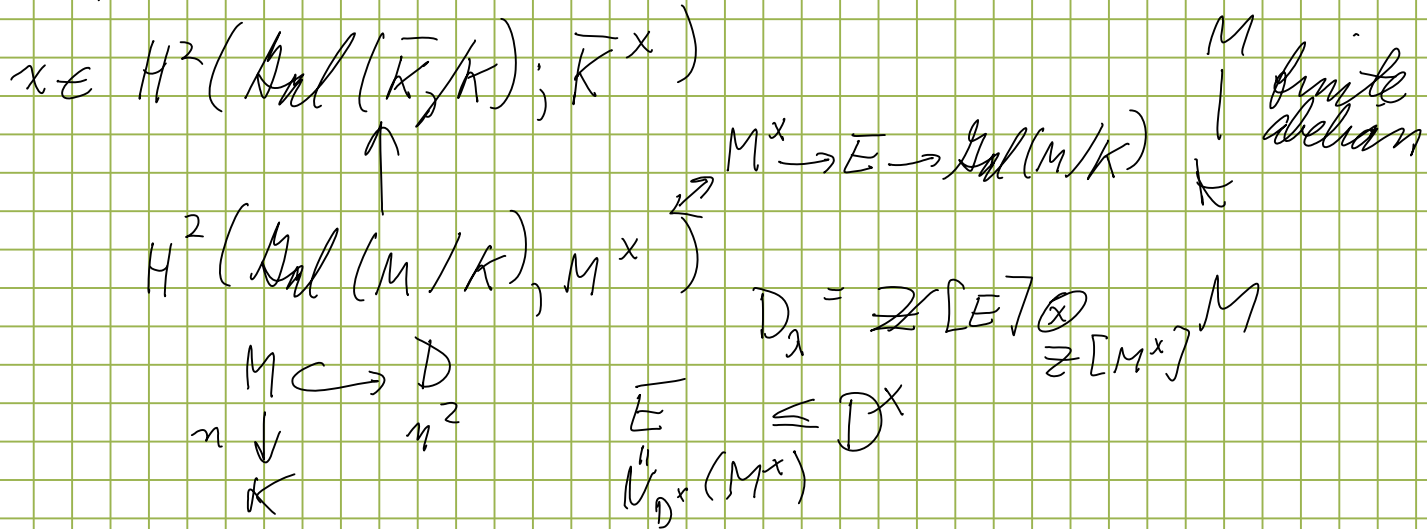
"TAF sees as much as EO_n "

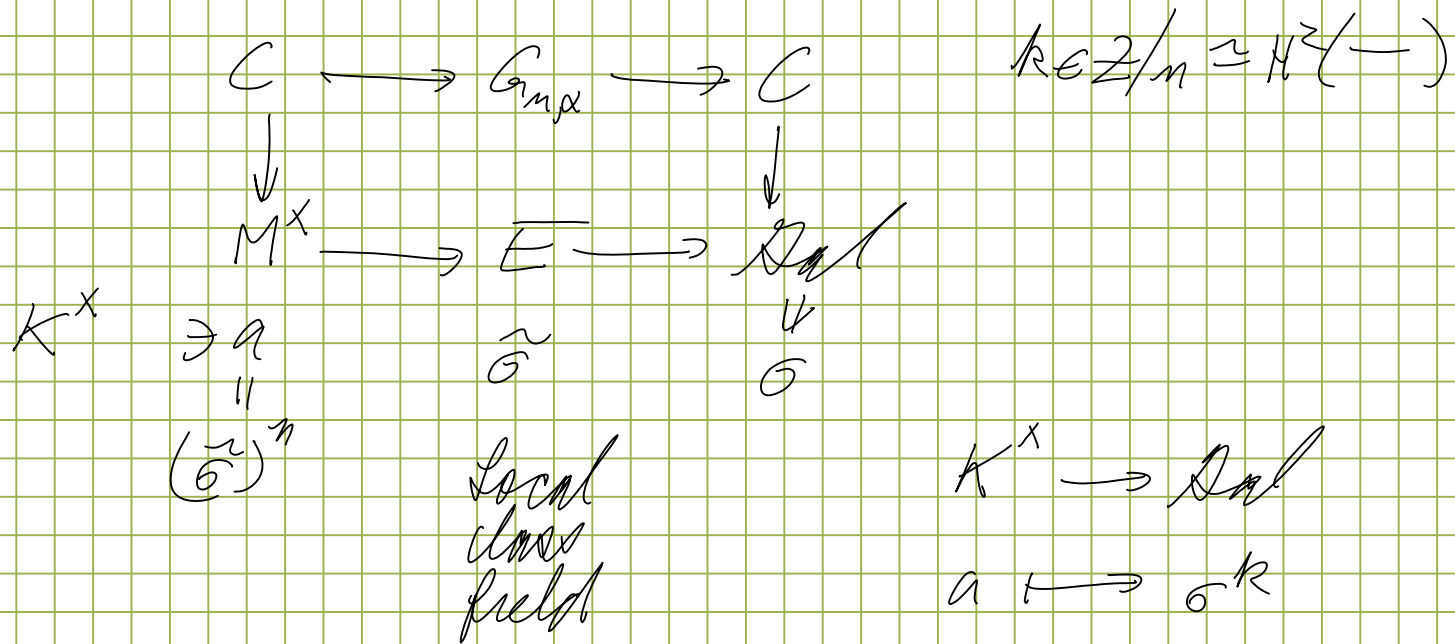
Method of proof: Use local/global class field theory

Idea: Explicit construction of division algebras

$$B_n(K) \cong H^2(\text{Gal}(\bar{K}/K), \bar{K}^\times)$$

Explicit construction due to (Serre 1950)





$F \supseteq \mathbb{Q}(\zeta_{p-1})$ We need F to contain $\mathbb{Q}(\zeta_{p-1})$
 so p must be small
 $\mathbb{Q} \xrightarrow{\quad} \mathbb{Q}(\zeta_{p-1})$
 \uparrow
 $(p-1)$ th root of unity

