BEHRENS: Real K-theories + TAF

Note Title 10/24/2009

Joint with MJH

Background

"v_n-periodic cohom theories"

v_1-periodicity + J-hom

$\mathbb{Z}$ $\mathbb{Z}/2$ $\mathbb{Z}/2$ 0 $\mathbb{Z}$ 0 0 0 $\mathbb{Z}$ . . . . .

		Global	Local
$n=1$	good	KU	$KU_p = E_1 \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1)_p$
	better	$KO = KU^{hC_2}$	$KO_p \simeq E_1^{hC_2}$
$n=2$	good	Ell_C for ell curve C	$E(2)$, $E_2 \simeq \bigvee E(2)_{K(2)} \otimes \mathbb{Z}_{p^2}$
	better	TMF $p=2,3$	$EO_2 = E_2^{hG} \underset{p=2,3}{\simeq} TMF_{K(2)}$
			$S_n \supset G$ finite
general n	good	"automorphic"	$E(n)$, $E_n = \bigvee \Sigma^{2i} E(n)_{K(n)} \otimes \mathbb{Z}_{p^n}$
		$\mathbb{Z}_{p^n} = W(\mathbb{F}_{p^n})$	
	better	TAF	$EO_n = E_n^{hG}$

Q What is relation between EO_n and $TAF_{K(n)}$?

More background

$$\mathbb{Z}_{p^n}[[u_1, \cdots u_{n-1}]] = \pi_0 E_n \qquad \tilde{H}_n = \text{Lubin-Tate universal deformation} \qquad \mathbb{G}_{E_n} = \tilde{H}_n$$

$$\begin{array}{c} \pi_0 E_n \\ | \\ \mathbb{F}_{p^n} \end{array} \qquad H_n = \text{Honda height } n \text{ FGL}$$

$\mathbb{S}_n = \mathrm{Aut}(H_n)$ = Morava stabilizer

$$S_n \curvearrowright \pi_0 E_n \curvearrowleft Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \qquad \mathbb{G}_n = \mathbb{S}_n \rtimes Gal$$

Hopkins–Miller: $G \overset{\exists \text{ action}}{\curvearrowright} E_n$ for $G \subset \mathbb{S}_n$ maximal finite

$$EO_n := E_n^{hG} \qquad \text{e.g. } EO_1 = \begin{cases} KO_2 & \text{for } p=2 \\ E(1)_p & p \text{ odd} \end{cases}$$

For $(p-1) \mid n$ and $n > 1$, the maximal finite subgp is not unique + must be chosen

<u>Thm</u> (T. Hewett) Let $n = (p-1)p^{\alpha-1}s$ with $p \nmid s$

Then $\exists!\ G_{n,\alpha} \subset \mathbb{S}_n$ for $0 \le \alpha \le n$ (or $1 \le \alpha \le n$ for $p=2$)

s.t. $G_{n,\alpha}$ has elts with maximal exponent p^α

TAF?

n	alg group	Moduli space	Spectrum
1	GL_1	one pt	KO
2	GL_2	$\mathcal{M}_{Ell}$	TMF
n	$U(1,n-1)$	Sh "Shimura Variety"	TAF (Can do this for ~ 1/2 of primes)

What is the moduli space $\underline{Sh}$?

Data $\begin{array}{c} F \\ 2\,| \\ \mathbb{Q} \end{array}$ $\quad \begin{array}{c} u\ \bar{u} \\ | \\ p \end{array}$ $\quad F$ = quadratic extension of $\mathbb{Q}$

p = prime which splits

V = n dim vector space / F

$\langle -, - \rangle$ = alt Hermitian form on V with sign $(1, n-1)$

$Sh = \{(A, i, \lambda)$ Polarization is an iso with dual of A

(A, λ) = polarized abelian vty of dim n

$i : \mathcal{O}_F \to \mathrm{End}(A)$ ring-map (complex multiplication)

$\hat{A}$ = formal completion of A

$= \hat{A}_u \oplus \hat{A}_{\bar{u}}$ where $\dim \hat{A}_u = 1$

Thm (B-Lawson)

$$TAF \simeq \left(\bigvee_{(A,i,\lambda) \in Sh^{[n]}(\overline{\mathbb{F}}_p)} E_n^{hAut(A,i,\lambda)} \right)^{hGal_{\mathbb{F}_{p^n}}}$$

ht of $\hat{A}_u$ is n

$Aut(A,i,\lambda) < S_n$
finite

$$\begin{tikzcd}
\mathbb{Z}_p^{nr} \arrow[r, dash] \arrow[d, dash] & \mathbb{Q}_p^{nr} \arrow[d, dash] \\
\mathbb{Z}_{p^n} \arrow[r, hook] \arrow[d, dash] & \mathbb{Q}_{p^n} \arrow[d, dash] \\
\mathbb{Z}_p \arrow[r, hook] & \mathbb{Q}_p
\end{tikzcd}$$

$\mathbb{Q}_p^{nr}$ = maximal nonram. extension

The indexing set is like a class number.

When is $Aut(A,i,\lambda)$ a maximal finite subgp?

TAF depends on choice of F and $\langle -,- \rangle$

New Question

Given $G_{n,\alpha} \subset \mathbb{S}_n$, $\exists ?$ choice of $(F, \langle -,- \rangle)$ s.t. $\exists\, (A, i, \lambda) \in Sh^{[n]}(\bar{\mathbb{F}}_p)$ s.t.

$$Aut(A, i, \lambda) = G_{n,\alpha} \;?$$

Answer: Thm (B-Hopkins) Yes if $p \le 7$ and $n = (p-1)p^{r-1}$ and $\alpha = 1$

No if p odd, n not as above.

Unsure if $p = 2$, n not as above

Consolation prize: for all $n = (p-1)p^{r-1}$ we can find an (A, i, λ) for which $Aut(\;)$ has an elt of order p^r. (Possibly true for all n)

"TAF sees as much as EO_n"

<u>Method of proof</u>: Use local/global class field theory

Idea: Explicit construction of division algebras.

$$Br(K) \cong H^2(Gal(\bar{K}/K), \bar{K}^\times)$$

Explicit construction due to (Serre 1950)

$$\begin{tikzcd} x \in H^2(Gal(\bar{K}/K); \bar{K}^\times) \\ H^2(Gal(M/K), M^\times) \arrow[u] \arrow[r, phantom, "\cong"] & M^\times \to E \to Gal(M/K) \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[d, dash, "\text{finite abelian}"] \\ K \end{tikzcd}$$

$$D_x = \mathbb{Z}[E] \otimes_{\mathbb{Z}[M^\times]} M$$

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "n"'] & D \quad n^2 \\ K \end{tikzcd}$$

$$E \le D^\times$$

$$E = N_{D^\times}(M^\times)$$

$$k \in \mathbb{Z}/n \cong H^2(\;-\;)$$

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & G_{m,x} \arrow[r] & C \arrow[d] \\
M^\times \arrow[r] & \overline{E} \arrow[r] & \mathrm{Gal}
\end{tikzcd}$$

$K^\times \ni a = (\tilde{\sigma})^n$, $\quad \tilde{\sigma}$, $\quad \mathrm{Gal} \ni \sigma$

Local class field

$$K^\times \longrightarrow \mathrm{Gal}$$

$$a \longmapsto \sigma^k$$

$F \supseteq \mathbb{Q}(\zeta_{p-1})$

$\supseteq \mathbb{Q}_p$

$(p-1)$th root of unity

We need F to contain $\mathbb{Q}(\zeta_{p-1})$ so p must be small