BEHRENS ANSS

Note Title 10/28/2010

ANSS + Ravenel's <u>very</u> odd $(p\geq 5)$ primary Kervaire thm

The $p=3$ problem is still open

EO_{p-1} is similar to ΣL as in EO_4

ANSS

Let E be a strictly assoc latts comm ring spectrum with E_*E flat over E_*

For any X we have

$$\begin{tikzcd} X \arrow[r] & E\wedge X \arrow[r, shift left=2] \arrow[r, shift right=2] & E\wedge E\wedge X \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & E\wedge E\wedge E \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] & \cdots \end{tikzcd}$$

Bousfield-Kan cosimp. resoln

$X \longrightarrow \mathrm{Tot}(E^{\wedge\bullet+1}\wedge X) = X^{\wedge}_E$ = E-nilp completion of X

There is a map $X_E \longrightarrow X^{\wedge}_E$ sometimes an equiv

BKSS: $E_2^{s,t} = H^s(\pi_t\, E^{\bullet+1}\wedge X) \Longrightarrow \pi_{t-s} X^{\wedge}_E$

This is the E-based ANSS

Identification of E_2

$$\begin{tikzcd} E \arrow[r, bend left, "\eta_L"] \arrow[r, bend right, "\eta_R"'] & E\wedge E \arrow[l, "\varepsilon"'] \arrow[r, "\psi"] & E\wedge E\wedge E \arrow[d, equal] \arrow[r] & \cdots \\ & & (E\wedge E)\wedge_E(E\wedge E) & \end{tikzcd}$$

$$\begin{tikzcd} E_* \arrow[r, bend left] \arrow[r, bend right] & E_*E \arrow[l] \arrow[r, "\psi"] & (E_*E)\otimes_{E_*} E_*E \arrow[r] & \cdots \end{tikzcd}$$

$(E_*, E_*E) = (A, \Gamma)$ is a Hopf algebroid

$$\begin{array}{ccc} E\wedge X & \longrightarrow & E\wedge E\wedge X \\ & & \parallel \\ & & (E\wedge E)\wedge_E (E\wedge X) \end{array}$$

This makes $E_*X = M$ a comodule over (A, Γ)

$$E_* X \longrightarrow (E_*E)\otimes_{E_*} E_*X$$

For $M = E_*X$ we get cobar cx $C_\Gamma(M)$

$$\begin{array}{ccccccc} M & \longrightarrow & \Gamma\otimes_A M & \longrightarrow & \Gamma\otimes_A \Gamma\otimes_A M & \longrightarrow & \cdots \\ & & \gamma\otimes m & & \gamma_1\otimes\gamma_2\otimes m & & \\ & & \parallel & & & & \\ & & [\gamma]\, m & & [\gamma_1|\gamma_2]\ m & & \end{array}$$

$$E_2^{s,t} = H^*(C_\Gamma(M)) = \mathrm{Ext}_\Gamma^{s,t}(A, M) =: \mathrm{Ext}^{s,t}(M)$$

for $M = E_*X$

$$\mathrm{Ext}^{s,t}(M) \Downarrow \pi_{t-s}(X^\wedge_E)$$

Examples $E = H\mathbb{F}_2$, $X = S$ = sphere spectrum

Classical ASS for $p=2$

$\hat{S}_{H\mathbb{F}_2} = \hat{S}_2$ = 2-adic completion

$E_*E = A^*$ = dual Steenrod algebra

$= \mathbb{F}_2[\xi_1, \xi_2 \dots]$ with $|\xi_i| = 2^i - 1$

$[\xi_1^{2^j}] = h_j \in \mathrm{Ext}^1$ converging to Hopf invariant one element

$h_0 \Rightarrow 2 \in \pi_0$, $h_1 \Rightarrow \eta \in \pi_1$, $h_2 \Rightarrow \nu \in \pi_3$

$h_3 \Rightarrow \sigma \in \pi_7$, $d_2 h_j = h_0 h_{j-1}^2$ for $j \geq 4$

$[\xi_1^{2^j} | \xi_1^{2^j}] = h_j^2 \in \mathrm{Ext}^{2, 2^{j+1}} \overset{?}{\Longrightarrow} \theta_j$ Kervaire class

<u>Example</u> $E = H\mathbb{F}_p$ for p odd

$E_* E$ = dual to mod p Steenrod algebra

$= \mathbb{F}_p[\xi_1, \xi_2 \cdots] \otimes E(\tau_0, \tau_1, \cdots)$

$|\xi_i| = |\tau_i| - 1 = 2(p^i - 1)$

$h_i = [\xi_1^{p^i}] \in \mathrm{Ext}^1 \Longrightarrow$ "odd primary Hopf invariant one"

$h_0 \Longrightarrow \alpha_1 \in \pi_{2p-3}$ Liulevicius

$d_2(h_i) = v_0 b_{i-1}$ for $i \geq 1$ "

$h_1^2 = 0$

$$b_i = \sum_{0<t<p} p^{-1}\binom{p}{t}\left[\xi_1^{tp^i} \,\middle|\, \xi_1^{(p-t)p^i}\right] \in \mathrm{Ext}^2$$

$$b_0 \Longrightarrow \beta_1 \in \pi_{2p^2-2p-2}$$

$b_1 \Longrightarrow$ "odd primary Kervaire class"

Thm (Ravenel) b_i is **not** a perm. cycle for $i \geq 1$ and $p \geq 5$.

Example $E = BP$ ANSS

$BP_* = \mathbb{Z}_{(p)}[v_1, v_2 \cdots]$ $\quad |v_i| = 2(p^i - 1) = |t_i|$

$BP_* BP = BP_*[t_1, t_2, \cdots]$

$S_{BP} = S_{(p)}$ $\qquad \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Longrightarrow \pi_{t-s} S_{(p)}$

Interesting elements: Greek letter elements.

<u>Notation</u> For $x \in R$ not a zero divisor,

let $R/x^\infty = \varinjlim R/x^i \ni \frac{z}{x^i}$

e.g. $\mathbb{Z}/p^\infty \ni \frac{m}{p^i}$ which depends only on $m \bmod p^i$

Short exact sequences

$$0 \to \frac{BP_*}{(p^\infty, \dots v_{n-1}^\infty)} \to \frac{BP_*[v_n^{-1}]}{(p^\infty, \dots v_{n-1}^\infty)} \to \frac{BP_*}{(p^\infty, \dots v_n^\infty)} \to 0$$

analogous to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

This leads to LES of Ext gps and a connecting hom

$$\delta_n : \mathrm{Ext}^s\left(\frac{BP_*}{(p^\infty, \cdots v_n^\infty)}\right) \longrightarrow \mathrm{Ext}^{s+1}\left(\frac{BP_*}{(p^\infty, \cdots v_{n-1}^\infty)}\right)$$

Can iterate these and set

$$\delta^n : \mathrm{Ext}^s\left(\frac{BP_*}{(p^\infty, \cdots v_{n-1}^\infty)}\right) \longrightarrow \mathrm{Ext}^{s+n}(BP_*)$$

Consider the case $s = 0$. The source is the invariant subgp of $\frac{BP_*}{(p^\infty, \cdots v_{n-1}^\infty)}$.

e.g. $\frac{v_n^{i_n}}{p^{i_0} \cdots v_{n-1}^{i_{n-1}}} + \frac{x}{p^{i_0'} \cdots v_{n-1}^{i_{n-1}'}}$ with $(i_0' \cdots i_{n-1}') < (i_0 \cdots i_{n-1})$ w.r. to right lex ordering

$\delta^n(\ ) = \alpha^{(n)}_{i_n/(i_{n-1},\dots i_0)} \in \mathrm{Ext}^n$ note reversal

$= n$ th Greek letter element

$\alpha^{(n)} = n$ th letter of Greek alphabet

$\alpha^{(1)} = \alpha \qquad \alpha^{(2)} = \beta$ etc

Example $n=1$ $\quad \alpha_{i/j} \in \mathrm{Ext}^1$. These are permanent cycles related to Im J

$\alpha_{1/1} = \alpha_1 = h_0$ in ASS

$n=2 \qquad \beta_{i/j,k} \in \mathrm{Ext}^2$

$\beta_{1/(1,1)} = \beta_1 = b_0$ in ASS

$\beta_{p^i/(p^i,1)} \longleftrightarrow b_i$ in ASS

Relation between $E = BP$ and $E = H\mathbb{F}_p$

$$\begin{tikzcd}
BP \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d] & BP \wedge BP \arrow[l] \arrow[d] & \cdots \\
H\mathbb{F}_p \arrow[r, shift left=2] \arrow[r, shift right=2] & H\mathbb{F}_p \wedge H\mathbb{F}_p \arrow[l] & \cdots
\end{tikzcd}$$

Thom reduction

Map of SSs

$$\begin{tikzcd}
\beta_{p^i/p^i} \arrow[d] \\
b_i
\end{tikzcd}$$

(not the only element in its bidegree

b_i = only elt in its bidegree

$$\left\{ \beta_{p^i/p^i},\ \underbrace{v_2^{p^{i-1}(p+1)} \beta_{p^{i-2}/p^{i-2}},\ v_2^{-} \beta_{p^{i-4}/p^{i-4}} \cdots}_{\text{spurious elements}} \right\}$$

is complete list in bidegree

others map to 0 under Thom reduction

Suppose θ_j exists

$\Rightarrow$ b_j is perm cycle in ASS

$\Rightarrow$ θ_j detected in $\mathrm{Ext}^{\leq 2}$ ANSS

but $\mathrm{Ext}^{\leq 1} = 0$ in this dim

$\Rightarrow$ θ_j detected by $\beta_{p^j/p^j} + x$

for x a linear comb of spurious elements

Revisionist "Proof"

E_n = Morava E-theory

$\pi_* E_n = W(\mathbb{F}_{p^n})[[u_1, \cdots u_{n-1}]][u^{\pm 1}]$ $\quad |u_i| = 0$

$|u| = 2$

This has action of $\mathbb{S}_n = \mathcal{O}_D^\times$ = Morava stabilizer gp

$\hookrightarrow D_n$

D_n = division algebra / $\mathbb{Q}_p$ with Hasse invariant $1/n$

It contains all degree n field extensions of $\mathbb{Q}_p$

e.g. $\mathbb{Q}_p(\zeta_p) \subset D_{p-1}$

$$\begin{tikzcd}\mathbb{Q}_p(\zeta_p) \arrow[d, dash, "p-1"] \\ \mathbb{Q}_p\end{tikzcd}$$

so $\underset{\underset{\langle \zeta_p \rangle}{\shortparallel}}{C_p} \subset S_{p-1}$ acting on E_{p-1}

E_{p-1} is analogous to $\tilde{\Omega}$

$EO_{p-1} = E_{p-1}^{hC_p}$ ″ Ω

Map of ANSS

$$S \longrightarrow E_{p-1}^{hC_p}$$

$$\begin{tikzcd}
\mathrm{Ext}(BP_*) \arrow[r, Rightarrow] \arrow[d, "\varphi_*"] & \pi_{*-s} S \arrow[d, "\varphi"] \\
H^s(C_p; \pi_* E_n) \arrow[r, Rightarrow] \arrow[d, phantom, "\cap"] & \pi_{*-s} EO_{p-1} \\
P(\beta) \otimes E(\alpha) &
\end{tikzcd}$$

$$\begin{tikzcd}
\beta_1 \arrow[d, mapsto] & \alpha_1 \arrow[d, mapsto] \\
\beta & \alpha
\end{tikzcd}$$

There is the Toda differential

$$d_{2p-1}(\beta_{p/p}) = \alpha_1 \beta_1^p$$

$$d(\varphi_* \beta_{p/p}) = \alpha \beta^p$$

$\Delta \in H^0$ invertible perm cycle

$\parallel$

$u^N + \cdots$

$\varphi_* \quad \beta_{p^i/p^i} \neq 0$

$= \tilde{\Delta} \quad \varphi_* (\beta_{p/p})$

so each β_{p^i/p^i} support a diff

What about the spurious β elements x ?

$R = \underbrace{\mathbb{Z}_p[\zeta_p]}_{A}[u^{\pm}]$

$\pi = \zeta - 1$

$\pi^{p-1} = \text{unit} \cdot p$

A carries a ht 1 Lubin-Tate formal A-module

$\log x = \sum \frac{x^{p^i}}{\pi^i}$

$$\begin{tikzcd}
BP_* \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & R \\
& E_{p-1} \arrow[ur] &
\end{tikzcd}$$

$\varphi(p) = \text{unit } \pi^{p-1}$

$\varphi(v_1) = \text{"} \ \pi^{p-2}$

$\varphi(v_2) = \text{"} \ \pi^{p-3}$

$\varphi(v_{p-1}) = \text{unit}$

$\varphi(v_2^2) = \text{unit } \pi^{2p-6}$

$$\begin{tikzcd}
v_2^n \beta_{p^i/p^i} \arrow[d, equal] \arrow[rrr, bend left, "0", "\text{for } n \geq 2"'] & \mathrm{Ext}(BP_*) \arrow[rr] \arrow[dr] & & H^*(C_p; R) \\
x & & H^*(C_p; \mathbb{Z}_{p-1}) \arrow[ur] &
\end{tikzcd}$$

$$\beta_{p^i/p^i} \longrightarrow \neq 0$$