

THE P_2^1 MARGOLIS HOMOLOGY OF CONNECTIVE TOPOLOGICAL MODULAR FORMS

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ABSTRACT. The element P_2^1 of the mod 2 Steenrod algebra \mathcal{A} has the property $(P_2^1)^2 = 0$. This property allows one to view P_2^1 as a differential on $H_*(X, \mathbb{F}_2)$ for any spectrum X . Homology with respect to this differential, $\mathcal{M}(X, P_2^1)$, is called the P_2^1 Margolis homology of X . In this paper we give a complete calculation of the P_2^1 Margolis homology of the 2-local spectrum of topological modular forms tmf and identify its \mathbb{F}_2 basis via an iterated algorithm. We apply the same techniques to calculate P_2^1 Margolis homology for any smash power of tmf .

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Convention. Throughout this paper we work in the stable homotopy category of spectra localized at the prime 2.

1. INTRODUCTION

The connective E_∞ ring *spectrum of topological modular forms* tmf has played a vital role in computational aspects of chromatic homotopy theory over the last two decades [Goe10], [DFHH14]. It is essential for detecting information about the chromatic height 2, and it has the rare quality of having rich Hurewicz image. There is a $K(2)$ -local equivalence [HM14]

$$L_{K(2)}tmf \simeq E_2^{hG_{48}}$$

where E_2 is the second Morava E -theory at $p = 2$ and G_{48} is the maximal finite subgroup of the Morava stabilizer group \mathbb{G}_2 . The spectrum $E_2^{hG_{48}}$ can be used to build the $K(2)$ -local sphere spectrum (see [BG18]). The homotopy groups of tmf approximate both the stable homotopy groups of spheres and the ring of integral modular forms. In many senses, tmf is the chromatic height 2 analogue of connective real K -theory *ko*. Further, the homotopy groups of tmf are completely known [Bau08].

Let us now recall the definition of the element $P_2^1 \in \mathcal{A}$. Milnor described the mod 2 dual Steenrod algebra \mathcal{A}_* as the graded polynomial algebra [Mil58, App. 1]

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots],$$

where $|\xi_i| = 2^i - 1$. The Steenrod algebra \mathcal{A} has an \mathbb{F}_2 -basis dual to the monomial basis of \mathcal{A}_* . The elements of the \mathbb{F}_2 -basis of \mathcal{A} which are dual to $\xi_t^{2^s}$ are denoted by P_t^s , and the elements P_t^0 are denoted by Q_{t-1} . When $s < t$, the elements P_t^s are exterior power generators, i.e. $(P_t^s)^2 = 0$. Thus, any left \mathcal{A} -module K can be regarded as a complex with differential given by the left multiplication by P_t^s (for $s < t$). This leads to the following definition.

Definition 1.1 ([Mar83]). Let K be any left \mathcal{A} -module and $0 \leq s < t$. Let

$${}^L\mathcal{P}_t^s : K \longrightarrow K$$

denote the left action by P_t^s . The *left P_t^s Margolis homology group* of K , $\mathcal{M}^L(K, P_t^s)$, is defined as

$$\mathcal{M}^L(K, P_t^s) := \frac{\text{Ker } {}^L\mathcal{P}_t^s : K \rightarrow K}{\text{Im } {}^L\mathcal{P}_t^s : K \rightarrow K}.$$

For a right \mathcal{A} -module K , one can similarly define the *right P_t^s Margolis homology group* of K as

$$\mathcal{M}^R(K, P_t^s) := \frac{\text{Ker } {}^R\mathcal{P}_t^s : K \rightarrow K}{\text{Im } {}^R\mathcal{P}_t^s : K \rightarrow K}$$

where ${}^R\mathcal{P}_t^s$ is the right action by P_t^s on K .

Notation 1.2. For a spectrum X , $\mathcal{M}(X, P_t^s)$ will denote $\mathcal{M}^L(H^*(X), P_t^s)$ or equivalently $\mathcal{M}^R(H_*(X), P_t^s)$.

Computations of Margolis homology underly many essential computations in homotopy theory. For example, Adams work on $BP\langle 1 \rangle$ cooperations [Ada74] relies on the computations of $\mathcal{M}(BP\langle 1 \rangle, Q_i)$ for $i = 0, 1$. Calculations like $\mathcal{M}(bo, Q_i)$ for $i = 0, 1$ are essential ingredients in the work of Mahowald on bo -resolutions [Mah81]. More recently, Culver described $BP\langle 2 \rangle$ resolutions [Cul] by understanding $\mathcal{M}(BP\langle 2 \rangle, Q_i)$ for $i = 0, 1, 2$. Computation of $\mathcal{M}(tmf^{\wedge n}, Q_2)$ is an essential ingredient in [BBB⁺b].

The element Q_i is primitive for all $i \in \mathbb{N}$. In other words, the comultiplication map Δ on \mathcal{A} sends Q_i to

$$(1.3) \quad \Delta(Q_i) = Q_i \otimes 1 + 1 \otimes Q_i.$$

Consequently, Q_i acts on $H_*(X)$ as a derivation, namely it follows the Leibniz rule

$$Q_i(xy) = Q_i(x) \cdot y + x \cdot Q_i(y),$$

whenever X is a ring spectrum. The Leibniz rule implies the Künneth isomorphism [Mar83, Proposition 17, pg 343]

$$\mathcal{M}(X \otimes Y, Q_i) \cong \mathcal{M}(X, Q_i) \otimes \mathcal{M}(Y, Q_i)$$

and hence, $\mathcal{M}(X, Q_i)$ is an \mathbb{F}_2 algebra whenever X is a ring spectrum. As a result, computation of Q_i Margolis homology and its description is often fairly straightforward.

On the other hand, for $s > 0$, P_t^s is not a primitive element of \mathcal{A} . In particular,

$$\Delta(P_2^1) = P_2^1 \otimes 1 + Q_1 \otimes Q_1 + 1 \otimes P_2^1$$

and its action on $H_*(X)$ for a ring spectrum X , does not follow the Leibniz rule. Instead, we have

$$(1.4) \quad P_2^1(xy) = P_2^1(x)y + Q_1(x)Q_1(y) + xP_2^1(y).$$

As a result, the product of two P_2^1 cycles may not necessarily be a P_2^1 cycle, hence $\mathcal{M}(X, P_2^1)$ may not admit any multiplicative structure even if X is a ring spectrum. This is the main reason why the P_2^1 Margolis homology calculations are significantly more complicated.

Let us now consider the spectrum tmf . It is well-known ([HM14], [Mat16]) that

$$H_*(tmf; \mathbb{F}_2) \cong \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots] \subset \mathcal{A}_*$$

is a subalgebra of \mathcal{A}_* . Here the elements ζ_i are the images of ξ_i under the antipode of the Hopf algebra \mathcal{A}_* (see Section 2). The right action of Q_i is given by the formula (see [Cul, §2] for details)

$$Q_i(\zeta_n) = \zeta_{n-i-1}^{2^{i+1}}.$$

Then, since the Q_i are derivations, it can be easily seen that

$$(1.5) \quad \mathcal{M}(tmf, Q_0) = \mathbb{F}_2[\zeta_1^8, \zeta_2^4]$$

$$(1.6) \quad \mathcal{M}(tmf, Q_1) = \frac{\mathbb{F}_2[\zeta_1^8, \zeta_3^2, \zeta_4, \dots]}{\langle \zeta_3^4, \zeta_4^4, \dots \rangle}$$

$$(1.7) \quad \mathcal{M}(tmf, Q_2) = \frac{\mathbb{F}_2[\zeta_2^4, \zeta_3^2, \zeta_4, \dots]}{\langle \zeta_2^8, \zeta_3^8, \zeta_4^8, \dots \rangle}.$$

In this paper, we give a complete calculation of $\mathcal{M}(tmf^{\wedge r}, P_2^1)$ for arbitrary $r \geq 1$. In fact, the calculation for $r > 1$ follows from the case $r = 1$, because after forgetting the internal grading one can construct a non-canonical isomorphism (see Section 4)

$$\mathcal{M}(tmf^{\wedge r}, P_2^1) \cong \mathcal{M}(tmf, P_2^1).$$

For the case $r = 1$, we give an iterated algorithm (see Definition 3.16) that constructs an \mathbb{F}_2 -basis of $\mathcal{M}(tmf, P_2^1)$. We give a complete description of $\mathcal{M}(tmf, P_2^1)$ in Theorem 3.18 which is the main result of this paper. Although $\mathcal{M}(tmf, P_2^1)$ is not an algebra, we notice that $\mathcal{M}(tmf, P_2^1)$ is a module over an infinitely generated exterior algebra \mathcal{S} (see Lemma 3.1 for a description of \mathcal{S}). Theorem 3.18 also describes $\mathcal{M}(tmf, P_2^1)$ as an \mathcal{S} -module.

The key tool we use is the *length spectral sequence* (2.14), which we define in Section 2. The length spectral sequence admits a d_0 differential and a d_2 differential and collapses at the E_3 page. The Leibniz rule does hold for the d_0 , but not for d_2 . In order to work around this issue, we notice that the E_2 page admits an action of \mathcal{S} (i.e. d_2 are \mathcal{S} linear) and we use it to simplify the computation of $E_\infty = E_3$.

We also notice that almost identical calculations lead to a complete description of $\mathcal{M}((B\mathbb{Z}/2^{\times n})_+, P_2^1)$. The methods developed in this paper can be considered as a blueprint for computations of P_t^1 Margolis homology of a variety of other \mathcal{A} -modules.

Our calculations of $\mathcal{M}(tmf^{\wedge r}, P_2^1)$ have many applications, as the spectrum tmf has a wide range of applications, particularly in chromatic homotopy theory. First

note that the cohomology of tmf , as a module over the Steenrod algebra \mathcal{A} , is isomorphic to (see [HM14], [Mat16])

$$(1.8) \quad H^*(tmf; \mathbb{F}_2) \cong \mathcal{A} // \mathcal{A}(2)$$

where $\mathcal{A}(2)$ is the subalgebra of \mathcal{A} generated by Sq^1, Sq^2 and Sq^4 . This, and a change of rings isomorphism, imply that the E_2 page of the Adams spectral sequence converging to $tmf_* X$ (for a spectrum X) is

$$(1.9) \quad E_2^{s,t} := Ext_{\mathcal{A}(2)}^{s,t}(H^*(X), \mathbb{F}_2).$$

One can detect infinite families in the E_2 page via the map

$$q : Ext_{\mathcal{A}(2)}^{s,t}(H^*(X), \mathbb{F}_2) \longrightarrow Ext_{\Lambda(\mathbb{P}_2^1)}^{s,t}(H^*(X), \mathbb{F}_2).$$

The codomain of q can be understood by calculating $\mathcal{M}(X, \mathbb{P}_2^1)$. Note that

$$Ext_{\Lambda(\mathbb{P}_2^1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{2,1}],$$

where $|h_{2,1}| = (1, 6)$ and

$$\mathbb{F}_2[h_{2,1}] \otimes \mathcal{M}(X, \mathbb{P}_2^1) \subset Ext_{\Lambda(\mathbb{P}_2^1)}^{s,t}(H^*(X), \mathbb{F}_2)$$

accounts for all the elements with positive s filtration. This shows that the knowledge of $\mathcal{M}(X, \mathbb{P}_2^1)$ is crucial in detecting patterns in the E_2 -page of (1.9).

Motivation I - Towards homotopy groups of $K(2)$ -local sphere. Computation of the homotopy groups of $L_{K(n)}S^0$ — the sphere spectrum localized with respect to Morava K -theories $K(n)$ at various primes p and heights n — is the central question of chromatic homotopy theory. It is sometimes easier to compute $\pi_* L_{K(n)}X$ for finite complexes other than the sphere, although very little data like this is known at $n = p = 2$ anyway. Recently, Bhattacharya and Egger introduced a family of finite spectra Z [BEa], and $\pi_* L_{K(2)}Z$ has been computed [BBB⁺b, BEb], the first example of a finite complex at $p = 2$ whose $K(2)$ -local homotopy groups are completely determined. The finite complex Z can be constructed from the sphere spectrum, by a succession of cofiber sequences of self-maps (see [BEa], the last one of which is

$$\Sigma^5 A_1 \wedge C\nu \xrightarrow{w} A_1 \wedge C\nu \longrightarrow Z.$$

In a quest to leverage the knowledge of $\pi_* L_{K(2)}Z$ to $\pi_* L_{K(2)}S^0$, one must first attempt to compute the $K(2)$ -local homotopy groups of $A_1 \wedge C\nu$. Very briefly, our strategy is to use the v_2 -local tmf -based Adams spectral sequence

$$E_1^{r,t} = v_2^{-1} \pi_t(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu) \implies \pi_{t-r}(L_{K(2)}A_1 \wedge C\nu)$$

and compare it with that of Z . One can identify the E_1 -page of the above spectral sequence using the classical Adams spectral sequence

$$(1.10) \quad E_2^{s,t} = Ext_{\mathcal{A}}^{s,t}(H^*(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu), \mathbb{F}_2) \implies \pi_{t-s}(tmf \wedge \overline{tmf}^{\wedge r} \wedge A_1 \wedge C\nu).$$

Because of (1.8) and the fact that $H^*(A_1 \wedge C\nu) \cong \mathcal{A}(2) // \Lambda(\mathbb{Q}_2, \mathbb{P}_2^1)$, and the change of rings isomorphism, the E_2 -page of the spectral sequence (1.10) has the form

$$Ext_{\Lambda(\mathbb{Q}_2, \mathbb{P}_2^1)}^{s,t}(H^*(\overline{tmf}^{\wedge r}), \mathbb{F}_2)$$

Hence, computation of $\mathcal{M}(tmf^{\wedge r}, \mathbb{P}_2^1)$ is essential for understanding the E_2 -page of (1.10).

Motivation II - tmf resolution of the sphere spectrum. The connective spectrum bo is not a flat ring spectrum, hence the E_2 page of the bo -based Adams spectral sequence does not have a straightforward expression like the classical Adams spectral sequence. However, Lellmann and Mahowald [LM87] were able to calculate the d_1 differentials (also see [BBB⁺a]) and gave a description of the “ v_1 -periodic part” of the E_2 -page. They identified the free Eilenberg–MacLane summand of $bo^{\wedge r}$. To identify this free summand one needs to identify the $\mathcal{A}(1)$ free summand of

$$H^*(bo^{\wedge r}) \cong \mathcal{A} // \mathcal{A}(1)^{\otimes r}.$$

This can be done by calculating $\mathcal{M}(bo^{\wedge r}, \mathbb{Q}_0)$ and $\mathcal{M}(bo^{\wedge r}, \mathbb{Q}_1)$ and using the following theorem due to Margolis.

Theorem 1.11 ([Mar83, Chapter 19, Theorem 6]). *An $\mathcal{A}(n)$ -module K is free if and only if $\mathcal{M}(K, \mathbb{P}_i^s) = 0$ whenever $s + t \leq n + 1$ with $s < t$.*

To emulate the strategy of Lellmann and Mahowald to understand the tmf -based Adams spectral sequence for S^0 one needs to first identify the $\mathcal{A}(2)$ -free part of

$$H^*(tmf^{\wedge r}) \cong (\mathcal{A} // \mathcal{A}(2))^{\otimes r}.$$

Potentially, this can be identified using the knowledge of $\mathcal{M}(tmf^{\wedge r}, \mathbb{Q}_i)$ for $i = 0, 1, 2$ and $\mathcal{M}(tmf^{\wedge r}, \mathbb{P}_2^1)$, along with Theorem 1.11.

Motivation III - Infinite loop space of tmf . There are \mathcal{A} -modules $J(k)$, called Brown–Gitler modules [BG73], which assemble into a doubly graded \mathcal{A} -algebra, denoted here by $J(*)^*$. Moreover, there is an \mathcal{A} -module isomorphism $J(*)^* \cong \mathbb{F}_2[x_1, x_2, \dots]$ where $x_i \in J(2^i)^1$ and the left \mathcal{A} action on $J(*)^*$ is [Sch94]

$$\text{Sq}(x_i) = x_i + x_{i-1}^2.$$

In fact, $J(k)^*$ can be thought of as inheriting this action by virtue of being a subobject of \mathcal{A} . Because of this, minor modifications to methods of this paper apply to the calculation of $\mathcal{M}(J(k), \mathbb{P}_2^1)$. By [KM13] there is a spectral sequence, obtained by studying Goodwillie towers, relating the knowledge of $H_*(tmf; \mathbb{F}_2)$ to that of $H_*(\Omega^\infty tmf; \mathbb{F}_2)$ (also see [HM16] which provides a spectral sequence relating the cohomology of tmf to the cohomology of its infinite loop-space $H^*(\Omega^\infty tmf; \mathbb{F}_2)$). Roughly speaking, this relies on computing certain derived functors, usually labeled Ω_s^∞ , in the category of unstable modules over \mathcal{A} . It turns out that there is an isomorphism (see [Goe86] or [HK00])

$$\Omega_s^\infty \Sigma^{-t} (\mathcal{A} // \mathcal{A}(2))_* \cong \text{Ext}_{\mathcal{A}(2)}^{s,t}(\mathbb{F}_2, J(*)^*),$$

so that these computations require an understanding of the $J(k)$ as modules over $\mathcal{A}(2)$, the hardest part of which is understanding how \mathbb{P}_2^1 acts.

Organization of the paper. In Section 2, we recall some facts about the Steenrod algebra and its dual. We introduce the spectral sequence (2.14), which computes the \mathbb{P}_2^1 Margolis homology of tmf , and discuss the d_0 differentials in it.

In Section 3, we compute the $E_3 = E_\infty$ page of the spectral sequence (2.14). We do that by introducing building blocks M_J and computing $\mathcal{M}(M_J, \mathbb{P}_2^1)$. Then we establish the relationship between $\mathcal{M}(tmf, \mathbb{P}_2^1)$ and $\mathcal{M}(M_J, \mathbb{P}_2^1)$ in Theorem 3.18.

In Section 4, we show how to apply the same methods to calculate \mathbb{P}_2^1 Margolis homology for $tmf^{\wedge r}$ and $(B\mathbb{Z}/2^{\times k})_+$. Theorem 3.18 essentially gives complete answer in these cases.

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2. ACTION OF \mathbb{P}_2^1 AND THE LENGTH SPECTRAL SEQUENCE

The dual Steenrod algebra $\mathcal{A}_* = \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$ has the structure of a graded commutative algebra which Milnor [Mil58] showed to be a polynomial algebra

$$\mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots]$$

where $|\xi_i| = 2^i - 1$. Milnor defined $\text{Sq}(r_1, r_2, \dots) \in \mathcal{A}$ as the dual of $\xi_1^{r_1} \xi_2^{r_2} \dots$ and showed that they form an \mathbb{F}_2 basis of the Steenrod algebra \mathcal{A} , known as the Milnor basis. The \mathbb{P}_t^s elements are defined as

$$\mathbb{P}_t^s = \text{Sq}(r_1, \dots), \text{ where } r_i = \begin{cases} 0, & i \neq t \\ 2^s, & i = t. \end{cases}$$

The action of an element $a \in \mathcal{A}$ on an \mathcal{A} -algebra follows the product rule given by the Cartan formula, i.e.

$$a(x \cdot y) = \sum_i a'_i(x) \cdot a''_i(y).$$

where $\Delta(a) = \sum_i a'_i \otimes a''_i$ is the comultiplication in the Hopf algebra \mathcal{A} .

Remark 2.1. We would like to note that standard commonly used notation for the generators of the dual Steenrod algebra at $p = 2$ differs from the notation in the original paper [Mil58], and we are grateful to John Rognes for explaining this to us. In [Mil58, Appendix 1], Milnor denotes the polynomial generators of the dual Steenrod algebra at $p = 2$ by ζ_i , so that $\mathcal{A}_* \cong \mathbb{F}_2[\zeta_1, \zeta_2, \dots]$ and defines $\text{Sq}(r_1, r_2, \dots)$ as dual to the element $\zeta_1^{r_1} \zeta_2^{r_2} \dots$. It has since become standard in the literature [MT68, Ada74, Mar83] to use a different notation and to denote the polynomial generators which were denoted by ζ_i in [Mil58, Appendix 1] by ξ_i , in order to match the notation for the odd primary Steenrod algebra. Hence in current standard notation $\text{Sq}(r_1, r_2, \dots)$ is dual to $\xi_1^{r_1} \xi_2^{r_2} \dots$. The symbol ζ_i is now usually used to denote the image of ξ_i under the antipode of the Hopf algebra $\chi : \mathcal{A}_* \rightarrow \mathcal{A}_*$, induced by the ‘flip map’ on $H\mathbb{F}_2 \wedge H\mathbb{F}_2$. The elements $\zeta_i = \chi(\xi_i)$ can be computed recursively using the formula $\sum_{i+j=k} \xi_i^{2^j} \chi(\xi_j) = 0$, together with the assumption that $\xi_0 = 1$ and $\xi_i = 0$ when $i < 0$.

The homology of tmf is the subalgebra of \mathcal{A}_* ([HM14], [Mat16, Theorem 5.13])

$$(2.2) \quad \mathfrak{T} := H_*(tmf; \mathbb{F}_2) \cong (\mathcal{A} // \mathcal{A}(2))_* = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots].$$

Thus the action of \mathcal{A} on \mathfrak{T} is simply the restriction of the action of \mathcal{A} on \mathcal{A}_* .

The right action of \mathcal{A} on \mathcal{A}_* is determined by the action of the total squaring operation $\text{Sq} = 1 + \sum_{i>0} \text{Sq}^i$ [Pea14, Lemma 3.6]

$$(2.3) \quad (\zeta_i) \text{Sq} = \zeta_i + \zeta_{i-1}^2 + \zeta_{i-2}^4 + \cdots + \zeta_1^{2^{i-1}} + 1$$

which is a ring homomorphism.

Remark 2.4 (Action of the total squaring operation). There are multiple ways to define the action of \mathcal{A} on \mathcal{A}_* . While we will be using the action defined by (2.3), we would like to collect other commonly used actions here. By [Mah81], the right and left actions of Sq on ξ_i are given by the formulas

$$\begin{aligned} \text{Sq}(\xi_i) &= \xi_i + \xi_{i-1}^2 \\ (\xi_i) \text{Sq} &= \xi_i + \xi_{i-1}, \end{aligned}$$

while the left action on ζ_i is

$$\text{Sq}(\zeta_i) = \zeta_i + \zeta_{i-1} + \cdots + \zeta_1 + 1.$$

From these formulas we can derive

$$\begin{aligned} \mathbf{Q}_{i-1}(\xi_n) &= \xi_{n-i}^{2^i} \\ (\zeta_n) \mathbf{Q}_{i-1} &= \zeta_{n-i}^{2^i}; \end{aligned}$$

the second equation can also be found in [Cul].



Important Notation 2.5. Since we only work with the right action of Sq in this paper, we will write $a(x)$ to denote the *right* action of $a \in \mathcal{A}$ on $x \in H_*(tmf)$ for the rest of the paper. Thus, from now on

$$a(x) := (x)a.$$

We now focus on the action of $\mathbf{P}_2^1 = \text{Sq}(0, 2) = \text{Sq}^2 \text{Sq}^4 + \text{Sq}^4 \text{Sq}^2$ on \mathfrak{X} . From (2.3), one can easily see that Sq^{2^i} acts trivially on ζ_n , when $i > 0$ and $n \neq 1$. It follows immediately that

$$\mathbf{P}_2^1(\zeta_i) = 0.$$

Beware! This *does not* mean that $\mathbf{P}_2^1(\zeta_i \zeta_j) = 0$, as the Leibniz rule does not hold. Since $\Delta(\mathbf{P}_2^1) = \mathbf{P}_2^1 \otimes 1 + \mathbf{Q}_1 \otimes \mathbf{Q}_1 + 1 \otimes \mathbf{P}_2^1$, we obtain the product formula

$$(2.6) \quad \mathbf{P}_2^1(xy) = \mathbf{P}_2^1(x)y + \mathbf{Q}_1(x) \mathbf{Q}_1(y) + x \mathbf{P}_2^1(y).$$

Using $\mathbf{Q}_1(\zeta_i) = \zeta_{i-2}^4$, we get

$$(2.7) \quad \mathbf{P}_2^1(\zeta_i \zeta_j) = \zeta_{i-2}^4 \zeta_{j-2}^4 \quad \mathbf{P}_2^1(\zeta_i^2) = \zeta_{i-2}^8.$$

Formulas become more complicated for triple products, e.g.

$$\mathbf{P}_2^1(\zeta_i \zeta_j \zeta_k) = \zeta_{i-2}^4 \zeta_{j-2}^4 \zeta_k + \zeta_{i-2}^4 \zeta_j \zeta_{k-2}^4 + \zeta_i \zeta_{j-2}^4 \zeta_{k-2}^4,$$

and in general we have the following result.

Lemma 2.8. *The action of \mathbf{P}_2^1 on \mathfrak{X} is given by the formula*

$$\begin{aligned} \mathbf{P}_2^1(\zeta_{i_1} \cdots \zeta_{i_n}) &= \sum_{1 \leq j < k \leq n} \frac{\zeta_{i_1} \cdots \zeta_{i_n}}{\zeta_{i_j} \zeta_{i_k}} \mathbf{Q}_1(\zeta_{i_j}) \mathbf{Q}_1(\zeta_{i_k}) \\ &= \sum_{1 \leq j < k \leq n} \zeta_{i_1} \cdots \zeta_{i_{j-1}} \zeta_{i_j-2}^4 \zeta_{i_{j+1}} \cdots \zeta_{i_{k-1}} \zeta_{i_k-2}^4 \zeta_{i_{k+1}} \cdots \zeta_{i_n}, \end{aligned}$$

where indices are allowed to repeat.

Proof. Follows from an inductive argument on n , using (2.6) and the facts that $P_2^1(\zeta_i) = 0$ and $Q_1(\zeta_i) = \zeta_{i-2}^4$. \square

The technique developed in this paper begins with the following observation. Consider the subalgebra

$$\mathcal{E} := \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4^2, \zeta_5^2, \dots] \subset \mathfrak{T} = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \dots]$$

which we will call the *even* subalgebra of \mathfrak{T} , as every element in \mathcal{E} has even grading. Since $|Q_1| = 3$ and every element in \mathcal{E} has even grading, Q_1 must act trivially on \mathcal{E} . Thus, P_2^1 restricted to \mathcal{E} follows the Leibniz rule, hence $\mathcal{M}(\mathcal{E}^{\otimes r}, P_2^1)$ is an algebra. Using (2.7) and the Künneth isomorphism for a derivation, we can easily deduce the following result.

Lemma 2.9. *The P_2^1 Margolis homology of \mathcal{E} is given by*

$$\mathcal{M}(\mathcal{E}, P_2^1) \cong \Lambda(\zeta_2^4, \zeta_3^4, \zeta_4^4, \dots).$$

Moreover

$$\mathcal{M}(\mathcal{E}^{\otimes r}, P_2^1) \cong \mathcal{M}(\mathcal{E}, P_2^1)^{\otimes r} \cong (\Lambda(\zeta_2^4, \zeta_3^4, \zeta_4^4, \dots))^{\otimes r}.$$

Now consider the quotient $\mathcal{K} := \mathfrak{T} // \mathcal{E} \cong \mathbb{F}_2 \otimes_{\mathcal{E}} \mathfrak{T}$. We have an isomorphism $\mathcal{K} \cong \Lambda(\zeta_4, \zeta_5, \dots)$, and the induced action of Q_1 and P_2^1 on \mathcal{K} is trivial. For a set A , we let $\mathbb{F}_2\langle A \rangle$ denote the \mathbb{F}_2 -vector space which has the generating set A . The algebra \mathcal{K} admits a natural increasing filtration

$$G^p(\mathcal{K}) := \mathbb{F}_2\langle \zeta_{i_1} \dots \zeta_{i_k} \mid k \leq p \rangle.$$

induced by the length of the monomials. We call it the *length filtration*.

This length filtration on \mathcal{K} induces an increasing filtration $\{G^p(\mathfrak{T})\}_{p \geq 0}$ on \mathfrak{T} , where $G^p(\mathfrak{T})$ is the pullback of $G^p(\mathcal{K})$ (in vector spaces) along the quotient map $\mathfrak{T} \rightarrow \mathcal{K}$

$$\begin{array}{ccc} G^p(\mathfrak{T}) & \longrightarrow & \mathfrak{T} \\ \downarrow & & \downarrow \\ G^p(\mathcal{K}) & \longrightarrow & \mathcal{K}. \end{array}$$

Definition 2.10. Let I be a finite tuple of natural numbers, and for $I = \{i_1, \dots, i_n\}$ let ζ^I denote the monomial $\zeta_1^{i_1} \dots \zeta_n^{i_n}$. Then the *length* L of ζ^I is defined by

$$L(\zeta^I) = \sum_{j=1}^{|I|} (i_j \bmod 2).$$

In other words, $L(\zeta^I)$ counts the number of odd exponents in ζ^I . Then $G^p(\mathfrak{T})$ is the span of monomials ζ^I of length less than or equal to p

$$G^p(\mathfrak{T}) \cong \mathbb{F}_2\langle \zeta^I \mid L(\zeta^I) \leq p \rangle.$$

The length function L measures “how far” a given monomial in \mathfrak{T} is from the even subalgebra \mathcal{E} . Since there is an \mathbb{F}_2 -vector space isomorphism

$$\mathfrak{T} \cong \mathcal{E} \otimes \mathfrak{T} // \mathcal{E} = \mathcal{E} \otimes \mathcal{K}$$

any monomial $m \in \mathfrak{T}$ can be uniquely written as $e \cdot k$ where $e \in \mathcal{E}$ and $k \in \mathcal{K}$.

Example 2.11. If $m = \zeta_3^4 \zeta_5^5 \zeta_8^3$, then there is a unique expression $m = e \cdot k$, where $e = \zeta_3^4 \zeta_5^4 \zeta_8^2 \in \mathcal{E}$ and $k = \zeta_5 \zeta_8 \in \mathcal{K}$.

The following lemma shows that the action of Q_1 and P_2^1 preserves the length filtration.

Lemma 2.12. *Let $m \in \mathfrak{T}$ be any monomial.*

- (i) *If $m \in \mathcal{E}$, then $Q_1(m) = 0$ and $P_2^1(m) \in \mathcal{E}$.*
- (ii) *If $m \notin \mathcal{E}$, then $Q_1(m) \in G^{L(m)-1}(\mathfrak{T})/G^{L(m)-2}(\mathfrak{T})$ and*

$$P_2^1(m) = m_L + m_{L-2},$$

where $m_L \in G^{L(m)}(\mathfrak{T})/G^{L(m)-1}(\mathfrak{T})$ and $m_{L-2} \in G^{L(m)-2}(\mathfrak{T})/G^{L(m)-3}(\mathfrak{T})$.

Proof. When $m \in \mathcal{E}$, $Q_1(m) = 0$ by the Leibniz rule. Using Lemma 2.8 we have $P_2^1(m) \in \mathcal{E}$ and $L(P_2^1(m)) = L(m) = 0$.

Now assume $m \notin \mathcal{E}$, which means $m = e \cdot k$ for some $e \in \mathcal{E}$ and some $1 \neq k \in \mathcal{K}$. Note that k is of the form $\zeta_{i_1} \dots \zeta_{i_n}$, where indices do not repeat.

The action of Q_1 is given by the formula

$$Q_1(\zeta_{i_1} \dots \zeta_{i_n}) = \sum_{k=1}^n \zeta_{i_1} \dots \zeta_{i_{k-1}} \zeta_{i_k}^4 \zeta_{i_{k+1}} \dots \zeta_{i_n}$$

where we allow repetition of indices. Since Q_1 acts trivially on \mathcal{E} , it follows that

$$Q_1(e \cdot k) = e \cdot Q_1(k).$$

From the formula above we see that $Q_1(k) \neq 0$ and $L(Q_1(k)) = L(k) - 1$. Hence,

$$L(Q_1(m)) = L(e \cdot Q_1(k)) = L(Q_1(k)) = L(k) - 1 = L(e \cdot k) - 1 = L(m) - 1.$$

Next, note that

$$P_2^1(m) = P_2^1(e) \cdot k + Q_1(e) \cdot Q_1(k) + e \cdot P_2^1(k) = P_2^1(e) \cdot k + e \cdot P_2^1(k)$$

From Lemma 2.8, we see that $L(P_2^1(k)) = L(P_2^1(k)) - 2$ assuming $P_2^1(k) \neq 0$. Now set $m_L = P_2^1(e) \cdot k$ and $m_{L-2} = e \cdot P_2^1(k)$ \square

Lemma 2.13. *The Hopf algebra $\Lambda(Q_1, P_2^1)$ is commutative and cocommutative.*

Proof. Commutativity follows from the fact that P_2^1 and Q_1 commute [AM71, Lemma 1.3(2)] (in the notation of [AM71], $P_2^1 = P_2(2)$ and $Q_1 = P_2(1)$). Cocommutativity follows from the fact that the diagram

$$\begin{array}{ccc} \Lambda(Q_1, P_2^1) & \xrightarrow{\Delta} & \Lambda(Q_1, P_2^1) \otimes \Lambda(Q_1, P_2^1) \\ & \searrow \Delta & \downarrow \text{flip} \\ & & \Lambda(Q_1, P_2^1) \otimes \Lambda(Q_1, P_2^1) \end{array}$$

commutes, because of (1.3) and (1.4). \square

If M is a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module then let \mathcal{C}_M^\bullet denote the periodic chain complex

$$\dots \xrightarrow{\mathbb{P}_2^1} M \xrightarrow{\mathbb{P}_2^1} M \xrightarrow{\mathbb{P}_2^1} \dots$$

Its homology groups are isomorphic in each degree, i.e.

$$H_i(\mathcal{C}_M^\bullet) \cong H_j(\mathcal{C}_M^\bullet)$$

for all $i, j \in \mathbb{Z}$. We use $\mathcal{M}(M, \mathbb{P}_2^1)$ to denote this common homology group. When $M = \mathfrak{T}$, the filtration $G^\bullet(\mathfrak{T})$ induces a filtration on $\mathcal{C}_\mathfrak{T}^\bullet$. By Lemma 2.12, \mathbb{P}_2^1 respects the length filtration. This means we have a short exact sequence of chain complexes

$$0 \longrightarrow \bigoplus_{p \in \mathbb{Z}} G^{p-1}(\mathcal{C}_\mathfrak{T}^\bullet) \longrightarrow \bigoplus_{p \in \mathbb{Z}} G^p(\mathcal{C}_\mathfrak{T}^\bullet) \longrightarrow \bigoplus_{p \in \mathbb{Z}} \frac{G^p(\mathcal{C}_\mathfrak{T}^\bullet)}{G^{p-1}(\mathcal{C}_\mathfrak{T}^\bullet)} \longrightarrow 0.$$

Upon taking the homology, this short exact sequence of chain complexes produces an exact couple, resulting in a spectral sequence

$$E_1^{p,q} := H^q \left(\frac{G^p(\mathcal{C}_\mathfrak{T}^\bullet)}{G^{p-1}(\mathcal{C}_\mathfrak{T}^\bullet)} \right) \Rightarrow H^q(\mathcal{C}_\mathfrak{T}^\bullet).$$

We rewrite this spectral sequence as

$$(2.14) \quad E_1^p := \mathcal{M} \left(\frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})}, \mathbb{P}_2^1 \right) \Rightarrow \mathcal{M}(tmf, \mathbb{P}_2^1).$$

and we call it the *length spectral sequence*.

The E_1 page of (2.14) is easy to calculate. Note that the associated graded

$$\bigoplus_{p \geq 0} \frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})} \cong \mathcal{E} \otimes \mathcal{K}$$

as an \mathbb{F}_2 -algebra. The action of $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ on $\mathcal{E} \otimes \mathcal{K}$ is defined using the Cartan formula as in the definition below.

Definition 2.15 ([Mar83], p.186). Let Γ be any Hopf algebra. For two Γ -modules M and N , the underlying \mathbb{F}_2 vector space of $M \otimes N$ is simply $M \otimes_{\mathbb{F}_2} N$, and Γ acts via the diagonal map, i.e.

$$a(m \otimes n) = \sum_i a_i(m) \otimes a'_i(n).$$

where $a \in \Gamma$ and $\Delta(a) = \sum_i a_i \otimes a'_i$, where Δ is the coproduct of the Hopf algebra.

Now we describe the action of \mathbb{P}_2^1 on a monomial $m \in \bigoplus_{p \geq -1} \frac{G^p(\mathfrak{T})}{G^{p-1}(\mathfrak{T})}$. Write $m = e \otimes k$ for some $e \in \mathcal{E}$ and $k \in \mathcal{K}$. By Definition 2.15

$$\mathbb{P}_2^1(m) = \mathbb{P}_2^1(e \otimes k) = \mathbb{P}_2^1(e) \otimes k.$$

Since the length filtration $G^\bullet(\mathfrak{T})$ is multiplicative, i.e.

$$G^p(\mathfrak{T}) \cdot G^{p'}(\mathfrak{T}) \subset G^{p+p'}(\mathfrak{T}),$$

and \mathbb{P}_2^1 restricted to \mathcal{E} follows the Leibniz rule, the E_1 page of (2.14) is an \mathbb{F}_2 -algebra and isomorphic to

$$E_1^* \cong \mathcal{M}(\mathcal{E} \otimes \mathcal{K}, \mathbb{P}_2^1) \cong \mathcal{M}(\mathcal{E}, \mathbb{P}_2^1) \otimes \mathcal{K} \cong \Lambda(\zeta_2^4, \zeta_3^4, \dots) \otimes \Lambda(\zeta_4, \zeta_5, \dots).$$

In order to avoid confusion regarding the multiplicative structure of E_1^* , it is convenient to rename the generators.

Notation 2.16. We set $x_i := \zeta_{i+3}$ and $t_i := \zeta_{i+1}^4$. Further, for finite subsets $I = \{i_1, \dots, i_n\} \subset \mathbb{N}$ and $J = \{j_1, \dots, j_m\} \subset \mathbb{N}$, we let t_I and x_I denote the monomials $t_{i_1} \dots t_{i_n}$ and $x_{j_1} \dots x_{j_m}$ respectively. We use $t_I x_J$ to denote the tensor product $t_I \otimes x_J$.

Lemma 2.12 and Lemma 2.13 imply that we have a commutative diagram of chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_p G^p(C_{\mathfrak{I}}^\bullet) & \longrightarrow & \bigoplus_p G^{p+1}(C_{\mathfrak{I}}^\bullet) & \longrightarrow & \bigoplus_p \frac{G^{p+1}(C_{\mathfrak{I}}^\bullet)}{G^p(C_{\mathfrak{I}}^\bullet)} \longrightarrow 0 \\ \parallel & & \downarrow \mathbf{Q}_1(\cdot) & & \downarrow \mathbf{Q}_1(\cdot) & & \downarrow \mathbf{Q}_1(\cdot) \\ 0 & \longrightarrow & \bigoplus_p G^{p-1}(C_{\mathfrak{I}}^\bullet) & \longrightarrow & \bigoplus_p G^p(C_{\mathfrak{I}}^\bullet) & \longrightarrow & \bigoplus_p \frac{G^p(C_{\mathfrak{I}}^\bullet)}{G^{p-1}(C_{\mathfrak{I}}^\bullet)} \longrightarrow 0 \end{array}$$

Consequently there is an action of \mathbf{Q}_1 on each page of (2.14), which shifts the length filtration by -1 . In particular, we note $\mathbf{Q}_1(x_i) = t_i$ and in general

$$(2.17) \quad \mathbf{Q}_1(t_I x_J) = \sum_{j \in J} t_j t_I x_{J-\{j\}}.$$

Let $m \in \mathfrak{I}$ be any monomial, m_L and m_{L-2} be as in Lemma 2.12, and let $[m]$ denote the equivalence class in the E_1 page of (2.14) represented by m . Lemma 2.12 implies that the d_1 differential of (2.14) is trivial,

$$d_2([m]) = [m_{L-2}]$$

for the class of the monomial $m \in \mathfrak{I}$ in the E_1 page, and the spectral sequence (2.14) collapses at the E_3 page. If we write $m \in \mathfrak{I}$ as $m = e \cdot k$, where $e \in \mathcal{E}$ and $k \in \mathcal{K}$, then

$$d_2([m]) = [e \cdot \mathbf{P}_2^1(k)] = [e] \cdot [\mathbf{P}_2^1(k)].$$

This means that the d_2 differential of (2.14) is $\mathcal{M}(\mathcal{E}, \mathbf{P}_2^1)$ -linear. It follows from the formula of Lemma 2.8 that

$$(2.18) \quad d_2(t_I x_J) = \sum_{K \in J[2]} t_K t_I x_{J-K},$$

where $J[2]$ is the set of subsets of J which contain two elements.

The formula for the d_2 differentials is intimately related to the action of \mathbf{Q}_1 on the E_2 page of (2.14). The $\Lambda(\mathbf{Q}_1)$ -module structure on E_2^\bullet (see (2.17)) can extend it to the $\Lambda(\mathbf{Q}_1, \mathbf{P}_2^1)$ -module structure using the algebra structure of E_2^\bullet and the product formula (2.6), along with the initial condition

$$(2.19) \quad \mathbf{P}_2^1(x_i) = \mathbf{P}_2^1(t_i) = 0.$$

The action of \mathbf{P}_2^1 that results from this procedure is

$$(2.20) \quad \mathbf{P}_2^1(t_I x_J) = \sum_{K \in J[2]} t_K t_I x_{J-K}$$

on the monomial basis, which can be extended to all of E_2^\bullet using \mathbb{F}_2 -linearity. Notice that the action we obtain through this process coincides with the formula for the d_2 differentials (2.18).

3. THE REDUCED LENGTH

For convenience, we denote the E_2 -page of (2.14) by

$$\mathcal{R} = \Lambda(t_i : i \geq 1) \otimes \Lambda(x_i : i \geq 1),$$

which is an \mathbb{F}_2 -algebra, as well as a $\Lambda(Q_1, P_2^1)$ -module, where the actions of Q_1 and P_2^1 are given by (2.17) and (2.20) respectively. In this section we analyze the $\Lambda(Q_1, P_2^1)$ -module structure of \mathcal{R} , which leads us to a description of

$$E_\infty^\bullet \cong \dots \cong E_3^\bullet \cong H(E_2^\bullet, d_2) \cong \mathcal{M}(\mathcal{R}, P_2^1).$$

The main idea here is to notice that the action of P_2^1 is linear with respect to the subalgebra

$$\mathcal{S} := \Lambda(t_i x_i | i \in \mathbb{N}_+) \subset \mathcal{R},$$

which implies that $\mathcal{M}(\mathcal{R}, P_2^1)$ admits an \mathcal{S} -module structure.

Lemma 3.1. *The subalgebra $\mathcal{S} \subset \mathcal{R}$ is a trivial $\Lambda(Q_1, P_2^1)$ -submodule which splits off as a $\Lambda(Q_1, P_2^1)$ -module.*

Proof. For any element $t_I x_I \in \mathcal{S}$, it is clear from (2.17) and (2.18) that

$$Q_1(t_I x_I) = 0 = P_2^1(t_I x_I).$$

Thus \mathcal{S} is a trivial submodule.

Now observe from (2.17) and (2.18) that none of the monomials $t_I x_I \in \mathcal{S}$ is a summand of $Q_1(t_{I'} x_{J'})$ or $P_2^1(t_{I'} x_{J'})$ for any choice of I' and J' . Hence, \mathcal{S} is a split summand. \square

Corollary 3.2. *Every element of \mathcal{S} is a nonzero cycle in the $\mathcal{M}(\mathcal{R}, P_2^1)$.*

Lemma 3.3. *The action of P_2^1 on \mathcal{R} is \mathcal{S} -linear.*

Proof. It is enough to show that

$$(3.4) \quad P_2^1(t_i x_i \cdot t_I x_J) = (t_i x_i) \cdot P_2^1(t_I x_J).$$

If $i \in I$, then $t_i t_I = 0$. Hence both the LHS and the RHS of (3.4) are zero.

If $i \in J$, then $x_i x_J = 0$, hence LHS of (3.4) is zero. On the other hand,

$$\begin{aligned} RHS &= t_i x_i \cdot \sum_{K \in J[2]} t_K t_I x_{J-K} \\ &= \sum_{i \in K \in J[2]} t_i t_K t_I x_i x_{J-K} + t_i \cdot \left(\sum_{i \notin K \in J[2]} t_i t_I x_i x_{J-K} \right) = 0, \end{aligned}$$

as $t_i t_K = 0$ when $i \in K$ and $x_i x_{J-K} = 0$ when $i \notin K$.

Now consider the case when $i \notin I \cup J$. Let $I' = I \cup \{i\}$ and $J' = J \cup \{i\}$. Then,

$$\begin{aligned}
P_2^1(t_i x_i \cdot t_I x_J) = P_2^1(t_{I'} x_{J'}) &= \sum_{K \in J'[2]} t_K t_{I'} x_{J'-K} \\
&= \sum_{i \in K \in J'[2]} t_K t_{I'} x_{J'-K} + \sum_{i \notin K \in J'[2]} t_K t_{I'} x_{J'-K} \\
&= \sum_{i \notin K \in J'[2]} t_K t_{I'} x_{J'-K} \\
&= t_i x_i \cdot \sum_{K \in J[2]} t_K t_I x_{J-K} \\
&= t_i x_i \cdot P_2^1(t_I x_J). \quad \square
\end{aligned}$$

Remark 3.5. While the E_2 page of (2.14) admits an \mathbb{F}_2 -algebra structure, the E_3 page does not admit any multiplicative structure. This is because the d_2 differentials do not follow Leibniz rule and the product of d_2 cycles may not be a cycle. For example, x_i for all $i \in \mathbb{N}_+$, is a d_2 -cycle, whereas $x_i x_j$ for $i \neq j$ supports a differential $d_2(x_i x_j) = t_i t_j$ by (2.18). Even if α, β and $\alpha \cdot \beta$ are P_2^1 cycles it is unclear that the pairing $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ is well-defined in the E_3 page.

Corollary 3.6. $\mathcal{M}(\mathcal{R}, P_2^1)$ is a module over the ring \mathcal{S} .

Proof. By Lemma 3.3, there exists a pairing $\mu : \mathcal{S} \otimes \mathcal{R} \rightarrow \mathcal{R}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{S} \otimes \mathcal{R} & \xrightarrow{\mu} & \mathcal{R} \\
1 \otimes P_2^1 \downarrow & & \downarrow P_2^1 \\
\mathcal{S} \otimes \mathcal{R} & \xrightarrow{\mu} & \mathcal{R}.
\end{array}$$

commutes. It follows that $\mathcal{M}(\mathcal{R}, P_2^1)$ is an \mathcal{S} module. \square

As a result, we only need to understand the action of P_2^1 on the generators of \mathcal{R} when viewed as an \mathcal{S} -module. In order to approach this problem we introduce the notion of *reduced length*.

Definition 3.7. For any monomial $t_I x_J \in \mathcal{R}$ the *reduced length* ℓ is

$$\ell(t_I x_J) = |J - I| = |J \cap I^c| = |J| - |J \cap I|,$$

where I^c denotes the complement of I .

Note that the length of $t_I x_J \in \mathcal{R}$ is given by the formula $L(t_I x_J) = |J|$; in other words, it is counting the number of factors of x_J . Whereas, $\ell(t_I x_J)$ counts only those factors x_j in x_J for which t_j is not a factor of t_I . For example,

$$\ell(x_1) = \ell(t_1 x_1 x_2) = \ell(t_1 t_2 x_1 x_2 x_3) = \ell(t_1 t_2 t_3 x_4) = 1$$

$$\ell(x_1 x_2) = \ell(t_1 x_1 x_2 x_3) = \ell(t_1 t_2 t_3 t_4 x_5 x_6) = 2.$$

Remark 3.8. The reduced length function ℓ measures “how far” a given monomial in \mathcal{R} is from the subalgebra \mathcal{S} .

For each $i \in \mathbb{N}_+$, let $M_i := \Lambda(\mathbb{Q}_1)\{x_i\} \subset \mathcal{R}$ denote the $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -submodule isomorphic to $\Lambda(\mathbb{Q}_1)$ and generated by x_i . For an indexing set $K \subset_{\text{finite}} \mathbb{N}_+$, let

$$M_K := \bigotimes_{j \in K} M_j \subset \mathcal{R}$$

with the convention that $M_\emptyset := \mathbb{F}_2$. If the indexing set is $[n] = \{1, \dots, n\} \subset \mathbb{N}_+$, then we write $M_{[n]}$ to denote $M_{\{1, \dots, n\}}$.

In Figure 1, Figure 2 and Figure 3 we present M_i , $M_{\{1,2\}}$ and $M_{\{1,2,3\}}$ respectively as a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module. In these figures the blue curved lines depict the action of \mathbb{Q}_1 and red boxed lines depict the action of \mathbb{P}_2^1 .

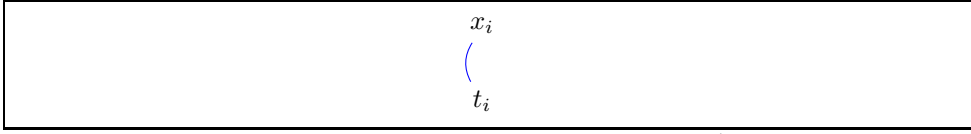


Figure 1: M_i as a module over $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$

Note that the set $\mathcal{W} := \{t_I x_J \in \mathcal{R} \mid I \cap J = \emptyset\}$ forms a generating set for \mathcal{R} as an \mathcal{S} -module as any monomial $t_I x_J \in \mathcal{R}$ can be uniquely written as a product of an element of \mathcal{W} and a monomial in \mathcal{S} :

$$t_I x_J = t_{I \cap J} x_{I \cap J} \cdot t_{I - (I \cap J)} x_{J - (I \cap J)}.$$

For any finite subset $K \subset \mathbb{N}_+$,

$$\mathcal{W}_K := \{t_I x_J \mid I \cup J = K, I \cap J = \emptyset\} \subset \mathcal{W}$$

forms an \mathbb{F}_2 -basis for M_K , i.e. $\mathbb{F}_2\langle \mathcal{W}_K \rangle = M_K$. Since

$$\mathcal{W} = \bigsqcup_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{W}_K$$

and $\mathbb{F}_2\langle \mathcal{W}_K \rangle = M_K$ is closed under the action of \mathbb{Q}_1 and \mathbb{P}_2^1 (these actions preserve the total indexing set K , by (2.17) and (2.18)), we learn that

$$\mathcal{R} // \mathcal{S} \cong \mathbb{F}_2 \otimes_{\mathcal{S}} \mathcal{R} \cong \mathcal{R} \otimes_{\mathcal{S}} \mathbb{F}_2 \cong \bigoplus_K M_K$$

is an isomorphism of $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -modules. Consequently, we have the following lemma.

Lemma 3.9. *Let $\mathcal{S}_K \subset \mathcal{S}$ denote the subalgebra $\Lambda(t_I x_I \mid I \subset \mathbb{N}_+ - K)$. There is a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module isomorphism*

$$\bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{S}_K \otimes M_K \cong \mathcal{R}.$$

Proof. Consider the \mathbb{F}_2 -vector space isomorphism

$$\iota : \mathcal{R} \longrightarrow \bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{S}_K \otimes M_K$$

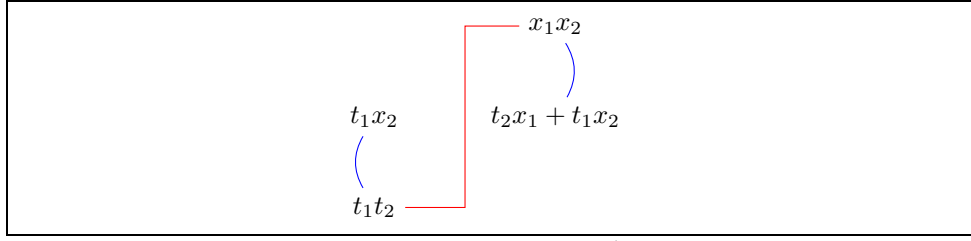


Figure 2: $M_{[2]}$ as a module over $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$, where $[2] = \{1, 2\}$

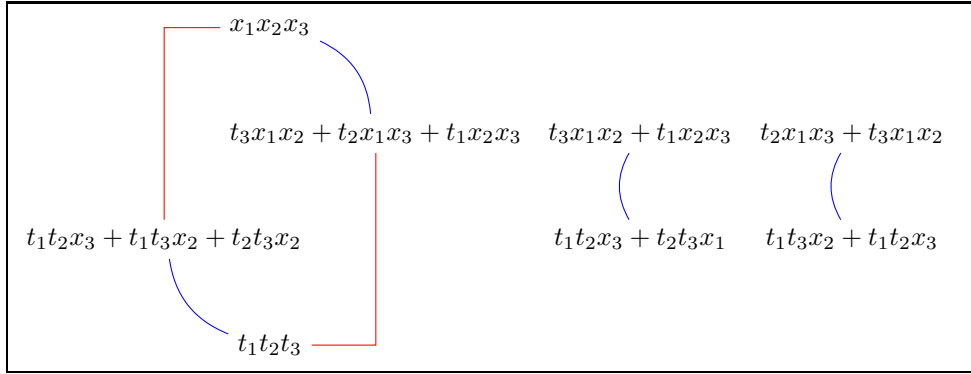


Figure 3: $M_{[3]}$ as a module over $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$, where $[3] = \{1, 2, 3\}$

which sends

$$t_I x_J \mapsto t_{I \cap J} x_{I \cap J} \otimes t_{I - (I \cap J)} x_{J - (I \cap J)} \in \mathcal{S}_K \otimes M_K,$$

where $K = I \cup J - I \cap J$. The map ι^{-1} sends

$$t_B x_B \otimes t_{K-A} x_A \mapsto t_{B \cup (K-A)} \cdot x_{B \cup A},$$

where $A \subset K$. This map is also a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module isomorphism as $\mathcal{S}_K \subset \mathcal{S}$ is a trivial $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module by Lemma 3.1. \square

Hence we can reduce the problem of computing $\mathcal{M}(\mathcal{R}, \mathbb{P}_2^1)$ to computing $\mathcal{M}(M_K, \mathbb{P}_2^1)$ for various finite subsets K of \mathbb{N}_+ . Thus, we first need to understand the structure of M_K as a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module.

Remark 3.10. Let $[n]$ denote the indexing set $\{1, \dots, n\} \subset \mathbb{N}_+$. If $|K| = n$, then there exists the unique order preserving bijection

$$\iota : [n] \longrightarrow K$$

and it induces an isomorphism $\iota : M_{[n]} \xrightarrow{\cong} M_K$. Thus it is enough to understand $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module structure of $M_{[n]}$ for all $n \in \mathbb{N}_+$.

As depicted in Figure 3, $M_{[3]}$ splits as a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module

$$(3.11) \quad M_{[3]} \cong \begin{aligned} & \Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)\{x_1x_2x_3\} \\ & \oplus \Lambda(\mathbb{Q}_1)\{t_3x_1x_2 + t_1x_2x_3\} \\ & \oplus \Lambda(\mathbb{Q}_1)\{t_2x_1x_3 + t_3x_1x_2\} \end{aligned}$$

as summand of a free $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module and two copies of $\Lambda(\mathbb{Q}_1)$.

Remark 3.12. The splitting of (3.11) is a consequence of Lemma 2.13. Since $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ is cocommutative, for any $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module M and $\sigma \in \mathbb{F}_2[\Sigma_n]$, the induced map

$$\sigma : M^{\otimes n} \longrightarrow M^{\otimes n}$$

is a map of $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -modules. Note that in the group ring $\mathbb{F}_2[\Sigma_3]$, the identity element can be written as a sum of idempotent elements

$$\mathbb{1} = e + f_1 + f_2.$$

For example, one can choose $e = \mathbb{1} + (1\ 2\ 3) + (1\ 2\ 3)$, $f_1 = \mathbb{1} + (1\ 2) + (1\ 3) + (1\ 3\ 2)$ and $f_2 = \mathbb{1} + (1\ 2) + (1\ 3) + (1\ 2\ 3)$. Then we have

$$M^{\otimes 3} \cong e(M^{\otimes 3}) \oplus f_1(M^{\otimes 3}) \oplus f_2(M^{\otimes 3}).$$

When $M \cong \Lambda(\mathbb{Q}_1)$, we get the decomposition of of (3.11).

The splitting of (3.11), along with the following fact about finite dimensional Hopf algebras, is the key to understanding the structure of M_K .

Theorem 3.13 ([NZ89]). *If \mathcal{H} is a finite dimensional connected Hopf algebra over a field \mathbb{F} , then for any \mathcal{H} -module M , $\mathcal{H} \otimes M$ is a free \mathcal{H} -module.*

Let us denote by A the $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module isomorphic to $\Lambda(\mathbb{Q}_1)$ and let $B := A \otimes A$.

Then using (3.11) and Theorem 3.13, we notice that

$$\begin{aligned} M_{[3]} &\cong B \otimes A \cong \{\text{Free}\} \oplus A^{\oplus 2} \\ M_{[4]} &\cong \{\text{Free}\} \oplus B^{\oplus 2} \\ M_{[5]} &\cong \{\text{Free}\} \oplus A^{\oplus 4} \end{aligned}$$

where $\{\text{Free}\}$ denotes a free $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ module. This iterative process can be continued as described in Lemma 3.14 below. We use $A\{y\}$, resp. $B\{y\}$, to specify that y generates A , resp. B , as a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ module. For example, $M_i \cong A\{x_i\}$.

Lemma 3.14. *As a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module,*

$$M_{[2r+1]} \cong \{\text{Free}\} \oplus \left(\bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\} \right),$$

where $\ell(h_{2r+1,i}) = r + 1$.

As a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ -module,

$$M_{[2r]} \cong \{\text{Free}\} \oplus \left(\bigoplus_{i=1}^{2^{r-1}} B\{h_{2r,i}\} \right),$$

where $\ell(h_{2r,i}) = r + 1$.

Proof. Our proof is by induction on r . From Figure 1, Figure 2 and Figure 3, the claim is true for $k = 1, 2, 3$. Note that

$$h_{1,1} = x_1 \quad h_{2,1} = x_1 x_2 \quad h_{3,1} = (t_3 x_1 + x_3 t_1) x_2 \quad h_{3,2} = (t_2 x_3 + t_3 x_2) x_1$$

Now assume that the result is true for $2r - 1$, i.e.

$$M_{[2r-1]} \cong \{\text{Free}\} \oplus \left(\bigoplus_{1 \leq i \leq 2^{r-1}} A\{h_{2r-1,i}\} \right)$$

where $\ell(h_{2r-1,i}) = r$ and $\{\text{Free}\}$ is a free $\Lambda(\mathbf{Q}_1, \mathbf{P}_2^1)$ -module. It follows that

$$M_{[2r]} \cong M_{[2r-1]} \otimes M_{2r} \cong (\{\text{Free}\} \otimes A\{x_{2r}\}) \oplus \bigoplus_{1 \leq i \leq 2^{r-1}} B\{h_{2r-1,i} \cdot x_{2r}\}.$$

By Theorem 3.13, the first summand is, again, a free module. Set

$$h_{2r,i} = h_{2r-1,i} \cdot x_{2r}$$

and notice $\ell(h_{2r-1,i}x_{2r}) = \ell(h_{2r-1,i}) + \ell(x_{2r}) = r + 1$.

To complete the inductive argument, observe

$$\begin{aligned} M_{[2r+1]} &\cong M_{[2r-1]} \otimes B\{x_{2r}x_{2r+1}\} \\ &\cong (\{\text{Free}\} \oplus \bigoplus_{1 \leq i \leq 2^{r-1}} A\{h_{2r-1,i}\}) \otimes B\{x_{2r}x_{2r+1}\} \\ &\cong \{\text{Free}\} \oplus \bigoplus_{1 \leq i \leq 2^{r-1}} (A\{h_{2r+1,2i-1}\} \oplus A\{h_{2r+1,2i}\}), \end{aligned}$$

where one can define the generators $h_{2r-1,j}$ from Figure 3 by replacing x_1, x_2, x_3 with $h_{2r-1,i}, x_{2r}$ and x_{2r+1} respectively. More specifically, one can define

$$\begin{aligned} h_{2r+1,2i-1} &= \mathbf{Q}_1(h_{2r-1,i} \cdot x_{2r+1}) \cdot x_{2r} \\ h_{2r+1,2i} &= h_{2r-1,i} \cdot \mathbf{Q}_1(x_{2r}x_{2r+1}). \end{aligned}$$

It is easy to check that $\ell(h_{2r+1,j}) = r + 1$. \square

Following the proof of Lemma 3.14, we can provide an explicit basis of $\mathcal{M}(M_K, \mathbf{P}_2^1)$. By Remark 3.10 it suffices to provide a basis for $\mathcal{M}(M_{[n]}, \mathbf{P}_2^1)$ for all $n \geq 1$. We do so inductively (see Definition 3.16), however we must treat the odd and the even case separately, essentially because of Lemma 3.14. Since A is a trivial $\Lambda(\mathbf{P}_2^1)$ -module, $\mathcal{M}(A, \mathbf{P}_2^1) \cong A$, and we get

$$\mathcal{M}(M_{[2r+1]}, \mathbf{P}_2^1) \cong \mathcal{M}\left(\bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}, \mathbf{P}_2^1\right) \cong \bigoplus_{i=1}^{2^r} A\{h_{2r+1,i}\}.$$

Thus the collection

$$\{h_{2r+1,i} : 1 \leq i \leq 2^r\} \cup \{\mathbf{Q}_1(h_{2r+1,i}) : 1 \leq i \leq 2^r\}$$

is an \mathbb{F}_2 -basis of $\mathcal{M}(M_{[2r+1]}, \mathbf{P}_2^1)$. When n is even, say $n = 2r$, then

$$\mathcal{M}(M_{[2r]}, \mathbf{P}_2^1) \cong \bigoplus_{i=1}^{2^{r-1}} \mathcal{M}(B\{h_{2r,i}\}, \mathbf{P}_2^1).$$

Now note that, if $B\{x \otimes y\} = A\{x\} \otimes A\{y\}$ (where x and y are generators), then

$$\{\mathbf{Q}_1(x) \otimes y, x \otimes \mathbf{Q}_1(y)\}$$

is an \mathbb{F}_2 -basis of $\mathcal{M}(B\{x \otimes y\}, \mathbf{P}_2^1)$. Using the fact that

$$h_{2r,i} = h_{2r-1,i} \cdot x_{2r},$$

we get Corollary 3.15 and Definition 3.16 thereafter.

Corollary 3.15. *Let $\mathcal{M}(M_K, \mathbb{P}_2^1)_l = \{x \in \mathcal{M}(M_K, \mathbb{P}_2^1) | \ell(x) = l\}$.*

If $|K| = 2r + 1$, then

$$\dim \mathcal{M}(M_K, \mathbb{P}_2^1)_l = \begin{cases} 2^r, & \text{if } l = r, r + 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $|K| = 2r$, then

$$\dim \mathcal{M}(M_K, \mathbb{P}_2^1)_l = \begin{cases} 2^r, & \text{if } l = r \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Lemma 3.14 implies

$$M_{[2r+1]} \cong \{\text{Free}\} \oplus \bigoplus_{1 \leq i \leq 2^r} A\{h_{2r+1,i}\}$$

where $\ell(h_{2r+1,i}) = r + 1$. By Lemma 2.12 we have $\ell(\mathbb{Q}_1(h_{2r+1,i})) = r$. Thus $\{h_{2r+1,i}\}$ is the basis for $\mathcal{M}(M_{[2r+1]}, \mathbb{P}_2^1)_{r+1}$ and $\{\mathbb{Q}_1(h_{2r+1,i})\}$ is the basis for $\mathcal{M}(M_{[2r+1]}, \mathbb{P}_2^1)_r$. Applying Remark 3.10 we deduce the statement about dimension for any M_K with $|K| = 2r + 1$.

For the even case we have from Lemma 3.14

$$M_{[2r]} \cong \{\text{Free}\} \oplus \bigoplus_{1 \leq i \leq 2^{r-1}} B\{h_{2r,i}\}$$

where $\ell(h_{2r,i}) = r + 1$. Then for each i , $\mathcal{M}(B\{h_{2r,i}\}, \mathbb{P}_2^1) = \mathcal{M}(B\{h_{2r,i}\}, \mathbb{P}_2^1)_r$ is an \mathbb{F}_2 vector space of dimension 2 generated by $\{h_{2r-1,i} \cdot t_{2r}, \mathbb{Q}_1(h_{2r-1,i}) \cdot x_{2r}\}$. \square

Definition 3.16. We define the basis $\mathcal{B}_{[n],l}$ of $\mathcal{M}(M_{[n]}, \mathbb{P}_2^1)_l$ for $0 \leq l \leq n$ inductively starting with $\mathcal{B}_{[1],0} = \{t_1\}$ and $\mathcal{B}_{[1],1} = \{x_1\}$. Suppose we have defined

$$\mathcal{B}_{[2r-1],l} := \begin{cases} \{h_{2r-1,1}, \dots, h_{2r-1,2^{r-1}}\} & \text{if } l = r \\ \{\mathbb{Q}_1(h_{2r-1,1}), \dots, \mathbb{Q}_1(h_{2r-1,2^{r-1}})\} & \text{if } l = r - 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Then define:

$$\mathcal{B}_{[2r],r} := \{h_{2r-1,1} \cdot t_{2r}, \dots, h_{2r-1,2^{r-2}} \cdot t_{2r}\} \cup \{\mathbb{Q}_1(h_{2r-1,1}) \cdot x_{2r}, \dots, \mathbb{Q}_1(h_{2r-1,2^{r-2}}) \cdot x_{2r}\}$$

and set $\mathcal{B}_{[2r],l} := \emptyset$ if $l \neq r$.

Now define $h_{2r+1,2i-1} = \mathbb{Q}_1(h_{2r-1,i}) \cdot (x_{2r+1} \cdot x_{2r})$ and $h_{2r+1,2i} = h_{2r-1,i} \cdot \mathbb{Q}_1(x_{2r} \cdot x_{2r+1})$ and set

$$\mathcal{B}_{[2r+1],l} := \begin{cases} \{h_{2r+1,1}, \dots, h_{2r+1,2^{r-2}}\} & \text{if } l = r + 1 \\ \{\mathbb{Q}_1(h_{2r+1,1}), \dots, \mathbb{Q}_1(h_{2r+1,2^{r-2}})\} & \text{if } l = r \\ \emptyset & \text{otherwise.} \end{cases}$$

We let $\mathcal{B}_{[n]}$ denote the union $\bigcup_l \mathcal{B}_{[n],l}$. Let \mathcal{B}_K denote the image of the $\mathcal{B}_{[n]}$ under the isomorphism $\iota : M_{[n]} \rightarrow M_K$ of Remark 3.10.

Example 3.17 (Examples of \mathcal{B}_K). We explicitly identify $\mathcal{B}_{[n]}$ using Definition 3.16 for $n \leq 4$, and for $n = 1, 2, 3$ we can compare to Figures 1, 2 and 3, to see that $\mathcal{B}_{[n]}$ is indeed the basis for $\mathcal{M}(M_{[n]}, \mathbb{P}_2^1)$.

- $\mathcal{B}_{[1]} = \{t_1, x_1\}$,
- $\mathcal{B}_{[2]} = \{t_1 x_2, t_2 x_1\}$,

- $\mathcal{B}_{[3]} = \{t_1 t_2 x_3 + t_1 t_3 x_2, t_1 t_2 x_3 + t_2 t_3 x_1\} \cup \{t_3 x_1 x_2 + t_2 x_1 x_3, t_3 x_1 x_2 + t_1 x_2 x_3\}$,
and,
- $\mathcal{B}_{[4]} = \{t_1 t_2 x_3 x_4 + t_1 t_3 x_2 x_4, t_1 t_2 x_3 x_4 + t_2 t_3 x_1 x_4, t_3 t_4 x_1 x_2 + t_2 t_4 x_1 x_3, t_3 t_4 x_1 x_2 + t_1 t_4 x_2 x_3\}$.

Let $P_K := \mathbb{F}_2\langle \mathcal{B}_K \rangle \subset M_K$. By construction, P_K is a trivial split $\Lambda(\mathbb{P}_2^1)$ -submodule of M_K . The splitting of P_K follows from the fact that its complement is free and the fact that $\Lambda(\mathbb{P}_2^1)$, like any other finite dimensional connected Hopf algebra, is self-injective [Mar83, Ch. 12, §2].

Theorem 3.18. *Let K be a finite subset of \mathbb{N}_+ . Let*

$$\mathcal{SB}_K := \{t_I x_I \cdot b \mid I \cap K = \emptyset \text{ and } b \in \mathcal{B}_K\} \subset \mathcal{R}.$$

Then

$$\mathcal{B} := \bigsqcup_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{SB}_K$$

forms a basis of the \mathbb{F}_2 -vector space $\mathcal{M}(tmf, \mathbb{P}_2^1)$ and

$$\mathcal{M}(tmf, \mathbb{P}_2^1) \cong \bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathbb{F}_2\langle \mathcal{SB}_K \rangle \cong \bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{S}_K \otimes \mathcal{M}(M_K, \mathbb{P}_2^1).$$

is an isomorphism of \mathbb{F}_2 -vector spaces.

Proof. By Lemma 3.9, we have a $\Lambda(\mathbb{Q}_1, \mathbb{P}_2^1)$ module isomorphism

$$\mathcal{R} \cong \bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{S}_K \otimes M_K.$$

Therefore, the linearity of the action of \mathbb{P}_2^1 (see Corollary 3.6) with respect to elements in \mathcal{S} gives us

$$\begin{aligned} \mathcal{M}(tmf, \mathbb{P}_2^1) &\cong \mathcal{M}(\mathcal{R}, \mathbb{P}_2^1) \\ &\cong \mathcal{M}\left(\bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{S}_K \otimes M_K, \mathbb{P}_2^1\right) \\ &\cong \bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{S}_K \otimes \mathcal{M}(M_K, \mathbb{P}_2^1) \\ &\cong \bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathcal{S}_K \otimes P_K \\ &\cong \bigoplus_{\substack{K \subset \mathbb{N}_+ \\ \text{finite}}} \mathbb{F}_2\langle \mathcal{SB}_K \rangle. \quad \square \end{aligned}$$

Remark 3.19. Let e denote the exchange map $e : \mathcal{R} \rightarrow \mathcal{R}$ which sends

$$e : t_I x_J \mapsto t_J x_I.$$

It seems to be the case that $[m] \in \mathcal{M}(tmf, \mathbb{P}_2^1)$ if and only if $[e(m)] \in \mathcal{M}(tmf, \mathbb{P}_2^1)$. The source of such symmetry is unclear to the authors, although it might be related to Spanier–Whitehead duality.

Finally, we would like to say a word about the module structure of $\mathcal{M}(tmf, \mathbb{P}_2^1)$ over \mathcal{S} . Note that the collection of elements

$$\mathcal{B}_{\mathcal{S}} := \{t_I x_I \mid I \subset_{\text{finite}} \mathbb{N}_+\}$$

forms an \mathbb{F}_2 -basis of \mathcal{S} . The \mathcal{S} -module structure on $\mathcal{M}(tmf, \mathbb{P}_2^1)$ is extended from a pairing at the level of bases

$$\begin{aligned} \mathcal{B}_{\mathcal{S}} \otimes \mathcal{S}\mathcal{B}_K &\xrightarrow{\mu} \mathcal{S}\mathcal{B}_K \\ s \otimes (s' \cdot b) &\mapsto \begin{cases} (s \cdot s') \cdot b, & \text{if } I \cap K = \emptyset \\ 0, & \text{if } I \cap K \neq \emptyset. \end{cases} \end{aligned}$$

Remark 3.20. Recall that $H_*(tmf)$ was described in terms of ζ_i . We can convert an element of the Margolis homology expressed in terms of t_i and x_i back to an expression involving ζ_i using the identifications of Notation 2.16. For example,

$$t_4 t_9 x_2 x_6 + t_2 t_9 x_4 x_6$$

can be identified with the class represented by element $\zeta_5^5 \zeta_{10}^4 \zeta_9 + \zeta_3^4 \zeta_{10}^4 \zeta_7 \zeta_9 \in \mathfrak{T}$.

4. \mathbb{P}_2^1 MARGOLIS HOMOLOGY OF $tmf^{\wedge r}$ AND $B(\mathbb{Z}/2^{\times n})_+$

4.1. \mathbb{P}_2^1 Margolis homology of $tmf^{\wedge r}$. Note that

$$H_*(tmf^{\wedge r}) \cong H_*(tmf)^{\otimes r} \cong \mathfrak{T}^{\otimes r}.$$

We first extend the notion of length to $\mathfrak{T}^{\otimes r}$. For a monomial $\zeta^{I_1} | \dots | \zeta^{I_r}$ for $\zeta^{I_i} \in \mathfrak{T}^{\otimes r}$, which is a tensor product of monomials in \mathfrak{T} , we define

$$L(\zeta^{I_1} | \dots | \zeta^{I_r}) = L(\zeta^{I_1}) + \dots + L(\zeta^{I_r}).$$

We define the even subalgebra \mathbb{E}_r of $\mathfrak{T}^{\otimes r}$ as the span of those monomials in $\mathfrak{T}^{\otimes r}$ whose lengths are zero. Observe that,

$$\mathbb{E}_r \cong \mathcal{E}^{\otimes r}.$$

Notion of length leads to an increasing filtration on $\mathfrak{T}^{\otimes r}$, call it the length filtration, by setting

$$G^p(\mathfrak{T}^{\otimes r}) = \{(\zeta^{I_1} | \dots | \zeta^{I_r}) \mid L(\zeta^{I_1} | \dots | \zeta^{I_r}) \leq p\}.$$

Let $\mathbb{K}_r = \mathcal{K}^{\otimes r}$, where \mathcal{K} is as defined in Section 2. Just like in the case $r = 1$, we get a length spectral sequence and its E_1 page is

$$(4.1) \quad E_1^\bullet \cong \mathcal{M}(\mathbb{E}_r, \mathbb{P}_2^1) \otimes \mathbb{K}_r \Rightarrow \mathcal{M}(tmf^{\wedge r}, \mathbb{P}_2^1).$$

Since action of \mathbb{P}_2^1 follows the Leibniz rule when restricted to \mathcal{E} , we get

$$\mathcal{M}(\mathbb{E}_r, \mathbb{P}_2^1) \cong \mathcal{M}(\mathcal{E}, \mathbb{P}_2^1)^{\otimes r}.$$

Notation 4.2. For shorthand, we denote $x_{i,j} = \underbrace{(1 | \dots | 1)}_{j-1} | \zeta_{i+3} | \underbrace{(1 | \dots | 1)}_{r-j}$ and $t_{i,j} =$

$\underbrace{(1 | \dots | 1)}_{j-1} | \zeta_{i+1}^4 | \underbrace{(1 | \dots | 1)}_{r-j}$. With this notation we have

$$\mathbb{Q}_1(x_{i,j}) = t_{i,j}.$$

Using Notation 4.2, we see that the E_1 page of the length spectral sequence (4.1), as an algebra, is isomorphic to

$$\mathcal{R}_r := \Lambda(t_{i,j} : i \in \mathbb{N} - \{0\}, 1 \leq j \leq r) \otimes \Lambda(x_{i,j} : i \in \mathbb{N} - \{0\}, 1 \leq j \leq r).$$

It is easy to see that the map induced by the reindexing map

$$\iota : (i, j) \mapsto r(i - 1) + j,$$

produces a (non-canonical) isomorphism of algebras between \mathcal{R}_r (the E_2 page of (4.1)) and \mathcal{R} (the E_2 page of (2.14)), after forgetting the internal grading. This is also an isomorphism of $\Lambda(\mathbb{Q}_1, P_2^1)$ -modules. Thus we have an isomorphism

$$\iota_* : \mathcal{M}(tmf, P_2^1) \xrightarrow{\cong} \mathcal{M}(tmf^{\wedge r}, P_2^1)$$

induced by the ι . Therefore, Theorem 3.18 essentially gives a complete calculation of $\mathcal{M}(tmf^{\wedge r}, P_2^1)$.

Example 4.3. For example, let us assume $r = 3$. Then the element $t_2 t_4 x_6 x_9 + t_2 t_6 x_4 x_9 \in \mathcal{M}(tmf, P_2^1)$ (see Example 3.17) corresponds to the element

$$t_{1,2} t_{2,1} x_{2,3} x_{3,3} + t_{1,2} t_{2,3} x_{2,1} x_{3,3} \in \mathcal{M}(tmf^{\wedge 3}, P_2^1)$$

under the bijection obtained from the above reindexing. When expressed in terms of ζ_i s (see Notation 4.2), the same element can be expressed as

$$\zeta_3^4 |\zeta_2^4 | \zeta_5 \zeta_6 | 1 + \zeta_5 |\zeta_2^4 | \zeta_3^4 \zeta_6 | 1.$$

Remark 4.4 (P_2^1 Margolis homology of Brown–Gitler spectra). It is well-known that

$$H_*(tmf) \cong \bigoplus_{i \geq 0} H_*(\Sigma^{8i} bo_i)$$

where bo_i are certain Brown–Gitler spectra associated with bo . In [Mah81] Mahowald defined a multiplicative weight function, which is given by $w(\zeta_i) = 2^{i-1}$. $H_*(\Sigma^{8i} bo_i)$ is the summand of $H_*(tmf)$ which consists of elements of Mahowald weight exactly equal to $8i$. We assign Mahowald weight of $t_{i,j}$ and $x_{i,j}$ as

$$w(t_{i,j}) = w(x_{i,j}) = 2^{i+1}.$$

It follows that the Margolis homology $\mathcal{M}(bo_{q_1} \wedge \cdots \wedge bo_{q_r}, P_2^1)$ is a summand of $\mathcal{M}(tmf^{\wedge r}, P_2^1)$. It consists of all polynomials of $\mathcal{M}(tmf^{\wedge r}, P_2^1)$ expressed in terms of $x_{i,j}$ and $t_{i,j}$ such that $w(x_{i,j}) = w(t_{i,j}) = 4q_j$.

Remark 4.5. While it is true that $\mathcal{R}_r \cong \mathcal{R}^{\otimes r}$, as an \mathbb{F}_2 -algebra as well as an $\Lambda(\mathbb{Q}_1, P_2^1)$ -module, it is not useful for the purposes of calculating $\mathcal{M}(\mathcal{R}_r, P_2^1)$. This is because P_2^1 does not obey Leibniz rule and

$$\mathcal{M}(\mathcal{R}_r, P_2^1) \not\cong \mathcal{M}(\mathcal{R}, P_2^1)^{\otimes r}.$$

However we overcome this difficulty by producing an $\Lambda(\mathbb{Q}_1, P_2^1)$ -module isomorphism ι_* at the expense of forgetting the internal grading.

4.2. \mathbb{P}_2^1 **Margolis homology of $(B\mathbb{Z}/2^{\times k})_+$.** The space $B\mathbb{Z}/2$ is also known as $\mathbb{R}\mathbb{P}^\infty$, the real infinite-dimensional projective space. It is well-known that

$$H^*((B\mathbb{Z}/2)_+, \mathbb{F}_2) \cong \mathbb{F}_2[x]$$

and therefore

$$H^*((B\mathbb{Z}/2^{\times k})_+, \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots, x_k].$$

It can be seen that $\mathbb{P}_2^1(x_i) = 0$ and $\mathbb{Q}_1(x_i) = x_i^4$. We again define the length function on the monomials in the usual way

$$L(x_1^{i_1} \dots x_k^{i_k}) = (i_1 \bmod 2) + \dots + (i_k \bmod 2).$$

The even complex \mathcal{E} , which is the span of elements of length zero, is isomorphic to

$$\mathcal{E} = \mathbb{F}_2[x_1^2, \dots, x_k^2].$$

It can be seen that $\mathbb{P}_2^1(x_i^2) = x_i^8$. Now observe that \mathbb{Q}_1 acts trivially on \mathcal{E} , hence \mathbb{P}_2^1 acts as a derivation and, therefore,

$$\mathcal{M}(\mathcal{E}, \mathbb{P}_2^1) \cong \Lambda(x_1^4, \dots, x_k^4).$$

Now the length function gives us an increasing length filtration

$$G^p(\mathbb{F}_2[x_1, \dots, x_k]) = \mathbb{F}_2\langle x_1^{i_1} \dots x_k^{i_k} : L(x_1^{i_1} \dots x_k^{i_k}) \leq p \rangle.$$

This results in a length spectral sequence which only has d_0 and d_2 differentials. If we denote x_i^4 by t_i for convenience, we can see that the length spectral sequence

$$E_1^\bullet = \Lambda(t_1, \dots, t_k) \otimes \Lambda(x_1, \dots, x_k) \Rightarrow \mathcal{M}((B\mathbb{Z}/2^{\times k})_+, \mathbb{P}_2^1)$$

is a sub spectral sequence of (2.14) and is, in fact, isomorphic to it when $k = \infty$. Thus, when k is finite, we can recover a complete description of $\mathcal{M}((B\mathbb{Z}/2^{\times k})_+, \mathbb{P}_2^1)$ from Theorem 3.18. More precisely, we obtain

$$\mathcal{M}((B\mathbb{Z}/2^{\times k})_+, \mathbb{P}_2^1) \cong \bigoplus_{K \subset [k]} S_K \otimes \mathcal{M}(M_K, \mathbb{P}_2^1),$$

where $S_K = \Lambda(t_i x_i \mid i \in [k] - K)$. $\mathcal{M}((B\mathbb{Z}/2^{\times k})_+, \mathbb{P}_2^1)$ is a module over $\mathcal{S}_{[k]}$.

Example 4.6. $\mathcal{M}(\mathbb{R}\mathbb{P}_+^\infty, \mathbb{P}_2^1) \cong \mathbb{F}_2\langle x_1, t_1, t_1 x_1 \rangle$, where the internal degrees of x_1 and t_1 are 1 and 4 respectively and $\mathcal{S}_{[1]} = \Lambda(t_1 x_1)$. Similarly,

$$\begin{aligned} \mathcal{M}((\mathbb{R}\mathbb{P}^\infty \times \mathbb{R}\mathbb{P}^\infty)_+, \mathbb{P}_2^1) \cong & \mathbb{F}_2\langle x_1, x_2, t_1, t_2, t_1 x_1, t_2 x_2, \\ & t_1 x_2, t_2 x_1, t_1 x_1 x_2, t_2 x_2 x_1, t_1 t_2 x_2, t_1 t_2 x_1 \rangle \end{aligned}$$

where the internal degrees of x_i and t_i are 1 and 4 respectively. Here $\mathcal{S}_{[2]} = \Lambda(t_1 x_1, t_2 x_2)$. If we denote

$$H^*((\mathbb{R}\mathbb{P}^\infty \times \mathbb{R}\mathbb{P}^\infty)_+) \cong \mathbb{F}_2[y, z]$$

where $|y| = |z| = 1$, then one may choose $x_1 = [x]$, $x_2 = [y]$, $t_1 = [x^4]$ and $t_2 = [y^4]$.

REFERENCES

- [Ada74] J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1974.
- [AM71] J. F. Adams and H. R. Margolis. Modules over the Steenrod algebra. *Topology*, 10:271–282, 1971.
- [Bau08] T. Bauer. Computation of the homotopy of the spectrum tmf . In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 11–40. Geom. Topol. Publ., Coventry, 2008.
- [BBB⁺a] Agnes Beaudry, Mark Behrens, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu. On the E_2 -term of the bo -Adams spectral sequence. Available at <https://arxiv.org/abs/1702.00230>.
- [BBB⁺b] Agnes Beaudry, Mark Behrens, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu. On the tmf -resolution of \tilde{Z} . In preparation.
- [BEa] Prasit Bhattacharya and Philip Egger. A class of 2-local finite spectra which admit v_2^1 -self-map. Available at <https://arxiv.org/abs/1608.06250>.
- [BEb] Prasit Bhattacharya and Philip Egger. Towards the $K(2)$ -local homotopy groups of Z . Available at <https://arxiv.org/abs/1706.06170>.
- [BG73] Edgar H. Brown, Jr. and Samuel Gitler. A spectrum whose cohomology is a certain cyclic module over the Steenrod algebra. *Topology*, 12:283–295, 1973.
- [BG18] Irina Bobkova and Paul G. Goerss. Topological resolutions in $K(2)$ -local homotopy theory at the prime 2. *Journal of Topology*, 11(4):917–956, 2018.
- [Cul] D. Culver. On $BP\langle 2 \rangle$ -cooperations. Available at <https://arxiv.org/abs/1708.03001>.
- [DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill, editors. *Topological modular forms*, volume 201 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2014.
- [Goe86] Paul G. Goerss. Unstable projectives and stable Ext: with applications. *Proc. London Math. Soc. (3)*, 53(3):539–561, 1986.
- [Goe10] P. G. Goerss. Topological modular forms [after Hopkins, Miller and Lurie]. *Astérisque*, (332):Exp. No. 1005, viii, 221–255, 2010. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011.
- [HK00] David J. Hunter and Nicholas J. Kuhn. Characterizations of spectra with U -injective cohomology which satisfy the Brown-Gitler property. *Trans. Amer. Math. Soc.*, 352(3):1171–1190, 2000.
- [HM14] Michael J. Hopkins and Mark Mahowald. From elliptic curves to homotopy theory. In *Topological modular forms*, volume 201 of *Math. Surveys Monogr.*, pages 261–285. Amer. Math. Soc., Providence, RI, 2014.
- [HM16] Rune Haugseng and Haynes Miller. On a spectral sequence for the cohomology of infinite loop spaces. *Algebr. Geom. Topol.*, 16(5):2911–2947, 2016.
- [KM13] Nicholas Kuhn and Jason McCarty. The mod 2 homology of infinite loopspaces. *Algebr. Geom. Topol.*, 13(2):687–745, 2013.
- [LM87] Wolfgang Lellmann and Mark Mahowald. The bo -Adams spectral sequence. *Trans. Amer. Math. Soc.*, 300(2):593–623, 1987.
- [Mah81] Mark Mahowald. bo -resolutions. *Pacific J. Math.*, 92(2):365–383, 1981.
- [Mar83] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category.
- [Mat16] Akhil Mathew. The homology of tmf . *Homology Homotopy Appl.*, 18(2):1–29, 2016.
- [Mil58] John Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.
- [MT68] Robert E. Mosher and Martin C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row, Publishers, New York-London, 1968.
- [NZ89] Warren D. Nichols and M. Bettina Zoeller. A Hopf algebra freeness theorem. *Amer. J. Math.*, 111(2):381–385, 1989.
- [Pea14] Paul Thomas Pearson. The connective real K -theory of Brown-Gitler spectra. *Algebr. Geom. Topol.*, 14(1):597–625, 2014.
- [Sch94] Lionel Schwartz. *Unstable modules over the Steenrod algebra and Sullivan’s fixed point set conjecture*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1994.

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