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The topological modular forms of $\mathbb{R}P^2$ and $\mathbb{R}P^2 \wedge \mathbb{C}P^2$

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Abstract

We study the elliptic spectral sequence computing $tmf_*(\mathbb{R}P^2)$ and $tmf_*(\mathbb{R}P^2 \wedge \mathbb{C}P^2)$. Specifically, we compute all differentials and resolve exotic extensions by 2, η, and ν. For $tmf_*(\mathbb{R}P^2 \wedge \mathbb{C}P^2)$, we also compute the effect of the v_1-self maps of $\mathbb{R}P^2 \wedge \mathbb{C}P^2$ on tmf-homology.

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Contents

1 | INTRODUCTION

1.1 | Motivation

Topological modular forms (tmf) are ubiquitous in algebraic topology and homotopy theory. The goal of this paper is to compute the tmf-homology of two spaces, namely, $\mathbb{R}P^2$ and $\mathbb{R}P^2 \wedge \mathbb{C}P^2$, and to determine the differentials and extensions in their elliptic spectral sequences.

We approach this problem from the point of view of stable homotopy theory. As is common, we let $V(0)$ denote the cofiber of multiplication by 2 on the sphere spectrum. Then

$$V(0) \simeq \Sigma^{-1}\Sigma^{\infty}\mathbb{R}P^2$$

and, via the suspension isomorphism, computing $tmf_*V(0) \cong \pi_*tmf \wedge V(0)$ is equivalent to computing the tmf-homology of $\mathbb{R}P^2$. Similarly, let Y be the smash product of $V(0)$ with C_η, the cofiber of the stable Hopf map η. Then

$$Y \simeq \Sigma^{-3}\mathbb{R}P^2 \wedge \mathbb{C}P^2$$

and computing tmf_*Y is equivalent to computing the tmf-homology of $\mathbb{R}P^2 \wedge \mathbb{C}P^2$. In this paper, we compute the elliptic spectral sequence for both $tmf \wedge V(0)$ and $tmf \wedge Y$. From this computation, we deduce $tmf_*V(0)$ and tmf_*Y and provide information about their module structure over tmf_*. In particular, we resolve all exotic $2, \eta, \nu$ extensions as as compute the effect of v_1-self maps of Y on tmf_*Y. Note that determining the tmf_*-module structure is much less straightforward than a simple degree-wise computation of $tmf_*V(0)$ or tmf_*Y.

Knowing the homology of basic spaces is part of a full understanding of any generalized homology theory. So, we see these computations as having independent and fundamental interest. They are, at the very least, an addition to the slim bank of examples of computations in tmf-homology theory of spaces and finite spectra.

However, our motivation for doing this runs deeper and this computation is part of a more ambitious program, coming from chromatic homotopy theory. Specifically, our main goal in doing this computation is not just to understand the structure of $tmf_*V(0)$ and tmf_*Y as tmf_*-modules, but more-so *to fully compute their elliptic spectral sequences*. To explain this, we let $K(2)$ denote the Morava K-theory spectrum and E_2 the Lubin–Tate spectrum (also often called Morava E-theory).

In the sequence of papers [22–26, 28, 29], Goerss, Henn, Karamanov, Mahowald, and Rezk carry out a program for studying $K(2)$–local homotopy theory at $p = 3$ using the theory of *finite resolutions*. These are sequences of spectra built from the $K(2)$-localization of tmf (and tmf with level structures) that resolve the $K(2)$-local sphere. Finite resolutions give rise to Bousfield–Kan spectral sequences. Let us call these *finite resolution spectral sequences*. The input is $K(2)$-local tmf-homology, possibly with level structures, and the output is $K(2)$-local homotopy groups. The ultimate goal is to use finite resolutions to compute $\pi_*L_{K(2)}S^0$, but an intermediate step is the computations of the homotopy groups of $L_{K(2)}F$ for some key finite spectra F, such as the prime 3 Moore spectrum $V(0)$ [29] and the cofiber of its v_1-self map, commonly denoted $V(1)$ [23]. So, to use the finite resolution approach to $K(2)$-local homotopy, a key input is $\pi_*L_{K(2)}(tmf \wedge F)$. This can be computed via the $K(2)$-local E_2-based Adams–Novikov spectral sequence (which can also be cast as a homotopy fixed point spectral sequence). This spectral sequence receives a map from the elliptic spectral sequence of $tmf \wedge F$. Understanding the elliptic spectral sequence of $tmf \wedge F$ thus provides key input for $K(2)$-local computations.

Recently, there have been significant advancements towards carrying out an analogous program at the prime $p = 2$ (see [3, 4, 8, 14]). But the program is still in progress. For example, the only complete computation of the $K(2)$-local homotopy groups of a finite spectrum at $p = 2$ is the computation of $\pi_* L_{K(2)} Z$ for $Z \in \mathcal{Z}$, where $\mathcal{Z}$ is the class of Bhattacharya–Egger spectra admitting a v_2-self map (see [10, 11] and also [5]). The motivation for this project is to add to this bank of computations, namely, to study $L_{K(2)} V(0)$, $L_{K(2)} Y$, but also $L_{K(2)} A_1$ where A_1 is the cofiber of a v_1-self map of Y. For this, we found the need to understand the elliptic spectral sequence of $tmf \wedge V(0)$, $tmf \wedge Y$ and $tmf \wedge A_1$. In [33], the third author computes a $K(2)$-local E_2-based Adams–Novikov spectral sequence converging to $\pi_* L_{K(2)}(tmf \wedge A_1)$. From this computation, one can deduce that of the elliptic spectral sequence of $tmf \wedge A_1$.

Here, we study the elliptic spectral sequences of $tmf \wedge V(0)$ and $tmf \wedge Y$. For F either $V(0)$ or Y, $tmf_* F = 0$ for $* < 0$ and $tmf_* F$ is determined by its values in the range $0 \leqslant * < 192$. In this paper, we obtain the following result, where the definition of what we mean by *exotic extensions* is given in Definition 2.20.

Theorem 1.1. *The elliptic spectral sequence for $tmf \wedge V(0)$ is depicted in Figures 4–7 and [6].*

$$tmf_* V(0) \cong \widetilde{tmf}_{*+1} \mathbb{R}P^2,$$

together with ***all exotic*** $2, \eta$ ***and*** ν ***extensions*** *in the corresponding elliptic spectral sequence is as displayed in Figures 8 and 9 in degrees* $0 \leqslant * < 192$.

Similarly, the elliptic spectral sequence for $tmf \wedge Y$ is depicted in Figures 14–20 and [7].

$$tmf_* Y \cong \widetilde{tmf}_{*+3} \mathbb{R}P^2 \wedge \mathbb{C}P^2,$$

together with ***all exotic*** $2, \eta$ ***and*** ν ***extensions and almost all exotic*** v_1***-extensions*** *in the corresponding elliptic spectral sequence is as displayed in Figures 22 and 23 in degrees* $0 \leqslant * < 192$. *In particular,*

$$2(\widetilde{tmf}_*(\mathbb{R}P^2 \wedge \mathbb{C}P^2)) = 0.$$

Remark 1.2.

(1) In addition to charts in this paper, large, full range charts of the elliptic spectral sequences can be found in [6, 7].
(2) Computing exotic extensions in this sense of Definition 2.20 can (and does in some places here) leave ambiguity about the module structure. However, this definition of exotic extensions, which we borrowed from [30], is very standard in these kinds of large spectral sequence computations.

1.2 | Methods and comparison with existing work

To say a few words about our techniques, the major input in our computation is the elliptic spectral sequence of tmf, which was first computed by Hopkins and Mahowald [21, chapter 15], and later by Bauer [2]. The computation of the spectral sequence for $tmf_* V(0)$ is straightforward given that data, while that of $tmf_* Y$ is more intricate. The technique we use for the latter relies on an observation of the third author from [33]. For both $V(0)$ and Y, computation of the exotic extensions requires work and new input. Several techniques are used to achieve this, and the most interesting among these is probably the Brown-Comenetz 'self-duality' of $tmf_*V(0)$ and tmf_*Y (see Theorem 2.7).

In [19], Bruner and Rognes do a thorough investigation of the classical Adams sequence of tmf and some of its modules. (Note that the study of the classical Adams spectral sequence of tmf probably goes back to Hopkins and Mahowald, and later to Henriques in [21, chapter 13].) Among many other topics, including duality for topological modular forms which is relevant for our approaches, they study the classical Adams spectral sequence of tmf smashed with many finite spectra, including a study of tmf smashed with $V(0)$. In particular, they also compute $tmf_*V(0)$, determining all but a few $2, \eta, \nu$-multiplications as well as v_1^4-multiplications. Here, we deliberately use the word *multiplication* in contrast to the word *extension* discussed above to emphasize that Bruner–Rognes name all classes, which leads them to a more precise determination of multiplicative relations. During the writing of this paper, Bruner and Rognes shared their charts and an advanced copy of some of the chapters of their book with us. However, our results were obtained independently from theirs and via different methods. So, the two approaches complement one another. We also use a few results on the classical Adams spectral sequence of tmf_* which we verified against both [21, chapter 13] and [19, Chapters 5 and 9]. Furthermore, [19, Theorem 10.6] gives the key result on Brown–Commenetz duality for tmf (see Theorem 2.7), which we use to resolve extensions.

Finally, we reiterate that for our applications, namely, as an input in the finite resolution approach to $K(2)$-local homotopy theory, it is important to understand specifically the elliptic spectral sequence instead of the classical Adams spectral sequence because of its close relationship to the homotopy fixed point spectral sequence, a key tool in chromatic homotopy theory (see the discussion above).

1.3 | Organization of the paper

In Section 2, we discuss the elliptic spectral sequences and other key tools used later in the paper. In Section 3, we review the computation of the E_2-term of the elliptic spectral sequence for $tmf \wedge V(0)$. In Section 4, we compute the differentials and some exotic extensions. In Section 5, we turn to the computation of the E_2-term of the elliptic spectral sequence for $tmf \wedge Y$ and in Section 6, we compute the differentials and exotic extensions.

2 | BACKGROUND

Conventions 2.1. In this paper, all spectra are 2-local, in particular we will write tmf to mean $tmf_{(2)}$. All spectral sequence charts are drawn in Adams notation: for a spectral sequence $E_2^{s,t} = \mathrm{Ext}^{s,t}(\ldots)$ the x-axis represents $t-s$ and the y-axis represents s.

2.1 | (co)Truncated spectral sequences

In Section 6, we will use the (co-)truncation of the spectral sequence associated to a tower of cofibrations. We will now recall the constructions and their basic properties. Let

$$\begin{tikzcd}
X_0 & X_1 \arrow[l] & X_2 \arrow[l] & X_3 \arrow[l] & \dots \arrow[l] & X_{n-1} \arrow[l] & X_n \arrow[l] & \dots \arrow[l] \\
I_0 \arrow[ur, dotted] \arrow[from=u] & I_1 \arrow[ur, dotted] \arrow[from=u] & I_2 \arrow[ur, dotted] \arrow[from=u] & I_3 \arrow[ur, dotted] \arrow[from=u] & & I_{n-1} \arrow[ur, dotted] \arrow[from=u] & &
\end{tikzcd}$$

be a tower of cofibrations of spectra. Let $(E_r^{*,*}, d_r)_{r\geqslant 1}$ be the associated spectral sequence.

Let X_i/X_n be the cofiber of the evident map $X_n \to X_i$. For any $n \in \mathbb{N}$, there is a tower of fibrations, which we call the *n-truncated tower*:

$$\begin{tikzcd}
X_0/X_n & X_1/X_n \arrow[l] & X_2/X_n \arrow[l] & \dots \arrow[l] & X_{n-1}/X_n \arrow[l] & \mathrm{pt} \arrow[l] \\
I_0 \arrow[ur, dotted] \arrow[from=u] & I_1 \arrow[ur, dotted] \arrow[from=u] & I_2 \arrow[ur, dotted] \arrow[from=u] & & I_{n-1} \arrow[ur, dotted] \arrow[from=u] &
\end{tikzcd}$$

We denote the terms of the resulting spectral sequence by $E_{r,<n}^{s,t}$. This spectral sequence computes the homotopy groups of

$$\mathrm{sk}_{n-1}X_0 := X_0/X_n.$$

There is a natural map from the original tower to the n-truncated tower. Let

$$T_r^{s,t} : E_r^{s,t} \to E_{r,<n}^{s,t}$$

be the induced map between the respective E_r-terms. Then $E_{2,<n}^{s,t} = 0$ for $s \geqslant n$, while $T_2^{s,t}$ is an isomorphism if $s < n-1$ and an injection if $s = n-1$. More generally, we have:

Lemma 2.2. *For every $r \geqslant 2$, the map $T_r^{s,t}$ has the following properties:*

(i) *$T_r^{s,t}$ is injective for $s \leqslant n-1$, and*
(ii) *$T_r^{s,t}$ is bijective for $s \leqslant n-1-(r-1)$.*

Proof. We prove this by induction on the r. From the above discussion, (i) and (ii) hold for $r = 2$. Suppose both hold for some $r \geqslant 2$.

We prove that (i) holds at E_{r+1}. Let $[x] \in E_{r+1}^{s,t}$ be represented by an element $x \in E_r^{s,t}$ such that $s \leqslant n-1$ and $T_{r+1}^{s,t}([x]) = 0$. So, $T_r^{s,t}(x)$ is the target of a d_r-differential. That is, there exists $y \in E_{r,<n}^{s-r,t-r-1}$ such that $d_r(y) = T_r^{s,t}(x)$. Since $s-r \leqslant n-r$, $T_r^{s-r,*}$ is bijective by the induction hypothesis. It follows that there exists $\overline{y} \in E_r^{s-r,t-r-1}$ such that $T_r^{s-r,t-r-1}(\overline{y}) = y$. So, by naturality and the hypothesis that $T_r^{s,t}$ is injective, $d_r(\overline{y}) = x$. This means that $[x] = 0$, and hence $T_{r+1}^{s,t}$ is injective when $s \geqslant n-1$.

Now, we prove that (ii) holds at E_{r+1}. Let $[x] \in E_{r+1,<n}^{s,t}$ with $s \leqslant n-r-1$. We need to show that $[x]$ is in the image of $T_{r+1}^{s,t}$. By the induction hypothesis, there is a class $\overline{x} \in E_r^{s,t}$ such that $T_r^{s,t}(\overline{x}) = x$. It suffices to prove that $\overline{x}$ is a d_r-cycle. By naturality,

$$T_r^{s+r,t+r-1}(d_r(\overline{x})) = d_r(T_r^{s,t}(\overline{x})) = d_r(x) = 0.$$

Since $d_r(x) \in E_{r,<n}^{s+r,t+r-1}$ and $s+r \leqslant n-1$, the induction hypothesis implies that $d_r(\overline{x}) = 0$. $\square$

Next, we look at the co-truncated spectral sequence. Consider the following tower of fibrations, which we call the *n-co-truncated tower*,

$$\begin{tikzcd}
Y_0 \arrow[d] & Y_1 \arrow[l, "\mathrm{id}"'] \arrow[d] & \cdots \arrow[l, "\mathrm{id}"'] & Y_n = X_n \arrow[l, "\mathrm{id}"'] \arrow[d] & X_{n+1} \arrow[l] \arrow[d] & X_{n+2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
J_0 \arrow[ur, dotted] & J_1 \arrow[ur, dotted] & & J_n = I_n \arrow[ur, dotted] & I_{n+1} \arrow[ur, dotted] & I_{n+2} \arrow[ur, dotted] &
\end{tikzcd}$$

where $Y_0 = \cdots = Y_n = X_n$ and $J_0 = \cdots = J_{n-1} = pt$. We denote by $E^{s,t}_{r,\geqslant n}$ the r-term of the spectral sequence associated to this tower. There is an obvious map from the n-co-truncated tower to the original one. This map induces a map of spectral sequences:

$$cT^{s,t}_r : E^{s,t}_{r,\geqslant n} \to E^{s,t}_r.$$

We observe that $E^{s,t}_{r,\geqslant n} = 0$ for $s < n$, and that $cT^{s,*}_2$ is a bijection for $s \geqslant n+1$ and a surjection for $s = n$. The following lemma is proved as in Lemma 2.2.

Lemma 2.3. *For every $r \geqslant 2$, the map $cT^{s,t}_r$ has the following properties:*

(i) *$cT^{s,t}_r$ is surjective for $s \geqslant n$, and*
(ii) *$cT^{s,t}_r$ is bijective for $s \geqslant n + r - 1$.*

2.2 | The elliptic spectral sequence

In this section, we will introduce our main spectral sequence. Let

$$(A, \Lambda) = (\mathbb{Z}[a_1, a_2, a_3, a_4, a_6], \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, s, r, t])$$

with

$$|a_i| = 2i, |r| = 4, |s| = 2, |t| = 6$$

be the Hopf algebroid of Weierstrass elliptic curves. Then the elliptic spectral sequence has the form [2]

$$E_2^{s,t-s} = \mathrm{Ext}^{s,t}_\Lambda(A, A) \Longrightarrow \pi_{t-s} tmf.$$

Consider the map

$$\Omega SU(4) \to \Omega SU \simeq BU$$

induced by the usual inclusion $SU(4) \to SU$. Let $X(4)$ be the Thom spectrum of the associated virtual vector bundle over $\Omega SU(4)$. These spectra play a crucial role in the study of nilpotence and periodicity in chromatic homotopy theory, in particular, in the work of Ravenel [36]. As outlined in [21, chapter 9], the elliptic spectral sequence is the $X(4)$-based Adams spectral sequence for tmf (see also [37]).

Let us spell this out. We let $R = tmf$ and $E = tmf \wedge X(4)$. Then

$$E \wedge_R E \simeq tmf \wedge X(4) \wedge X(4).$$

Let $\overline{E}$ be the fiber of the unit map $R \to E$. For any tmf-module M, one can construct the Adams tower

$$\begin{tikzcd}
M \arrow[d] & \overline{E} \wedge_R M \arrow[l] \arrow[d] & \overline{E} \wedge_R \overline{E} \wedge_R M \arrow[l] \arrow[d] & \cdots \arrow[l] \\
E \wedge_R M \arrow[ur, dotted] & E \wedge_R \overline{E} \wedge_R M \arrow[ur, dotted] & E \wedge_R \overline{E} \wedge_R \overline{E} \wedge_R M \arrow[ur, dotted] &
\end{tikzcd}$$

by splicing together the cofiber sequences

$$\overline{E}^{\wedge_R (n+1)} \wedge_R M \to \overline{E}^{\wedge_R n} \wedge_R M \to E \wedge_R \overline{E}^{\wedge_R n} \wedge_R M.$$

We abbreviate

$$X_k := \overline{E}^{\wedge_R k} \wedge_R M \simeq \overline{X(4)}^{\wedge k} \wedge M,$$

$$I_k := E \wedge_R \overline{E}^{\wedge_R k} \wedge_R M \simeq X(4) \wedge \overline{X(4)}^{\wedge k} \wedge M,$$

where $\overline{X(4)}$ is the fiber of the unit map $S^0 \to X(4)$. As a consequence, the associated spectral sequence is identified with the $X(4)$-based Adams spectral sequence for M.

However, we have that the Hopf algebroid

$$(\pi_*(E), \pi_*(E \wedge_R E)) = (\pi_*(tmf \wedge X(4)), \pi_*(tmf \wedge X(4) \wedge X(4)))$$

is isomorphic to (A, Λ). In particular, it is flat. Therefore, the E_2-term of the associated spectral sequence is identified with

$$E^{s,t}(M) \cong \mathrm{Ext}_{\Lambda}^{s,t}(A, \pi_*(E \wedge_R M)). \tag{2.4}$$

See [1]. When $M = tmf$, this is precisely the elliptic spectral sequence, and more generally, this is the elliptic spectral sequence for the tmf-module M.

By [15, Theorem 6.5], since $X(4)$ is connected and $\pi_0(X(4)) \cong \mathbb{Z}$, if M is connective, then $L_{X(4)}M \simeq M$ and the spectral sequence (2.4) converges strongly in the sense of [13] to $\pi_*(M)$. In this paper, we will be working with modules M of the form $tmf \wedge F$ (where $F = V(0)$ or Y) and with the elliptic spectral sequence which reads as

$$E_2^{s,t-s} = \mathrm{Ext}_{\Lambda}^{s,t}(A, \pi_*(tmf \wedge X(4) \wedge F)) \Longrightarrow \pi_{t-s}(tmf \wedge F).$$

To simplify the notation, we put

$$\mathcal{F}_*(F) := \pi_*(tmf \wedge X(4) \wedge F)$$

noting that this is a Λ-comodule.

The spectra we will be working with in this paper are 2-local. As described in [2, section 7], one can simplify the computation of the cohomology of the Weierstrass Hopf algebroid

$$(A_{(2)}, \Lambda_{(2)}) \cong (A \otimes \mathbb{Z}_{(2)}, \Lambda \otimes \mathbb{Z}_{(2)})$$

as follows. Let A' denote $\mathbb{Z}_{(2)}[a_1, a_3]$ and $f : A \to A'$ the evident projection. Let Λ' denote $A' \otimes_A \Lambda \otimes_A A'$, which is isomorphic to $A'[s,t]/\sim$, where the relations $\sim$ are generated by

$$s^4 - 6st + a_1 s^3 - 3a_1 t - 3a_3 s = 0$$

$$s^6 - 27t^2 + 3a_1 s^5 - 9a_1 s^2 t + 3a_1^2 s^4 - 9a_1^2 st + a_1^3 s^3 - 27a_3 t = 0.$$

The map between Hopf algebroids

$$f : (A_{(2)}, \Lambda_{(2)}) \to (A', \Lambda')$$

induces an equivalence of the associated categories of comodules [2, sections 2 and 7], where

$$N \mapsto A' \otimes_{A_{(2)}} N$$

for an $(A_{(2)}, \Lambda_{(2)})$-comodule N. When F is the 2-localization of a finite spectrum, the E_2-term of the elliptic spectral sequence for

$$tmf \wedge F \simeq tmf_{(2)} \wedge F$$

is isomorphic to

$$E_2^{s,t}(tmf \wedge F) \cong \mathrm{Ext}^{s,t}_{\Lambda'}(A', A' \otimes_A \mathcal{F}_*(F)).$$

Remark 2.5. The spectrum $tmf \wedge X(4)$ is a complex oriented ring spectrum (for example, $A = \pi_*(tmf \wedge X(4))$ is concentrated in even degrees). Let us denote by

$$H : MU \to tmf \wedge X(4)$$

the map of ring spectra inducing the complex orientation of $tmf \wedge X(4)$ given by the completion of the universal Weierstrass curve at the origin. Then H induces a homomorphism of Hopf algebroids

$$H_* : (MU_*, MU_*MU) \to ((tmf \wedge X(4))_*, (tmf \wedge X(4) \wedge X(4))_*) = (A, \Lambda).$$

Recall that $MU_* \cong \mathbb{Z}[x_1, x_2, ...]$ with $|x_i| = 2i$ and $MU_*MU \cong MU_*[m_1, m_2, ...]$ with $|m_i| = 2i$. We note that $H_*(x_1) = \pm a_1$. This is discussed in [2, (3.2)].

The map H also induces a map from the Adams–Novikov spectral sequence for $\pi_*(F)$ to the elliptic spectral sequence for $\pi_*(tmf \wedge F)$, which converges to the Hurewitz map $h : \pi_*(F) \to \pi_*(tmf \wedge F)$. Moreover, the induced map at the E_2-term is induced by H_*.

2.3 Duality

In this section, we discuss Brown–Comenetz duality for tmf. This will be used for determining some of the exotic extensions in our spectral sequences. First, we introduce the following notation.

Notation 2.6. Let A be a graded module over a graded commutative ring S and $x \in S$. We let $\Sigma^r A$ be the module determined by $(\Sigma^r A)_t = A_{t-r}$. We denote by $\Gamma_x A$ the x-power torsion of A, that is,

$$\Gamma_x A = \{m \in A \mid x^i m = 0, i \gg 0\},$$

and by $A/(x^\infty)$ the module that fits into the exact sequence of S-modules

$$A \to A\left[\frac{1}{x}\right] \to A/(x^\infty) \to 0.$$

We will also denote by $A^\vee$ the Pontryagin dual of A, that is,

$$(A^\vee)_* = \mathrm{Hom}((A)_{-*}, \mathbb{Q}/\mathbb{Z})$$

with the S-module structure given by $(r.f)(m) = (-1)^{|r||f|} f(rm)$ for every $r \in S_{|r|}$, $f \in (A^\vee)_{|f|}$ and $m \in A_{|m|}$.

Now suppose that R is a commutative ring spectrum (for example, $R = tmf$) and M is a R-module. For any $x \in \pi_*(R)$, we define $M[\frac{1}{x}]$ to be

$$M\left[\frac{1}{x}\right] = \mathrm{hocolim}(M \xrightarrow{x} \Sigma^{-|x|} M \xrightarrow{x} \Sigma^{-2|x|} M \xrightarrow{x} \dots).$$

We define $M/(x^\infty)$ to be the cofiber of the natural map $M \to M[\frac{1}{x}]$. Inductively, if $(x_1, x_2, \dots, x_n)$ is a sequence of elements of $\pi_* R$, then we define

$$M/(x_1^\infty, x_2^\infty, \dots, x_n^\infty) = (M/(x_1^\infty, x_2^\infty, \dots, x_{n-1}^\infty))/(x_n^\infty).$$

With this notation, using the long exact sequence on homotopy groups, we see that the cofiber sequence

$$M \to M\left[\frac{1}{x}\right] \to M/(x^\infty)$$

gives rise to the short exact sequence of $\pi_*(R)$-modules

$$0 \to \pi_*(M)/(x^\infty) \to \pi_*(M/(x^\infty)) \to \Gamma_x(\pi_{*-1}(M)) \to 0.$$

Let $I_{\mathbb{Q}/\mathbb{Z}}$ be the spectrum representing the Pontryagin dual of stable homotopy groups, so that for a spectrum X,

$$I^q_{\mathbb{Q}/\mathbb{Z}}(X) := \mathrm{Hom}(\pi_q X, \mathbb{Q}/\mathbb{Z}).$$

Then the Brown–Comenetz dual of a spectrum X is defined to be

$$I_{\mathbb{Q}/\mathbb{Z}}(X) = F(X, I_{\mathbb{Q}/\mathbb{Z}}).$$

The literature contains a variety of references and methods for studying dualities of tmf and related spectra. To name a few, we note work of Mahowald and Rezk [31], Stojanoska [39, 40], Greenlees [27], and Bruner and Rognes [19, chapter 10].

Recall that throughout this paper tmf denotes the 2-localization $tmf_{(2)}$, according to Conventions 2.1.

Theorem 2.7 [19, Theorem 10.6]. *There is an equivalence of tmf-modules*

$$I_{\mathbb{Q}/\mathbb{Z}}(tmf/(2^\infty, c_4^\infty, \Delta^\infty)) \simeq \Sigma^{20} tmf.$$

Remark 2.8. Here and below, by '$-/\Delta^\infty$', we really mean '$-/(\Delta^8)^\infty$' as Δ is an element of the E_2-term of the elliptic spectral sequence but it does not survive to the E_∞-term. However, Δ^8 survives and detects a class in $\pi_{192} tmf$. Note also that the class $c_4 \in \pi_8 tmf$ reduces to $v_1^4 \in tmf \wedge V(0)$ and so c_4-power torsion is the same as v_1-power torsion when the latter makes sense.

Corollary 2.9. *There are equivalences of tmf-modules*

(1) $I_{\mathbb{Q}/\mathbb{Z}}(tmf \wedge V(0)/(2^\infty, c_4^\infty, \Delta^\infty)) \simeq \Sigma^{19} tmf \wedge V(0)$, *and*
(2) $I_{\mathbb{Q}/\mathbb{Z}}(tmf \wedge Y/(2^\infty, c_4^\infty, \Delta^\infty)) \simeq \Sigma^{17} tmf \wedge Y$.

Lemma 2.10. *For $\mathcal{X} = tmf \wedge V(0)$ or $tmf \wedge Y$, Δ^8 acts injectively on $(\pi_* \mathcal{X})/(c_4^\infty)$.*

Remark 2.11. The proof makes use of the structure of the E_∞-terms of the elliptic spectral sequences (see Figures 8, 9, 22, and 23 and, for a single large chart, [6, 7]). So, this is a bit premature but we want to have this result here to gather all our techniques in one place. We note that the logic of the argument is not circular as the determination of the E_∞-terms does not require this lemma; it is needed in the proof of Corollary 2.12, which will be used to establish exotic extensions in the elliptic spectral sequences.

Proof of Lemma 2.10. In this proof, write $M = \pi_* \mathcal{X}$ and $\overline{M} = \pi_* \mathcal{X}/\Gamma_{c_4}(\pi_* \mathcal{X})$. For any $\pi_* tmf$-module N, write $T_{\Delta^8}(N)$ to denote the submodule consisting of elements that are Δ^8-torsion.

Our goal is to show that $T_{\Delta^8}(M/c_4^\infty) = 0$. But the quotient map $M \to \overline{M}$ induces an isomorphism $M[c_4^{-1}] \xrightarrow{\cong} \overline{M}[c_4^{-1}]$, and hence an isomorphism

$$M/c_4^\infty \xrightarrow{\cong} \overline{M}/c_4^\infty.$$

So, it is equivalent to prove that $T_{\Delta^8}(\overline{M}/c_4^\infty)$ is zero, and we show that below.

The snake lemma applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \overline{M} \arrow[r] \arrow[d, "\Delta^8"] & \overline{M}[c_4^{-1}] \arrow[r] \arrow[d, "\Delta^8"] & \overline{M}/c_4^\infty \arrow[r] \arrow[d, "\Delta^8"] & 0 \\ 0 \arrow[r] & \overline{M} \arrow[r] & \overline{M}[c_4^{-1}] \arrow[r] & \overline{M}/c_4^\infty \arrow[r] & 0 \end{tikzcd}$$

gives an exact sequence

$$T_{\Delta^8}(\overline{M}[c_4^{-1}]) \to T_{\Delta^8}(\overline{M}/c_4^\infty) \to \overline{M}/\Delta^8 \to \overline{M}[c_4^{-1}]/\Delta^8$$

Therefore, if

(1) $T_{\Delta^8}(\overline{M}[c_4^{-1}]) = 0$, and
(2) $\overline{M}/\Delta^8 \to \overline{M}[c_4^{-1}]/\Delta^8$ is injective,

then we can conclude that $T_{\Delta^8}(\overline{M}/c_4^\infty) = 0$.

We will explain why the conditions (1) and (2) hold for $\mathcal{X} = tmf \wedge Y$. The argument for $tmf \wedge V(0)$ is more cumbersome, but can be adapted from this one.

All classes of $\pi_*(tmf \wedge Y)$ detected in positive filtration in the elliptic spectral sequence are c_4-power torsion. Indeed, they are c_4-power torsion at E_∞ and the spectral sequence has a horizontal vanishing line. All elements in filtration zero are c_4-free. From this, it follows that the edge homomorphism $M \to E_\infty^{0,*}$ (to the zero line of the spectral sequence) induces an isomorphism $\overline{M} \cong E_\infty^{0,*}$. But $E_\infty^{0,*}$ is a free module over $\mathbb{F}_2[c_4, \Delta^8]$ and so the conditions (1) and (2) follow. □

Corollary 2.12. *We have the following isomorphisms of $\pi_* tmf$-modules*

(1) $\Gamma_{c_4}(\pi_*(tmf \wedge V(0))/(\Delta^\infty))^\vee \cong \Gamma_{c_4}(\pi_{*-21}(tmf \wedge V(0)))$, *and*
(2) $\Gamma_{c_4}(\pi_*(tmf \wedge Y)/(\Delta^\infty))^\vee \cong \Gamma_{c_4}(\pi_{*-19}(tmf \wedge Y))$.

Proof. In this proof, we let $\mathcal{X} = tmf \wedge V(0)$. Since $\pi_*\mathcal{X}$ is 2-power torsion, we have $\mathcal{X}[1/2] \simeq *$. Thus,

$$\mathcal{X}/(2^\infty) \simeq \Sigma\mathcal{X}. \tag{2.13}$$

The long exact sequence in homotopy associated to the cofiber sequence

$$\mathcal{X}/(2^\infty) \to \mathcal{X}/(2^\infty)\left[\frac{1}{c_4}\right] \to \mathcal{X}/(2^\infty, c_4^\infty),$$

gives an exact sequence

$$0 \to (\pi_*\mathcal{X}/(2^\infty))/(c_4^\infty) \to \pi_*(\mathcal{X}/(2^\infty, c_4^\infty)) \to \Gamma_{c_4}\pi_{*-1}(\mathcal{X}/(2^\infty)) \to 0. \tag{2.14}$$

By (2.13), we have

$$(\pi_*(\mathcal{X}/(2^\infty)))/(c_4^\infty) \cong (\pi_{*-1}\mathcal{X})/(c_4^\infty)$$

and

$$\Gamma_{c_4}(\pi_{*-1}(\mathcal{X}/(2^\infty))) \cong \Gamma_{c_4}(\pi_{*-2}\mathcal{X}).$$

Since Δ^8 acts injectively on $\pi_*\mathcal{X}$, it also acts injectively on $\Gamma_{c_4}(\pi_{*-2}\mathcal{X})$. Moreover, Δ^8 acts injectively on $(\pi_*\mathcal{X})/(c_4^\infty)$ by Lemma 2.10. The short exact sequence (2.14) then shows that Δ^8 acts injectively on $\pi_*(\mathcal{X}/(2^\infty, c_4^\infty))$. Therefore, we have that

$$\pi_*(\mathcal{X}/(2^\infty, c_4^\infty, \Delta^\infty)) \cong (\pi_*\mathcal{X}/(2^\infty, c_4^\infty))/(\Delta^\infty).$$

The 9-lemma then implies that the following is a short exact sequence of $\pi_* tmf$-modules:

$$0 \to (\pi_{*-1}\mathcal{X})/(c_4^\infty, \Delta^\infty) \to \pi_*(\mathcal{X}/(2^\infty, c_4^\infty, \Delta^\infty)) \to \Gamma_{c_4}(\pi_{*-2}\mathcal{X})/(\Delta^\infty) \to 0. \tag{2.15}$$

By applying $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ to this exact sequence, we obtain

$$0 \to (\Gamma_{c_4}(\pi_{*-2}\mathcal{X})/(\Delta^\infty))^\vee \to \pi_*(\mathcal{X}/(2^\infty, c_4^\infty, \Delta^\infty))^\vee \to ((\pi_{*-1}\mathcal{X})/(c_4^\infty, \Delta^\infty))^\vee \to 0,$$

is an exact sequence of $\pi_* tmf$-modules.

We see that the right most term is c_4-free and the left most term is c_4-torsion. In particular, it follows that

$$\begin{aligned}(\Gamma_{c_4}(\pi_{*-2}\mathcal{X})/(\Delta^\infty))^\vee &\cong \Gamma_{c_4}(\pi_*(\mathcal{X}/(2^\infty, c_4^\infty, \Delta^\infty))^\vee) \\ &\cong \Gamma_{c_4}(\pi_* I_{\mathbb{Q}/\mathbb{Z}}(\mathcal{X}/(2^\infty, c_4^\infty, \Delta^\infty))),\end{aligned}$$

where the second isomorphism comes from the definition of the Brown-Comenetz dual $I_{\mathbb{Q}/\mathbb{Z}}(\mathcal{X}/(2^\infty, c_4^\infty, \Delta^\infty))$. Together with Corollary 2.9, we obtain an isomorphism of $\pi_* tmf$-modules

$$\begin{aligned}(\Gamma_{c_4}(\pi_*\mathcal{X})/(\Delta^\infty))^\vee &\cong \Sigma^2(\Gamma_{c_4}(\pi_{*-2}\mathcal{X})/(\Delta^\infty))^\vee \\ &\cong \Sigma^2\Gamma_{c_4}\pi_*(I_{\mathbb{Q}/\mathbb{Z}}(\mathcal{X}/(2^\infty, c_4^\infty, \Delta^\infty))) \\ &\cong \Sigma^2\Sigma^{19}\Gamma_{c_4}(\pi_*\mathcal{X}) \\ &\cong \Sigma^{21}\Gamma_{c_4}(\pi_*\mathcal{X}).\end{aligned}$$

Substituting $\mathcal{X}$ for $tmf \wedge Y$ and this last 19 with 17 gives the result for Y. □

Remark 2.16. We will explain how to use Corollary 2.12 to compute extensions. Continue to let $\mathcal{X} = tmf \wedge V(0)$. Let K denote the kernel of the homomorphism induced by multiplication by Δ^8 on $\Gamma_{c_4}(\pi_*\mathcal{X})/(\Delta^\infty)$. Since multiplication by Δ^8 induces an isomorphism

$$\Gamma_{c_4}(\pi_*\mathcal{X}) \xrightarrow{\cong} \Gamma_{c_4}(\pi_{*+192}\mathcal{X}) \tag{2.17}$$

for $* \geqslant 0$, we see that, for $-192 \leqslant t < 0$,

$$K_t \cong \Gamma_{c_4}(\pi_*\mathcal{X})/(\Delta^\infty)_t.$$

The snake lemma applied to the following diagram

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & \Gamma_{c_4}(\pi_*\mathcal{X})\left[\frac{1}{\Delta^8}\right] \arrow[r, "\Delta^8"] \arrow[d] & \Gamma_{c_4}(\pi_*\mathcal{X})\left[\frac{1}{\Delta^8}\right] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & \Gamma_{c_4}(\pi_*\mathcal{X})/(\Delta^\infty) \arrow[r, "\Delta^8"] & \Gamma_{c_4}(\pi_*\mathcal{X})/(\Delta^\infty) & \end{tikzcd}$$

gives rise to the exact sequence

$$0 \to \Gamma_{c_4}(\pi_*\mathcal{X}) \xrightarrow{\Delta^8} \Gamma_{c_4}(\pi_{*+192}\mathcal{X}) \to K \to 0.$$

Note that the injective map is an isomorphism for $* \geqslant 0$ and the surjective map is an isomorphism for $* < 0$. Using (2.17) again, the homomorphism $\Gamma_{c_4}(\pi_{*+192}\mathcal{X}) \to K$ in the above short exact sequence induces an isomorphism

$$\Gamma_{c_4}(\pi_*\mathcal{X})_t \to K_{t-192} \cong \Gamma_{c_4}\pi_*(\mathcal{X}/(\Delta^\infty))_{t-192}$$

for $0 \leqslant t < 192$.

Now let r be an element of $\pi_l(tmf)$. If $0 \leqslant k < 192 - l$, multiplication by r induces a commutative diagram

$$\begin{tikzcd}
\Gamma_{c_4}(\pi_*\mathcal{X})_k \arrow[r, "\cong"] \arrow[d, "r"] & K_{k-192} \cong \Gamma_{c_4}(\pi_*\mathcal{X})/(\Delta^\infty)_{k-192} \arrow[d, "r"] \\
\Gamma_{c_4}(\pi_*\mathcal{X})_{k+l} \arrow[r, "\cong"] & K_{k+l-192} \cong \Gamma_{c_4}(\pi_*\mathcal{X})/(\Delta^\infty)_{k+l-192}.
\end{tikzcd}$$

By applying the Pontryagin dual to this commutative diagram, together with Corollary 2.12, we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\Gamma_{c_4}(\pi_*\mathcal{X})_k, \mathbb{Q}/\mathbb{Z}) & \Gamma_{c_4}(\pi_*\mathcal{X})_{171-k} \arrow[l, "\cong"'] \\
\mathrm{Hom}(\Gamma_{c_4}(\pi_*\mathcal{X})_{k+l}, \mathbb{Q}/\mathbb{Z}) \arrow[u, "r^\vee"] & \Gamma_{c_4}(\pi_*\mathcal{X})_{171-k-l}. \arrow[l, "\cong"'] \arrow[u, "r"']
\end{tikzcd}$$

As a consequence, the cardinality of the image of

$$r : \Gamma_{c_4}(\pi_*\mathcal{X})_k \to \Gamma_{c_4}(\pi_*\mathcal{X})_{k+l}$$

is the same as that of

$$r : \Gamma_{c_4}(\pi_*\mathcal{X})_{171-k-l} \to \Gamma_{c_4}(\pi_*\mathcal{X})_{171-k}.$$

In particular, this means that a non-trivial multiplication by r on stem k forces a non-trivial multiplication by r on stem $171 - k - l$.

Similarly, for $tmf \wedge Y$ we obtain that a non-trivial multiplication by r on stem k forces a non-trivial multiplication by r on stem $173 - k - l$.

2.4 | The geometric boundary theorem

We also make use of the following result, due to Bruner [17]. A standard reference is [35, Theorem 2.3.4]. We apply this theorem to the $X(4)$-based Adams–Novikov spectral sequence and the cofiber sequence

$$tmf \wedge S^0 \xrightarrow{2} tmf \wedge S^0 \xrightarrow{i} tmf \wedge V(0) \xrightarrow{p} tmf \wedge S^1.$$

Using $X(4)_* tmf \cong A$ and $X(4)_*(tmf \wedge V(0)) \cong A/2$, we have $X(4)_* p = 0$ and hence a short exact sequence

$$0 \to A \xrightarrow{2} A \to A/2 \to 0. \tag{2.18}$$

Theorem 2.19 (Geometric Boundary Theorem). *There are maps*

$$\delta_r : E_r^{s,t}(V(0)) \to E_r^{s+1,t}(S^0)$$

such that

$$\delta_2 = \delta : E_2^{s,t}(V(0)) \to E_2^{s+1,t}(S^0)$$

is the connecting homomorphism arising from (2.18)*. For all r,*

$$\delta_r d_r = d_r \delta_r$$

and δ_{r+1} is induced by δ_r. Furthermore, δ_∞ is a filtered form of

$$p_* : \pi_* tmf \wedge V(0) \to \pi_{*+1} tmf.$$

2.5 | Further observations on extensions

Here, we collect a few classical but useful extension results. Note that, in this paper, we use [30, Definition 2.10] as our definition of an *exotic extension*. See Subsection 2.1 of that reference for a detailed discussion. However, briefly, we have

Definition 2.20. [30, Definition 2.10] Let $\alpha \in \pi_* tmf$ be an element detected by a on the E_∞-term of the elliptic spectral sequence for tmf. An *exotic extension by* α is a pair of elements b and c on the E_∞-term of the elliptic spectral sequence for M (where M is a tmf-module) such that

(1) $ab = 0$ on the E_∞-term,
(2) there is an element β detected by b such that $\alpha\beta$ is detected by c,
(3) if an element β' detected by b' is such that $\alpha\beta'$ is detected by c, then the filtration of b' is less than or equal to that of b.

Note that this implies that if both $\alpha\beta$ and $\alpha\beta'$ are detected by c as in Figure 1, there is no exotic extension from b' to c.

Lemma 2.21. *Let X be a spectrum. Consider the long exact sequence in homotopy*

$$\ldots \to \pi_n X \xrightarrow{i} \pi_n(X \wedge V(0)) \xrightarrow{p} \pi_{n-1} X \xrightarrow{2} \ldots$$

associated to the cofiber sequence $X \xrightarrow{2} X \to X \wedge V(0)$. Let $a \in \pi_{n-1}X$ be an element of order 2. If $a' \in \pi_n(X \wedge V(0))$ is such that $p_*(a') = a$, then

$$2a' = i_*(\eta a) \in \pi_n X \wedge V(0).$$

Proof. This is a classical result (see, for example, [8, Lemma 3.1.5]). □

Remark 2.22. Lemma 2.21 will be used with $X = tmf$ and $tmf \wedge C_\eta$ where C_η is the cofiber of the Hopf map $\eta : S^1 \to S^0$. This gives all exotic 2-extensions in the elliptic spectral sequences for $tmf \wedge V(0)$ and $tmf \wedge Y$, since $Y \simeq C_\eta \wedge V(0)$.

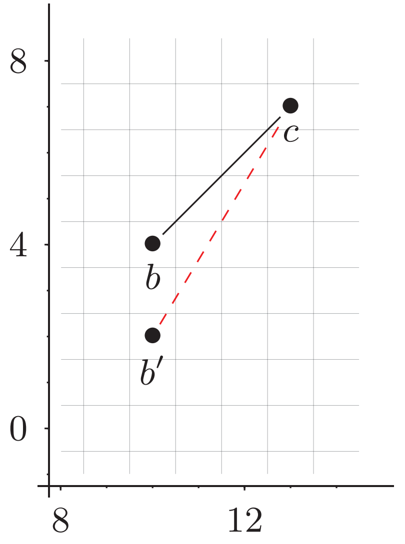


FIGURE 1 Here, there is no exotic extensions from b' to c, and so the dashed line would not be drawn.

Finally, we have the following classical result which is an analogue of Lemma 2.21.

Lemma 2.23. *Let $b \in \pi_n X$ be such that $\eta b = 0$. If $b' \in \pi_{n+2}(C_\eta \wedge X)$ is such that $p_* b' = b$ in the long exact sequence on homotopy groups associated to*

$$\Sigma X \xrightarrow{\eta} X \xrightarrow{i} X \wedge C_\eta \xrightarrow{p} \Sigma^2 X,$$

then $\eta b' = i_*(\nu b)$.

Proof. First, consider $b = \iota \in \pi_0 C_\eta$ given by the inclusion $S^0 \to C_\eta$ of the bottom cell. We have a cofiber sequence

$$C_\eta \xrightarrow{i} C_\eta \wedge C_\eta \xrightarrow{p} \Sigma^2 C_\eta$$

which is not split because of the non-triviality of Sq^4 in $H^*(C_\eta \wedge C_\eta, \mathbb{Z}/2)$. We get a diagram

$$\begin{tikzcd}
 & \pi_2 C_\eta \arrow[r, "i_*"] \arrow[d, "\eta"] & \pi_2 C_\eta \wedge C_\eta \arrow[r, "p_*"] \arrow[d, "\eta"] & \pi_2 \Sigma^2 C_\eta \arrow[r] & 0 \\
\pi_3 \Sigma C_\eta \arrow[r, "\eta"] & \pi_3 C_\eta \arrow[r, "i_*"] & \pi_3 (C_\eta \wedge C_\eta) & &
\end{tikzcd}$$

For any $b' \in \pi_2(C_\eta \wedge C_\eta)$ such that $p_* b' = \iota$, we must have $\eta b' \neq 0$, else we could split the above cofiber sequence. Since $\eta\iota = 0$, $\eta b' \in i_*(\pi_3 C_\eta)$, where $\pi_3 C_\eta \cong \mathbb{Z}/4\{\nu\iota\}$. But in $\pi_2 C_\eta$

$$2\nu\iota \in \langle \eta, 2, \eta \rangle = \langle \iota, \eta, 2 \rangle \eta,$$

hence $2\nu \in \eta_*(\pi_3 \Sigma C_\eta)$ and $i_*(2\nu) = 0$ and $\eta b' = i_*(\nu\iota)$

For the general case, note that any class $b : S^n \to X$ such that $\eta b = 0$ can be extended to a map $\bar{b} : \Sigma^n C_\eta \to X$. The claim then follows from the commutativity of the following diagram

$$\begin{tikzcd}
\Sigma^n C_\eta \arrow[r] \arrow[d, "\bar{b}"] & \Sigma^n C_\eta \wedge C_\eta \arrow[r] \arrow[d, "\bar{b}\wedge C_\eta"] & \Sigma^{n+2} C_\eta \arrow[d, "\Sigma^2 \bar{b}"] \\
X \arrow[r, "i"] & X \wedge C_\eta \arrow[r, "p"] & \Sigma^2 X
\end{tikzcd}$$

Then $b' = (\bar{b} \wedge C_\eta)_* \iota$ satisfies $\eta b' = i_*(\nu b)$. Now, suppose that $p_* \tilde{b}' = b$. Then $\tilde{b}' - b' \in \ker p_* = \operatorname{im} i_*$. Therefore, $\eta(\tilde{b}' - b') = 0$ so, $\eta \tilde{b}' = i_*(\nu b)$ as well. □

2.6 | Self-maps and their cofibers

It is well-known that $V(0)$ admits v_1^4 self-maps, that is, maps $\Sigma^8 V(0) \to V(0)$ which induce multiplication by v_1^4 in $K(1)$-homology for $K(1)$ the first Morava K-theory. The map on MU-homology is given by multiplication by $x_1^4 \in MU_8$. Under the map from the Adams–Novikov spectral sequence of $V(0)$ to that of the elliptic spectral sequence of $tmf \wedge V(0)$, x_1 maps to v_1 on the E_2-term. See the discussion surrounding (3.3). Any v_1^4 self-map is detected by the same-named element. The spectral sequence inherits an action of v_1^4 and the differentials are v_1^4-linear.

Recall that we let Y be the spectrum $V(0) \wedge C_\eta$. In [20], Davis and Mahowald show that there exist v_1 self-maps of Y, that is, maps $\Sigma^2 Y \to Y$ which induce multiplication by v_1 in $K(1)_* Y$. Any of these is detected by the element v_1 on the E_2-term of elliptic spectral sequence for $tmf \wedge Y$ and the differentials are v_1-linear.

In Lemma 6.41, we will be studying the v_1-multiplication in $tmf_* Y$. Some of the answers will *depend on the choice of* v_1*-self map*, so we give a bit of background here on this subject. This material can be found in [20].

In [20], the authors show that there are in fact eight v_1-self maps of Y. They also show that a v_1-self map of Y is detected in the Adams spectral sequence by an element of $\mathrm{Ext}_{\mathcal{A}}^{1,3}(H^*(Y), H^*(Y))$, where $\mathcal{A}$ denotes the Steenrod algebra at $p = 2$.

A class of $\mathrm{Ext}_{\mathcal{A}}^{1,3}(H^*(Y), H^*(Y))$ is represented by a short sequence of $\mathcal{A}$-modules:

$$0 \to \Sigma^2 H^*(Y) \to M \to H^*(Y) \to 0.$$

Let $\mathcal{A}(1)$ be the sub-algebra of the Steenrod algebra generated by Sq^1 and Sq^2. We know that $\mathrm{Ext}_{\mathcal{A}(1)}^{1,3}(H^*(Y), H^*(Y)) \cong \mathbb{F}_2$ and its unique non-trivial class is represented by the short exact sequence of $\mathcal{A}(1)$-module

$$0 \to \Sigma^2 H^*(Y) \to A(1) \to H^*(Y) \to 0,$$

where $A(1)$ is isomorphic to $\mathcal{A}(1)$ as an $\mathcal{A}(1)$-module, thus the notation. Davis and Mahowald showed that a class of $\mathrm{Ext}_{\mathcal{A}}^{1,3}(H^*(Y), H^*(Y))$ which detects a v_1-self map of Y is sent to the unique non-trivial class of $\mathrm{Ext}_{\mathcal{A}(1)}^{1,3}(H^*(Y), H^*(Y))$ (via the map induced by the inclusion $\mathcal{A}(1) \subset \mathcal{A}$).

To put an $\mathcal{A}$-module structure on $A(1)$, it suffices to specify the Sq^4 action. Indeed, the action of Sq^k, for $k \geqslant 8$ on $A(1)$ is trivial for degree reasons. By the Adem relations, there must be a non-trivial Sq^4 on the class of degree one of $A(1)$. There are possibilities for a non-trivial action of Sq^4 on the classes of degrees zero and two, giving rise to four different $\mathcal{A}$-module structures on $A(1)$.

This implies, in particular, that

$$\mathrm{Ext}^{1,3}_{\mathcal{A}}(H^*(Y), H^*(Y)) \cong \mathbb{F}_2^{\oplus 3}.$$

Computing the first three stems of $\mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(Y), H^*(Y))$, we see that

$$\mathrm{Ext}^{s,s+2}_{\mathcal{A}}(H^*(Y), H^*(Y)) \cong \begin{cases} \mathbb{F}_2 & \text{if } s = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We deduce that there are eight homotopy classes of maps $\Sigma^2 Y \to Y$ detected in $\mathrm{Ext}^{1,3}_{\mathcal{A}}(H^*(Y), H^*(Y))$ and mapping non-trivially to $\mathrm{Ext}^{1,3}_{\mathcal{A}(1)}(H^*(Y), H^*(Y))$. These are the v_1 self-maps of Y.

It is somewhat surprising that out of eight v_1-self-maps, there are only four homotopy types which are distinguished by their cohomology, as is shown [20].

The singular cohomology of the cofiber of each of the v_1-self maps on Y is isomorphic to one of the four $\mathcal{A}(1)$s as an $\mathcal{A}$-module. We denote the cofibers realizing the four choices of Steenrod algebra structure by $A_1[ij]$, with $i, j \in \{0, 1\}$. Here, $A_1[ij]$ means that the cohomology of the spectrum $A_1[ij]$ has a non-trivial Sq^4 on the class of degree 0 (if $i = 1$) or 2 (if $j = 1$). For more details and diagrams indicating the Steenrod algebra action, see [12, figure 1]. We use the notation A_1, for short, when we mean any or all of the four models.

3 | $tmf_*V(0)$: THE E_2-PAGE

From now on, we will be working exclusively with 2-local spectra. We will write tmf for $tmf_{(2)}$ to simplify the notation. Furthermore, we will be considering only elliptic spectral sequences for $M = tmf \wedge F$ for F a finite spectrum and so shorten our notation even more to

$$E_2^{s,t}(F) := \mathrm{Ext}^{s,t}_{\Lambda'}(A', A' \otimes_A \mathcal{F}_*(F)).$$

The map $S^0 \xrightarrow{\times 2} S^0$ induces multiplication by 2 on $\mathcal{F}_*(S^0) \cong A$, which is injective. Thus, the cofiber sequence

$$S^0 \xrightarrow{2} S^0 \to V(0)$$

gives rise to a short exact sequence of Λ'-comodules

$$0 \to A' \xrightarrow{\times 2} A' \to A' \otimes_A \mathcal{F}_*(V(0)) \to 0. \tag{3.1}$$

It follows that $A' \otimes_A \mathcal{F}_*(V(0))$ is isomorphic to $A'/(2)$ as a Λ'-comodule. Since $(2) \subseteq A'$ is a Λ'-invariant ideal, we have that

$$\mathrm{Ext}^{s,t}_{\Lambda'}(A', A'/(2)) \cong \mathrm{Ext}^{s,t}_{\Lambda'/(2)}(A'/(2), A'/(2)).$$

See, for example, [35, Proposition A1.2.16]. So, we have a spectral sequence

$$E_2^{s,t}(V(0)) = \mathrm{Ext}^{s,t}_{\Lambda'/(2)}(A'/(2), A'/(2)) \Longrightarrow \pi_* tmf \wedge V(0). \tag{3.2}$$

A computation of the cohomology of $(A'/(2), \Lambda'/(2))$ is originally due to Hopkins and Mahowald and can be found in [2, section 7; 21, chapter 15, section 7] Let us describe the answer here and introduce some notation.

Classical computations of modular forms yield

$$\mathrm{Ext}^{0,*}_{\Lambda'}(A', A') \cong \mathbb{Z}_{(2)}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - (12)^3\Delta),$$

where

$$\begin{aligned} c_4 &= a_1^4 - 24a_1a_3 \\ c_6 &= -a_1^6 + 36a_1^3a_3 - 216a_3^2 \\ \Delta &= a_1^3a_3^3 - 27a_3^4 \end{aligned}$$

as well as

$$\mathrm{Ext}^{0,*}_{\Lambda'/(2)}(A'/(2), A'/(2)) \cong \mathbb{Z}/2[a_1, \Delta].$$

See, for example, [2; 38, III.1]. The map on $\mathrm{Ext}^{0,*}$ induced by the mod 2 reduction $(A', \Lambda') \to (A'/(2), \Lambda'/(2))$ sends $c_4 \mapsto a_1^4$ and $c_6 \mapsto a_1^6$.

There are also maps of Adams–Novikov spectral sequences, where H and h are as in Remark 2.5:

$$\begin{tikzcd}
\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*V(0)) \arrow[r] & \pi_*V(0) \\
\mathrm{Ext}^{*,*}_{MU_*MU}(MU_*, MU_*V(0)) \arrow[r] \arrow[u, "\cong"] \arrow[d, "H"] & \pi_*V(0) \arrow[u, "\cong"'] \arrow[d, "h"] \\
\mathrm{Ext}^{*,*}_{\Lambda'/(2)}(A'/(2), A'/(2)) \arrow[r] & \pi_*tmf \wedge V(0)
\end{tikzcd}$$

Further,

$$\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*V(0)) \cong \mathbb{F}_2[v_1];$$

see [35, Theorem 4.3.2].

So, we have $a_1 \in \mathrm{Ext}^{0,2}_{\Lambda'/(2)}(A'/(2), A'/(2))$, $v_1 \in \mathrm{Ext}^{0,2}_{BP_*BP}(BP_*, BP_*V(0))$ and $x_1 \in \mathrm{Ext}^{0,2}_{MU_*MU}(MU_*, MU_*V(0))$, and

$$v_1 \leftarrow\!\shortmid x_1 \mapsto a_1. \tag{3.3}$$

Note that v_1 detects either of the two classes in $\pi_2V(0) \cong \mathbb{Z}/4$ which map to $\eta \in \pi_1V(0)$ under the homomorphism $\pi_2V(0) \to \pi_1S^0$ in the long exact sequence in homotopy. We fix a choice and call it $v_1 \in \pi_2V(0)$. It follows that a_1 survives to detect the image of $v_1 \in \pi_2V(0)$ in $\pi_2tmf \wedge V(0)$. From now on, in mod 2 computations, we abuse notation and denote all classes we have named a_1 by v_1.

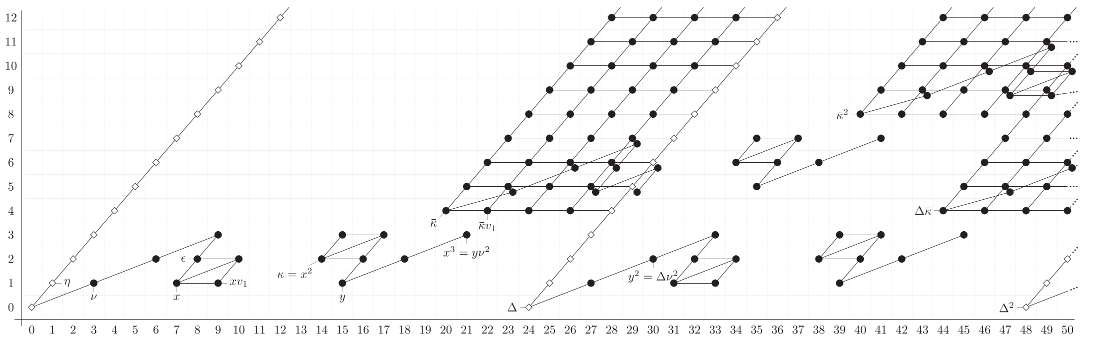


FIGURE 2 The E_2-term of the elliptic spectral sequence for $tmf \wedge V(0)$ in the range $0 \leqslant t-s \leqslant 50$. A bullet • denotes $\mathbb{F}_2$ and a diamond ⋄ denotes a copy of $\mathbb{F}_2[v_1]$. The lines of slope 1 denote multiplication by η, and the lines of slope 1/3 denote multiplication by ν. Horizontal lines are v_1-multiplications.

Now we will present the E_2 page of (3.2) as computed in [2, p.26; 21, p. 270; 40, figure. 5] (see Figure 2). Even if the elliptic spectral sequence for $V(0)$ is not multiplicative, $E_2(V(0))$ is a ring and we can completely describe the algebraic relations (which also follow from [2]). The ring structure will be used in our computation of $E_2(Y)$ below.

Recall that $\delta = \delta_2$ was defined in Theorem 2.19. In the theorem below, $\kappa \in E_2^{2,16}(S^0)$ is the unique non-zero element.

Theorem 3.4 (Figure 2). *The ring $E_2(V(0))$ is isomorphic to*

$$\mathbb{F}_2[v_1, \Delta, \bar{\kappa}, \eta, \nu, x, y]/(\sim)$$

for elements

$$\eta \in \mathrm{Ext}^{1,2}, \quad \nu \in \mathrm{Ext}^{1,4}, \quad \bar{\kappa} \in \mathrm{Ext}^{4,24}, \quad \Delta \in \mathrm{Ext}^{0,24}$$

in the image of $E_2(S^0) \to E_2(V(0))$, as well as elements

$$v_1 \in \mathrm{Ext}^{0,2}, \quad x \in \mathrm{Ext}^{1,8}, \quad y \in \mathrm{Ext}^{1,16}$$

in the image of $\delta_2 : E_2(V(0)) \to E_2(S^0)$ where

$$\delta_2(v_1) = \eta, \;\; \delta_2(x) = \nu^2, \;\; \delta_2(y) = \kappa.$$

The relations $(\sim)$ is the ideal generated by

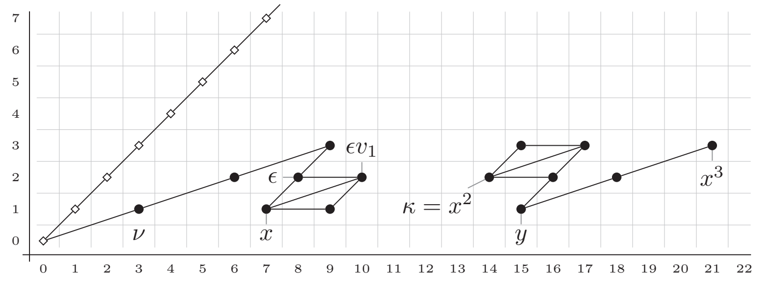
$(s=1)$	$v_1\nu$	$v_1^2 x$	$v_1 y$		
$(s=2)$	$\nu\eta$	$\nu x - v_1\eta x$	$\eta y - v_1 x^2$	xy	$y^2 - \nu^2\Delta$
$(s=3)$	$\eta^2 x - \nu^3$	$x^3 - \nu^2 y$			
$(s=4)$	$\eta^4\Delta - v_1^4\bar{\kappa}.$				

Furthermore, we have $\kappa = x^2$ and $\delta_2(\nu^2 y) = 4\bar{\kappa}$.

Remark 3.5. The algebraic structure in Theorem 3.4 can also be deduced from the appendix of [4].

Remark 3.6. The element Δ is detected by v_2^4 in the Bockstein spectral sequence computation of [21, II.2.7].

Remark 3.7. Let P denote the following pattern:



Then $E_2^{*,*}(V(0))$ can be summarized additively as

$$E_2^{*,*}(V(0)) = P[\bar{\kappa}, \Delta]/(\Delta\eta^4 - \bar{\kappa}v_1^4).$$

4 | $tmf_*V(0)$: THE DIFFERENTIALS AND EXTENSIONS

We begin with an observation that $V(0)$ has a v_1^4 self-map, hence all differentials d_r for $r \geqslant 3$ are v_1^4 linear. Since η, ν, $\bar{\kappa}$ and Δ^8 are permanent cycles, all differentials are linear with respect to multiplication by these elements. Note that there are no even length differentials due to sparseness.

We will use the following methods when computing differentials in this section.

(1) The map of spectra

$$i : tmf \to tmf \wedge V(0)$$

induces a map of spectral sequences. Let d_r^{tmf} denote the differentials in the spectral sequence for tmf. We can import the differentials $d_r^{tmf}(a) = b$ from the spectral sequence for tmf if the images of a and b are both non-trivial on the E_r page of the spectral sequence for $tmf \wedge V(0)$. Note also that the elliptic spectral sequence for $tmf \wedge V(0)$ is a module over the elliptic spectral sequence for the ring spectrum tmf. For $a \in E_r(S^0)$, let $\overline{a} \in E_r(V(0))$ denote $i^*(a)$ where $i^* : E_r(S^0) \to E_r(V(0))$ is induced from the unit map $i : S^0 \to V(0)$. Then, for $a \in E_r(S^0)$ and $x \in E_r(V(0))$ we have

$$d_r(ax) = \overline{d_r^{tmf}(a)}x + ad_r(x). \tag{4.1}$$

(2) The long exact sequence in homotopy groups associated to the fiber sequence

$$tmf \xrightarrow{2} tmf \to tmf \wedge V(0)$$

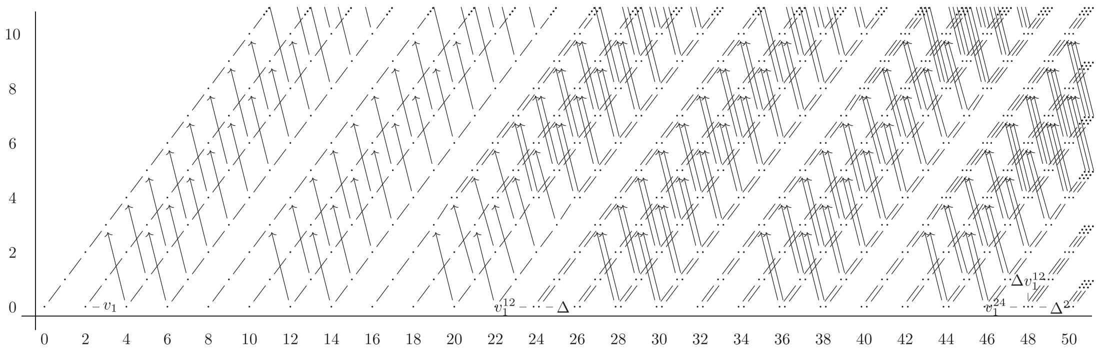


FIGURE 3 The η-towers and the d_3 differentials between them

gives short exact sequences

$$0 \to (\pi_i tmf)/2 \to \pi_i(tmf \wedge V(0)) \to \ker_2(\pi_{i-1} tmf) \to 0,$$

where $\ker_2(\pi_{i-1} tmf)$ is the subgroup of elements of order 2. This allows us to compute the rank of $\pi_i(tmf \wedge V(0))$ and forces certain differentials by various dimension count arguments.

(3) The Geometric Boundary Theorem, stated in Theorem 2.19.

For convenience of the reader, the large charts of the elliptic spectral sequence for $tmf_* V(0)$ can be found in [6].

4.1 | The d_3-differentials

Lemma 4.2 (Figure 3). *The d_3-differentials are Δ and v_1^4-linear. They are determined by this linearity, the differentials*

$$d_3(v_1^2) = \eta^3; \quad d_3(v_1^3) = v_1\eta^3,$$

and the module structure over the elliptic spectral sequence for tmf.

Proof. These differentials follow from the differential $d_3(a_1^2\eta) = \eta^4$ in the elliptic spectral sequence for tmf. Both the source and the target are not η-torsion on the E_3 page, so we can divide by η to get the first differential. Alternatively, the two listed d_3-differentials occur in the Adams–Novikov spectral sequence computing $\pi_* V(0)$ so happen here also by naturality (see, for example, [34, Theorem 5.13 (a)]).

Since Δ is a d_3-cycle in the elliptic spectral sequence computing $\pi_* tmf$ and the elliptic spectral sequence for $V(0)$ is a module over this spectral sequence, the d_3-differentials are Δ-linear. For degree reasons (making use of Δ and $\bar{\kappa}$-linearity), these determine all d_3-differentials. $\square$

The effect of the d_3 differentials is truncating the η-towers on the E_3 page. Figure 3 illustrates this process. This figure contains only the η-towers and omits the other classes. It does contain all the d_3 differentials.

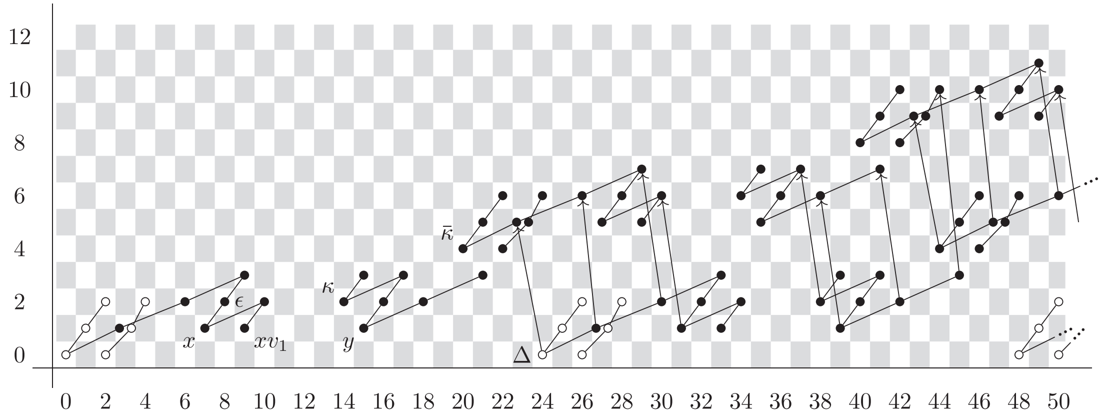


FIGURE 4 d_5 and d_7-differentials in stems 0-48. A $\circ$ denotes $\mathbb{F}_2[v_1^4]$.

Remark 4.3. On the E_5-page, all classes in filtrations $s \geqslant 3$ are v_1^4-torsion. The v_1^4-free classes are concentrated in stems $t - s \not\equiv 5, 6, 7 \mod 8$.

4.2 | The d_5-differentials

Lemma 4.4 (Figure 4). *The d_5-differentials are Δ^2-linear. They are determined by this linearity, the differential*

$$d_5(\Delta) = \bar{\kappa}\nu \qquad\qquad (24,0) \mapsto (23,5)$$

and the module structure over the elliptic spectral sequence for tmf.

Proof. The differential

$$d_5(\Delta) = \bar{\kappa}\nu$$

occurs in the spectral sequence for $\pi_* tmf$.

Linearity (4.1) over the spectral sequence for $\pi_* tmf$ gives us, for $x \in E_5^{*,*}(V(0))$

$$d_5(\Delta^2 x) = d_5^{tmf}(\Delta^2)x + \Delta^2 d_5(x) = \overline{2\Delta\bar{\kappa}\nu} x + \Delta^2 d_5(x) = \Delta^2 d_5(x). \qquad \square$$

4.3 | Higher differentials

Since all the classes in filtrations 4 and above are in the ideal generated by $\bar{\kappa}$, the differentials that have sources in filtrations 0–3 generate the other differentials with respect to the module structure over the elliptic spectral sequence for tmf (denoted $E_r^{*,*}(S^0)$). We focus on these differentials in the narrative (see Figures 4, 5, 6, and 7).

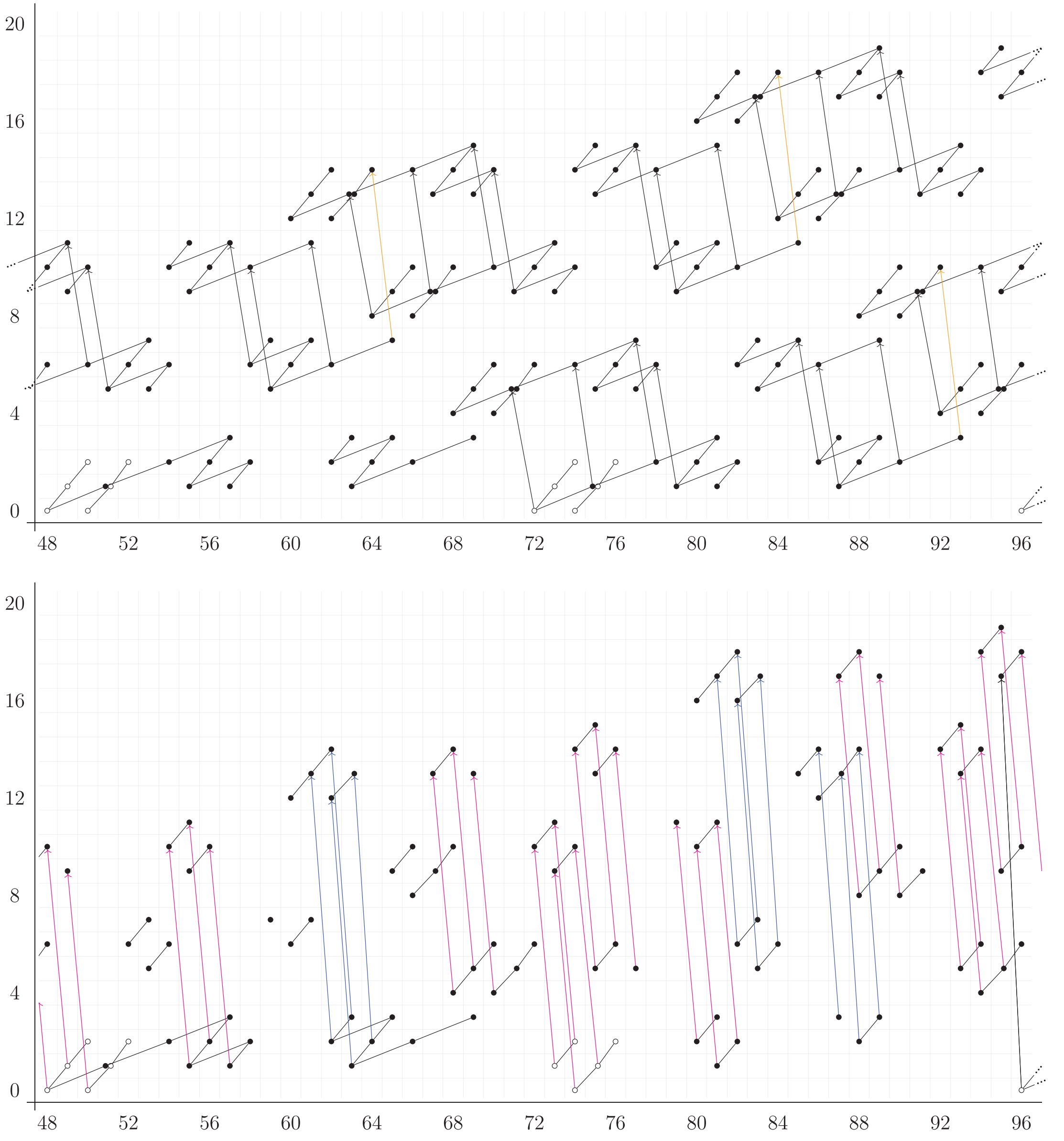


FIGURE 5 Differentials in stems 48 to 96

Lemma 4.5. *The d_7-differentials are Δ^4-linear and determined by*

$$d_7(\Delta\nu^2 y) = \bar{\kappa}^2\eta^2 v_1 \qquad (45, 3) \mapsto (44, 10)$$

$$d_7(\Delta^3\nu^2 y) = \Delta^2\bar{\kappa}^2\eta^2 v_1 \qquad (93, 3) \mapsto (92, 10)$$

and the module structure over the elliptic spectral sequence for tmf.

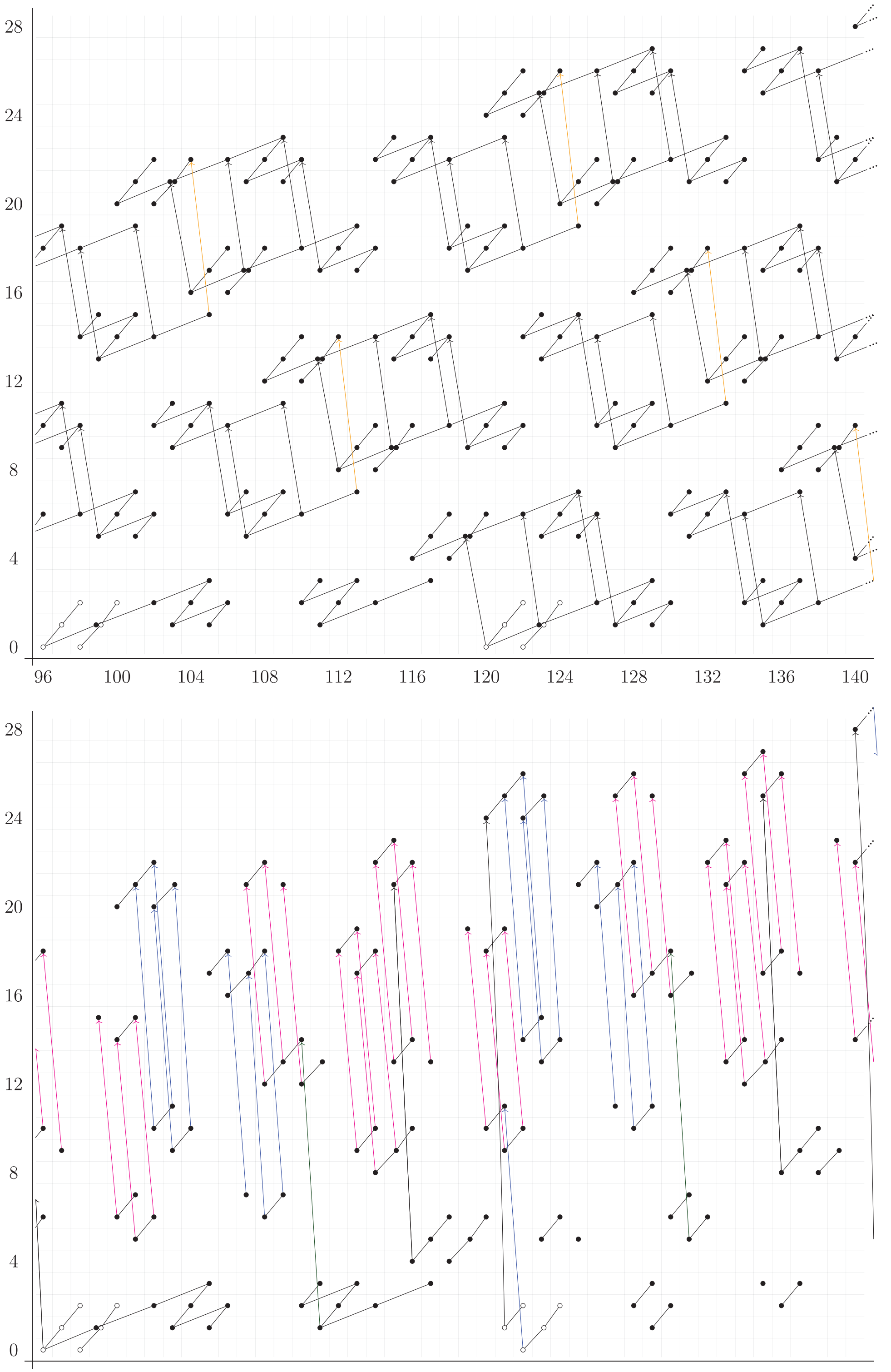
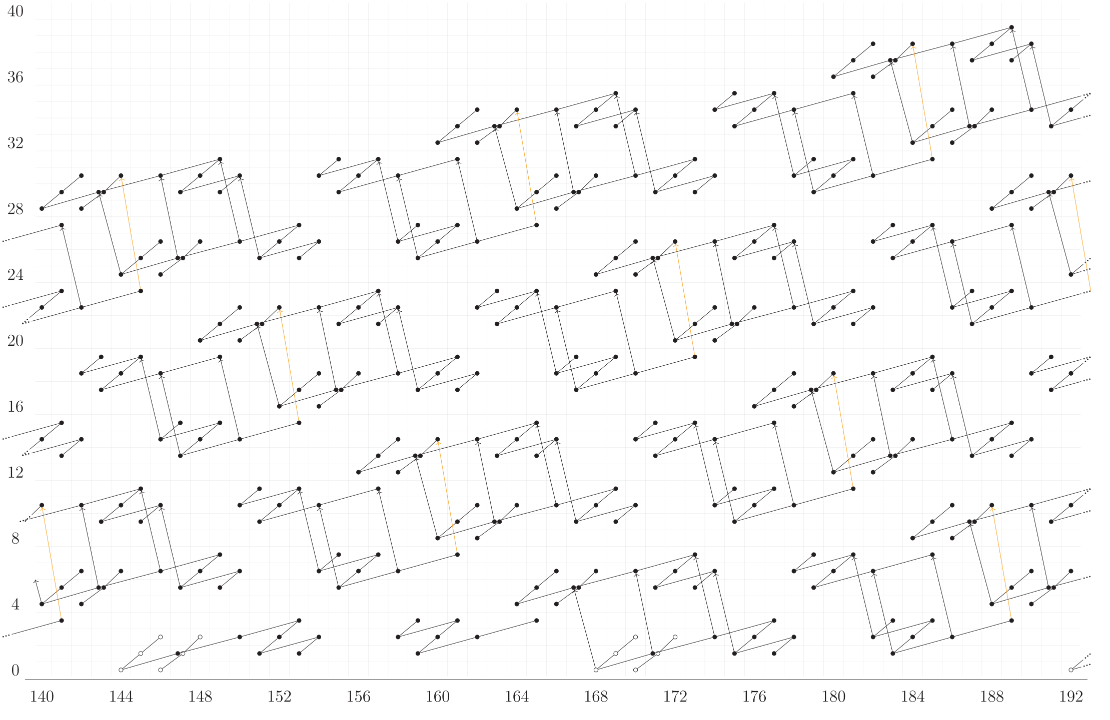


FIGURE 6 Differentials in stems 96 to 140



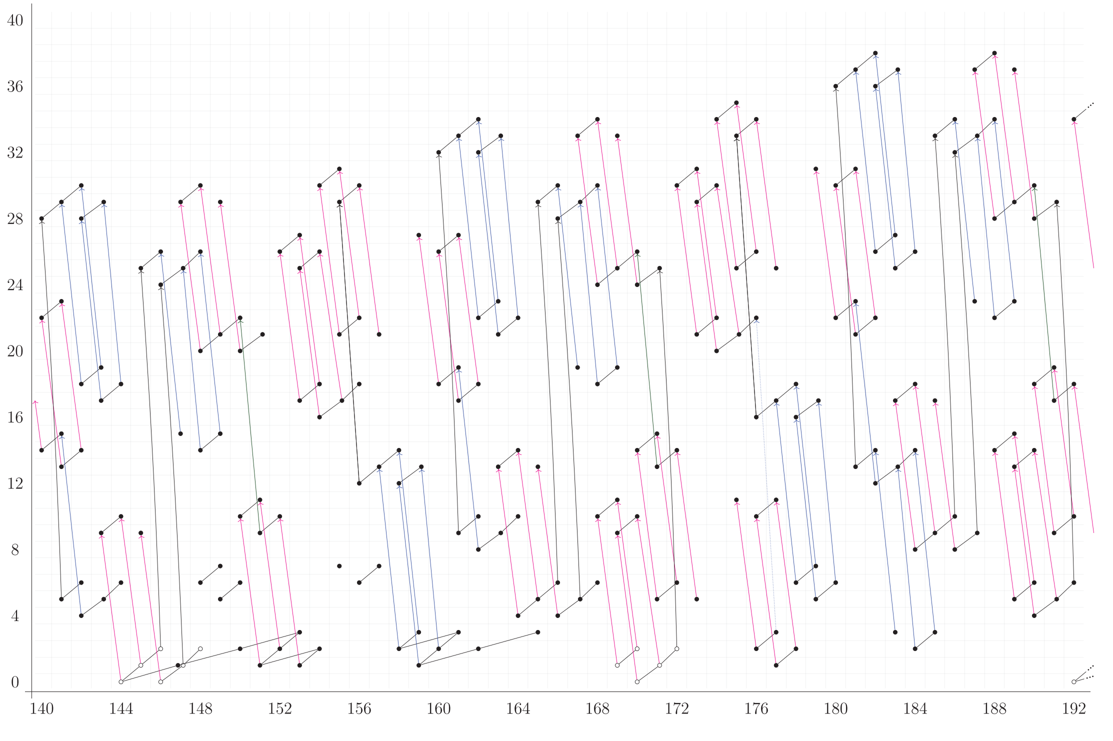


FIGURE 7 Differentials in stems 140 to 192

Proof. First, note that $d_7^{tmf}(\Delta^4) = \Delta^3\eta^3\bar{\kappa}$ in the spectral sequence for tmf. Therefore, using (4.1), for any $a \in E_7(V(0))$ we have

$$d_7(\Delta^4 a) = \overline{\Delta^3\eta^3\bar{\kappa}}a + \Delta^4 d_7(a).$$

Since $\overline{\Delta^3\eta^3\bar{\kappa}} = 0$, we get Δ^4-linearity.

We give a proof for the differential $d_7(\Delta\nu^2 y) = \bar{\kappa}^2\eta^2 v_1$. The proof for the other differential is similar. In the spectral sequence for tmf, we have

$$d_7(\Delta 4\bar{\kappa}) = \eta^3\bar{\kappa}^2.$$

But, for $\delta_2 : E_2^{s,t}(V(0)) \to E_2^{s+1,t}(S^0)$ the connecting homomorphism, we have

$$\delta_2(\Delta\nu^2 y) = \Delta 4\bar{\kappa}$$

and

$$\delta_2(\bar{\kappa}^2\eta^2 v_1) = \bar{\kappa}^2\eta^3.$$

The differential when $i = 0$ then follows from Theorem 2.19.

Making use of the module structure over the spectral sequence for tmf, the only other possible d_7-differential for degree reasons is on $\Delta^2\nu^2 y$. But this class is in fact a d_7-cycle since $\Delta^2 y$ is a d_7-cycle by sparseness. □

Lemma 4.6. *Using the module structure over the elliptic spectral sequence for tmf, the d_9-differentials are determined by the following differentials with $i = 0, 1$:*

(1) $d_9(\Delta^{2+4i}) = \Delta^{4i}\bar{\kappa}^2 x$	$(48 + 96i, 0) \mapsto (47 + 96i, 9)$
(2) $d_9(\Delta^{2+4i} x) = \Delta^{4i}\bar{\kappa}^2\kappa$	$(55 + 96i, 1) \mapsto (54 + 96i, 10)$
(3) $d_9(\Delta^{3+4i}\eta) = \Delta^{1+4i}\bar{\kappa}^2\epsilon$	$(73 + 96i, 1) \mapsto (72 + 96i, 10)$
(4) $d_9(\Delta^{3+4i}\epsilon) = \Delta^{1+4i}\kappa\bar{\kappa}^2\eta$	$(80 + 96i, 2) \mapsto (79 + 96i, 11)$
(5) $d_9(\Delta^{2+4i} v_1) = \Delta^{4i}\bar{\kappa}^2 v_1 x$	$(50 + 96i, 0) \mapsto (49 + 96i, 9)$
(6) $d_9(\Delta^{2+4i} v_1 x) = \Delta^{4i}\bar{\kappa}^2\eta y$	$(57 + 96i, 1) \mapsto (56 + 96i, 10)$
(7) $d_9(\Delta^{3+4i} v_1) = \Delta^{1+4i}\bar{\kappa}^2 v_1 x$	$(74 + 96i, 0) \mapsto (73 + 96i, 9)$
(8) $d_9(\Delta^{3+4i} v_1 x) = \Delta^{1+4i}\bar{\kappa}^2\eta y$	$(81 + 96i, 1) \mapsto (80 + 96i, 10)$

Proof. We prove the claim for $i = 0$. To prove $i = 1$, one uses exactly the same arguments in later stems.

To show (1), note that Δ^2 cannot support any d_r for $r < 9$ by sparseness. Then we have the differential from the elliptic spectral sequence for tmf

$$d_9(\Delta^2\eta) = \bar{\kappa}^2\epsilon$$

and this differential becomes η divisible in the spectral sequence for $tmf \wedge V(0)$. For (2), we use the same argument with the differential $d_9(\Delta^2\epsilon) = \Delta\bar{\kappa}^2\kappa\eta$ from the elliptic spectral sequence for tmf.

The differentials (3) and (4) are the images of the same differentials in the elliptic spectral sequence for tmf. The differentials (5)–(8) are proved using Theorem 2.19. For example, the differential $d_9(\Delta^2\eta) = \bar{\kappa}^2\epsilon$ and the facts that $\delta(v_1) = \eta$ and $\delta(v_1 x) = \epsilon$ together imply (5). The others are similar.

It remains to argue that there are no other generating d_9-differentials. As noted above, it suffices to determine this on classes in filtration less than four.

Combining a comparison with the spectral sequence for tmf and sparseness, we see that the only question is whether or not the classes $\Delta^4 x$ and $\Delta^4 v_1 x$ support non-trivial d_9s. However, a differential $d_9(\Delta^4 x) = \Delta^2\kappa\bar{\kappa}^2$ together with η-linearity would imply the differential $d_9(\Delta^4\epsilon) = \Delta^2\kappa\bar{\kappa}^2\eta$. In the latter differential both source and target are in the image of the unit map from the elliptic spectral sequence for tmf, hence this would also imply a differential $d_9^{tmf}(\Delta^4\epsilon) = \Delta^2\kappa\bar{\kappa}^2\eta$ in the elliptic spectral sequence for tmf, which does not happen. The same argument works for $\Delta^4 v_1 x$.

We will also see in the next lemma that the possible targets of these d_9s are the sources of $\bar{\kappa}$-multiples of the d_{11}-differentials (1) and (3) of Lemma 4.7. □

Lemma 4.7. *Using the module structure over the elliptic spectral sequence for tmf, the d_{11}-differentials are determined by the following differentials with $i = 0, 1$:*

(1) $d_{11}(\Delta^{2+4i}\kappa) = \Delta^{4i}\bar{\kappa}^3\eta$ $\quad (62 + 96i, 2) \mapsto (61 + 96i, 13)$

(2) $d_{11}(\Delta^{3+4i}\kappa\eta) = \Delta^{1+4i}\bar{\kappa}^3\eta^2$ $\quad (87 + 96i, 3) \mapsto (86 + 96i, 14)$

(3) $d_{11}(\Delta^{2+4i}y) = \Delta^{4i}\bar{\kappa}^3 v_1$ $\quad (63 + 96i, 1) \mapsto (62 + 96i, 12)$

(4) $d_{11}(\Delta^{3+4i}\kappa v_1) = \Delta^{1+4i}v_1\bar{\kappa}^3\eta$ $\quad (88 + 96i, 2) \mapsto (87 + 96i, 13)$

(5) $d_{11}(\Delta^5 v_1) = \Delta^3\bar{\kappa}^2\nu^3$ $\quad (122, 0) \mapsto (121, 11).$

Proof. The differentials (1) and (2) are images of the same differentials in the spectral sequence for tmf. The differentials (3) and (4) follow from (1) and (2), respectively, using Theorem 2.19. The differential (5) follows from the fact that $\pi_{121}(tmf \wedge V(0))$ does not contain v_1^4-torsion, which can be verified by comparing with $\pi_* tmf$ using the long exact sequence on π_*.

Sparseness and multiplicative structure guarantees that these are all the generating d_{11}-differentials, except for a possible d_{11} on $\Delta^7\eta^2 v_1$. However, $\delta_2(\Delta^7\eta^2 v_1) = 0$ but δ_2 of the possible target of this d_{11} is non-zero. □

Lemma 4.8. *The d_{13}-differentials are determined by*

$$d_{13}(\Delta^4 y) = \Delta^2\bar{\kappa}^3\eta^2 \qquad (111, 1) \mapsto (110, 14).$$

There are no d_{15}-differentials and the d_{17}-differentials are determined by

$$d_{17}(\Delta^4) = \bar{\kappa}^4 y \qquad (96,0) \mapsto (95,17).$$

The d_{19}-differentials are determined by

$$d_{19}(\Delta^7\nu^3) = \bar{\kappa}^5\Delta^3 v_1\eta^2 \qquad (177,3) \mapsto (176,22).$$

Proof. The first and second differentials follow from the facts that

$$\pi_{110}(tmf \wedge V(0)) = \mathbb{Z}/2 \text{ and } \pi_{95}(tmf \wedge V(0)) = 0,$$

respectively. The d_{19}-differential follows from the fact that the there is no v_1^4-torsion in $\pi_{177}(tmf \wedge V(0))$.

There are no d_{15} differentials and no other d_{17} and d_{19} for degree reasons. The only argument needed beyond sparseness and multiplicative structure to show that there are no other d_{13}-differentials is as follows. There are possible d_{13}s on $\Delta^3\nu^3$ and $\Delta^7\nu^3$. These classes are in the image of the tmf spectral sequence. For tmf, $d_{13}(\Delta^3\nu^3) = 2\bar{\kappa}^4$ and the target maps to zero in the spectral sequence for $tmf \wedge V(0)$ and similarly for $\Delta^7\nu^3$. □

Warning 4.9. The d_{13} differential above is in fact equivalent to the 2-extension in $\pi_{110}tmf$. For the reader familiar with names of classes, this corresponds to $2\kappa_4 = \eta_1\bar{\kappa}^3$. For a recent detailed treatment of this extension, see [19, chapter 9].

Lemma 4.10. *There are no d_{21}-differentials. The d_{23}-differentials are determined by:*

(1) $d_{23}(\Delta^5\eta) = \bar{\kappa}^6$ $\qquad (121,1) \mapsto (120,24)$
(2) $d_{23}(\Delta^6\eta^2) = \bar{\kappa}^6\Delta\eta$ $\qquad (146,2) \mapsto (145,25)$
(3) $d_{23}(\Delta^6\eta v_1) = \bar{\kappa}^6\Delta v_1$ $\qquad (147,1) \mapsto (146,24)$
(4) $d_{23}(\Delta^7\eta^2 v_1) = \bar{\kappa}^6\Delta^2\eta v_1$ $\qquad (172,2) \mapsto (171,25)$.

Proof. The differentials (1) and (2) occur in the elliptic spectral sequence for tmf. The differential (3) is the geometric boundary of (2) as in Theorem 2.19. The last differential is forced by the fact that the v_1^4-torsion in $\pi_{171}(tmf \wedge V(0))$ is trivial. Another way to see differentials (3) and (4) is to note that they follow from (1) and (2) using the module structure over the spectral sequence for tmf and the fact that $\Delta^5\eta \in E_{23}(S^0)$. We thank the referee for pointing this out. There are no d_{21} or other d_{23}-differentials for degree reasons. □

The following is now immediate.

Lemma 4.11. *The spectral sequence (3.2) computing $tmf_*V(0)$ collapses at E_{24} with a horizontal vanishing line at $s = 22$, that is, $E_\infty^{s,t}(V(0)) = 0$ for $s \geqslant 22$.*

4.4 | Exotic extensions

We list the exotic extensions that do occur. All other possibilities can be ruled out using algebraic structure and duality. We bring to the attention of the reader the precise meaning of exotic exten-

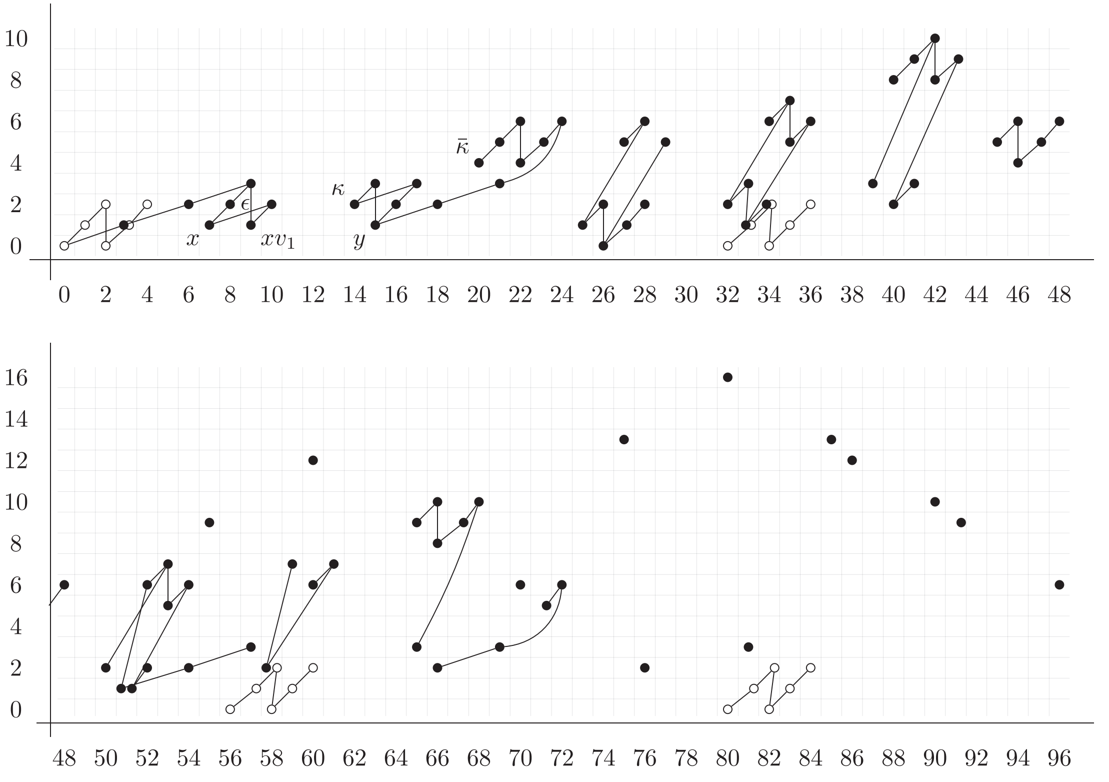


FIGURE 8 Exotic extensions in the elliptic spectral sequence for $tmf \wedge V(0)$ in stems 0 to 96. This records $tmf_*V(0) \cong \widetilde{tmf}_{*+1}\mathbb{R}P^2$.

sions given in Definition 2.20. Note also that all exotic 2-extensions are deduced from Lemma 2.21. We do not discuss 2-extensions further but include them in our figures.

Lemma 4.12 (Figure 8). *In stems 0 to 45, there are exotic extensions:*

$$
\begin{aligned}
&[\Delta\eta]\nu = \bar{\kappa}\epsilon && \textit{from } (25, 1) \textit{ to } (28, 6)\\
&[\Delta\epsilon]\nu = \kappa\bar{\kappa}\eta && \textit{from } (32, 2) \textit{ to } (35, 7)\\
&[\Delta\kappa\eta]\nu = \bar{\kappa}^2\eta^2 && \textit{from } (39, 3) \textit{ to } (42, 10)\\
&[\Delta v_1]\nu = \bar{\kappa}v_1x && \textit{from } (26, 0) \textit{ to } (29, 5)\\
&[\Delta v_1x]\nu = \kappa\bar{\kappa}v_1 && \textit{from } (33, 1) \textit{ to } (36, 6)\\
&[\Delta\kappa v_1]\nu = \bar{\kappa}^2\eta v_1 && \textit{from } (40, 2) \textit{ to } (43, 9)\\
&[y\nu^2]\nu = \bar{\kappa}v_1\eta^2 && \textit{from } (21, 3) \textit{ to } (24, 6).
\end{aligned}
$$

Proof. The first three extensions are between elements from π_*tmf, see [2]. The next three are forced by the fact that the connecting homomorphism in the long exact sequence on homotopy groups is a map of $\pi_* S^0$-modules, the geometric boundary theorem, and the fact that under the map

$$\delta : E_2^{s,t}(V(0)) \to E_2^{s+1,t}(S^0)$$

we have $\delta(v_1) = \eta$ (and so $\delta(xv_1) = \epsilon$, $\delta(\kappa v_1) = \eta\kappa$, etc.).

The last extension follows from duality and the fact that there is a ν multiplication between stems 147 and 150 (already present on the E_2-page). □

Lemma 4.13 (Figure 8). *In stems 46 to 96, there are exotic extensions:*

(1) $[\Delta^2\eta^2]\nu = \Delta\bar{\kappa}\nu^3$ *from* $(50, 2)$ *to* $(53, 7)$
(2) $[\Delta^2\nu]\eta = \Delta\bar{\kappa}\epsilon$ *from* $(51, 1)$ *to* $(52, 6)$
(3) $[\Delta^2 v_1\eta]\nu = \Delta\bar{\kappa}x\nu$ *from* $(51, 1)$ *to* $(54, 6)$
(4) $[\Delta^2 x\nu]\eta = \Delta\bar{\kappa}\kappa\eta$ *from* $(58, 2)$ *to* $(59, 7)$
(5) $[\Delta^2 x\nu]\nu = \Delta\bar{\kappa}\kappa\eta v_1$ *from* $(58, 2)$ *to* $(61, 7)$
(6) $[\Delta^2\kappa\nu]\nu = \Delta\bar{\kappa}^2\eta^2 v_1$ *from* $(65, 3)$ *to* $(68, 10)$
(7) $[\Delta^2 y\nu^2]\nu = \Delta^2\bar{\kappa}v_1\eta^2$ *from* $(69, 3)$ *to* $(72, 6)$.

Proof. The first two extensions (1) and (2) are multiplicative relations that hold in $\pi_* tmf$. Extension (3) follows from (1) and Theorem 2.19. Extension (4) is dual to the algebraic η multiplication from stem 112 to 113, and similarly for (5). Extension (6) involves classes in the image of i_* and this extension happens in tmf_*. Finally, (7) is dual to the algebraic ν multiplication from stem 99 to 102. □

Remark 4.14. Looking at the charts in [2], one might have expected extensions $[\Delta^2\kappa\nu]\eta = \Delta\bar{\kappa}^2\eta^2$ and, by the Geometric Boundary Theorem, $[\Delta^2 y\nu]\eta = \Delta\bar{\kappa}^2\eta v_1$. However, these are not exotic extensions according to Definition 2.20.

We also note that $[\Delta^2 c_4]\nu \neq [\Delta\bar{\kappa}\kappa\eta]$ and $[\Delta^3 c_4 v_1]\nu \neq [\Delta\bar{\kappa}^3\eta]$. The first comes from the fact that in $\pi_* tmf$, there is no such extension. (This can be seen, for example, from the Adams spectral sequence of tmf.) The second follows from the fact that the target has a non-trivial $\bar{\kappa}$-multiple and $\bar{\kappa}\nu = 0$.

Lemma 4.15 (Figure 9). *In stems 97 to 144, there are exotic extensions:*

(1) $[\Delta^4\eta]\nu = \bar{\kappa}^5$ *from* $(97, 1)$ *to* $(100, 20)$
(2) $[\Delta^4\nu]\eta = \bar{\kappa}^5$ *from* $(99, 1)$ *to* $(100, 20)$
(3) $[\Delta^4\bar{\kappa}\epsilon]\eta = \Delta\bar{\kappa}^5\eta$ *from* $(124, 6)$ *to* $(125, 21)$
(4) $[\Delta^5\eta^2]\nu = \Delta\bar{\kappa}^5\eta$ *from* $(122, 2)$ *to* $(125, 21)$
(5) $[\Delta^5\epsilon]\nu = \Delta^4\kappa\bar{\kappa}\eta$ *from* $(128, 2)$ *to* $(131, 7)$
(6) $[\Delta^5\kappa\eta]\nu = \Delta^4\bar{\kappa}^2\eta^2$ *from* $(135, 3)$ *to* $(138, 10)$
(7) $[\Delta^4\bar{\kappa}xv_1]\eta = \Delta\bar{\kappa}^5 v_1$ *from* $(125, 5)$ *to* $(126, 20)$
(8) $[\Delta^5 xv_1]\nu = \Delta^4\kappa\bar{\kappa}v_1$ *from* $(129, 1)$ *to* $(132, 6)$
(9) $[\Delta^5\kappa v_1]\nu = \Delta^4\bar{\kappa}^2\eta v_1$ *from* $(136, 2)$ *to* $139, 9)$
(10) $[\Delta^5\epsilon v_1]\eta = \Delta^2\bar{\kappa}^4 v_1\eta$ *from* $(130, 2)$ *to* $(131, 17)$
(11) $[\Delta^4 v_1]\nu = \Delta^3\bar{\kappa}\nu^3$ *from* $(98, 0)$ *to* $(101, 7)$
(12) $[\Delta^5\kappa\eta]\eta = \Delta^3\bar{\kappa}^3\eta^2 v_1$ *from* $(135, 3)$ *to* $(136, 14)$
(13) $[\Delta^4 y\nu^2]\nu = [\Delta^4\bar{\kappa}v_1\eta^2]$ *from* $(117, 3)$ *to* $(120, 6)$.

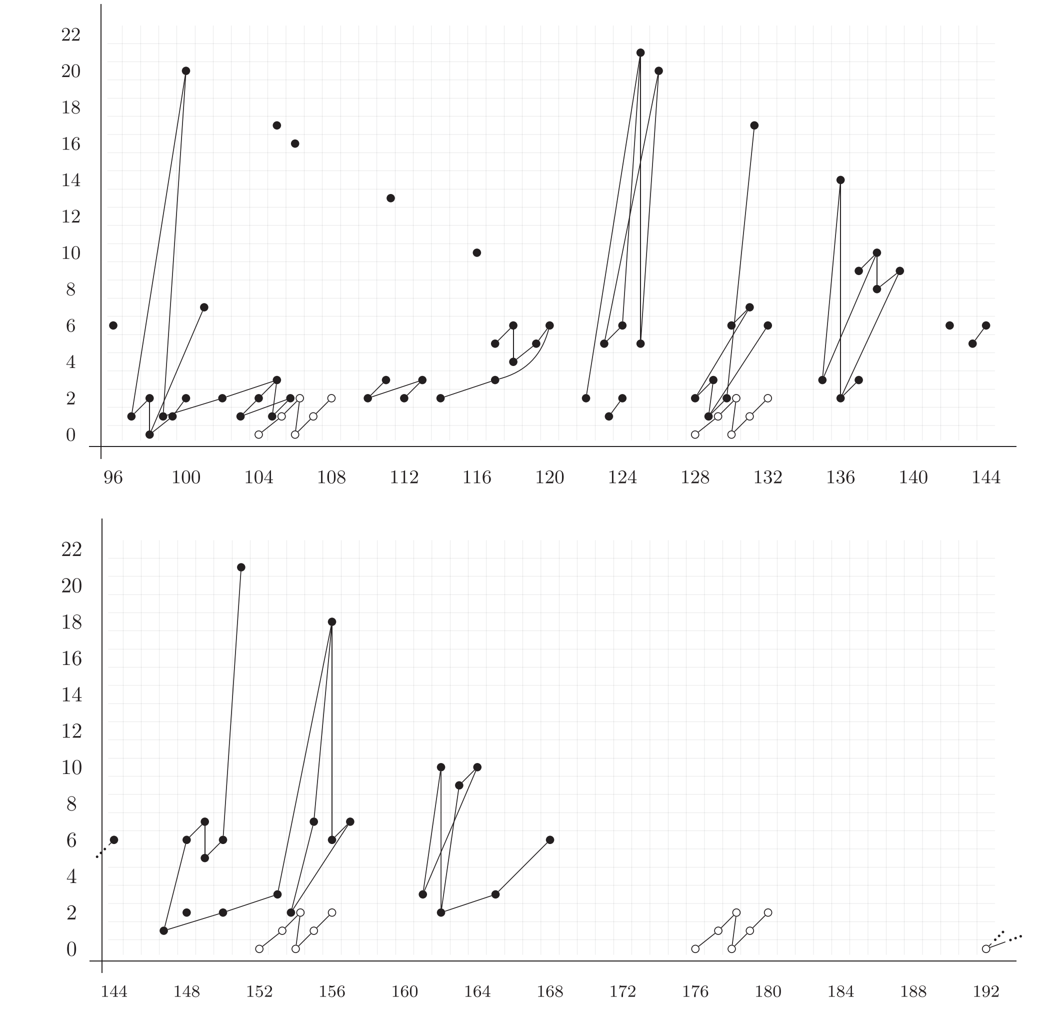


FIGURE 9 Exotic extensions in the elliptic spectral sequence of $tmf \wedge V(0)$ in stems 96 to 192, recording $tmf_*V(0) \cong \widetilde{tmf}_{*+1}\mathbb{R}P^2$

Proof. Extensions (1)–(6) follow from studying $i_* : tmf_* \to tmf_*V(0)$. Note that (4) is missing from the [2] charts, but it is the $[\Delta\eta]$- multiple of the extension $[\Delta^4\eta]\nu = \bar{\kappa}^5$ as computed there. We thank the referee for pointing this out. Extensions (7), (8), and (9) follow from (3), (5), and (6), respectively, using Theorem 2.19.

For (11), note that by Theorem 2.19, $[\Delta^4 v_1]$ has geometric boundary $[\Delta^4\eta]$. Since $[\Delta^4\eta]\nu \neq 0$, $[\Delta^4 v_1]\nu \neq 0$ and this extension is the only choice. For (12), use Remark 2.16 and the algebraic η multiplication between $\pi_{35} tmf \wedge V(0)$ and $\pi_{36} tmf \wedge V(0)$. A similar argument applies for (13). □

Remark 4.16. There is no exotic ν-extension on $[\Delta^5 c_4]$ since the potential target is not annihilated by $\bar{\kappa}$.

Lemma 4.17 (Figure 9). *In stems* 145 *to* 191*, there are exotic extensions:*

(1) $[\Delta^6\nu]\eta = [\Delta^5\bar{\kappa}\epsilon]$ (*from* $(147, 1)$ *to* $(148, 6)$)
(2) $[\Delta^6\kappa\nu]\eta = [\Delta^5\bar{\kappa}^2\eta^2]$ (*from* $(161, 3)$ *to* $(162, 10)$)
(3) $[\Delta^5\bar{\kappa}\kappa\eta]\eta = [\Delta^3\bar{\kappa}^4\eta^2 v_1]$ (*from* $(155, 7)$ *to* $(156, 18)$)
(4) $[\Delta^6 y\nu]\eta = [\Delta^5\bar{\kappa}^2 v_1\eta]$ (*from* $(162, 2)$ *to* $(163, 9)$)
(5) $[\Delta^5\bar{\kappa}\epsilon v_1]\eta = [\Delta^2\bar{\kappa}^5\eta v_1]$ (*from* $(150, 6)$ *to* $(151, 21)$)
(6) $[\Delta^6\nu^3]\nu = \Delta^3\bar{\kappa}^4 v_1\eta^2$ (*from* $(153, 3)$ *to* $(156, 18)$)
(7) $[\Delta^6\epsilon v_1]\eta = \Delta^5\bar{\kappa}\kappa\eta$ (*from* $(154, 2)$ *to* $(155, 7)$)
(8) $[\Delta^6\epsilon v_1]\nu = \Delta^5\bar{\kappa}\kappa\nu$ (*from* $(154, 2)$ *to* $(157, 7)$)
(9) $[\Delta^6\kappa\nu]\nu = \Delta^5\bar{\kappa}^2 v_1\eta^2$ (*from* $(161, 3)$ *to* $(164, 10)$)
(10) $[\Delta^6 y\nu^2]\nu = [\Delta^6\bar{\kappa}v_1\eta^2]$ (*from* $(165, 3)$ *to* $(168, 6)$)

Proof. The first two extensions occur in tmf_*. The third is also an extension in tmf_*, namely, $[\Delta^5\bar{\kappa}\kappa\eta]\eta = [\Delta^4 2\bar{\kappa}^3]$, but the image of the class $[\Delta^4 2\bar{\kappa}^3]$ is detected by $[\Delta^3\bar{\kappa}^4\eta^2 v_1]$ in $tmf_* V(0)$. Extension (4) follows from (2) and Theorem 2.19. This result also implies (5) from the extensions $[\Delta^5\bar{\kappa}\nu^3]\eta = [\Delta^2\bar{\kappa}^5\eta^2]$ in tmf_*. All the extensions (6)–(10) follow from Corollary 2.12 and Remark 2.16 and the data for algebraic multiplications in the range $3 \leqslant t - s \leqslant 20$. □

5 | $tmf_* Y$: THE E_2-PAGE

Let C_η be the cofiber of the Hopf map η, so that there is an exact triangle

$$S^1 \xrightarrow{\eta} S^0 \to C_\eta \to S^2. \tag{5.1}$$

We define the spectrum Y to be $V(0) \wedge C_\eta$. which can be built from two different cofiber sequences

$$C_\eta \xrightarrow{2} C_\eta \to Y,$$

which uses the multiplication by 2 on C_η, and

$$\Sigma V(0) \xrightarrow{\eta} V(0) \to Y,$$

which uses the multiplication by η on $V(0)$. Depending on the situation it will be more advantageous to use either the former or the latter fiber sequence. We abbreviate

$$\mathcal{F}_*(F) := \pi_*(tmf \wedge X(4) \wedge F),$$

where F will be one of the finite spectra of interest.

We now proceed to compute the E_2-term of the elliptic spectral sequence computing the tmf-homology of Y, namely, $\mathrm{Ext}^{*,*}_{\Lambda'}(A', A' \otimes_A \mathcal{F}_*(Y))$.

Let us first describe $\mathcal{F}_*(C_\eta)$. Since $\pi_*(tmf \wedge X(4)) \cong A$ is concentrated in even degrees, the cofiber sequence (5.1) induces a short exact sequence on $tmf \wedge X(4)$-homology

$$0 \to A \to \mathcal{F}_*(C_\eta) \to \Sigma^2 A \to 0.$$

This splits as a sequence of A-modules so that

$$\mathcal{F}_*(C_\eta) \cong A \oplus \Sigma^2 A.$$

Multiplication by 2 on C_η induces multiplication by 2 on $tmf \wedge X(4)$-homology, which is injective because $\mathcal{F}_*(C_\eta)$ is torsion-free. Thus, the cofiber sequence

$$C_\eta \xrightarrow{2} C_\eta \to Y$$

induces a short exact sequence in $tmf \wedge X(4)$-homology

$$0 \to \mathcal{F}_*(C_\eta) \to \mathcal{F}_*(C_\eta) \to \mathcal{F}_*(Y) \to 0,$$

and it follows that

$$\mathcal{F}_*(Y) \cong A/(2) \oplus \Sigma^2 A/(2) \tag{5.2}$$

as an $A/(2)$-module.

Likewise, since $\mathcal{F}_*(V(0))$ is concentrated in even degrees, the induced map on $tmf \wedge X(4)$-homology of the cofiber sequence

$$\Sigma V(0) \xrightarrow{\eta} V(0) \to Y$$

is trivial. It follows that there is a short exact sequence of Λ-comodules

$$0 \to A/(2) \to \mathcal{F}_*(Y) \to \Sigma^2 A/(2) \to 0.$$

This short exact sequence of A-modules splits because of (5.2). Tensoring it with A' over A, we obtain a short exact sequence of Λ'-comodules, which splits as a sequence of A'-modules

$$0 \to A'/(2) \to A' \otimes_A \mathcal{F}_*(Y) \to \Sigma^2 A'/(2) \to 0. \tag{5.3}$$

As $\mathcal{F}_*(Y)$ is 2-torsion, (5.3) is a short exact sequence of $A'/(2)$-module, and hence splits as such. Therefore, applying $\mathrm{Ext}^{*,*}_{\Lambda'}(A', -)$ to (5.3), we get a long exact sequence of $\mathrm{Ext}^{*,*}_{\Lambda'}(A', A'/(2))$-modules (see, for example, [16, p. 110, (3.3)]). Its connecting homomorphism

$$\delta : \mathrm{Ext}^{s,t}_{\Lambda'}(A', A'/(2)) \to \mathrm{Ext}^{s+1,t+2}_{\Lambda'}(A', A'/(2)) \tag{5.4}$$

is given by multiplication with $\eta \in \mathrm{Ext}^{1,2}_{\Lambda'}(A', A'/(2))$. Here, as is often the case, we denote by η the class in Ext which detects the same-named homotopy class.

We present the effect of the connecting homomorphism separately for the v_1-power torsion and for the v_1-free classes of $E_2(V(0))$ in Figures 10 and 11, respectively.

In Figure 11, a $\circ$ denotes a copy of $\mathbb{F}_2[v_1]$, a bullet denotes a copy of $\mathbb{F}_2$ and a line of slope 1 denotes, as usual, multiplication by η. Note that we have $\bar{\kappa} v_1^4 = \Delta\eta^4$, hence $\bar{\kappa} v_1^4 = 0$ in $E_2(Y)$, while v_1 itself is not nilpotent and Δ^i is not v_1 torsion.

Proposition 5.5 (Figure 12). *As a module over $E_2(V(0))$, $E_2(Y)$ is generated by classes*

$$a[0,0], a[5,1], a[17,3].$$

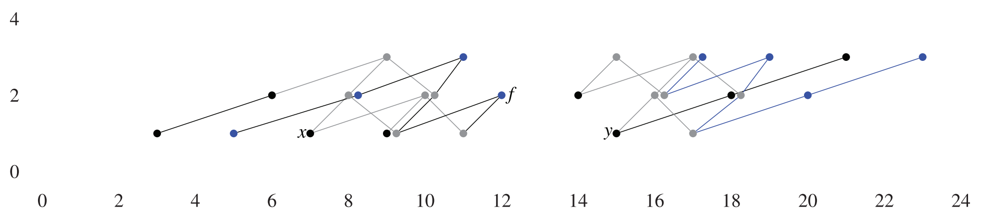


FIGURE 10 The connecting homomorphism (5.4) for the v_1-power torsion classes

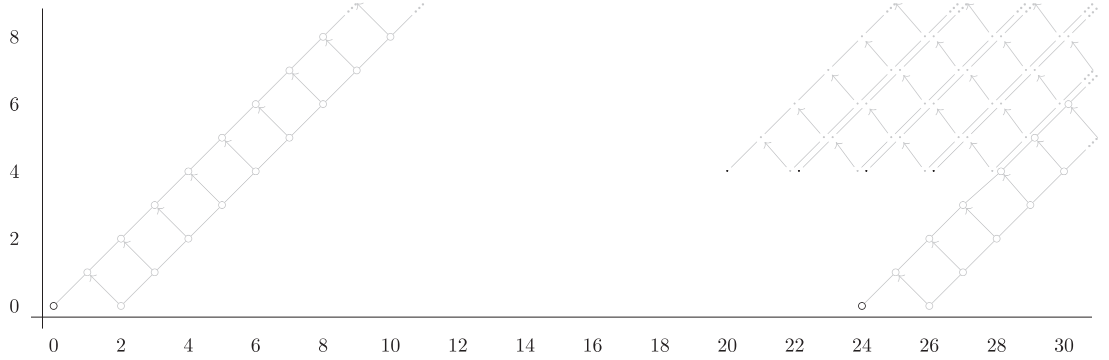


FIGURE 11 The connecting homomorphism (5.4) for the v_1-free classes

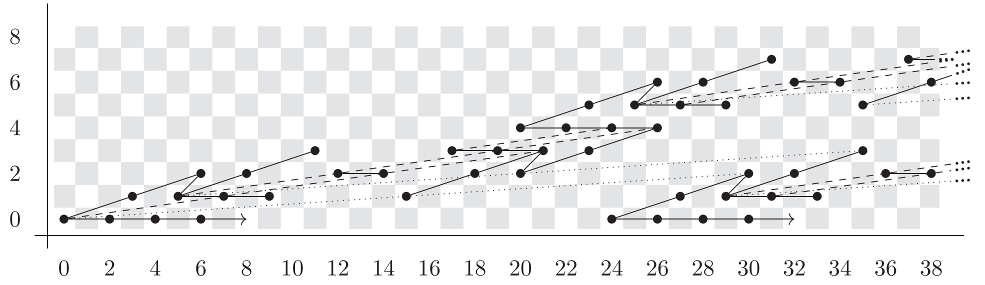


FIGURE 12 $E_2(Y)$ as a module over $E_2(V(0))$. The dashed lines are x-multiplications and dotted lines y-multiplications. Other structure lines are as in Figure 2.

The submodule generated by $a[0,0]$ *is isomorphic to* $E_2(V(0))/\eta$. *There are Massey products*

$$a[5,1] = \langle \nu, \eta, a[0,0] \rangle, \quad a[17,3] = \langle \eta x^2, \eta, a[0,0] \rangle$$

and these classes are subject to the following relations. On the new classes, we have v_1 *multiplications*

$$v_1 a[5,1] = x a[0,0] \quad v_1 a[17,3] = x^2 a[5,1] \quad v_1^2 a[17,3] = \nu^2 y a[0,0],$$

η *and* ν *multiplications*

$$\eta a[5,1] = \nu^2 a[0,0], \quad \eta a[17,3] = \nu a[17,3] = y a[17,3] = 0$$

as well as

$$\nu^2 y a[5,1] = v_1^3 \bar{\kappa} a[0,0].$$

Proof. Using the description of $E_2(V(0))$, the effect of the connecting homomorphism δ of (5.4) is straightforward to compute. The cokernel is simply $E_2 V(0)/\eta$ as an $E_2(V(0))$-module. Using the multiplication on $E_2(V(0))$, the kernel is generated by classes $a[5,1]$ and $a[17,3]$ defined, without ambiguity, by

$$p_*(a[5,1]) = \nu \quad p_*(a[17,3]) = \eta x^2,$$

where p_* is induced by the map $A' \otimes_A \mathcal{F}_*(Y) \to \Sigma^2 A'/(2)$ of (5.3).

We now show the relations on the generators. Since $\eta a[0,0] = 0$ and $\nu^2 a[0,0] \neq 0$, the Juggling formula

$$\nu^2 a[0,0] = \langle \eta, \nu, \eta \rangle a[0,0] = \eta \langle \nu, \eta, a[0,0] \rangle$$

implies that $\langle \nu, \eta, a[0,0] \rangle \neq 0$. The Massey product $\langle \nu, \eta, a[0,0] \rangle$ has zero indeterminacy, hence by sparseness,

$$a[5,1] = \langle \nu, \eta, a[0,0] \rangle$$

and

$$\eta a[5,1] = \nu^2 a[0,0].$$

We have that $v_1 \nu = 0 \in \mathrm{Ext}_\Lambda^{*,*}(A/(2), A/(2))$. As a consequence,

$$\begin{aligned} v_1 a[5,1] &= v_1 \langle \nu, \eta, a[0,0] \rangle \\ &= \langle v_1, \nu, \eta \rangle a[0,0] \text{ (by juggling formula)} \\ &= x a[0,0] \text{ (by [2, Formula 7.5])} \end{aligned}$$

The equation

$$v_1 a[17,3] = x^2 a[5,1]$$

follows from the fact that

$$v_1 \eta x^2 = x^2 \nu$$

in $E_2(V(0))$ and the definition of $a[5,1]$ and $a[17,3]$ as the pre-image of ν and ηx^2 by p_*, respectively. It follows then that

$$\begin{aligned} v_1^2 a[17,3] &= v_1 x^2 a[5,1] \text{ (because } v_1 a[17,3] = x^2 a[5,1]) \\ &= x^3 a[0,0] \text{ (because } v_1 a[5,1] = x a[0,0]) \\ &= \nu^2 y a[0,0] \text{ (because } x^3 = \nu^2 y, \text{ cf.} \mathit{Theorem} 3.4) \end{aligned}$$

The relations $\eta a[17,3] = \nu a[17,3] = y a[17,3] = 0$ follows for degree reasons.

It remains to verify that $\nu^2 y a[5,1] = v_1^3 \bar{\kappa} a[0,0]$. A juggling of Massey products gives

$$y\nu^2\langle \nu, \eta, a[0,0]\rangle = \langle y\nu^2, \nu, \eta\rangle a[0,0].$$

The relation $\nu^2 y a[5,1] = v_1^3 \bar{\kappa} a[0,0]$ then follows by Lemma 5.6 and the fact that $\eta a[0,0] = 0$. □

Lemma 5.6. *In* $\mathrm{Ext}^{*,*}_{\Lambda'}(A', A'/(2))$*, the Massey product* $\langle y\nu^2, \nu, \eta\rangle$ *contains* $\bar{\kappa}v_1^3$*. Furthermore, its indeterminacy is equal to* $\eta\,\mathrm{Ext}^{3,28}_{\Lambda'}(A', A'/(2))$*, which does not contain* $\bar{\kappa}v_1^3$*.*

Proof. By [2, formula 7.9], $\bar{\kappa}v_1^2 = \langle \eta, \kappa\eta, x\rangle$, and so

$$\bar{\kappa}v_1^3 = v_1\langle \eta, \kappa\eta, x\rangle \subset \langle v_1\eta, \kappa\eta, x\rangle \subset \langle \eta, v_1\kappa\eta, x\rangle = \langle \eta, \eta^2 y, x\rangle.$$

Here, we used the relation $v_1\kappa\eta = \eta^2 y$. It follows that

$$\bar{\kappa}v_1^4 \in v_1\langle \eta, \eta^2 y, x\rangle \subset \langle v_1\eta, \eta^2 y, x\rangle$$

The indeterminacy of the latter is equal to

$$v_1\eta\,\mathrm{Ext}^{3,28}_{\Lambda'}(A', A'/(2)) + x\,\mathrm{Ext}^{3,24}_{\Lambda'}(A', A'/(2)) = \mathbb{F}_2\{v_1^{12}\eta^4\},$$

which does not contain $\bar{\kappa}v_1^4$, so $\langle v_1\eta, \eta^2 y, x\rangle$ does not contain zero.

Now consider

$$\langle v_1\eta, y, \nu^3\rangle = \langle v_1\eta, y, \eta^2 x\rangle \subset \langle v_1\eta, \eta^2 y, x\rangle.$$

The indeterminacy of $\langle v_1\eta, y, \nu^3\rangle$ is $v_1\eta Ext^{3,28}_{\Lambda'}(A', A'/(2)) + \nu^3 Ext^{3,21}_{\Lambda'}(A', A'/(2)) = \mathbb{F}_2\{v_1^{12}\eta^4\}$, which is the same as the indeterminacy of $\langle v_1\eta, \eta^2 y, x\rangle$, hence $\langle v_1\eta, y, \nu^3\rangle$ does not contain zero and contains $\bar{\kappa}v_1^4$.

Moreover, since

$$\langle v_1\eta, y, \nu^3\rangle \subset \langle v_1\eta, y\nu, \nu^2\rangle$$

and the indeterminacy of the latter is equal to $\eta v_1\,\mathrm{Ext}^{3,28}_{\Lambda'}(A', A'/(2)) + \nu^2\,\mathrm{Ext}^{2,24}_{\Lambda'}(A', A'/(2)) = \mathbb{F}_2\{v_1^{12}\eta^4\}$, which does not contain $\bar{\kappa}v_1^4$,

$$\langle v_1\eta, y\nu, \nu^2\rangle = \bar{\kappa}v_1^4 + \mathbb{F}_2\{v_1^{12}\eta^4\}.$$

Finally, since

$$\langle v_1\eta, y\nu, \nu^2\rangle \supseteq \langle v_1\eta, \nu, \nu^2 y\rangle \supseteq v_1\langle \eta, \nu, y\nu^2\rangle$$

and multiplication by v_1 induces an injective homomorphism on $\mathrm{Ext}^{4,30}_{\Lambda'}(A', A'/(2))$, we obtain that

$$\bar{\kappa}v_1^3 + \mathbb{F}_2\{v_1^{11}\eta^4\} \supset \langle \eta, \nu, y\nu^2\rangle = \langle y\nu^2, \nu, \eta\rangle.$$

The conclusion of the lemma follows by observing that

$$\bar{\kappa}v_1^3 \notin \mathrm{Ind}(\langle y\nu^2, \nu, \eta\rangle) = \eta\,\mathrm{Ext}^{3,28}_{\Lambda'}(A', A'/(2)) \supset \mathbb{F}_2\{v_1^{11}\eta^4\}.$$ □

Remark 5.7. In $E_2^{s,t}(Y)$, there is at most one non-zero element in any bi-degree (s,t) with filtration $s > 0$. There is also a unique non-zero element in bi-degree (0,0). So, for $s > 0$ or $(s,t) = (0,0)$, we often denote by $a[t-s,s] \in E_r^{s,t}(Y)$ the non-zero element, if it exists. Furthermore, when $s = 0$ and $t > 0$, we let $a[t,0]$ denote the element of $E_2^{0,t}(Y)$ which is divisible by the largest power of Δ. For example, $E_2^{0,52}(Y) \cong \mathbb{F}_2\{v_1^{26}a[0,0], v_1^{14}\Delta a[0,0], v_1^2\Delta^2 a[0,0]\}$ and $a[52,0] = v_1^2\Delta^2 a[0,0]$.

For our purposes, we also need a partial knowledge of $\mathrm{Ext}_{\Lambda'}^{*,*}(A'/(2), \mathcal{M})$, where

$$\mathcal{M} := A'/(2, a_1) \otimes_{A/(2)} \mathcal{F}_*(Y).$$

Since $a_1 \in A'/(2)$ is a Λ'-primitive, $A'/(2, a_1)$ is a Λ'-comodule. By tensoring (5.3) with $A'/(2, a_1)$ over $A'/(2)$, we obtain a diagram of short exact sequences of Λ'-comodules

$$\begin{tikzcd} 0 \arrow[r] & A'/(2) \arrow[r] \arrow[d] & A'/(2) \otimes_{A/(2)} \mathcal{F}_*(Y) \arrow[r] \arrow[d] & \Sigma^2 A'/(2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A'/(2, a_1) \arrow[r] & A'/(2, a_1) \otimes_{A/(2)} \mathcal{F}_*(Y) \arrow[r] & \Sigma^2 A'/(2, a_1) \arrow[r] & 0 \end{tikzcd} \tag{5.8}$$

We consider the long exact sequence derived from the bottom short exact sequence of the diagram (5.8). The cohomology ring $\mathrm{Ext}_{\Lambda'/(2,a_1)}^{*,*}(A'/(2, a_1), A'/(2, a_1))$ is computed in [2, section 7]. With our notation,

$$\mathrm{Ext}_{\Lambda'/(2,a_1)}^{*,*}(A'/(2, a_1), A'/(2, a_1)) \cong \mathbb{F}_2[\eta, \nu, \bar{\kappa}, v_2]/(v_2\eta^3 - \nu^3, \eta\nu),$$

where v_2 is represented by the Λ'-primitive a_3. The bottom short exact sequence of the diagram (5.8) splits as a sequence of $A'/(2, a_1)$-modules. However, it does not split as a one of $\Lambda'/(2, a_1)$-comodules, as it represents the element $0 \neq \eta \in \mathrm{Ext}_{\Lambda'/(2,a_1)}^{1,2}(A'/(2, a_1), A'/(2, a_1))$. Therefore, the connecting homomorphism

$$\mathrm{Ext}_{\Lambda'/(2,a_1)}^{s,t}(A'/(2, a_1), A'/(2, a_1)) \to \mathrm{Ext}_{\Lambda'/(2,a_1)}^{s+1,t+2}(A'/(2, a_1), A'/(2, a_1)) \tag{5.9}$$

of the induced long exact sequence in $\mathrm{Ext}_{\Lambda'/(2,a_1)}^{*,*}(A'/(2, a_1), -)$ is given by multiplication by η. We obtain:

Lemma 5.10. *As a module over the ring $\mathbb{F}_2[\eta, \nu, \bar{\kappa}, v_2]/(v_2\eta^3 - \nu^3, \eta\nu)$, the cohomology group*

$$\mathrm{Ext}_{\Lambda'/(2,a_1)}^{*,*}(A'/(2, a_1), A'/(2, a_1) \otimes_{A/(2)} \mathcal{F}_*(Y))$$

is generated by $a[0,0] \in \mathrm{Ext}^{0,0}$ and $a[5,1] \in \mathrm{Ext}^{1,6}$ with the relations

$$\eta a[0,0] = 0, \qquad \eta a[5,1] = \nu^2 a[0,0].$$

Proof. By the description of the connecting homomorphism (5.9), we see that

$$\mathrm{Ext}_{\Lambda'/(2,a_1)}^{*,*}(A'/(2, a_1), A'/(2, a_1) \otimes_{A/(2)} \mathcal{F}_*(Y)) \cong \mathbb{F}_2[\nu, \bar{\kappa}, v_2]/(\nu^3)\{a[0,0], a[5,0]\}$$

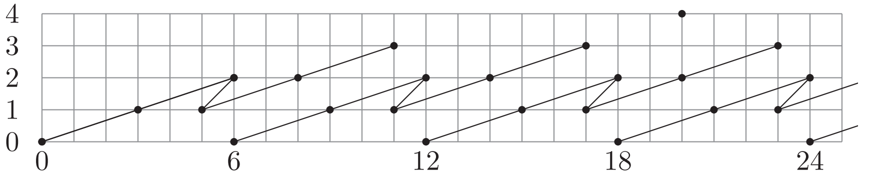


FIGURE 13 $\mathrm{Ext}^{s,t}_{\Lambda'/(2,a_1)}(A'/(2,a_1), A'/(2,a_1) \otimes_{A/(2)} \mathcal{F}_*(Y))$ depicted in the coordinates $(t-s, s)$

as an $\mathbb{F}_2[\nu, \bar{\kappa}, v_2]/(\nu^3)$-module. Next, we determine the action of η. We see easily that $\eta a[0,0] = 0$. To calculate $\eta a[5,1]$, we remark that

$$\nu^2 a[0,0] = \langle \eta, \nu, \eta \rangle a[0,0] = \eta \langle \nu, \eta, a[0,0] \rangle,$$

where the first equality comes from the Massey product $\nu^2 = \langle \eta, \nu, \eta \rangle$ and the second is a shuffle. As $\nu^2 a[0,0] \neq 0$, $\langle \nu, \eta, a[0,0] \rangle$ is not trivial and must be equal to $a[5,1]$ by sparseness. Hence, $\nu^2 a[0,0] = \eta a[5,1]$. □

Remark 5.11. This calculation will be used in Lemma 6.25 in order to prove Proposition 6.24. It has also an independent interest being the E_2-term of the elliptic spectral sequence for $tmf \wedge A_1$, see Subsection 6.4 for a discussion on A_1.

Although Proposition 5.5 gives us a very compact description of $E_2(Y)$, the elliptic spectral sequence of $tmf \wedge Y$ is not a module over the elliptic spectral sequence of $tmf \wedge V(0)$ as the latter is not even a multiplicative spectral sequence. However, the elliptic spectral sequence of $tmf \wedge Y$ is a module over the elliptic spectral sequence of tmf. In fact, we get even more structure than that from the fact that Y has v_1-self maps. As explained in Subsection 2.6, we have:

Lemma 5.12 (v_1-linearity). *The differentials in the elliptic spectral sequence for $tmf \wedge Y$ are v_1-linear.*

We state the following 'intermediate' result for convenience of reference in the computations below. The module structure of the elliptic spectral sequence spectral sequence of $tmf \wedge Y$ over that of tmf is richer than what is stated here but that information can be read off of Proposition 5.5.

Corollary 5.13. *As a module over*

$$\mathbb{F}_2[v_1, \nu, \bar{\kappa}, \Delta]/(v_1\nu, \nu^3, v_1^4\bar{\kappa}),$$

$E_2(Y)$ is generated by

$$a[0,0],\ a[5,1],\ a[12,2],\ a[15,1],\ a[17,3],\ a[20,2]$$

subject to the relations generated by

$$v_1^3 a[5,1] = v_1^2 a[12,2] = v_1 a[15,1] = \nu a[12,2] = \nu a[17,3] = 0$$

and

$$\nu^2 a[15,1] = v_1^2 a[17,3], \quad \nu^2 a[20,2] = v_1^3 \bar{\kappa} a[0,0].$$

Furthermore, the differentials are $\mathbb{F}_2[v_1, \nu, \bar{\kappa}, \Delta^8]/(v_1\nu, \nu^3, v_1^4\bar{\kappa})$*-linear.*

Proof. This follows from the results of this section and the fact that Δ^8 is a permanent cycle in the elliptic spectral sequence spectral sequence of tmf. ☐

6 | tmf_*Y: THE DIFFERENTIALS AND EXTENSIONS

Our approach to computing the differentials of the elliptic spectral sequence for $\pi_*(tmf \wedge Y)$ is based largely on the analysis of the action of $\bar{\kappa}$. More precisely, since $\bar{\kappa}$ is a permanent cycle in the elliptic spectral sequence for tmf, the elliptic spectral sequence for $tmf \wedge Y$ is a spectral sequence of modules over $\mathbb{F}_2[\bar{\kappa}]$, meaning that every term is a $\mathbb{F}_2[\bar{\kappa}]$-module and the differentials are maps of $\mathbb{F}_2[\bar{\kappa}]$-modules. Note that the E_∞-term is $\bar{\kappa}$-torsion, since $\bar{\kappa}$ is nilpotent in π_*tmf. But all the intermediate terms E_r for $r \leqslant 23$ do contain non-trivial $\bar{\kappa}$-free elements, that is, those elements whose multiplication with $\bar{\kappa}^i$ is non-trivial for all $i \in \mathbb{N}$.

Lemma 6.1. *The E_r-term of the elliptic spectral sequence for $tmf \wedge Y$ has the following properties.*

(1) *All classes in filtration greater than $(r-1)$ are $\bar{\kappa}$-free.*
(2) *All classes in filtration greater than or equal to 4 are divisible by $\bar{\kappa}$.*

Proof. We prove these two properties by induction on $r \geqslant 2$. For $r = 2$, this follows from Proposition 5.5. Suppose now that $r > 2$. Let a be a d_{r-1}-cycle and $[a] \in E_r^{s,t}$ the corresponding class. Suppose that a lives in filtration s with $s > (r-1)$. We have that $\bar{\kappa}[a] = 0$ if and only if there exists $b \in E_{r-1}$ such that $d_{r-1}(b) = \bar{\kappa}a$. Then, b must live in filtration $(4+s)-(r-1) > 4$. By the second property, b is divisible by $\bar{\kappa}$, that is, there exists $c \in E_{r-1}$ such that $\bar{\kappa}c = b$. As a consequence of the $\bar{\kappa}$-linearity, $\bar{\kappa}d_{r-1}(c) = d_{r-1}(b) = \bar{\kappa}a$, and so $\bar{\kappa}(d_{r-1}(c) - a) = 0$. Since $(d_{r-1}(c) - a) \in E_{r-1}$ lives in filtration s greater than $r-2$, it is $\bar{\kappa}$-free by the second property. It follows that $d_{r-1}(c) = a$, and so $[a] = 0$. Therefore, the E_r-term has the first property.

For the second property, suppose that a lives in filtration greater than or equal to 4. By the second property for E_{r-1}, there exists $b \in E_{r-1}$ such that $\bar{\kappa}b = a$. It suffices to prove that b is a d_{r-1}-cycle. Suppose that $d_{r-1}(b) = c$. The latter implies that c lives in filtration greater that $(r-2)$, hence is $\bar{\kappa}$-free by the first property. Since a is a d_{r-1}-cycle by assumption, we have, by $\bar{\kappa}$-linearity, that

$$0 = d_{r-1}(a) = d_{r-1}(\bar{\kappa}b) = \bar{\kappa}c.$$

This means that $c = 0$ and so b is a d_{r-1}-cycle, as required. ☐

Terminology.

(1) For a class $x \in E_r$ having filtration less than four, we call the subset $\{\bar{\kappa}^i x | i \in \mathbb{N}\} \subset E_r$ the $\bar{\kappa}$-family of x. A $\bar{\kappa}$-family is called free if it contains infinitely many elements and is called torsion otherwise.

(2) Let $x, y \in E_2$ have filtration less than four. We say that a $\bar{\kappa}$-family of x is truncated by the $\bar{\kappa}$-family of y if there exists r such that $d_r(\bar{\kappa}^n y) = \bar{\kappa}^{n+l} x$ for all $n \in \mathbb{N}$.

By part (2) of the above lemma, at any term of the spectral sequence, every class belongs to some $\bar{\kappa}$-family. The following corollary tells us how these $\bar{\kappa}$-families are organized.

Corollary 6.2.

(1) *At any term of the spectral sequence, all non-zero $\bar{\kappa}$-power torsion classes survive to the E_∞-term.*
(2) *Every $\bar{\kappa}$-free family consisting of permanent cycles is truncated uniquely by another $\bar{\kappa}$-free family. More precisely, if $0 \neq a \in E_r$ is a permanent cycle which generates a $\bar{\kappa}$-free family, then there exists a unique integer $r' \geqslant r$ for which there exists $b \in E_{r'}$ having filtration less than four, such that*

$$d_{r'}(\bar{\kappa}^n b) = \bar{\kappa}^{n+l} a$$

for all $n \in \mathbb{N}$, where l is determined by r', the filtration of a and that of b, and moreover, $\{\bar{\kappa}^i a | 0 \leqslant i \leqslant l-1\}$ consists of non-trivial permanent cycles surviving to the E_∞-term.

Proof. For part (1), let $a \in E_r$ be a non-zero $\bar{\kappa}$-power torsion class. By part (1) of Lemma 6.1, a is in filtration less than or equal to $r-1$. It follows that a cannot be hit by any differential from the E_r-term onwards. Moreover, by part (1) of Lemma 6.1 again, the possible targets of $d_{r'}(a)$, $r' \geqslant r$ are $\bar{\kappa}$-free classes. Since $a \in E_r$ is $\bar{\kappa}$-power torsion, it is a permanent cycle, by $\bar{\kappa}$-linearity. Therefore, a persists to the E_∞-term.

For part (2), let a be a permanent cycle of filtration strictly less than four which is $\bar{\kappa}$-free at the E_r-term. Then the $\bar{\kappa}$-family of a consists of permanent cycles. Since $\bar{\kappa}$ is nilpotent at the E_∞-term of the elliptic spectral sequence for tmf, some power of $\bar{\kappa}$-multiple of a must be hit by a differential. Thus, there exists a smallest integer $r' \geqslant r$ and a smallest $l \in \mathbb{N}$ for which there exists $b \in E_{r'}$ such that $d_{r'}(b) = \bar{\kappa}^l a$. By the minimality of r', a is $\bar{\kappa}$-free at the $E_{r'}$-term, so is b, because by $\bar{\kappa}$-linearity, $d_{r'}(\bar{\kappa}^n b) = \bar{\kappa}^{l+n} a$ for all $n \in \mathbb{N}$. It also follows from the latter that all the classes $\bar{\kappa}^k a$ for $k \leqslant l-1$ are non-zero $\bar{\kappa}$-power torsion classes on the $E_{r'+1}$-term, hence survives to the E_∞-term by part (1).

Finally, we claim that b has filtration less than four. If b had filtration greater than or equal to 4, then b would be divisible by $\bar{\kappa}$, that is, there would exist $c \in E_{r'}$ such that $\bar{\kappa} c = b$, by Lemma 6.1 part (2). By $\bar{\kappa}$-linearity, we have that $\bar{\kappa}^l a = d_{r'}(b) = \bar{\kappa} d_{r'}(c)$, and so $\bar{\kappa}(\bar{\kappa}^{l-1} a - d_{r'}(c)) = 0$. This implies that $d_{r'}(c) = \kappa^{l-1} a$ because $d_{r'}(c) - \bar{\kappa}^{l-1} a$, having filtration at least r', is $\bar{\kappa}$-free, by Lemma 6.1 part (1). This contradicts the minimality of ℓ, so b has filtration less than four. □

Slogan 6.3. The $\bar{\kappa}$-free families at the E_r-page come in pairs. The first member of the pair is a family consisting of permanent cycles. The second member is a family which eventually supports differentials (that is, possibly at a later page) to truncate the first family.

Corollary 6.4. *At the E_r-term, we have the following.*

(1) *The homomorphism $E_r^{s,t} \to E_r^{s,t+192}$ induced by multiplication by Δ^8 is an injection for all s and t.*
(2) *If a is a class of the E_2-term such that $\Delta^8 a$ is a d_r-cycle, then a is also a d_r-cycle.*

Proof. We prove part (1) by induction on $r \geqslant 2$. For $r = 2$, this can be seen from the explicit structure of the E_2-term. Suppose the $E_{r'}$-term has these properties for $r' < r$. Let us prove part (1) for E_r. Let $a \in E_{r-1}$ represent a class of E_r. If $\Delta^8[a] = 0 \in E_r$. This means that there exists $b \in E_{r-1}$ such that $d_{r-1}(b) = \Delta^8 a$. It is obvious that b lives in stem at least 192, hence there exists $c \in E_{r-1}$ such that $b = \Delta^8 c$, by the induction hypothesis. It follows that $\Delta^8(d_{r-1}(c) - a) = 0$, and so $d_{r-1}(c) = a$ because of part (1) of the induction hypothesis. Thus, $[a] = 0 \in E_r$, as needed.

For part (2), by induction, suppose that a is a d_{r-1}-cycle. We need to prove that a is a d_r-cycle. In effect, if $d_r(a) = b$, then

$$0 = d_r(\Delta^8 a) = \Delta^8 d_r(a) = \Delta^8 b.$$

By part (1), $b = 0$, and so $d_r(a) = 0$, as needed. □

Finally, we will also use the following result to establish the differentials.

Lemma 6.5 (Vanishing line). *The spectral sequence for $\pi_* tmf \wedge Y$ degenerates at the E_{24}-term and has a horizontal vanishing line at $s = 24$, that is, $E_{24}^{s,t} = E_\infty^{s,t} = 0$ for $s \geqslant 24$.*

Proof. We know that $\bar{\kappa}^6$ is hit by a differential d_{23} in the elliptic spectral sequence for tmf, see [2]. This means that at the E_{24}-term of the elliptic spectral sequence for $tmf \wedge Y$, all the classes are annihilated by $\bar{\kappa}^6$, hence are $\bar{\kappa}$-power torsion. Therefore, by Lemma 6.1, all the classes in the E_{24}-term are in filtrations less than 24, meaning that the spectral sequence has the horizontal vanishing line at $s = 24$, that is, $E_r^{s,t} = 0$ for $s \geqslant 24$ and $r \geqslant 24$. □

Remark 6.6. The cofiber sequence

$$V(0) \xrightarrow{i} Y \xrightarrow{p} \Sigma^2 V(0) \xrightarrow{\eta} \Sigma V(0)$$

gives rise to maps of spectral sequences

$$i_* : E_2^{s,t}(V(0)) \to E_2^{s,t}(Y), \quad p_* : E_2^{s,t}(Y) \to E_2^{s,t-2}(V(0))$$

as well as a long exact sequence

$$\ldots \to tmf_{*+1}V(0) \xrightarrow{\eta} tmf_* V(0) \xrightarrow{i_*} tmf_* Y \xrightarrow{p_*} tmf_{*-1} V(0) \to \ldots \tag{6.7}$$

6.1 | The d_3, d_5 and d_7-differentials

Note that for r even, $E_r(Y) \cong E_{r+1}(Y)$ since the spectral sequence is concentrated in bi-degrees (s,t) with t even. The differentials in this section are depicted in Figures 14, 15, 16, and 17. In addition, large charts of the elliptic spectral sequence for $tmf_* Y$ can be found in [7].

Proposition 6.8. *There is no non-trivial d_3-differential, and so $E_3(Y) \cong E_5(Y)$.*

Proof. Since Δ is a d_3-cycle in the elliptic spectral sequence of tmf, the d_3-differentials are $\mathbb{F}_2[v_1, \nu, \bar{\kappa}, \Delta]/(v_1\nu, \nu^3, v_1^4\bar{\kappa})$-linear. All the generators listed in Corollary 5.13 are d_3-cycles for degree reasons. □

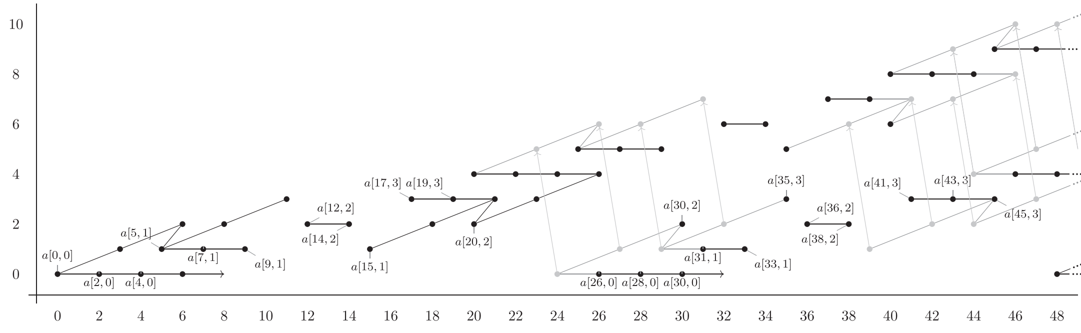


FIGURE 14 d_5-differentials in stems 0 to 48 and $\bar{\kappa}$-free generators at E_9

We then get the following result for degree reasons.

Corollary 6.9. *The classes in stems $t - s < 24$ are permanent cycles.*

Lemma 6.10. *The d_5-differentials are linear with respect to $\bar{\kappa}, \nu, v_1, \Delta^2$ and are determined by*

$$d_5(\Delta) = \nu\bar{\kappa}, \qquad d_5(\Delta a[5,1]) = \nu\bar{\kappa} a[5,1]$$

$$d_5(\Delta a[15,1]) = \nu\bar{\kappa} a[15,1], \qquad d_5(\Delta a[20,2]) = \nu\bar{\kappa} a[20,2]$$

under multiplication by elements of $\mathbb{F}_2[\Delta^2, \bar{\kappa}, \nu, v_1]/(v_1\nu, \nu^3, \bar{\kappa} v_1^4)$.

Proof. For linearity, we only need to prove the Δ^2-linearity. Note that $d_5(\Delta) = \nu\bar{\kappa}$ in the elliptic spectral sequence of tmf. By Leibniz rule and the fact that $E_2(Y)$ is 2-torsion,

$$d_5(\Delta^2 x) = 2\Delta d_5(\Delta)x + \Delta^2 d_5(x) = \Delta^2 d_5(x).$$

Using the module structure over the elliptic spectral sequence of tmf, we get

$$d_5(\Delta a[5,1]) = d_5(\Delta)a[5,1] + \Delta d_5(a[5,1]) = \nu\bar{\kappa} a[5,1].$$

The other arguments are similar. □

Lemma 6.11. *There are no non-trivial d_7-differentials.*

Proof. This is an immediate consequence of sparseness. □

The following observation will be crucial for our computation and is motivated by Slogan 6.3.

Corollary 6.12 (Figure 14). *The $\bar{\kappa}$-free families on the E_9-term of the elliptic spectral sequence of $tmf \wedge Y$ in stems $0 \leqslant t - s < 48$ are given by the following 24 classes*

$$a[0,0] \qquad a[2,0] = v_1 a[0,0] \qquad a[4,0] = v_1^2 a[0,0]$$

$$a[5,1] \qquad a[7,1] = v_1 a[5,1] \qquad a[9,1] = v_1^2 a[5,1]$$

$a[12,2]$ $\quad a[14,2] = v_1 a[12,2]$ $\quad a[15,1]$

$a[17,3]$ $\quad a[19,3] = v_1 a[17,3]$ $\quad a[20,2]$

$a[26,0] = \Delta v_1 a[0,0]$ $\quad a[28,0] = \Delta v_1^2 a[0,0]$ $\quad a[30,0] = \Delta v_1^3 a[0,0]$

$a[30,2] = \Delta \nu^2 a[0,0]$ $\quad a[31,1] = \Delta v_1 a[5,1]$ $\quad a[33,1] = \Delta v_1^2 a[5,1]$

$a[35,3] = \Delta \nu^2 a[5,1]$ $\quad a[36,2] = \Delta a[12,2]$ $\quad a[38,2] = \Delta v_1 a[12,2]$

$a[41,3] = \Delta a[17,3]$ $\quad a[43,3] = \Delta v_1 a[17,3]$ $\quad a[45,3] = \Delta v_1^2 a[17,3]$

All $\bar{\kappa}$-free families at E_9 are given by these classes and their Δ^2-multiples. All the elements in filtrations four and above are $\bar{\kappa}$-multiples of these generators.

The generators of the $\bar{\kappa}$-free families in stems $0 \leqslant t-s < 48$ are presented in Figure 14. The $\bar{\kappa}$-free generators in the range $0 \leqslant t-s < 192$ are given by products of these with $1, \Delta^2, \Delta^4$ and Δ^6 and all other $\bar{\kappa}$-free generators are products of the latter ones with the powers of Δ^8. By Corollary 6.2, each $\bar{\kappa}$-free family consisting of permanent cycles is truncated by one other $\bar{\kappa}$-free family, and so by exactly one because of sparseness — any two distinct $\bar{\kappa}$-free families have different bi-degrees. Thus, using the Δ^8-linearity and Corollary 6.4, we see that the 24×4 $\bar{\kappa}$-free generators in the range $0 \leqslant t-s < 192$ organize themselves as follows. Exactly half of them are permanent cycles and the other half are not. The $\bar{\kappa}$-family of each non-permanent $\bar{\kappa}$-free generator supports a differential that hits the $\bar{\kappa}$-family of exactly one of the other permanent generators. Note that the truncation must begin in stems less than four by Corollary 6.2. This allows us to determine longer differentials before settling shorter ones.

All 24 $\bar{\kappa}$-free generators in the range $0 \leqslant t-s < 48$ are permanent cycles due to sparseness and in the next section we will find their 'partners'.

6.2 | The d_9-differentials

To analyze the d_9-differentials, we make the following observation, which, in some sense, is a very small part of the geometric boundary theorem as in [9, appendix 4].

Lemma 6.13. *Let $a \in E_r^{s,t}(Y)$ so that $p_*(a) \in E_r^{s,t-2}(V(0))$. Suppose $p_*(a)$ persists to the $E_{r'}$-term for some $r' \geqslant r$ and that there is a non-trivial differential, $d_{r'}(p_* a) \neq 0$. Then $d_{r''}(a) \neq 0$ for some $r'' \leqslant r'$.*

Proof. This is a straightforward application of naturality. Indeed, the assumptions imply that a cannot be hit by a differential $d_{r''}$ for $r'' \leqslant r'$. Furthermore, if a persists to the $E_{r'}$ term, then $d_{r'}(a) = b$ for b such that $p_*(b) = d_{r'}(p_*(a))$. □

Lemma 6.14 (Figures 15, 16, and 17). *There are d_9-differentials, for $i = 0, 1$,*

(1) $d_9(\Delta^{4i+2} a[0,0]) = \bar{\kappa}^2 \Delta^{4i} v_1 a[5,1]$

(2) $d_9(\Delta^{4i+2} a[5,1]) = \bar{\kappa}^2 \Delta^{4i} a[12,2]$

(3) $d_9(\Delta^{4i+3} v_1 a[0,0]) = \bar{\kappa}^2 \Delta^{4i+1} v_1^2 a[5,1]$

(7) $d_9(\Delta^{4i+2} a[12,2]) = \bar{\kappa}^2 \Delta^{4i} v_1 a[17,3]$

(8) $d_9(\Delta^{4i+3} v_1 a[5,1]) = \bar{\kappa}^2 \Delta^{4i+1} v_1 a[12,2]$

(9) $d_9(\Delta^{4i+2} v_1 a[0,0]) = \bar{\kappa}^2 \Delta^{4i} v_1^2 a[5,1]$

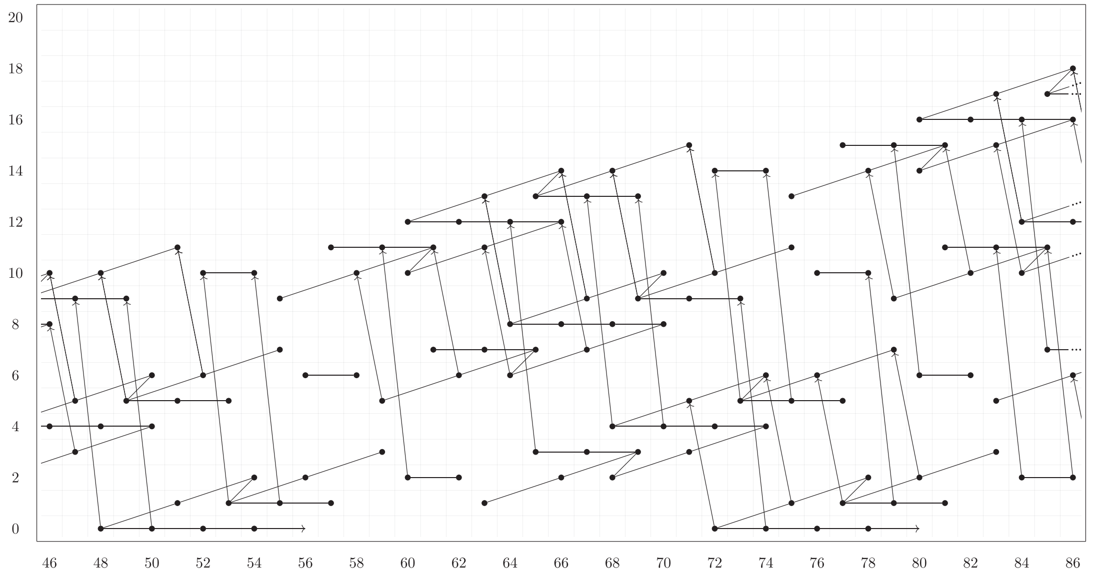


FIGURE 15 d_5 and d_9 differentials in stems 46 to 86

(4) $d_9(\Delta^{4i+2}a[17,3]) = \bar{\kappa}^3\Delta^{4i}v_1^2a[0,0]$

(5) $d_9(\Delta^{4i+3}a[17,3]) = \bar{\kappa}^3\Delta^{4i+1}v_1^2a[0,0]$

(6) $d_9(\Delta^{4i+3}a[12,2]) = \bar{\kappa}^2\Delta^{4i+1}v_1a[17,3]$

(10) $d_9(\Delta^{4i+2}v_1a[5,1]) = \bar{\kappa}^2\Delta^{4i}v_1a[12,2]$

(11) $d_9(\Delta^{4i+3}v_1a[17,3]) = \bar{\kappa}^3\Delta^{4i+1}v_1^3a[0,0]$

(12) $d_9(\Delta^{4i+3}v_1a[12,2]) = \bar{\kappa}^2\Delta^{4i+1}v_1^2a[17,3]$

Proof. Let $i = 0$. The differentials (1) and (3) are the image of a differential in $E_2(V(0))$ under i_*. The second differential (2) follows v_1-linearity and from the fact that $d_9(\Delta^{4i+2}x) = \bar{\kappa}^2\Delta^{4i}\kappa$ in $E_2(V(0))$, $i_*(x) = v_1a[5,1]$ and $i_*(\kappa) = v_1a[12,2]$.

For (4), we use Lemma 6.13. In $E_*(V(0))$, we have $d_{11}(\Delta^2\eta\kappa) = \eta^2\bar{\kappa}^3$. Since $p_*(\Delta^2a[17,3]) = \Delta^2\eta\kappa$, $\Delta^2a[17,3]$ supports a differential of length at most 11. This d_9 is the only choice. The argument for (5) is the same, with one more power of Δ.

For (6), note that $p_*(\Delta^3a[12,2]) = \Delta^3v_1\eta x$. Since $d_9(\eta v_1x) = \nu\kappa\bar{\kappa}^2\Delta$, the class $\Delta^3a[12,2]$ supports a differential of length at most 9. This is the only choice.

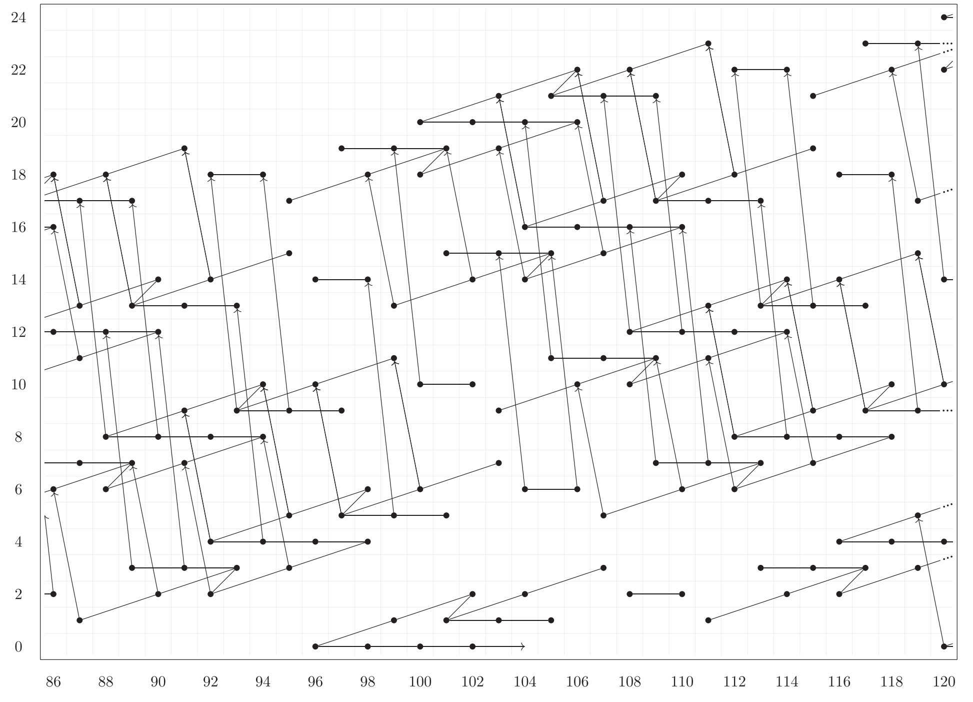
The arguments (1)–(6) when $i = 1$ are the same as those for $i = 0$.

For (7)–(8), note that from our computation above, $tmf_{59}Y \cong \mathbb{Z}/2$. This forces (7) when $i = 0$. Arguing in a similar way, $tmf_{79}Y = 0$, $tmf_{155}Y \cong \mathbb{Z}/2$ and $tmf_{175}Y = 0$ imply the other d_9s.

The d_9-differentials (9)–(12) follow from those of (1), (2), (5), (6), respectively, by v_1-linearity. □

Remark 6.15. It turns out these are all the d_9-differentials. For degree reasons, there can be very few other d_9s. The class $\Delta^5v_1a[0,0]$ is the image of a d_9-cycle in $E_9(V(0))$ so does not support a d_9. The only other possible d_9 differentials for degree reasons are as follows.

- A nontrivial d_9 on $\Delta^5a[17,3]$. This does not happen since it implies a nontrivial d_9 on $v_1\Delta^5a[17,3] = \Delta^4a[43,3]$, but this family has already been paired: it is truncated by $\Delta^6a[36,2]$.
- A nontrivial d_9 on $\Delta^4a[17,3]$, truncating the $\bar{\kappa}$-family of $\Delta^2a[4,0]$. We will see below that this does not happen, but at this point, we leave this undecided.



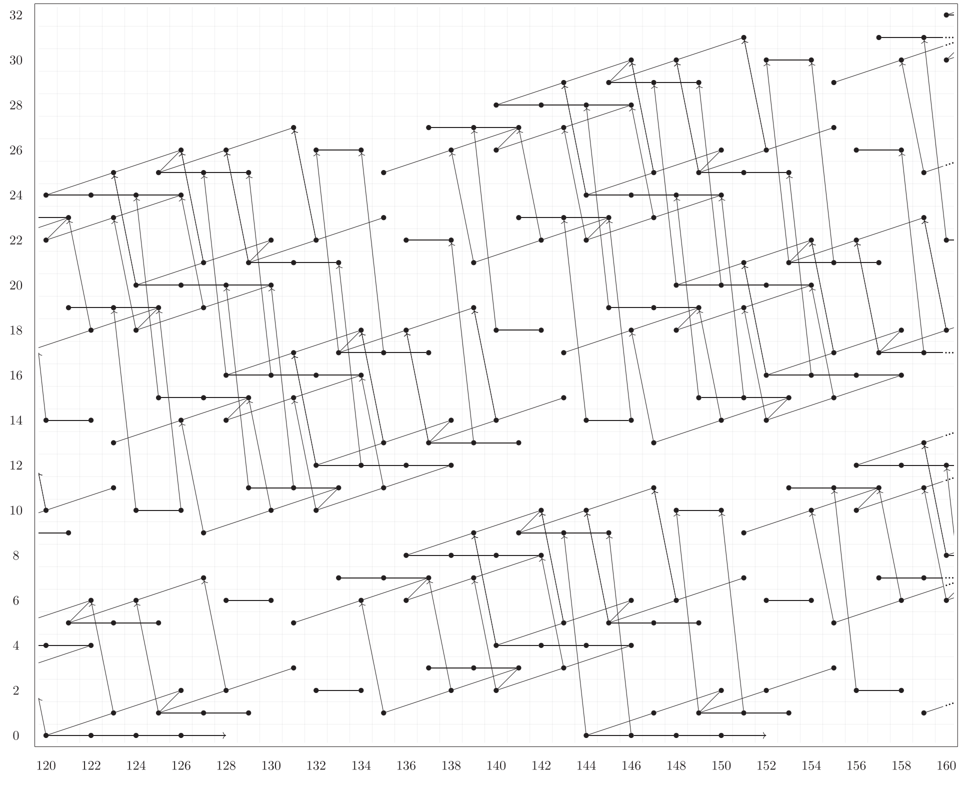


FIGURE 16 d_5 and d_9 differentials in stems 86 to 160

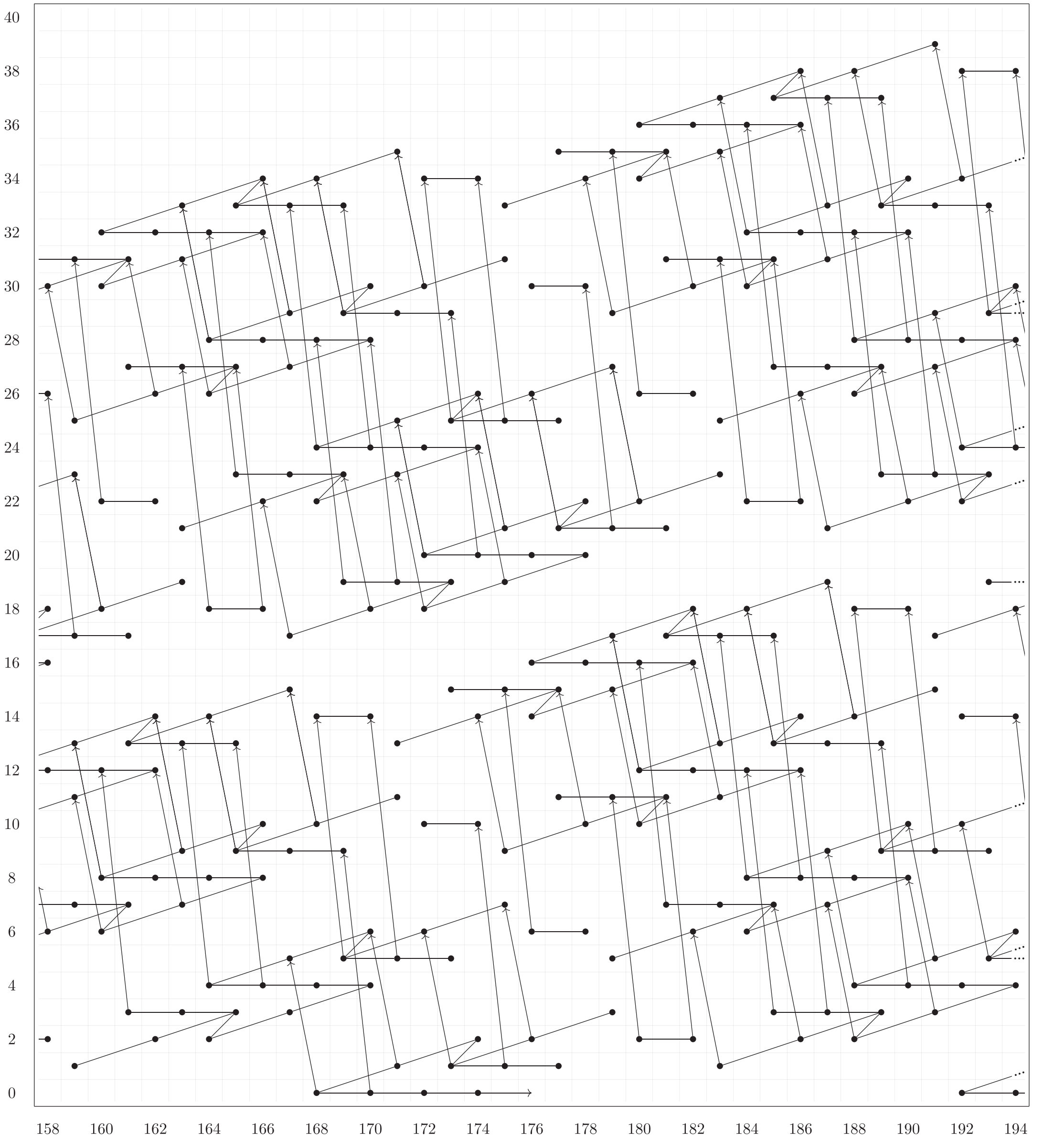


FIGURE 17 d_5 and d_9 differentials in stems 160 to 194

6.3 | Higher differentials

We begin our analysis using Slogan 6.3. The reader should remember that we only need to analyze the generators of the $\bar{\kappa}$-free families, which are in filtration less than four. All differentials discussed in this section are depicted in Figures 18 and 20.

Lemma 6.16. *There are differentials*

(1) $d_{19}(\Delta^4 a[5,1]) = \bar{\kappa}^5 a[0,0]$
(2) $d_{19}(\Delta^5 v_1 a[5,1]) = \bar{\kappa}^5 \Delta v_1 a[0,0]$

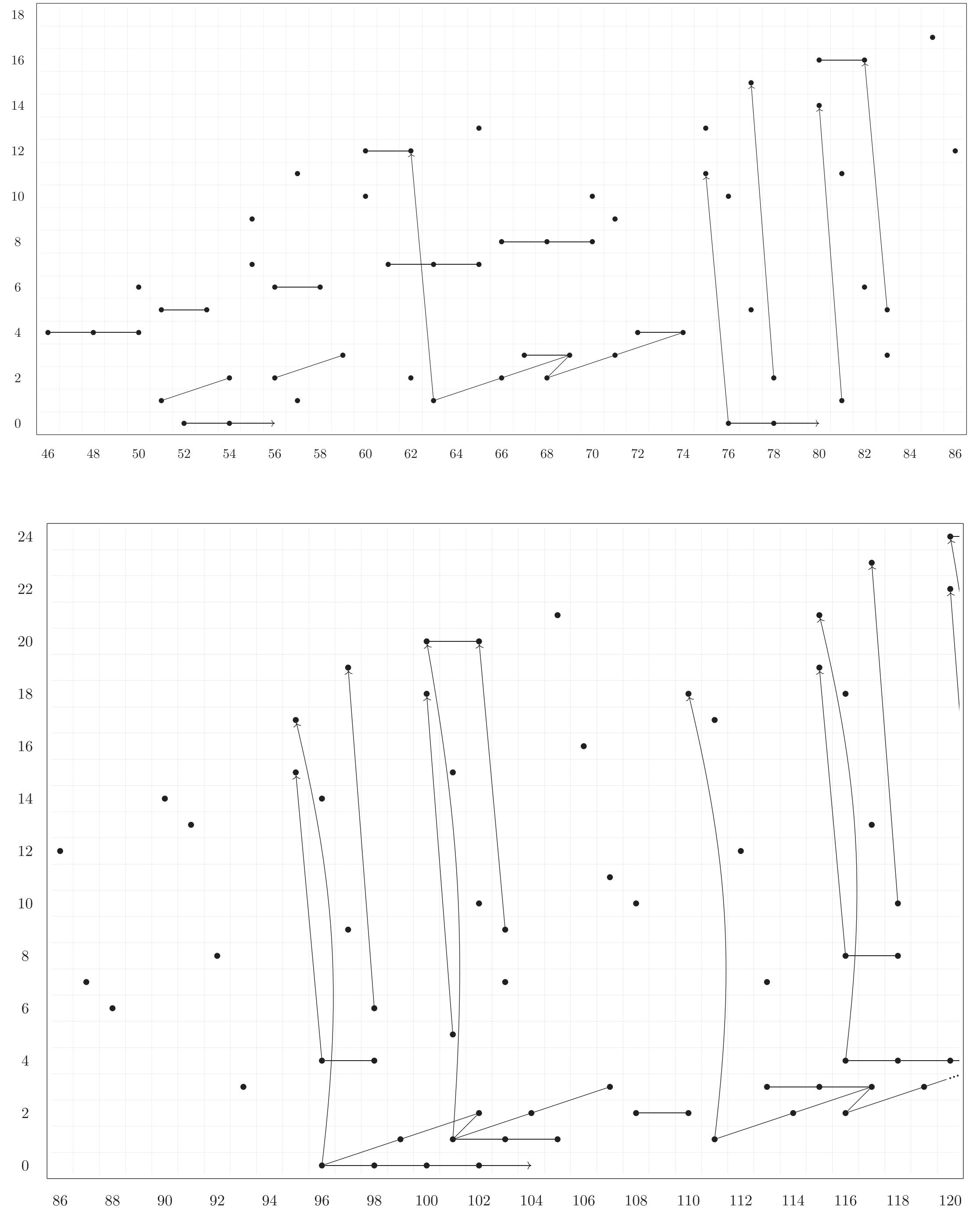


FIGURE 18 d_{11} to d_{23} differentials in stems 46 to 120

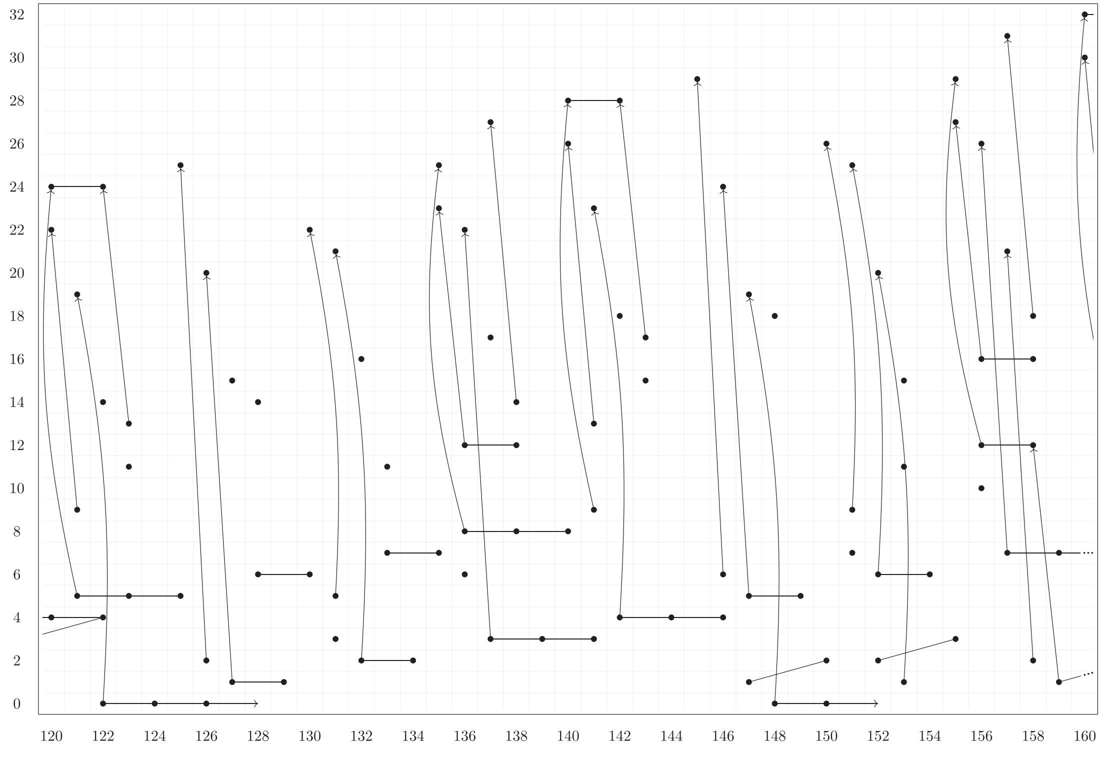


FIGURE 19 d_{11} to d_{23} differentials in stems 120 to 160

(3) $d_{19}(\Delta^4 a[36,2] = \bar{\kappa}^5 a[31,1]$
(4) $d_{19}(\Delta^4 a[41,3]) = \bar{\kappa}^5 a[36,2]$
(5) $d_{19}(\Delta^4 a[26,0]) = \bar{\kappa}^4 a[41,3]$

Proof. For (1), since the element $\bar{\kappa}^4 \in \pi_{80}(tmf \wedge V(0))$ is not divisible by η and $\bar{\kappa}^5 \in \pi_{100}(tmf \wedge V(0))$ is divisible by η, the $\bar{\kappa}$-family of $a[0,0]$ in the elliptic spectral sequence for $tmf \wedge Y$ must be truncated at $\bar{\kappa}^5 a[0,0]$. Remembering that the source has to have filtration less than four, the only possibility is this differential.

Inspection then shows that the differentials (2)–(4) are the only possibilities to satisfy Slogan 6.3. □

Lemma 6.17. *There are differentials*

(1) $d_{17}(\Delta^4 a[0,0]) = \bar{\kappa}^4 a[15,1]$ *and*
(2) $d_{17}(\Delta^4 a[15,1]) = \bar{\kappa}^4 a[30,2]$

Proof. For (1), note that in $\pi_*(tmf \wedge V(0))$, $\bar{\kappa}^3 y$ is not divisible by η and $\bar{\kappa}^4 y = 0$. The class y maps to $a[15,1]$ under i_* so it follows that the $\bar{\kappa}$-family of $a[15,1]$ is truncated at $\bar{\kappa}^4 a[15,1]$. The only possibility is this differential.

For (2), using the long exact sequence, we obtain that $\pi_{111}(tmf \wedge Y) = \mathbb{Z}/2$. By part Lemma 6.16 (3), the class $\bar{\kappa}^4 a[31,1] \in E_5^{17,128}$ survives the spectral sequence and so detects the

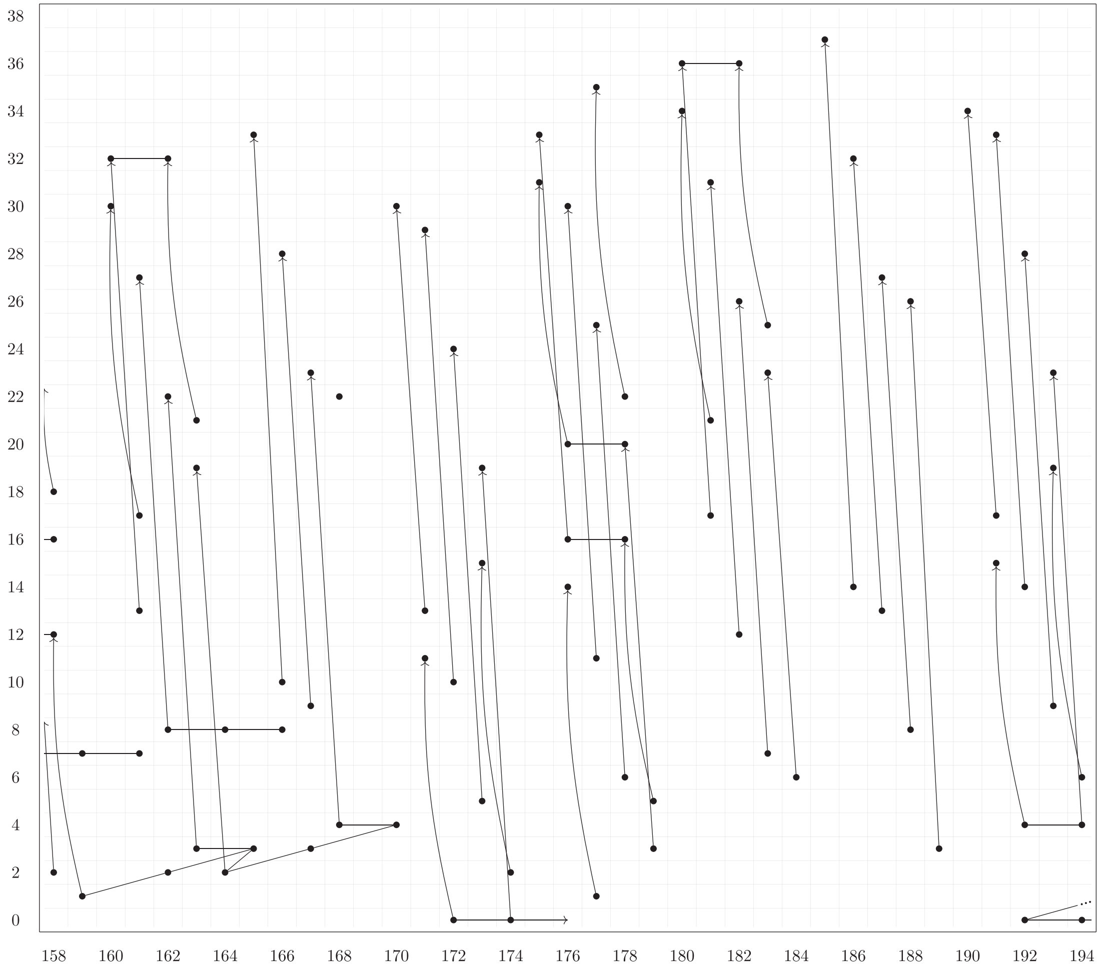


FIGURE 20 d_{11} to d_{23} differentials in stems 160 to 194

unique non-trivial class of $\pi_{111}(tmf \wedge Y)$. This implies that the class $\Delta^4 a[15,1] \in E_5^{1,112}$ must support a differential. Taking into account the d_9 differentials proves (2). □

Lemma 6.18. *There is a differential* $d_{23}(\Delta^4 a[30,2]) = \bar{\kappa}^6 a[5,1]$.

Proof. By inspection, taking into account the d_9s, the only generators that can be paired with $a[5,1]$ are $\Delta^4 a[30,2]$ and $\Delta^4 a[30,0]$. However, it cannot be $\Delta^4 a[30,0]$ because such a differential would have length 25, contradicting Lemma 6.5. □

Lemma 6.19. *For* $i = 0, 1$, *there are differentials:*

(1) $d_{11}(\Delta^{4i+2} a[15,1]) = \bar{\kappa}^3 \Delta^{4i} a[2,0]$ *and*
(2) $d_{11}(\Delta^{4i+2} a[28,0]) = \bar{\kappa}^2 \Delta^{4i} a[35,3]$

Proof. In (1), for both $i = 0, 1$, these are the image of differentials in the spectral sequence $E_*(V(0))$. Both source and targets survive to $E_{11}(Y)$ and so these two differentials occur.

For (2), the long exact sequence shows that $\pi_{75}(tmf \wedge Y) = \mathbb{Z}/2$. Lemma 6.17 (1) implies that the class $\bar{\kappa}^3 a[15,1] \in E_7^{13,88}$ survives the spectral sequence and detects the unique non-trivial element of the $\pi_{75}(tmf \wedge Y)$. On the other hand, the class $\bar{\kappa}^2 \Delta \nu^2 a[5,1] \in E_7^{11,86}$ is a permanent cycle. Thus, it must be hit by a differential and this is the possibility.

For $i = 1$, by taking into account the d_9-differentials and the d_{17}-differential Lemma 6.17 (2), we see that $\Delta^4 a[35,3]$ is a permanent cycle, which is $\bar{\kappa}$-free at the E_{11}-term. By inspection, the only class which can truncate its κ-family is $\Delta^6 a[28,0]$ by the indicated d_{11}-differential. □

Lemma 6.20. *There are differentials:*

(1) $d_{13}(\Delta^2 a[30,2]) = \bar{\kappa}^3 a[17,3]$ *and*
(2) $d_{13}(\Delta^2 a[33,1]) = \bar{\kappa}^3 a[20,2]$

Proof. For (1), it follows from (6.7) that $\pi_{78}(tmf \wedge Y) \cong \mathbb{Z}/2$. By sparseness, either $\Delta^2 a[30,2]$ or $\Delta^2 a[30,0]$ is a permanent cycle detecting the non-zero element of $\pi_{78}(tmf \wedge Y)$. Suppose that

$$\Delta^2 a[30,2] = \Delta^3 \nu^2 a[0,0]$$

is a permanent cycle detecting a class $\alpha \in \pi_{78}(tmf \wedge Y)$. At E_2, $\Delta^3 \nu^2 a[0,0]$ is in the image of $i_* : E_2(V(0)) \to E_2(Y)$ and so $p_*(\Delta^3 \nu^2 a[0,0]) = 0$. However, since $\pi_{78}(tmf \wedge V(0)) = 0$, $p_* \alpha \neq 0$ in $\pi_{76}(tmf \wedge V(0))$ and so is detected by a non-zero class in filtration $s > 2$, but such a class does not exist. We conclude that $\Delta^2 a[30,0]$ is a permanent cycle and that $\Delta^2 a[30,2]$ supports the stated differential.

For (2), by inspection, only $\Delta^2 a[33,1]$ and $\Delta^4 a[5,1]$ can support differentials truncating the $\bar{\kappa}$-family of $a[20,2]$. But $\Delta^4 a[5,1]$ is already paired with $a[0,0]$. □

Proposition 6.21. *The following classes are $\bar{\kappa}$-free permanent cycles:*

$$(A): \quad \begin{matrix} \Delta^2 a[4,0] & \Delta^2 a[9,1] & \Delta^2 a[14,2] & \Delta^2 a[19,3] & \Delta^2 a[20,2] \\ \Delta^2 a[30,0] & \Delta^2 a[35,3] & \Delta^2 a[45,3] & \Delta^4 a[17,3] & \Delta^4 a[20,2] \end{matrix}$$

and the following classes are not permanent cycles:

$$(B): \quad \begin{matrix} \Delta^6 a[4,0] & \Delta^6 a[9,1] & \Delta^6 a[14,2] & \Delta^6 a[19,3] & \Delta^6 a[20,2] \\ \Delta^6 a[30,0] & \Delta^6 a[30,2] & \Delta^6 a[33,1] & \Delta^6 a[35,3] & \Delta^6 a[45,3] \end{matrix}$$

Consequently, in the elliptic spectral sequence for $tmf \wedge Y$, each generator in (B) truncates some $\bar{\kappa}$-multiple of one and only one generator in (A).

Proof. These are the remaining generators of $\bar{\kappa}$-free families. No class in (B) can be a permanent cycle because the $\bar{\kappa}$-family of a class of (B) cannot be truncated. This means that all the 10 classes of (B) are non-permanent cycles, and so all the 10 classes of (A) are permanent cycles. □

Lemma 6.22. *We have the following differentials:*

(1) $d_{19}(\Delta^6 a[4,0]) = \bar{\kappa}^4 \Delta^2 a[19,3]$
(2) $d_{19}(\Delta^6 a[9,1]) = \bar{\kappa}^5 \Delta^2 a[4,0]$
(5) $d_{17}(\Delta^6 a[20,2]) = \bar{\kappa}^4 \Delta^2 a[35,3]$
(6) $d_{13}(\Delta^6 a[33,1]) = \bar{\kappa}^3 \Delta^4 a[20,2]$

(3) $d_{19}(\Delta^6 a[14,2]) = \bar{\kappa}^5\Delta^2 a[9,1]$

(4) $d_{19}(\Delta^6 a[19,3]) = \bar{\kappa}^5\Delta^2 a[14,2]$

(7) $d_{17}(\Delta^6 a[35,3]) = \bar{\kappa}^5\Delta^2 a[30,0]$

(8) $d_{23}(\Delta^6 a[45,3]) = \bar{\kappa}^6\Delta^2 a[20,2]$

Proof. Taking into account the differentials shown above, these are the only possible pairings remaining between the classes in (B) which are the sources in (1)–(8) and classes of (A). □

Remark 6.23. There are only two generators in (B) left living in the same topological degree, namely, $\Delta^6 a[30,0]$ and $\Delta^6 a[30,2]$. Each of these supports a differential truncating the $\bar{\kappa}$-families of either $\Delta^4 a[17,3]$ or $\Delta^2 a[45,3]$ and one differential determines the other.

Determining the last differential pattern turns out to be unfortunately tricky (as far as we know). A crucial step towards settling the last differentials is to establish the following extension in the E_∞-term of the elliptic spectral sequence for $tmf \wedge Y$.

Proposition 6.24. *There is an exotic extension*

$$\nu^2(\nu\Delta^6 a[0,0]) = \bar{\kappa}^2\Delta^4 a[17,3].$$

To prove this extension, we need some intermediate results.

Lemma 6.25. *In* $\mathrm{Ext}^{*,*}_{\Lambda'}(A', A'/(2,a_1)\otimes \mathcal{F}_*(Y))$*, there is a Massey product*

$$\langle \eta, \nu, \Delta^4 a[12,2]\rangle = \Delta^4 a[17,3].$$

Proof. Since $\Delta^4 a[12,2] = \eta\Delta^4 a[11,1]$ (see Lemma 5.10, also Figure 13), we have that

$$\langle \eta, \nu, \Delta^4 a[12,2]\rangle = \langle \eta, \nu, \eta\Delta^4 a[11,1]\rangle \supseteq \langle \eta, \nu, \eta\rangle \Delta^4 a[11,1] = \nu^2 a[11,1] = a[17,1].$$

The indeterminacy is zero since

$$\eta\,\mathrm{Ext}^{2,114}_{\Lambda'}(A', A'/(2,a_1)\otimes \mathcal{F}_*(Y)) + \mathrm{Ext}^{1,6}_{\Lambda'}(A', A'/(2))\Delta^4 a[12,2] = 0.$$

□

Proposition 6.26. *In* $\mathrm{Ext}^{*,*}_{\Lambda'}(A', \mathcal{F}_*(Y))$*, there is a Massey product*

$$\langle \eta, \nu, \Delta^4 a[12,2]\rangle = \Delta^4 a[17,3].$$

Proof. Let $f_*: \mathrm{Ext}^{*,*}_{\Lambda'}(A', \mathcal{F}_*(Y)) \to \mathrm{Ext}^{*,*}_{\Lambda'}(A', A'/(2,a_1)\otimes \mathcal{F}_*(Y))$ be the map induced by the Λ-comodule homomorphism $\mathcal{F}_*(Y) \to A'/(2,a_1)\otimes \mathcal{F}_*(Y)$. By naturality of Massey products, we have that

$$f_*(\langle \eta, \nu, \Delta^4 a[12,2]\rangle) \subseteq \langle \eta, \nu, f_*(\Delta^4 a[12,2])\rangle.$$

Further, $f_*(\Delta^4 a[12,2]) = \Delta^4 a[12,2]$. By Lemma 6.25, the above equation gives

$$f_*(\langle \eta, \nu, \Delta^4 a[12,2]\rangle) = \Delta^4 a[17,3].$$

The pre-image of $\Delta^4 a[17,3]$ is the same-named class. The indeterminacy is zero. □

Lemma 6.27. *There is an element of $\pi_{108}(tmf \wedge Y)$ detected by $\Delta^4 a[12,2]$ and annihilated by $\bar{\kappa}^2$.*

Proof. We have already determined $E_\infty(Y)$ in stems $t-s = 108, 148$. We see that there is a short exact sequence

$$0 \to \mathbb{Z}/2\{\bar{\kappa}^2\Delta^2 a[20,2]\} \to G \to \mathbb{Z}/2\{\Delta^4 a[12,2]\} \to 0,$$

where $G \subseteq \pi_{108}(tmf \wedge Y)$ is the subgroup of elements detected in positive filtration. At the E_∞-term in stem $t-s = 148$, the only non-zero class in positive filtration is $\bar{\kappa}^4\Delta^2 a[20,2]$. In particular, $\bar{\kappa}^2\Delta^4 a[12,2] = 0$. So, one of the classes detected by $\Delta^4 a[12,2]$ satisfies the claim. □

We will denote also by $\Delta^4 a[12,2]$ the element in $\pi_{108}(tmf \wedge Y)$, which is detected by $\Delta^4 a[12,2]$ and is annihilated by $\bar{\kappa}^2$.

Proposition 6.28. *There are the following relations in $\pi_*(tmf \wedge Y)$:*

(1) $\nu^2[\nu\Delta^6 a[0,0]] \neq 0$ *and*
(2) $\eta[\nu\Delta^6 a[0,0]] = 0$

Proof. The class detected by $\nu\Delta^6 a[0,0]$ lifts to $\pi_*(tmf \wedge V(0))$ and there is a lift detected by $\nu\Delta^6$. But in $\pi_*(tmf \wedge V(0))$, $\nu^2[\nu\Delta^6]$ is not divisible by η. □

Now, we use the truncated spectral sequences of Subsection 2.1, applied to the elliptic spectral sequence of $tmf \wedge Y$. As in Subsection 2.1, let

$$\mathrm{sk}_{16}(tmf \wedge Y) = X_0/X_{17}$$

for X_n the nth term of the $X(4)$-Adams tower of $tmf \wedge Y$. Then $E^{*,*}_{r,<17}(Y)$ as in Subsection 2.1 is a spectral sequence computing $\pi_*\mathrm{sk}_{16}(tmf \wedge Y)$, and it satisfies $E^{s,*}_{r,<17}(Y) = 0$ for $s \geqslant 17$. Furthermore, we have a map of spectral sequences

$$T^{s,t}_r : E^{s,t}_r(Y) \to E^{s,t}_{r,<17}(Y).$$

Proposition 6.29. *In $\pi_*(\mathrm{sk}_{16}(tmf \wedge Y))$, we have*

$$\langle \eta, \nu, \Delta^4 a[12,2]\rangle = \Delta^4 a[17,3].$$

Proof. In $\pi_*(tmf \wedge Y)$, the product $\nu\Delta^4 a[12,2]$, if not trivial, is detected in filtration 17. It follows that $\nu\Delta^4 a[12,2]$ is equal to zero in $\pi_*(\mathrm{sk}_{16}(tmf \wedge Y))$. Thus, the Toda bracket $\langle \eta, \nu, \Delta^4 a[12,2]\rangle$

can be formed. Proposition 6.26 means that in $E_{2,<17}^{s,t}(Y)$, there is Massey product

$$\langle \eta, \nu, \Delta^4 a[12,2]\rangle = \Delta^4 a[17,3].$$

The conditions of the Moss Convergence Theorem [32] are satisfied, so the Toda bracket $\langle \eta, \nu, \Delta^4 a[12,2]\rangle$ contains $\Delta^4 a[17,3]$ and the indeterminacy is zero. □

Proposition 6.30. *In the elliptic spectral sequence for $tmf \wedge Y$, there is an exotic extension*

$$\eta a[152,2] = \bar{\kappa}^2 \Delta^4 a[17,3].$$

Proof. Since $\bar{\kappa}^2\Delta^4 a[17,3]$ lives in filtration $s = 11$, it suffices to prove that extension in the E_∞-term of the spectral sequence for $\mathrm{sk}_{16}(tmf \wedge Y)$. The above proposition and the choice of $\Delta^4 a[12,2]$ imply that

$$\bar{\kappa}^2\Delta^4 a[17,3] = \langle \eta, \nu, \Delta^4 a[12,2]\rangle \bar{\kappa}^2 = \eta\langle \nu, \Delta^4 a[12,2], \bar{\kappa}^2\rangle.$$

Since $\bar{\kappa}^2\Delta^4 a[17,3] \neq 0$ at E_∞, $\langle \nu, \Delta^4 a[12,2], \bar{\kappa}^2\rangle$ must be non-trivial, and it must be detected by a class which is not in the kernel of η. This forces $\langle \nu, \Delta^4 a[12,2], \bar{\kappa}^2\rangle$ to be detected by $a[152,2]$, and so $\eta a[152,2]$ is detected by $\bar{\kappa}^2\Delta^4 a[17,3]$. □

Proof of Proposition 6.24. Let $\beta = [\nu\Delta^6 a[0,0]]$. By Proposition 6.28, $\eta\beta = 0$ and we can form the Toda bracket $\langle \nu, \eta, \beta\rangle$. Then

$$\eta\langle \nu, \eta, \beta\rangle = \langle \eta, \nu, \eta\rangle\beta = \nu^2\beta.$$

On the other hand, $\nu^2\beta \neq 0$ by Proposition 6.28. It follows that $\langle \nu, \eta, \beta\rangle \neq 0$. We see that it must be detected by $a[152,2]$. So, $\eta a[152,2] = \nu^2\beta$ and Proposition 6.30 implies that $\nu^2\beta$ is detected by $\bar{\kappa}^2\Delta^4 a[17,3]$. □

Lemma 6.31. *There are differentials:*

(1) $d_{13}(\Delta^6 a[30,2]) = \bar{\kappa}^3\Delta^4 a[17,3]$ *and*
(2) $d_{19}(\Delta^6 a[30,0]) = \bar{\kappa}^4\Delta^2 a[45,3]$

Proof. Let

$$tmf \wedge Y \leftarrow (tmf \wedge Y)_1 \leftarrow (tmf \wedge Y)_2 \leftarrow \ldots$$

be the Adams tower associated to the $X(4)$-based resolution of $tmf \wedge Y$. We consider its 1-co-truncated tower and the induced map of spectral sequences

$$cT_r^{s,t} : E_{r,\geqslant 1}^{s,t} \to E_r^{s,t}.$$

By Lemma 2.3, $cT_r^{s,t}$ is surjective for $s \geqslant 1$.

Let $a = \nu^2\Delta^6 a[0,0] \in E_2^{2,150+2}$. This is a permanent cycle representing a unique non-zero element of $\pi_{150}(tmf \wedge Y)$, which in this proof we denote by α. Since a has positive filtration, there is a class $\bar{a} \in E_{2,\geqslant 1}^{2,150+2}$ such that $cT_2(\bar{a}) = a$ and the surjectivity of cT_∞ guarantees that we can choose $\bar{a}$ to be a permanent cycle. It then detects classes $\bar{\alpha} \in \pi_{150}((tmf \wedge Y)_1)$ that map to α.

Since $\nu\alpha$ is detected by $b = \bar{\kappa}^2\Delta^4 a[17,3] \in E_\infty^{11,153+11}$ (Proposition 6.24), $\nu\bar{\alpha}$ must be detected in $E_\infty^{s,153+s}(cT_1)$ for $3 \leqslant s \leqslant 11$. Since $E_2^{s,153+s}(cT_1) = 0$ for $3 \leqslant s \leqslant 10$ (this is true for $E_2^{*,*}$), $\nu\bar{\alpha}$ must be detected by a lift $\bar{b}$ of b.

The relation $\bar{\kappa}\nu = 0 \in \pi_* tmf$ implies that $\bar{\kappa}\nu\bar{\alpha} = 0 \in \pi_*((tmf \wedge Y)_1)$. This implies that $d_r(\bar{c}) = \bar{\kappa}\bar{b}$ for some non-trivial element $\bar{c} \in E_{r,\geqslant 1}^{15-r,174+(15-r)}$. As $E_{2,\geqslant 1}^{0,*} = 0$, $\bar{c}$ must live in filtration $1 \leqslant s \leqslant 13$, and hence so does $cT_r(\bar{c})$. In particular, $cT_r(\bar{c}) \neq \Delta^6 a[30,0]$. However, we find that

$$d_r(cT_r(\bar{c})) = cT_r(\bar{\kappa}\bar{b}) = \bar{\kappa} \cdot cT_r(\bar{b}) = \bar{\kappa}^3\Delta^4 a[17,3].$$

The only way for this to make sense is if $cT_r(\bar{c})$ is equal to $\Delta^6 a[30,2]$ and this is the desired differential (1).

This differential then determines (2) as noted in Remark 6.23. □

Remark 6.32. From this discussion, we also learn that there is a non-trivial class in $i_*\pi_{150}V(0)$ which is detected by $a[153,11]$.

6.4 | Exotic extensions

In this section, we resolve the exotic 2, η, ν and v_1 extensions in the elliptic spectral sequence of $tmf \wedge Y$. The extensions are depicted in Figures 22 and 23.

We begin with the exotic η-extensions, which are few. To determine them, we use the following strategies. First, the long exact sequence

$$\dots \to tmf_{*+1}V(0) \xrightarrow{\eta} tmf_*V(0) \xrightarrow{i_*} tmf_*Y \xrightarrow{p_*} tmf_{*-1}V(0) \to \dots \tag{6.33}$$

We use the following basic, but useful facts.

Lemma 6.34. *For $a \in tmf_*Y$ and $b \in tmf_*V(0)$,*

(1) *if $a = i_*b$, then $\eta a = i_*\eta b = 0$,*
(2) *$p_*\eta a = \eta p_* a = 0$, and*
(3) *$v_1\eta a = \eta v_1 a$.*

Proof. These are easy consequences of the long exact sequence on homotopy groups (6.33) combined with the fact that composition as well as the smash product induces the π_*S^0-module structure in the stable homotopy category. □

Note further that Corollary 2.12 as described in Remark 2.16 gives a way to relate extensions in different stems between the v_1-power torsion classes. We also use Lemmas 2.21 and 2.23.

A stem-by-stem analysis using the above techniques then allows us to determine that the only non-trivial exotic η-extensions are as follows:

Lemma 6.35. *In the elliptic spectral sequence of Y, there are exotic extensions*

(1) $\eta[\Delta^2\nu a[5,1]] = \bar{\kappa}^2 a[17,3]$
(2) $\eta[\Delta^4\nu a[5,1]] = \bar{\kappa}^5 a[5,1]$
(3) $\eta[\Delta^6\nu a[5,1]] = \bar{\kappa}^2[\Delta^4 a[17,3]]$
(4) $\eta[\Delta^6\nu a[20,2]] = \bar{\kappa}^5[\Delta^2 a[20,2]]$

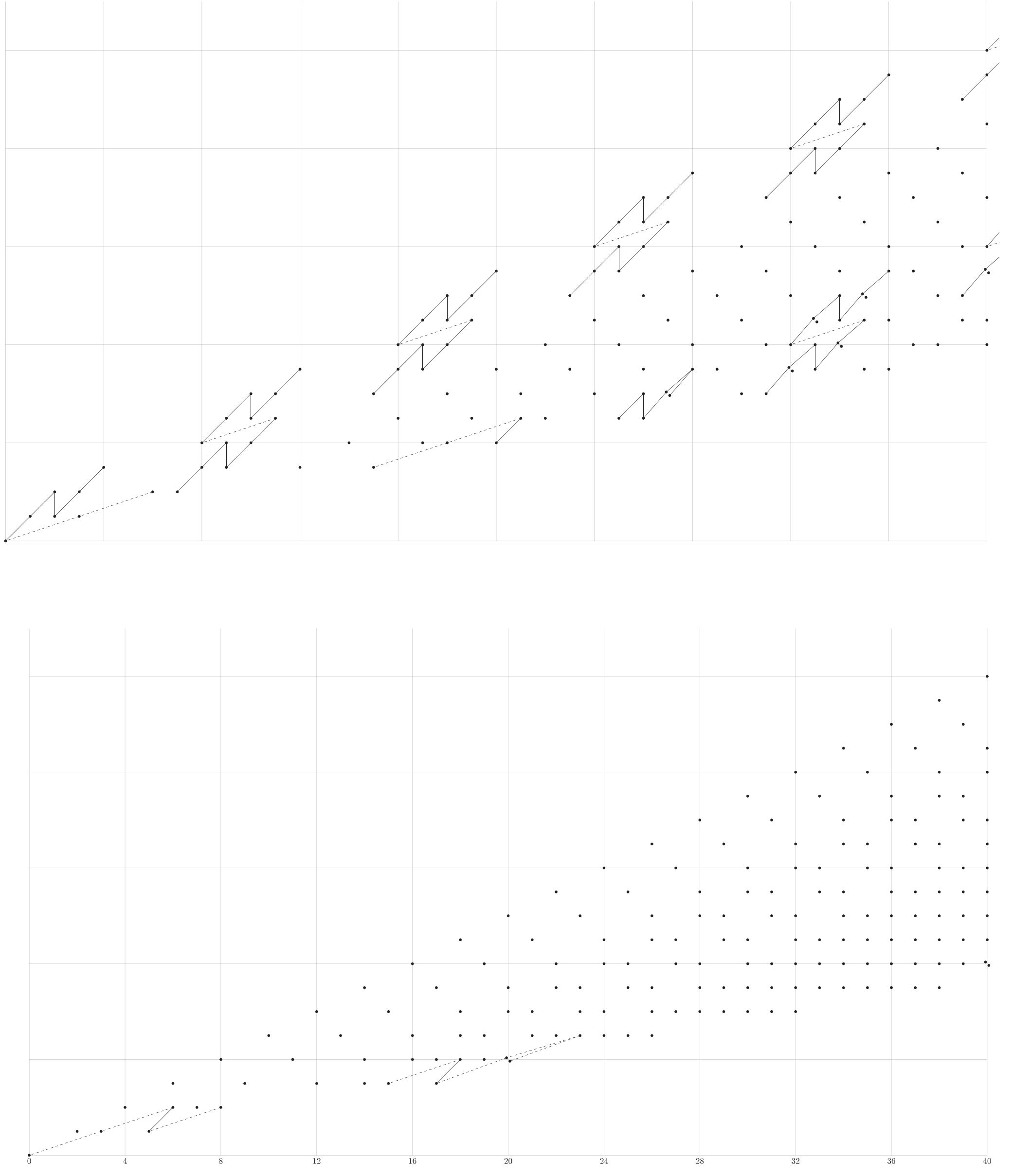


FIGURE 21 Classical Adams spectral sequence E_2-pages for $tmf \wedge V(0)$ (top) and $tmf \wedge Y$ (bottom) computed with Bruner's Ext-program [18]

There are no other exotic η-extensions.

Proof. Extension (1) follows from Lemma 2.23. Extensions (2) and (4) follow from duality: (2) from $\eta[\Delta^2 a[20,2]] = [\Delta^2 v_1^2 a[17,3]]$ and (4) from $\eta a[5,1] = \nu^2 a[0,0]$. Finally, (3) is Proposition 6.30.

All possible exotic η-extensions are shown not to occur using Lemma 6.34, duality and Lemma 2.23. In particular, the possible η-extensions with source in stems $52 \leqslant t-s \leqslant 57$ are shown not to occur using Lemma 2.23 and v_1-linearity. □

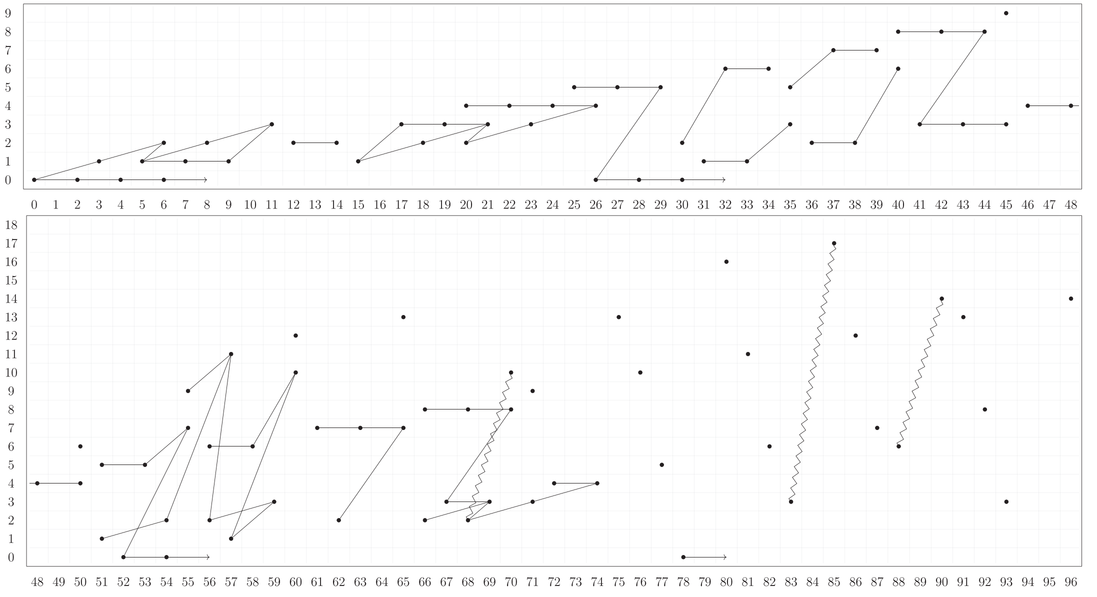


FIGURE 22 Exotic extensions in the elliptic spectral sequence of $tmf \wedge Y$. This records $tmf_*Y \cong \widetilde{tmf}_{*+3}(\mathbb{R}P^2 \wedge \mathbb{C}P^2)$. The zigzags denote exotic v_1-extensions that occur only for certain choices of v_1 self-maps.

Now, we turn to the exotic 2-extensions.

Theorem 6.36. *There are no exotic 2-extensions in the elliptic spectral sequence for Y and, consequently,*

$$2(\pi_* tmf \wedge Y) = 0.$$

Proof. Since we have a cofiber sequence

$$tmf \wedge C_\eta \xrightarrow{2} tmf \wedge C_\eta \xrightarrow{j} tmf \wedge Y \xrightarrow{q} \Sigma tmf \wedge C_\eta,$$

we can apply Lemma 2.21 with $X = tmf \wedge C_\eta$, $i = j$ and $p = q$. From this, we deduce that if $a' \in \pi_* tmf \wedge Y$ is in the image of j_*, then it has order 2 and that if $q_*(a') = a$, then $2a' = j_*(\eta a)$. It follows that if $2a' \neq 0$, then $2a'$ is divisible by η.

This leaves one possible extension in stem 57. But such a 2-extension would lead, by duality, to a 2-extension in stem 116. However, there are no η-divisible classes in that stem. Since the E_2-term was 2-torsion and there are no exotic 2-extensions, $\pi_* tmf \wedge Y$ is annihilated by 2. $\square$

Next, we turn to the ν extensions.

Remark 6.37. We will use without mention that $\bar{\kappa}\nu = 0$ in tmf_*-modules. This allows us to eliminate many possible exotic ν-extensions.

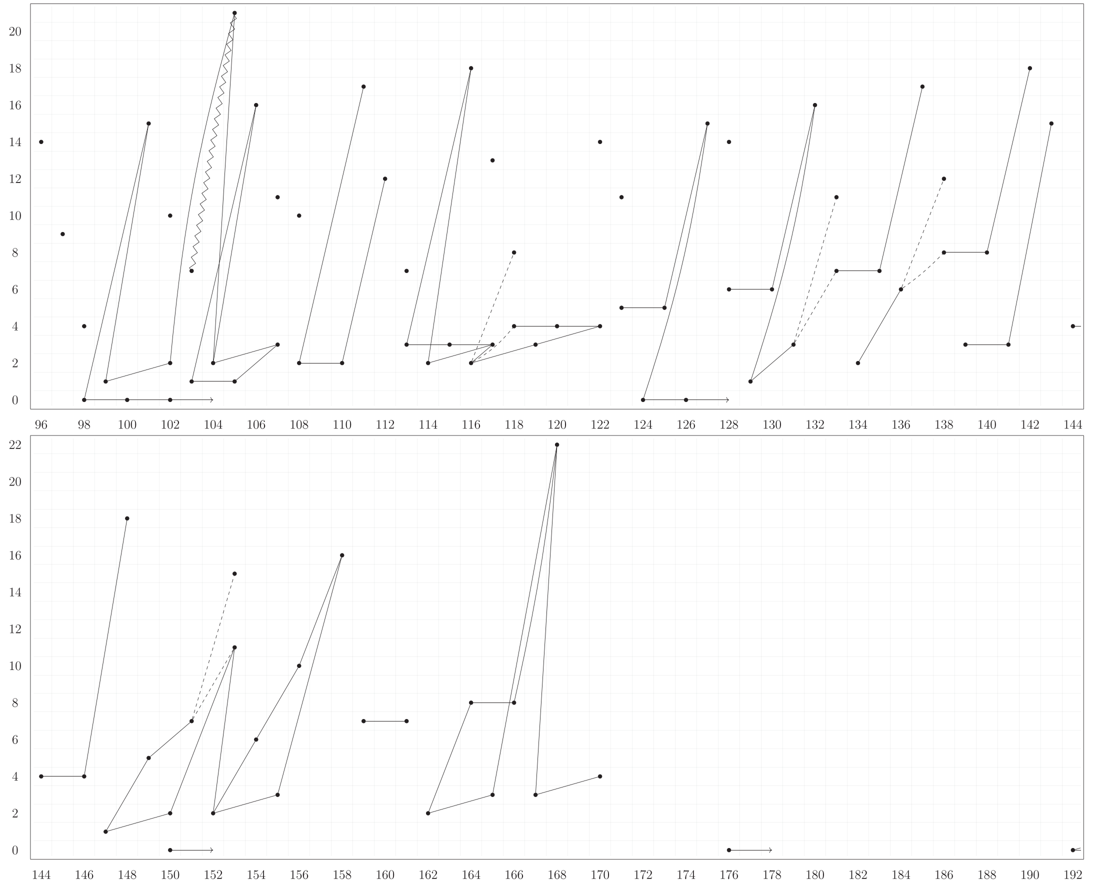


FIGURE 23 Exotic extensions in the elliptic spectral sequence of $tmf \wedge Y$. This records $tmf_* Y \cong \widetilde{tmf}_{*+3}(\mathbb{R}P^2 \wedge \mathbb{C}P^2)$. The zigzags denote exotic v_1-extensions that occur only for certain choices of v_1 self-maps.

Lemma 6.38. *In the elliptic spectral sequence of Y, there are exotic extensions*

(1) $\nu a[26,0] = a[29,5]$
(2) $\nu a[41,3] = a[44,8]$
(3) $\nu a[52,0] = a[55,7]$
(4) $\nu a[54,2] = \bar{\kappa}^2 a[17,3]$
(5) $\nu a[67,3] = \bar{\kappa}^2 a[30,0]$
(6) $\nu a[98,0] = a[101,15]$
(7) $\nu a[102,2] = \bar{\kappa}^5 a[5,1]$
(8) $\nu a[103,1] = a[106,16]$
(9) $\nu a[124,0] = a[127,15]$
(10) $\nu a[129,1] = a[132,16]$
(11) $\nu a[150,2] = a[153,11]$
(12) $\nu a[155,3] = a[158,16]$
(13) $\nu a[165,3] = a[168,22]$

Proof. Extensions (1) and (6) follow from the extensions $\nu a[26,0] = a[29,5]$ and $\nu a[98,0] = a[101,7]$, respectively, in $\pi_* tmf \wedge V(0)$ by applying i_*. Extensions (2), (3), (5), and (9) follow from examining the effect of p_* and the extensions $\nu a[39,3] = a[42,10]$, $\nu a[50,2] = a[53,7]$, $\nu a[65,3] = a[68,10]$ and $\nu a[122,2] = a[125,21]$ in $\pi_* tmf \wedge V(0)$, respectively.

Extensions (4), (7), (12), and (13) are obtained by duality from algebraic extensions. Extensions (10) and (8) follow by duality from (2) and (5).

Extension (11) is proved in Proposition 6.24. □

Lemma 6.39. *In the elliptic spectral sequence of Y, there are exotic extensions*

(1) $\nu a[57,1] = \bar{\kappa}^2 a[20,2]$ *and*
(2) $\nu a[62,2] = \bar{\kappa} a[45,3]$
Dually, we have
(3) $\nu a[108,2] = a[111,17]$
(4) $\nu a[113,3] = a[116,18]$

Together with Lemma 6.38, there are no other non-trivial exotic ν-extensions.

To prove Lemma 6.39, we use the tmf-based Atiyah–Hirzebruch spectral sequence for Y, whose filtration comes from the cellular filtration of Y. To set up notation, we have the E_1-page of this spectral sequence

$$E_1 = \oplus_{n=0}^{3} \pi_* tmf \Longrightarrow \pi_{*+n} tmf \wedge Y.$$

For a homotopy class β in $\pi_* tmf \wedge Y$, we denote by $\alpha[n]$ the element that detects it in the E_1-page of the tmf-based Atiyah–Hirzebruch spectral sequence, where n is the Atiyah–Hirzebruch filtration of β, and α is a class in $\pi_* tmf$. The stem of β is then the stem of α plus n.

Proof of Lemma 6.39. In our Atiyah–Hirzebruch notation, we can rewrite the two ν-extensions of Lemma 6.39 as

(1) $\nu \cdot \bar{\kappa}^2 \kappa[3] = \Delta \eta \kappa \bar{\kappa}[1]$,
(2) $\nu \cdot \bar{\kappa}^3[2] = \Delta^2 \nu \kappa[0]$.

We first prove (2), namely, that $\nu \cdot \bar{\kappa}^3[2] = \Delta^2 \nu \kappa[0]$. In $\pi_* tmf \wedge C_\eta$, we have

$$\nu \cdot \bar{\kappa}^3[2] = \langle \nu, \bar{\kappa}^3, \eta \rangle [0]$$

by [41, Lemma 5.3]. By Moss's theorem and the differential $d_{11}(\Delta^2 \kappa) = \eta \bar{\kappa}^3$ in the elliptic spectral sequence of tmf, we have

$$\langle \nu, \bar{\kappa}^3, \eta \rangle = \Delta^2 \nu \kappa.$$

Mapping this relation along the inclusion $C_\eta \to Y$ gives us (2).

For (1), note that in $\pi_* tmf \wedge \Sigma C_\eta$, we have

$$\nu \cdot \bar{\kappa}^2 \kappa[3] = \langle \nu, \bar{\kappa}^2 \kappa, \eta \rangle [1]$$

by [41, Lemma 5.3]. Since $\bar{\kappa}^2 \kappa$ is ν-divisible in $\pi_* tmf$, we may shuffle

$$\langle \nu, \bar{\kappa}^2 \kappa, \eta \rangle = \langle \bar{\kappa}^2 \kappa, \nu, \eta \rangle.$$

By Moss's theorem and the differential $d_5(\Delta\kappa\bar{\kappa}) = \nu\bar{\kappa}^2\kappa$ in tmf, we have

$$\langle \bar{\kappa}^2\kappa, \nu, \eta\rangle = \Delta\eta\kappa\bar{\kappa}.$$

Pulling back this relation along the quotient map $Y \to \Sigma C_\eta$ gives (1).

Extensions (3) and (4) follow by duality. The fact that there are no other exotic ν-extensions is discussed below. □

Most possibilities for other exotic ν-extensions are ruled out in a straightforward way by analyzing i_* and p_*, duality, the fact that $\bar{\kappa}\nu = 0$. However, the following two extensions require us to analyze the classical Adams spectral sequence. The following proof depends on checking algebraic extensions in

$$\mathrm{Ext}_{\mathcal{A}}((H\mathbb{F}_2)^*(tmf \wedge Y), (H\mathbb{F}_2)^*)$$

using Bruner's Ext-program [18]. See Figure 21 for classical Adams E_2-charts for $tmf \wedge V(0)$ and $tmf \wedge Y$, and see [21, chapter 13] for tmf.

Lemma 6.40. *In $\pi_* tmf \wedge Y$,*

(1) $\nu a[31,1] = 0$,
(2) $\nu a[36,2] = 0$.
Dually, we have,
(3) $\nu a[134,2] = 0$
(4) $\nu a[139,3] = 0$

Proof. To show this, we need to prove that

(1) $\nu a[31,1] \neq a[34,6]$,
(2) $\nu a[36,2] \neq a[39,7]$.

In our Atiyah–Hirzebruch notation, we can rewrite these extensions as

(1) $\nu \cdot \kappa^2[3] \neq \kappa\bar{\kappa}[0]$,
(2) $\nu \cdot \Delta\nu^3[3] \neq \Delta\eta\kappa[0]$.

We give a proof for (1) that $\nu \cdot \kappa^2[3] \neq \kappa\bar{\kappa}[0]$ using the classical Adams spectral sequence. We consider the Adams spectral sequence for $tmf \wedge Y$ and its subquotients. We will show that the Adams filtration of $\kappa^2[3]$ is 7 and the Adams filtration of $\kappa\bar{\kappa}[0]$ is 8. The fact that there is no such ν-extension follows from the algebraic fact that on the Adams E_2-page, the h_2-multiple of the first element is not the second element, which is checked by a computer program.

For the class $\kappa\bar{\kappa}[0]$, it is clear that the Adams filtration of $\kappa\bar{\kappa}$ in $\pi_{34} tmf$ is 8, (it is detected by the element $d_0 g$,) and it maps nontrivially on the Adams E_2-pages along the map $tmf \to tmf \wedge Y$. The image under this map, which we denoted by $d_0 g[0]$, is a permanent cycle. It cannot be killed due to filtration reasons. Therefore, the class $\kappa\bar{\kappa}[0]$ is detected by $d_0 g[0]$ and, in particular, it has Adams filtration 8.

For the class $\kappa^2[3]$, we first consider the class $\kappa^2[1]$ in $\pi_{29} tmf \wedge V(0)$. Since $\pi_{29} tmf = 0, \pi_{30} tmf = 0$, we have $\pi_{30} tmf \wedge V(0) = 0$. This forces three non-zero Adams differentials eliminating the three elements in the Adams E_2-page for $tmf \wedge V(0)$. In particular, we learn that $\kappa^2[1]$

in $\pi_{29}tmf \wedge V(0)$ is detected by the only remaining element $j[0]$ in Adams filtration 7, and that there is a non-zero d_3-differential from $(t-s,s)$-bi-degrees (31,6) to (30,9).

Considering the quotient map $tmf \wedge Y \to tmf \wedge \Sigma^2 V(0)$, we learn that $\kappa^2[3]$ is detected in Adams filtration at most 7. Considering the induced map on the Adams E_2-pages, we also learn that it is an isomorphism on the $(t-s,s)$-bi-degrees (31,6) and (30,9). So, in particular, the element in $(t-s,s)$-bi-degree (31,6) does not survive. Therefore, $\kappa^2[3]$ is detected in Adams filtration exactly 7.

For (2), that $\nu \cdot \Delta\nu^3[3] \neq \Delta\eta\kappa[0]$, we use the Adams spectral sequence again in a very similar way. We will show that the Adams filtration of $\Delta\nu^3[3]$ is 8 and the Adams filtration of $\Delta\eta\kappa[0]$ is 9. The fact that there is no such extensions then follows as in (1).

For the class $\Delta\eta\kappa[0]$, it is clear that the Adams filtration of $\Delta\eta\kappa$ in $\pi_{39}tmf$ is 9, (it is detected by the element u,) and it maps nontrivially on the Adams E_2-pages along the map $tmf \to tmf \wedge Y$. The image under this map, which we denoted by $d_0g[0]$, is a permanent cycle. It cannot be killed due to filtration reasons. Therefore, the class $\Delta\eta\kappa[0]$ is detected by $u[0]$, and in particular it has Adams filtration 9.

For the class $\Delta\nu^3[3]$, we first consider the class $\Delta\nu^3$ in $\pi_{33}tmf$. The class $\Delta\nu^3$ in $\pi_{33}tmf$ is detected in the Adams filtration 8. Considering the quotient map $tmf \wedge Y \to \Sigma^3 tmf$, we learn that $\Delta\nu^3[3]$ is detected in Adams filtration at most 8. To show that it is detected in Adams filtration 8, we will show that the only other element in lower filtration, the class in $(t-s,s)$-bi-degree (36,7), supports a non-zero d_2-differential.

The maps in the zigzag

$$tmf \wedge S^1 \longleftarrow tmf \wedge V(0) \longrightarrow tmf \wedge Y$$

are isomorphisms in $(t-s,s)$-bi-degrees (36,7) and (35,9) on Adams E_2-pages. So, the claimed non-zero d_2-differential follows from the one in the Adams spectral sequence of tmf, from $(t-s,s)$-bi-degrees (35,7) and (34,9). $\square$

We now turn to the study of the v_1-extensions. First, recall the discussion on v_1-self maps and A_1 from Subsection 2.6. The homotopy groups of $tmf \wedge A_1$ are studied by the third author in [33]. Furthermore, the knowledge of the homotopy groups of $tmf \wedge A_1$ is sufficient to allow us to deduce much of the action of v_1 on the homotopy groups of $tmf \wedge Y$, via the long exact sequence on homotopy of the cofiber sequence

$$tmf \wedge \Sigma^2 Y \xrightarrow{v_1} tmf \wedge Y \to tmf \wedge A_1.$$

Since the outcome depends on the choice of the v_1-self-map, we call a v_1-self-map of type I if its cofiber is $A_1[01]$ or $A_1[10]$ and of type II, otherwise. Again, see Subsection 2.6 for the definition of $A_1[ij]$.

Lemma 6.41.

(a) *For all v_1-self maps of Y, there are the following exotic v_1-extensions, and those induced from these by $\bar{\kappa}$-linearity:*
   (1) $v_1 a[9,1] = a[11,3]$
   (2) $v_1 a[15,1] = a[17,3]$
   (3) $v_1 a[30,2] = \bar{\kappa} a[12,2]$
   (4) $v_1 a[33,1] = a[35,3]$

(5) $v_1 a[38,2] = \bar{\kappa} a[20,2]$
(6) $v_1 \Delta^2 a[9,1] = \Delta^2 a[11,3]$
(7) $v_1 a[99,1] = \bar{\kappa}^3 a[21,3]$
(8) $v_1 a[104,2] = \bar{\kappa}^4 a[26,0]$
(9) $v_1 a[105,1] = a[107,3]$
(10) $v_1(v_1 a[108,2]) = \bar{\kappa}^3 a[52,0]$
(11) $v_1 a[114,2] = \bar{\kappa}^4 a[36,2]$
(12) *either* $v_1 a[116,2] = \bar{\kappa}^2 a[78,0]$ *or* $v_1 a[116,2] = \bar{\kappa} a[98,0]$
(13) $v_1 \bar{\kappa} a[105,1] = \bar{\kappa}^3 a[67,3]$
(14) $v_1 a[129,1] = a[131,3]$
(15) *either* $v_1 a[131,3] = \bar{\kappa}^2 a[93,3]$ *or* $v_1 a[131,3] = \bar{\kappa} a[113,3]$
(16) $v_1 a[134,2] = \bar{\kappa} a[116,2]$
(17) $v_1 \bar{\kappa} a[115,3] = \bar{\kappa} a[117,13]$
(18) $v_1(v_1 a[139,3]) = \bar{\kappa}^3 a[83,3]$
(19) $v_1 \bar{\kappa} a[120,3] = \bar{\kappa} a[122,14]$
(20) $v_1(v_1 \bar{\kappa} a[124,0]) = \bar{\kappa}^4 a[68,2]$
(21) $v_1 a[147,1] = \bar{\kappa} a[129,1]$;
(22) $v_1 a[152,2] = \bar{\kappa} a[134,2]$
(23) $v_1 a[156,10] = a[158,16]$
(24) $v_1 a[162,2] = \bar{\kappa}^2 a[124,0]$

(b) *For v_1-self-maps of type I, there are also the following v_1-extensions, and those induced from these by $\bar{\kappa}$-linearity:*
(1) $v_1 a[68,2] = \bar{\kappa}^2 a[30,2]$ *and*
(2) $v_1 a[83,3] = \bar{\kappa}^4 a[15,1]$

Proof. For all parts, except for (9), (12), (15), we see, by inspecting the relevant parts of the homotopy groups of appropriate $tmf \wedge A_1[ij]$, that the targets of the stated v_1 extensions are sent to zero via the natural map

$$\pi_*(tmf \wedge Y) \to \pi_*(tmf \wedge A_1[ij]).$$

Therefore, they are in the image of a v_1-multiplication and the stated v_1-extensions are the only possibilities.

For part (9), consider

$$\mathrm{sk}_4(tmf \wedge Y) = (tmf \wedge Y)/(tmf \wedge Y)_5,$$

where $(tmf \wedge Y)_5$ is the fifth term in the $X(4)$-Adams tower of $tmf \wedge Y$. It is a module over $\mathrm{sk}_4(tmf)$. Since $\Delta^4 \in \pi_{96}(\mathrm{sk}_4(tmf))$, this element acts on $\pi_*\mathrm{sk}_4(tmf \wedge Y)$. We see that the induced map $\pi_*(tmf \wedge Y) \to \pi_*\mathrm{sk}_4(tmf \wedge Y)$ sends $a[9,1]$ and $a[11,3]$ to non-trivial elements, which we denote by the same names. Furthermore, it sends $a[105,1]$ and $a[107,3]$ to elements detected by the products $\Delta^4 a[9,1]$ and $\Delta^4 a[11,3]$. Since $v_1 a[9,1] = a[11,3]$ by part (1),

$$v_1 \Delta^4 a[9,1] = \Delta^4 v_1 a[9,1] = \Delta^4 a[11,1]$$

in $\pi_*\mathrm{sk}_4(tmf \wedge Y)$. It follows that $v_1 a[105,1]$ must be detected by $a[107,3]$ in the E_∞-term of the elliptic spectral sequence of $tmf \wedge Y$. □

Remark 6.42. We are left with two undecided v_1-extensions, namely, (12) and (13) in Lemma 6.41, which we were unable to resolve.

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JOURNAL INFORMATION

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