BAILEY on TMF

DL, MW, FC, NR, PS, JH
TW

Note Title 10/9/2009

I What is a stack?

<u>Def</u> A stack is a sheaf of groupoids which satisfies "effective descent" (a way of gluing isomorphisms

<u>Example</u> X = topological space

contravariant functor $U \subseteq X$

$U \longmapsto \mathrm{Bund}_n(U)$ Object n plane bundle over U

Morphisms: isomorphisms of these bundles

More structure:

$\{V_i\}$ open cover for $\mathcal{U}$

ξ_i over V_i

$$\xi_i|_{V_i \cap V_j} \xrightarrow[\phi_{i,j}]{\cong} \xi_j|_{V_i \cap V_j}$$

If the $\phi_{i,j}$ satisfy co-cycle conditions, then they can glue, and we get a global bundle ξ such that

$$\phi_i : \xi|_{V_i} \longrightarrow \xi_i$$

Caveat We only have a functor up to bundle isomorphisms.

The functor is really defined on ?

A stack is a "category fibered in groupoids"

"stackification"

$$\{\text{stacks}\} \xrightarrow{\text{forgetful}} \{\text{sheaves of groupoids}\}$$

e.g over Rings

[stacks generalize schemes]

This functor has a left adjoint which is stackification

II We will regard Spec(R) as a covariant functor

$$\text{Rings}_c \xrightarrow{\text{Spec}(R)} \text{Sets}$$

$$S \longrightarrow \text{Hom}(R, S)$$

"Grothendieck topology" on a category C

Category $\text{Cov}(C)$

families of morphisms $\{\varphi_i : S_i \to S\}$ in C

with nice properties

<u>Example</u> Flat topology

$\{\text{Spec}(U_i) \to \text{Spec}(R)\}$ is a cover if

(1) each map $R \to U_i$ is flat $\forall i$

(2) If $M \otimes_R U_i = 0$ $\forall i$, then $M = 0$.

To define a general scheme. It is a functor

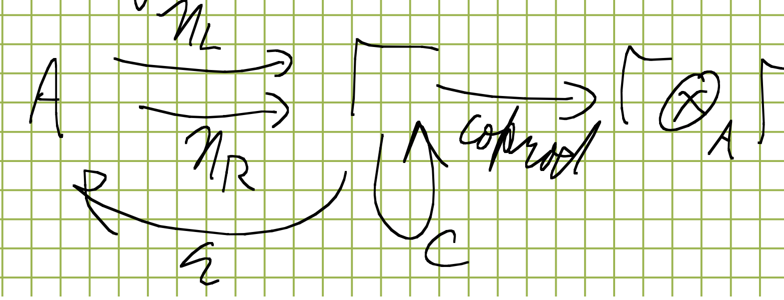
$X : \mathrm{Rings}_C \longrightarrow \mathrm{Sets}$

(1) X is a sheaf in the Zariski Top

(2) X has an open cover by affine scheme.

III Hopf algebroids

A pair of commutative rings (A, Γ) with structure maps



These make a groupoid valued functor G_Γ on rings R

$$Ob(G_\Gamma(R)) = Hom(A, R)$$

$$Mor(G_\Gamma(R)) = Hom(\Gamma, R)$$

$$Spec(R) \longrightarrow G_\Gamma(R)$$

is a sheaf in any topology.

Call $M(A, \Gamma)$ the associated stack, the stackification of this sheaf.

Example (formal group laws FGLs ungraded

Let $L = \mathbb{Z}[a_1, a_2, \dots]$ = Lazard ring.

If $F \in R[[x, y]]$ is a FGL, it is induced by a map $L \to R$.

Objects of this category = $\mathrm{Hom}(L, R)$

$W = L[b_0^{\pm 1}, b_1, b_2, \dots]$ representing (non strict) isomorphisms between FGLs.

Morphisms in category = $\mathrm{Hom}(W, R)$

$$\mathcal{M}(L, W) = \mathcal{M}_{FGL} \longleftrightarrow (L, W)$$

$L \cong MU_*$ by Quillen $= \pi_* MU)$

$W \cong MU_* MU$ $= \pi_*(MU \wedge MU)$

Example Elliptic curves

Every elliptic curve over R is isomorphic to one of the form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \longrightarrow R$$

Automorphisms are given by base change

$$x \longrightarrow \lambda^2 x + r \qquad r, s, t \in R$$

$$y \longrightarrow \lambda^3 y + s\lambda^2 x + t \qquad \lambda \in R^\times$$

$$\begin{bmatrix} \lambda^2 & 0 & r \\ s\lambda^2 & \lambda^3 & t \\ 0 & 0 & 1 \end{bmatrix}$$ Let $A[r, s, t, \lambda^{\pm 1}] =: \Gamma$

The stack $\mathcal{M}(A, \Gamma) = \mathcal{M}_{Ell}$ is the moduli stack of elliptic curves.

TMF comes from $\mathcal{M}_{Ell}$.

$\mathcal{M}_{Ell}$ needs to be topologized and given a sheaf of E_∞-ring spectra.

An algebraic property of stacks.

(homotopy) Pull-back of groupoids

$$\begin{array}{ccc} G_1 \times_H G_2 & \longrightarrow & G_2 \\ \downarrow & & \downarrow f_2 \\ G_1 & \xrightarrow[f_1]{} & H \end{array}$$

$$Ob(G_1 \times_H G_2) = \left\{ (x, y, \varphi) \;\middle|\; \begin{array}{l} x \in G_1 \quad y \in G_2 \\ \varphi : f_1(x) \xrightarrow{\cong} f_2(y) \end{array} \right\}$$

$$Mor(\qquad) = \left\{ (\gamma_1, \gamma_2) : (x, y, \varphi) \longrightarrow (x', y', \varphi') \right\}$$

$$\begin{array}{ccc} f_1(x) & \xrightarrow{\gamma_1} & f_1(x') \\ \downarrow \varphi & & \downarrow \varphi' \\ f_2(y) & \xrightarrow[\gamma_2]{} & f_2'(y') \end{array} \qquad \begin{array}{l} \gamma_1 : x \xrightarrow{\cong} x' \\ \gamma_2 : y \xrightarrow{\cong} y' \end{array}$$

A representable morphism of stacks

$$\mathcal{M} \xrightarrow{f} \mathcal{N}$$

Spec (R) can be viewed as a stack with $A = \Gamma = R$ so Objs and Morphisms are the same. The only morphisms are identity maps.

$$\begin{array}{ccc} \operatorname{Spec}(S) \underset{(*)}{\cong} \operatorname{Spec}(R) \times_{\mathcal{N}} \mathcal{M} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow f \\ \operatorname{Spec}(R) & \longrightarrow & \mathcal{N} \end{array}$$

f is <u>representable</u> if $\exists$ S such that (*)

$M \to N$ has property P if

$\operatorname{Spec}(S) \to \operatorname{Spec}(R)$