LOCALISATION AND COMPLETION

with an
addendum on the use of
Brown-Peterson homology
in stable homotopy

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FORWARD

In spring 1973 Frank Adams gave a course at the University of Chicago on localisation and completion. This was in the very early days of the subject, which arose from disparate constructions of Quillen, Sullivan, Mimura, Nishida, Toda and others during the period 1969-1971. In those days one usually assumed the spaces one was localising or completing were simply connected. There were various proposals for extending the domain of definition of these constructions to more general spaces, but there was no clear consensus on how to proceed.

In his lectures Adams gave a lucid and compelling analysis of the properties one would want of such constructions. He set up an elegant axiomatic treatment of localisation and completion in the framework of category theory and proposed a vast generalisation of the existing constructions. Unfortunately Adams' program for constructing these localisation functors with respect to arbitrary generalised homology theories ran into a serious difficulty during the course of these lectures. His proposal involved the use of the Brown Representability Theorem to construct his localisation functors, but he was unable to show that the relevant representable functors were set-valued rather than class-valued. Subsequent work by Bousfield established the existence of these generalised localisation functors, using more technical simplicial methods. These functors are now an essential tool in homotopy theory.

At that time I was a graduate student at Chicago and was charged with the responsibility of taking notes for Adams' lectures. The resulting notes were briefly available in mimeographed form from the University of Chicago Mathematics Department. However these notes were never published in a more formal venue due to this apparent flaw in the proof.[1] The notes also contain an addendum devoted to establishing that a certain element in the gamma family of the stable homotopy groups of spheres is nonzero, using Brown-Peterson (co)homology. At that time this was a matter of controversy, as Oka and Toda claimed to have proved the contrary result.

I thought I had lost my only copy of these notes a long time ago, but I recently rediscovered them, and I want to make them publicly available again. Besides being of historical interest, these notes give a very readable intro-

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[1]An announcement of this work was published in [3].

duction to localisation and completion, with minimal prerequisites. I was long aware that the gap in Adams' proof was easily mendable, so I have supplemented these notes to include an epilogue explaining this and have made a few other minor editorial changes. Thus it can now be seen in retrospect that Adams amazingly succeeded in his project of *"constructing localizations and completions without doing a shred of work"* (cf. [9]).

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CONTENTS

1   Introduction to Localisation

> *"Il y a là possibilité d'une étude locale (au sens arithmetique) des groupes d'homotopie"*

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J. P. Serre [17]

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In homotopy theory we have known for a long time that it is sufficient to attack problems one prime at a time. This insight goes back to the pioneering work of J.-R Serre [17].

More recently we have gained a particularly convenient language and some particularly convenient machinery for exploiting this insight. This language and machinery was introduced following an analogy from commutative algebra. In commutative algebra we attack our problems one prime at a time by using the method of localisation. Thus we seek a comparable method in homotopy theory.

The earliest reference I have which develops such a method is Sullivan [19]. This was certainly very influential. At this point perhaps we should also mention Mimura-Nishida-Toda [12], Mimura-O'Neill-Toda [13], Mimura-Toda [14], and Zabrodsky [23]. Another reference we might suggest is Quillen [16].

In the first part of these lectures, I want to present a simple and uniform method of constructing all functors in homotopy theory which have formal properties similar to those of Sullivan's localisation functor. This opens the way to a study of such functors along axiomatic lines. I may also say something about Sullivan's completion functor. However it is clear from Sullivan's work that the completion functor enters it for a visibly good and sufficient reason which is particular to that piece of work. The localisation functor, however, is of very general use, and every graduate student of topology should learn about it.

I must begin by sketching some background, and I start with commutative algebra. Let R be a commutative ring with 1 and $S \subset R$ be a multiplicatively closed subset (i. e. a subset closed under finite products, such that $1 \in S$). For example, if $R = \mathbf{Z}$ we may take

$$S = \{1, 2, 4, 8, \ldots, 2^n, \ldots\}$$

or
$$S = \{1, 3, 5, \ldots 2m + 1, \ldots\}.$$

Let M be an R-module. We say M is S-*local* if the map $M \to M$ given by multiplication by s, i. e., $m \mapsto ms$, is an isomorphism for any $s \in S$. To every R-module M we can find a map $f : M \to M'$ so that

(i) M' is S-local

(ii) f is universal with respect to (i). That is, if $g : M \to M''$ is another map such that M'' is S-local, then there is a unique map $h : M' \to M''$ which makes the following diagram commute

Such a map f is called a localisation map; we say f localises M at S.

The usual construction of M' is as a module of fractions. We first take pairs (m, s), $m \in M$, $s \in S$. We then define an equivalence relation on pairs:

$$(m, s) \sim (m', s') \quad \Longleftrightarrow \quad \exists s'' \in S \quad \ni ms's'' = m'ss''$$

We define $S^{-1}M$ to be the set of all equivalence classes. The fraction $\frac{m}{s}$ is the equivalence class containing (m, s). We make $S^{-1}M$ into an R-module in the obvious way. We define the map $f : M \to S^{-1}M$ by $f(m) = \frac{m}{1}$. We see that $S^{-1}M$ is S-local and the map f is universal.

Since the ring R is an R-module, $S^{-1}R$ is defined; we can make it into a ring so that the canonical map $R \to S^{-1}R$ is a map of rings:

$$\left(\frac{r}{s}\right)\left(\frac{r'}{s'}\right) = \frac{rr'}{ss'}$$

Similarly $S^{-1}M$ becomes a module over $S^{-1}R$:

$$\left(\frac{m}{s'}\right)\left(\frac{r}{s}\right) = \frac{mr}{s's}$$

Moreover we obtain a commutative diagram

$$
\begin{array}{ccc}
M \otimes_R S^{-1}R & & \\
& \searrow^{\cong} & \\
\cong \Big\downarrow & & S^{-1}M \\
& \nearrow_{\cong} & \\
S^{-1}M \otimes_R S^{-1}R & &
\end{array}
$$

We often use this fact just as a matter of notation when we have a convenient name for $S^{-1}R$. For example, suppose $R = \mathbf{Z}$ and

$$S = \{1, 2, 4, 8, \ldots, 2^n, \ldots\}$$

so that $S^{-1}R = \mathbf{Z}\left[\frac{1}{2}\right]$. If M is a $\mathbf{Z}$ module, we would usually write $M \otimes \mathbf{Z}\left[\frac{1}{2}\right]$ for $S^{-1}M$.

The most common example of a multiplicatively closed subset S is the complement of a prime ideal P. If $S = \mathcal{C}P$ we write M_P for $S^{-1}M$. For example, if $R = \mathbf{Z}$, $P = (2)$,

$$\mathcal{C}P = \{\pm 1, \pm 3, \pm 5, \ldots, \pm(2m+1), \ldots\},$$

then $\mathbf{Z}_{(2)}$ is the set of fractions

$$\left\{\frac{a}{2b+1}\right\} \subset \mathbf{Q}.$$

Often algebraists omit the parenthesis and write $\mathbf{Z}_2$.

The construction of localisation in commutative algebra has many good properties. Most of them need not delay us now; however, it is essential to know that localisation preserves exactness. If

$$L \xrightarrow{i} M \xrightarrow{i} N$$

is exact, then so is

$$S^{-1}L \xrightarrow{i} S^{-1}M \xrightarrow{i} S^{-1}N$$

Example (a) Take the two localisation functors on $\mathbf{Z}$-modules

$$- \otimes \mathbf{Z}\left[\frac{1}{2}\right] \quad \text{and} \quad - \otimes \mathbf{Z}_{(2)}$$

These two functors commute up to isomorphism and for any $\mathbf{Z}$-module M we have a commutative diagram

$$
\begin{array}{ccc}
M & \longrightarrow & M \otimes \mathbf{Z}\left[\tfrac{1}{2}\right] \\
\downarrow & & \downarrow \\
& & \left(M \otimes \mathbf{Z}\left[\tfrac{1}{2}\right]\right) \otimes \mathbf{Z}_{(2)} \\
& & \downarrow{\scriptstyle\cong} \\
M \otimes \mathbf{Z}_{(2)} & \longrightarrow & (M \otimes \mathbf{Z}_2) \otimes \mathbf{Z}\left[\tfrac{1}{2}\right] \cong M \otimes \mathbf{Q}
\end{array}
$$

This diagram is both a pullback and pushout.

Conversely, if we are given : M' a $\mathbf{Z}\left[\tfrac{1}{2}\right]$-module, M'' a $\mathbf{Z}_{(2)}$-module , M''' a $\mathbf{Q}$-module, and localising maps $f : M' \to M'''$, $g : M'' \to M'''$, then in the pullback diagram

$$
\begin{array}{ccc}
M & \overset{h_1}{\dashrightarrow} & M' \\
{\scriptstyle h_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
M'' & \underset{g}{\longrightarrow} & M'''
\end{array}
$$

h_1 and h_2 are also localising maps.

Now I want to recall the basic theorem about Sullivan's localisation functor and give one example of its use to show what it is meant for.

Before we do this, however, we must consider the category on which it is to be defined. If we stick to simply-connected CW-complexes, everyone will feel happy and secure. Moreover, it might be a matter of debate exactly how far we might wish to enlarge the domain of definition of our functor; and if we take the domain too large, there might be more than one functor extending the functor we all agree about for simply-connected spaces, and it might be a matter of debate which extension is best.

So let us take $\mathcal{C}$ to be the category whose objects are 1-connected CW-complexes with basepoint and whose morphisms are homotopy classes of maps, with both maps and homotopies preserving basepoints.

Let $S \subset \mathbf{Z}$ be a multiplicatively closed subset. Then Sullivan showed there is a functor $\mathcal{C} \to \mathcal{C}$ which at the level of homotopy and homology performs localisation at S.

Theorem 1.1 (Sullivan). *The following conditions on a map $f : X \to Y$ in $\mathcal{C}$ are equivalent:*

> *(i) $f_\# : \pi_n(X) \to \pi_n(Y)$ localises at S for each $n \geq 1$.*

> *(ii) $f_* : H_n(X) \to H_n(Y)$ localises at S for each $n \geq 1$.*

Moreover there is a functor $E : \mathcal{C} \to \mathcal{C}$ and a natural transformation $\eta : 1 \to E$ so that for each X, $\eta_X : X \to EX$ satisfies both (i) and (ii).

Before we go on, let me comment. Theorems like this usually tell us that there is something with a stated property when it is not obvious that there is; but it is obvious that if there is anything with that property, then the property characterises it. On the face of it this is not a theorem of that form; the properties do characterise EX and η_X, but it is not obvious they do. Of course, these are comments on the way I've stated the theorem not on the way Sullivan states it. Still we may make a note to look for other forms of the statement.

Following the analogy from algebra we use the notation X_S or $X \otimes \mathbf{Z}_S$ for EX. We defer discussing the properties of E until we are forced to do so by examples.

As an application, I recall that at one time there was a conjecture of the following sort.

Conjecture. *Any finite CW-complex which is an H-space is homotopy-equivalent to a product of spaces from the classical list: S^7, $\mathbf{R}P^7$, compact Lie groups.*

This conjecture must always have looked optimistic, and it is now known to be false. The first counterexample was due to Hilton and Roitberg. Let us see how we get one by Zabrodsky's method of mixing homotopy groups, expressed in the language of localisation.

The classical list contains two entries $S^3 \times S^7$ and $Sp(2)$. These are different at the prime 2 and also at the prime 3 (e.g. because $\pi_6(S^3 \times S^7) = \mathbf{Z}/12\mathbf{Z}$, $\pi_6(Sp(2)) = O$.) However, if we apply the localisation functor $- \otimes \mathbf{Q}$ they become the same:

$$(S^3 \times S^7) \otimes \mathbf{Q} = K(\mathbf{Q}, 3) \times K(\mathbf{Q}, 7) = Sp(2) \otimes \mathbf{Q}.$$

So consider for example

$$Sp(2) \otimes \mathbf{Z} \left[\tfrac{1}{2}\right]$$

$$\downarrow$$

$$Sp(2) \otimes \mathbf{Z} \left[\tfrac{1}{2}\right] \otimes \mathbf{Z}_{(2)}$$

$$\Big\downarrow {\scriptstyle\cong}$$

$$(S^3 \times S^7) \otimes \mathbf{Z}_{(2)} \longrightarrow (S^3 \times S^7) \otimes \mathbf{Z}_{(2)} \otimes \mathbf{Z} \left[\tfrac{1}{2}\right]$$

There should be a space X such that

$$\begin{aligned}
X \otimes \mathbf{Z} \left[\frac{1}{2}\right] &\simeq SP(2) \otimes \mathbf{Z} \left[\frac{1}{2}\right] \\
X \otimes \mathbf{Z}_{(2)} &\simeq Sp(2) \otimes \mathbf{Z}_{(2)}
\end{aligned}$$

It is easy to construct a candidate for X. Suppose we are given a diagram

$$
\begin{array}{ccc}
& & X' \\
& & \downarrow{\scriptstyle p} \\
X'' & \xrightarrow{\ g\ } & X'''
\end{array}
$$

then the weak pull-back is the space of triples (x', x'', ω), where $\omega : I \to X'''$ is a path from $p(x')$ to $g(x'')$. Strictly I shall take a weakly equivalent CW-complex; anyway I get a diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow & & \downarrow{\scriptstyle p} \\
X'' & \xrightarrow{\ g\ } & X'''
\end{array}
$$

and an exact homotopy sequence

$$\ldots \longrightarrow \pi_{n+1}(X''') \longrightarrow \pi_n(X) \longrightarrow \pi_n(X') \oplus \pi_n(X'') \longrightarrow \pi_n(X''') \longrightarrow \ldots$$

If we apply this to our case we get a diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Sp(2) \otimes \mathbf{Z} \left[\tfrac{1}{2}\right] \\
\downarrow & & \downarrow \\
(S^3 \times S^7) \otimes \mathbf{Z}_{(2)} & \longrightarrow & K(\mathbf{Q}, 3) \times K(\mathbf{Q}, 7)
\end{array}
$$

and we see that X is 1-connected,

$$\pi_n(X) \longrightarrow \pi_n(SP(2) \otimes \mathbf{Z}\left[\frac{1}{2}\right]$$

localises at $\mathbf{Z}\left[\frac{1}{2}\right]$ and

$$\pi_n(X) \longrightarrow \pi_n\left(\left(S^3 \times S^7\right) \otimes \mathbf{Z}_{(2)}\right)$$

localises at $\mathbf{Z}_{(2)}$. In particular

$$\pi_6(X) = \mathbf{Z}/4\mathbf{Z}$$

Therefore X is not equivalent to $Sp(2)$ and it is not equivalent to $S^3 \times S^7$; it isn't equivalent to $Sp(2)$ at the prime 2 and it is not equivalent to $S^3 \times S^7$ at the prime 3. Also it certainly isn't equivalent to anything else in the classical list.

Of course I should still convince you that X is an H-space and that it is equivalent to a finite complex. To do the former, we need to note that localisation commutes with products. More precisely, we have the projections

$$X \times Y \longrightarrow X, \qquad X \times Y \longrightarrow Y.$$

Localising we get

$$(X \times Y)_S \longrightarrow X_S, \qquad (X \times Y)_S \longrightarrow Y_S.$$

With these components we get

$$(X \times Y)_S \longrightarrow X_S \times Y_S$$

This is an equivalence, because on homotopy it induces

$$\left(\pi_*(X) \oplus \pi_{(}Y)\right)_S \longrightarrow \pi_*(X)_S \oplus \pi_*(Y)_S.$$

The product map

$$\left(S^3 \times S^7\right)^2 \longrightarrow \left(S^3 \times S^7\right)$$

gives

$$\left(\left(S^3 \times S^7\right) \otimes \mathbf{Z}_{(2)}\right)^2 \longrightarrow \left(S^3 \times S^7\right) \otimes \mathbf{Z}_{(2)}$$

and
$$\left((S^3 \times S^7) \otimes \mathbf{Q}\right)^2 \longrightarrow \left(S^3 \times S^7\right) \otimes \mathbf{Q}.$$
Similarly, the product map
$$(Sp(2))^2 \longrightarrow Sp(2)$$
gives
$$\left(Sp(2) \otimes \mathbf{Z}\left[\tfrac{1}{2}\right]\right)^2 \longrightarrow Sp(2) \otimes \mathbf{Z}\left[\tfrac{1}{2}\right]$$
and
$$(Sp(2) \otimes \mathbf{Q})^2 \longrightarrow Sp(2) \otimes \mathbf{Q}.$$
The product maps on $Sp(2) \otimes \mathbf{Q} \simeq (S^3 \times S^7) \otimes \mathbf{Q} \simeq K(\mathbf{Q}, 3) \times K(\mathbf{Q}, 7)$ agree, since there is only one class of maps
$$(K(\mathbf{Q}, 3) \times K(\mathbf{Q}, 7))^2 \longrightarrow K(\mathbf{Q}, 3) \times K(\mathbf{Q}, 7)$$
which has the basepoint as unit.

Hence we have the following diagram

Since the right-hand square is a weak pull-back, there is a map $\mu : X^2 \to X$ making the diagram commute. Moreover, consider the map
$$X \times \mathrm{pt} \longrightarrow X \times X \overset{\mu}{\longrightarrow} X$$
The induced map of homotopy groups becomes the identity if we apply $- \otimes \mathbf{Z}\left[\tfrac{1}{2}\right]$ or $- \otimes \mathbf{Z}_{(2)}$. Therefore it is the identity. In particular this composite is an equivalence by the theorem of J. H.C. Whitehead. Similarly
$$\mathrm{pt} \times X \longrightarrow X \times X \overset{\mu}{\longrightarrow} X$$

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is an equivalence. Hence we can alter μ so that these become the identity and X becomes an H-space.

Of course if we knew that X was a strict pullback we wouldn't need that last argument, but who cares.

Finally, we need to see that X is equivalent to a finite complex. Since X is 1-connected it is sufficient to show that $\oplus_n H_n(X)$ is finitely-generated. From the fact that the maps

$$
\begin{array}{ccc}
X & \longrightarrow & Sp(2) \otimes \mathbf{Z}\left[\frac{1}{2}\right] \\
\downarrow & & \\
(S^3 \times S^7) \otimes \mathbf{Z}_{(2)} & &
\end{array}
$$

induce localisation on homotopy, we infer they induce localisation on homology. Then we obtain

$$
H_*(X) \cong H_*(Sp(2)) \cong H_*(S^3 \times S^7).
$$

Before we go on, let me make one comment on the above example. If we want to construct a fake Lie group then the original method of Hilton and Roitberg is simple and explicit, and why should we use any other? This is certainly a reasonable objection. On the other hand, if we want to state and prove that

$$
(F/PL) \otimes \mathbf{Z}\left[\frac{1}{2}\right] \simeq BO \otimes \mathbf{Z}\left[\frac{1}{2}\right]
$$

then the method of Hilton and Roitberg can't do us much good. Since I actually want to talk about localisation, I wanted to give a minimal example which would smuggle in a few of the things I wanted to smuggle in. Thus I gave a minimal example in the direction of Zabrodsky mixing.

2 Idempotent Functors

I want to study functors in homotopy theory with the same formal properties as localisation, so I'd better say what those properties are.

Suppose we are given a category C, a functor $E : C \to C$ and a natural transformation $\eta : 1 \to E$. On these I'm going to put two axioms:

Axiom 2.1. $E\eta_X = \eta_{EX} : EX \to E^2 X$

Axiom 2.2. The common value of $E\eta_X$ and η_{EX} is an equivalence from EX to $E^2 X$.

These axioms say that the functor E is idempotent (in a particular way). The categorists have already considered this axiom system, and they call (E, η) an idempotent triple or idempotent monad.

Example 2.3. $C = R$-modules, $EM = S^{-1}M$ for a fixed S, $\eta_M : M \to S^{-1}M$ is the canonical map $m \mapsto \frac{m}{1}$. Then

$$\eta_{EM} \text{ is the map } \frac{m}{s} \mapsto \frac{\frac{m}{s}}{1}$$

$$E\eta_M \text{ is the map } \frac{m}{s} \mapsto \frac{\frac{m}{1}}{s}$$

Example 2.4. C is the category in which the objects are metric spaces and the maps are uniformly continuous functions. EX is the completion $\hat{X}$ of X, constructed for example by taking equivalence classes of Cauchy sequences. Also $\eta_X : X \to \hat{X}$ is the canonical map $x \mapsto \{x, x, \ldots, x, \ldots\}$.

η_{EX} is the map $\{x_1, x_2, \ldots\} \mapsto \{\{x_1, x_2, \ldots\}, \{x_1, x_2, \ldots\}, \ldots\}$

$E\eta_X$ is the map $\{x_1, x_2, \ldots\} \mapsto \{\{x_1, x_1, \ldots\}, \{x_2, x_2, \ldots\}, \ldots\}$

Since 2.1 and 2.2 are very simple formal properties, valid in Examples 2.3 and 2.4, we may expect to see them hold in any case which presents a valid analogy with 2.3 or 2.4. Equivalently suppose that for some C, E, η either 2.1 or 2. 2 is found to fail; then that by itself would tend to discredit any

analogy with 2.3 or 2.4 to the point where we would hesitate to use the word "localisation" or "completion" for such a functor E.

Our programme is now as follows. First we must explore the consequences of our axioms. More particularly we must understand how to characterise the map $\eta_X : X \to EX$ by universal properties. For this purpose we need to introduce two constructions. First one needs to introduce a subset D of the objects of C and we must do it in such a way so that for Example 2.4, D becomes the subset of complete metric spaces (not just the spaces which arise as $\hat{X}$ for some particular choice of completion). There are two equivalent definitions

Definition 2.5.

(i) $X \in D$ if $X \simeq EY$ for some Y in C.

(ii) $X \in D$ if $\eta_X : X \to EX$ is an equivalence.

Clearly (ii) implies (i) (take $Y = X$). Also (i) implies (ii). For if $X \xrightarrow{f} EY$ is an equivalence, then Ef is an equivalence and we have

$$
\begin{array}{ccc}
EX & \xrightarrow[\simeq]{EF} & E^2Y \\
\eta_X \uparrow & & \simeq \uparrow \eta_{EY} \\
X & \xrightarrow[\simeq]{f} & EY
\end{array}
$$

so η_X is also an equivalence.

We must also define a subclass S of the morphisms of C.

Definition 2.6. The map $f : X \to Y$ lies in S if and only if $Ef : EX \to EY$ is an equivalence.

The notation shows that this subclass of morphisms in a category is to be thought of as analogous to a multiplicatively closed subset in a ring.

Example 2.7. Let C be the category in which the objects are all CW-complexes with basepoint and the morphisms are homotopy classes of maps. When we construct Postnikov systems, we may choose for each complex X a complex $EX = X(1, 2, ..., n)$ and a map $\eta_X : X \to EX$ such that

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(i) $(\eta_X)_\# : \pi_i(X) \to \pi_i(EX)$ is an isomorphism for $i \leq n$

(ii) $\pi_i(EX) = 0$ for $i > n$.

There is then a unique way to define Ef for maps $f : X \to Y$ so that E becomes a functor $C \to C$ and η becomes a natural transformation. Then Axioms 2.1 and 2.2 are satisfied. We may call this E the n-type functor. This shows that even in homotopy theory there are functors E satisfying 2.1 and 2.2 which are very different from localisation. In this example it is particularly easy to describe the subsets S and D. A map $f : X \to Y$ is in S if and only if

$$f_\# : \pi_i(X) \to \pi_i(X) \text{ is an isomorphism for } i \leq n.$$

An object X is in D if and only if

$$\pi_i(X) = 0 \text{ for } i > n$$

Our plan is now this. For those categories C which concern us, we will characterise by axioms those subclasses S which arise from pairs (E, η) by Definition 2.6. In the applications we can then proceed as follows: first we write a definition which defines a subclass S; secondly we check that this subclass S satisfies the axioms; and finally we apply the main theorem to deduce that this subset S arises from a pair (E, η) satisfying 2. 1 and 2. 2.

Before we go on, let me make a few comments on why topologists should study idempotent functors on categories which interest them. First the general method of algebraic topology has the following pattern. We wish to study a category C which is not properly known, like the homotopy category. So we apply functors like homotopy and homology which go from C to categories which are known a bit better. Now if we are ever to get to knowing C, certainly we must begin by knowing some of those full subcategories D which are a bit simpler. Once we do, it becomes fruitful to consider functors on C taking values in D and apply our standard methods.

Secondly, by studying idempotent functors on C we gain a great deal of understanding of the structure of C. This is in good analogy with the study of idempotents in a ring or idempotents acting on a module.

Thirdly, the examples suggest that this study is reasonable.

Let us now define the notion of equivalence for idempotent functors. The appropriate equivalence relation is as follows:

Definition 2.8. If (E, η), (E', η') are two idempotent functors on a category C we say that $(E, \eta) \simeq (E', \eta')$, if there is a natural equivalence $\epsilon : E \to E'$ such that $\eta' = \epsilon\eta$, i. e.

$$
\begin{array}{ccc}
 & & EX \\
 & \overset{\eta_X}{\nearrow} & \big| \epsilon_X \\
X & & \big\downarrow \\
 & \underset{\eta'_X}{\searrow} & \\
 & & E'X
\end{array}
$$

If $(E, \eta) \simeq (E', \eta')$, then $S = S'$, $D = D'$; it is trivial to check it.

The categorists know some things about idempotent monads, and I must run through some of them. The first thing you know about any monad is that the functor factors through a pair of adjoint functors. In our case it is very easy to display the factorisation. On the one hand we can regard E as taking values in the full subcategory D

$$ C \xrightarrow{E} D \qquad D \xrightarrow{I} C $$

and on the other hand we have the inclusion of D in C. I must prove that these are adjoint functors.

Lemma 2.9. *If $X \in C$, $Y \in D$ then*

$$ [X, Y] \xleftarrow{\eta_X^*} [EX, Y] $$

is an isomorphism.

Proof. (i) η_X^* is epi: Suppose $f : X \to Y$. From the diagram

$$
\begin{array}{ccc}
EX & \xrightarrow{EF} & EY \\
\eta_X \big\uparrow & & \simeq \big\uparrow \eta_Y \\
X & \xrightarrow{f} & Y
\end{array}
$$

we see that f factors through η_X.

(ii) η_X^* is mono: Take $f, g : EX \to Y$ and assume that $f \cdot \eta_X = g \cdot \eta_X$. Then $Ef \cdot E\eta_X = Eg \cdot E\eta_X$. But since $\eta_{EX} = E\eta_X$, it follows from the diagram

$$
\begin{array}{ccc}
E^2 X & \xrightarrow{\ \ Ef\ \ \atop \ \ Eg\ \ } & EY \\[2pt]
\Big\uparrow{\scriptstyle \eta_{EX}} & & \Big\uparrow{\scriptstyle \eta_Y} \\[2pt]
EX & \xrightarrow{\ \ f\ \ \atop \ \ g\ \ } & Y
\end{array}
$$

that $\eta_Y \cdot f = \eta_Y \cdot g$. Since η_Y is an equivalence, $f = g$. $\qquad\square$

The next proposition shows how S and D determine each other.

Proposition 2.10. *(i) Suppose $f : X \to Y$ is a morphism in C. Then f lies in S if and only if*

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is an isomorphism for all Z in D.
(ii) If Z is an object in C, then Z is in D if and only if

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is an isomorphism for all morphisms $f : X \to Y$ in S. Indeed it is sufficient to check that f^* is epi for all f in S.

Proof. We must prove that if $f : X \to Y$ lies in S and Z lies in D, then

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is an isomorphism. This is immediate from the following commutative diagram

$$
\begin{array}{ccc}
[Y, Z] & \xrightarrow{\ \ f^*\ \ } & [X, Z] \\[2pt]
{\scriptstyle \eta_Y^*}\Big\uparrow{\scriptstyle \simeq} & & {\scriptstyle \eta_X^*}\Big\uparrow{\scriptstyle \simeq} \\[2pt]
[EY, Z] & \xrightarrow{\ \ (Ef)^*\ \ } & [EX, Z]
\end{array}
$$

and since η_Y^* and η_X^* are isomorphisms by 2.9, while $(Ef)^*$ is an isomorphism because $f \in S$.

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(i) Conversely suppose

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is an isomorphism for all Z in D; we wish to show that $f \in S$. By the same diagram as above, we see that

$$(Ef)^* : [EY, Z] \longrightarrow [EX, Z]$$

s an isomorphism for all $Z \in D$. Now we argue in the standard fashion. Taking $Z = EX$ we see that there is a $g : EY \to EX$ such that

$$g \cdot Ef = 1_{EX}.$$

Then

$$Ef \cdot g \cdot Ef = Ef \cdot 1_{EX} = 1_{EY} \cdot Ef.$$

But taking $Z = EY$ and using the fact that $(Ef)^*$ is an isomorphism we get

$$Ef \cdot g = 1_{EY}.$$

Hence Ef is an equivalence and $f \in S$.

(ii) Suppose $f^* : [Y, Z] \to [X, Z]$ is epi for all f in S, we wish to show $Z \in D$. Take $f = \eta_Z : Z \to EZ$. This lies in S because $Ef = E\eta_Z$ is an equivalence by Axiom 2.2. Hence

$$\eta_Z^* : [EZ, Z] \to [Z, Z]$$

is epi and there is a map $\zeta : EZ \to Z$ such that

$$\zeta \cdot \eta_Z = 1_Z.$$

Then

$$\eta_Z \cdot \zeta \eta_Z = \eta_Z \cdot 1_Z = 1_Z \cdot \eta_Z$$

But

$$\eta_Z^* : [EZ, EZ] \to [Z, EZ]$$
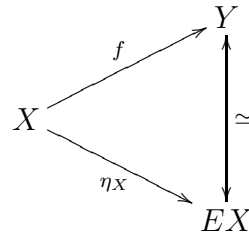
is an isomorphism by 2.9, since $EZ \in D$. Therefore

$$\eta_Z \cdot \zeta = 1_{EZ},$$

η_Z is an equivalence, and $Z \in D$. □

The next proposition shows that the pair (E, η) is determined up to equivalence by either S or D.

Proposition 2.11. *The following conditions on a map $f : X \to Y$ are equivalent.*
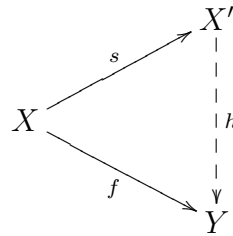
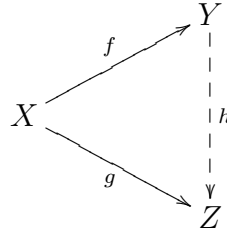(i) *There is a commutative diagram*

In other words, f is a map η_X up to equivalence.
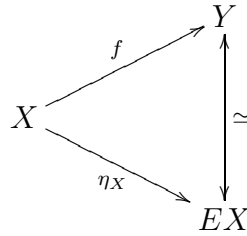
(ii) *f is in S and Y is in D.*

(iii) *f is in S and is couniversal with respect to that property. That is, given a map $s : X \to X'$ in S, there is a unique map $h : X' \to Y$ which makes the following diagram commute.*
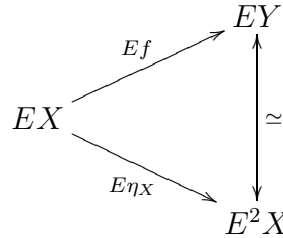

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(iv) Y *is in* D *and is universal with respect to that property. That is, if* $g : X \to Z$ *is a map with* $Z \in D$*, then there is a unique map* $h : Y \to Z$ *which makes the following diagram commute.*

$$
\begin{array}{ccc}
 & & Y \\
 & \nearrow^{f} & \vdots \\
X & & \vdots h \\
 & \searrow_{g} & \vdots \\
 & & Z
\end{array}
$$

Proof. (i) $\Longrightarrow$ (ii) Suppose we have a commutative diagram

$$
\begin{array}{ccc}
 & & Y \\
 & \nearrow^{f} & \big\updownarrow \simeq \\
X & & \\
 & \searrow_{\eta_X} & \\
 & & EX
\end{array}
$$

Applying E we obtain the commutative diagram

$$
\begin{array}{ccc}
 & & EY \\
 & \nearrow^{Ef} & \big\updownarrow \simeq \\
EX & & \\
 & \searrow_{E\eta_X} & \\
 & & E^2X
\end{array}
$$

But $E\eta_X$ is an equivalence by 2.2. Therefore Ef is an equivalence and $f \in S$. Also $Y \in D$ since $Y \simeq EX$.

(ii)$\Longrightarrow$ (iii) Suppose f is in S and Y is in D. Suppose $s : X \to X'$ is in S. Then

$$
s^* : [X', Y] \longrightarrow [X, Y]
$$

is an isomorphism by 2.10, so f has the couniversal property required.

(ii) $\Longrightarrow$ (iv) Suppose f is in S and $Y \in D$. Let $g : X \to Z$ with $Z \in D$ be given. Then

$$
f^* : [Y, Z] - + [X, Z]
$$

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is an isomorphism by 2.10, so f has the universal property required.

(iii) $\Longrightarrow$ (i) Suppose f is in S and has the couniversal property. We know η_X is in S and we have shown it has the couniversal property, so f and η_X differ by a canonical equivalence:

$$
\begin{array}{ccc}
 & & Y \\
 & \nearrow^{f} & \uparrow \\
X & & \simeq \\
 & \searrow_{\eta_X} & \downarrow \\
 & & EX
\end{array}
$$

Similarly for (iv) $\Longrightarrow$ (i). $\hfill\square$

Remark. In Example 2.7, I gave a description of the n-type functor. If you look back you will see that this is a description of the type 2.11(ii) – η_X is in S and EX is in D. In the situation we wish to study, such a description is adequate to characterise E and η.

You might think this was enough of §2, and so did I when I first wrote this lecture, but then I found there was some more material I needed here. In fact, I must first recall some of the standard material on the category of fractions.

Suppose we are given a category C and a subset S of the morphisms in C. Then there is a category $S^{-1}C$ and a functor $Q : C \to S^{-1}C$ with the following properties

(i) If $s \in S$, Qs is invertible in $S^{-1}C$;

(ii) Q is universal with respect to property (i). That is, suppose we are given a functor $T : C \to C'$ such that Ts is invertible for any $s \in S$; then there exists a unique functor $U : S^{-1}C \to C'$ such that $T = UQ$,

i. e.,

$$
\begin{array}{ccc}
 & & S^{-1}C \\
 & \nearrow^{Q} & \vdots \\
C & & \vdots\, U \\
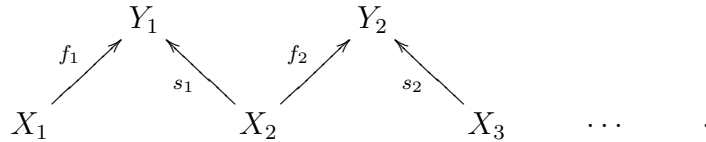 & \searrow_{T} & \vdots \\
 & & C'
\end{array}
$$
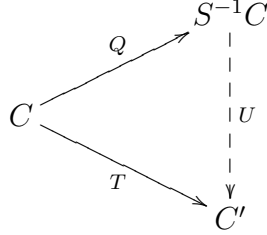
In fact, to construct these things, you take $S^{-1}C$ to have the same objects as C and you take Q to be the identity on objects. To construct the morphisms of $S^{-1}C$ you start with long words

$$\ldots (Qs_2)^{-1}(Qf_2)(Qs_1)^{-1}(Qf_1)$$

which are grammatical, i. e.,

$$
\begin{array}{ccccccc}
 & & Y_1 & & & Y_2 & \\
 & {}^{f_1}\nearrow & \nwarrow^{s_1} & & {}^{f_2}\nearrow & \nwarrow^{s_2} & \\
X_1 & & & X_2 & & & X_3 \qquad \cdots \qquad .
\end{array}
$$

You then divide these words into equivalence classes in an obvious way and check that everything works.

N. B. In practice, however, the collection of morphisms S we wish to invert is not a set but a class. In this case there are serious set-theoretic objections against the legitimacy of our constructing the category $S^{-1}C$, e. g. for general classes S and $X, Y \in C$, $\mathrm{Hom}_{S^{-1}C}(QX, QY)$ need not be a set. To overcome these objections we need a condition analogous to the solution set condition in Freyd's adjoint functor theorem.

Lemma 2.12. *Let $T_0, T_1 : S^{-1}C \to \Gamma$ be functors and let $\beta : T_0Q \to T_1Q$ be a natural transformation. Then there is one and only one natural transformation $\alpha : T_0 \to T_1$ such that $\beta_X = \alpha_{QX}$ for each X in C.*

Proof. (1) First proof. Since Q is a one-to-one correspondence on objects, the equation $\beta_X = \alpha_{QX}$ defines α; we have only to check that it is natural.

We are given that the diagram

$$
\begin{array}{ccc}
T_0QX & \xrightarrow{\;\alpha_{QX}\;} & T_1QX \\
{\scriptstyle T_0g}\big\downarrow & & \big\downarrow{\scriptstyle T_1g} \\
T_0QY & \xrightarrow{\;\alpha_{QX}\;} & T_1QY
\end{array}
$$

commutes when g is of the form Qf, f in C. Therefore it commutes when g is of the form $(Qs)^{-1}$. Therefore it commutes when g is of the form

$$
\dots (Qs_2)^{-1}(Qf_2)(Qs_1)^{-1}(Qf_1)
$$

(2) Second proof: We construct a category Γ^I as follows: An object in Γ^I is a map $f : X_0 \to X_1$ in Γ. A morphism in Γ^I from $f : X_0 \to X_1$ to $g : Y_0 \to Y_1$ is a square diagram of the following form

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\;f\;} & X_1 \\
{\scriptstyle h_0}\big\downarrow & & \big\downarrow{\scriptstyle h_1} \\
Y_0 & \xrightarrow{\;g\;} & Y_1
\end{array}
$$

There are functors $\pi_i : \Gamma^I \to \Gamma$ for $i = 0, 1$; the value of π_i on the square diagram displayed above is $h_i : X_i \to Y_i$. The natural transformation $\beta : T_0Q \to T_1Q$ may be interpreted as a functor $\beta : C \to \Gamma^I$ such that $\pi_i\beta = T_iQ$. Similarly, a natural transformation $\alpha : T_0 \to T_1$ may be interpreted as a functor $\alpha : S^{-1}C \to \Gamma^I$ such that $\pi_i\alpha = T_i$. It can be checked that our problem is now to factorise the functor β in the form $\beta = \alpha Q$. This factorisation is possible and unique by the universal property of Q. More precisely, let $s \in S$; then T_iQs is invertible in Γ. That is, βs is a square diagram whose two vertical arrows are invertible in Γ; so βs is invertible in Γ^I. Therefore β factors uniquely through Q. $\qquad\square$

Conditions are known under which we can get a much better hold on $S^{-1}C$. Among them are the following:

Axiom 2.13. The set S is closed under finite compositions.

Axiom 2.14. Given any diagram

$$
\begin{array}{ccc}
 & & X \\
 & & \big\uparrow{\scriptstyle s} \\
W & \xrightarrow{\;f\;} & Y
\end{array}
$$

with s in S, there is a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;g\;} & Z \\
\uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} \\
W & \xrightarrow{\;f\;} & Y
\end{array}
$$

with $t \in S$ and $gs = tf$.

Axiom 2.14 allows us to rewrite $(Qf)(Qs)^{-1}$ as $(Qt)^{-1}(Qg)$; with the use of Axiom 2.13 it allows us to reduce every long word

$$\ldots (Qs_2)^{-1}(Qf_2)(Qs_1)^{-1}(Qf_1)$$

to a short word

$$(Qs)^{-1}(Qf)$$

so that every morphism in $S^{-1}C$ can be represented as a short word.

Axiom 2.15. Given any diagram

$$W \xrightarrow{\;s\;} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

with $s \in S$ and $fs = gs$, there exists a diagram

$$W \xrightarrow{\;s\;} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\;t\;} Z$$
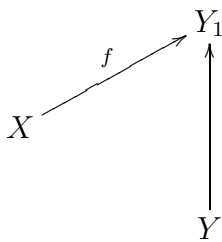
with $t \in S$ and to $tf = tg$.

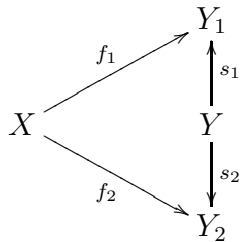Using Axiom 2.15 we can tell if two short words are equivalent. Suppose the short words are $(Qs_1)^{-1}(Qf_1)$, $(Qs_2)^{-1}(Qf_2)$ so that

Then the condition is that there exists $s_3 : Y \to Y_3$ in S and maps $g_i : Y_i \to Y_3$ for $i = 1, 2$ such that

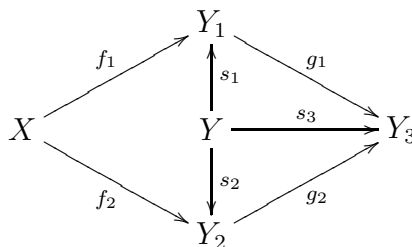$$g_1 f_1 = g_2 f_2 \text{ and } g_1 s_1 = g_3 = g_2 s_2.$$

For our purposes, it is sufficient to assume that S satisfies 2.13, 2.14 and 2.15; hence we can construct the category of fractions $S^{-1}C$ as follows: Take as objects the objects of C. For morphisms from X to Y, first take the diagrams

$$\begin{array}{ccc} & & Y_1 \\ & \overset{f}{\nearrow} & \uparrow \\ X & & \\ & & Y \end{array}$$

Define two diagrams

$$\begin{array}{ccc} & \overset{f_1}{\nearrow} & Y_1 \\ X & & \uparrow s_1 \\ & & Y \\ & \underset{f_2}{\searrow} & \downarrow s_2 \\ & & Y_2 \end{array}$$

to be equivalent if there is a diagram

$$\begin{array}{ccccc} & \overset{f_1}{\nearrow} & Y_1 & \overset{g_1}{\searrow} & \\ X & & \uparrow s_1 & \overset{s_3}{\longrightarrow} & Y_3 \\ & \underset{f_2}{\searrow} & Y & & \\ & & \downarrow s_2 & \underset{g_2}{\nearrow} & \\ & & Y_2 & & \end{array}$$

such that $g_1 f_1 = g_2 f_2$ and $g_1 s_1 = g_3 = g_2 s_2$. We check that this is an equivalence relation and take the morphisms from X to Y to be the equivalence classes. We define composition appropriately, check that we get a category. Finally we define Q in the obvious way and check that it has the stated properties. See Gabriel and Zisman [7].

3   Axiomatic Characterisation of Classes S

Suppose we are given a subclass S of the morphisms in a category C. Then we inquire whether it satisfies the following six axioms, and I intend to prove that they characterise the subsets S which arise by Definition 2.6 at least for the categories which concern us.

Axiom 3.1. The class S is closed under finite compositions.

Axiom 3.2. Given any diagram

$$
\begin{array}{c}
X \\
\big\uparrow{\scriptstyle s} \\
W \xrightarrow{\ f\ } Y
\end{array}
$$

with s in S, there is a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ g\ } & Z \\
{\scriptstyle s}\big\uparrow & & \big\uparrow{\scriptstyle t} \\
W & \xrightarrow{\ f\ } & Y
\end{array}
$$

with $t \in S$ and $gs = tf$.

Axiom 3.3. Given any diagram

$$
W \xrightarrow{\ s\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y
$$

with $s \in S$ and $fs = gs$, there exists a diagram

$$
W \xrightarrow{\ s\ } X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\ t\ } Z
$$

with $t \in S$ and to $tf = tg$.

The next axiom is a solution set condition necessary for constructing the quotient category $S^{-1}C$.

Axiom 3.4. To each object Y in C there is a set of arrows $\left\{ Y \xrightarrow{s_\alpha} Z_\alpha \right\}$ in S which are cofinal in S, i. e., given any $s : Y \to Z$, there is an arrow $s_\alpha : Y \to Z_\alpha$ and a map $f : Z \to Z_\alpha$ such that $fs = s_\alpha$
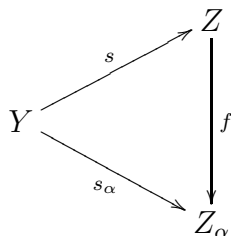
In algebra several subsets of a ring may give the same localisation. For instance if $R = \mathbf{Z}$ and

$$\begin{aligned} S_1 &= \{1, 2, 4, 8, \ldots, 2^n, \ldots\} \\ S_2 &= \{1, 4, 16, 64, \ldots, 4^n, \ldots\} \end{aligned}$$

then $S_1^{-1}R = S_2^{-1}R$. To simplify matters we usually consider the biggest such S. The following axiom imposes a condition of a similar sort on our class S.

Axiom 3.5. If f is a morphism in C such that Qf is invertible in $S^{-1}C$, then $f \in S$.

Remark. Strictly speaking, Axiom 3.5 renders Axiom 3.1 redundant, but 3.1 is so elementary it seems foolish not to state it first.

Axiom 3.5 also admits the following equivalent formulation.

Axiom 3.5*. Given maps

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

with gf and hg in S, then $g \in S$.

This version of 3.5 has the advantage that it makes perfect sense without assuming 3.4.

For the next axiom, I assume that my category C has arbitrary coproducts; since I am interested in the case of homotopy theory, I write the coproduct of factors $\{X_\alpha\}$ as $\bigvee_\alpha X_\alpha$.

Axiom 3. 6. If $s_\alpha : X_\alpha \to Y_\alpha$ lies in S for each α, then

$$\bigvee_\alpha s_\alpha : \bigvee_\alpha X_\alpha \longrightarrow \bigvee_\alpha Y_\alpha$$

lies in S.

The easier half of our work is as follows.

Proposition 3.7. *Let C be a category with arbitrary coproducts, (E, η) a pair satisfying 2.1, 2.2, and D, S as constructed in Definitions 2.5-2.6. Then S satisfies 3.1 to 3. 6 inclusive. Moreover, E factors as*

$$C \xrightarrow{Q} S^{-1}C \longrightarrow D \xrightarrow{I} C$$

where $S^{-1}C \to D$ is an equivalence.

The harder half is as follows.

Theorem 3.8. *Let C be the category in which the objects are connected CW-complexes with basepoint and the maps are homotopy classes. Let S be a subclass of the morphisms of C, satisfying 3.1 to 3.6 inclusive. Then S arises by 2.6 from a pair (E, η) satisfying 2.1 and 2.2.*

Note. I really need only one thing about C. I want to use Brown's Representability Theorem. Therefore the same result is true if we take C to be the category of 1-connected complexes or the category of spectra, etc.

In the categories which categorists usually consider one can probably get through even more easily by using standard adjoint functor theorems, but one presumably has to modify the axioms accordingly.

Proof of Proposition 3.7. To prove Axiom 3.1 suppose

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow \ldots \longrightarrow X_n \xrightarrow{f_n} X_{n+1}$$

all lie in S, i. e., Ef_1, Ef_22 ...,Ef_n are all equivalences. Then

$$E(f_nf_{n-1}\ldots f_2f_1) = (Ef_n)(Ef_{n-1})\ldots(Ef_2)(Ef_1)$$

is an equivalence, i.e , $f_nf_{n-1}\ldots f_2f_1$ lies in S.

(ii) Suppose we are given the diagram

with s in S. Then we can form the diagram

Then $(Ef)(Es)^{-1}\eta_X : X \to EY$ is a map such that

$$\left[(Ef)(Es)^{-1}\eta_X\right] s = \eta_Y f,$$

and $\eta_Y \in S$, since $E\eta_Y$ is an equivalence.

(iii) To prove Axiom 3.3, suppose we are given

with $s \in S$ and $fs = gs$. Applying E we obtain $(Ef)(Es) = (Eg)(Es)$. Since Es is an equivalence, $Ef = Eg$. But then $(Ef)\eta_X = (Eg)\eta_X$ so by the diagrams

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we obtain $\eta_Y f = \eta_Y g$. Since $\eta_Y \in S$, this proves 3.3.

(iv) Take the set $\{s_\alpha : Y \to Z_\alpha\}$ to consist of the single map $\eta_Y : Y \to EY$. Then given any $s : Y \to Z$ in S, we have the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\;\eta_Z\;} & EZ \\
{\scriptstyle s}\big\uparrow & & \simeq\big\uparrow{\scriptstyle Es} \\
Y & \xrightarrow{\;\eta_Y\;} & EY
\end{array}
$$

whence $\eta_Y = (Es)^{-1}\eta_Z s$. Since $\eta_Y \in S$, $\eta_Y : Y \to EY$ is cofinal in S and S satisfies Axiom 3.4.

(v) To prove Axiom 3.5, we first note that by definition, if $s \in S$, then Es is invertible. Hence E factors through $S^{-1}C$:

$$
\begin{array}{ccc}
C & \xrightarrow{\quad E \quad} & C \\
& {\scriptstyle Q}\searrow \quad \nearrow{\scriptstyle R} & \\
& S^{-1}C &
\end{array}
$$

Now suppose f is a morphism in C such that Qf is invertible. Then $RQf = Ef$ is invertible, so $f \in S$.

(vi) To prove Axiom 3.6 suppose that $s_\alpha : X_\alpha \to Y_\alpha$ lies in S for each α. Then

$$
s_\alpha^* : [Y_\alpha, Z] \longrightarrow [X_\alpha, Z]
$$

is an isomorphism for any $Z \in D$, by 2.10. Consider the following diagram

$$
\begin{array}{ccc}
[\bigvee_\alpha Y_\alpha, Z] & \xrightarrow{(\bigvee_\alpha s_\alpha)^*} & [\bigvee_\alpha X_\alpha, Z] \\
{\scriptstyle \simeq}\big\downarrow & & \big\downarrow{\scriptstyle \simeq} \\
\prod_\alpha [Y_\alpha, Z] & \xrightarrow[\;\simeq\;]{\prod_\alpha s_\alpha^*} & \prod_\alpha [X_\alpha, Z]
\end{array}
$$

We conclude that $(\bigvee_\alpha s_\alpha)^*$ is an isomorphism for any $Z \in D$. Therefore $\bigvee_\alpha s_\alpha$ lies in S, by 2.10.

It is clear that the functor $E : C \to C$ factors through D considered as a full subcategory of C. Since $f \in S$ implies that Ef is invertible the functor $C \to D$ factors through $S^{-1}C$. It remains to prove that $S^{-1}C \to D$ is an equivalence of categories.

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(i) Certainly every object of D is equivalent to an object EX, i.e., an object in the image of C or $S^{-1}C$.

(ii) Take a map $f : EX \to EY$ in D. Then we have a diagram

$$
\begin{array}{ccc}
E^2X & \xrightarrow{\;Ef\;} & E^2Y \\
{\scriptstyle E\eta_X=E\eta_X}\Big\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\Big\uparrow{\scriptstyle E\eta_Y=E\eta_Y} \\
EX & \xrightarrow{\;f\;} & EY
\end{array}
$$

Therefore

$$
\begin{array}{ccc}
EX & \xrightarrow{\;f\;} & EY \\
{\scriptstyle \eta_X}\Big\uparrow & & \Big\uparrow{\scriptstyle \eta_Y} \\
X & & Y
\end{array}
$$

represents an element of $S^{-1}C$ whose image in D is the given element.

(iii) Take two morphisms of $S^{-1}C$ which have the same images in D, say represented by diagrams

I claim that without loss of generality we may suppose $s_1 = s_2$. For by 3.2 we can construct the following diagram

$$
\begin{array}{ccc}
Y_1 & \longrightarrow & Y_3 \\
{\scriptstyle s_1}\Big\uparrow & {\scriptstyle s}\nearrow & \Big\uparrow{\scriptstyle s_3} \\
Y & \xrightarrow{\;s_2\;} & Y_2
\end{array}
$$

with $s_3 \in S$ so $s = s_3 s_2 \in S$. Then the two elements are equally well represented by

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So suppose $s_1 = s_2$ and revert to the notation f_1, f_2. Now the condition that the two elements have the same images in D is

$$(Es)^{-1}(Ef_1) = (Es)^{-1}(Ef_2)$$

whence

$$Ef_1 = Ef_2$$

Now we can make the argument that

$$\eta_{Y_1} f_1 = (Ef_1)\eta_X = (Ef_2)\eta_X = \eta_{Y_1} f_2$$

and $\eta_{Y_1} \in S$ whence the diagram

shows that the two given elements are equal in $S^{-1}C$. $\qquad\square$

Before we start the proof of Theorem 3.8 let's just observe one thing. When we proved Axiom 3.2, we took t to be the map η_Y independent of all the other data. But when we want to use Axiom 3.2, it only delivers a map t depending on all the other data. However, we can use Axiom 3.6 to mitigate this effect.

Lemma 3.9. *Suppose there is given a set of diagrams*

with a common Y and s_α in S. Then there exists a single map $t : Y \to Z$ in S so that we can complete all the diagrams

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Proof. Consider the diagram

$$\bigvee_\alpha X_\alpha$$
$$\bigvee_\alpha s_\alpha \uparrow$$
$$\bigvee_\alpha W_\alpha \xrightarrow{\{f_\alpha\}} Y$$

By 3.6, $\bigvee_\alpha s_\alpha$ lies in S, so by 3.2 we can fill the diagram to get

$$\begin{array}{ccc} \bigvee_\alpha X_\alpha & \xrightarrow{\;g_\alpha\;} & Z \\ \bigvee_\alpha s_\alpha \uparrow & & \uparrow t \\ \bigvee_\alpha W_\alpha & \xrightarrow{\{f_\alpha\}} & Y \end{array}$$

It is clear that t has the required properties. □

Corollary 3.10. *Suppose there is given a set of maps $\{s_\alpha : Y \to X_\alpha\}$ in S. Then there exists a single map $t : Y \to Z$ in S which factors through each s_α.*

Proof. This is a special case of 3.9 with $W_\alpha = Y$ and $f_\alpha = 1_Y$ each α. □

We operate similarly on Axiom 3.3.

Lemma 3.11. *Suppose there is given a set of diagrams*

$$\left\{ X_\alpha \overset{f_\alpha}{\underset{g_\alpha}{\rightrightarrows}} Y \right\}$$

such that $Qf_\alpha = Qg_\alpha$ in $S^{-1}C$ for each α. Then there exists a single map $t : Y \to Z$ in S such that $tf_\alpha = tg_\alpha$ for each α.

Proof. There is for each α a map $s_\alpha : Y \to Z_\alpha$ such that $sf_\alpha = sg\alpha$. By Corollary 3.10 there is a single map $t : Y \to Z$ in S which factors through each s_α. Then $tf_\alpha = tg_\alpha$ for each α. □

Lemma 3.12. *A coproduct $\bigvee_\alpha X_\alpha$ in C remains a coproduct in $S^{-1}C$.*

Proof. Let $i_\alpha : X_\alpha \to \bigvee_\alpha X_\alpha$ be the canonical injections; then $Qi_\alpha : QX_\alpha \to Q\left(\bigvee_\alpha X_\alpha\right)$ induces a map

$$(Qi_\alpha)^* : \left[Q\left(\bigvee_\alpha X_\alpha\right), Y\right]_{S^{-1}C} \longrightarrow [QX_\alpha, Y]_{S^{-1}C}$$

for any Y in $S^{-1}C$. We have to show that

$$\{(Qi_\alpha)^*\} : \left[Q\left(\bigvee_\alpha X_\alpha\right), Y\right]_{S^{-1}C} \longrightarrow \prod_\alpha [QX_\alpha, Y]_{S^{-1}C}$$

is an isomorphism.

First we prove that the map is epi. Suppose we are given diagrams in C

representing elements of $[QX_\alpha, Y]_{S^{-1}C}$. By 3.10 there is a single map $t : Y \to Z$ in S through which all the s_α factor. Then the elements in $[QX_\alpha, Y]_{S^{-1}C}$ are equally well represented by diagrams

Then

represents an element in $[Q\left(\bigvee_\alpha X_\alpha\right), Y]_{S^{-1}C}$ which restricts to the given elements.

It remains to show that the function is mono. Suppose we are given two elements in $[Q\left(\bigvee_\alpha X_\alpha\right), Y]_{S^{-1}C}$ with the same image in $\prod_\alpha [QX_\alpha, Y]_{S^{-1}C}$.

They may be represented as diagrams

$$Y_1 \qquad\qquad Y_2$$

with maps f, s_1 to Y_1 from $\bigvee_\alpha X_\alpha$ and Y; and g, s_2 to Y_2 from $\bigvee_\alpha X_\alpha$ and Y.

By the same argument as was used in the latter part of the proof of Proposition 3.7, we may assume $s_1 = s_2$. Our data now says that for each α, the components

$$X_\alpha \mathrel{\substack{f_\alpha \\ \longrightarrow \\ g_\alpha}} Y$$

satisfy $(Qs_1)^{-1}(Qf_\alpha) = (Qs_1)^{-1}(Qg_\alpha)$, i. e. $Qf_\alpha = Qg_\alpha$. Now Lemma 3.11 states that there is a single map $t : Y \to Z$ in S such that $tf_\alpha = tg_\alpha$ for each α. Then we conclude that $tf = tg$ in C, and so $Qf = Qg$. Hence

$$(Qs_1)^{-1}(Qf) = (Qs_1)^{-1}(Qg)$$

in $S^{-1}C$. This completes the proof of Lemma 3.12. □

The above results were derived for any category with arbitrary coproducts, with S satisfying 3.1-3.6. Now we restrict our attention to the category whose objects are connected CW-complexes with basepoint and whose morphisms are homotopy classes.

Lemma 3.13. *The diagram*

$$
\begin{array}{ccc}
U & \xrightarrow{\ i_1\ } & U \cup V \\
{\scriptstyle j_1}\big\uparrow & & \big\uparrow{\scriptstyle i_2} \\
U \cap V & \xrightarrow{\ j_2\ } & V
\end{array}
$$

remains a weak pushout in $S^{-1}C$.

Proof. Let Y be an object in $S^{-1}C$ and let the diagrams

$$Y_1 \qquad\qquad Y_1$$

with maps f_1, s_1 to Y_1 from U and Y; and f_2, s_2 to Y_1 from V and Y.

with s_1, s_2 in S represent morphisms in $S^{-1}C$ such that

$$(Qs_1)^{-1}(Qf_1)(Qj_1) = (Qs_2)^{-1}(Qf_2)(Qj_2)$$

Now as noted previously we may assume $s_1 = s_2$. Then $(Qf_1)(Qj_1) = (Qf_2)(Qj_2)$ Hence by 3.11 there is a $t : Y_1 \to Z$ such that $tf_1j_1 = tf_2j_2$. Since the diagram is a weak pushout in C there is a map $f : U \cup V \to Z$ such that $fi_1 = tf_1$, $fi_2 = tf_2$. Then

$$
\begin{aligned}
\left([Q(ts_1)]^{-1}Qf\right)Qi_1 &= [Q(ts_1)]^{-1}Q(fi_1) \\
&= [Q(ts_1)]^{-1}Q(tf_1) \\
&= (Qs_1)^{-1}Qf_1
\end{aligned}
$$

Similarly

$$\left([Q(ts_1)]^{-1}Qf\right)Qi_2 = (Qs_1)^{-1}Qf_2$$

$\square$

Lemma 3.14. *The functor $Q : C \to S^{-1}C$ has a right adjoint $R : S^{-1}C \to C$.*

Proof. Consider $[QX, Y]_{S^{-1}C}$ where X varies over C and Y stays fixed in $S^{-1}C$. We get a contravariant functor from C to sets. I claim it satisfies the hypotheses of E. H. Brown's Representability Theorem. In fact, I have just proved in Lemma 3.12 that it satisfies the Wedge Axiom, and in Lemma 3.13 that it satisfies the Mayer-Vietoris Axiom. We conclude that for fixed Y in $S^{-1}C$ there is an object RY in C and an isomorphism

$$[QX, Y]_{S^{-1}C} \longleftrightarrow [X, RY]_C$$

natural for maps of X.

It is now standard that there is just one way to define R on maps so that R becomes a functor and the above isomorphism becomes natural for maps of Y. This proves 3.14. $\square$

Proof of Theorem 3.8. We now define E to be the composite

$$C \xrightarrow{\ Q\ } S^{-1}C \xrightarrow{\ R\ } C$$

We define η as follows. Since there is an isomorphism

$$[QX, QX]_{S^{-1}C} \longleftrightarrow [X, RQX]_C,$$

let $\eta_X \in [X, RQX]$ be the element corresponding to 1_{QX}. It is standard that η is natural.

We have to check that (E, η) satisfies 2.1 and 2.2, and that they yield (via 2.6) the same class S we started from.

Let us use the isomorphism

$$[QRY, Y]_{S^{-1}C} \longleftrightarrow [RY, RY]_C$$

to define $\xi_Y \in [QRY, Y]_{S^{-1}C}$ as the map corresponding to 1_{RY}. Then it is standard that ξ is natural and ξ, η satisfy the following identities

(3.15) $\xi_{QX} \cdot Q\eta_X = 1_{QX}$

(3.16) $R\xi_Y \cdot \eta_{RY} = 1_{RY}$

Now $Q\eta_X : QX \to QRQX$ is a natural transformation. By Lemma 2.12 there is one and only one natural transformation

$$\zeta_Y : Y \longrightarrow QRY$$

natural for maps of Y in $S^{-1}C$ and such that

$$\zeta_{QX} = Q\eta_X.$$

Now (3.15) gives $\xi_{QX} \cdot \zeta_{QX} = 1_{QX}$. Since Q is a one-to-one correspondence on objects, this is the same as

$$\xi_Y \cdot \zeta_Y = 1_Y, \qquad Y \in S^{-1}C.$$

By the naturality of ζ we have the following commutative diagram

$$
\begin{array}{ccc}
QRY & \xrightarrow{\;\;\xi_Y\;\;} & Y \\
\downarrow{\scriptstyle \zeta_{QRY}} & & \downarrow{\scriptstyle \zeta_Y} \\
QRQRY & \xrightarrow{\;\;QR\xi_Y\;\;} & QRY
\end{array}
$$

But applying Q to (3.16) we get

$$QR\xi_Y \cdot Q\eta_{RY} = 1_{QRY}$$

whence

$$1_{QRY} = QR\xi_Y \cdot \zeta_{QRY} = \zeta_Y \cdot \xi_Y.$$

We thus conclude that ξ and ζ are mutually inverse natural equivalences. Applying R to (3.15) we get

$$R\xi_{QX} \cdot \eta_{RQX} = 1_{RQX};$$

substituting $Y = QX$ in (3.16) we get

$$R\xi_{QX} \cdot \eta_{RQX} = 1_{RQX}$$

Then $E\eta_X = RQ\eta_X$ and $\eta_{EX} = \eta_{RQX}$ are both inverses to the natural equivalence $R\xi_{QX}$ so they must be equal and natural equivalences. This proves (E, η) satisfy 2.1 and 2.2.

If $f \in S$, then Qf is invertible so $Ef = RQf$ is invertible. Conversely, suppose there is given $f : X \to Y$ in C such that $Ef = RQf$ is invertible. Then $QRQf$ is invertible. However, we have the following commutative diagram

$$
\begin{array}{ccc}
QX & \xleftarrow{\;\simeq\;} & QRX \\
{\scriptstyle Qf}\downarrow & & \simeq\downarrow{\scriptstyle QRQf} \\
QY & \xleftarrow{\;\simeq\;} & QRQY
\end{array}
$$

the horizontal arrows are given by ξ or ζ. Therefore Qf is invertible. Now Axiom 3.5 shows $f \in S$.

This proves that S does arise from (E, η), by 2.6. This completes the proof of Theorem 3,8. $\qquad\square$

4 A Further Axiom

In the applications, Axioms 3.1, 3.5 and 3.6 are very easy to verify, as we will see. However, 3.2 and 3.3 (as well as 3.4) are less convenient because they are existence statements and they leave us to construct

$$
\begin{array}{ccc}
X & \overset{g}{\dashrightarrow} & Z \\
 & & \wedge \\
 & & \vert\, t \\
 & & \vert \\
 & & Y
\end{array}
$$

for 3.2, or

$$
X \overset{t}{\dashrightarrow} Z
$$

for 3.3. It is much more convenient to be given a definite map and be told to verify that it lies in S (cf. 3. 6). We therefore introduce the following axiom. For it we assume that $U \cup V$ is a complex which is the union of subcomplexes U, V.

Axiom 4.1. If $i : U \cap V \longrightarrow U$ is in S, then $j : V \longrightarrow U \cup V$ is in S.

Remarks. This is a sort of excision axiom (cf. following page). To show $j \in S$ it is sufficient to verify that

$$
[U \cup V, Z] \overset{j^*}{\longrightarrow} [V, Z]
$$

is a bijection for all $Z \in D$. We now show j^* is automatically surjective. Hence to check that S satisfies 4.1, it suffices to verify that j^* is injective.

For if $i \in S$ and $Z \in D$ the map

$$
i^* : [U, Z] \longrightarrow [U \cap V, Z]
$$

is a bijection. Hence given $g \in [V, Z]$ there is a map $f \in [U, Z]$ such that $gi' = fi$ where $i' : U \cap V \longrightarrow U$. Since the following diagram is a weak pushout

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there is a map $h : U \cup V \longrightarrow Z$ such that $hj = g$. Hence j^* is always surjective.

Proposition 4.2. *If S satisfies 3.1, 3.4, 3.5, 3.6 and 4.1, then S satisfies 3.2 and 3.3.*

Proposition 4.3. *There exist subclasses S in C satisfying Axioms 3.1 to 3.6 inclusive but not 4.1.*

Proof of 4.2. **(i)** We wish to prove Axiom 3.2. Suppose we are given a diagram

$$
\begin{array}{c}
X \\
\uparrow {\scriptstyle s} \\
W \xrightarrow{\ f\ } Y
\end{array}
$$

with s in S. Without loss of generality we may suppose s and f are injections of complexes. For we may represent s and f as cellular maps and then take the corresponding mapping cylinders X', Y'. The diagram

$$
\begin{array}{ccc}
X & \xleftarrow{\ \simeq\ } & X' \\
& {\scriptstyle s}\searrow \quad \nearrow & \\
& W &
\end{array}
$$

and Axiom 3.5 imply the inclusion $W \hookrightarrow X'$ also is in S. (More generally, 3.5 allows us to change maps in S by equivalences without leaving S.)

We now take $Z = X \cup_W Y$. Then we get a commutative diagram

$$
\begin{array}{ccc}
X & \hookrightarrow & Z \\
\uparrow {\scriptstyle s} & & \uparrow \\
W & \xrightarrow{\ f\ } & Y
\end{array}
$$

Since $W \hookrightarrow X$ is in S, by Axiom 4.1 $Y \hookrightarrow Z$ is in S.

If you don't wish to replace s and f by inclusions, but merely to suppose they are cellular maps, you can construct Z as

$$
X \cup_s \left(\frac{I \times W}{I \times \mathrm{pt}} \right) \cup_f Y
$$

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and then of course you can divide this into two parts U and V at $\frac{1}{2} \times W$ and argue similarly. $\qquad\square$

Before we go on with the proof of 4.2, we need the following lemma. We assume S satisfies 3.1, 3.4, 3.5, 3.6, and 4.1.

Lemma 4.4. *Suppose the inclusion $A \longrightarrow X$ is in S. Then the inclusion*

$$(0 \times X) \cup \frac{I \times A}{I \times pt} \cup (1 \times X) \longrightarrow \frac{I \times X}{I \times pt}$$

is also in S.

Proof. The inclusion

$$(0 \times A) \vee (1 \times A) \longrightarrow (0 \times X) \vee (1 \times X)$$

is in S by Axiom 3.6. Let

$$
\begin{aligned}
U &= (0 \times X) \vee (1 \times X) \\
V &= \frac{I \times A}{I \times \mathrm{pt}}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
U \cap V &= (0 \times A) \vee (1 \times A) \\
U \cup V &= (0 \times X) \cup \frac{I \times A}{I \times \mathrm{pt}} \cup (1 \times X)
\end{aligned}
$$

so the inclusion $U \cap V \longrightarrow U$ is in S. By 4.1, the inclusion $V \longrightarrow U \cup V$ is in S. Now since the inclusion $A \longrightarrow X$ is in S, it follows from the diagram

and 3.5 that the inclusion $V \longrightarrow \frac{I \times X}{I \times \mathrm{pt}}$ is in S. Now in the following diagram

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two of the three maps have been shown to be in S. By 3.5 it follows that the third is also in S. □

Proof of 4. 2 (ii) We wish to prove Axiom 3.3. Suppose we are given a diagram

$$W \xrightarrow{s} X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y$$

of maps rather than homotopy classes with $f_0 s \sim f_1 s$. Again we may assume s is an inclusion of complexes. Now the homotopy $f_0 s \sim f_1 s$ provides a map $h : (0 \times X) \cup \frac{I \times W}{I \times \mathrm{pt}} \cup (1 \times X) \longrightarrow Y$ and we get the diagram

$$
\begin{array}{ccc}
\frac{I \times X}{I \times \mathrm{pt}} & \dashrightarrow & Z \\
\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle t} \\
(0 \times X) \cup \frac{I \times A}{I \times \mathrm{pt}} \cup (1 \times X) & \xrightarrow{h} & Y
\end{array}
$$

Now i is in S by 4.4. Hence by Axiom 3.2 which we have already proved, we can fill in the above diagram. Then $t f_0 \sim t f_1$. □

Proof of 4.3. We need a counterexample. Let k be a field such that the product map $k \otimes_{\mathbf{Z}} k \longrightarrow k$ is an isomorphism; this holds precisely for the prime fields $k = \mathbf{Z}_p, \mathbf{Q}$. Passing to direct sums, we see that for any vector space V over k the product map $k \otimes_{\mathbf{Z}} V \longrightarrow V$ is an isomorphism. So if Y is an Eilenberg-MacLane space of type (V, n) we have

$$H_n(Y; k) \cong V$$

as a k-module. Now I recall that maps $f : X \to Y$ are in one-to-one correspondence with k-linear maps

$$H_n(X; k) \longrightarrow H_n(Y; k)$$

under $f \mapsto f_*$. So given X, choose EX to be an Eilenberg-MacLane space of type (V, n) such that

$$H_n(EX; k) \cong H_n(X; k)$$

and let the map $\eta_X : X \to EX$ realise the isomorphism. Clearly we have an idempotent triple. A map $f : X \to Y$ is in S if and only if $f_* : H_n(EX; k) \longrightarrow H_n(Y; k)$ is an isomorphism.

Now let us U, V denote the upper and lower hemispheres of the n-sphere, i. e.,

$$
\begin{aligned}
U &= E_+^n \\
V &= E_-^n \\
U \cap V &= S^{n-1} \\
U \cup V &= S^n
\end{aligned}
$$

Then $U \cap V \longrightarrow U$ lies in S, but $V \longrightarrow U \cup V$ does not lie in S.

Example 4.5. Let K_* be a generalised homology theory satisfying the usual axioms including

$$
colim_\alpha K_*(X_\alpha) \xrightarrow{\cong} K_*(X)
$$

where X_α runs over the finite subcomplexes of X. Define S as follows: a map $f : X \to Y$ is to lie in S if and only if $f_* : K_*(X) \longrightarrow K_*(Y)$ is an isomorphism. I claim this satisfies all the axioms, except possibly 3.4. We check 3.1. Suppose

$$
X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow \ldots \longrightarrow X_n \xrightarrow{f_n} X_{n+1}
$$

are maps such that $(f_i)_* : K_*(X_i) \longrightarrow K_*(X_{i+1})$ are isomorphisms. The composite

$$
X_1 \xrightarrow{f_n f_{n-1}\ldots f_2 f_1} X_{n+1}
$$

has the same property.

We check 3.5. By construction, the functor $C \xrightarrow{K_*}$ (graded groups) has the property that $f \in S$ implies $K_*(f)$ is invertible. Therefore K_* factors through $S^{-1}C$:

$$
\begin{array}{ccc}
C & \xrightarrow{\quad K_* \quad} & \text{graded groups} \\
& Q \searrow \quad \nearrow T & \\
& S^{-1}C &
\end{array}
$$

Now suppose $f : X \to Y$ in C is such that Qf is invertible. Then TQf is invertible, i. e., $K_*(f)$ is invertible so $f \in S$.

We check 3.6. Suppose $f_\alpha : X_\alpha \to Y_\alpha$ is in S for each α, i. e. $(f_\alpha)_* : K_*(X_\alpha) \to K_*(Y_\alpha)$ is an isomorphism for all α. Then we obtain the diagram

$$
\begin{array}{ccc}
\widetilde{K}_* \left(\bigvee_\alpha X_\alpha \right) & \xrightarrow{\ \left(\bigvee_\alpha f_\alpha \right)_* \ } & \widetilde{K}_* \left(\bigvee_\alpha Y_\alpha \right) \\
\Big\uparrow{\cong} & & \Big\uparrow{\cong} \\
\bigoplus_\alpha \widetilde{K}_* (X_\alpha) & \xrightarrow[\cong]{\ \oplus_\alpha (f_\alpha)_* \ } & \bigoplus_\alpha \widetilde{K}_* (Y_\alpha)
\end{array}
$$

whence $\left(\bigvee_\alpha f_\alpha \right)_*$ is an isomorphism and $\bigvee_\alpha f_\alpha$ lies in S.

We check 4.1. Suppose $i : U \cap V \longrightarrow U$ lies in S, i. e.,

$$ i_* : K_*(U \cap V) \longrightarrow K_*(U) $$

is an isomorphism. By exactness this is the same as saying $K_*(U, U \cap V) = 0$. By excision this is the same as saying $K_*(U \cup V, V) = 0$. Then by exactness again

$$ j_* : K_*(V) \longrightarrow K_*(U \cup V) $$

is an isomorphism, so $j \in S$. $\qquad\square$

Remark. Unfortunately there seems to be no way in the above example to verify that S satisfies the set-theoretic condition expressed in Axiom 3.4. Until we get around to verifying 3.4 under suitable extra assumptions or proving the existence of the desired idempotent functors (E, η) in some other way, the following two conjectures remain pious hopes or indications of what we wish to prove.

Editorial Note. The pessimism expressed in the above remark proved to be unwarranted. It turns out that there is a simple alternative to Axiom 3.4 which allows us to prove the the existence of idempotent functors (E, η) for these classes S. See the epilogue for details. The editor has thus taken the liberty of upgrading the following two statements in the original manuscript from conjectures to theorem/corollary and made a few other revisions in some subsequent statements referring to the conjectural status of these results.

Theorem 4.6. *Suppose there is given a generalised homology theory K_*. Then for each $X \in C$, there exists an object EX and a map $\eta_X : X \to EX$ such that*

(i) $(\eta_X)_* : K_*(X) \longrightarrow K_*(EX)$ *is an isomorphism,*

(ii) η_X *is couniversal with respect to (i).*

This follows from the slight revision of the main theorem 3.8, discussed in the epilogue, and from 2.11. In this case the category of fractions $S^{-1}C$ gives you as much of homotopy theory, as you can see through the eyes of K_*-theory. Up to equivalence, it would be embedded in C as the full subcategory D.

Corollary 4.7. *Take $K_*(-)$ to be $H_*(-;A)$ where A is a subring of $\mathbf{Q}$, i. e., A is obtained by localising $\mathbf{Z}$. In this way we get a localisation functor defined without restrictions on $\pi_*(X)$.*

For the remainder of this section, let (E, η) and S be as in Corollary 4.7, i. e., $f : X \to Y$ is in S if and only if $f_* : H_*(X;A) \longrightarrow H_*(Y;A)$ is an isomorphism. Let D be the corresponding subclass of C, i. e., $Z \in D$ if and only if

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is an isomorphism for all $f \in S$.

Lemma 4.8. *Let $Z \in D$ and let Γ be a subgroup of $\pi_1(Z)$ such that $(\Gamma/[\Gamma, \Gamma]) \otimes A = 0$. Then $\Gamma = 0$.*

Proof. Let $\{\gamma_\alpha\}$ be a set of generators for Γ. Since $[\gamma_\alpha] \otimes 1$ is zero in $(\Gamma/[\Gamma, \Gamma]) \otimes A$, there exists an integer n which is invertible in A such that

$$\gamma_\alpha^n = 1 \text{ in } \Gamma/[\Gamma, \Gamma]$$

Then γ_α^n is a product of commutators, say

$$\gamma_\alpha^n = \prod_\beta [\delta_{\alpha\beta}, \epsilon_{\alpha\beta}], \ \delta_{\alpha\beta}, \epsilon_{\alpha\beta} \in \Gamma$$

Define a map $\bigvee_\alpha S^1 \longrightarrow Z$ such that

$$\iota_\alpha \mapsto \gamma_\alpha,$$

where ι_α is a generator of the α-th free summand of $\pi_1 \left(\bigvee_\alpha S^1\right)$. Then

$$\pi_1 \left(\bigvee_\alpha S^1\right) \longrightarrow \Gamma \subseteq \pi_1(Z)$$

is epi. Choose elements, $\overline{\delta}_{\alpha\beta}$, $\overline{\epsilon}_{\alpha\beta}$ in $\pi_1 \left(\bigvee_\alpha S^1\right)$ mapping onto $\delta_{\alpha\beta}$, $\epsilon_{\alpha\beta}$. Form a space X by attaching to $\bigvee_\alpha S^1$ a 2-cell e_α^2, one for each α, by a map in the class

$$\iota_\alpha^{-n} \prod_\beta [\overline{\delta}_{\alpha\beta}, \overline{\epsilon}_{\alpha\beta}].$$

Then the map $\bigvee_\alpha S^1 \longrightarrow Z$ extends to give a map $X \to Z$ and the attaching maps induce isomorphisms on $H_*(-;A)$. Computing with the cellular chain complex of X we obtain $\widetilde{H}_*(X;A) = 0$, and so the constant map $X \to \mathrm{pt}$ lies in S. Since $Z \in D$, the map $X \to Z$ factors to give

$$\begin{array}{ccc} X & \longrightarrow & Z \\ & \searrow \quad \nearrow & \\ & \mathrm{pt} & \end{array}$$

Therefore the map $\pi_1(X) \to \pi_1(Z)$ is zero. But it mapped onto Γ, so $\Gamma = 0$. $\qquad\square$

Corollary 4.9. *If $\pi_1(X) = 0$, then $\pi_1(EX) = 0$.*

Proof. If $\pi_1(X) = 0$, then $H_1(X;A) = 0$; therefore $H_1(EX;A) = 0$ since $\eta_X \in S$. Now apply Lemma 4.8 with $Z = EX$, $\Gamma = \pi_1(Z)$. $\qquad\square$

Next we show that the homotopy groups of EX are always local.

Proposition 4.10. *If $Z \in D$ and n is invertible in A, then division by n is possible and unique in each homotopy group $\pi_i(Z)$ (including $i = 1$).*

Proof. Let $n : S^i \to S^i$ be the map which induces multiplication by n on the homotopy group $\pi_i(S^i)$. Then $n_* : \widetilde{H}_*(S^i;A) \longrightarrow \widetilde{H}_*(S^i;A)$ also is

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multiplication by n and hence is an isomorphism, so $n \in S$. Hence

$$\pi_i(Z) = [S^i, Z] \longrightarrow [S^i, Z] = \pi_i(Z),$$

which is again multiplication by n, is an isomorphism.

Proposition 4.11. *Suppose $\pi_1(Y) = 0$. Then there is a map $s : Y \to Y^{(\infty)}$ in S with $\pi_1\left(Y^{(\infty)}\right) = 0$ and with all $\pi_i\left(Y^{(\infty)}\right)$ being A-modules.*

Proof. We define inductively a sequence of spaces $Y^{(i)}$ and maps $s_i : Y^{(i)} \longrightarrow Y^{(i+1)}$ in S such that $\pi_j\left(Y^{(i)}\right)$ is an A-module and $(s_i)_\# : \pi_j\left(Y^{(i)}\right) \longrightarrow \pi_j\left(Y^{(i+1)}\right)$ is an isomorphism for $j \leq i$. We choose $Y^{(1)} = Y$.

Having chosen $Y^{(i)}$, let W be a Moore space of type $\pi_{i+1}\left(Y^{(i)}\right)$ in dimension $i + 1$, and let $f : W \longrightarrow Y^{(i)}$ be a map inducing an isomorphism $\pi_{i+1}(W) \overset{\cong}{\longrightarrow} \pi_{i+1}\left(Y^{(i)}\right)$. Now embed W in a Moore space X of type $\pi_{i+1}\left(Y^{(i)}\right) \otimes A$ in dimension $i + 1$, so that the map $e : W \to X$ is in S. Then we have a commutative diagram

with $e, j \in S$. If we let $Y^{(i+1)} = X \cup \frac{I \times W}{I \times \mathrm{pt}} \cup Y^{(i)}$ and $s_i = j$, we see that $s_i : Y^{(i)} \longrightarrow Y^{(i+1)}$ and $Y^{(i+1)}$ have all the required properties.

Having defined the $Y^{(i)}$'s, define $Y^{(\infty)}$ to be the colimit of the $Y^{(i)}$'s. Clearly the induced map $s : Y = Y^{(1)} \to Y^{(\infty)}$ is in S, $\pi_1\left(Y^{(\infty)}\right) = 0$ and $\pi_i\left(Y^{(\infty)}\right)$ is an A-module for all i. $\qquad \square$

We note that the construction of Proposition 4.11 is precisely Sullivan's cellular construction of his localisation functor for simply connected spaces. We now show that it is equivalent to our localisation functor.

Proposition 4.12. *If $\pi_1(Z)$ acts trivially on $\pi_i(Z)$ (which implies $\pi_1(Z)$ is abelian) and each $\pi_i(Z)$ is an A-module, then $Z \in D$.*

Proof. By 2.10 it suffices to show that whenever $f : X \to Y$ is in S, then

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is epi. We may assume that f is an inclusion.

Now let $g : X \to Z$ be a map. Since f is in $H_*(Y, X; A) = 0$. Hence $H^{i+1}(Y, X; \pi_i(Z)) = 0)$. Therefore by obstruction theory, g extends to a map $\tilde{g} : Y \to Z$, i. e., $\tilde{g}f = g$. Hence f^* is epi and $Z \in D$. $\qquad\square$

Proposition 4.13. *For simply connected spaces Y the localisation $\eta_Y : Y \to EY$ of Corollary 4.7 is equivalent to the construction of Proposition 4.11 (i. e. Sullivan's localisation). Moreover $(\eta_Y)_\# : \pi_i(Y) \to \pi_i(EY)$ is a localising map for all i.*

Proof. By 4.11 $s : Y \to Y^{(\infty)}$ is in s, $\pi_1\left(Y^{(\infty)}\right) = 0$ and $\pi_i\left(Y^{(\infty)}\right)$ are A-modules for all i. By Proposition 4.12 it follows that $Y^{(\infty)} \in D$. By Proposition 2.11 $EY \simeq Y^{(\infty)}$ and we have a commutative diagram

$$
\begin{array}{ccc}
 & & Y^{(\infty)} \\
 & \nearrow^{s} & \uparrow^{\simeq} \\
Y & & \\
 & \searrow_{\eta_Y} & \\
 & & EY
\end{array}
$$

Thus EY is simply connected and the homotopy groups of EY are A-local.

By definition of S

$$(\eta_Y)_* : H_*(Y; A) \longrightarrow H_*(EY; A)$$

is an isomorphism. But since Y and EY are simply connected, then by Serre's C-theory this is equivalent to saying that

$$(\eta_Y)_\# \otimes 1 : \pi_i(Y) \otimes A \longrightarrow \pi_i(EY) \otimes A$$

is an isomorphism for each i (Spanier [18, p. 512, Thm. 22]).

Now consider the following diagram

$$
\begin{array}{ccc}
\pi_i(Y) & \xrightarrow{\ (\eta_Y)_\#\ } & \pi_i(EY) \\
\downarrow & & \downarrow^{\cong} \\
\pi_i(Y) \otimes A & \xrightarrow[\cong]{(\eta_Y)_\# \otimes 1} & \pi_i(EY) \otimes A
\end{array}
$$

We have just shown the bottom arrow is an isomorphism. The right-hand arrow is an isomorphism since the homotopy groups of EY are A-modules. The left-hand arrow is a localising map. Therefore the top arrow is also a localising map. $\square$

5 Behaviour of Idempotent Functors with Respect to Fiberings; Construction of Localisation Using Postnikov Decomposition

Assume C is the homotopy category of connected CW-complexes and S is class of morphisms which satisfies 3.1, 3.5, 3.6 and 4.1 but not necessarily 3.4. We define D to be the class of all spaces Z such, that whenever $f : X \to Y$ is in S then

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is an isomorphism.

If we knew that S defined an idempotent functor (E, η), then by 2.10 in order to show that $Z \in D$ it would suffice to check that f^* is epi for all $f \in S$. Since we do not require S to satisfy 3.4, we need the following lemma.

Lemma 5.1. *A space Z is in D if and only if*

$$f^* : [Y, Z] \longrightarrow [X, Z]$$

is epi for all $f \in S$.

Proof. First note that if $i : X \to Y$ is an inclusion and if $i^* : [Y, Z] \longrightarrow [X, Z]$ is epi, then each map $g : X \to Z$ can be extended over Y. For if this is the case, some homotopic map g' extends to $h' : Y \to Z$ and we can use the homotopy extension property to extend g.

Now assume $f^* : [Y, Z] \longrightarrow [X, Z]$ is epi for all $f \in S$. Assume $f : X \to Y$ is in S. We wish to show $f^* : [Y, Z] \longrightarrow [X, Z]$ is an isomorphism. Without loss of generality we may suppose f is an inclusion.

We already know that f^* is epi, so we only have to check that it is also mono. Suppose we are given $g_0, g_1 : Y \to Z$ and a homotopy $g_0|X \simeq g_1|X$. This defines a map $h : 0 \times Y \cup \frac{I \times X}{I \times \mathrm{pt}} \cup 1 \times Y \longrightarrow Z$. Now by Lemma 4.4, the inclusion

$$0 \times Y \cup \frac{I \times X}{I \times \mathrm{pt}} \cup 1 \times Y \longrightarrow \frac{I \times Y}{I \times \mathrm{pt}}$$

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lies in S. So by our assumption h extends to a map H over $\frac{I \times Y}{I \times \text{pt}}$

$$
\begin{array}{c}
0 \times Y \cup \frac{I \times X}{I \times \text{pt}} \cup 1 \times Y \\[2mm]
\downarrow \quad\quad\quad \searrow^{h} \\[2mm]
\quad\quad\quad\quad\quad\quad\quad Z \\[2mm]
\quad\quad\quad {}^{H}\,\nearrow \\[2mm]
\frac{I \times Y}{I \times \text{pt}}
\end{array}
$$

which gives a homotopy $g_0 \simeq g_1$. Hence f^* is mono and therefore an isomorphism. $\qquad\square$

Proposition 5.2. *Suppose we are given a diagram*

$$
\begin{array}{ccc}
 & & V \\
 & & \downarrow{\scriptstyle q} \\
U & \xrightarrow{\ p\ } & W
\end{array}
$$

and form the homotopy pullback

$$
\begin{array}{ccc}
F & \longrightarrow & V \\
\downarrow & & \downarrow{\scriptstyle q} \\
U & \xrightarrow{\ p\ } & W
\end{array}
$$

If U, V, and W are in D, then F is (weakly) in D.

Note. By the standard construction, a point in F is a triple (u, v, ω) where $u \in U$, $v \in V$, and $\omega : I \to W$ is a path from $\omega(0) = p(u)$ to $\omega(1) = q(v)$. So on the face of it, F is not a CW-complex; we should simply assert that any weakly equivalent CW-complex is in D.

Proof of 5.2. We want to prove $f^* : [Y, F] \longrightarrow [X, F]$ is epi for all $f : X \to Y$ in S. Without loss of generality we can suppose f is an inclusion.

Suppose we are given a map $X \to F$, that is, a pair of maps

$$
g' : X \longrightarrow U, \quad g'' : X \longrightarrow V
$$

and a specific homotopy

$$h : \frac{I \times X}{I \times \text{pt}} \longrightarrow W$$

between pg' and qg''. Since $X \hookrightarrow Y$ is an inclusion in S and U, V are in D, we can extend g', g'' to $\tilde{g}'$, $\tilde{g}''$ over Y. Now we have a map

$$\tilde{h} : 0 \times Y \cup \frac{I \times X}{I \times \text{pt}} \cup 1 \times Y \longrightarrow W$$

defined by $p\tilde{g}'$, $q\tilde{g}''$ and h. By 4.4, the inclusion

$$0 \times Y \cup \frac{I \times X}{I \times \text{pt}} \cup 1 \times Y \hookrightarrow \frac{I \times Y}{I \times \text{pt}}$$

lies in S. Since W is in D, we can extend h to a map H over $\frac{I \times Y}{I \times \text{pt}}$. Now the maps $\tilde{g}'$, $\tilde{g}''$ and H give a map $Y \to F$ extending the given map $X \to F$. This proves the proposition. $\qquad\square$

Remark. The study of homotopy pullbacks includes that of fiberings as the special case:

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & B \end{array}$$

The more general study is desirable, e. g., for the study of Postnikov decompositions in the non-simply connected case.

Now we introduce another axiom.

Axiom 5.3. Suppose we are given a strictly commutative diagram of maps

and form the homotopy pullback. If u, v, w are in S, then f is in S.

Corollary 5.4. *If E exists and S satisfies 4.1 and 5.3, then E preserves homotopy pullbacks. More precisely, if EU, EV and EW exist, then EF exists and is their homotopy pullback.*

Proof. Take a diagram

If you wish we can suppose the diagram is strictly commutative (replace EU, EV by mapping cylinders). Complete the diagram of homotopy pullbacks. Then f is in S by Axiom 5.3 and F' is in D by Proposition 5.2, so up to equivalence f is the map η_F. $\qquad\qquad\square$

Proposition 5.5. *The localisation functor of Corollary 4.7 preserves homotopy pullbacks with 1-connected base W. More precisely, if $\pi_1(W) = 0$,*

then

$$EF \longrightarrow EV$$
$$\downarrow \qquad\qquad \downarrow$$
$$EU \longrightarrow EW$$

is a homotopy pullback.

Proof. If W is 1-connected, then EW is 1-connected by 4.9. So in order to apply the argument of Corollary 5.4, it is sufficient to check Axiom 5.3 with W and W' 1-connected.

Consider first the diagram

$$
\begin{array}{ccc}
G & \overset{g}{\dashrightarrow} & G' \\
\downarrow & & \downarrow \\
V & \overset{v}{\longrightarrow} & V' \\
q\downarrow & & \downarrow q' \\
W & \overset{w}{\longrightarrow} & W'
\end{array}
$$

Let us supply homotopy fibres G, G'. In the Serre spectral sequence the operations of $\pi_1(W)$, $\pi_1(W')$ on $H_*(G;A)$, $H_*(G';A)$ are trivial, because $\pi_1(W) = 0$, $\pi_1(W') = 0$. We are given

$$v_* : H_*(V;A) \longrightarrow H_(V';A), \quad w_* : H_*(W;A) \longrightarrow H_(W';A)$$

are isomorphisms. Therefore the comparison theorem for spectral sequences allows one to prove that

$$g_* : H_*(G;A) \longrightarrow H_*(G';A)$$

is an isomorphism. But now consider

$$
\begin{array}{ccc}
G & \overset{g}{\dashrightarrow} & G' \\
\downarrow & & \downarrow \\
F & \overset{f}{\longrightarrow} & F' \\
\downarrow & & \downarrow \\
U & \overset{u}{\longrightarrow} & U'
\end{array}
$$

Let us supply the homotopy fibres; they are G, G' up to equivalence carrying the map we want into g. In the Serre spectral sequence the actions of $\pi_1(U)$, $\pi_1(U')$ on $H_*(G; A)$, $H_*(G'; A)$ are trivial, since they factor through $\pi_1(W)$, $\pi_1(W')$. Now the Serre spectral sequence shows that

$$f_* : H*(F; A) \longrightarrow H_*(F'; A)$$

is an isomorphism. This concludes the proof. $\qquad\qquad\qquad\qquad\square$

Proposition 5.5. *Let π be an abelian group. Then the localisation of $K(\pi, n)$ is $K(\pi \otimes A, n)$. More precisely, the following diagram is commutative*

$$
\begin{array}{ccc}
\tilde{H}_*\left(K(\pi, n)\right) & \longrightarrow & \tilde{H}_*\left(K(\pi \otimes A, n)\right) \\
\downarrow & & \downarrow{\scriptstyle\cong} \\
\tilde{H}_*\left(K(\pi, n); A\right) & \xrightarrow{\ \cong\ } & \tilde{H}_*\left(K(\pi \otimes A, n); A\right)
\end{array}
$$

Proof. First we tackle the case $n = 1$. Suppose $\pi = \mathbf{Z}$. Then we get 0 except in dimension 1, and there

$$
\begin{array}{ccc}
\mathbf{Z} & \longrightarrow & A \\
\downarrow & & \downarrow \\
A & \longrightarrow & A \otimes_{\mathbf{Z}} A
\end{array}
$$

with the obvious maps.

Suppose $\pi = \mathbf{Z}_{p^k}$, p invertible in A. Then we have 0 in even dimensions, while in odd ones

$$
\begin{array}{ccc}
\mathbf{Z}_{p^k} & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
$$

Suppose $\pi = \mathbf{Z}_{p^k}$, p not invertible in A. Then we have 0 in even dimensions, while in odd ones

$$
\begin{array}{ccc}
\mathbf{Z}_{p^k} & \longrightarrow & \mathbf{Z}_{p^k} \\
\downarrow & & \downarrow \\
\mathbf{Z}_{p^k} & \longrightarrow & \mathbf{Z}_{p^k}
\end{array}
$$

all the maps being isomorphisms.

Suppose $\pi \cong \pi' \oplus \pi''$ and the result is true for π', π''. The map

$$K(\pi', 1) \times K(\pi'', 1) \longrightarrow K(\pi' \otimes A, 1) \times K(\pi'' \otimes A, 1)$$

induces an isomorphism of $\tilde{H}_*(-; A)$ by the Künneth theorem. Thus we have

$$
\begin{array}{ccc}
\tilde{H}_* \left(K(\pi', 1) \times K(\pi'', 1) \right) & \longrightarrow & \tilde{H}_* \left(K(\pi' \otimes A, 1) \times K(\pi'' \otimes A, 1) \right) \\
\downarrow & & \downarrow \\
\tilde{H}_* \left(K(\pi', 1) \times K(\pi'', 1); A \right) & \overset{\cong}{\longrightarrow} & \tilde{H}_* \left(K(\pi' \otimes A, 1) \times K(\pi'' \otimes A, 1); A \right)
\end{array}
$$

For the right-hand vertical arrow, suppose m is invertible in A. Then it acts by an isomorphism on $\tilde{H}_i \left(K(\pi' \otimes A, 1) \right)$ and $\tilde{H}_J \left(K(\pi'' \otimes A, 1) \right)$ and therefore on the tensor and torsion products of these; so by the Künneth formula $\tilde{H}_* \left(K(\pi', 1) \times K(\pi'', 1); A \right)$ is a module over A, and the right-hand vertical arrow is an isomorphism. So the result is true for $\pi \cong \pi' \oplus \pi''$.

At this point we have proved the result where π is finitely-generated and $n = 1$ by the structure theorem for finitely-generated abelian groups.

The result for general π and $n = 1$ follows by passing to colimits.

Now we prove the general result by induction over n. Assume it for n. Regard $K(\pi, n)$ as the space of loops on $K(\pi, n+1)$ and similarly with π replaced by $\pi \otimes A$. If $K(\pi, n) \longrightarrow K(\pi \otimes A, n)$ induces an isomorphism of $\tilde{H}_*(-; A)$, then so does $K(\pi, n+1) \longrightarrow K(\pi \otimes A, n+1)$ by the Rothenberg-Steenrod spectral sequence. Similarly if $\tilde{H}_* \left(K(\pi \otimes A, n) \right)$ is a module over A, then so is $\tilde{H}_* \left(K(\pi \otimes A, n+1) \right)$ by the Rothenberg-Steenrod spectral sequence. $\square$

Theorem 5.7. *Let (E, η) be the localisation of Corollary 4.7. Let X be a space such that $\pi_1(X)$ is nilpotent and acts trivially on the higher homotopy groups. Then*

(i) *the homology groups $\tilde{H}_*(EX)$ are A-modules (then of course $(\eta_X)_* : \tilde{H}_*(X) \longrightarrow \tilde{H}_*(EX)$ localises).*

(ii) *$(\eta_X)_\# : \pi_i(X) \longrightarrow \pi_i(EX)$ is a localising map. If $i = 1$ this means that the induced map on the subquotients of the lower central series is a localising map.)*

Proof. First suppose that X has all but a finite number of its homotopy groups zero. Then X can be built up by a finite sequence of fiberings with 1-connected base which is of type $K(\pi, n+1)$

$$
\begin{array}{ccc}
X_n & \xdashrightarrow{\ \ \eta_{X_n}\ \ } & F = EX_n \\
\downarrow & & \vdots \\
X_{n-1} & \xrightarrow{\ \ \eta_{X_{n-1}}\ \ } & EX_{n-1} \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\
B = K(\pi, n+1) & \longrightarrow & K(\pi \otimes A, n+1)
\end{array}
$$

Now we assume as an induction hypothesis that (i) and (ii) hold for $\eta_{X_{n-1}} : X_{n-1} \longrightarrow EX_{n-1}$. By 5.5, $EB = K(\pi \otimes A, n+1)$ and $p' = Ep$. By Corollary 5.4, EX_n is (up to equivalence) the homotopy fibre F of p'. Now the homology groups $\tilde{H}_*(EX_n)$ are A-modules by a spectral sequence argument, and $(\eta_{X_n})_\# : \pi_i(X_n) \longrightarrow \pi_i(EX_n)$ is a localising map.

It remains to pass to the general case.

Suppose we are given a diagram

$$
Z_1 \xleftarrow{\ p_1\ } Z_2 \xleftarrow{\ p_2\ } Z_3 \xleftarrow{\ p_3\ } Z_4 \xleftarrow{\ p_4\ } \cdots
$$

I define the homotopy inverse-limit Z_∞ as follows: an element of Z_∞ is a sequence of functions $\omega_n : [n, n+1] \longrightarrow Z_n$ such that

$$
\omega_n(n+1) = p_n \omega_{n+1}(n).
$$

To conclude the proof, I first prove:

Lemma 5.8. *If all the objects Z_n are in D, then Z_∞ is in D.*

Proof. Suppose given $f : X \to Y$ in S. Without loss of generality we can assume f is an inclusion. Suppose we are given a map $g : X \to Z_\infty$; equivalently, it is a sequence of maps

$$
g_n : \frac{[n, n+1] \times X}{[n, n+1] \times \text{pt}} \longrightarrow Z_n.
$$

Since $X \hookrightarrow Y$ is in S, I can extend g_n from $n \times X$ to $n \times Y$. Of course, the extension of g_{n+1} forces the extension $p_n g_{n+1}$ of g_n on $(n+1) \times Y$. Now since

the inclusion

$$n \times Y \cup \frac{[n, n+1] \times X}{[n, n+1] \times \mathrm{pt}} \cup (n+1) \times Y \longrightarrow \frac{[n, n+1] \times Y}{[n, n+1] \times \mathrm{pt}}$$

is in S by 4.1, we can extend g_n over $\frac{[n,n+1] \times Y}{[n,n+1] \times \mathrm{pt}}$. This extends g to a map $Y \to Z$. This concludes the proof of the lemma. $\qquad\square$

Conclusion of the Proof of 5.7 To apply the lemma, consider the diagram

where the left-hand column is the Postnikov system of X. Take the homotopy inverse limit of the right-hand side and call it Z_∞. Since the diagram is at least homotopy commutative we have a map $X \to Z_\infty$ (not necessarily unique at this stage).

Now Z_∞ lies in D by 5.8. Since the $\pi_i(EX)$ are at least groups, it makes sense to observe that there is an exact sequence

$$0 \longrightarrow \varprojlim_n {}^1 \pi_{i+1}(EX_n) \longrightarrow \pi_i(Z\infty) \longrightarrow \varprojlim_n \pi_i(EX_n) \longrightarrow 0$$

But by construction $\pi_i(EX_n)$ is constant for $n \gg i$; so we get $\pi_i(Z_\infty) \xrightarrow{\cong} \pi_i(EX_n)$ for $n \gg i$. The homotopy diagram shows $\pi_i(X) \longrightarrow \pi_i(Z_\infty)$ localises. Also

for $n \gg i$ we get a diagram

$$
\begin{array}{ccc}
H_i(X; A) & \longrightarrow & H_i(Z_\infty; A) \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
H_i(X_n; A) & \xrightarrow{\ \cong\ } & H_i(EX_n; A)
\end{array}
$$

Hence $X \to Z_\infty$ lies in S. Also $\tilde{H}_i(Z_\infty) \xrightarrow{\cong} \tilde{H}_i(EX)$ for $n \gg i$ so $\tilde{H}_*(Z_\infty)$ is an A-module. This concludes the proof of 5.7. $\qquad\square$

Corollary 5.9. (After Serre). *Let X and Y be spaces such that $\pi_1(X)$, $\pi_1(Y)$ are nilpotent and act trivially on $\pi_n(X)$, $\pi_n(Y)$ for $n > 1$. Then $f \in S$, i. e.,*

$$
f_* : H_*(X; A) \longrightarrow H_*(Y; A)
$$

is an isomorphism, if and only if

$$
f_\# : \pi_i(X) \otimes A \longrightarrow \pi_i(Y) \otimes A
$$

is an isomorphism. (For $i = 1$ this means that the subquotients of the central series map isomorphically after tensoring with A.)

First proof. (i) Suppose $f_\# : \pi_i(X) \otimes A \longrightarrow \pi_i(Y) \otimes A$ is an isomorphism for all i. Then 5.5 shows that for all i

$$
(*) \qquad\qquad \tilde{H}_* \left(K(\pi_i(X), i); A \right) \longrightarrow \tilde{H}_* \left(K(\pi_i(U), i); A \right)
$$

is an isomorphism. Now we work up the Postnikov system by induction. For if we assume

$$
H_* \left(X(1, \ldots, n - 1); A \right) \longrightarrow H_* \left(Y(1, \ldots, n - 1); A \right)
$$

is an isomorphism, then applying $(*)$ with $i = n$ and the Serre spectral sequence, we conclude that

$$
H_* \left(X(1, \ldots, n); A \right) \longrightarrow H_* \left(Y(1, \ldots, n); A \right)
$$

is an isomorphism. Of course, to deal with π_1 we take as many steps as there are subquotients in the central series. By taking n sufficiently large we see that

$$
H_m(X; A) \longrightarrow H_m(Y; A)
$$

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is an isomorphism.

(ii) Conversely, suppose $H_*(X;A) \longrightarrow H_*(Y;A)$ is an isomorphism. Suppose we have proved that

$$\pi_i(X) \otimes A \longrightarrow \pi_i(Y) \otimes A$$

is an isomorphism for $i < n$. Then as in part (i),

$$H_* \left(X(1, \ldots, n-1); A \right) \longrightarrow H_* \left(Y(1, \ldots, n-1); A \right)$$

is an isomorphism. By the five-lemma

$$H_* \left(X(1, \ldots, n-1), X; A \right) \longrightarrow H_* \left(Y(1, \ldots, n-1), Y; A \right)$$

is an isomorphism. But since π_1 operates trivially on π_n we have

$$
\begin{array}{ccc}
H_{n+1}\left(X(1,\ldots,n-1),X;A\right) & \xrightarrow{\ \cong\ } & H_{n+1}\left(Y(1,\ldots,n-1),Y;A\right) \\
\uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
\pi_{n+1}\left(X(1,\ldots,n-1),X\right)\otimes A & \longrightarrow & \pi_{n+1}\left(Y(1,\ldots,n-1),Y\right)\otimes A \\
\uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
\pi_n(X)\otimes A & \longrightarrow & \pi_n(X)\otimes A
\end{array}
$$

This completes the induction. Of course, for $n = 1$ we need as many steps as there are in your central series, and the argument requires slight adaptation (replace π_n by the relevant subquotient). □

Second proof. We have the localisation functor already. So

$$f_* : H_*(X;A) \longrightarrow H_*(Y;A)$$

is an isomorphism if and only if $f \in S$, if and only if Ef is an equivalence if and only if
$$(Ef)_\# : \pi_*(EX) \longrightarrow \pi_*(EY)$$
is an isomorphism, if and only if

$$f_\# : \pi_*(X) \otimes A \longrightarrow \pi_*(Y) \otimes A$$

is an isomorphism. □

Corollary 5.10. *Let Y be a space such that $\pi_1(Y)$ is nilpotent and acts trivially on $\pi_n(Y)$ for $n > 1$. Then $Y \in D$ if and only if $\tilde{H}_*(Y)$ is local, if and only if $\pi_*(Y)$ is local.*

Proof. Consider

$$
\begin{array}{ccc}
\tilde{H}_*(Y) & \xrightarrow{\ b\ } & \tilde{H}_*(EY) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \cong} \\
\tilde{H}_*(Y;A) & \xrightarrow{\ \cong\ } & \tilde{H}_*(EY;A)
\end{array}
\qquad
\begin{array}{ccc}
\pi_*(Y) & \xrightarrow{\ d\ } & \pi_*(EY) \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \cong} \\
\pi_*(Y) \otimes A & \xrightarrow{\ \cong\ } & \pi_*(EY) \otimes A
\end{array}
$$

Then $\tilde{H}_*(Y)$ is local iff a is an isomorphism, iff b is an isomorphism, iff c is an isomorphism (5.9 with $A = \mathbf{Z}$, iff $Y \in D$, iff d is an isomorphism, iff $\pi_*(Y)$ is local. $\qquad\square$

Corollary 5.11. *Let X, Y be spaces such that π_1 is nilpotent and acts trivially on π_n for $n > 1$. Then the following conditions on $f : X \to Y$ are equivalent.*

(i) There is a commutative diagram

(iia) $f_{\#} : \pi_*(X) \longrightarrow \pi_*(Y)$ is a localising map .

(iib) $f_* : \tilde{H}_*(X) \longrightarrow \tilde{H}_*(Y)$ is a localising map.

(iiia) $f_{\#} : \pi_*(X) \otimes A \longrightarrow \pi_*(Y) \otimes A$ is an isomorphism and is couniversal with that property (in the category of such X, Y).

(iiib) $f_* : H_*(X;A) \longrightarrow H_*(Y;A)$ is an isomorphism and is couniversal with that property (in the category of such X, Y).

(iva) $\pi_*(Y)$ is local (i. e., an A-module) and f is universal with that property (in the category of such X, Y).

(ivb) $\tilde{H}_*(Y)$ *is local (i. e., an A-module) and f is universal with that property (in the category of such X, Y).*

This is precisely the statement of 2.11 with the alternative descriptions of S and D inserted from 5.9, 5.10.

[A section suggesting possible directions for further work on idempotent functors is omitted here.]

Editorial Note. The results of this section can be strengthened with a little additional effort. Namely one can weaken the hypothesis that the fundamental group is nilpotent and acts trivially on the higher homotopy groups to requiring that the fundamental group act nilpotently on the higher homotopy groups. This means that each higher homotopy group has a finite filtration such that the fundamental group acts trivially on the filtration quotients. Such spaces are called nilpotent spaces. The proof is along exactly the same lines as above, except that one refines the Postnikov tower of the space by a larger tower of fibrations corresponding to the filtration quotients in the higher homotopy groups. See [10] for details.

6 Profinite Completion

We start with the most classical example.

Example 6.1. Let G be a topological group. Then we can consider all homomorphisms from G to finite groups F_α (with the discrete topology). For our purposes we can replace F_α by the image of $G \to F_\alpha$; so it will be sufficient to consider all closed normal subgroups $N_\beta \subset G$ of finite index and consider the projections $G \to G/N_\beta$. Such normal subgroups are ordered by inclusion, $N_\beta \subset N_\gamma$, so we get an inverse system of finite discrete groups, $G/N_\beta \to G/N_\gamma$ We define the profinite completion of G to be $\widehat{G} = \varprojlim_\beta G/N_\beta$.

It comes equipped with a compact Hausdorff topology, the topology of the inverse limit; and there is a canonical map $G \to \widehat{G}$.

In this way we get an idempotent functor on the category of topological groups.

Of course, if you start with a discrete group G and complete it, you get a topological group, say CoG. You can then apply the functor F which forgets the topology. But the composite functor FCo is not idempotent.

Example 6.2. Take V to be a vector space over $\mathbf{Z}_p$, with the discrete topology. Then $CoV = \widehat{V} \cong V^{**}$, the double dual of V, with the topology of the double dual. So $FCoV \cong V^{**}$ with the discrete topology. So $FCoFCoV \cong V^{****}$ which is not isomorphic to V^{**} if V is infinite-dimensional.

Now let me try to explain Sullivan's concerns. Suppose X is say an affine algebraic variety $\subseteq \mathbf{C}^n$, defined by polynomial equations with integer coefficients. Then the automorphism $z \mapsto \overline{z}$ of $\mathbf{C}$ defines an automorphism of $\mathbf{C}^n$ and an automorphism of X. So it gives an automorphism of $H_*(X)$ or of any other topological invariant of X.

Now suppose we have an invariant $K(X)$ which can be defined by purely algebraic means starting from the equations which define X. Then,we reason that any algebraic automorphism of $\mathbf{C}$ which leaves $\mathbf{Z}$ fixed should induce an automorphism of $K(X)$. In other words, the Galois group $\mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ should act on $K(X)$. Now, this is a monstrous big symmetry group. However, most of it should be irrelevant. Purely algebraic constructions should hardly

require more of $\mathbf{C}$ than $\overline{\mathbf{Q}}$, the algebraic closure of $\mathbf{Q}$. So we might expect the action of $\mathrm{Gal}(\mathbf{C}/\mathbf{Q})$ to factor through the quotient group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. But this is still a pretty large group, and such symmetry is very useful.

This still leaves open, of course the question of what invariants of X can be defined by purely algebraic means; and in particular, how close we can come to the usual topological invariants. This question has been much studied, and the original studies concerned the fundamental group. It is equivalent to study the covering spaces of X. It turns out that what you can do by purely algebraic means is to construct the finite covering spaces of X. What you cannot do by purely algebraic means is to construct any infinite covering spaces of X. More precisely, let $p : \widetilde{X} \to X$ be a regular map between algebraic varieties; if $x_0 \in X$, the counterimage $p^{-1}x_0$ has to be an algebraic subset of X, so if it is 0-dimensional it has to be finite. Therefore, what you can learn about the fundamental group $\pi_1(X)$ is exactly the finite quotients $\pi_1(X)/N_\beta$; so you can learn the profinite completion $\widehat{\pi_1(X)}$, which indeed serves to summarize the information about the finite quotients.

The further developments of this story are the business of étale homotopy theory, and I don't claim to understand them. I am credibly informed that what you can do by purely algebraic means is the following: you can recover $[X, Z]$ whenever Z is a complex all of whose homotopy groups are finite. For example, taking Z to be an Eilenberg-MacLane space of type $(\pi, 1)$, where π is finite, you can recover the homomorphisms $\pi_1(X) \to \pi$.

What Sullivan now does is to construct a functor $\widehat{X}$ of X which summarizes all the information you can obtain from $[X, Y]$ where Y runs over complexes with finite homotopy groups.

You might think that we already have a plausible candidate for such a functor. That is, we define a subset S as follows: a map $f : X \to Y$ is to lie in S if and only if $f^* : [Y, Z] \longrightarrow [X, Z]$ is an isomorphism whenever Z has finite homotopy groups. We can replace this by two conditions

(i) $f^* : \mathrm{Hom}\,(\pi_1(Y), F) \longrightarrow \mathrm{Hom}\,(\pi_1(X), F)$ is an isomorphism whenever F is a finite group,

(ii) $f^* : H^*(Y; A) \longrightarrow H^*(X; A)$ is an isomorphism whenever A is a finite abelian group with operations from $\pi_1(Y)$.

The necessity of (i) is seen by taking Z to be an Eilenberg-MacLane space

of type $(F, 1)$. The necessity of (ii) is seen by taking Z to be the obvious fibering

$$K(A, n) \longrightarrow Z \longrightarrow K(\mathrm{Aut}(A), 1).$$

(Since A is a finite group, $\mathrm{Aut}(A)$ is a finite group.)

The sufficiency of (i) plus (ii) is seen by obstruction theory.

The subset S clearly satisfies 3.1, 3.5 and 3.6. We see it satisfies 4.1 by checking (i) and (ii); here (i) comes easily from Van Kampen's Theorem and (ii) comes by the same argument we used in 4.5.

Modulo 3.4, we get a functor EX. It is plausible to conjecture that it coincides with Sullivan's $\widehat{X}$ in the main cases of interest. In fact, Sullivan only proves that his functor $\widehat{X}$ has good properties under rather restrictive conditions ($\pi_1(X) = 0$ and $\pi_n(X)$ finitely generated for each n; see Sullivan [19, Theorem 3.9, p. 330].) But since Sullivan indicates that these conditions can be weakened, let us disregard the conditions and turn to Sullivan's conclusions. The conclusion 3.9(iv) certainly implies that Sullivan's map $X \to \widehat{X}$ lies in our class S; and whenever this conclusion holds, then Sullivan's $\widehat{X}$ is equivalent to our EX. It may even be the case that Sullivan's $\widehat{X}$ is equivalent to our EX whenever X is a finite complex. This would be completely satisfactory, because the main application arises when X is an algebraic variety, and algebraic varieties tend to be equivalent to finite complexes. It is equally certain that I can't prove it because of genuine difficulties with the fundamental group. For example, one might like to know that if G is a finitely presented group, such as $\pi_1(X)$, then any homomorphism from $\widehat{G}$ to a finite group is necessarily continuous. So far as I know, this is not known.

There are reasons why $\widehat{X}$ and EX should not be equivalent in general. The first is this Sullivan 's approach assumes that you are given X and know $[X, Z]$ whenever Z has finite homotopy. The construction of EX tends to assume that you know $[Y, Z]$ whenever Z has finite homotopy groups and Y runs over a lot of other things as well as X. Clearly if you take Y to be something which is not an algebraic variety you are feeding in information not accessible to algebraic geometry, so in this sense it is not obvious that algebraic geometry will suffice to construct EX. The second reason is this. Sullivan's approach does not give an idempotent functor defined on the category C we have been considering. If it is to have any chance to be idempotent it must be defined on a different category.

To explain this, I need the notion of a topologised object in a category. One way to define a group object in a category is as follows. It is an object G such that the set $[X, G]$ is given the structure of a group for each X so that the induced functions $f^* : [Y, G] \longrightarrow [X, G]$ are group homomorphisms. Similarly, we say that T is a topologised object in a category C if for each X in C, $[X, T]$ is given the structure of a topological space, and for each $f : X \to Y$ in C, the induced function $f^* : [Y, T] \longrightarrow [X, T]$ is continuous. Let T and U be topologised objects in C, we say that a map $g : T \to U$ in C is a map topologised objects if. $g^* : [X, T] \longrightarrow [X, U]$ is, continuous for each X.

Example 5.3. Let π be a compact (Hausdorff) abelian group, for example, S^1 or the p-adic integers. Then the Eilenberg-MacLane space $K(\pi, n)$ is a topologised object in the homotopy category. In fact, for any X we topologise $C^q(X; \pi)$ with the product topology; we take the obvious topology on cocycles, coboundaries and $H^n(X; \pi)$. This gives a topology in $[X, K(\pi, n)]$ for each n (indeed making it a compact group), and the induced functions are continuous. A continuous homomorphism $\theta : \pi_1 \to \pi_2$ induces a map of the topologised objects $K(\pi_1, n) \longrightarrow K(\pi_2, n)$.

I will say that T is a compact Hausdorff object in C if it is topologised and the topology on $[X, T]$ is compact and Hausdorff for each X. Example 5.3 is of this kind. However, Sullivan speaks of a compact representable functor; that is, he identifies T with the representable functor $[X, T]$.

In particular, let C be the homotopy category, and let C^T be the category of topologised objects in C. Then our ignorance of C^T is really deplorable; in fact, it is one of the objects of this lecture to persuade some of you to give a little thought to C^T. Anyway, Sullivan's profinite completion functor is a functor Co from C to C^T. Of course, we can compose it with the functor $F : C^T \to C$, which forgets the topology; but the composite FCo is not idempotent. (Take X to be an Eilenberg-MacLane space of type (V, n) where V is as in Ex. 5.2.) Therefore, the only chance of making Sullivan's profinite completion idempotent is to set it up as a functor from C^T to C^T (compare 5.1). I do not know whether Sullivan's profinite completion can be set up as a functor from C^T to C^T; a fortiori I do not know if it can be set up as an idempotent functor from C^T to C^T.

At this point, I should start to do the work. What follows is my interpre-

tation of a letter from Sullivan.

Proposition 6.4. *Let $Z \in C$ be an object with finite homotopy groups. Then there is one and only one way to give Z the structure of a compact Hausdorff object. Any map $Z \to Z'$ between such objects is a map of topologised objects.*

Proof. First let X be a finite complex. Then $[X, Z]$ is a finite set, and there is one and only one way to make it compact Hausdorff space.

Next let X be a general complex, and define $H(X) = \varprojlim_{\alpha}[X_\alpha, Z]$, where X_α runs over all the finite subcomplexes of X. We topologise $\varprojlim_{\alpha}[X_\alpha, Z]$ with the inverse-limit topology. Clearly H is a contravariant functor from the category C to the category of compact Hausdorff spaces. I claim H is representable. To check the wedge axiom is automatic. To check the Mayer-Vietoris axiom is straightforward, but one uses in an essential way the fact that the inverse limit of a system of nonempty finite sets is nonempty. We conclude that $H(X) \xleftarrow{\cong} [X, \overline{Z}]$. Now we clearly have a natural transformation

$$[X, Z] \longrightarrow \varprojlim_{\alpha}[X_\alpha, Z] = H(X) \longleftrightarrow [X, \overline{Z}]$$

This must be induced by a map $Z \to \overline{Z}$. Now if X is a sphere then clearly $[X, Z] \longrightarrow \varprojlim_{\alpha}[X_\alpha, Z]$ is an isomorphism because the limit is a attained. Therefore $Z \to \overline{Z}$ induces isomorphisms of homotopy groups, and is an equivalence. So for any X, the obvious map $[X, Z] \longrightarrow \varprojlim_{\alpha}[X_\alpha, Z]$ is an isomorphism. We already topologised $\varprojlim_{\alpha}[X_\alpha, Z]$, so this puts a compact Hausdorff topology on $[X, Z]$. We check that we have made Z a compact Hausdorff object, and such a structure on Z is unique. For any map $Z \to Z'$ the map $[X_\alpha, Z] \longrightarrow [X_\alpha, Z']$ is continuous, so $\varprojlim_{\alpha}[X_\alpha, Z] \longrightarrow \varprojlim_{\alpha}[X_\alpha, Z']$ is continuous. $\square$

Now we turn to the construction of Sullivan's profinite completion $Y \to \widehat{Y}$.

Let Y be a fixed object in C. We consider a category: the objects are maps $g_\alpha : Y \to Z_\alpha$ such that Z_α has finite homotopy groups; the arrows are diagrams

$$
\begin{array}{ccc}
& Z_\alpha & \\
\nearrow^{g_\alpha} & & \searrow \\
Y & \xrightarrow[g_\beta]{} & Z_\beta
\end{array}
$$
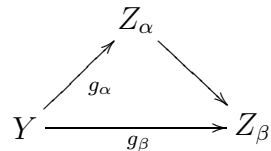
We check that this is a directed category (directed away from infinity). Given two objects, I have to find one further from infinity than each. This is easy: given

$$
\begin{array}{ccc}
& & Z_\alpha \\
& \nearrow^{g_\alpha} & \\
Y & & \\
& \searrow_{g_\beta} & \\
& & Z_\beta
\end{array}
$$

I construct

$$
\begin{array}{ccc}
& & Z_\alpha \\
& \xrightarrow{g_\alpha} & \\
Y \rightrightarrows Z_\alpha \times Z_\beta & & \\
& \searrow_{g_\beta} & \\
& & Z_\beta
\end{array}
$$
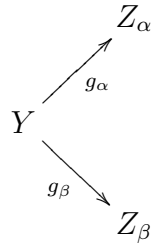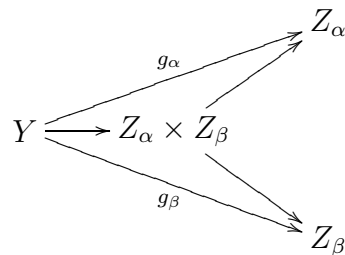
and $Z_\alpha \times Z_\beta$ has finite homotopy groups. Secondly, suppose we are given a diagram

$$
\bullet \rightrightarrows \bullet \; ;
$$
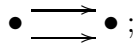
I have to construct

$$
\dashrightarrow \bullet \rightrightarrows \bullet \; .
$$

So suppose given

$$
\begin{array}{ccc}
& Z_\alpha & \\
\nearrow & \big\downarrow h \quad \big\downarrow k & \\
Y & & \\
\searrow & Z_\beta & \\
\end{array}
$$

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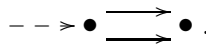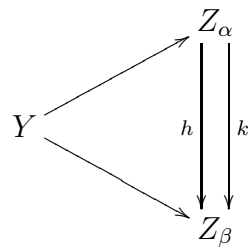
Form

$$
\begin{array}{c}
Z_\alpha \\
\downarrow {\scriptstyle (1,k)} \\
Z_\alpha \xrightarrow{\;(1,h)\;} Z_\alpha \times Z_\beta
\end{array}
$$

and take the homotopy pullback

$$
\begin{array}{ccc}
F & \longrightarrow & Z_\alpha \\
\downarrow & & \downarrow {\scriptstyle (1,k)} \\
Z_\alpha & \xrightarrow{\;(1,h)\;} & Z_\alpha \times Z_\beta
\end{array}
$$

(If I want to stay inside the category of connected spaces, I take the component of the basepoint.) Then F has finite homotopy groups, and it is easy to show that there is a map $Y \to F$ which does as required.

It is very tempting to modify this construction, taking Y to be a fixed object in C^T and considering maps $g_\alpha : Y \to Z_\alpha$ in C^T. The difficulty comes at the last step above: how can we know that we can, choose the map $Y \to F$ to be a map of topologised objects?

Now that we have a directed category, we form $H(X) = \varprojlim_\alpha [X, Z_\alpha]$ Each set $[X, Z_\alpha]$ is a compact Hausdorff space by 5.4, so we give $\varprojlim_\alpha [X, Z_\alpha]$ the inverse-limit topology, and it becomes a compact Hausdorff space. Thus H becomes a contravariant functor from C to compact Hausdorff spaces.

Lemma 6.5. *H is representable.*

Proof. The proof of the wedge axiom is automatic. The proof of the Mayer-Vietoris axiom is straightforward, but one makes essential use of the fact that an inverse limit of nonempty compact Hausdorff spaces is nonempty. $\square$

So we have $\varprojlim_\alpha [X, Z_\alpha] \longleftrightarrow [X, \widehat{Y}]$ for an object $\widehat{Y}$ in C. Since we have given $\varprojlim_\alpha [X, Z_\alpha]$ a compact Hausdorff topology, $\widehat{Y}$ is a compact Hausdorff object in C. We have an obvious natural transformation.

$$
[X, Y] \longrightarrow \varprojlim_\alpha [X, Z_\alpha] \longleftrightarrow [X, \widehat{Y}];
$$

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this must be induced by a map $Y \to \widehat{Y}$.

It is rather clear that the homotopy groups of $\widehat{Y}$ are profinite: we have $[S^n, \widehat{Y}] \longleftrightarrow \varprojlim_{\alpha}[S^n, Z_\alpha]$ where $[S^n, Z_\alpha]$ is finite.

The following are among the main properties of Sullivan's profinite completion.

Theorem 6.6. (Sullivan [19, Theorem 3.9, p. 330]) *Suppose Y is simply connected and its homotopy groups $\pi_n(Y)$ are finitely-generated for each n. Then*

(i) *The homotopy groups of $\widehat{Y}$ are profinite and universal with this property: if*

$$\begin{array}{ccc} & & \widehat{Y} \\ & \nearrow & \\ Y & & \\ & \searrow & \\ & & Z \end{array}$$
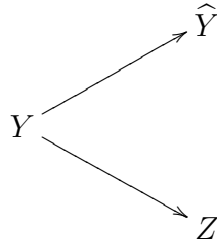
and the homotopy groups of Z are profinite, then

$$\begin{array}{ccc} & & \widehat{Y} \\ & \nearrow & \vdots \\ Y & & \vdots \; \exists t \\ & \searrow & \vdots \\ & & Z \end{array}$$
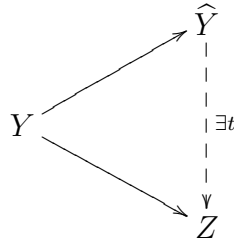
(ii) *More precisely, $\widehat{\pi_i(Y)} \xrightarrow{\cong} \pi_i(\widehat{Y})$.*

(iii) *The topological structure of $\widehat{Y}$ is determined by the underlying object. In fact*

$$CoFCoY \cong CoY.$$

The proof is to see what profinite completion does to the Postnikov system.

Editorial Note. It turns out that Sullivan's profinite completion functor, for spaces satisfying the hypothesis of Theorem 6.6, is a special case of the localisation functor of Theorem 4.6. Namely if one takes as S the class of morphisms which induce isomorphisms in $H_*(-; A)$, where A is a torsion abelian group which has p-torsion for all primes p (e.g. $A = \oplus_p \mathbf{Z}/p\mathbf{Z}$), then the corresponding idempotent functor EY is naturally equivalent to $\widehat{Y}$. Again a good reference is [10].

7 Use of Brown-Peterson Homology in Stable Homotopy

You may have heard that we have a contradiction in homotopy theory these days. There is supposed to be a certain map γ_1; and Zahler and Thomas claim to have proved that $\gamma_1 \neq 0$, while Oka and Toda claim to have proved that $\gamma_1 = 0$. The situation reminds me of these words of André Weil [22] about the need for rigour in mathematics: *"It is possible for the advancing army to outrun its services of supply and incur disaster unless it waits for the quartermaster to perform his inglorious but indispensable tasks."* In the present situation the duty of the quartermaster is clearly to call in both parties to the dispute and audit their books.

I have tried to repeat the work of Zahler and Thomas. What I would like you to do is to watch with the most critical attention while I try to perform it in public. I don't think I've got anything up my sleeve, but someone, somewhere has got a joker hidden away; probably not because he put it there, but more likely because he accepted someone else's work in too trusting and uncritical a fashion.

I must begin by explaining the element γ_1 and for that I need the spaces $V(n)$ of Toda and others. Let p be an odd prime. Let us recall that according to Milnor, the mod p homology of the Eilenberg-MacLane spectrum $H\mathbf{Z}_p$ is the tensor product of an exterior algebra and a polynomial algebra

$$(H\mathbf{Z}_p)_*(H\mathbf{Z}_p) = \Lambda[\tau_0, \tau_1, \ldots] \otimes \mathbf{Z}_p[\xi_1, \xi_2, \ldots].$$

The Brown-Peterson spectrum BP admits a map $f : BP \to H\mathbf{Z}_p$ so that

$$f_* : (H\mathbf{Z}_p)_*(BP) \to (H\mathbf{Z}_p)_*(H\mathbf{Z}_p)$$

is mono, and its image is $\mathbf{Z}_p[\xi_1, \xi_2, \ldots]$. Equivalently $(H\mathbf{Z}_p)_*(BP)$ is a cyclic module over the mod p Steenrod algebra A^* generated by one generator f and is isomorphic to the quotient $A^*/A^*\beta_p A^*$, where β_p denotes the Bockstein operation.

Similarly, we would admit X as a spectrum of type $V(\infty)$ if it had a map $f : X \to H\mathbf{Z}_p$ such that

$$f_* : (H\mathbf{Z}_p)_*(X) \to (H\mathbf{Z}_p)_*(H\mathbf{Z}_p)$$

is mono and its image is $\Lambda[\tau_0, \tau_1, \ldots]$. It would be well to specify also that X should be connected and $(H\mathbf{Z}_q)_*(X) = 0$ for q prime to p. If such a spectrum existed, we could form

$$V(\infty) \wedge BP \longrightarrow H\mathbf{Z}_p \wedge H\mathbf{Z}_p \longrightarrow H\mathbf{Z}_p,$$

and this would induce an isomorphism of $(H\mathbf{Z}_p)_*$ so that $V(\infty) \wedge BP$ would be an Eilenberg-MacLane spectrum $H\mathbf{Z}_p$. However, no such spectrum $V(\infty)$ is known to exist, so we turn to finite approximations to it.

We admit X is a spectrum of type $V(n)$ if it has a map $f : X \to H\mathbf{Z}_p$ such that
$$f_* : (H\mathbf{Z}_p)_*(X) \to (H\mathbf{Z}_p)_*(H\mathbf{Z}_p)$$
is mono and its image is $\Lambda[\tau_0, \tau_1, \ldots, \tau_n]$. Essentially we are prescribing $(H\mathbf{Z}_p)_*(X)$ as a module over the mod p Steenrod algebra. Again we specify that X should be connected and $(H\mathbf{Z}_q)_*(X) = 0$ for q prime to p.

Example. $V(0)$ exists; we can take $V(0) = S^0 \cup_p e^1$.

Example. $V(1)$ exists. More precisely, there is a map $\alpha : S^{2p-2} \longrightarrow S^1$ such that the operation P^1 is non-zero in $(H\mathbf{Z}_p)_*(S^1 \cup_\alpha e^{2p-1})$. Since $p\alpha = 0$ and $p, \alpha, p = 0$, we can find a map A to fill the diagram below

$$
\begin{array}{ccc}
S^{2p-1} \cup_p e^{2p-1} & \xdashrightarrow{\quad A \quad} & S^0 \cup_p e^1 \\
\uparrow & & \downarrow \\
S^{2p-2} & \xrightarrow{\quad \alpha \quad} & S^1
\end{array}
$$

The mapping cone of A is a complex

$$V(1) = V(0) \cup_A CS^{2p-2}V(0) = S^0 \cup_p e^1 \cup_\alpha e^{2p-1} \cup_p e^{2p}$$

in which the operation $\beta P^1 \beta$ is non-zero.

It is clearly stated by Toda that

$V(2)$ exists for $p \geq 5$

$V(3)$ exists for $p \geq 7$

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(H. Toda [21, pp. 53, Thm. 1.1].) He also asserts that these $V(n)$ are unique up to homotopy equivalence (Theorem 4.1, p. 57). This is not actually essential for our purposes, but it is convenient.

Now we want to see how the spaces $V(n)$ give rise to systematic families of elements in homotopy theory. We have seen that there is a map

$$A : S^{2(p-1)}V(0) \longrightarrow V(0).$$

By iterating it we can construct

$$S^{2r(p-1)}V(0) \longrightarrow \cdots \longrightarrow S^{4(p-1)}V(0) \xrightarrow{S^{2(p-1)}A} S^{2(p-1)}V(0) \xrightarrow{A} V(0)$$

$$S^{2r(p-1)} \xrightarrow{\qquad\qquad \alpha_r \qquad\qquad} S^1$$

So we construct $\alpha_r \in \pi^s_{2r(p-1)-1}$. It is usual to write $q = 2(p-1)$; then α_r is of degree $rq - 1$.

Toda clearly asserts that there is a map

$$B : S^{2(p^2-1)}V(1) \longrightarrow V(1) \quad \text{for } p \geq 5$$

whose mapping cone is $V(2)$, and a map

$$C : S^{2(p^3-1)}V(2) \longrightarrow V(2) \quad \text{for } p \geq 7$$

whose mapping cone is $V(3)$ (Toda [21, Corollary 4.3, p. 58]).

By iterating B we can construct

$$S^{2r(p^2-1)}V(1) \longrightarrow \cdots \longrightarrow S^{4(p^2-1)}V(1) \xrightarrow{S^{2(p^2-1)}B} S^{2(p^2-1)}V(1) \xrightarrow{B} V(1)$$

$$S^{2r(p^2-1)} \xrightarrow{\qquad\qquad \beta_r \qquad\qquad} S^{2p}$$

β_r is of degree $2r(p^2-1) - 2p = r(p+1)q - q - 2$. By iterating C we can construct

$$S^{2r(p^3-1)}V(2) \longrightarrow \cdots \longrightarrow S^{4(p^3-1)}V(2) \xrightarrow{S^{2(p^3-1)}C} S^{2(p^3-1)}V(2) \xrightarrow{C} V(2)$$

$$S^{2r(p^3-1)} \xrightarrow{\qquad\qquad \gamma_r \qquad\qquad} S^{(p+2)q+3}$$

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γ_r has degree $2r(p^3 - 1) - (p + 2)q - 3 = r(p^2 + p + 1)q - (p + 2)q - 3$. In particular, γ_1 has degree $(p^2 - 1)q - 3$.

Now, of course, if C and $V(3)$ do not exist, then there is nothing to discuss, but it is generally supposed that they do exist.

Now we must discuss the BP-homology and cohomology of our spaces $V(n)$. It is necessary to recall that $\pi_*(BP)$ is a polynomial algebra over $\mathbf{Z}_{(p)}$ on generators $v_1, v_2, v_3, \ldots, v_i, \ldots$, of dimension $2(p^i - 1)$. We will make the choice of generators v_i more precise later. From the cofibering

$$S^0 \xrightarrow{p} S^0 \longrightarrow V(0) \longrightarrow S^1 \xrightarrow{p} S^1$$

we see that $BP_*(V(0)) \cong \pi_*(BP)/p\pi_*(BP)$ on one generator of dimension 0, coming from the injection $S^0 \to V(0)$. Similarly,

$$BP^*(V(0)) \cong \pi_*(BP)/p\pi_*(BP)$$

on one generator of dimension 1, coming from the projection $V(0) \to S^1$.

We now observe that

$$(H\mathbf{Z}_p)_*(V(n) \wedge BP) \longrightarrow (H\mathbf{Z}_p)_*(H\mathbf{Z}_p)$$

is mono, and epi in dimension $< 2p^{n+1} - 1$. Therefore

$$\pi_i(V(n) \wedge BP) = 0 \quad \text{for } 0 < i < 2p^{n+1} - 2$$

i. e.,
$$BP_i(V(n)) = 0 \quad \text{for } 0 < i < 2p^{n+1} - 2.$$

In particular, if $g_n \in BP_0(V(n))$ comes from the injection $S^0 \to V(n)$, then we have
$$v_i g_n = 0 \quad \text{for } i \leq n$$

Consider now the cofibering

$$S^q V(0) \xrightarrow{A} V(0) \longrightarrow V(1) \longrightarrow S^{q+1} V(0) \xrightarrow{SA} SV(0)$$

We have $v_1 g_0 \mapsto 0$ in $BP_q(V(1))$, so we conclude

$$A_* g_0 = c_1 v_1 g_0, \quad c_1 \in \mathbf{Z}_p$$

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with $c_1 \neq 0$. Then the cofibering gives

$$BP_*(V(1)) \cong \pi_*(BP)/(p, v_1)$$

generated by g_1. Similarly, we see that

$$B_*g_1 = c_2 v_2 g_1, \quad c_1 \in \mathbf{Z}_p$$

with $c_2 \neq 0$, that

$$BP_*(V(2)) \cong \pi_*(BP)/(p, v_1, v_2)$$

generated by g_2, that

$$C_*g_2 = c_3 3 v_3 g_2, \quad c_3 \in \mathbf{Z}_p$$

with $c_3 \neq 0$, and that

$$BP_*(V(3)) \cong \pi_*(BP)/(p, v_1, v_2, v_3)$$

generated by g_3.

Similar results are true for BP-cohomology. To deduce them, recall that

$$BP^*(X) \cong BP_*(DX)$$

where DX is the Spanier-Whitehead dual of X. But the Spanier-Whitehead dual of a space $V(n)$ is again a space $V(n)$. Therefore we have

$$
\begin{aligned}
BP^*(V(0)) &\cong \pi_*(BP)/(p) \\
BP^*(V(1)) &\cong \pi_*(BP)/(p, v_1) \\
BP^*(V(2)) &\cong \pi_*(BP)/(p, v_1, v_2) \\
BP^*(V(3)) &\cong \pi_*(BP)/(p, v_1, v_z, v_3)
\end{aligned}
$$

on generators g_0, g_1, g_2, g_3 of degrees 1, $1 + (2p - 1)$, $1 + (2p - 1) + (2p^2 - 1)$, $1 + (2p - 1) + (2p^2 - 1) + (2p^3 - 1)$ coming from the projections $V(0) \to S^1$, $V(1) \to S^{2p}$, etc. Similarly, we have

$$
\begin{aligned}
A^*g_0 &= c_1' v_1 g_0 \\
B^*g_1 &= c_2' v_2 g_1 \\
C^*g_2 &= c_3' v_3 g_2
\end{aligned}
$$

where $c'_1, c'_2, c'_3 \in \mathbf{Z}_p$ and are non-zero.

Without loss of generality we may replace A by an integer multiple mA where $m \not\equiv 0 \bmod p$. Similarly for B and C. Therefore we may suppose that

$$
\begin{aligned}
A^* g_0 &= v_1 g_0 \\
B^* g_1 &= v_2 g_1 \\
C^* g_2 &= v_3 g_2
\end{aligned}
$$

This is the normalisation usually adopted by those who work with BP-cohomology. Now I want to explain the nature of the proof that $\gamma_1 \neq 0$. First we form the complex

$$
X_3 = S^0 \cup_{\gamma_1} e^{(p^2-1)q-2}
$$

Then we inspect the cofibering

$$
S^{(p^2-1)q-3} \xrightarrow{\gamma_1} S^0 \xrightarrow{i} X_3 = S^0 \cup_{\gamma_1} e^{(p^2-1)q-2} \longrightarrow S^{(p^2-1)q-2} \longrightarrow S^1
$$

Here $BP^*(S^0)$ is non-zero only in dimensions $\equiv 0 \bmod q$, and $BP^*(S^{(p^2-1)q-3})$ is non-zero only in dimensions $\equiv -3 \bmod q$, and we may certainly suppose $q \geq 8$, so $BP^*(\gamma_1) = 0$. There is a unique element $h \in BP^0(X_3)$ projecting to the generator of $BP^0(S^0)$ and we take k to be the image of the generator in $BP^{(p^2-1)q-2}\left(S^{(p^2-1)q-2}\right)$. We will produce a cohomology operation χ in BP-cohomology such that χh is a non-zero multiple of k. (By "multiple" we mean a multiple by an element of $\pi_*(BP)$.) This will contradict the statement $\gamma_1 = 0$, which implies $X_3 \simeq S^0 \vee S^{(p^2-1)q-2}$. For if so, we would have a map $j : X_3 \to S^0$ such that $ji = 1$; then $i^* j^* i^* h = i^* h$ and $i^* : BP^0(X_3) \longrightarrow BP^0(S^0)$ is an isomorphism, so $h = j^* i^* h$. Then

$$
\chi h \supseteq j^* i^* \chi h \supseteq j^* i^* (\mu k) \supseteq j^* 0 = 0.
$$

The only snag is that χ is a tertiary operation. However, when we see it we will be able to check that it is defined on h and therefore also on $i^* h$, and that its indeterminacy in X_3 is manageable and its indeterminacy in S^0 is zero. I must now explain how to set up χ and calculate it.

For this I must explain about primary operations on BP-cohomology and the relations between them. In [2], it is stated that

$$
BP_*(BP) = \pi_*(BP)[t_1, t_2, t_3, \ldots]
$$

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where the t_i are well-defined elements such that deg $t_i = 2(p^i - 1)$ (Thm 16.1, p. 97). It follows that in $BP^*(BP)$ we have unique elements $R_{i_1 i_2 \ldots i_n}$ such that

$$< R_{i_1 i_2 \ldots i_n}, t_1^{j_1} t_2^{j_2} \ldots t_n^{j_n} > \; = \; \delta_{i_1 j_1} \delta_{i_2 j_2} \ldots \delta_{i_n j_n}.$$

In particular, we will need R_1, R_p and R_{01}, which are the elements of the "dual base" corresponding to t_1, t_1^p, t_2. They are of degree q, pq, $(p+1)q$.

Lemma 7.1. *We have*

$$
\begin{aligned}
R_1 R_p - R_p R_1 &= R_{01} \\
R_1 R_{01} - R_{01} R_1 &= 0 \\
R_p R_{01} - R_{01} R_p &= 0
\end{aligned}
$$

Proof. According to [1, Prop. 3, p. 73], the composition product in $BP^*(BP)$ is determined by the diagonal map in $BP_*(BP)$, according to the following formula. Suppose

$$\psi x = \sum_i e_i \otimes x_i$$

Then

$$< ab, x > \; = \; \sum_i (-1)^{|b| \, |e_i|} < a, e_i < b, x_i >> .$$

According to [2, Th. 16.1, p. 98], the diagonal map in $BP_*(BP)$ is given by the following formulae

$$
\begin{aligned}
\psi t_1 &= t_1 \otimes 1 + 1 \otimes t_2 \\
\psi t_2 &= t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - \sum_{\substack{i+j=p \\ i>0, j>0}} \frac{(p-1)!}{i! j!} v_1 t_1^i \otimes t_1^j.
\end{aligned}
$$

Here I need the fact that

$$v_1 = [CP^{p-1}] = p m_{p-1}$$

in the notation of the above. Moreover, I need to know that if I have a monomial $t^I =, t_1^{i_1} t_2^{i_2} \ldots t_n^{i_n}$ of degree say $2d$, then ψt^I has the form $\sum c_{JK} t^J \otimes t^K$,

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where $0 \leq \deg c_{JK} \leq \frac{2d}{p}$, $\frac{2d(p-1)}{p} < \deg(t^J \otimes t^K) \leq 2d$. So in evaluating a product $R^I \otimes R^J$ of degree $2e$, it will be sufficient to consider ψt^I with $\deg t^I < \frac{2ep}{p-1}$. Hence in proving this lemma, we need not worry about monomials containing $t_3, t_4, \ldots$. The following tables give enough values of the pairing $< ab, t^I >$ to prove the lemma.

	t_1	t_2	t_1^{p+1}	$t_1 t_2$
$R_1 R_p$	$p+1$	0	1	$-v_1$
$R_p R_1$	$p+1$	0	0	$-v_1$

	t_1^j	$t_1 t_2$	t_2
$R_1 R_{01}$	0	1	0
$R_{01} R_1$	0	1	0

	t_1^j	$t_1^p t_2$	$t_1^{p+1} t_2$	$t_1^{p+2} t_2$	t_2	$t_1 t_2^2$
$R_p R_{01}$	0	1	0	0	0	0
$R_{01} R_p$	0	1	0	0	0	0

$\square$

Multiplying out these relations, I get:

$$R_p R_1^2 - 2R_1 R_p R_1 + R_1^2 R_p = 0$$
$$R_p^2 R_1 - 2R_p R_1 R_p + R_1 R_p^2 = 0.$$

I now construct maps of spectra

$$S^{(2p+2)q} BP \xleftarrow{d_2} S^{(p+2)q} BP \vee S^{(2p+1)q} BP \xleftarrow{d_1} S^q BP \vee S^{pq} \xleftarrow{d_0} BP$$

with components

$$d_0 = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

$$d_1 = \begin{bmatrix} R_p R_1 - 2R_1 R_p & R_1 \\ R_p^2 & R_1 R_p \end{bmatrix}$$

$$d_2 = \begin{bmatrix} R_p & R_1 \end{bmatrix}$$

Then by construction we have

$$d_2 d_1 \simeq 0 \qquad d_1 d_0 \simeq 0$$

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Consider also $h : X_3 \to BP$. Then I claim we have $d_0 h \simeq 0$. in fact, the maps

$$BP^q(X_3) \xrightarrow{\quad i^* \quad} BP^q(S^0)$$

$$BP^{pq}(X_3) \xrightarrow{\quad i^* \quad} BP^{pq}(S^0)$$

are isomorphisms and

$$[S^0, S^q BP] = 0$$
$$[S^0, S^{pq} BP] = 0$$

so $BP^q(X_3) = 0$, $BP^{pq}(X_3) = 0$. Hence the composite

$$S^0 \longrightarrow X_3 \xrightarrow{\ h\ } BP \xrightarrow{\ d_0\ } S^q BP \vee S^{pq}$$

is zero, and so is $d_0 h$. For the sake of convenience I will call the spectra

$$C_3 \xleftarrow{\ d_2\ } C_2 \xleftarrow{\ d_1\ } C_1 \xleftarrow{\ d_0\ } C_0$$

Let me now nail my colours to the mast. From now on set $r = (p^2 - 1)q$.

Theorem 7.2. *Let $X_3 = S^0 \cup_{\gamma_1} e^{r-2}$. Then*

$$BP^*(X_3) \cong \pi_*(BP) \text{ on generator } h \text{ of degree } 0, \text{ restricting to the}$$
$$\text{generator of } S^0$$
$$\oplus \pi_*(BP) \text{ on generator } \ell \text{ of degree } r - 2, \text{ coming from the}$$
$$\text{generator of } S^{r-2}$$

The operation $\{d_2, d_1, d_0, h\}$ is defined, and is a coset of maps

$$S^2 X_3 \longrightarrow C_3$$

and it is $-2v_2^{p-3}\ell \mod (p, v_1)\ell$.

At this time we only prove part of the theorem. We have already noted that the facts about $BP^0(X_3)$ follow immediately from the cofibering

$$S^{r-3} \xrightarrow{\ \gamma_1\ } S^0 \longrightarrow X_3 \longrightarrow S^{r-2} \longrightarrow S^1$$

We first check that the operation is defined. As I stated above $d_0 h \simeq 0$. In fact, we showed

$$[X_3, S^q BP] \xrightarrow{\cong} [S^0, S^q BP] = 0$$

$$[X_3, S^{pq} BP] \xrightarrow{\cong} [S^0, S^{pq} BP] = 0$$

So we will certainly be able to form the Toda brackets

$$\{d_2, d_1, d_0\}, \quad \{d_1, d_0, h\}$$

The first is a set of maps $SC_0 \to C_3$ but all such maps are null-homotopic because $BP^*(BP) = 0$ in dimensions $\equiv -1$ mod q. Similarly, the second is a set of maps $SX_3 \to C_2$; but all such maps are null-homotopic because $BP^*(X_3) = 0$ in dimensions $\equiv -1$ mod q. So these brackets are zero mod zero. Therefore the quadruple Toda bracket

$$\{d_2, d_1, d_0, h\}$$

is defined, and it is a set of maps $S^2 X_3 \to C_3$.

Similarly, in the sphere S^0, $\{d_2, d_1, d_0, hi\}$ is defined, and it is a set of maps $S^2 \longrightarrow S^{(2p+2)q} BP$, i. e., it is zero mod zero.

Let us consider the indeterminacy of $\{d_2, d_1, d_0, h\}$. We can vary the null-homotopies $d_2 d_1 \simeq 0$, $d_1 d_0 \simeq 0$, $d_0 h \simeq 0$ by maps

$$SC_1 \longrightarrow C_3, \quad SC_0 \longrightarrow C_2, \quad X_3 \longrightarrow C_1$$

All such maps are null-homotopic, the first two because $BP^*(BP) = 0$ in dimensions $\equiv -1$ mod q, the third because $BP^*(X_3) = 0$ in dimensions $\equiv -1$ mod q. We can vary the null-homotopies

$$\{d_2, d_1, d_0\} \simeq 0, \quad \{d_1, d_0, h\} \simeq 0$$

by maps

$$S^2 C_0 \longrightarrow C_3, \quad S^2 X_3 \longrightarrow C_2.$$

Any map of the first sort is null-homotopic because $BP^*(BP) = 0$ in dimensions $\equiv -2$ mod q. Maps of the second sort may be non-zero, but

$$[S^r, C_2] \xrightarrow{j^*} [S^2 X_3, C_2]$$

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is an isomorphism. We conclude that the indeterminacy of our operation is precisely

$$d_2[S^r, C_2]j$$

I now seek to prove that this indeterminacy is contained in the set $(p, v_1)\ell$. First I observe that the ideal $(p, v_1) \subseteq \pi_*(BP)$ is closed under all primary operations. This is clear, because it is the kernel of an induced homomorphism

$$\pi_*(BP) = BP^*(S^{2p}) \longrightarrow BP^*(V(1)) = \frac{\pi_*(BP)}{(p, v_1)}$$

Next we observe that any element of $[S^r, C_2]$ has the form

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

with $a, b \in \pi_*(BP)$ of degrees $(p^2 - p - 3)q$, $(p^2 - 2p - 2)q$. So they must be of the form $a = v_1^p c$, $b = v_1 d$ with c, d of degree $(p - 3)(p + 1)q$. So they lie in (p, v_1), and so

$$d_2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} R_p & R_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

lies in (p, v_1). It remains only to evaluate the operation, and this will take a bit more work.

First perhaps, you would like me to give you a bit more information on the action of operations R_I on the coefficient ring $\pi_*(BP)$. We have

$$R_I(xy) = \sum_{J+K=I} (R_J x)(R_K y)$$

so it is sufficient to give R_I on the generators.

Lemma 7.3. *Let the generators v_i be defined by the formulae of Hazewinkel. Then*

$$\begin{aligned}
R_0 v_1 &= v_1 \\
R_1 v_1 &= p \\
R_I v_i &= 0 \ for \ |I| > 1 \\
R_0 v_2 &= v_2
\end{aligned}$$

$$
\begin{aligned}
R_1 v_2 &= -(p+1)v_1^p \\
R_i v_2 &\equiv 0 \; mod \; p^i v_i^{p+1-i} \; for \; 1 < i < p \\
R_p v_2 &\equiv 0 \; mod \; p^{p-1} v_1 \\
R_{p+1} v_2 &\equiv 0 \; mod \; p^p \\
R_{01} v_2 &= p \\
R_I v_2 &= 0 \; for \; |I| > p + 1 \\
R_1 v_3 &= -v_2^p \; mod \; (p, v_1) \\
R_p v_3 &\equiv 0 \; mod \; (p, v_1)
\end{aligned}
$$

Proof. The formulae of Hazewinkel are as follows. We write m_i for what is called m_{p^i-1} in [2], i. e., $m_i = \frac{[CP^{p^i-1}]}{p^i}$. Then

(i) $v_1 = pm_2$

(ii) $v_2 = pm_2 - v_1^p m_1$

(iii) $v_3 = pm_3 - v_1^{p^2} m_2 - v_2^p m_1$

Perhaps we should try to justify these formulae. First consider the homomorphism

$$
Q_{2n}(\pi_*(BP)) \longrightarrow Q_{2n}(H_*(BP))
$$

where Q_{2n} means the indecomposable quotient in dimension $2n$. It is known that its image is the subgroup of index p if $n = p^i - 1$ ([2, Lemma 8.10, p. 58]). So we must have

$$
v_i = \lambda p m_i \; \text{mod decomposables}
$$

where $\lambda = \frac{a}{b}$ with a and b prime to p, and we may as well normalise by taking $\lambda = 1$. This disposes of v_1. For v_2 we must have

$$
v_2 = pm_2 + \lambda v_1^{p+1}
$$

where λ is rational; and since $p^2 m_2 = [CP^{p^2-1}]$ is integral, we must have $p\lambda \in \mathbf{Z}_{(p)}$. Of course, we get an equally good generator by changing mod p. So we may as well set

$$
v_2 = pm_2 + \mu v_1^p m_1
$$

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where $\mu \in \mathbf{Z}_{(p)}$ and we want to determine μ mod p.

Consider now the Kronecker product $< R_p \otimes R_{p^2-p}, \psi t_3 >$. It must lie in $\pi_*(BP)$. But according to [2, Thm 16.1, p. 98], we have

$$\psi t_3 + m_1(\psi t_2)^p + m_2(\psi t_1)^{p^2}$$

$$= t_3 \otimes 1 + t_2 \otimes t_1^{p^2} + t_1 \otimes t_2^p + 1 \otimes t_3 + m_1 t_2^p \otimes 1 + m_1 t_1^p \otimes t_1^{p^2}$$

$$+ m_1 1 \otimes t_2^p + m_2 t_1^{p^2} \otimes 1 + m_2 1 \otimes t_1^{p^2}$$

Consider the pairing of these various terms with $R_p \otimes R_{p^2-p}$. On the right-hand side we get O. On the left-hand side, the term $m_2(\psi t_1)^{p^2}$ gives

$$m_2 \frac{p^2)!}{p!(p^2 - p)!} \equiv pm_2 \text{ mod } p^3 m_2.$$

We have

$$\psi t_2 = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1(t_1 \otimes t_1^{p-1} + \ldots + t_1^{p-1} \otimes t_i)$$

Therefore
$$< (\psi t_2)^p, R_p \otimes R_{p^2-p} > = (-1)^p v_1^p = -v_1^p$$

We conclude that $pm_2 - m_1 v_1^p$ lies in $\pi_*(BP)$. So we may as well choose it for v_2.

We now conclude that the coefficient of $t_1^p \otimes t_1^{p^2-p}$ in ψt_3 is $-v_2$ mod $p^3 m_2$. For $0 \le i < p$ the coefficient of $t_1^i \otimes t_1^{p^2-1}$ in ψt_3 is 0 mod $p^2 m_2$.

Let us now proceed similarly with the Kronecker product $< R_{p^2} \otimes R_{p^3-p^2}, \psi t_4 >$. The result is an element of $\pi_*(BP)$. By the same reference we have

$$\psi t_4 + m_1(\psi t_3)^p + m_2(\psi t_2)^{p^2} + m_3(\psi t_1)^{p^3} = \ldots,$$

where every term on the right-hand side yields 0 when paired with $R_{p^2} \otimes R_{p^3-p^2}$. The term $m_3(\psi t_1)^{p^3}$ gives

$$m_3 \frac{(p^3)!}{(p^2)!(p^3 - p^2)!} \equiv pm_3 \text{ mod } p^3 m_3.$$

The term $m_2(\psi t_2)^{p^2}$ gives $m_2(-v_1)^{p^2} = -m_2 v_1^{p^2}$. The term $m_1(\psi t_3)^p$ gives

$$m_1 \left((-v_2)^p \text{ mod } (p^2 m_2) \right).$$

We conclude that there is an element of $\pi_*(BP)$ of the form

$$pm_3 - m_2 v_1^{p^2} - m_1 a$$

where $a \equiv v_2^p \bmod (p^2 m_2)$.

By the same reference, the effect of the operation R_I on the generators m_i is

$$R_0 m_1 = m_1, R_1 m_1 = 1, R_I m_1 = 0 \text{ otherwise}$$

$$R_0 m_2 = m_2, R_p m_2 = m_1, R_{01} m_2 = 1, R_I m_2 = 0 \text{ otherwise}$$

$$R_0 m_3 = m_3, R_{p^2} m_3 = m_2, R_{0p} m_3 = m_1, R_{001} m_3 = 1, R_I m_3 = 0 \text{ otherwise.}$$

So we get

$$R_1 v_1 = R_1(pm_1) = pR_1(m_1) = p$$

$$R_1 v_2 = R_1 \left(pm_2 - \frac{v_1^{p+1}}{p} \right) = -\frac{p+1}{p}(R_1 v_1)v_1^p = -(p+1)v_1^p$$

$$R_i v_2 = R_i \left(pm_2 - \frac{v_1^{p+1}}{p} \right) = -\frac{(p+1)!}{i!(p+1-i)!}\frac{1}{p} p^i v_1^{p+1-i} \quad (i \neq p)$$

and for $1 < i < p$ the binomial coefficient contains a factor p.

$$R_p v_2 = R_p \left(pm_2 - \frac{v_1^{p+1}}{p} \right) = pm_1 - (p+1)\frac{1}{p}p^p v_1 = v_1 \bmod p^{p-1}v_1$$

and

$$R_{01} v_2 = R_{01} \left(pm_2 - \frac{v_1^{p+1}}{p} \right) = p \cdot 1$$

Now take

$$v_3 = pm_3 - m_2 v_1^{p^2} - m_1 a$$

where

$$a = v_2^p + (p^2 m_2)b$$

Apply R_1. We get

$$
\begin{aligned}
R_1 v_3 &= -m_2 p^2 p v_1^{p^2-1} - a - m_1 R_1 a \\
&= -(m_2 p^2)pv_1^{p^2-1} - v_2^2 - (p^2 m_2)b - m_1(R_1 v_2^p) - m_1(R_1 p^2 m_2)b \\
&\quad -m_1(p^2 m_2)R_1 b \\
&= -\left(pv_2 + v_1^{p+1}\right)pv_1^{p^2-1} - v_2 - \left(pv_2 + v_1^{p+1}\right)b + m_1 p(p+1)v_1 v_2^{p-1} \\
&\quad -0 - m_1 \left(pv_2 + v_1^{p+1}\right)R_1 b.
\end{aligned}
$$

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Now $R_1 v_3$ is certainly integral, and all the terms on the right are integrai except possibly $-\!-m_1 v_1^{p+1} R_1 b$. We conclude that $R_1 b$ must oft be divisible by p, and the last term is $v_1 \left(p v_2 + v_1^{p+1} \right) c$. Thus

$$R_1 v_3 \equiv -v_2^p \bmod (p, v_1^{p+1})$$

For reasons of degree, we have

$$R_p v_3 \in (v_1^2)$$

This proves Lemma 7.3. □

Now I have to explain how I compute the fourfold bracket. Let me begin by explaining it in a different way: γ_1 is constructed as the following composite (where $r = (p^2 - 1)q$),

Here D induces a non-zero homomorphism of BP^*, that is, it carries the generator to $v_3 g_2$. The other three maps induce the zero map of BP^*, but they give known elements in $\mathrm{Ext}^1_{BP^*(BP)}$. Therefore, the composite γ_1 corresponds to a calculable element of $\mathrm{Ext}^3_{BP^*(BP)}$, namely the Yoneda product.

Now I don't propose to do it exactly that way, but in a parallel way. That is, using a relation of the Peterson-Stein type, I will reduce the calculation of our tertiary operation in $X_3 = S^0 \cup_{\gamma_1} CS^{r-3}$ to the calculation of a secondary operation in $X_2 = S^0 \cup CS^{r-3} V(0)$. (The attaching map gives an element of Ext^2.) Using another relation of the Peterson-Stein type, I will reduce the calculation of the secondary operation in X_2 to the calculation of a primary operation in $X_1 = S^0 \cup CS^{r-3} V(1)$. (The attaching map then gives an element of Ext^1.) Finally, I calculate this primary operation using the known behaviour of i_2, D.

In practice, of course, I proceed from the known to the unknown, and calculate first in X_1, then in X_2, then in X_3.

In calculating with composite maps, we always use the following lemma

Lemma 7.4. (Verdier's Axiom) *Given a composition*

$$X \xrightarrow{\;f\;} Y \xrightarrow{\;g\;} Z \; ,$$

we can form the following diagram of cofiberings

Lemma 7.5. *Let $X_1 = S^0 \cup_{Di_2} CS^{r-3}V(1)$. Then*

$$BP^*(X_1) \cong \pi_*(BP) \text{ on a unique generator } h_1 \text{ of degree } 0$$

$$\text{restricting to the generator on } S^0$$

$$\oplus \frac{\pi_*(BP)}{(p, v_1)} \text{ on a generator } g_1 \text{ of degree } p^2 q \text{ coming}$$

$$\text{from the generator on } S^{r-2}V(1).$$

We have
$$d_0 h_1 = \left[\begin{array}{c} -v_2^{p-1} \\ 0 \end{array} \right]$$

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Proof. Take the diagram of Lemma 7.4 and make the following substitutions

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z$$
$$S^{r-3}V(1) \xrightarrow{\ i_2\ } S^{r-3}V(2) \xrightarrow{\ D\ } S^0$$

We obtain:

$$\frac{\pi_*(BP)}{(p,v_1,v_2)}$$
on generator
g_2 of degree
$(p^2+p+1)q$

$$\frac{\pi_*(BP)}{(p,v_1}$$
on generator
$\overline{g}_1$ of degree
$(p^2+p+1)q$

$$BP^*\left(S^{r-3}V(2)\right) \longleftarrow BP^*\left(S^{(p^2+p)q-2}V(1)\right)$$

$$BP^*(X_1) \xleftarrow{\ i^*\ } BP^*\left(S^{r-2}V(1)\right)$$

$$D_*$$

$$\frac{\pi_*(BP)}{(p,v_1)}$$
on generator
$\tilde{g}_1$ of degree
p^2q

$$BP^*(S^0)$$

$$\pi_*(BP)$$
on generator
of degree 0

$$BP^*\left(S^{r-2}V(2)\right)$$

Since i_2 induces the zero map of BP^*, Di_2 must also do so, and the exact sequence across the middle is short exact. Now it is clear that $BP^*(X_1)$ is as stated, except that we have to show the choice of h_1 is unique. Since there is no monomial $v_2^i v_3^j \ldots$ in $\frac{\pi_*(BP)}{(p,v_1)}$ of the relevant degree p^2q, this is

clear. The generator g_1 in $BP^*(X_1)$ comes from $BP^*\left(S^{r-2}V(1)\right)$, and since the right-hand column is the cofibering which constructs $V(2)$ we have

$$g_1 i = v_2 \bar{g}_1$$

Also h_1 maps to the generator in $BP^*(S^0)$ and then to v_3 in $BP^*\left(S^{r-3}V(2)\right)$; so

$$h_1 i = v_3 \bar{g}_1 \bmod (p, v_1, v_2).$$

But since $(p^2 + p + 1)q$ is not divisible by $p + 1$, no power $v_2^n \bar{g}_1$ lies in degree 0. Therefore

$$h_1 i = v_3 \bar{g}_1 \bmod (p, v_1),$$

i. e.,

$$h_1 i = v_3 \bar{g}_1.$$

Now in S^0 we have

$$\begin{bmatrix} R_1 \\ R_p \end{bmatrix} h_1 = 0$$

So in X_1

$$\begin{bmatrix} R_1 \\ R_p \end{bmatrix} h_1$$

must be a multiple of g_1. But on the summand $\frac{\pi_*(BP)}{(p,v_1)} g_1$, i^* is a monomorphism; so it is sufficient to calculate in

$$S^{(p^2+p)q-2}V(1)$$

We find

$$\begin{bmatrix} R_1 \\ R_p \end{bmatrix} h_1 i = \begin{bmatrix} R_1 \\ R_p \end{bmatrix} v_3 \bar{g}_1 = \begin{bmatrix} -v_2^p \\ 0 \end{bmatrix} \bar{g}_1 \bmod (p, v_1) \text{ (by 7.3)}$$

$$= \begin{bmatrix} -v_2^p \\ 0 \end{bmatrix} g_1 i$$

Therefore

$$\begin{bmatrix} R_1 \\ R_p \end{bmatrix} h_1 = \begin{bmatrix} -v_2^p \\ 0 \end{bmatrix} g_1$$

$\square$

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To get any further we need Peterson-Stein relations. For the first one we suppose again we have maps

$$C_0 \xrightarrow{\ d_0\ } C_1 \xrightarrow{\ d_1\ } C_2$$

Here d_0 and d_1 are in practice the maps specified above, but the precise spectra are not relevant. We interest ourselves in secondary operations of the form

$$\phi(\epsilon) = \{d_1, d_0, \epsilon\}$$

where $\epsilon : Y \to C_0$. We suppose we are given a cofibering

$$X \xrightarrow{\ i\ } Y \xrightarrow{\ j\ } Z \xrightarrow{\ k\ } SX \xrightarrow{\ Si\ } SY$$

with maps $\zeta : Z \to C_0$ (below Z) and $\xi : SX \to C_1$ (below SX), $C_0 \xrightarrow{\ d_0\ } C_1$.

and maps $\zeta : Z \to C_0$, $\xi : SX \to C_1$ such that $\xi k = d_0 \zeta$. Given ζ, two such ξ differ by an element of

$$[SY, C_1] Si$$

We have $d_0 \zeta j \simeq \xi k j \simeq 0$ so

$$\{d_1, d_0, \zeta j\} \in \frac{[SY, C_2]}{d_1 [SY, C_1] + [SC_0, C_2](S\zeta)(Sj)}$$

and

$$\{d_1, d_0, \zeta j\} Si \in \frac{[SX, C_2]}{d_1 [SY, C_1] Si}$$

Lemma 7.6. $\{d_1, d_0, \zeta j\} Si = -d_1 \xi \mod d_1 [SY, C_1] Si.$

Proof. This is simply the familiar identity

$$\{d_1, d_0, \zeta j\} Si = -d_1 \{d_0, \zeta ij, i\} \supseteq -d_1 \{d_0 \zeta, j, i\}$$

(The latter is actually an equality sign because the indeterminacies are the

same.) In fact, the prescription for $\{d_0\zeta, j, i\}$ is that you take

$$SX \xrightarrow{\ i\ } Z \cup_j CY \xrightarrow{\ \overline{d_0\zeta}\ } C_1$$

$$\|$$

$$(Y \cup_i CX) \cup CY$$

$$\Big\downarrow \simeq$$

$$SX$$

and the identity map qualities as i. $\square$

Lemma 7.7. *Let* $X_2 = S^0 \cup_{Di_2i_1} CS^{r-3}V(0)$. *Then*

$$BP^*(X_2) \cong \pi_*(BP) \text{ on a generator } h_2 \text{ of degree 0 restricting}$$

$$\text{to the generator on } S^0$$

$$\oplus \frac{\pi_*(BP)}{(p)} \text{ on a generator } g_0 \text{ coming}$$

$$\text{from the generator in } S^{r-2}V(1)$$

The operation $\{d_1, d_0, h_2\}$ *is defined and it is a coset of maps* $SX_2 \to C_2$ *and it is*

$$\begin{bmatrix} 2v_1^p v_2^{p-3} g_0 \\ 2v_1 v_2^{p-3} g_0 \end{bmatrix} \mod d_1[SX_2, C_1].$$

N.B. I want to leave the indeterminacy in this form because will disappear at the next step.

Exercise. Show directly that the value given does not lie in the indeterminacy.

Proof of 7.7. We consider Di_2i_1 as the composite $(Di_2)i_1$. We take the diagram of Lemma 7.4 and substitute

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\
\| & & \| & & \| \\
S^{r-3}V(0) & \xrightarrow{\ i_1\ } & S^{r-3}V(1) & \xrightarrow{\ Di_2\ } & S^0
\end{array}
$$

We get

$$\frac{\pi_*(BP)}{9p)}$$
on generator $\overline{g}_0$
of degree $p^2q - 1$

$$\|$$

$$BP^*\left(S^{p^2q-2}V(0)\right)$$

i_*

$$BP^*(X_2) \longleftarrow BP^*\left(S^{r-2}V(0)\right)$$

$$\|$$ $$\|$$

$$\pi_*(BP) \oplus \frac{\pi_*(BP)}{(p)}$$ $$\frac{\pi_*(BP)}{(p)}$$
on generators h_2, g_0 on generator g_0
of degrees 0, $r-1$ of degree $r-1$

$$BP^*(S^0)$$ j^* $$BP^*(S^1)$$

$$\pi_*(BP) \oplus \frac{\pi_*(BP)}{(p,v_1)}$$ $$\frac{\pi_*(BP)}{(p,v_1)}$$
on generators h_1, g_1 on generator g_1
of degrees 0, p^2q of degree p^2q

$$\|$$ $$\|$$

$$BP^*(X_1) \longleftarrow BP^*\left(S^{r-2}V(1)\right)$$

k^*

$$BP^*\left(S^{p^2q-1}V(0)\right)$$

$$\|$$

$$\frac{\pi_*(BP)}{(p)}$$
on generator $\overline{g}_0$
of degree p^2q

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Certainly the map of BP^* induced by Di_2i_1 is zero, for those induced by i_2 and i_1 are zero. So we have a short exact sequence for $BP^*(X_2)$. We may take the generator h_1 in $BP^*(X_1)$ and map it into X_2; this gives an element restricting to the generator in $BP^*(S^0)$; we call this element h_2. Now it is clear that the structure of $BP^*(X_2)$ is as stated.

The generator $\overline{g}_0$ at the bottom of the diagram maps g_1 in $S^{r-2}V(1)$ and to g_1 in X_1. That is $\overline{g}_0 h_1 = g_1$

By 7.5, the map

$$X_1 \xrightarrow{d_0 h_1} C_1$$

is

$$\begin{bmatrix} -v_2^{p-1} g_1 \\ 0 \end{bmatrix}$$

and it is now clear that it factors through k; in fact, it is ξk, where

$$\xi = \begin{bmatrix} -v_2^{p-1} \overline{g}_0 \\ 0 \end{bmatrix}$$

So

$$d_0 h_2 = d_0 h_1 j = \xi k j = 0,$$

and the bracket $\{d_1, d_0, h_2\}$ is defined. We must check its indeterminacy. Since there is no way to change the homotopy $d_1 d_0 \simeq 0$ the indeterminacy is $d_1 [SX_2, C_1]$.

We now proceed to apply Lemma 7.6 taking the cofibering in that lemma to be the one marked with i^*, j^*, k^* in the previous diagram. We take $\zeta = h_1$ and ξ as above. The lemma gives

$$
\begin{aligned}
\{d_1, d_0, h_2\} Si \;&=\; -d_1 \xi \bmod d_1 [SX_2, C_1] Si \\
&=\; \begin{bmatrix} R_p R_1 - 2R_1 R_p & R_1^2 \\ R_p^2 & -2R_p R_1 + R_1 R_p \end{bmatrix} \begin{bmatrix} -v_2^{p-1} \overline{g}_0 \\ 0 \end{bmatrix} \bmod d_1 [SX_2, C_1] Si
\end{aligned}
$$

Now

$$R_p R_1 - 2R_1 R_p = -R_{01} - R_1 R_p,$$

and

$$R_{01} v_2^{p-1} \overline{g}_0 = 0 \bmod p$$

$$R_p v_2^{p-1} \overline{g}_0 = (p-1) v_1 v_2^{p-2} \overline{g}_0 = -v_1 v_2^{p-2} \overline{g}_0 \bmod p$$

$$\begin{aligned}
-R_1 R_p v_2^{p-1} \overline{g}_0 &= R_1 (v_1 v_2^{p-2} \overline{g}_0 \bmod p \\
&= v_1 (p-2)(-1) v_1^p v_2^{p-3} \overline{g}_0 \bmod p \\
&= 2 v_1^{p+1} v_2^{p-3} \overline{g}_0 \bmod p
\end{aligned}$$

Similarly

$$\begin{aligned}
R_p v_2^{p-1} \overline{g}_0 &= -v_1 v_2^{p-2} \bmod p \text{ (as above)} \\
R_p^2 (v_2^{p-1} \overline{g}_0) &= -v_1 (p-2) v_1 v_2^{p-3} \overline{g}_0 \bmod p \\
&= 2 v_1^2 v_2^{p-3} \overline{g}_0 \qquad \bmod p
\end{aligned}$$

Hence we obtain

$$\{d_1, d_0, h_2\} Si = \left[\begin{array}{c} 2v_1^{p+1} v_2^{p-3} \overline{g}_0 \\ 2v_1^2 v_2^{p-3} \overline{g}_0 \end{array} \right] \bmod d_1 [SX_2, C_1] Si.$$

Now from the commutativity of the upper triangle we see that $g_0 i = \overline{g}_0$; so we obtain

$$\{d_1, d_0, h_2\} Si = \left[\begin{array}{c} 2v_1^p v_2^{p-3} g_0 \\ 2v_1 v_2^{p-3} g_0 \end{array} \right] \bmod d_1 [SX_2, C_1] Si.$$

The injection $S^0 \to X_2$ clearly annihilates $\{d_1, d_0, h_2\}$ (maps $S^1 \to C_2$ are zero), so it annihilates the indeterminacy $d_1 [SX_2, C_1]$, and it clearly annihilates

$$\left[\begin{array}{c} 2v_1^p v_2^{p-3} g_0 \\ 2v_1 v_2^{p-3} g_0 \end{array} \right]$$

But on the complementary summand $\frac{\pi_*(BP)}{(p)}$ $(Si)^*$ is mono, so

$$\{d_1, d_0, h_2\} = \left[\begin{array}{c} 2v_1^p v_2^{p-3} g_0 \\ 2v_1 v_2^{p-3} g_0 \end{array} \right] \bmod d_1 [SX_2, C_1].$$

This proves 7.7 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

For the second Peterson-Stein relation, suppose we are given maps

$$C_0 \xrightarrow{\ d_0\ } C_1 \xrightarrow{\ d_1\ } C_2 \xrightarrow{\ d_2\ } C_3$$

with $d_2 d_1 \simeq 0$, $d_1 d_0 \simeq 0$ (e.g. as above). We interest ourselves in the tertiary operations of the form $\{d_2, d_1, d_0, \epsilon\}$ where $\epsilon : Y \to C_0$, so we will have to give ourselves the sort of data which ensure that such operations are defined.

We suppose given 0. cofibering

$$X \xrightarrow{\ i\ } Y \xrightarrow{\ j\ } Z \xrightarrow{\ k\ } SX \xrightarrow{\ Si\ } SY$$

and two maps

$$
\begin{array}{ccc}
S^2 X & & Z \\
\big\downarrow{\scriptstyle \xi} & & \big\downarrow{\scriptstyle \zeta} \\
C_1 & & C_0
\end{array}
$$

We suppose $d_0 \zeta \simeq 0$. We also pick a homotopy $d_1 d_0 \simeq 0$; in the applications there is only one way to pick it. We suppose that for this homotopy we have simultaneously

(i) $\{d_2, d_1, d_0\} = 0 \bmod [SC_1, C_3] S d_0$

(ii) $\{d_1, d_0, \zeta\} = 0$ in $[SZ, C_2]/d_1[SZ, C_1]$,

We pick a homotopy $d_2 d_1 \simeq 0$ which, gives 0 as a representative map for $\{d_2, d_1, d_0\}$. After this we are only willing to change it by a map $\theta \in [SC_1, C_3]$ such that $\theta S d_0 \simeq 0$. We also pick a homotopy $d_0 \zeta \simeq 0$ which gives $\zeta S k$ as the representative map for $\{d_1, d_0, \zeta\}$. After this we are willing to vary this homotopy by an element $\phi \in [SZ, C_1]$ such that $d_1 \phi \simeq 0$.

Then $\{d_1, d_0, \zeta j\}$ is represented by $\xi \cdot Sk \cdot Sj = 0$, and we can form

$$
\begin{aligned}
\{d_2, d_1, d_0, \zeta\} \ \in\ & [S^2 Y, C_3]/ \big(d_2[S^2 C_0, C_2] + [S^2 C_0, C_2] S^2 \zeta \cdot S^2 j \\
& + \{\{d_2, d_1, \phi S j\}|\phi \in [SZ, C_1], d_1 \phi \simeq 0\} \\
& + \{\{\theta, S d_0, S\zeta \cdot S j\}|\theta \in [SC_1, C_3], \theta S d_0 \simeq 0\} \\
& + \{\theta' \cdot S\phi' \cdot S^2 j|\theta' S d_0 \simeq 0, d_1 \phi' \simeq 0\}\big)
\end{aligned}
$$

(here we must also include any indeterminacy implied by the Toda brackets which is not already accounted for). Then we may as well write $\{d_2, d_1, \phi\} S^2 j$

for $\{d_2, d_1, \phi \cdot Sj\}$ and $\{\theta, Sd_0, S\zeta\}S^2j$ for $\{\theta, Sd_0, S\zeta \cdot Sj\}$ because these are defined, are contained in what we wrote before, and the larger indeterminacy is contained in what we have already written.

In the applications, of course, the indeterminacy reduces to $d_2[S^2Y, SC_2]$, as we have seen.

In general $\{d_2, d_1, d_0, \zeta j\}S^2 i$ lies in $[S^2 X, C_3]/d_2[S^2Y, SC_2]S^2 i$.

Lemma 7.8.

$$\{d_2, d_1, d_0, \zeta j\}S^2 i = -d_2\xi : S^2 X \to C_3 \text{ mod } d_2[S^2Y, SC_2]S^2 i.$$

N.B. We do not allow ourselves to vary ξ, because the indeterminacy of the quadruple bracket when we vary everything in sight may be greater than what we have written.

Proof. In Lemma 7.6 we substitute

$$
\begin{array}{ll}
C_2 \mapsto C_3 & d_1 \mapsto d_2 \\
C_1 \mapsto C_2 & d_0 \mapsto \overline{d}_1 \\
C_0 \mapsto C_1 \cup_{d_0} CC_0 & \text{(the choice of null-homotopy } d_1 d_0 \simeq 0 \\
& \text{gives a specific extension of } d_1\text{)}
\end{array}
$$

Now the old condition $d_1 d_0 \simeq 0$ becomes the two new conditions $d_2 d_1 \simeq 0$ and $\{d_2, d_1, d_0\} = 0$ mod $[SC_1, C_3]Sd_0$. For the cofibering in 7.6, we substitute

$$SX \xrightarrow{Si} SY \xrightarrow{Sj} SZ \xrightarrow{Sk} S^2 X \xrightarrow{S^2 i} S^2 Y$$
$$\downarrow{\underline{\zeta}} \qquad \downarrow{\xi}$$
$$C_1 \cup_{d_0} CC_0 \xrightarrow{d_1} C_2$$

and for the maps, ξ and a coextension $\underline{\zeta}$ of $\zeta : Z \to C_0$. Here the composite

$$SZ \xrightarrow{\underline{\zeta}} C_1 \cup_{d_0} CC_0 \xrightarrow{d_1} C_1$$

of course represents $\{d_1, d_0, \zeta\}$, and we choose the coextension $\underline{\zeta}$ so that we actually get the element $\xi \cdot Sk$ in $[SZ, C_2]$. Then the diagram is commutative. The triple bracket $\{d_1, d_0, \underline{\zeta} \cdot Sj\}$ is now an alternative construction

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for $\{d_2, d_1, d_0, \zeta j\}$ and if you don't like quadruple brackets, take this as the definition. Applying Lemma 7.6, we come out with

$$\{d_2, d_1, d_0, \zeta j\} = -d_2\xi \mod d_2[S^2Y, C_2]S^2i.$$

$\square$

Proof of Theorem 7. 2 completed. We continue the process exhibited in 7.5, 7.7 applying Lemma 7.8. We consider $Di_2i_1i_0$ as the composite $(Di_2i_1)i_0$. We take the diagram of Lemma 7.4, and substitute

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\
\| & & \| & & \| \\
S^{r-3} & \xrightarrow{\ i_0\ } & S^{r-3}V(0) & \xrightarrow{Di_2i_1} & S^0
\end{array}
$$

We obtain

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We need some induced maps. The right-hand vertical column is induced by the cofibering

$$S^0 \xrightarrow{\ p\ } S^0 \longrightarrow V(0),$$

so $\bar{\ell}$ at bottom maps to g_0, and ℓ maps to $p\ell$. We now proceed to apply Lemma 7. 8 to the cofibering marked i^*, j^*, k^*. We take ζh_0 to be $h_2 : X_2 \to C_0$. That $d_0 h_2 = 0$ was already proved in Lemma 7.7. We now wish to produce ξ.

Now Lemma 7.6 contains an indeterminacy $d_1[SX_2 C_1]$. But every element of $[SX_2, C_1]$ lifts to $[S^{r-1}V(0, C_1]$ by 7.7; and every element of $[S^{r-1}V(0, C_1]$ clearly lifts to $[S^r, C_1]$. Therefore every element of $d_1[SX_2, C_1]$ lies in $d_1[S^r, C_1]Sk$. We can also lift the representative

$$\begin{bmatrix} 2v_1^p v_2^{p-3} g_0 \\ 2v_1 v_2^{p-3} g_0 \end{bmatrix}$$

by taking

$$\xi = \begin{bmatrix} 2v_1^p v_2^{p-3}\bar{\ell} \\ 2v_1 v_2^{p-3}\bar{\ell} \end{bmatrix}$$

So we see that we can in fact lift any element of $\{d_1, d_0, h_2\}$ by a lift congruent to

$$\begin{bmatrix} 2v_1^p v_2^{p-3}\bar{\ell} \\ 2v_1 v_2^{p-3}\bar{\ell} \end{bmatrix} \ \bmod\ d_1[S^R, C_1]$$

Now Lemma 7.8 states that

$$\{d_2, d_1, d_0, h_3\}S^2 i = -d_2\xi : S^2 X_3 \to C_3 \bmod d_2[S^2 X_3, C_2]S^2 i.$$

Here $d_2[S^2 X_3, C_2]$ is in our case exactly the indeterminacy of $\{d_2, d_1, d_0, h_3\}$ and we have already shown that it is contained in $(p, v_1)\ell$. Therefore

$$d_2[S^2 X_3, C_2]S^2 i \subseteq (p^2, pv_1)\bar{\ell}.$$

As for $-d_2\xi$ consider first the term

$$-d_2 d_1[S^r, C_1].$$

It is 0, because $d_2 d_1 \simeq 0$. Consider next the term

$$-\begin{bmatrix} R_p & R_1 \end{bmatrix} \begin{bmatrix} 2v_1^p v_2^{p-3}\bar{\ell} \\ 2v_1 v_2^{p-3}\bar{\ell} \end{bmatrix}$$

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We have
$$R_p\left(v_1^p v_2^{p-3}\right) = \sum_{i+j=p}(R_i v_1^p)(R_j v_2^{p-3}).$$

Here the first term
$$R_i v_1^p = \frac{p!}{i!(p-i)!}p^i v_i^{p-i}$$

lies in (p^2, pv_1) unless $i = 0$, in which case
$$R_p(v_2^{p-3}) = (p-3)v_1 v_2^{p-4} \bmod p^2$$

and we get
$$R_p\left(v_1^p v_2^{p-3}\right) = (p-3)v_1^{p+1} v_2^{p-4} \bmod (p^2, pv_1)$$

We have by 7.3

$$
\begin{aligned}
R_1(v_1 v_2^{p-3}) &= pv_2^{p-3} + v_1(p-3)(-(p+1))v_1^p v_2 \\
&= pv_2^{p-3} + (p-3)v_1^{p+1} v_2^{p-4} \bmod pv_1^{p+1}
\end{aligned}
$$

Totalling we get

$$-\begin{bmatrix} R_p & R_1 \end{bmatrix}\begin{bmatrix} 2v_1^p v_2^{p-3}\overline{\ell} \\ 2v_1 v_2^{p-3}\overline{\ell} \end{bmatrix} = -2pv_2^{p-3}\overline{\ell} \bmod (p^2, pv_1)$$

Since $\ell(S^2 i) = p\overline{\ell}$, we conclude that

$$\{d_2, d_1, d_0, h_3\} = -2v_2^{p-3}\ell \bmod (p, v_1).$$

This completes the proof, and shows that $\gamma_1 \neq 0$. □

Oka and Toda say they have not checked the assertion of Larry Smith that $\beta_p \neq 0$ and if this is false their proof collapses. So perhaps it will be as well to examine β_p. It turns out that by using an indirect method, we can avoid even mentioning secondary operations; we need some preliminary information first.

The symbol r is now freed for use, $q = 2(p-1)$.

Lemma 7.9. *Let X be a complex $X = S^0 \cup_f e^{rq}$ where $p^2 < r < p^2 + p$. Then*

$$BP^*(X) \; \cong \; \pi_*(BP) \; \text{on one generator } h \text{ of degree 0 restricting}$$
$$\text{to a generator on } S^0$$
$$\oplus \pi_*(BP) \; \text{on one generator } \ell \text{ of degree } rq \text{ coming from } S^{rq}$$

and

$$R_{p^2} h \equiv 0 \; mod \; p^p$$

(mod p suffices for what follows).

Proof. From the BP exact sequence of the cofibering, it is clear that we have a short exact sequence

$$0 \longrightarrow BP^*(S^{rq}) \longrightarrow BP^*(X) \longrightarrow BP^*(S^0) \longrightarrow 0$$

Everything that follows would be equally valid for a short exact sequence of modules over $BP^*(BP)$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & BP^*(S^{rq}) & \longrightarrow & M & \longrightarrow & BP^*(S^0) & \longrightarrow & 0 \\
& & \| & & & & \| & & \\
& & N & & & & L & &
\end{array}
$$

Such a module defines an element of $\mathrm{Ext}^i_{BP^*(BP)}(L, N)$. It is clear that the structure of M as a module over If $\pi_*(BP)$ is as stated in the lemma; if we choose a generator h, then the operations in M are given by a vector

$$
\begin{bmatrix} R_1 \\ R_p \\ R_{p^2} \end{bmatrix} h
$$

In $N^3 = N \oplus N \oplus N$. We may alter h to $h + a\ell$ ($a \in \pi_{rq}(BP)$); this alters

$$
\begin{bmatrix} R_1 \\ R_p \\ R_{p^2} \end{bmatrix} h
\qquad \text{to} \qquad
\begin{bmatrix} R_1 \\ R_p \\ R_{p^2} \end{bmatrix} h +
\begin{bmatrix} R_1 \\ R_p \\ R_{p^2} \end{bmatrix} a\ell
$$

We thus get a description of Ext^i as "cocycles" modulo "coboundaries" ; the cocycles are the vectors

$$\begin{bmatrix} R_1 \\ R_p \\ R_{p^2} \end{bmatrix} h$$

the coboundaries are the vectors

$$\begin{bmatrix} R_1 \\ R_p \\ R_{p^2} \end{bmatrix} a\ell$$

We first check that the result claimed does not depend on the choice of cocycle mod coboundaries. In fact, in the dimension considered a is a sum of monomials

$$v_1^i v_2^j \text{ with } j \leq p - 1$$

Consider $R_{p^2}(v_1^i v_2^j)\ell$ and expand it by the Cartan formula. We have $R_k v_2 = 0$ for $k > p + 1$ and $R_{p+1} v_2 = 0 \mod p^p$, which only leaves for consideration terms containing a factor $R_k(v_1^i)$ with $k \geq p$. Such a factor is $\equiv 0 \mod p^k$, i. e., mod p^p.

Now we have a choice of arguments. First, we assume it known that $\text{Ext}^i_{BP*(BP)}(\pi_*(BP), \pi_*(BP)))$ is in this dimension $\mathbf{Z}/p\mathbf{Z}$ generated by α_r. Then we can easily work out the operations in the corresponding complex. We apply 7.4 to the composite

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \\
\| & & \| & & \| \\
S^{rq-1} & \longrightarrow & S^{rq-1}V(0) & \longrightarrow & S^0
\end{array}
$$

We obtain the following diagram

$$\frac{\pi_*(BP)}{(p)} \qquad\qquad\qquad \pi_*(BP)$$
on generator g_0 on generator $\overline{\ell}$
of degree rq of degree rq

$$\|\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$BP^*\left(S^{rq-1}V(0)\right) \longleftarrow\qquad\qquad BP^*\left(S^{rq}\right)$$

$$i^* \nearrow$$

$$BP^*\left(S^0 \cup_{gf} e^{rq}\right) \longleftarrow\qquad\qquad BP^*\left(S^{rq}\right)$$

$$g^* \qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\pi_*(BP)$$
on generator ℓ
of degree rq

$$BP^*\left(S^0\right) \qquad\qquad BP^*\left(S^{rq}V(0)\right) \qquad BP^*\left(S^1\right)$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\pi_*(BP) \qquad\qquad\qquad\qquad\qquad \pi_*(BP)$$
on generator h on generator h
of degree 0 of degree 1
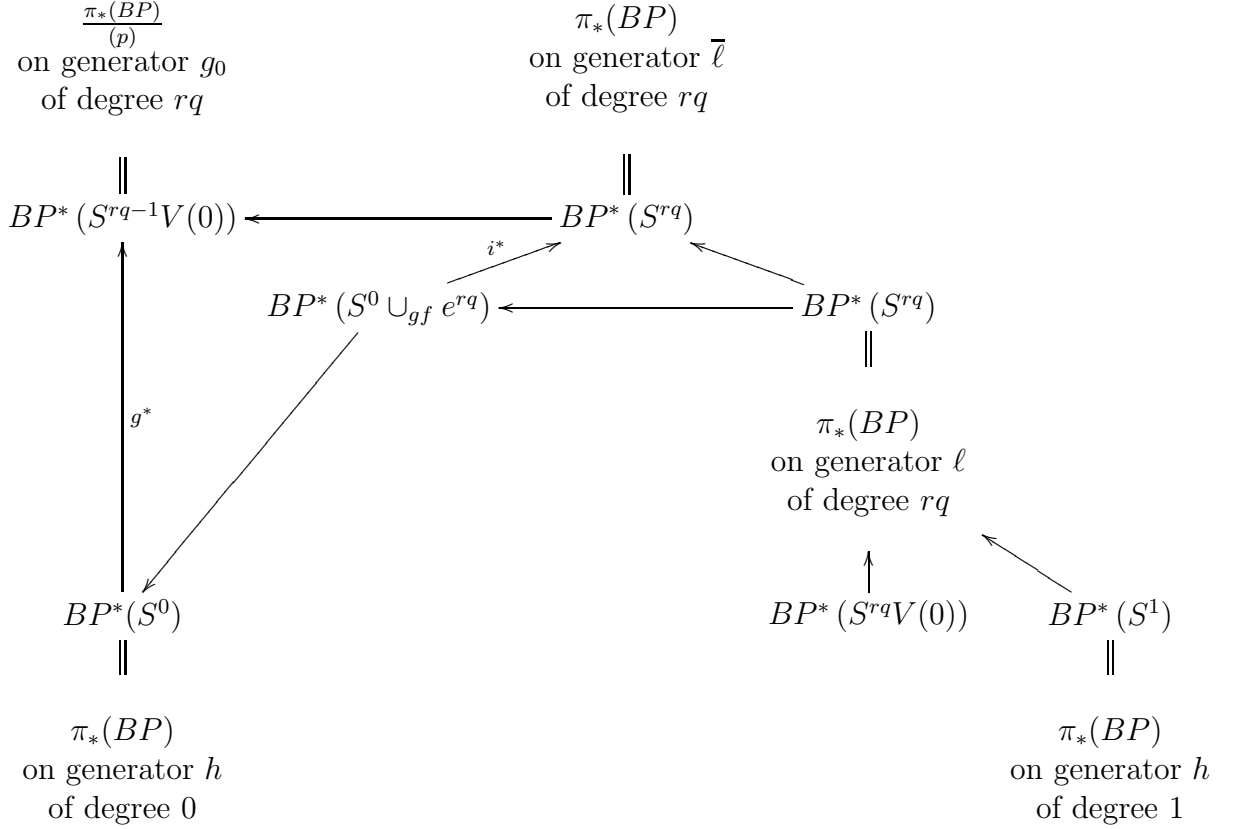
We see that

$$
\begin{aligned}
BP^*\left(S^0 \cup_{gf} e^{rq}\right) \;=\; & \pi_*(BP) \text{ on one generator } h \text{ of degree 0 restricting} \\
& \text{to a generator on } S^0 \\
& \oplus \pi_*(BP) \text{ on one generator } \ell \text{ of degree } rq \text{ coming} \\
& \text{from } S^{rq}
\end{aligned}
$$

as in the statement of the lemma. The map $S^{rq} \to S^{rq}$ is part of the cofibering defining $V(0)$, and has degree p, so ℓ maps to $p\overline{\ell}$.

Also g^* maps h to $v_1^r g_0$ (by construction); so i^* maps any choice of h to $v_1^r \overline{\ell} \bmod p$. But i^* maps ℓ to $p\overline{\ell}$, so we can change the choice of h to ensure

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that $hi = v_1^r\bar{\ell}$. Then

$$R_{p^2}hi = R_{p^2}v_1^r\bar{\ell} = cp^{p^2}v_1^{p-1}\bar{\ell} = cp^{p^2}v_1^{p-1}\ell i, \quad c \in \mathbf{Z}$$

Now $R_{p^2}h$ clearly maps to 0 on S^0 and on the complementary summand $\pi_*(BP)\ell$, i^* is mono; so with this choice of h we have

$$R_{p^2}h = cp^{p^2-1}v_1^{p-1}\ell.$$

Secondly, we can merely assume it known that $\mathrm{Ext}^i_{BP^*(BP)}(L, N)$ is finite. Let us denote

$$d_0 = \begin{bmatrix} R_1 \\ R_p \\ R_{p^2} \end{bmatrix}$$

Then in this case, for each cocycle d_0h there is an integer m such that md_0h is a coboundary $d_0(a\ell)$ where $a \in \pi_{qr}(BP)$. Now with $0 < j \le p-1$ the coboundary $d_0(v_1^i v_2^j \ell)$ is

$$\begin{bmatrix} -jv_1^{i+p}v_2^{j-1} & \mod p \\ jv_1^{i+1}v_2^{j-i} & \mod p^{p-1} \\ 0 & \mod p^p \end{bmatrix}\ell$$

while $d_0 * v_1^r\ell)$ is

$$\begin{bmatrix} prv_1^{r-1} & \\ 0 & \mod p^p \\ 0 & \mod p^{p^2} \end{bmatrix}\ell$$

Note $r \not\equiv 0 \mod p$.

It is now clear that if c, c_{ij} are rational coefficients, then the coboundary

$$d_0\left(cv_1^r\ell + \sum_{0<j\le p-1} c_{ij}v_1^i v_2^j\ell \right)$$

is integral only if the c_{ij} are integral (i. e.,lie in $\mathbf{Z}_{(p)}$) and pc is integral, say $c = \frac{m}{p}$. Therefore the most general cocycle is as obtained above.

Finally, you can explicitly obtain the cocycles as the kernel of d_1, and if you think this method is more reliable than the two already presented you are welcome to go ahead. $\quad\square$

Next I undertake to define an invariant defined on elements of order p in $\pi_{rq-2}(S^0)$, $p^2 < r < p^2 + p$. We know that β_p is an element of order p, because it extends over $S^{rq-2} \cup_p e^{rq-1} = S^{rq-2}V(0)$ and even over $S^{rq-2}V(1)$ by construction. Toda's elements ε_i of degree $(p^2 + i)q - 2$ $(1 \leq i \leq p-1)$ are also asserted to be of order p. So suppose given a map $f : S^{rq-2} \to S^0$ of order p, and extend it to $\overline{f} : S^{rq-2} \cup_p e^{rq-1} \longrightarrow S^0$. Form the mapping cone

$$X = S^0 \cup_{\overline{f}} C \left(S^{rq-2} \cup_p e^{rq-1} \right).$$

Then evidently we have a short exact sequence

$$0 \longrightarrow BP^* \left(S^{rq-2}V(0) \right) \longrightarrow BP^*(X) \longrightarrow BP^*(S^0) \longrightarrow 0$$

$$\pi_*(BP) \text{ on}$$
$$\text{generator } g_0$$
$$\text{of degree } qr$$

$$\frac{\pi_*(BP)}{(p)} \text{ on}$$
$$\text{generator } h$$
$$\text{of degree } 0$$

We take an element $h \in BP^0(X)$ projecting to the generator in $BP^*(S^0)$ and we consider $R_{p^2}h = ag_0$. Now changing h to $h + bg_0$ will of course change $R_{p^2}h$ to $R_{p^2}h + R_{p^2}bg_0$, but we have already checked that $R_{p^2}bg_0$ is zero mod p^p, and here we are working in $\frac{\pi_*(BP)}{(p)}$, so the resulting change is zero. We must also not forget that we can change the extension $\overline{f}$ of f by an element of $\pi_{qr-1}(S^0)$ but this changes our extension precisely by an element of the group $\text{Ext}^i_{BP^*(BP)}(L, N)$ which I so carefully discussed in Lemma 7.9, and by Lemma 7.9, the resulting change in $R_{p^2}h$ is zero mod p. This sets up our invariant.

It remains to calculate this invariant for β_p.

Theorem 7.10. *If $f = \beta_p$, we have*

$$R_{p^2}(h) = v_1^{p-1}g_0$$

and therefore $\beta_p \neq 0$.

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Proof. Factor β_p in the form

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z$$

$$\| \qquad\qquad \| \qquad\qquad \|$$

$$S^{rq-2}V(0) \qquad S^{rq-2}V(1) \qquad S^0$$

where $g^*(\text{generator}) = v_2^p g_1$. Apply 7.4. We obtain the following diagram

$$
\begin{array}{c}
\frac{\pi_*(BP)}{(p,v_1)} \\
\text{on generator } g_1 \\
\text{of degree} \\
(r+1)q
\end{array}
\qquad\qquad\qquad\qquad
\begin{array}{c}
\frac{\pi_*(BP)}{(p)} \\
\text{on generator } \overline{g}_0 \\
\text{of degree} \\
(r+1)q
\end{array}
$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\| $$

$$BP^*\left(S^{rq-2}V(1)\right) \longleftarrow\! BP^*\left(S^{(r+1)q-1}V(0)\right)$$

$$
BP^*\left(S^0 \cup_{gf} C\left(S^{rq-2}V(0)\right)\right) \xleftarrow{\ \ } BP^*\left(S^{rq-1}V(0)\right)
$$

$$\xrightarrow{\ i_*\ }$$

$$BP^*(S^0) \qquad\qquad\qquad\qquad\qquad
\begin{array}{c}
\frac{\pi_*(BP)}{(p)} \\
\text{on generator } \overline{g}_0 \\
\text{of degree} \\
(r+1)q
\end{array}
$$

The map $S^{rq-1}V(0) \longrightarrow S^{(r+1)q-1}V(0)$ comes from the cofibering defining $V(1)$ and therefore carries g_0 to $v_1\overline{g}_0$. So in our complex $S^0 \cup_{gf} C\left(S^{rq-2}V(0)\right)$ we have $g_0 i = v_1\overline{g}_0$.

Let us choose a generator h in $BP^*\left(S^0 \cup_{gf} C\left(S^{rq-2}V(0)\right)\right)$ so that it maps to a generator in S^0 and therefore to $v_2^p g_1$ in $S^{rq-2}V(1)$. So we must have

$$hi = v_2^p \overline{g}_0 \pmod{v_1}$$

But changing h by ag_0 changes hi by $v_1 a\overline{g}_0$ so we can choose the generator h so that it maps to $v_2^p \overline{g}_0$ exactly.

Now $R_{p^2}h$ clearly maps to zero on S^0. The complementary summand $\frac{\pi_*(BP)}{(p)}g_0$ maps monomorphically under i^*, so it is sufficient to calculate $R_{p^2}hi$. We find

$$
\begin{aligned}
R_{p^2}hi &= R_{p^2}v_2^p\overline{g}_0 \\
&= v_1^p\overline{g}_0 \bmod p^p \text{ at least} \\
&= v_1^{p-1}g_0i.
\end{aligned}
$$

We conclude that

$$R_{p^2}h = v_1^{p-1}g_0$$

in $\frac{\pi_*(BP)}{(p)}g_0$. $\qquad\square$

8 Epilogue

Thomas and Zahler published their proof that $\gamma_1 \neq 0$ in 1974, cf. [20]. Their proof uses a tertiary operation in Brown-Peterson cohomology to detect the nontriviality of the element. Oka and Toda tracked down the mistake in their original calculation and published an alternative proof of this result in 1975, cf. [15]. The methods they employ involve secondary compositions and extended powers of complexes. Later that year Miller, Ravenel and Wilson [11] announced a proof that $\gamma_t \neq 0$ for all $t > 0$ and all $p \geq 7$, using the Novikov spectral sequence.

In 1975 Bousfield published a proof of Theorem 4.6, cf. [5]. His proof uses simplicial methods and proves a stronger result. Bousfield constructs a strict functor on the category of simplicial sets which induces Adams' conjectured localisation upon passage to homotopy. However it should be noted that there is a very simple way of repairing Adams' original approach to localisation given in these notes, which we explain below.

We first introduce a new axiom on the class S of morphisms we are inverting.

Axiom 8.1. For each pair of objects X, Y in C there is a set of diagrams

$$\left\{ X \xrightarrow{f_\alpha} Z_\alpha \xleftarrow{s_\alpha} Y \right\}$$

with s_α in S such that for any diagram

$$X \xrightarrow{f} Z \xleftarrow{s} Y$$

with $s \in S$ we can find a morphism $t : Z_\alpha \longrightarrow Z$ in S so that the following diagram commutes

$$
\begin{array}{c}
Z_\alpha \\
f_\alpha \nearrow \quad \downarrow t \quad \nwarrow s_\alpha \\
X \xrightarrow{\ f\ } Z \xleftarrow{\ s\ } Y
\end{array}
\quad .
$$

Axiom 8.1 serves as a substitute for Axiom 3.4. Its virtue is that it is easily verified for the example of interest in these notes, in contrast to 3.4. We next prove the relevant revision of Theorem 3.8.

Theorem 8.2. *Let C be the category in which the objects are connected CW-complexes with basepoint and the maps are homotopy classes. Let S be a subclass of the morphisms of C, satisfying 3.1, 3.2, 3.3, 3.5, 3.6 and 8.1 (or alternatively 3.1, 3.5, 3.6, 4.1 and 8.1). Then S arises by 2.6 from a pair (E, η) satisfying 2.1 and 2.2.*

Proof. We first need to show that the quotient category $S^{-1}C$ is well defined, i. e. $[QX, QY]_{S^{-1}C}$ is a set for any pair of objects X, Y in C. We see this by showing that any morphism in $[QX, QY]_{S^{-1}C}$, represented by a diagram

$$X \xrightarrow{f} Z \xleftarrow{s} Y \; ,$$

in C has an equivalent representative

$$X \xrightarrow{f_\alpha} Z_\alpha \xleftarrow{s_\alpha} Y \; .$$

This is demonstrated by the following commutative diagram in C:

The rest of the proof is identical to that of Theorem 3.8. Briefly we use the Brown Representability Theorem to construct a right adjoint to $Q : C \to S^{-1}C$:

$$[QX, Y]_{S^{-1}C} \longleftrightarrow [X, RY]_C$$

and define E to be the composite

$$C \xrightarrow{Q} S^{-1}C \xrightarrow{R} C$$

$\square$

Finally we need to check that the class S of our main example satisfies 8.1. This is demonstrated by the following lemma, which is essentially Lemma 11.1 of [5].

Lemma 8.3. *Suppose that*

$$X \xrightarrow{f} Z \xleftarrow{s} Y$$

is a diagram in $\mathcal{C}$, the homotopy category of based connected CW complexes. Suppose that s induces an isomorphism with respect to a generalized homology theory K_*. Then there is a commutative diagram

such that s' and s'' induce isomorphisms with respect to K_* and such that

$$\#W \leq \max\{\#X, \#Y, \#K_*, \aleph_0\}.$$

Here $\#W$, $\#X$, $\#Y$ denote the cardinality of the sets of cells in these CW complexes and $\#K_*$ denotes the cardinality of $\oplus_{m\in\mathbf{Z}}K_m(pt)$.

Proof. Without loss of generality we may take f and s to be inclusions of complexes. We then construct an sequence of subcomplexes

$$W_0 \subset W_1 \subset W_2 \subset \ldots \subset Z$$

with the following properties:

(i) The inclusions $W_n \to Z$ induce epimorphisms in K_*.

(ii) $\ker(K_*(W_n) \to K_*(Z)) \subset \ker(K_*(W_n) \to K_*(W_{n+1}))$.

(iii) $\#W_n \leq \max\{\#X, \#Y, \#K_*, \aleph_0\}$.

We proceed by induction. We take $W_0 = X \vee Y$. Then (iii) is immediate and (i) follows from the fact that $Y \subset Z$ induces an isomorphism in K_*.

Having constructed W_n we note $K_*(Z) = \operatorname{colim}_\alpha K_*(Z_\alpha)$, where Z_α varies over all subcomplexes of Z obtained by attaching finitely many cells to W_n. Hence for each element $x \in \ker(K_*(W_n) \to K_*(Z))$ we can find such a subcomplex Z_{α_x} so that $x \in \ker(K_*(W_n) \to K_*(Z_{\alpha_x}))$. We define $W_{n+1} = \cup_x Z_{\alpha_x}$. Then (ii) follows from construction and (i) holds for W_{n+1}, since

it contains W_0. To check that (iii) holds for W_{n+1}, we first observe that it follows from the induction hypothesis and the Atiyah-Hirzebruch spectral sequence that $K_*(W_n)$, and hence also $\ker(K_*(W_n) \to K_*(Z))$, has cardinality bounded by $\max\{\#X, \#Y, \#K_*, \aleph_0\}$, which implies the same bound on $\#W_{n+1}$.

Finally we define $W = \cup_{n=1}^{\infty} W_n$. Since $K_*(W) = \text{colim}_n K_*(W_n)$, it follows from (ii) that $K_*(W) \to K_*(Z)$ is injective. Since W contains W_0, this map is also surjective, and hence an isomorphism. It follows that $Y \subset W$ is also an isomorphism in K_*. The bounds on $\#W_n$ imply the same bound on $\#W$. Thus the desired commutative diagram is provided by the inclusions $X \subset W \subset Z$ and $Y \subset W \subset Z$. $\qquad\square$

If we take S to be the class of morphisms in C which induce isomorphisms in K_*, as in Theorem 4.6, then Lemma 8.3 shows that S satisfies Axiom 8.1 (we may take the underlying sets of the Z_α in the axiom to be subsets of some fixed set of large enough cardinality). Thus Theorem 8.2 implies Theorem 4.6. Note that we do not need to prove that S satisfies Axiom 3.4. Indeed attempting to derive the proof of Theorem 4.6 from Axiom 3.4 appears to be a blind alley.

A natural follow up question, which Adams did not address in these notes, is whether there are localisations with respect to generalised cohomology theories. It is clear that if one takes S to be the class of morphisms which are inverted by a generalised cohomology theory, then S evidently satisfies Axioms 3.1, 3.5, 3.6, and 4.1. However the argument of Lemma 8.3 breaks down completely and there does not appear to be any alternative argument to show that $[QX, QY]_{S^{-1}C}$ is a set, and thus no apparent way to show the existence of localisation with respect to S. In [4] Bousfield shows that for ordinary cohomology theories (or more generally for anti-connective cohomology theories) such localisations exist because the class S is the same as the class of morphisms inverted by $H_*(-; A)$ for A a subring of $\mathbf{Q}$ or A a subgroup of $\oplus_{p \text{ prime}} \mathbf{Z}/p\mathbf{Z}$. In [8], Hovey similarly demonstrates the existence of localisations with respect to a larger class of generalised cohomology theories by constructing corresponding generalised homology theories with the same classes S of inverted morphisms. He moreover conjectures that for any generalised cohomology theory there is a corresponding generalised homology theory such that the class of morphisms S that they invert are the same. This conjecture is still open.

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