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Algebraic K-theory of topological K-theory

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Introduction

We are interested in the arithmetic of ring spectra.

To make sense of this we must work with structured ring spectra, such as S-algebras [EKMM], symmetric ring spectra [HSS] or Γ-rings [Ly]. We will refer to these as **S**-algebras. The commutative objects are then commutative **S**-algebras.

The category of rings is embedded in the category of **S**-algebras by the Eilenberg–Mac Lane functor $R \mapsto HR$. We may therefore view an **S**-algebra as a generalization of a ring in the algebraic sense. The added flexibility of **S**-algebras provides room for new examples and constructions, which may eventually also shed light upon the category of rings itself.

In algebraic number theory the arithmetic of the ring of integers in a number field is largely captured by its Picard group, its unit group and its Brauer group. These are

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in turn reflected in the algebraic K-theory of the ring of integers. Algebraic K-theory is defined also in the generality of **S**-algebras. We can thus view the algebraic K-theory of an **S**-algebra as a carrier of some of its arithmetic properties.

The algebraic K-theory of (connective) **S**-algebras can be closely approximated by diagrams built from the algebraic K-theory of rings [Du, §5]. Hence we expect that global structural properties enjoyed by algebraic K-theory as a functor of rings should also have an analogue for algebraic K-theory as a functor of **S**-algebras.

We have in mind, in particular, the étale descent property of algebraic K-theory conjectured by Lichtenbaum [Li] and Quillen [Qu2], which has been established for several classes of commutative rings [Vo], [RW], [HM2]. We are thus led to ask when a map of commutative **S**-algebras $A \to B$ should be considered as an étale covering with Galois group G. In such a situation we may further ask whether the natural map $K(A) \to K(B)^{hG}$ to the homotopy fixed-point spectrum for G acting on $K(B)$ induces an isomorphism on homotopy in sufficiently high degrees. These questions will be considered in more detail in [Ro3].

One aim of this line of inquiry is to find a conceptual description of the algebraic K-theory of the sphere spectrum, $K(S^0)=A(*)$, which coincides with Waldhausen's algebraic K-theory of the one-point space $*$. In [Ro2] the second author computed the mod 2 spectrum cohomology of $A(*)$ as a module over the Steenrod algebra, providing a very explicit description of this homotopy type. However, this result is achieved by indirect computation and comparison with topological cyclic homology, rather than by a structural property of the algebraic K-theory functor. What we are searching for here is a more memorable intrinsic explanation for the homotopy type appearing as the algebraic K-theory of an **S**-algebra.

More generally, for a simplicial group G with classifying space $X=BG$ there is an **S**-algebra $S^0[G]$, which can be thought of as a group ring over the sphere spectrum, and $K(S^0[G])=A(X)$ is Waldhausen's algebraic K-theory of the space X. When X has the homotopy type of a manifold, $A(X)$ carries information about the geometric topology of that manifold. Hence an étale descent description of $K(S^0[G])$ will be of significant interest in geometric topology, reaching beyond algebraic K-theory itself.

In the present paper we initiate a computational exploration of this 'brave new world' of ring spectra and their arithmetic.

Étale covers of chromatic localizations. We begin by considering some interesting examples of (pro-)étale coverings in the category of commutative **S**-algebras. For convenience we will choose to work locally, with **S**-algebras that are complete at a prime p. For the purpose of algebraic K-theory this is less of a restriction than it may seem at first. What we have in mind here is that the square diagram

$$\begin{array}{ccc} K(A) & \longrightarrow & K(A_p) \\ \downarrow & & \downarrow \\ K(\pi_0 A) & \longrightarrow & K(\pi_0 A_p) \end{array}$$

is homotopy Cartesian after p-adic completion [Du], when A is a connective $\mathbf{S}$-algebra, A_p its p-completion, $\pi_0 A$ its ring of path components and $\pi_0(A_p) \cong (\pi_0 A)_p$. This reduces the p-adic comparison of $K(A)$ and $K(A_p)$ to the p-adic comparison of $K(\pi_0 A)$ and $K(\pi_0 A_p)$, i.e., to a question about ordinary rings, which we view as a simpler question, or at least as one lying in better explored territory.

This leads us to study p-complete $\mathbf{S}$-algebras, or algebras over the p-complete sphere spectrum S^0_p. This spectrum is approximated in the category of commutative $\mathbf{S}$-algebras (or E_∞ ring spectra) by a tower of p-completed chromatic localizations [Ra1]

$$S^0_p \to \ldots \to L_n S^0_p \to \ldots \to L_1 S^0_p \to L_0 S^0_p = H\mathbf{Q}_p.$$

Here $L_n = L_{E(n)}$ is Bousfield's localization functor [Bou], [EKMM] with respect to the nth Johnson–Wilson theory with coefficient ring $E(n)_* = \mathbf{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$, and by $L_n S^0_p$ we mean $(L_n S^0)_p$. By the Hopkins–Ravenel chromatic convergence theorem [Ra3, §8], the natural map $S^0_p \to \operatorname{holim}_n L_n S^0_p$ is a homotopy equivalence. For each $n \geqslant 1$ there is a further map of commutative $\mathbf{S}$-algebras $L_n S^0_p \to L_{K(n)} S^0_p$ to the p-completed Bousfield localization with respect to the nth Morava K-theory with coefficient ring $K(n)_* = \mathbf{F}_p[v_n, v_n^{-1}]$. This is an equivalence for $n=1$, and $L_1 S^0_p \simeq L_{K(1)} S^0_p \simeq J_p$ is the non-connective p-complete image-of-J spectrum. See [Bou, §4].

There is a highly interesting sequence of commutative $\mathbf{S}$-algebras E_n constructed by Morava as spectra [Mo], by Hopkins and Miller [Re] as $\mathbf{S}$-algebras (or A_∞ ring spectra) and by Goerss and Hopkins [GH] as commutative $\mathbf{S}$-algebras (or E_∞ ring spectra). The coefficient ring of E_n is $(E_n)_* \cong W\mathbf{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$. As a special case $E_1 \simeq KU_p$ is the p-complete complex topological K-theory spectrum.

The cited authors also construct a group action on E_n through commutative $\mathbf{S}$-algebra maps, by a semidirect product $G_n = S_n \rtimes C_n$ where S_n is the nth (profinite) Morava stabilizer group [Mo] and $C_n = \operatorname{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p)$ is the cyclic group of order n. There is a homotopy equivalence $L_{K(n)} S^0_p \simeq E_n^{hG_n}$, where the homotopy fixed-point spectrum is formed in a continuous sense [DH], which reflects the Morava change-of-rings theorem [Mo].

Furthermore, the space of self-equivalences of E_n in the category of commutative $\mathbf{S}$-algebras is weakly equivalent to its group of path components, which is precisely G_n.

In fact the extension $L_{K(n)}S^0_p \to E_n$ qualifies as a pro-étale covering in the category of commutative $\mathbf{S}$-algebras, with Galois group weakly equivalent to G_n. The weak contractibility of each path component of the space of self-equivalences of E_n (over either S^0_p or $L_{K(n)}S^0_p$) serves as the commutative $\mathbf{S}$-algebra version of the unique lifting property for étale coverings. Also the natural map $\zeta\colon E_n \to THH(E_n)$ is a $K(n)$-equivalence, cf. [MS1, 5.1], implying that the space of relative Kähler differentials of E_n over $L_{K(n)}S^0_p$ is contractible. See [Ro3] for further discussion.

There are further étale coverings of E_n. For example there is one with coefficient ring $W\mathbf{F}_{p^m}[[u_1, \dots, u_{n-1}]][u, u^{-1}]$ for each multiple m of n. Let E_n^{nr} be the colimit of these, with $E^{nr}_{n*} = W\bar{\mathbf{F}}_p[[u_1, \dots, u_{n-1}]][u, u^{-1}]$. Then $\mathrm{Gal}(E_n^{nr}/L_{K(n)}S^0_p)$ is weakly equivalent to an extension of S_n by the profinite integers $\widehat{\mathbf{Z}} = \mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$. Let $\bar{E}_n$ be a maximal pro-étale covering of E_n, and thus of $L_{K(n)}S^0_p$. What is the absolute Galois group $\mathrm{Gal}(\bar{E}_n/L_{K(n)}S^0_p)$ of $L_{K(n)}S^0_p$?

In the case of Abelian Galois extensions of rings of integers in number fields, class field theory classifies these in terms of the ideal class group of the number field, which is basically K_0 of the given ring of integers. Optimistically, the algebraic K-theory of $\mathbf{S}$-algebras may likewise carry the corresponding invariants of a class theory for commutative $\mathbf{S}$-algebras. This gives us one motivation for considering algebraic K-theory.

Étale descent in algebraic K-theory. The p-complete chromatic tower of commutative $\mathbf{S}$-algebras induces a tower of algebraic K-theory spectra

$$K(S^0_p) \to \dots \to K(L_n S^0_p) \to \dots \to K(J_p) \to K(\mathbf{Q}_p)$$

studied in the p-local case by Waldhausen [Wa2]. The natural map

$$K(S^0_p) \to \operatorname*{holim}_n K(L_n S^0_p)$$

may well be an equivalence, see [MS2]. We are thus led to study the spectra $K(L_n S^0_p)$, and their relatives $K(L_{K(n)}S^0_p)$. (More precisely, Waldhausen studied finite localization functors L^f_n characterized by their behavior on finite CW-spectra. However, for $n=1$ the localization functors L_1 and L^f_1 agree, and this is the case that we will explore in the body of this paper. Hence we will suppress this distinction in the present discussion.)

Granting that $L_{K(n)}S^0_p \to E_n$ qualifies as an étale covering in the category of commutative $\mathbf{S}$-algebras, the descent question concerns whether the natural map

$$K(L_{K(n)}S^0_p) \to K(E_n)^{hG_n} \tag{0.1}$$

induces an isomorphism on homotopy in sufficiently high dimensions. We conjecture that it does so after being smashed with a finite p-local CW-spectrum of chromatic type $n+1$.

To analyze $K(E_n)$ we expect to use a localization sequence in algebraic K-theory to reduce to the algebraic K-theory of connective commutative $\mathbf{S}$-algebras, and to use the Bökstedt–Hsiang–Madsen cyclotomic trace map to topological cyclic homology to compute these [BHM]. The ring spectra E_n and $E(n)_p$ are closely related, and for $n\geqslant 1$ we expect that there is a cofiber sequence of spectra

$$K(BP\langle n-1\rangle_p)\to K(BP\langle n\rangle_p)\to K(E(n)_p) \tag{0.2}$$

analogous to the localization sequence $K(\mathbf{F}_p)\to K(\mathbf{Z}_p)\to K(\mathbf{Q}_p)$ in the case $n=0$. Something similar should work for E_n.

The cyclotomic trace map

$$\mathrm{trc}\colon K(BP\langle n\rangle_p)\to TC(BP\langle n\rangle_p;p)\simeq TC(BP\langle n\rangle;p)$$

induces a p-adic homotopy equivalence from the source to the connective cover of the target [HM1]. Hence a calculation of $TC(BP\langle n\rangle;p)$ is as good as a calculation of $K(BP\langle n\rangle_p)$, after p-adic completion. In this paper we present computational techniques which are well suited for calculating $TC(BP\langle n\rangle;p)$, at least when $BP\langle n\rangle_p$ is a commutative $\mathbf{S}$-algebra and the Smith–Toda complex $V(n)$ exists as a ring spectrum. In the algebraic case $n=0$, with $BP\langle 0\rangle=H\mathbf{Z}_{(p)}$, these techniques simultaneously provide a simplification of the argument in [BM1], [BM2] computing $TC(\mathbf{Z};p)$ and $K(\mathbf{Z}_p)$ for $p\geqslant 3$. Presumably the simplification is related to that appearing in different generality in [HM2].

It is also plausible that variations on these techniques can be made to apply when $V(n)$ is replaced by another finite type $n+1$ ring spectrum, and the desired commutative $\mathbf{S}$-algebra structure on $BP\langle n\rangle_p$ is weakened to the existence of an $\mathbf{S}$-algebra map from a related commutative $\mathbf{S}$-algebra, such as MU or BP.

Algebraic K-theory of topological K-theory. The first non-algebraic case occurs for $n=1$. Then $E_1\simeq KU_p$ has an action by $G_1=\mathbf{Z}_p^\times\cong\Gamma\times\Delta$. Here $\mathbf{Z}_p\cong\Gamma=1+p\mathbf{Z}_p\subset\mathbf{Z}_p^\times$, $\mathbf{Z}/(p-1)\cong\Delta\subset\mathbf{Z}_p^\times$ and $k\in\mathbf{Z}_p^\times$ acts on E_1 like the p-adic Adams operation ψ^k acts on KU_p.

Let $L_p=E_1^{h\Delta}$ be the p-complete Adams summand with coefficient ring $(L_p)_*=\mathbf{Z}_p[v_1,v_1^{-1}]$, so $L_p\simeq E(1)_p$. Then Γ acts continuously on L_p with $J_p\simeq L_p^{h\Gamma}$. Let l_p be the p-complete connective Adams summand with coefficient ring $(l_p)_*=\mathbf{Z}_p[v_1]$, so $l_p\simeq BP\langle 1\rangle_p$. We expect that there is a cofiber sequence of spectra

$$K(\mathbf{Z}_p)\to K(l_p)\to K(L_p).$$

The previous calculation of $TC(\mathbf{Z};p)$ [BM1], [BM2], and the calculation of $TC(l;p)$ presented in this paper, identify the p-adic completions of $K(\mathbf{Z}_p)$ and $K(l_p)$, respectively.

Given an evaluation of the transfer map between them, this presumably identifies $K(L_p)$. The homotopy fixed points for the Γ-action on $K(L_p)$ induced by the Adams operations ψ^k for $k\in 1+p\mathbf{Z}_p$ should then model $K(J_p)=K(L_1S_p^0)$.

This brings us to the contents of the present paper. In §1 we produce two useful classes λ_1^K and λ_2^K in the algebraic K-theory of l_p. In §2 we compute the $V(1)$-homotopy of the topological Hochschild homology of l, simplifying the argument of [MS1]. In §3 we present notation concerning topological cyclic homology and the cyclotomic trace map of [BHM]. In §4 we make preparatory calculations in the spectrum homology of the S^1-homotopy fixed points of $THH(l)$. These are applied in §5 to prove that the canonical map from the C_{p^n} fixed points to the C_{p^n} homotopy fixed points of $THH(l)$ induces an equivalence on $V(1)$-homotopy above dimension $2p-2$, using [Ts] to reduce to checking the case $n=1$. In §6 we inductively compute the $V(1)$-homotopy of all these (homotopy) fixed-point spectra, and their homotopy limit $TF(l;p)$. The action of the restriction map on this limit is then identified in §7. The pieces of the calculation are brought together in Theorem 8.4 of §8, yielding the following explicit computation of the $V(1)$-homotopy of $TC(l;p)$:

THEOREM 0.3. *Let $p\geqslant 5$. There is an isomorphism of $E(\lambda_1,\lambda_2)\otimes P(v_2)$-modules*

$$V(1)_*TC(l;p)\cong E(\lambda_1,\lambda_2,\partial)\otimes P(v_2)\oplus E(\lambda_2)\otimes P(v_2)\otimes \mathbf{F}_p\{\lambda_1 t^d \mid 0<d<p\}$$
$$\oplus E(\lambda_1)\otimes P(v_2)\otimes \mathbf{F}_p\{\lambda_2 t^{dp} \mid 0<d<p\}$$

with $|\lambda_1|=2p-1$, $|\lambda_2|=2p^2-1$, $|v_2|=2p^2-2$, $|\partial|=-1$ and $|t|=-2$.

The p-completed cyclotomic trace map

$$K(l_p)_p\to TC(l_p;p)\simeq TC(l;p)$$

identifies $K(l_p)_p$ with the connective cover of $TC(l;p)$. This yields the following expression for the $V(1)$-homotopy of $K(l_p)$, given in Theorem 9.1 of §9:

THEOREM 0.4. *Let $p\geqslant 5$. There is an exact sequence of $E(\lambda_1,\lambda_2)\otimes P(v_2)$-modules*

$$0\to\Sigma^{2p-3}\mathbf{F}_p\to V(1)_*K(l_p)\xrightarrow{\mathrm{trc}} V(1)_*TC(l;p)\to\Sigma^{-1}\mathbf{F}_p\to 0$$

taking the degree $2p-3$ generator in $\Sigma^{2p-3}\mathbf{F}_p$ to a class $a\in V(1)_{2p-3}K(l_p)$, and taking the class ∂ in $V(1)_{-1}TC(l;p)$ to the degree -1 generator in $\Sigma^{-1}\mathbf{F}_p$.

Chromatic red-shift. The $V(1)$-homotopy of any spectrum is a $P(v_2)$-module, but we emphasize that $V(1)_*TC(l;p)$ is a free finitely generated $P(v_2)$-module, and $V(1)_*K(l_p)$ is free and finitely generated except for the summand $\mathbf{F}_p\{a\}$ in degree $2p-3$. Hence both

$K(l_p)_p$ and $TC(l;p)$ are fp-spectra in the sense of [MR], with finitely presented mod p cohomology as a module over the Steenrod algebra. They both have fp-type 2, because $V(1)_*K(l_p)$ is infinite while $V(2)_*K(l_p)$ is finite, and similarly for $TC(l;p)$. In particular, $K(l_p)$ is closely related to elliptic cohomology.

More generally, at least if $BP\langle n\rangle_p$ is a commutative **S**-algebra and p is such that $V(n)$ exists as a ring spectrum, similar calculations to those presented in this paper show that $V(n)_*TC(BP\langle n\rangle;p)$ is a free $P(v_{n+1})$-module on $2^{n+2}+2^n(n+1)(p-1)$ generators. So algebraic K-theory takes such fp-type n commutative **S**-algebras to fp-type $n+1$ commutative **S**-algebras. If our ideas about localization sequences are correct then also $K(E_n)_p$ will be of fp-type $n+1$, and if étale descent holds in algebraic K-theory for $L_{K(n)}S^0_p \to E_n$ with $cd_p(G_n)<\infty$ then also $K(L_{K(n)}S^0_p)_p$ will be of fp-type $n+1$. The moral is that algebraic K-theory in many cases increases chromatic complexity by one, i.e., that it produces a constant red-shift of one in stable homotopy theory.

Notations and conventions. For an $\mathbf{F}_p$ vector space V let $E(V)$, $P(V)$ and $\Gamma(V)$ be the exterior algebra, polynomial algebra and divided power algebra on V, respectively. When V has a basis $\{x_1,\dots,x_n\}$ we write $E(x_1,\dots,x_n)$, $P(x_1,\dots,x_n)$ and $\Gamma(x_1,\dots,x_n)$ for these algebras. So $\Gamma(x)=\mathbf{F}_p\{\gamma_j(x)\mid j\geqslant 0\}$ with $\gamma_i(x)\cdot\gamma_j(x)=(i,j)\gamma_{i+j}(x)$. Let $P_h(x)=P(x)/(x^h=0)$ be the truncated polynomial algebra of height h. For $a\leqslant b\leqslant\infty$ let $P_a^b(x)=\mathbf{F}_p\{x^k\mid a\leqslant k\leqslant b\}$ as a $P(x)$-module.

By an infinite cycle in a spectral sequence we mean a class x such that $d^r(x)=0$ for all r. By a permanent cycle we mean an infinite cycle which is not a boundary, i.e., a class that survives to represent a nonzero class at E^∞. Differentials are often only given up to multiplication by a unit.

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1. Classes in algebraic K-theory

1.1. *E_∞ ring spectrum models.* Let p be an odd prime. Following the notation of [MS1], let $l=BP\langle 1\rangle$ be the Adams summand of p-local connective topological K-theory. Its homotopy groups are $l_*\cong\mathbf{Z}_{(p)}[v_1]$, with $|v_1|=q=2p-2$.

Its p-completion l_p with $l_{p*}\cong\mathbf{Z}_p[v_1]$ admits a model as an E_∞ ring spectrum, which can be constructed as the algebraic K-theory spectrum of a perfect field k'. Let g be a prime power topologically generating the p-adic units and let $k'=\operatorname{colim}_{n\geqslant 0}\mathbf{F}_{g^{p^n}}\subset\bar{k}$

be a $\mathbf{Z}_p$-extension of $k=\mathbf{F}_g$. Then $l_p=K(k')_p$ is an E_∞ ring spectrum model for the p-completed connective Adams summand [Qu1, p. 585].

Likewise $j_p=K(k)_p$ and $ku_p=K(\bar{k})_p$ are E_∞ ring spectrum models for the p-completed connective image-of-J spectrum and the p-completed connective topological K-theory spectrum, respectively. The Frobenius automorphism $\sigma_g(x)=x^g$ induces the Adams operation ψ^g on both l_p and ku_p. Then k is the fixed field of σ_g, and j_p is the connective cover of the homotopy fixed-point spectrum for ψ^g acting on either one of l_p or ku_p.

The E_∞ ring spectrum maps $S^0_p\to j_p\to l_p\to ku_p\to H\mathbf{Z}_p$ induce E_∞ ring spectrum maps on algebraic K-theory:

$$K(S^0_p)\to K(j_p)\to K(l_p)\to K(ku_p)\to K(\mathbf{Z}_p).$$

In particular, these are H_∞ ring spectrum maps [Ma].

1.2. *A first class in algebraic K-theory.* The Bökstedt trace map

$$\mathrm{tr}\colon K(\mathbf{Z}_p)\to THH(\mathbf{Z}_p)$$

maps onto the first p-torsion in the target, which is $THH_{2p-1}(\mathbf{Z}_p)\cong\mathbf{Z}/p\{e\}$ [BM1, 4.2]. Let $e^K\in K_{2p-1}(\mathbf{Z}_p)$ be a class with $\mathrm{tr}(e^K)=e$.

There is a $(2p-2)$-connected linearization map $l_p\to H\mathbf{Z}_p$ of E_∞ ring spectra, which induces a $(2p-1)$-connected map $K(l_p)\to K(\mathbf{Z}_p)$ [BM1, 10.9].

Definition 1.3. Let $\lambda_1^K\in K_{2p-1}(l_p)$ be a chosen class mapping to $e^K\in K_{2p-1}(\mathbf{Z}_p)$ under the map induced by linearization $l_p\to H\mathbf{Z}_p$.

The image $\mathrm{tr}(\lambda_1^K)\in THH_{2p-1}(l_p)$ of this class under the trace map

$$\mathrm{tr}\colon K(l_p)\to THH(l_p)$$

will map under linearization to $e\in THH_{2p-1}(\mathbf{Z}_p)$.

Remark 1.4. The class $\lambda_1^K\in K_{2p-1}(l_p)$ does not lift further back to $K_{2p-1}(S^0_p)$, since e^K has a nonzero image in π_{2p-2} of the homotopy fiber of $K(S^0_p)\to K(\mathbf{Z}_p)$ [Wa1]. Thus λ_1^K does not lift to $K_{2p-1}(j_p)$ either, because the map $S^0_p\to j_p$ is $(pq-2)$-connected. It is not clear if the induced action of ψ^g on $K(l_p)$ leaves λ_1^K invariant.

1.5. *Homotopy and homology operations.* For a spectrum X, let $D_pX=E\Sigma_p\ltimes_{\Sigma_p}X^{\wedge p}$ be its pth extended power. Part of the structure defining an H_∞ ring spectrum E is a map $\xi\colon D_pE\to E$. Then a mod p homotopy class $\theta\in\pi_m(D_pS^n;\mathbf{F}_p)$ determines a mod p homotopy operation

$$\theta^*\colon\pi_n(E)\to\pi_m(E;\mathbf{F}_p)$$

natural for maps of H_∞ ring spectra E. Its value $\theta^*(x)$ on the homotopy class x represented by a map $a: S^n \to E$ is the image of θ under the composite map

$$\pi_m(D_pS^n; \mathbf{F}_p) \xrightarrow{D_p(a)} \pi_m(D_pE; \mathbf{F}_p) \xrightarrow{\xi} \pi_m(E; \mathbf{F}_p).$$

Likewise the Hurewicz image $h(\theta) \in H_m(D_pS^n; \mathbf{F}_p)$ induces a homology operation

$$h(\theta)^*: H_n(E; \mathbf{F}_p) \to H_m(E; \mathbf{F}_p),$$

and the two operations are compatible under the Hurewicz homomorphisms.

For S^n with $n=2k-1$ an odd-dimensional sphere, the two lowest cells of D_pS^n are in dimensions $pn+(p-2)$ and $pn+(p-1)$, and are connected by a mod p Bockstein, cf. [Br2, 2.9(i)]. Hence the bottom two mod p homotopy classes of D_pS^n are in these two dimensions, and are called βP^k and P^k, respectively. Their Hurewicz images induce the Dyer–Lashof operations denoted βQ^k and Q^k in homology, cf. [Br2, 1.2].

For S^n with $n=2k$ an even-dimensional sphere, the lowest cell of D_pS^n is in dimension pn. The bottom homotopy class of D_pS^n is called P^k and induces the pth power operation $P^k(x)=x^p$ for $x \in \pi_{2k}(E)$. Its Hurewicz image is the Dyer–Lashof operation Q^k.

We shall make use of the following mod p homotopy Cartan formula.

LEMMA 1.6. *Let E be an H_∞ ring spectrum and let $x \in \pi_{2i}(E)$ and $y \in \pi_{2j-1}(E)$ be integral homotopy classes. Then*

$$(P^{i+j})^*(x \cdot y) = (P^i)^*(x) \cdot (P^j)^*(y)$$

in $\pi_{2p(i+j)-1}(E; \mathbf{F}_p)$. Here $(P^i)^*(x)=x^p$.

Proof. This is a lift of the Cartan formula for the mod p homology operation Q^{i+j} to mod p homotopy near the Hurewicz dimension. We use the notation in [Br1, §7]. Let $\delta: D_p(S^{2i} \wedge S^{2j-1}) \to D_pS^{2i} \wedge D_pS^{2j-1}$ be the canonical map. Then for $\alpha = P^{i+j} \in \pi_{2p(i+j)-1}(D_p(S^{2i} \wedge S^{2j-1}); \mathbf{F}_p)$ we have $\delta_*(\alpha) = P^i \wedge P^j$ in the image of the smash product pairing

$$\pi_{2pi} D_pS^{2i} \otimes \pi_{2pj-1}(D_pS^{2j-1}; \mathbf{F}_p) \xrightarrow{\wedge} \pi_{2p(i+j)-1}(D_pS^{2i} \wedge D_pS^{2j-1}; \mathbf{F}_p).$$

This is because the same relation holds in mod p homology, and the relevant mod p Hurewicz homomorphisms are isomorphisms in these degrees. The lemma then follows from [Br1, 7.3(v)]. □

1.7. *A second class in algebraic K-theory.* We use the H_∞ ring spectrum structure on $K(l_p)$ to produce a further element in its mod p homotopy.

Definition 1.8. Let $\lambda_2^K = (P^p)^*(\lambda_1^K) \in K_{2p^2-1}(l_p; \mathbf{F}_p)$ be the image under the mod p homotopy operation

$$(P^p)^*: K_{2p-1}(l_p) \to K_{2p^2-1}(l_p; \mathbf{F}_p)$$

of $\lambda_1^K \in K_{2p-1}(l_p)$.

Since the trace map tr: $K(l_p) \to THH(l_p)$ is an E_∞ ring spectrum map, it follows that $\mathrm{tr}(\lambda_2^K) \in THH_{2p^2-1}(l_p; \mathbf{F}_p)$ equals the image of $\mathrm{tr}(\lambda_1^K) \in THH_{2p-1}(l_p)$ under the mod p homotopy operation $(P^p)^*$. We shall identify this image in Proposition 2.8, and show that it is nonzero, which then proves that λ_2^K is nonzero.

Remark 1.9. It is not clear whether λ_2^K lifts to an integral homotopy class in $K_{2p^2-1}(l_p)$. The image of $e^K \in K_{2p-1}(\mathbf{Z}_p)$ in $K_{2p-1}(\mathbf{Q}_p; \mathbf{F}_p)$ is $v_1 d\log p$ for a class $d\log p \in K_1(\mathbf{Q}_p; \mathbf{F}_p)$ that maps to the generator of $K_0(\mathbf{F}_p; \mathbf{F}_p)$ in the K-theory localization sequence for $\mathbf{Z}_p$, cf. [HM2]. It appears that the image of λ_2^K in $V(1)_{2p^2-1}K(L_p)$ is $v_2 d\log v_1$ for a class $d\log v_1 \in V(1)_1 K(L_p)$ that maps to the generator of $V(0)_0 K(\mathbf{Z}_p)$ in the expected K-theory localization sequence for l_p. The classes λ_1^K and λ_2^K are therefore related to logarithmic differentials for poles at p and v_1, respectively, which partially motivates the choice of the letter 'λ'.

2. Topological Hochschild homology

Hereafter all spectra will be implicitly completed at p, without change in the notation.

The topological Hochschild homology functor $THH(-)$, as well as its refined versions $THH(-)^{C_{p^n}}$, $THH(-)^{hS^1}$, $TF(-;p)$, $TR(-;p)$ and $TC(-;p)$, preserve p-adic equivalences. Hence we will tend to write $THH(\mathbf{Z})$ and $THH(l)$ in place of $THH(\mathbf{Z}_p)$ and $THH(l_p)$, and similarly for the refined functors.

Algebraic K-theory does certainly not preserve p-adic equivalences, so we will continue to write $K(l_p)$ and $K(\mathbf{Z}_p)$ rather than $K(l)$ and $K(\mathbf{Z})$.

2.1. *Homology of* $THH(l)$. The ring spectrum map $l \to H\mathbf{F}_p$ induces an injection on mod p homology, identifying $H_*(l; \mathbf{F}_p)$ with the subalgebra

$$H_*(l; \mathbf{F}_p) = P(\bar{\xi}_k \mid k \geqslant 1) \otimes E(\bar{\tau}_k \mid k \geqslant 2)$$

of the dual Steenrod algebra A_*. Here $\bar{\xi}_k = \chi\xi_k$ and $\bar{\tau}_k = \chi\tau_k$, where ξ_k and τ_k are Milnor's generators for A_* and χ is the canonical involution. The degrees of these classes are $|\bar{\xi}_k| = 2p^k - 2$ and $|\bar{\tau}_k| = 2p^k - 1$.

There is a Bökstedt spectral sequence

$$E^2_{**} = HH_*(H_*(l; \mathbf{F}_p)) \implies H_*(THH(l); \mathbf{F}_p) \qquad (2.2)$$

with

$$E^2_{**} = H_*(l; \mathbf{F}_p) \otimes E(\sigma\bar{\xi}_k \mid k \geqslant 1) \otimes \Gamma(\sigma\bar{\tau}_k \mid k \geqslant 2).$$

See [HM1, §5.2]. Here $\sigma x \in HH_1(-)$ is represented by the cycle $1 \otimes x$ in degree 1 of the Hochschild complex. The inclusion of 0-simplices $l \to THH(l)$ and the S^1-action on $THH(l)$ yield a map $S^1_+ \wedge l \to THH(l)$, which when composed with the unique splitting of $S^1_+ \wedge l \to S^1 \wedge l \cong \Sigma l$ yields a map $\sigma\colon \Sigma l \to THH(l)$. The induced degree 1 map on homology takes x to σx.

By naturality with respect to the map $l \to H\mathbf{F}_p$, the differentials

$$d^{p-1}(\gamma_j(\sigma\bar{\tau}_k)) = \sigma\bar{\xi}_{k+1} \cdot \gamma_{j-p}(\sigma\bar{\tau}_k)$$

for $j \geqslant p$, found in the Bökstedt spectral sequence for $THH(\mathbf{F}_p)$, lift to the spectral sequence (2.2) above. See also [Hu]. Hence

$$E^p_{**} = H_*(l; \mathbf{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P_p(\sigma\bar{\tau}_k \mid k \geqslant 2),$$

and this equals the E^∞-term for bidegree reasons.

In $H_*(THH(l); \mathbf{F}_p)$ there are Dyer–Lashof operations acting, and $(\sigma\bar{\tau}_k)^p = Q^{p^k}(\sigma\bar{\tau}_k) = \sigma(Q^{p^k}(\bar{\tau}_k)) = \sigma\bar{\tau}_{k+1}$ for all $k \geqslant 2$ [St]. Thus as an $H_*(l; \mathbf{F}_p)$-algebra,

$$H_*(THH(l); \mathbf{F}_p) \cong H_*(l; \mathbf{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2). \tag{2.3}$$

Here $|\sigma\bar{\xi}_1| = 2p-1$, $|\sigma\bar{\xi}_2| = 2p^2-1$ and $|\sigma\bar{\tau}_2| = 2p^2$. Furthermore $Q^p(\sigma\bar{\xi}_1) = \sigma(Q^p(\bar{\xi}_1)) = \sigma\bar{\xi}_2$.

2.4. *$V(1)$-homotopy of $THH(l)$.* Let $V(n)$ be the nth Smith–Toda complex, with homology $H_*(V(n); \mathbf{F}_p) \cong E(\bar{\tau}_0, \ldots, \bar{\tau}_n)$. Thus $V(0)$ is the mod p Moore spectrum and $V(1)$ is the cofiber of the multiplication-by-v_1 map $\Sigma^q V(0) \to V(0)$, where $q = 2p-2$. There are cofiber sequences

$$S^0 \xrightarrow{p} S^0 \xrightarrow{i_0} V(0) \xrightarrow{j_0} S^1$$

and

$$\Sigma^q V(0) \xrightarrow{v_1} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0)$$

defining the maps labeled i_0, j_0, i_1 and j_1. When $p \geqslant 5$, $V(1)$ is a commutative ring spectrum [Ok].

For a spectrum X the rth (partially defined) v_1-Bockstein homomorphism $\beta_{1,r}$ is defined on the classes $x \in V(1)_*(X)$ with $j_1(x) \in V(0)_*(X)$ divisible by v_1^{r-1}. Then for $y \in V(0)_*(X)$ with $v_1^{r-1} \cdot y = j_1(x)$ let $\beta_{1,r}(x) = i_1(y) \in V(1)_*(X)$. So $\beta_{1,r}$ decreases degrees by $rq+1$.

Definition 2.5. Let $r(n)=0$ for $n\leqslant 0$, and let $r(n)=p^n+r(n-2)$ for all $n\geqslant 1$. Thus $r(2n-1)=p^{2n-1}+...+p$ (n odd powers of p) and $r(2n)=p^{2n}+...+p^2$ (n even powers of p). Note that $(p^2-1)r(2n-1)=p^{2n+1}-p$, while $(p^2-1)r(2n)=p^{2n+2}-p^2$.

PROPOSITION 2.6 (McClure–Staffeldt). *There is an algebra isomorphism*

$$V(1)_*THH(l)\cong E(\lambda_1,\lambda_2)\otimes P(\mu)$$

with $|\lambda_1|=2p-1$, $|\lambda_2|=2p^2-1$ *and* $|\mu|=2p^2$. *The* mod p *Hurewicz images of these classes are* $h(\lambda_1)=1\wedge\sigma\bar{\xi}_1$, $h(\lambda_2)=1\wedge\sigma\bar{\xi}_2$ *and* $h(\mu)=1\wedge\sigma\bar{\tau}_2-\bar{\tau}_0\wedge\sigma\bar{\xi}_2$. *There are* v_1*-Bocksteins* $\beta_{1,p}(\mu)=\lambda_1$, $\beta_{1,p^2}(\mu^p)=\lambda_2$ *and generally* $\beta_{1,r(n)}(\mu^{p^{n-1}})\neq 0$ *for* $n\geqslant 1$.

Proof. One proof proceeds as follows, leaving the v_1-Bockstein structure to the more detailed work of [MS1].

$H_*(THH(l);\mathbf{F}_p)$ is an A_*-comodule algebra over $H_*(l;\mathbf{F}_p)$. The A_*-coaction

$$\nu\colon H_*(THH(l);\mathbf{F}_p)\to A_*\otimes H_*(THH(l);\mathbf{F}_p)$$

agrees with the coproduct $\psi\colon A_*\to A_*\otimes A_*$ when both are restricted to the subalgebra $H_*(l;\mathbf{F}_p)\subset A_*$. Here

$$\psi(\bar{\xi}_k)=\sum_{i+j=k}\bar{\xi}_i\otimes\bar{\xi}_j^{p^i}\quad\text{and}\quad\psi(\bar{\tau}_k)=\sum_{i+j=k}\bar{\tau}_i\otimes\bar{\xi}_j^{p^i}+1\otimes\bar{\tau}_k.$$

Furthermore $\nu(\sigma x)=(1\otimes\sigma)\psi(x)$ and σ acts as a derivation. It follows that $\nu(\sigma\bar{\xi}_1)=1\otimes\sigma\bar{\xi}_1$, $\nu(\sigma\bar{\xi}_2)=1\otimes\sigma\bar{\xi}_2$ and $\nu(\sigma\bar{\tau}_2)=1\otimes\sigma\bar{\tau}_2+\bar{\tau}_0\otimes\sigma\bar{\xi}_2$.

Since $V(1)\wedge THH(l)$ is a module spectrum over $V(1)\wedge l\simeq H\mathbf{F}_p$, it is homotopy equivalent to a wedge of suspensions of $H\mathbf{F}_p$. Hence $V(1)_*THH(l)$ maps isomorphically to its Hurewicz image in

$$H_*(V(1)\wedge THH(l);\mathbf{F}_p)\cong A_*\otimes E(\sigma\bar{\xi}_1,\sigma\bar{\xi}_2)\otimes P(\sigma\bar{\tau}_2),$$

which consists of the primitive classes for the A_*-coaction. Let λ_1, λ_2 and μ in $V(1)_*THH(l)$ map to the primitive classes $1\wedge\sigma\bar{\xi}_1$, $1\wedge\sigma\bar{\xi}_2$ and $1\wedge\sigma\bar{\tau}_2-\bar{\tau}_0\wedge\sigma\bar{\xi}_2$, respectively. Then by a degree count, $V(1)_*THH(l)\cong E(\lambda_1,\lambda_2)\otimes P(\mu)$, as asserted. □

COROLLARY 2.7. $V(0)_tTHH(l)=0$ *and* $\pi_tTHH(l)=0$ *for all* $t\not\equiv 0,1$ mod $2p-2$, $t<2p^2+2p-2$.

Proof. This follows easily by a v_1-Bockstein spectral sequence argument applied to $V(1)_*THH(l)$ in low degrees. □

PROPOSITION 2.8. *The classes* $\lambda_1^K \in K_{2p-1}(l_p)$ *and* $\lambda_2^K \in K_{2p^2-1}(l_p; \mathbf{F}_p)$ *map under the trace map to integral and* mod p *lifts of*

$$\lambda_1 \in V(1)_{2p-1} THH(l) \quad \text{and} \quad \lambda_2 \in V(1)_{2p^2-1} THH(l),$$

respectively.

Proof. The Hurewicz and linearization maps

$$V(1)_{2p-1} THH(l) \to H_{2p-1}(V(1) \wedge THH(l); \mathbf{F}_p) \to H_{2p-1}(V(1) \wedge THH(\mathbf{Z}); \mathbf{F}_p)$$

are both injective. The mod p and v_1 reduction of the trace image $\mathrm{tr}(\lambda_1^K)$ and λ_1 are equal in $V(1)_{2p-1} THH(l)$, because both map to $1 \wedge \sigma\bar{\xi}_1$ in $H_{2p-1}(V(1) \wedge THH(\mathbf{Z}); \mathbf{F}_p)$.

The Hurewicz image in $H_{2p^2-1}(THH(l); \mathbf{F}_p)$ of $\mathrm{tr}(\lambda_2^K) = (P^p)^*(\mathrm{tr}(\lambda_1^K))$ equals the image of the homology operation Q^p on the Hurewicz image $\sigma\bar{\xi}_1$ of $\mathrm{tr}(\lambda_1^K)$ in $H_{2p-1}(THH(l); \mathbf{F}_p)$, which is $Q^p(\sigma\bar{\xi}_1) = \sigma Q^p(\bar{\xi}_1) = \sigma\bar{\xi}_2$. So the mod v_1 reduction of $\mathrm{tr}(\lambda_2^K)$ in $V(1)_{2p^2-1} THH(l)$ equals λ_2, since both classes have the same Hurewicz image $1 \wedge \sigma\bar{\xi}_2$ in $H_{2p^2-1}(V(1) \wedge THH(l); \mathbf{F}_p)$. □

3. Topological cyclotomy

We now review some terminology and notation concerning topological cyclic homology and the cyclotomic trace map. See [HM1] and [HM2] for more details.

3.1. *Frobenius, restriction, Verschiebung.* As already indicated, $THH(l)$ is an S^1-equivariant spectrum. Let $C_{p^n} \subset S^1$ be the cyclic group of order p^n. The Frobenius maps $F: THH(l)^{C_{p^n}} \to THH(l)^{C_{p^{n-1}}}$ are the usual inclusions of fixed-point spectra that forget part of the invariance. Their homotopy limit defines

$$TF(l;p) = \operatorname*{holim}_{n,F} THH(l)^{C_{p^n}}.$$

There are also restriction maps $R: THH(l)^{C_{p^n}} \to THH(l)^{C_{p^{n-1}}}$, defined using the cyclotomic structure of $THH(l)$, cf. [HM1]. They commute with the Frobenius maps, and thus induce a self-map $R: TF(l;p) \to TF(l;p)$. Its homotopy equalizer with the identity map defines the topological cyclic homology of l, which was introduced in [BHM]:

$$TC(l;p) \xrightarrow{\pi} TF(l;p) \underset{1}{\overset{R}{\rightrightarrows}} TF(l;p).$$

Hence there is a cofiber sequence

$$\Sigma^{-1} TF(l;p) \xrightarrow{\partial} TC(l;p) \xrightarrow{\pi} TF(l;p) \xrightarrow{1-R} TF(l;p),$$

which we shall use in §8 to compute $V(1)_*TC(l;p)$. There are also Verschiebung maps $V\colon THH(l)^{C_{p^{n-1}}} \to THH(l)^{C_{p^n}}$, defined up to homotopy in terms of the S^1-equivariant transfer.

3.2. *The cyclotomic trace map.* The Bökstedt trace map admits lifts

$$\mathrm{tr}_n\colon K(l_p) \to THH(l)^{C_{p^n}}$$

for all $n \geqslant 0$, with $\mathrm{tr} = \mathrm{tr}_0$, which commute with the Frobenius maps and homotopy commute with the restriction maps up to preferred homotopy. Hence the limiting map $\mathrm{tr}_F\colon K(l_p) \to TF(l;p)$ homotopy equalizes R and the identity map, and the resulting lift

$$\mathrm{trc}\colon K(l_p) \to TC(l;p)$$

is the Bökstedt–Hsiang–Madsen cyclotomic trace map [BHM].

3.3. *The norm-restriction sequences.* For each $n \geqslant 1$ there is a homotopy commutative diagram

$$\begin{array}{ccccc} & & K(l_p) & \xrightarrow{\mathrm{tr}_{n-1}} & \\ & & \downarrow{\scriptstyle \mathrm{tr}_n} & \searrow & \\ THH(l)_{hC_{p^n}} & \xrightarrow{N} & THH(l)^{C_{p^n}} & \xrightarrow{R} & THH(l)^{C_{p^{n-1}}} \\ \| & & \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle \hat{\Gamma}_n} \\ THH(l)_{hC_{p^n}} & \xrightarrow{N^h} & THH(l)^{hC_{p^n}} & \xrightarrow{R^h} & \hat{\mathbf{H}}(C_{p^n}, THH(l)). \end{array} \tag{3.4}$$

The lower part is the map of cofiber sequences that arises by smashing the S^1-equivariant cofiber sequence $ES^1_+ \to S^0 \to \widetilde{E}S^1$ with the S^1-equivariant map

$$THH(l) \to F(ES^1_+, THH(l))$$

and taking C_{p^n} fixed-point spectra. For closed subgroups $G \subseteq S^1$ recall that $THH(l)^{hG} = F(ES^1_+, THH(l))^G$ is the G homotopy fixed-point spectrum of $THH(l)$, and

$$\hat{\mathbf{H}}(G, THH(l)) = [\widetilde{E}S^1 \wedge F(ES^1_+, THH(l))]^G$$

is the G Tate construction on $THH(l)$. The remaining terms of the diagram are then identified by the canonical homotopy equivalences

$$THH(l)_{hC_{p^n}} \simeq [ES^1_+ \wedge THH(l)]^{C_{p^n}} \simeq [ES^1_+ \wedge F(ES^1_+, THH(l))]^{C_{p^n}}$$

and

$$THH(l)^{C_{p^{n-1}}} \simeq [\widetilde{E}S^1 \wedge THH(l)]^{C_{p^n}}.$$

(In each case there is a natural map which induces the equivalence.)

We call N, R, N^h and R^h the norm, restriction, homotopy norm and homotopy restriction maps, respectively. We call Γ_n and $\widehat{\Gamma}_n$ the canonical maps. The middle and lower cofiber sequences are the norm-restriction and homotopy norm-restriction sequences, respectively.

We shall later make particular use of the map

$$\widehat{\Gamma}_1 \colon THH(l) \simeq [\widetilde{E}S^1 \wedge THH(l)]^{C_p} \to [\widetilde{E}S^1 \wedge F(ES^1_+, THH(l))]^{C_p} = \widehat{\mathbf{H}}(C_p, THH(l)).$$

We note that $\widehat{\Gamma}_1$ is an S^1-equivariant map, and induces $\widehat{\Gamma}_{n+1} = (\widehat{\Gamma}_1)^{C_{p^n}}$ upon restriction to C_{p^n} fixed points.

By passage to homotopy limits over Frobenius maps we also obtain a limiting diagram

$$\begin{array}{ccccc} & & K(l_p) & & \\ & & \downarrow{\scriptstyle \mathrm{tr}_F} & \searrow^{\mathrm{tr}_F} & \\ \Sigma THH(l)_{hS^1} & \xrightarrow{N} & TF(l;p) & \xrightarrow{R} & TF(l;p) \\ \| & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \widehat{\Gamma}} \\ \Sigma THH(l)_{hS^1} & \xrightarrow{N^h} & THH(l)^{hS^1} & \xrightarrow{R^h} & \widehat{\mathbf{H}}(S^1, THH(l)). \end{array} \tag{3.5}$$

Implicit here are the canonical p-adic homotopy equivalences

$$\Sigma THH(l)_{hS^1} \simeq \operatorname*{holim}_{n,F} THH(l)_{hC_{p^n}},$$

$$THH(l)^{hS^1} \simeq \operatorname*{holim}_{n,F} THH(l)^{hC_{p^n}},$$

$$\widehat{\mathbf{H}}(S^1, THH(l)) \simeq \operatorname*{holim}_{n,F} \widehat{\mathbf{H}}(C_{p^n}, THH(l)).$$

4. Circle homotopy fixed points

4.1. *The circle trace map.* The circle trace map

$$\mathrm{tr}_{S^1} = \Gamma \circ \mathrm{tr}_F \colon K(l_p) \to THH(l)^{hS^1} = F(ES^1_+, THH(l))^{S^1}$$

is a preferred lift of the trace map $\mathrm{tr} \colon K(l_p) \to THH(l)$. We take S^∞ as our model for ES^1. Let

$$T^n = F(S^\infty / S^{2n-1}, THH(l))^{S^1}$$

for $n \geqslant 0$, so that there is a descending filtration $\{T^n\}_n$ on $T^0 = THH(l)^{hS^1}$, with layers $T^n/T^{n+1} \cong F(S^{2n+1}/S^{2n-1}, THH(l))^{S^1} \cong \Sigma^{-2n} THH(l)$.

4.2. *The homology spectral sequence.* Placing T^n in filtration $s = -2n$ and applying homology, we obtain a (not necessarily convergent) homology spectral sequence

$$E^2_{s,t} = H^{-s}(S^1; H_t(THH(l); \mathbf{F}_p)) \implies H_{s+t}(THH(l)^{hS^1}; \mathbf{F}_p) \tag{4.3}$$

with

$$E^2_{**} = P(t) \otimes H_*(l; \mathbf{F}_p) \otimes E(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2) \otimes P(\sigma\bar{\tau}_2).$$

Here t has bidegree $(-2, 0)$ while the other generators are located on the vertical axis. (No confusion should arise from the double usage of t as a polynomial cohomology class and the vertical degree in this or other spectral sequences.)

LEMMA 4.4. *There are differentials $d^2(\bar{\xi}_1) = t \cdot \sigma\bar{\xi}_1$, $d^2(\bar{\xi}_2) = t \cdot \sigma\bar{\xi}_2$ and $d^2(\bar{\tau}_2) = t \cdot \sigma\bar{\tau}_2$ in the spectral sequence* (4.3).

Proof. The d^2-differential

$$d^2_{0,t}: E^2_{0,t} \cong H_t(THH(l); \mathbf{F}_p)\{1\} \to E^2_{-2,t+1} \cong H_{t+1}(THH(l); \mathbf{F}_p)\{t\}$$

is adjoint to the S^1-action on $THH(l)$, hence restricts to σ on $H_t(l; \mathbf{F}_p)$. See [Ro1, 3.3]. □

4.5. *The $V(1)$-homotopy spectral sequence.* Applying $V(1)$-homotopy to the filtration $\{T^n\}_n$, in place of homology, we obtain a conditionally convergent $V(1)$-homotopy spectral sequence

$$E^2_{s,t}(S^1) = H^{-s}(S^1; V(1)_t THH(l)) \implies V(1)_{s+t} THH(l)^{hS^1} \tag{4.6}$$

with

$$E^2_{**}(S^1) = P(t) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu).$$

Again t has bidegree $(-2, 0)$ while the other generators are located on the vertical axis.

Definition 4.7. Let

$$\alpha_1 \in \pi_{2p-3}(S^0), \quad \beta'_1 \in \pi_{2p^2-2p-1} V(0) \quad \text{and} \quad v_2 \in \pi_{2p^2-2} V(1)$$

be the classes represented in their respective Adams spectral sequences by the cobar 1-cycles $h_{10} = [\bar{\xi}_1]$, $h_{11} = [\bar{\xi}_1^p]$ and $[\bar{\tau}_2]$. So $j_1(v_2) = \beta'_1$ and $j_0(\beta'_1) = \beta_1 \in \pi_{2p^2-2p-2}(S^0)$.

Consider the unit map $S^0 \to K(l_p) \to THH(l)^{hS^1}$, which is well defined after p-adic completion.

PROPOSITION 4.8. *The classes* $i_1 i_0(\alpha_1) \in \pi_{2p-3} V(1)$, $i_1(\beta_1') \in \pi_{2p^2-2p-1} V(1)$ *and* $v_2 \in \pi_{2p^2-2} V(1)$ *map under the unit map* $V(1)_* S^0 \to V(1)_* THH(l)^{hS^1}$ *to nonzero classes represented in* $E^\infty(S^1)$ *by* $t\lambda_1$, $t^p\lambda_2$ *and* $t\mu$, *respectively.*

Proof. Consider first the filtration subquotient $T^0/T^2 = F(S^3_+, THH(l))^{S^1}$. The unit map $V(1) \to V(1) \wedge (T^0/T^2)$ induces a map of Adams spectral sequences, taking the permanent 1-cycles $[\bar\xi_1]$ and $[\bar\tau_2]$ in the source Adams spectral sequence to infinite 1-cycles with the same cobar names in the target Adams spectral sequence. These are not 1-boundaries in the cobar complex

$$H_*(T^0/T^2; \mathbf{F}_p) \xrightarrow{d^0} \bar A_* \otimes H_*(T^0/T^2; \mathbf{F}_p) \xrightarrow{d^1} \dots$$

for the A_*-comodule $H_*(T^0/T^2; \mathbf{F}_p)$, because of the differentials $d^2(\bar\xi_1) = t \cdot \sigma\bar\xi_1$ and $d^2(\bar\tau_2) = t \cdot \sigma\bar\tau_2$ that are present in the 2-column spectral sequence converging to $H_*(T^0/T^2; \mathbf{F}_p)$. In detail, $H_{2p-2}(T^0/T^2; \mathbf{F}_p) = 0$ and $H_{2p^2-1}(T^0/T^2; \mathbf{F}_p)$ is spanned by the primitives $\sigma\bar\xi_2$ and $\bar\xi_1^p \cdot \sigma\bar\xi_1$.

Thus $[\bar\xi_1]$ and $[\bar\tau_2]$ are nonzero infinite cycles in the target Adams E_2-term. They have Adams filtration one, hence cannot be boundaries. Thus they are permanent cycles, and are nonzero images of the classes $i_1 i_0(\alpha_1)$ and v_2 under the composite $V(1)_* \to V(1)_*(T^0) \to V(1)_*(T^0/T^2)$. Thus they are also detected in $V(1)_*(T^0)$, in filtration $s \geqslant -2$. For bidegree reasons the only possibility is that $i_1 i_0(\alpha_1)$ is detected in the $V(1)$-homotopy spectral sequence $E^\infty(S^1)$ as $t\lambda_1$, and v_2 is detected as $t\mu$.

Next consider the filtration subquotient $T^0/T^{p+1} = F(S^{2p+1}_+, THH(l))^{S^1}$. Restriction across $S^{2p+1}_+ \to ES^1_+$ yields the second of two E_∞ ring spectrum maps:

$$S^0 \xrightarrow{\iota} THH(l)^{hS^1} \xrightarrow{\varrho} T^0/T^{p+1}.$$

The composite map $\varrho\iota$ takes $\alpha_1 \in \pi_{2p-3}(S^0)$ to a product $t \cdot \lambda_1$ in $\pi_{2p-3}(T^0/T^{p+1})$, where $t \in \pi_{-2}(T^0/T^{p+1})$ and $\lambda_1 \in \pi_{2p-1}(T^0/T^{p+1})$. Here t and λ_1 are represented by the classes with the same names in the integral homotopy spectral sequence:

$$E^2_{s,t} = \begin{cases} H^{-s}(S^1; \pi_t THH(l)), & -2p \leqslant s \leqslant 0, \\ 0, & \text{otherwise}, \end{cases} \quad \Rightarrow \quad \pi_{s+t}(T^0/T^{p+1}).$$

By Proposition 2.6 and Corollary 2.7 we have $\pi_t THH(l) = 0$ for $0 < t < 2p-2$ and for $2p-1 < t < 4p-4$, so the class t is a permanent cycle for bidegree reasons, and the factorization $\varrho\iota(\alpha_1) = t \cdot \lambda_1$ holds strictly, not just modulo lower filtrations. We know from Proposition 2.8 that $\lambda_1 = \mathrm{tr}_{S^1}(\lambda_1^K)$ is an integral homotopy class.

Now we apply naturality and the mod p homotopy Cartan formula in Lemma 1.6, to see that $\beta_1' = (P^{p-1})^*(\alpha_1)$ in $\pi_{2p^2-2p-1}(S^0; \mathbf{F}_p)$ maps under $\varrho\iota$ to

$$(P^{p-1})^*(t \cdot \lambda_1) = (P^{-1})^*(t) \cdot (P^p)^*(\lambda_1) = t^p \cdot \lambda_2$$

in $\pi_{2p^2-2p-1}(T^0/T^{p+1};\mathbf{F}_p)$. Hence $i_1(\beta_1')$ maps to the infinite cycle $t^p\lambda_2$ in $E^\infty(S^1)$, which cannot be a boundary for bidegree reasons. Thus $t^p\lambda_2$ is a permanent cycle. □

5. The homotopy limit property

5.1. *Homotopy fixed-point and Tate spectral sequences.* For closed subgroups $G\subseteq S^1$ we will consider the (second quadrant) G homotopy fixed-point spectral sequence

$$E^2_{s,t}(G)=H^{-s}(G,V(1)_t THH(l)) \Rightarrow V(1)_{s+t}THH(l)^{hG}.$$

We also consider the (upper half-plane) G Tate spectral sequence

$$\widehat{E}^2_{s,t}(G)=\widehat{H}^{-s}(G,V(1)_t THH(l)) \Rightarrow V(1)_{s+t}\widehat{\mathbf{H}}(G,THH(l)).$$

When $G=S^1$ we have

$$E^2_{**}(S^1)=E(\lambda_1,\lambda_2)\otimes P(t,\mu)$$

since $H^*(S^1;\mathbf{F}_p)=P(t)$, and

$$\widehat{E}^2_{**}(S^1)=E(\lambda_1,\lambda_2)\otimes P(t,t^{-1},\mu)$$

since $\widehat{H}^*(S^1;\mathbf{F}_p)=P(t,t^{-1})$. When $G=C_{p^n}$ we have

$$E^2_{**}(C_{p^n})=E(u_n,\lambda_1,\lambda_2)\otimes P(t,\mu)$$

since $H^*(C_{p^n};\mathbf{F}_p)=E(u_n)\otimes P(t)$, while

$$\widehat{E}^2_{**}(C_{p^n})=E(u_n,\lambda_1,\lambda_2)\otimes P(t,t^{-1},\mu)$$

since $\widehat{H}^*(C_{p^n};\mathbf{F}_p)=E(u_n)\otimes P(t,t^{-1})$. In all cases u_n has bidegree $(-1,0)$, t has bidegree $(-2,0)$, λ_1 has bidegree $(0,2p-1)$, λ_2 has bidegree $(0,2p^2-1)$ and μ has bidegree $(0,2p^2)$.

All of these spectral sequences are conditionally convergent by construction, and are thus strongly convergent by [Boa, 7.1], since the E^2-terms are finite in each bidegree.

The homotopy restriction map R^h induces a map of spectral sequences

$$E^*(R^h): E^*(G)\to\widehat{E}^*(G),$$

which on E^2-terms inverts t, identifying $E^2(G)$ with the restriction of $\widehat{E}^2(G)$ to the second quadrant.

The Frobenius and Verschiebung maps F and V are compatible under $\widehat{\Gamma}_{n+1}$ and $\widehat{\Gamma}_n$ with homotopy Frobenius and Verschiebung maps F^h and V^h that induce maps of Tate spectral sequences

$$\widehat{E}^*(F^h)\colon \widehat{E}^*(C_{p^{n+1}}) \to \widehat{E}^*(C_{p^n})$$

and

$$\widehat{E}^*(V^h)\colon \widehat{E}^*(C_{p^n}) \to \widehat{E}^*(C_{p^{n+1}}).$$

Here $\widehat{E}^2(F^h)$ is induced by the natural map $\widehat{H}^*(C_{p^{n+1}};\mathbf{F}_p)\to\widehat{H}^*(C_{p^n};\mathbf{F}_p)$ taking t to t and u_{n+1} to 0. It thus maps the even columns isomorphically and the odd columns trivially. On the other hand, $\widehat{E}^2(V^h)$ is induced by the transfer map $\widehat{H}^*(C_{p^n};\mathbf{F}_p)\to\widehat{H}^*(C_{p^{n+1}};\mathbf{F}_p)$ taking t to 0 and u_n to u_{n+1}. It thus maps the odd columns isomorphically and the even columns trivially.

This pattern persists to higher E^r-terms, until a differential of odd length appears in either spectral sequence. More precisely, we have the following lemma:

LEMMA 5.2. *Let $d^r(G)$ denote the differential acting on $\widehat{E}^r(G)$. Choose $n_0\geqslant 1$, and let $r_0\geqslant 3$ be the smallest odd integer such that there exists a nonzero differential*

$$d^{r_0}_{s,*}(C_{p^{n_0}})\colon \widehat{E}^{r_0}_{s,*}(C_{p^{n_0}}) \to \widehat{E}^{r_0}_{s-r_0,*}(C_{p^{n_0}})$$

with s odd. (If $\widehat{E}^*_{**}(C_{p^{n_0}})$ has no nonzero differentials of odd length from an odd column, let $r_0=\infty$.) Then:

(a) *For all $2\leqslant r\leqslant r_0$ and $n\geqslant n_0$ the terms $\widehat{E}^r(C_{p^n})$ and $\widehat{E}^r(C_{p^{n+1}})$ are abstractly isomorphic. Indeed, $F=\widehat{E}^r(F^h)\colon \widehat{E}^r_{s,*}(C_{p^{n+1}})\to\widehat{E}^r_{s,*}(C_{p^n})$ is an isomorphism if s is even and is zero if s is odd, while $V=\widehat{E}^r(V^h)\colon \widehat{E}^r_{s,*}(C_{p^n})\to\widehat{E}^r_{s,*}(C_{p^{n+1}})$ is an isomorphism if s is odd and is zero if s is even.*

(b) *For all odd r with $3\leqslant r\leqslant r_0$ and $n\geqslant n_0$ the differential $d^r_{s,*}(C_{p^n})$ is zero, unless $r=r_0$, $n=n_0$ and s is odd.*

Proof. We consider the two (superimposed) commuting squares

$$\begin{array}{ccc} \widehat{E}^r_{s,*}(C_{p^{n+1}}) & \underset{V}{\overset{F}{\rightleftarrows}} & \widehat{E}^r_{s,*}(C_{p^n}) \\ {\scriptstyle d^r_{s,*}(C_{p^{n+1}})}\downarrow & & \downarrow{\scriptstyle d^r_{s,*}(C_{p^n})} \\ \widehat{E}^r_{s-r,*}(C_{p^{n+1}}) & \underset{V}{\overset{F}{\rightleftarrows}} & \widehat{E}^r_{s-r,*}(C_{p^n}). \end{array}$$

The following statements then follow in sequence by increasing induction on r, for $2\leqslant r\leqslant r_0$ and $n\geqslant n_0$.

(1) $F: \widehat{E}^r_{s,*}(C_{p^{n+1}}) \to \widehat{E}^r_{s,*}(C_{p^n})$ is an isomorphism for all s even, and is zero for s odd.

(2) $V: \widehat{E}^r_{s,*}(C_{p^n}) \to \widehat{E}^r_{s,*}(C_{p^{n+1}})$ is an isomorphism for all s odd, and is zero for s even.

(3) $d^r_{s,*}(C_{p^n}) \circ F = F \circ d^r_{s,*}(C_{p^{n+1}})$ with F an isomorphism for all s even and $r < r_0$ even.

(4) $d^r_{s,*}(C_{p^{n+1}}) \circ V = V \circ d^r_{s,*}(C_{p^n})$ with V an isomorphism for all s odd and $r < r_0$ even.

(5) $d^r_{s,*}(C_{p^n}) = 0$ for all s even and $r \leqslant r_0$ odd.

(6) $d^r_{s,*}(C_{p^{n+1}}) = 0$ for all s odd and $r \leqslant r_0$ odd. □

The lemma clearly also applies to the system of homotopy fixed-point spectral sequences $E^*(C_{p^n})$.

5.3. *Input for Tsalidis' theorem.*

Definition 5.4. A map $A_* \to B_*$ of graded groups is k-coconnected if it is an isomorphism in all dimensions greater than k and injective in dimension k.

THEOREM 5.5. *The canonical map*

$$\widehat{\Gamma}_1 : THH(l) \to \widehat{\mathbf{H}}(C_p, THH(l))$$

induces a $(2p-2)$*-coconnected map on* $V(1)$*-homotopy, which factors as the localization map*

$$V(1)_* THH(l) \to \mu^{-1} V(1)_* THH(l) \cong E(\lambda_1, \lambda_2) \otimes P(\mu, \mu^{-1}),$$

followed by an isomorphism

$$\mu^{-1} V(1)_* THH(l) \cong V(1)_* \widehat{\mathbf{H}}(C_p, THH(l)).$$

Proof. Consider diagram (3.4) in the case $n=1$. The classes $i_1 i_0(\alpha_1)$, $i_1(\beta_1')$ and v_2 in $V(1)_*$ map through $V(1)_* K(l_p)$ and $\Gamma_1 \circ \mathrm{tr}_1$ to classes in $V(1)_* THH(l)^{hC_p}$ that are detected by $t\lambda_1$, $t^p \lambda_2$ and $t\mu$ in $E^\infty(C_p)$, respectively. Continuing by R^h to $V(1)_* \widehat{\mathbf{H}}(C_p, THH(l))$ these classes factor through $V(1)_* THH(l)$, where they pass through zero groups. Hence the images of $t\lambda_1$, $t^p\lambda_2$ and $t\mu$ in $\widehat{E}^\infty(C_p)$ must be zero, i.e., these infinite cycles in $\widehat{E}^2(C_p)$ are boundaries. For dimension reasons the only possibilities are

$$d^{2p}(t^{1-p}) = t\lambda_1,$$
$$d^{2p^2}(t^{p-p^2}) = t^p \lambda_2,$$
$$d^{2p^2+1}(u_1 t^{-p^2}) = t\mu.$$

The classes $i_1 i_0(\lambda_1^K)$ and $i_1(\lambda_2^K)$ in $V(1)_* K(l_p)$ map by $\Gamma_1 \circ \mathrm{tr}_1$ to classes in $V(1)_* THH(l)^{hC_p}$ that have Frobenius images λ_1 and λ_2 in $V(1)_* THH(l)$, and hence survive as permanent cycles in $E^\infty_{0,*}(C_p)$. Thus their images λ_1 and λ_2 in $\widehat{E}^*(C_p)$ are infinite cycles.

Hence the various E^r-terms of the C_p Tate spectral sequence are

$$\begin{aligned}
\widehat{E}^2(C_p) &= E(u_1,\lambda_1,\lambda_2)\otimes P(t,t^{-1},t\mu),\\
\widehat{E}^{2p+1}(C_p) &= E(u_1,\lambda_1,\lambda_2)\otimes P(t^p,t^{-p},t\mu),\\
\widehat{E}^{2p^2+1}(C_p) &= E(u_1,\lambda_1,\lambda_2)\otimes P(t^{p^2},t^{-p^2},t\mu),\\
\widehat{E}^{2p^2+2}(C_p) &= E(\lambda_1,\lambda_2)\otimes P(t^{p^2},t^{-p^2}).
\end{aligned}$$

For bidegree reasons there are no further differentials, so $\widehat{E}^{2p^2+2}(C_p)=\widehat{E}^\infty(C_p)$ and the classes λ_1, λ_2 and $t^{\pm p^2}$ are permanent cycles.

On $V(1)$-homotopy the map $\widehat{\Gamma}_1\colon THH(l)\to \widehat{\mathbf{H}}(C_p, THH(l))$ induces the homomorphism

$$E(\lambda_1,\lambda_2)\otimes P(\mu)\to E(\lambda_1,\lambda_2)\otimes P(t^{p^2},t^{-p^2})$$

that maps $\lambda_1\mapsto\lambda_1$, $\lambda_2\mapsto\lambda_2$ and $\mu\mapsto t^{-p^2}$. For the classes $i_1 i_0(\lambda_1^K)$ and $i_1(\lambda_2^K)$ in $V(1)_* K(l_p)$ map by tr to λ_1 and λ_2 in $V(1)_* THH(l)$, and by $R^h\circ\Gamma_1\circ\mathrm{tr}_1$ to the classes in $V(1)_*\widehat{\mathbf{H}}(C_p, THH(l))$ represented by λ_1 and λ_2. The class μ in $V(1)_* THH(l)$ must have nonzero image in $V(1)_*\widehat{\mathbf{H}}(C_p, THH(l))$, since its pth v_1-Bockstein $\beta_{1,p}(\mu)=\lambda_1$ has nonzero image there. Thus μ maps to the class represented by t^{-p^2}, up to a unit multiple which we ignore. So $V(1)_*\widehat{\Gamma}_1$ is an isomorphism in dimensions greater than $|\lambda_1\lambda_2 t^{p^2}|=2p-2$, and is injective in dimension $2p-2$. □

5.6. *The homotopy limit property.*

THEOREM 5.7. *The canonical maps*

$$\begin{aligned}
\Gamma_n &: THH(l)^{C_{p^n}}\to THH(l)^{hC_{p^n}},\\
\widehat{\Gamma}_n &: THH(l)^{C_{p^{n-1}}}\to \widehat{\mathbf{H}}(C_{p^n}, THH(l))
\end{aligned}$$

and

$$\begin{aligned}
\Gamma &: TF(l;p)\to THH(l)^{hS^1},\\
\widehat{\Gamma} &: TF(l;p)\to \widehat{\mathbf{H}}(S^1, THH(l))
\end{aligned}$$

all induce $(2p-2)$-coconnected maps on $V(1)$-homotopy.

Proof. The claims for Γ_n and $\widehat{\Gamma}_n$ follow from Theorem 5.5 and a theorem of Tsalidis [Ts]. The claims for Γ and $\widehat{\Gamma}$ follow by passage to homotopy limits, using the p-adic homotopy equivalence $THH(l)^{hS^1}\simeq \mathrm{holim}_{n,F}\, THH(l)^{hC_{p^n}}$ and its analogue for the Tate constructions. □

6. Higher fixed points

Let $[k]=1$ when k is odd, and $[k]=2$ when k is even. Let $\lambda'_{[k]}=\lambda_{[k+1]}$, so that $\{\lambda_{[k]},\lambda'_{[k]}\}=\{\lambda_1,\lambda_2\}$ for all k. We write $v_p(k)$ for the p-valuation of k, i.e., the exponent of the greatest power of p that divides k. By convention, $v_p(0)=+\infty$. Recall the integers $r(n)$ from Definition 2.5.

THEOREM 6.1. *In the C_{p^n} Tate spectral sequence $\widehat{E}^*(C_{p^n})$ there are differentials*

$$d^{2r(k)}(t^{p^{k-1}-p^k})=\lambda_{[k]}t^{p^{k-1}}(t\mu)^{r(k-2)}$$

for all $1\leqslant k\leqslant 2n$, and

$$d^{2r(2n)+1}(u_n t^{-p^{2n}})=(t\mu)^{r(2n-2)+1}.$$

The classes λ_1, λ_2 and $t\mu$ are infinite cycles.

We shall prove this by induction on n, the case $n=1$ being settled in the previous section. Hence we assume that the theorem holds for one $n\geqslant 1$, and we will establish its assertions for $n+1$.

The terms of the Tate spectral sequence are

$$\begin{aligned}\widehat{E}^{2r(m)+1}(C_{p^n})&=E(u_n,\lambda_1,\lambda_2)\otimes P(t^{p^m},t^{-p^m},t\mu)\\&\quad\oplus\bigoplus_{k=3}^{m}E(u_n,\lambda'_{[k]})\otimes P_{r(k-2)}(t\mu)\otimes\mathbf{F}_p\{\lambda_{[k]}t^i\mid v_p(i)=k-1\}\end{aligned}$$

for $1\leqslant m\leqslant 2n$. To see this, note that the differential $d^{2r(k)}$ only affects the summand $E(u_n,\lambda_1,\lambda_2)\otimes P(t\mu)\otimes\mathbf{F}_p\{t^i\mid v_p(i)=k-1\}$, and here its homology is

$$E(u_n,\lambda'_{[k]})\otimes P_{r(k-2)}(t\mu)\otimes\mathbf{F}_p\{\lambda_{[k]}t^i\mid v_p(i)=k-1\}.$$

Next

$$\begin{aligned}\widehat{E}^{2r(2n)+2}(C_{p^n})&=E(\lambda_1,\lambda_2)\otimes P_{r(2n-2)+1}(t\mu)\otimes P(t^{p^{2n}},t^{-p^{2n}})\\&\quad\oplus\bigoplus_{k=3}^{2n}E(u_n,\lambda'_{[k]})\otimes P_{r(k-2)}(t\mu)\otimes\mathbf{F}_p\{\lambda_{[k]}t^i\mid v_p(i)=k-1\}.\end{aligned}$$

For bidegree reasons the remaining differentials are zero, so $\widehat{E}^{2r(2n)+2}(C_{p^n})=\widehat{E}^{\infty}(C_{p^n})$, and the classes $t^{\pm p^{2n}}$ are permanent cycles.

PROPOSITION 6.2. *The associated graded of $V(1)_*\widehat{\mathbf{H}}(C_{p^n},THH(l))$ is*

$$\begin{aligned}\widehat{E}^{\infty}(C_{p^n})&=E(\lambda_1,\lambda_2)\otimes P_{r(2n-2)+1}(t\mu)\otimes P(t^{p^{2n}},t^{-p^{2n}})\\&\quad\oplus\bigoplus_{k=3}^{2n}E(u_n,\lambda'_{[k]})\otimes P_{r(k-2)}(t\mu)\otimes\mathbf{F}_p\{\lambda_{[k]}t^i\mid v_p(i)=k-1\}.\end{aligned}$$

Comparing $E^*(C_{p^n})$ with $\widehat{E}^*(C_{p^n})$ via the homotopy restriction map R^h, we obtain

PROPOSITION 6.3. *In the C_{p^n} homotopy fixed-point spectral sequence $E^*(C_{p^n})$ there are differentials*

$$d^{2r(k)}(t^{p^{k-1}}) = \lambda_{[k]} t^{p^k+p^{k-1}} (t\mu)^{r(k-2)}$$

for all $1 \leqslant k \leqslant 2n$, and

$$d^{2r(2n)+1}(u_n) = t^{p^{2n}} (t\mu)^{r(2n-2)+1}.$$

The classes λ_1, λ_2 and $t\mu$ are infinite cycles.

Let G be a closed subgroup of S^1. We will also consider the (strongly convergent) G homotopy fixed-point spectral sequence for $\widehat{\mathbf{H}}(C_p, THH(l))$ in $V(1)$-homotopy

$$\mu^{-1}E^2_{s,t}(G) = H^{-s}(G; V(1)_t \widehat{\mathbf{H}}(C_p, THH(l))) \implies V(1)_{s+t} \widehat{\mathbf{H}}(C_p, THH(l))^{hG}.$$

By Theorem 5.5 its E^2-term $\mu^{-1}E^2(G)$ is obtained from $E^2(G)$ by inverting μ. Therefore we shall denote this spectral sequence by $\mu^{-1}E^*(G)$, and refer to it as the μ-inverted spectral sequence, even though the later terms $\mu^{-1}E^r(G)$ are generally not obtained from $E^r(G)$ by simply inverting μ. For each r the natural map $E^r(G) \to \mu^{-1}E^r(G)$ is an isomorphism in total degrees greater than $2p-2$, and an injection in total degree $2p-2$.

PROPOSITION 6.4. *In the μ-inverted spectral sequence $\mu^{-1}E^*(C_{p^n})$ there are differentials*

$$d^{2r(k)}(\mu^{p^k-p^{k-1}}) = \lambda_{[k]} (t\mu)^{r(k)} \mu^{-p^{k-1}}$$

for all $1 \leqslant k \leqslant 2n$, and

$$d^{2r(2n)+1}(u_n \mu^{p^{2n}}) = (t\mu)^{r(2n)+1}.$$

The classes λ_1, λ_2 and $t\mu$ are infinite cycles.

The terms of the μ-inverted spectral sequence are

$$\mu^{-1}E^{2r(m)+1}(C_{p^n}) = E(u_n, \lambda_1, \lambda_2) \otimes P(\mu^{p^m}, \mu^{-p^m}, t\mu) \\ \oplus \bigoplus_{k=1}^{m} E(u_n, \lambda'_{[k]}) \otimes P_{r(k)}(t\mu) \otimes \mathbf{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j) = k-1\}$$

for $1 \leqslant m \leqslant 2n$. Next

$$\mu^{-1}E^{2r(2n)+2}(C_{p^n}) = E(\lambda_1, \lambda_2) \otimes P_{r(2n)+1}(t\mu) \otimes P(\mu^{p^{2n}}, \mu^{-p^{2n}}) \\ \oplus \bigoplus_{k=1}^{2n} E(u_n, \lambda'_{[k]}) \otimes P_{r(k)}(t\mu) \otimes \mathbf{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j) = k-1\}.$$

Again $\mu^{-1}E^{2r(2n)+2}(C_{p^n}) = \mu^{-1}E^\infty(C_{p^n})$ for bidegree reasons, and the classes $\mu^{\pm p^{2n}}$ are permanent cycles.

PROPOSITION 6.5. *The associated graded $E^\infty(C_{p^n})$ of $V(1)_*THH(l)^{hC_{p^n}}$ maps by a $(2p-2)$-coconnected map to*

$$\mu^{-1}E^\infty(C_{p^n}) = E(\lambda_1,\lambda_2)\otimes P_{r(2n)+1}(t\mu)\otimes P(\mu^{p^{2n}},\mu^{-p^{2n}}) \\ \oplus \bigoplus_{k=1}^{2n} E(u_n,\lambda'_{[k]})\otimes P_{r(k)}(t\mu)\otimes \mathbf{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j)=k-1\}.$$

Proof of Theorem 6.1. By our inductive hypothesis, the abutment $\mu^{-1}E^\infty(C_{p^n})$ contains summands

$$P_{r(2n-1)}(t\mu)\{\lambda_1\mu^{p^{2n-2}}\},\quad P_{r(2n)}(t\mu)\{\lambda_2\mu^{p^{2n-1}}\}\quad\text{and}\quad P_{r(2n)+1}(t\mu)\{\mu^{p^{2n}}\}$$

representing elements in $V(1)_*THH(l)^{C_{p^n}}$. By inspection there are no classes in $\mu^{-1}E^\infty(C_{p^n})$ in the same total degree and of lower s-filtration than $(t\mu)^{r(2n-1)}\cdot\lambda_1\mu^{p^{2n-2}}$, $(t\mu)^{r(2n)}\cdot\lambda_2\mu^{p^{2n-1}}$ and $(t\mu)^{r(2n)+1}\cdot\mu^{p^{2n}}$, respectively. So the three homotopy classes represented by $\lambda_1\mu^{p^{2n-2}}$, $\lambda_2\mu^{p^{2n-1}}$ and $\mu^{p^{2n}}$ are v_2-torsion classes of order precisely $r(2n-1)$, $r(2n)$ and $r(2n)+1$, respectively.

Consider the commutative diagram

$$\begin{array}{ccccc} THH(l)^{hC_{p^n}} & \xleftarrow{\Gamma_n} & THH(l)^{C_{p^n}} & \xrightarrow{\hat\Gamma_{n+1}} & \hat{\mathbf{H}}(C_{p^{n+1}},THH(l)) \\ \downarrow F^n & & \downarrow F^n & & \downarrow F^n \\ THH(l) & \xleftarrow[=]{\Gamma_0} & THH(l) & \xrightarrow{\hat\Gamma_1} & \hat{\mathbf{H}}(C_p,THH(l)). \end{array}$$

Here F^n is the n-fold Frobenius map forgetting C_{p^n}-invariance. The right-hand square commutes because $\hat\Gamma_{n+1}$ is constructed as the C_{p^n}-invariant part of an S^1-equivariant model for $\hat\Gamma_1$.

The three classes in $V(1)_*THH(l)^{C_{p^n}}$ represented by $\lambda_1\mu^{p^{2n-2}}$, $\lambda_2\mu^{p^{2n-1}}$ and $\mu^{p^{2n}}$ map by the middle F^n to classes in $V(1)_*THH(l)$ with the same names, and by $\hat\Gamma_1$ to classes in $V(1)_*\hat{\mathbf{H}}(C_p,THH(l))$ represented by $\lambda_1 t^{-p^{2n}}$, $\lambda_2 t^{-p^{2n+1}}$ and $t^{-p^{2n+2}}$ in $\hat E^\infty(C_p)$, respectively. Hence they map by $\hat\Gamma_{n+1}$ to permanent cycles in $\hat E^*(C_{p^{n+1}})$ with these images under the right-hand F^n.

Once we show that there are no classes in $\hat E^\infty(C_{p^{n+1}})$ in the same total degree and with higher s-filtration than $\lambda_1 t^{-p^{2n}}$, $\lambda_2 t^{-p^{2n+1}}$ and $t^{-p^{2n+2}}$, then it will follow that these are precisely the permanent cycles that represent the images of $\lambda_1\mu^{p^{2n-2}}$, $\lambda_2\mu^{p^{2n-1}}$ and $\mu^{p^{2n}}$ under $\hat\Gamma_{n+1}$.

By Lemma 5.2 applied to the system of Tate spectral sequences $\hat E^*(C_{p^n})$ for $n\geqslant 1$, using the inductive hypothesis about $\hat E^*(C_{p^n})$, there are abstract isomorphisms

$\widehat{E}^r(C_{p^n}) \cong \widehat{E}^r(C_{p^{n+1}})$ for all $r \leqslant 2r(2n)+1$, by F in the even columns and V in the odd columns. This determines the d^r-differentials and E^r-terms of $\widehat{E}^*(C_{p^{n+1}})$ up to and including the E^r-term with $r=2r(2n)+1$:

$$\widehat{E}^{2r(2n)+1}(C_{p^{n+1}}) = E(u_{n+1}, \lambda_1, \lambda_2) \otimes P(t^{p^{2n}}, t^{-p^{2n}}, t\mu) \\ \oplus \bigoplus_{k=3}^{2n} E(u_{n+1}, \lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbf{F}_p\{\lambda_{[k]} t^i \mid v_p(i) = k-1\}.$$

By inspection there are no permanent cycles in the same total degree and of higher s-filtration in $\widehat{E}^*(C_{p^{n+1}})$ than $\lambda_1 t^{-p^{2n}}$, $\lambda_2 t^{-p^{2n+1}}$ and $t^{-p^{2n+2}}$, respectively. So the equivalence $\widehat{\Gamma}_{n+1}\Gamma_n^{-1}$ takes the homotopy classes represented by $\lambda_1 \mu^{p^{2n-2}}$, $\lambda_2 \mu^{p^{2n-1}}$ and $\mu^{p^{2n}}$ to homotopy classes represented by $\lambda_1 t^{-p^{2n}}$, $\lambda_2 t^{-p^{2n+1}}$ and $t^{-p^{2n+2}}$, respectively.

Since $\widehat{\Gamma}_{n+1}\Gamma_n^{-1}$ induces an isomorphism on $V(1)$-homotopy in dimensions greater than $2p-2$, it preserves the v_2-torsion order of these classes. Thus the infinite cycles

$$(t\mu)^{r(2n-1)} \cdot \lambda_1 t^{-p^{2n}}, \quad (t\mu)^{r(2n)} \cdot \lambda_2 t^{-p^{2n+1}} \quad \text{and} \quad (t\mu)^{r(2n)+1} \cdot t^{-p^{2n+2}}$$

are all boundaries in $\widehat{E}^*(C_{p^{n+1}})$. All these are $t\mu$-periodic classes in $\widehat{E}^r(C_{p^{n+1}})$ for $r= 2r(2n)+1$, hence cannot be hit by differentials from the $t\mu$-torsion classes in this E^r-term.

This leaves the $t\mu$-periodic part $E(u_{n+1}, \lambda_1, \lambda_2) \otimes P(t^{p^{2n}}, t^{-p^{2n}}, t\mu)$, where all the generators above the horizontal axis are infinite cycles. Hence the differentials hitting $(t\mu)^{r(2n-1)} \cdot \lambda_1 t^{-p^{2n}}$, $(t\mu)^{r(2n)} \cdot \lambda_2 t^{-p^{2n+1}}$ and $(t\mu)^{r(2n)+1} \cdot t^{-p^{2n+2}}$ must (be multiples of differentials that) originate on the horizontal axis, and by inspection the only possibilities are

$$d^{2r(2n+1)}(t^{-p^{2n}-p^{2n+1}}) = (t\mu)^{r(2n-1)} \cdot \lambda_1 t^{-p^{2n}},$$
$$d^{2r(2n+2)}(t^{-p^{2n+1}-p^{2n+2}}) = (t\mu)^{r(2n)} \cdot \lambda_2 t^{-p^{2n+1}},$$
$$d^{2r(2n+2)+1}(u_{n+1} t^{-2p^{2n+2}}) = (t\mu)^{r(2n)+1} \cdot t^{-p^{2n+2}}.$$

The algebra structure on $\widehat{E}^*(C_{p^{n+1}})$ lets us rewrite these differentials as the remaining differentials asserted by case $n+1$ of Theorem 6.1. □

Passing to the limit over the Frobenius maps, we obtain

THEOREM 6.6. *The associated graded of* $V(1)_* \widehat{\mathbf{H}}(S^1, THH(l))$ *is*

$$\widehat{E}^\infty(S^1) = E(\lambda_1, \lambda_2) \otimes P(t\mu) \\ \oplus \bigoplus_{k \geqslant 3} E(\lambda'_{[k]}) \otimes P_{r(k-2)}(t\mu) \otimes \mathbf{F}_p\{\lambda_{[k]} t^i \mid v_p(i) = k-1\}.$$

THEOREM 6.7. *The associated graded* $E^\infty(S^1)$ *of* $V(1)_*THH(l)^{hS^1}$ *maps by a* $(2p-2)$*-coconnected map to*

$$\mu^{-1}E^\infty(S^1) = E(\lambda_1,\lambda_2)\otimes P(t\mu) \oplus \bigoplus_{k\geqslant 1} E(\lambda'_{[k]})\otimes P_{r(k)}(t\mu)\otimes \mathbf{F}_p\{\lambda_{[k]}\mu^j \mid v_p(j)=k-1\}.$$

For a bigraded Abelian group E^∞_{**} let the (product) total group $\mathrm{Tot}^\Pi_*(E^\infty)$ be the graded Abelian group with

$$\mathrm{Tot}^\Pi_n(E^\infty) = \prod_{s+t=n} E^\infty_{s,t}.$$

Then each of the E^∞-terms above compute $V(1)_*TF(l;p)$ in dimensions greater than $2p-2$, by way of the $(2p-2)$-coconnected maps

$$\widehat{\Gamma}\colon V(1)_*TF(l;p)\to V(1)_*\widehat{\mathbf{H}}(S^1,THH(l))\cong \mathrm{Tot}^\Pi_*(\widehat{E}^\infty(S^1))$$

and

$$\Gamma\colon V(1)_*TF(l;p)\to V(1)_*THH(l)^{hS^1}\to \mathrm{Tot}^\Pi_*(\mu^{-1}E^\infty(S^1)),$$

respectively.

7. The restriction map

In this section we will evaluate the homomorphism

$$R_*\colon V(1)_*TF(l;p)\to V(1)_*TF(l;p)$$

induced on $V(1)$-homotopy by the restriction map R, in dimensions greater than $2p-2$.

The canonical map from C_{p^n} fixed points to C_{p^n} homotopy fixed points applied to the S^1-equivariant map $\widehat{\Gamma}_1\colon THH(l)\to\widehat{\mathbf{H}}(C_p,THH(l))$ yields a commutative square of ring spectrum maps

$$\begin{array}{ccc} THH(l)^{C_{p^n}} & \xrightarrow{\Gamma_n} & THH(l)^{hC_{p^n}} \\ \downarrow{\scriptstyle \widehat{\Gamma}_{n+1}} & & \downarrow{\scriptstyle (\widehat{\Gamma}_1)^{hC_{p^n}}} \\ \widehat{\mathbf{H}}(C_{p^{n+1}},THH(l)) & \xrightarrow{G_n} & \widehat{\mathbf{H}}(C_p,THH(l))^{hC_{p^n}}. \end{array}$$

The right-hand vertical map $(\widehat{\Gamma}_1)^{hC_{p^n}}$ induces the natural map

$$E^*(C_{p^n})\to\mu^{-1}E^*(C_{p^n})$$

of C_{p^n} homotopy fixed-point spectral sequences. By Theorem 5.7 and preservation of coconnectivity under passage to homotopy fixed points, all four maps in this square induce isomorphisms of finite groups on $V(1)$-homotopy in dimensions greater than $2p-2$.

Regarding G_n, more is true:

LEMMA 7.1. *G_n is a $V(1)$-equivalence.*

Proof. We proceed as in [HM1, p. 69]. The $d^{2r(2n)+1}$-differential in Theorem 6.1 implies a differential

$$d^{2r(2n)+1}(u_n t^{-p^{2n}} \cdot (t\mu)^{-r(2n-2)-1}) = 1$$

in the C_{p^n} Tate spectral sequence $\mu^{-1}\widehat{E}^*(C_{p^n})$ for $\widehat{\mathbf{H}}(C_p, THH(l))$. It follows that $\mu^{-1}\widehat{E}^{2r(2n)+2}_{**}(C_{p^n}) = 0$, so $V(1) \wedge \widehat{\mathbf{H}}(C_{p^n}, \widehat{\mathbf{H}}(C_p, THH(l))) \simeq *$.

Hence the C_{p^n} homotopy norm map for $\widehat{\mathbf{H}}(C_p, THH(l))$ is a $V(1)$-equivalence, and the canonical map G_n induces a split surjection on $V(1)$-homotopy. (Compare with (3.4).) Its source and target have abstractly isomorphic $V(1)$-homotopy groups of finite type, by Propositions 6.2 and 6.5, thus G_n induces an isomorphism of finite $V(1)$-homotopy groups in all dimensions. □

By passage to homotopy limits over the Frobenius maps we obtain the commutative square

$$\begin{array}{ccc} TF(l;p) & \xrightarrow{\Gamma} & THH(l)^{hS^1} \\ \Big\downarrow{\scriptstyle \hat{\Gamma}} & & \Big\downarrow{\scriptstyle (\hat{\Gamma}_1)^{hS^1}} \\ \widehat{\mathbf{H}}(S^1, THH(l)) & \xrightarrow{G} & \widehat{\mathbf{H}}(C_p, THH(l))^{hS^1}. \end{array}$$

Again, the map $(\hat{\Gamma}_1)^{hS^1}$ induces the natural map $E^*(S^1) \to \mu^{-1}E^*(S^1)$ of S^1-homotopy fixed-point spectral sequences. In each dimension greater than $2p-2$ it follows that $V(1)_* TF(l;p) \cong \lim_{n,F} V(1)_* THH(l)^{C_{p^n}}$ is a profinite group, and likewise for the other three corners of the square. Thus Γ, $\hat{\Gamma}$ and $(\hat{\Gamma}_1)^{hS^1}$ all induce homeomorphisms of profinite groups on $V(1)$-homotopy in each dimension greater than $2p-2$, while $G = \operatorname{holim}_{n,F} G_n$ induces such a homeomorphism in all dimensions by Lemma 7.1.

(An alternative proof that G is a $V(1)$-equivalence, not using Lemma 7.1, can be given by using that G_* is a ring homomorphism and an isomorphism in dimensions greater than $2p-2$.)

We can now study the restriction map R_* by applying $V(1)$-homotopy to the commutative diagram

$$\begin{array}{ccccc} TF(l;p) & \xrightarrow{R} & TF(l;p) & \xrightarrow{\Gamma} & THH(l)^{hS^1} \\ \Big\downarrow{\scriptstyle \Gamma} & & \Big\downarrow{\scriptstyle \hat{\Gamma}} & & \Big\downarrow{\scriptstyle (\hat{\Gamma}_1)^{hS^1}} \\ THH(l)^{hS^1} & \xrightarrow{R^h} & \widehat{\mathbf{H}}(S^1, THH(l)) & \xrightarrow{G} & \widehat{\mathbf{H}}(C_p, THH(l))^{hS^1}. \end{array}$$

The source and target of R_* are both identified with $V(1)_* THH(l)^{hS^1}$ via Γ_*. Then R_* is identified with the composite homomorphism $\Gamma_* \circ \hat{\Gamma}_*^{-1} \circ R^h_*$. We shall consider the factors R^h_* and $(\Gamma\hat{\Gamma}^{-1})_*$ in turn.

The homotopy restriction map R^h induces a map of spectral sequences

$$E^*(R^h)\colon E^*(S^1)\to \widehat{E}^*(S^1),$$

where the E^∞-terms are given in Theorems 6.6 and 6.7.

PROPOSITION 7.2. *In total dimensions greater than* $2p-2$ *the homomorphism* $E^\infty(R^h)$ *maps*

(a) $E(\lambda_1,\lambda_2)\otimes P(t\mu)$ *in* $E^\infty(S^1)$ *isomorphically to* $E(\lambda_1,\lambda_2)\otimes P(t\mu)$ *in* $\widehat{E}^\infty(S^1)$;

(b) $E(\lambda'_{[k]})\otimes P_{r(k)}(t\mu)\otimes \mathbf{F}_p\{\lambda_{[k]}\mu^{-dp^{k-1}}\}$ *in* $E^\infty(S^1)$ *onto* $E(\lambda'_{[k]})\otimes P_{r(k-2)}(t\mu)\otimes \mathbf{F}_p\{\lambda_{[k]}t^{dp^{k-1}}\}$ *in* $\widehat{E}^\infty(S^1)$, *for* $k\geqslant 3$ *and* $0<d<p$;

(c) *the remaining terms in* $E^\infty(S^1)$ *to zero.*

Proof. Case (a) is clear. For (b) and (c) note that $E^\infty(R^h)$ maps the term

$$E(\lambda'_{[k]})\otimes P_{r(k)}(t\mu)\otimes \mathbf{F}_p\{\lambda_{[k]}\mu^{-dp^{k-1}}\}$$

in $E^\infty(S^1)$ to the term

$$E(\lambda'_{[k]})\otimes P_{r(k-2)}(t\mu)\otimes \mathbf{F}_p\{\lambda_{[k]}t^{dp^{k-1}}\}$$

in $\widehat{E}^\infty(S^1)$. Here d is prime to p. For $d>p$ the source and target are in negative total dimensions, while for $d<0$ the source and target are concentrated in disjoint total dimensions. The cases $0<d<p$ remain, when the map is a surjection since $r(k)-dp^{k-1}>r(k-2)$. □

This identifies the image of R^h_*, by the following lemma extracted from [BM1, §2].

LEMMA 7.3. *The representatives in* $E^\infty(S^1)$ *of the kernel of* R^h_* *equal the kernel of* $E^\infty(R^h)$. *Hence the image of* R^h_* *is isomorphic to the image of* $\mathrm{Tot}^\Pi_*(E^\infty(R^h))$ *in* $\mathrm{Tot}^\Pi_*(\widehat{E}^\infty(S^1))$.

The composite equivalence $\Gamma\widehat{\Gamma}^{-1}$ does not induce a map of spectral sequences. Nonetheless it induces an isomorphism of $E(\lambda_1,\lambda_2)\otimes P(v_2)$-modules on $V(1)$-homotopy in dimensions greater than $2p-2$. Here v_2 acts by multiplication in $V(1)_*$, while multiplications by λ_1 and λ_2 are realized by the images of λ_1^K and λ_2^K, since both Γ and $\widehat{\Gamma}$ are ring spectrum maps.

PROPOSITION 7.4. *In dimensions greater than* $2p-2$ *the composite map* $(\Gamma\widehat{\Gamma}^{-1})_*$ *takes all classes in* $V(1)_*\widehat{\mathbf{H}}(S^1,THH(l))$ *represented by* $\lambda_1^{\varepsilon_1}\lambda_2^{\varepsilon_2}(t\mu)^m t^i$ *in* $\widehat{E}^\infty(S^1)$ *to classes in* $V(1)_*THH(l)^{hS^1}$ *represented by* $\lambda_1^{\varepsilon_1}\lambda_2^{\varepsilon_2}(t\mu)^m\mu^j$ *in* $E^\infty(S^1)$ *with* $i+p^2j=0$. *Here* $\varepsilon_1,\varepsilon_2\in\{0,1\}$ *and* $m\geqslant 0$.

Proof. We prove that G_* takes all classes represented by $\lambda_1^{\varepsilon_1}\lambda_2^{\varepsilon_2}(t\mu)^m t^i$ to classes in $V(1)_*\widehat{\mathbf{H}}(C_p,THH(l))^{hS^1}$ represented by $\lambda_1^{\varepsilon_1}\lambda_2^{\varepsilon_2}(t\mu)^m\mu^j$ in $\mu^{-1}E^\infty(S^1)$, with $i+p^2j=0$.

The assertion then follows by restriction to dimensions greater than $2p-2$, since the natural map $E^\infty(S^1)\to\mu^{-1}E^\infty(S^1)$ is an isomorphism in these dimensions.

The source and target groups of G_* are degreewise profinite $P(v_2)$-modules. An element in $V(1)_*\widehat{\mathbf{H}}(S^1,THH(l))$ is divisible by v_2 (i.e., in the image of multiplication by v_2) if and only if it is represented by a class in $\widehat{E}^\infty(S^1)$ that is divisible by $t\mu$, and similarly for $V(1)_*\widehat{\mathbf{H}}(C_p,THH(l))^{hS^1}$ and $\mu^{-1}E^\infty(S^1)$. Let (v_2) and $(t\mu)$ denote the closed subgroups of v_2-divisible and $t\mu$-divisible elements, respectively.

Then there are isomorphisms

$$\begin{aligned}V(1)_*\widehat{\mathbf{H}}(S^1,THH(l))/(v_2)&\cong \mathrm{Tot}^\Pi_*\widehat{E}^\infty(S^1)/(t\mu)\\&=E(\lambda_1,\lambda_2)\oplus\bigoplus_{k\geqslant 3}E(\lambda'_{[k]})\otimes\mathbf{F}_p\{\lambda_{[k]}t^i\mid v_p(i)=k-1\}\end{aligned}$$

and

$$\begin{aligned}V(1)_*\widehat{\mathbf{H}}(C_p,THH(l))^{hS^1}/(v_2)&\cong \mathrm{Tot}^\Pi_*\mu^{-1}E^\infty(S^1)/(t\mu)\\&=E(\lambda_1,\lambda_2)\oplus\bigoplus_{k\geqslant 1}E(\lambda'_{[k]})\otimes\mathbf{F}_p\{\lambda_{[k]}\mu^j\mid v_p(j)=k-1\}.\end{aligned}$$

Clearly G_* induces an isomorphism between these two groups, which by a dimension count must be given by

$$\lambda_1^{\varepsilon_1}\lambda_2^{\varepsilon_2}t^i\mapsto\lambda_1^{\varepsilon_1}\lambda_2^{\varepsilon_2}\mu^j$$

with $i+p^2j=0$. Hence the same formulas hold modulo multiples of v_2 on $V(1)$-homotopy. Taking the $P(v_2)$-module structure into account, the corresponding formulas including factors $(t\mu)^m$ must also hold. □

LEMMA 7.5. *In dimensions greater than* $2p-2$ *the restriction map*

$$R_*: V(1)_*TF(l;p)\to V(1)_*TF(l;p)$$

is continuous with respect to the profinite topology on $V(1)_*TF(l;p)$.

Proof. The filtration topologies on $V(1)_*THH(l)^{hS^1}$ and $V(1)_*\widehat{\mathbf{H}}(S^1,THH(l))$ associated to the spectral sequences $E^*(S^1)$ and $\widehat{E}^*(S^1)$, respectively, are equal to the profinite topologies, because both E^∞-terms are finite in each bidegree and are bounded to the right in each total dimension.

Since R^h induces a map of spectral sequences, R^h_* is continuous with respect to the filtration topologies. Hence $R_*=\widehat{\Gamma}_*^{-1}\circ R^h_*\circ\Gamma_*$ is continuous in dimensions greater than $2p-2$, where Γ_* and $\widehat{\Gamma}_*$ are homeomorphisms. □

We now decompose $E^\infty(S^1)$ as a sum of three subgroups.

Definition 7.6. Let $A=E(\lambda_1,\lambda_2)\otimes P(t\mu)$,

$$B_k=E(\lambda'_{[k]})\otimes P_{r(k)}(t\mu)\otimes \mathbf{F}_p\{\lambda_{[k]}\mu^{-dp^{k-1}}\mid 0<d<p\}\cap E^\infty(S^1)$$
$$=E(\lambda'_{[k]})\otimes \bigoplus_{0<d<p} P_{r(k)-dp^{k-1}}(t\mu)\otimes \mathbf{F}_p\{\lambda_{[k]}t^{dp^{k-1}}\}$$

and $B=\bigoplus_{k\geqslant 1}B_k$. Let C be the span of the remaining monomial terms $\lambda_1^{\varepsilon_1}\lambda_2^{\varepsilon_2}t^i\mu^j$ in $E^\infty(S^1)$. Then $E^\infty(S^1)=A\oplus B\oplus C$.

THEOREM 7.7. *In dimensions greater than* $2p-2$ *there are closed subgroups* $\tilde{A}=E(\lambda_1,\lambda_2)\otimes P(v_2)$, $\widetilde{B}_k$ *and* $\widetilde{C}$ *of* $V(1)_*TF(l;p)$ *represented by* A, B_k *and* C *in* $E^\infty(S^1)$, *respectively, such that*

(a) R_* *is the identity on* $\tilde{A}$;

(b) R_* *maps* $\widetilde{B}_{k+2}$ *onto* $\widetilde{B}_k$ *for all* $k\geqslant 1$;

(c) R_* *is zero on* $\widetilde{B}_1$, $\widetilde{B}_2$ *and* $\widetilde{C}$.

In these dimensions $V(1)_*TF(l;p)=\tilde{A}\oplus\widetilde{B}\oplus\widetilde{C}$, *with* $\widetilde{B}=\prod_{k\geqslant 1}\widetilde{B}_k$.

Proof. At the level of $E^\infty(S^1)$, the composite map $(\Gamma\widehat{\Gamma}^{-1})_*\circ E^\infty(R^h)$ is the identity on A, maps B_{k+2} onto B_k for all $k\geqslant 1$, and is zero on B_1, B_2 and C, by Propositions 7.2 and 7.4. The task is to find closed lifts of these groups to $V(1)_*TF(l;p)$ such that R_* has similar properties.

Let $\tilde{A}=E(\lambda_1,\lambda_2)\otimes P(v_2)\subset V(1)_*TF(l;p)$ be the (closed) subalgebra generated by the images of the classes λ_1^K, λ_2^K and v_2 in $V(1)_*K(l_p)$. Then $\tilde{A}$ lifts A and consists of classes in the image from $V(1)_*K(l_p)$. Hence R_* is the identity on $\tilde{A}$.

By Proposition 7.2 (c) we have $C\subset\ker E^\infty(R^h)$. Thus by Lemma 7.3 there is a closed subgroup $\widetilde{C}$ in $\ker(R_*)\cong\ker(R_*^h)$ represented by C. Then R_* is zero on $\widetilde{C}$.

The closed subgroups $\mathrm{im}(R_*)$ and $\ker(R_*)$ span $V(1)_*TF(l;p)$. For by Proposition 7.2 the representatives of $\mathrm{im}(R_*)$ span $A\oplus B$, and the representatives of the subgroup $\widetilde{C}$ in $\ker(R_*)$ span C. Thus the classes in $\mathrm{im}(R_*)$ and $\ker(R_*)$ have representatives spanning $E^\infty(S^1)$. Both $\mathrm{im}(R_*)$ and $\ker(R_*)$ are closed by Lemma 7.5, hence they span all of $V(1)_*TF(l;p)$.

It follows that the image of R_* on $V(1)_*TF(l;p)$ equals the image of its restriction to $\mathrm{im}(R_*)$.

Consider the finite subgroup

$$B_k^0=B_k\cap\ker E^\infty(R^h)=E(\lambda'_{[k]})\otimes\bigoplus_{0<d<p}P^{r(k)-dp^{k-1}-1}_{r(k-2)}(t\mu)\otimes\mathbf{F}_p\{\lambda_{[k]}t^{dp^{k-1}}\}$$

of $E^\infty(S^1)$ contained in the image of $(\Gamma\widehat{\Gamma}^{-1})_*\circ E^\infty(R^h)$ and the kernel of $E^\infty(R^h)$. It can be lifted to $\mathrm{im}(R_*)$ by Proposition 7.2, and to $\ker(R_*)$ by Lemma 7.3. We claim that it can be simultaneously lifted to a finite subgroup of $\mathrm{im}(R_*)\cap\ker(R_*)$.

(It suffices to lift a monomial basis for B_k^0 to $\mathrm{im}(R_*)\cap\ker(R_*)$ and take its span in $V(1)_*TF(l;p)$. To lift a basis element x in B_k^0, first lift it to a class $\tilde{x}$ in $\mathrm{im}(R_*)$, with $\Gamma_*(\tilde{x})$ represented by x. Then $R_*(\tilde{x})$ might not be zero, but $\widehat{\Gamma}_*R_*(\tilde{x})$ is represented by a class $y\in\widehat{E}^\infty(S^1)$ of strictly lower s-filtration than x. By Theorem 6.6 and Proposition 7.2 (b), y is in the image of $E^\infty(R^h)$, with $y=E^\infty(R^h)(z)$ for a class $z\in E^\infty(S^1)$ of strictly lower s-filtration than x. By Proposition 7.2 (b) and Proposition 7.4 we may assume that z is in the image of $E^\infty(R^h)$ followed by $(\Gamma\widehat{\Gamma}^{-1})_*$. Thus we can lift z to a class $\tilde{z}\in\mathrm{im}(R_*)$. Then $\widehat{\Gamma}_*R_*(\tilde{z})$ is represented by y. Replacing $\tilde{x}$ by $\tilde{x}-\tilde{z}$ keeps $\tilde{x}$ in $\mathrm{im}(R_*)$ as a lift of x, and strictly reduces the s-filtration of $R_*(\tilde{x})$. Iterating, and using strong convergence of $E^\infty(S^1)$, ensures that we can find a lift $\tilde{x}$ in $\mathrm{im}(R_*)\cap\ker(R_*)$, as desired.)

Let $\widetilde{B}_k^0\subset\mathrm{im}(R_*)\cap\ker(R_*)$ be such a lift.

Inductively for $n\geqslant 1$ let $B_k^n\subset B_{k+2n}\subset E^\infty(S^1)$ be the finite subgroup generated by the monomials mapped by $E^\infty(R^h)$ and $(\Gamma\widehat{\Gamma}^{-1})_*$ to the monomials generating B_k^{n-1}. Then B_k is the span of all B_{k-2n}^n for $n\geqslant 0$.

Suppose inductively that we have chosen a lift $\widetilde{B}_k^n\subset\mathrm{im}(R_*)$ of B_k^n which maps by R_* to $\widetilde{B}_k^{n-1}$ for $n\geqslant 1$, and to zero for $n=0$. Then choose monomial classes in $\mathrm{im}(R_*)$ mapping by R_* to generators of $\widetilde{B}_k^n$, and let $\widetilde{B}_k^{n+1}$ be the finite subgroup they generate. Then $\widetilde{B}_k^{n+1}$ is a lift of B_k^{n+1} by Proposition 7.2 (b) and Proposition 7.4.

Let $\widetilde{B}_k\subset V(1)_*TF(l;p)$ be the span of all $\widetilde{B}_{k-2n}^n$ for $n\geqslant 0$. Then $\widetilde{B}_k$ is represented by all of B_k, R_* maps $\widetilde{B}_{k+2}$ onto $\widetilde{B}_k$ for $k\geqslant 1$, and $\widetilde{B}_1$ and $\widetilde{B}_2$ lie in $\ker(R_*)$. □

8. Topological cyclic homology

We apply $V(1)$-homotopy to the cofiber sequence in §3.1 to obtain a long exact sequence

$$\ldots\xrightarrow{\partial}V(1)_*TC(l;p)\xrightarrow{\pi}V(1)_*TF(l;p)\xrightarrow{R_*-1}V(1)_*TF(l;p)\xrightarrow{\partial}\ldots\,. \tag{8.1}$$

PROPOSITION 8.2. *In dimensions greater than $2p-2$ there are isomorphisms*

$$\ker(R_*-1)\cong E(\lambda_1,\lambda_2)\otimes P(v_2)\oplus E(\lambda_2)\otimes P(v_2)\otimes\mathbf{F}_p\{\lambda_1t^d\mid 0<d<p\}$$
$$\oplus E(\lambda_1)\otimes P(v_2)\otimes\mathbf{F}_p\{\lambda_2t^{dp}\mid 0<d<p\}$$

and

$$\mathrm{cok}(R_*-1)\cong E(\lambda_1,\lambda_2)\otimes P(v_2).$$

Proof. By Theorem 7.7 the homomorphism R_*-1 is zero on $\tilde{A}=E(\lambda_1,\lambda_2)\otimes P(v_2)$, and an isomorphism on $\widetilde{C}$. The remainder of $V(1)_*TF(l;p)$ decomposes as

$$\widetilde{B}=\prod_{k\text{ odd}}\widetilde{B}_k\oplus\prod_{k\text{ even}}\widetilde{B}_k,$$

and R_* takes $\widetilde{B}_{k+2}$ to $\widetilde{B}_k$ for $k \geqslant 1$, forming two sequential limit systems. Hence there is an exact sequence

$$0 \to \lim_{k \text{ odd}} \widetilde{B}_k \to \prod_{k \text{ odd}} \widetilde{B}_k \xrightarrow{R_* - 1} \prod_{k \text{ odd}} \widetilde{B}_k \to \lim_{k \text{ odd}}{}^1 \widetilde{B}_k \to 0,$$

and a corresponding one for k even. The right derived limit vanishes since each $\widetilde{B}_k$ is finite. Hence it remains to prove that in dimensions greater than $2p-2$,

$$\lim_{k \text{ odd}} \widetilde{B}_k \cong E(\lambda_2) \otimes P(v_2) \otimes \mathbf{F}_p\{\lambda_1 t^d \mid 0 < d < p\}$$

and

$$\lim_{k \text{ even}} \widetilde{B}_k \cong E(\lambda_1) \otimes P(v_2) \otimes \mathbf{F}_p\{\lambda_2 t^{dp} \mid 0 < d < p\}.$$

Each $\widetilde{B}_k \cong B_k$ is a sum of $2p-2$ finite cyclic $P(v_2)$-modules. The restriction homomorphisms R_* respect this sum decomposition, and map each cyclic module surjectively onto the next. Hence their limit is a sum of $2p-2$ cyclic modules, and it remains to check that these are infinite cyclic, i.e., not bounded above.

For k odd the 'top' class $\lambda_1 \lambda_2 (t\mu)^{r(k)-1} \mu^{-dp^{k-1}}$ in B_k is in dimension $2p^{k+1}(p-d)$. For k even the corresponding class in B_k is in dimension $2p^{k+1}(p-d)+2p-2p^2$. In both cases the dimension grows to infinity for $0<d<p$ as k grows.

For k odd each infinite cyclic $P(v_2)$-module is generated by a class in non-negative degree with nonzero image in $\widetilde{B}_1 \cong B_1$, namely the classes $\lambda_1 t^d$ and $\lambda_1 \lambda_2 t^d$ for $0<d<p$. Hence we take these as generators for $\lim_{k \text{ odd}} \widetilde{B}_k$. Likewise there are generators in non-negative degrees for $\lim_{k \text{ even}} \widetilde{B}_k$ with nonzero image in $\widetilde{B}_2 \cong B_2$, namely the classes $\lambda_2 t^{dp}$ and $\lambda_1 \lambda_2 t^{dp}$ for $0<d<p$. □

Let $e \in \pi_{2p-1} TC(\mathbf{Z};p)$ be the image of $e^K \in K_{2p-1}(\mathbf{Z}_p)$, and let $\partial \in \pi_{-1} TC(\mathbf{Z};p)$ be the image of $1 \in \pi_0 TF(\mathbf{Z};p)$ under $\partial\colon \Sigma^{-1} TF(\mathbf{Z};p) \to TC(\mathbf{Z};p)$. We recall from [BM1], [BM2] the calculation of the mod p homotopy of $TC(\mathbf{Z};p)$.

THEOREM 8.3 (Bökstedt–Madsen).

$$V(0)_* TC(\mathbf{Z};p) \cong E(e, \partial) \otimes P(v_1) \oplus P(v_1) \otimes \mathbf{F}_p\{et^d \mid 0 < d < p\}.$$

Hence

$$V(1)_* TC(\mathbf{Z};p) \cong E(e, \partial) \oplus \mathbf{F}_p\{et^d \mid 0 < d < p\}.$$

The $(2p-2)$-connected map $l_p \to H\mathbf{Z}_p$ induces a $(2p-1)$-connected map $K(l_p) \to K(\mathbf{Z}_p)$, and thus a $(2p-1)$-connected map $TC(l;p) \to TC(\mathbf{Z};p)$ after p-adic completion, by [Du]. This brings us to our main theorem.

THEOREM 8.4. *There is an isomorphism of* $E(\lambda_1, \lambda_2) \otimes P(v_2)$*-modules*

$$V(1)_* TC(l;p) \cong E(\lambda_1, \lambda_2, \partial) \otimes P(v_2) \oplus E(\lambda_2) \otimes P(v_2) \otimes \mathbf{F}_p\{\lambda_1 t^d \mid 0 < d < p\}$$
$$\oplus E(\lambda_1) \otimes P(v_2) \otimes \mathbf{F}_p\{\lambda_2 t^{dp} \mid 0 < d < p\}$$

with $|\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, $|v_2| = 2p^2-2$, $|\partial| = -1$ *and* $|t| = -2$.

Proof. This follows in dimensions greater than $2p-2$ from Proposition 8.2 and the exact sequence (8.1). It follows in dimensions less than or equal to $2p-2$ from Theorem 8.3 and the $(2p-1)$-connected map $V(1)_* TC(l;p) \to V(1)_* TC(\mathbf{Z};p)$. It remains to check that the module structures are compatible for multiplications crossing dimension $2p-2$.

The classes

$$E(\lambda_1) \oplus \mathbf{F}_p\{\lambda_1 t^d \mid 0 < d < p\}$$

in $V(1)_* TC(l;p)$ map to

$$E(e) \oplus \mathbf{F}_p\{e t^d \mid 0 < d < p\}$$

in $V(1)_* TC(\mathbf{Z};p)$, and map by $\Gamma \circ \pi$ to classes with the same names in the S^1 homotopy fixed-point spectral sequence for $THH(\mathbf{Z})$. By naturality, the given classes in $V(1)_* TC(l;p)$ map by $\Gamma \circ \pi$ to classes with the same names in $E^\infty(S^1)$. Here these classes generate free $E(\lambda_2) \otimes P(t\mu)$-modules. For degree reasons multiplication by λ_1 is zero on each $\lambda_1 t^d$. Hence the $E(\lambda_1, \lambda_2) \otimes P(v_2)$-module action on the given classes is as claimed.

Finally the class ∂ in $V(1)_{-1} TC(l;p)$ is the image under the connecting homomorphism ∂ of the class 1 in $V(1)_* TF(l;p)$, which generates the free $E(\lambda_1, \lambda_2) \otimes P(v_2)$-module $\mathrm{cok}(R_* - 1)$ of Proposition 8.2. Hence also the module action on ∂ and $\lambda_1 \partial$ is as claimed. □

A very important feature of this calculational result is that $V(1)_* TC(l;p)$ is a finitely generated free $P(v_2)$-module. Thus $TC(l;p)$ is an fp-spectrum of fp-type 2 in the sense of [MR]. Notice that $V(1)_* TF(l;p)$ is not a free $P(v_2)$-module. On the other hand, we have the following calculation for the companion functor $TR(l;p) = \mathrm{holim}_{n,R} THH(l)^{C_{p^n}}$, showing that $V(1)_* TR(l;p)$ is a free but not finitely generated $P(v_2)$-module.

THEOREM 8.5. *There is an isomorphism of $E(\lambda_1,\lambda_2)\otimes P(v_2)$-modules*

$$V(1)_*TR(l;p)\cong E(\lambda_1,\lambda_2)\otimes P(v_2)\oplus\prod_{n\geqslant 1}E(u,\lambda_2)\otimes P(v_2)\otimes \mathbf{F}_p\{\lambda_1 t^d\mid 0<d<p\}$$
$$\oplus\prod_{n\geqslant 1}E(u,\lambda_1)\otimes P(v_2)\otimes \mathbf{F}_p\{\lambda_2 t^{dp}\mid 0<d<p\}.$$

The classes $u^\delta\lambda_1\lambda_2^{\varepsilon_2}t^d$ and $u^\delta\lambda_1^{\varepsilon_1}\lambda_2 t^{dp}$ in the n-th factors, for $\delta,\varepsilon_1,\varepsilon_2\in\{0,1\}$ and $0<d<p$, are detected in $V(1)_*THH(l)^{C_{p^n}}$ by classes that are represented by $u_n^\delta\lambda_1\lambda_2^{\varepsilon_2}t^d$ and $u_n^\delta\lambda_1^{\varepsilon_1}\lambda_2 t^{dp}$ in $E^\infty(C_{p^n})$, respectively.

We omit the proof. Compare [HM1, Theorem 5.5] and [HM2, 6.1.2] for similar results.

9. Algebraic K-theory

We are now in a position to describe the $V(1)$-homotopy of the algebraic K-theory spectrum of the p-completed Adams summand of connective topological K-theory, i.e., $V(1)_*K(l_p)$. We use the cyclotomic trace map to largely identify it with the corresponding topological cyclic homology. Hence we will identify the algebraic K-theory classes λ_1^K and λ_2^K with their cyclotomic trace images λ_1 and λ_2, in this section.

THEOREM 9.1. *There is an exact sequence of $E(\lambda_1,\lambda_2)\otimes P(v_2)$-modules*

$$0\to\Sigma^{2p-3}\mathbf{F}_p\to V(1)_*K(l_p)\xrightarrow{\text{trc}}V(1)_*TC(l;p)\to\Sigma^{-1}\mathbf{F}_p\to 0$$

taking the degree $2p-3$ generator in $\Sigma^{2p-3}\mathbf{F}_p$ to a class $a\in V(1)_{2p-3}K(l_p)$, and taking the class ∂ in $V(1)_{-1}TC(l;p)$ to the degree -1 generator in $\Sigma^{-1}\mathbf{F}_p$. Hence

$$\begin{aligned}V(1)_*K(l_p)\cong{}&E(\lambda_1,\lambda_2)\otimes P(v_2)\oplus P(v_2)\otimes\mathbf{F}_p\{\partial\lambda_1,\partial v_2,\partial\lambda_2,\partial\lambda_1\lambda_2\}\\&\oplus E(\lambda_2)\otimes P(v_2)\otimes\mathbf{F}_p\{\lambda_1t^d\mid 0<d<p\}\\&\oplus E(\lambda_1)\otimes P(v_2)\otimes\mathbf{F}_p\{\lambda_2t^{dp}\mid 0<d<p\}\\&\oplus\mathbf{F}_p\{a\}.\end{aligned}$$

Proof. By [HM1] the map $l_p\to H\mathbf{Z}_p$ induces a map of horizontal cofiber sequences of p-complete spectra:

$$\begin{array}{ccccc}K(l_p)_p & \xrightarrow{\text{trc}} & TC(l;p) & \longrightarrow & \Sigma^{-1}H\mathbf{Z}_p\\ \downarrow & & \downarrow & & \|\\ K(\mathbf{Z}_p)_p & \xrightarrow{\text{trc}} & TC(\mathbf{Z};p) & \longrightarrow & \Sigma^{-1}H\mathbf{Z}_p.\end{array}$$

Here $V(1)_*\Sigma^{-1}H\mathbf{Z}_p$ is $\mathbf{F}_p$ in degrees -1 and $2p-2$, and 0 otherwise. Clearly ∂ in $V(1)_*TC(l;p)$ maps to the generator in degree -1, since $K(l_p)_p$ is a connective spectrum. The connecting map in $V(1)$-homotopy for the lower cofiber sequence takes the generator in degree $2p-2$ to the nonzero class $i_1(\partial v_1)$ in $V(1)_{2p-3}K(\mathbf{Z}_p)$. By naturality it factors through $V(1)_{2p-3}K(l_p)$, where we let a be its image. □

Hence also $K(l_p)_p$ is an fp-spectrum of fp-type 2. By [MR, 3.2] its mod p spectrum cohomology is finitely presented as a module over the Steenrod algebra, hence is induced up from a finite module over a finite subalgebra of the Steenrod algebra. In particular, $K(l_p)_p$ is closely related to elliptic cohomology.

9.2. *The* mod p *homotopy of* $K(l_p)$. We would now like to use the v_1-Bockstein spectral sequence to determine the mod p homotopy of $K(l_p)$ from its $V(1)$-homotopy, and then to use the usual p-primary Bockstein spectral sequence to identify $\pi_*K(l_p)_p$. We shall see in Lemma 9.3 that the $P(v_2)$-module generators of $V(1)_*K(l_p)$ all lift to mod p homotopy. In Lemma 9.4 this gives us formulas for the primary v_1-Bockstein differentials $\beta_{1,1}$. But there also appear to be higher-order v_1-Bockstein differentials, as indicated in Lemma 9.5, which shows that the general picture is rather complicated.

For any X, classes in the image of $i_1: V(0)_*X \to V(1)_*X$ are called mod p classes, while classes in the image of $i_1 i_0: \pi_*X_p \to V(1)_*X$ are called integral classes.

LEMMA 9.3. *The classes* 1, $\partial\lambda_1$, λ_1 *and* $\lambda_1 t^d$ *for* $0<d<p$ *are integral classes both in* $V(1)_*K(l_p)$ *and* $V(1)_*TC(l;p)$. *Also* ∂ *is integral in* $V(1)_*TC(l;p)$, *while* a *and* ∂v_2 *are integral in* $V(1)_*K(l_p)$.

The classes $\partial\lambda_2$, λ_2, $\partial\lambda_1\lambda_2$, $\lambda_1\lambda_2$, $\lambda_1\lambda_2 t^d$, $\lambda_2 t^{dp}$ *and* $\lambda_1\lambda_2 t^{dp}$ *for* $0<d<p$ *are* mod p *classes in both* $V(1)_*K(l_p)$ *and* $V(1)_*TC(l;p)$.

We are not excluding the possibility that some of the mod p classes are actually integral classes.

Proof. Each v_1-Bockstein $\beta_{1,r}$ lands in a trivial group when applied to the classes ∂, 1, a and $\lambda_1 t^d$ for $0<d<p$ in $V(1)_*K(l_p)$ or $V(1)_*TC(l;p)$. Hence these are at least mod p classes.

Since 1 maps to an element of infinite order in $\pi_0 TC(\mathbf{Z};p)\cong\mathbf{Z}_p$ and the other classes sit in odd degrees, all mod p^r Bocksteins on these classes are zero. Hence they are integral classes. The class λ_1 is integral by construction, hence so is the product $\partial\lambda_1$.

The primary v_1-Bockstein $\beta_{1,1}$ applied to ∂v_2 in $V(1)_*K(l_p)$ is zero, because it lands in degree $2p^2-2p-2$ of $\mathrm{im}(\partial)=\mathrm{cok}(R_*-1)$, which by Proposition 8.2 is zero in this degree. The higher v_1-Bocksteins $\beta_{1,r}(\partial v_2)$ all land in zero groups, so ∂v_2 admits a mod p lift. Again, all mod p^r Bocksteins on this lift land in a zero group, so ∂v_2 must be an integral class.

The mod p homotopy operation $(P^{p-d})^*$ takes $\lambda_1 t^d$ in integral homotopy to $\lambda_2 t^{dp}$ in mod p homotopy, for $0<d<p$. Hence these are all mod p classes, as is λ_2 by construction. The remaining classes listed are then products of established integral and mod p classes, and are therefore mod p classes. □

The classes listed in this lemma generate $V(1)_*K(l_p)$ and $V(1)_*TC(l;p)$ as $P(v_2)$-modules. But v_2 itself is not a mod p class.

LEMMA 9.4. *Let x be a* mod p *(or integral) class of $V(1)_*K(l_p)$ or $V(1)_*TC(l;p)$, and let $t\geqslant 0$. Then*

$$\beta_{1,1}(v_2^t\cdot x) = tv_2^{t-1}i_1(\beta_1')\cdot x.$$

In particular, $i_1(\beta_1')\cdot 1 = t^p\lambda_2$ and $i_1(\beta_1')\cdot\lambda_1 = t^p\lambda_1\lambda_2$.

We expect that $i_1(\beta_1')\cdot t^{p^2-p}\lambda_2 = \partial\lambda_2$ and $i_1(\beta_1')\cdot t^{p^2-p}\lambda_1\lambda_2 = \partial\lambda_1\lambda_2$, by duality and symmetry considerations.

Proof. The v_1-Bockstein $\beta_{1,1} = i_1 j_1$ acts as a derivation by [Ok]. By definition $j_1(v_2) = \beta_1' = [h_{11}]$, which is detected as $t^p\lambda_2$ by Proposition 4.8. Clearly $j_1(x) = 0$ for mod p classes x. □

In $V(1)_*$ the powers v_2^t support nonzero differentials $\beta_{1,1}(v_2^t) = tv_2^{t-1}i_1(\beta_1')$ for $p\nmid t$. The first nonzero differential on v_2^p is $\beta_{1,p}$:

LEMMA 9.5. *$\beta_{1,p}(v_2^p) = [h_{12}] \neq 0$ in $V(1)_*$.*

We refer to [Ra2, §4.4] for background for the following calculation.

Proof. In the BP-based Adams–Novikov spectral sequence for $V(0)$ the relation $j_1(v_2^p) = v_1^{p-1}\beta_{p/p}'$ holds, where $\beta_{p/p}'$ is the class represented by $h_{12} + v_1^{p^2-p}h_{11}$ in degree 1 of the cobar complex. Its integral image $\beta_{p/p} = j_0(\beta_{p/p}')$ is represented by b_{11}, and supports the Toda differential $d_{2p-1}(\beta_{p/p}) = \alpha_1\beta_1^p$. This differential lifts to $d_{2p-1}(\beta_{p/p}') = v_1\beta_1^p$ in the Adams–Novikov spectral sequence for $V(0)$. Consider the image of $\beta_{p/p}'$ under i_1 in the Adams–Novikov spectral sequence for $V(1)$, which is represented by h_{12} in the cobar complex. Then $d_{2p-1}(i_1(\beta_{p/p}')) = i_1(v_1\beta_1^p) = 0$. By sparseness and the vanishing line there are no further differentials, and $i_1(\beta_{p/p}') = [h_{12}]$ represents a nonzero element of $V(1)_*$. Hence $\beta_{1,p}(v_2^p) = [h_{12}]$, as claimed. □

To determine the mod p homotopy groups of $TC(l;p)$ or $K(l_p)$ by means of the v_1-Bockstein spectral sequence we must first compute the remaining products $i_1(\beta_1')\cdot x$ in Lemma 9.4. Next we must identify the image of $\beta_{1,p}(v_2^p) = [h_{12}]$ in $V(1)_*TC(l;p)$. Imaginably this equals the generator $v_2^{p-1}\lambda_1 t$ of $V(1)_*TC(l;p)$ in this degree. If so, much of the great complexity of the v_1-Bockstein spectral sequence for the sphere spectrum also carries over to the v_1-Bockstein spectral sequence for $TC(l;p)$. We view this as justification for stating the result of our calculations in terms of $V(1)$-homotopy instead.

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