STRATIFIED NONCOMMUTATIVE GEOMETRY

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ABSTRACT. We introduce a theory of stratifications of noncommutative stacks (i.e. presentable stable ∞-categories), and we prove a reconstruction theorem that expresses them in terms of their strata and gluing data. This reconstruction theorem is compatible with symmetric monoidal structures, and with more general operadic structures such as $\mathbb{E}_n$-monoidal structures. We also provide a suite of fundamental operations for constructing new stratifications from old ones: restriction, pullback, quotient, pushforward, and refinement. Moreover, we establish a dual form of reconstruction; this is closely related to Verdier duality and reflection functors, and gives a categorification of Möbius inversion.

Our main application is to equivariant stable homotopy theory: for any compact Lie group G, we give a symmetric monoidal stratification of genuine G-spectra. In the case that G is finite, this expresses genuine G-spectra in terms of their geometric fixedpoints (as homotopy-equivariant spectra) and gluing data therebetween (which are given by proper Tate constructions).

We also prove an adelic reconstruction theorem; this applies not just to ordinary schemes but in the more general context of tensor-triangular geometry, where we obtain a symmetric monoidal stratification over the Balmer spectrum. We discuss the particular example of chromatic homotopy theory.

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0. Introduction

0.1. **Overview.** In this paper, we develop a theory of ***stratified noncommutative stacks***. We take the term *noncommutative stack* to mean a presentable stable ∞-category, as explained in Remark 0.1.2.[1] We suggestively refer to the objects of a noncommutative stack as its *quasicoherent sheaves*.[2] Our novel contribution is a theory of *stratifications*.[3] In short, a stratification of a noncommutative stack $\mathcal{X}$ is a filtration by noncommutative substacks $\{\mathcal{Z}_p\}_{p\in\mathsf{P}}$ indexed by a poset P that satisfies certain natural geometrically-inspired conditions; for each $p \in \mathsf{P}$, the p^{th} ***stratum*** of the stratification is the associated-graded $\mathcal{X}_p := \mathcal{Z}_p/\mathcal{Z}_{<p}$.[4]

The primary purpose of stratifications is that they provide ***reconstruction theorems***, in a way that can be summarized informally as follows.[5]

Slogan 0.1.1. *Let $\mathcal{X}$ be a noncommutative stack equipped with a stratification over a poset P.*

(1) **macrocosm:** *The noncommutative stack $\mathcal{X}$ can be reconstructed from the strata*
$$\{\mathcal{X}_p \subseteq \mathcal{X}\}_{p\in\mathsf{P}}$$
along with gluing data between them.

(2) **microcosm:** *Each quasicoherent sheaf $\mathcal{F} \in \mathcal{X}$ can be reconstructed from its geometric localizations*
$$\{\Phi_p(\mathcal{F}) \in \mathcal{X}_p\}_{p\in\mathsf{P}}$$
along with gluing data between them.

The simplest interesting example of a stratification is when $\mathsf{P} = \{0 < 1\}$: in this case we recover the data of a *recollement* (which we review for the reader's convenience in §1.1).[6]

Our main application is a symmetric monoidal reconstruction theorem for genuine G-spectra, which has particularly simple strata.[7] The eager reader may turn directly to §5.3 to see specific examples of this reconstruction theorem in action:

- genuine G-spectra where G is one of the cyclic groups C_p, C_{p^2}, and C_{pq} (for distinct primes p and q) or the symmetric group S_3, and
- proper-genuine $\mathbb{T}$-spectra, where $\mathbb{T}$ denotes the circle group.

In [AMGRd], we build on this last example to provide a symmetric monoidal reconstruction theorem for cyclotomic spectra. This improves on the foundational work [NS18] of Nikolaus–Scholze, in that it applies to all cyclotomic spectra (instead of only eventually-connective ones) and specifies its canonical symmetric monoidal structure. In particular, it provides a universal mapping-in property

[1]Our results apply equally well to pretriangulated dg-categories admitting all direct sums (or more precisely, to their underlying $\Bbbk$-linear presentable stable ∞-categories).

[2]In particular, an ordinary scheme or stack X has an underlying noncommutative stack $\mathsf{QC}(X)$, its presentable stable ∞-category of quasicoherent sheaves.

[3]This builds on work of Glasman and others, as described in §0.2.

[4]As we explain in §§1.2-1.3, a stratification of an ordinary scheme X determines a stratification of $\mathsf{QC}(X)$ via set-theoretic support on closures of strata, whose strata are closely related to those of X. (On the other hand, in general not all stratifications of $\mathsf{QC}(X)$ arise from stratifications of X.)

[5]Our terminology for the two parts of Slogan 0.1.1 is inspired by the "macrocosm/microcosm principle", which asserts e.g. that it is precisely a monoidal structure on a category that enables one to speak of algebra objects in that category. In the present situation, macrocosm reconstruction for the noncommutative stack $\mathcal{X}$ enables microcosm reconstruction for each quasicoherent sheaf $\mathcal{F} \in \mathcal{X}$. This is a familiar phenomenon from classical sheaf theory: categories of globally-defined sheaves can be reconstructed from categories of locally-defined sheaves, and so globally-defined sheaves can be reconstructed from locally-defined sheaves.

[6]The French word *recollement* translates to "regluing".

[7]At the microcosm level, this presents a genuine G-spectrum in terms of its geometric fixedpoints (as opposed to its presentation in terms of its categorical fixedpoints as a spectral Mackey functor [GM, Bar17]).

at the level of objects, which we use to obtain the cyclotomic trace map

$$\mathsf{K} \longrightarrow \mathsf{TC}$$

from algebraic K-theory to topological cyclic homology in [AMGRc]. In a different direction, in [AMGRa] we apply our reconstruction theorem to compute the C_{p^n}-equivariant cohomology of a point for any odd prime p.

We also set up an $\mathcal{O}$-monoidal enhancement of our theory, where $\mathcal{O}$ denotes any ∞-operad satisfying certain mild conditions (e.g. $\mathbb{E}_n$ for $1 \leq n \leq \infty$); this accounts for the symmetric monoidality of our reconstruction theorem for genuine G-spectra. In this vein, we make contact with the world of tensor-triangular geometry, by showing that under mild hypotheses a presentably symmetric monoidal stable ∞-category admits a canonical *adelic stratification*, which is a symmetric monoidal stratification over the specialization poset of its Balmer spectrum. The adelic stratification of $\mathsf{Mod}_{\mathbb{Z}}$ recovers the classical *arithmetic fracture square*, which is the natural pullback square for any $M \in \mathsf{Mod}_{\mathbb{Z}}$ that is indicated in Figure 1. More generally, for any scheme X satisfying mild finiteness

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \mathbb{Q} \otimes_{\mathbb{Z}} M \arrow[d] \\ \prod\limits_{p \text{ prime}} M_p^{\wedge} \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod\limits_{p \text{ prime}} M_p^{\wedge} \right) \end{tikzcd} \tag{0.1.1}$$

FIGURE 1. The arithmetic fracture square is a natural pullback square that reconstructs any $M \in \mathsf{Mod}_{\mathbb{Z}}$ from its rationalization, its p-completions, and gluing data between them.

hypotheses, the adelic stratification of $\mathsf{QC}(X)$ leads to an *adelic reconstruction theorem*, which bears a close relationship to existing such formalisms of Beilinson and others. Moreover, the *chromatic stratification* of the ∞-category $\mathcal{S}\mathsf{p}$ of spectra organizes the fundamental objects of chromatic homotopy theory and recovers integral (i.e. not p-local) and higher-dimensional variants of the chromatic fracture square, as described in Example 4.3.8.[8]

In a different direction, we introduce the theory of ***reflection***. This affords a dual form of reconstruction; applied to $\mathsf{Mod}_{\mathbb{Z}}$, this yields the *reflected arithmetic fracture square*, which is the natural pushout square for any $M \in \mathsf{Mod}_{\mathbb{Z}}$ that is indicated in Figure 2.[9] In particular, we establish a precise relationship between the gluing data and the reflected gluing data; in the case of $\mathsf{Mod}_{\mathbb{Z}}$, this specializes to the remarkable equivalence

$$\mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod_p M_p^{\wedge} \right) \simeq \Sigma \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}} \left(\mathbb{Q}, \left(\bigoplus_p M_p^{\mathsf{tors}} \right) \right) .$$

Specialized to the poset $\mathsf{P} = [1]$, reflection recovers the theory of reflection functors (which explains our choice of terminology). More generally, it gives a categorification of the Möbius inversion formula and is closely related to Verdier duality.

[8] This is closely related to its adelic stratification, which is described in Example 4.6.13.

[9] This particular example can be seen as a consequence of Greenlees–May duality (or even of local duality for $\mathsf{Spec}(\mathbb{Z})$).

[10] We write $M_p^{\mathsf{tors}} := \mathsf{fib}(M \to M \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}])$ for the p-torsionification of M, in analogy with the notation $M_p^{\wedge}$ for its p-completion.

$$\begin{tikzcd}
\underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}}\left(\mathbb{Q}, \left(\bigoplus_{p \text{ prime}} M_p^{\mathsf{tors}}\right)\right) \arrow[r] \arrow[d] & \underline{\mathsf{hom}}_{\mathsf{Mod}_{\mathbb{Z}}}(\mathbb{Q}, M) \arrow[d] \\
\bigoplus_{p \text{ prime}} M_p^{\mathsf{tors}} \arrow[r] & M
\end{tikzcd}$$

FIGURE 2. The reflected arithmetic fracture square is a natural pushout square that reconstructs any $M \in \mathsf{Mod}_{\mathbb{Z}}$ from its corationalization, its p-torsionifications, and gluing data between them.[10]

We give a detailed overview of our work in §1, which begins with some recollections and motivation. Our main theorems (which are stated more precisely therein) may be summarized as follows.

- Theorem A is our ***reconstruction theorem*** for stratified noncommutative stacks, a precise articulation of Slogan 0.1.1. In fact, it provides a universal mapping-in property – that is, a limit-type description – both at the macrocosm level (for noncommutative stacks) and at the microcosm level (for their quasicoherent sheaves).

- Theorem B provides a suite of ***fundamental operations*** for constructing new stratifications from old ones: restriction, pullback, quotient, pushforward, and refinement.

- Theorem C is our ***$\mathcal{O}$-monoidal reconstruction theorem***, an enhancement of Theorem A. At the macrocosm level, this provides universal mapping-in properties for presentably $\mathcal{O}$-monoidal stable ∞-categories as such.

- Theorem D establishes the symmetric monoidal ***adelic stratification*** of a presentably symmetric monoidal stable ∞-category satisfying mild finiteness hypotheses over (the specialization poset of) its Balmer spectrum.

- Theorem E establishes the symmetric monoidal ***geometric stratification*** of the presentably symmetric monoidal stable ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ of genuine G-spectra, where G is any compact Lie group. This has the following features:

  – its strata are the presentably symmetric monoidal stable ∞-categories
  $$\mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} := \mathsf{Fun}(\mathsf{BW}(H), \mathcal{S}\mathsf{p})$$
  of homotopy $\mathsf{W}(H)$-spectra, where H is a closed subgroup of G and $\mathsf{W}(H)$ denotes its Weyl group;

  – its geometric localization functors are the geometric fixedpoints functors
  $$\mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{\Phi^H} \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \ ;$$
  and

  – its gluing functors are given by a version of the Tate construction.

  As explained in Remark 1.7.2, this provides a sense in which genuine G-spectra are the quasicoherent sheaves on a "nearly commutative" stack.

- Theorem F establishes the theory of ***reflection***, which affords a dual form of reconstruction for stratified noncommutative stacks.

In §1 we also discuss a number of additional applications of our work: constructible sheaves; categorified Möbius inversion; naive G-spectra; t-structures; and additive and localizing invariants.

Remark 0.1.2. The philosophy of noncommutative algebraic geometry can be traced back to Gabriel's thesis [Gab62], in which he proved that one can reconstruct a scheme from its abelian category of quasicoherent sheaves. Following this, Manin proposed that arbitrary abelian categories might therefore be thought of as categories of quasicoherent sheaves on "noncommutative schemes" [Man88, §12.6]. This proposal has since been developed further by many authors, notably Rosenberg [Ros98b, Ros98a], as well as Kontsevich–Rosenberg [KR00] and Kontsevich–Soibelman [KS09] from a more derived perspective. Our usage of the term "noncommutative stack" to mean a presentable stable ∞-category is inspired by this trajectory.

0.2. **Relations with existing literature.** A number of distinct narrative threads converge in the present work, some of which we discuss here. However, the literature is vast, and we make no attempt to be comprehensive.

0.2.1. *Recollements and semiorthogonal decompositions.* Stratifications admit a rich history: they generalize recollements (which are stratifications over $[1]$) and more generally semiorthogonal decompositions (which are stratifications over $[n]$).[11] Recollements were originally introduced by Beilinson–Bernstein–Deligne in their study of perverse sheaves [BBD82]. A fruitful source of semiorthogonal decompositions is exceptional collections; this technique first appeared in Beilinson's calculation of the derived category of $\mathbb{P}^n$ [Beĭ78], and was pursued more systematically by Bondal–Kapranov [BK89]. Semiorthogonal decompositions continue to be a highly active area of research, especially in connection with algebraic geometry; see e.g. [Kuz14] for more in this direction.

0.2.2. *Adelic reconstruction.* As explained in §1.6, given a scheme X, our work provides a decomposition of $\mathsf{QC}(X)$ in adelic terms; this generalizes the arithmetic fracture square (0.1.1), which corresponds to the case that $X = \mathsf{Spec}(\mathbb{Z})$. This is quite similar to prior adelic reconstruction results in the literature, e.g. [Par76, Beĭ80, Hub91, Gro17, HPV]. However, there is a subtle difference, even in the case of $X = \mathsf{Spec}(\mathbb{Z})$: we recover the arithmetic fracture square (0.1.1) for all $\mathbb{Z}$-modules M, despite the fact that two of its terms don't commute with filtered colimits in the variable M. In the specific context of tensor-triangulated geometry, [BG20] provides a symmetric monoidal macrocosm-type reconstruction theorem.

0.2.3. *Chromatic homotopy theory.* Reconstruction has long been a guiding principle in homotopy theory, going back to Sullivan's influential lecture notes [Sul05]. The chromatic approach to stable homotopy theory grew out of Ravenel's work [Rav84] and the resulting nilpotence and periodicity theorems of Devinatz–Hopkins–Smith [DHS88, HS98], along with the extensive axiomatic treatment of Hovey–Palmieri–Strickland [HPS97] – all pointing to the chromatic fracture squares as essential from the perspective of reconstruction. More recently, higher-dimensional chromatic fracture cubes for p-local spectra – and indeed, corresponding macrocosm reconstruction theorems – appear e.g. in [Glab, Examples 3.14 and 3.31] and [ACB22].

0.2.4. *Reconstruction for genuine G-spectra.* The idea that genuine G-spectra can be expressed in terms of their geometric fixedpoints stems from the work of Greenlees and May; see in particular [Gre, GM95]. There is also much work on similar expressions of rational G-spectra (which are simpler because the relevant Tate constructions vanish rationally), notably the reconstruction results of Greenlees–Shipley [GS18]. More recent works in this direction include [MNN17, Glab]; see also [NS18, Remark II.4.8].

[11] In the present discussion we do not distinguish between the small and presentable settings.

0.2.5. *Glasman's theory of stratifications.* Theorems A and E are directly inspired by Glasman's paper [Glab], as we now explain.

In [Glab, Definition 3.5], Glasman introduces a notion of a stratification of a stable ∞-category (not assumed to be presentable). His definition is phrased in terms of the strata (in the sense of Definition 2.4.6) for all convex subsets $\mathsf{C} \subseteq \mathsf{P}$ of the stratifying poset. He proves a reconstruction result for his stratifications [Glab, Theorem 3.21], and for any finite group G he provides a stratification of the ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ genuine G-spectra over the poset P_G of conjugacy classes of subgroups of G [Glab, Proposition 3.18].

By contrast, we work primarily in the setting of presentable stable ∞-categories. This enables us to give a relatively simple definition of a stratification, in terms of closed subcategories indexed by the poset P itself (rather than by its poset of convex subsets): we recover the strata as presentable quotients. (These notions are summarized in §1.3.) On the other hand, using this we also provide a theory of stratifications of stable ∞-categories (see §7.2).[12] This effectively recovers Glasman's theory of stratifications, and offers a substantial refinement of his reconstruction theorem as well (which is a version of our microcosm reconstruction).

0.3. **Outline.** This paper is organized as follows.

- §1: We give a detailed overview of our work, and explain a number of fundamental examples and applications.
- §2: We introduce closed subcategories and stratifications. We prove that the macrocosm reconstruction theorem (Theorem A(2)) follows from the metacosm reconstruction theorem (Theorem A(1)).
- §3: We establish our fundamental operations on stratifications (Theorem B). We accomplish this by studying the phenomenon of *alignment*.
- §4: We introduce $\mathcal{O}$-monoidal stratifications and prove the $\mathcal{O}$-monoidal reconstruction theorem (Theorem C). We also establish the adelic stratification (Theorem D), which we unpack in the setting of chromatic homotopy theory.
- §5: We review the ∞-category of genuine G-spectra and establish its geometric stratification (Theorem E). We record a few facts about its gluing functors, which are essentially given by proper Tate constructions. Using these facts, we unpack a number of examples of reconstruction for genuine G-spectra. We also give a formula for categorical fixedpoints in terms of the geometric stratification, and we explain how this interacts with restriction and transfer.
- §6: We prove the metacosm reconstruction theorem (Theorem A(1)).
- §7: We prove a number of variants of the metacosm reconstruction theorem, notably our dual form of reconstruction and the theory of reflection (Theorem F).
- §A: We review the theory of lax modules and lax limits, and record a number of results that we need. This material is used systematically throughout the main body of the paper, but this usage is confined to proofs (rather than assertions) to the greatest extent possible.
- §B: We establish the necessary background regarding $(\infty, 2)$-categories, particularly the theory of lax functors and natural transformations as well as the theory of adjunctions. This material primarily supports §A.

0.4. **Notation and conventions.**

[12] As a matter of convenience, we restrict our attention to idempotent-complete stable ∞-categories.

(1) We work within the context of ∞-categories, taking [Lur09] and [Lur] as our standard references. We work model-independently (for instance, we make no reference to the simplices of a quasicategory), and we omit all technical uses of the word "essentially" (for instance, we shorten the term "essentially surjective" to "surjective"). We also make some light use of the theory of $(\infty, 2)$-categories; §B is devoted to its relevant aspects.

(2) We use the following decorations for our functors.[13]

- The arrow in the notation
$$\mathcal{C} \hookrightarrow \mathcal{D}$$
denotes a monomorphism, i.e. the inclusion of a subcategory: a functor which is fully faithful on equivalences and induces inclusions of path components (i.e. monomorphisms) on all hom-spaces.

- The arrow in the notation
$$\mathcal{C} \overset{\mathsf{f.f.}}{\hookrightarrow} \mathcal{D}$$
denotes a fully faithful functor. (However, the notation "f.f." is merely emphasis: one should not take its absence to mean that the indicated monomorphism is not fully faithful.)

- The arrow in the notation
$$\mathcal{C} \twoheadrightarrow \mathcal{D}$$
denotes a surjection.

- The arrow in the notation
$$\mathcal{C} \downarrow \mathcal{D}$$
denotes a functor $\mathcal{C} \to \mathcal{D}$ considered as an object of the overcategory $\mathsf{Cat}_{/\mathcal{D}}$ of its target (which will often be some sort of fibration).

More generally, we use the notation $X \downarrow Y$ to denote a morphism in any ∞-category $\mathcal{C}$ that we consider as defining an object in the overcategory $\mathcal{C}_{/Y}$.

(3) Given some datum in an ∞-category (such as an object or morphism), for clarity we may use the superscript $(-)^{\circ}$ to denote the corresponding datum in the opposite ∞-category.

(4) Given a functor F, we write F^* for pullback along it, and we respectively write $F_!$ and F_* for left and right Kan extension along it.

(5) We write Cat for the ∞-category of ∞-categories, $\mathcal{S}$ for the ∞-category of spaces, and $\mathcal{S}\mathsf{p}$ for the ∞-category of spectra. These are related by the various adjoint functors

$$\begin{tikzcd}
\mathsf{Cat} \arrow[r, bend left=40, "|-|"] \arrow[r, bend right=40, "\iota_0"'] & \mathcal{S} \arrow[l, hook'] \arrow[l, phantom, shift right=2.5, "\perp"] \arrow[l, phantom, shift left=2.5, "\perp"] \arrow[r, shift left, "\Sigma^\infty_+"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p} \arrow[l, shift left, "\Omega^\infty"]
\end{tikzcd} .$$

(6) We define the commutative diagram of monomorphisms among ∞-categories

$$\begin{tikzcd}
\mathsf{Pr}^L \arrow[r, hook, two heads] & \mathsf{Pr} & \mathsf{Pr}^R \arrow[l, hook', two heads] \\
\mathsf{Pr}^L_{\mathsf{st}} \arrow[u, hook, "\mathsf{f.f.}"] \arrow[r, hook, two heads] & \mathsf{Pr}_{\mathsf{st}} \arrow[u, hook] & \mathsf{Pr}^R_{\mathsf{st}} \arrow[u, hook, "\mathsf{f.f.}"'] \arrow[l, hook', two heads]
\end{tikzcd}$$

as follows:

[13] These are only for emphasis: the absence of such a decoration should not be taken to imply that the corresponding adjective does not apply.

- objects in the upper rows are presentable ∞-categories,
- objects in the lower rows are presentable stable ∞-categories,
- morphisms in the left column are left adjoint functors,
- morphisms in the middle column are accessible functors, with morphisms in $\mathsf{Pr}_{\mathsf{st}}$ additionally required to be exact, and
- morphisms in the right column are right adjoint functors.

So, passing to adjoints determines equivalences $\mathsf{Pr}^L \simeq (\mathsf{Pr}^R)^{\mathsf{op}}$ and $\mathsf{Pr}^L_{\mathsf{st}} \simeq (\mathsf{Pr}^R_{\mathsf{st}})^{\mathsf{op}}$, and both squares are pullbacks. Moreover, we define the ∞-categories appearing in the outer two columns of the commutative squares

$$\begin{tikzcd} \mathsf{Pr}^{L,\omega} \arrow[r, hook] & \mathsf{Pr}^L \\ \mathsf{Pr}^{L,\omega}_{\mathsf{st}} \arrow[r, hook] \arrow[u, hook, "\mathsf{f.f.}"] & \mathsf{Pr}^L_{\mathsf{st}} \arrow[u, hook, "\mathsf{f.f.}"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{Pr}^{R} & \mathsf{Pr}^{R,\omega} \arrow[l, hook'] \\ \mathsf{Pr}^{R}_{\mathsf{st}} \arrow[u, hook, "\mathsf{f.f.}"] & \mathsf{Pr}^{R,\omega}_{\mathsf{st}} \arrow[l, hook'] \arrow[u, hook, "\mathsf{f.f.}"'] \end{tikzcd}$$

as follows:

- in all cases, their objects are additionally required to be compactly generated,
- for those in the left square, their morphisms are additionally required to preserve compact objects, and
- for those in the right square, their morphisms are additionally required to preserve filtered colimits.

So, passing to adjoints determines equivalences $\mathsf{Pr}^{L,\omega} \simeq (\mathsf{Pr}^{R,\omega})^{\mathsf{op}}$ and $\mathsf{Pr}^{L,\omega}_{\mathsf{st}} \simeq (\mathsf{Pr}^{R,\omega}_{\mathsf{st}})^{\mathsf{op}}$, and both squares are pullbacks.

(7) For a base ∞-category $\mathcal{B}$, we define the commutative diagrams of monomorphisms among ∞-categories

$$\begin{tikzcd} \mathsf{coCart}_{\mathcal{B}} \arrow[r, hook, two heads] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}} \arrow[d, hook, "\mathsf{f.f.}"] \\ \mathsf{loc.coCart}_{\mathcal{B}} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{loc.cocart}/\mathcal{B}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{Cart}_{\mathcal{B}} \arrow[r, hook, two heads] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{Cat}_{\mathsf{cart}/\mathcal{B}} \arrow[d, hook, "\mathsf{f.f.}"] \\ \mathsf{loc.Cart}_{\mathcal{B}} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{loc.cart}/\mathcal{B}} \end{tikzcd}$$

as follows:

- objects in the upper rows are co/cartesian fibrations over $\mathcal{B}$,
- objects in the lower rows are locally co/cartesian fibrations over $\mathcal{B}$,
- morphisms in the left columns are functors over $\mathcal{B}$ which preserve co/cartesian morphisms, and
- morphisms in the right columns are arbitrary functors over $\mathcal{B}$.

We write

$$\begin{tikzcd} \mathsf{coCart}_{\mathcal{B}} \simeq \mathsf{Fun}(\mathcal{B}, \mathsf{Cat}) \simeq \mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}} \arrow[loop, phantom] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{coCart}_{\mathcal{B}} \arrow[r, bend left, "(-)^{\vee}_{\mathsf{cocart}}"] & \mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}} \arrow[l, bend left, "(-)^{\vee}_{\mathsf{cart}}"] \end{tikzcd}$$

for the composite equivalences, and refer to them as the *cocartesian dual* and *cartesian dual* functors (named for their respective sources) [BGN18]. We respectively write

$$\mathsf{LFib}_{\mathcal{B}} \subseteq \mathsf{coCart}_{\mathcal{B}} \qquad \text{and} \qquad \mathsf{RFib}_{\mathcal{B}} \subseteq \mathsf{Cart}_{\mathcal{B}}$$

for the full subcategories on the left and right fibrations. When the base ∞-category $\mathcal{B}$ is understood, for any ∞-category $\mathcal{C}$ we write

$$\underline{\mathcal{C}} := \mathcal{C} \times \mathcal{B}$$

for the product, generally considered as an object of $\mathsf{Cat}_{/\mathcal{B}}$ via the projection functor

$$\underline{\mathcal{C}} := \mathcal{C} \times \mathcal{B} \xrightarrow{\mathsf{pr}} \mathcal{B}\ .[14]$$

(8) We make use of the theory of *exponentiable fibrations* of [AF20] (see also [Lur, §B.3] and [AFR18, §5]), an ∞-categorical analog of the "Conduché fibrations" of [Gir64, Con72]: these are the objects $(\mathcal{E} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$ satisfying the condition that there exists a right adjoint

$$\begin{tikzcd} \mathsf{Cat}_{/\mathcal{B}} \arrow[r, shift left, "- \underset{\mathcal{B}}{\times} \mathcal{E}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{/\mathcal{B}} \arrow[l, shift left, dashed, "\mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E}, -)"] \end{tikzcd}$$

to the pullback; by the adjoint functor theorem, these can be equivalently characterized as those objects for which the proposed left adjoint preserves colimits. We refer to this right adjoint as the ***relative functor ∞-category*** construction; it is analogous to the internal hom of presheaves. Thus, for any target object $(\mathcal{F} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$ and any test object $(\mathcal{K} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}}$, a lift

$$\begin{tikzcd} & \mathsf{Fun}^{\mathsf{rel}}_{/\mathcal{B}}(\mathcal{E}, \mathcal{F}) \arrow[d] \\ \mathcal{K} \arrow[r] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

is equivalent data to a functor

$$\begin{tikzcd} \mathcal{E}_{|\mathcal{K}} \arrow[rr, dashed] \arrow[dr] & & \mathcal{F}_{|\mathcal{K}} \arrow[dl] \\ & \mathcal{K} & \end{tikzcd}$$

between pullbacks over $\mathcal{K}$. We write

$$\mathsf{EFib}_{\mathcal{B}} \subseteq \mathsf{Cat}_{/\mathcal{B}}$$

for the full subcategory on the exponentiable fibrations, and we note once and for all that cocartesian fibrations and cartesian fibrations are exponentiable.

[14] This notation is meant to be suggestive of the idea that $\underline{\mathcal{C}}$ is the "constant pre(co)sheaf" at $\mathcal{C}$.

0.5. **Acknowledgments.** This paper builds upon a broad array of mathematics developed over the past few decades; so, we owe a substantial intellectual debt to the community at large, all of the individual contributors among which it would be impossible to name here. Nevertheless, we would like to highlight the works of Saul Glasman [Glab] and of Akhil Mathew, Niko Naumann, and Justin Noel [MNN17], which were particularly influential to us. We also extend our gratitude to an anonymous referee for a helpful report.

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1. Detailed overview and fundamental examples

In this section, we give an informal overview of our work. In addition to giving somewhat more precise statements of our main theorems (which we only informally described in §0.1), we place our work within a broader mathematical narrative and collect key examples and applications.

In contrast with the present section, the main body of the paper (i.e. all the material beyond §1) is almost entirely devoted to proofs of the main theorems.[15] (So for instance, we will not revisit any discussion of sheaves.)

Local Notation 1.0.1. Throughout this section, we fix a scheme X,[16] a noncommutative stack $\mathcal{X}$ (i.e. a presentable stable ∞-category), and a poset P.

This section is organized as follows.

- §1.1: We recall the notion of a recollement of $\mathcal{X}$ and the fact that a closed-open decomposition of X determines a recollement of $\mathsf{QC}(X)$.
- §1.2: We generalize closed-open decompositions of X to stratifications of X.
- §1.3: We define stratifications of $\mathcal{X}$ and state our main reconstruction theorem (Theorem A). We also explain how a stratification of X determines a stratification of $\mathsf{QC}(X)$; in retrospect, §1.1 describes the special case of this phenomenon when $\mathsf{P} = [1]$.
- §1.4: To address certain subtleties arising in Theorem A, we indicate our fundamental operations on stratifications (Theorem B).
- §1.5: We describe our theory of $\mathcal{O}$-monoidal stratifications and state our $\mathcal{O}$-monoidal reconstruction theorem (Theorem C).
- §1.6: We begin by describing the adelic stratification of $\mathsf{QC}(X)$. We unpack in detail the example of $X = \mathsf{Spec}(\mathbb{Z})$, which nicely illustrates essentially all of the material surveyed up to this point, and which ultimately recovers the arithmetic fracture square (0.1.1). We conclude by generalizing adelic stratifications to the setting of tensor-triangular geometry (Theorem D).
- §1.7: We describe the geometric stratification of genuine G-spectra (Theorem E).
- §1.8: Given a P-stratified topological space, we obtain stratifications over P^{op} of its ∞-categories of sheaves, constructible sheaves, and P-constructible sheaves.

[15] However, our specific examples of reconstruction for genuine G-spectra are collected in §5.3, and we defer a discussion of the chromatic and adelic stratifications of spectra to Examples 4.3.8 and 4.6.13.

[16] More precisely, in order to simplify our exposition, we tacitly assume that our scheme X is finite-dimensional and noetherian. The utility of these assumptions is explained in Footnotes 21, 29, and 54.

§1.9: As a special case of a general construction, we obtain a stratification of naive G-spectra, which is closely related to the geometric stratification of genuine G-spectra.

§1.10: We explain the theory of reflection (Theorem F) and indicate a number of examples, notably its close relationship with Verdier duality.

§1.11: We explain how to use stratifications to build t-structures.

§1.12: We explain the relationship between stratifications and additive and localizing invariants (such as (resp. connective and nonconnective) algebraic K-theory).

1.1. **Closed-open decompositions and recollements.** We begin by recalling the theory of recollements (in the context of presentable stable ∞-categories).

Definition 1.1.1. A ***recollement*** of the noncommutative stack (i.e. presentable stable ∞-category) $\mathcal{X}$ is a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, bend left=50, "i_L"] \arrow[r, bend right=50, "i_R"'] & \mathcal{X} \arrow[l, "y" description] \arrow[r, bend left=50, "p_L"] \arrow[r, bend right=50, "p_R"'] & \mathcal{U} \arrow[l, "\nu" description] \end{tikzcd} \tag{1.1.1}$$

of adjunctions among presentable stable ∞-categories such that there are equalities

$$\mathsf{im}(i_L) = \mathsf{ker}(p_L) \ , \qquad \mathsf{im}(\nu) = \mathsf{ker}(y) \ , \qquad \text{and} \qquad \mathsf{im}(i_R) = \mathsf{ker}(p_R) \tag{1.1.2}$$

among full subcategories of $\mathcal{X}$.[17]

Given a recollement (1.1.1) of $\mathcal{X}$, it is not hard to check that for each $\mathcal{F} \in \mathcal{X}$ we obtain a canonical pullback square

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\eta_{p_L \dashv \nu}(\mathcal{F})"] \arrow[d, "\eta_{y \dashv i_R}(\mathcal{F})"'] & \nu p_L \mathcal{F} \arrow[d, "\nu p_L(\eta_{y \dashv i_R}(\mathcal{F}))"] \\ i_R y \mathcal{F} \arrow[r, "\eta_{p_L \dashv \nu}(i_R y \mathcal{F})"'] & \nu p_L i_R y \mathcal{F} \end{tikzcd} \ .^{18} \tag{1.1.3}$$

Hence, the object $\mathcal{F} \in \mathcal{X}$ is recorded by the lower right cospan. However, to record the object $\mathcal{F} \in \mathcal{X}$ we may actually record less data than this cospan: its lower morphism is the unit of the adjunction $p_L \dashv \nu$, and so is canonically determined by its source $i_R y \mathcal{F} \in \mathcal{X}$. Noting further that the functors i_R and ν are fully faithful, we find that the object $\mathcal{F} \in \mathcal{X}$ can be reconstructed from the data of the object $y\mathcal{F} \in \mathcal{Z}$, the object $p_L \mathcal{F} \in \mathcal{U}$, and the morphism

$$p_L \mathcal{F} \xrightarrow{p_L(\eta_{y \dashv i_R}(\mathcal{F}))} p_L i_R y \mathcal{F} \tag{1.1.4}$$

in $\mathcal{U}$. This observation forms the basis of an equivalence

$$\mathcal{X} \xrightarrow{\sim} \mathsf{lim}^{\mathsf{r.lax}} \left(\mathcal{Z} \xrightarrow{p_L i_R} \mathcal{U} \right) := \mathsf{lim} \left(\begin{tikzcd} & \mathsf{Fun}([1], \mathcal{U}) \arrow[d, "t"] \\ \mathcal{Z} \arrow[r, "p_L i_R"'] & \mathcal{U} \end{tikzcd} \right) \ ,^{19} \tag{1.1.5}$$

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[17]We have chosen the notation "y" because this is the restricted Yoneda functor (with respect to the inclusion i_L), and the notation "ν" because this is the inclusion of the full subcategory of objects whose restricted Yoneda functors are null.

[18]Indeed, taking fibers of the vertical morphisms reduces us to the case where $\mathcal{F} \in \mathsf{ker}(y) = \mathsf{im}(\nu) \subseteq \mathcal{X}$, in which case the claim is immediate.

[19]Right-lax limits will be explained further in Remarks 1.3.7 and 1.3.8.

which is given by the formula

$$\mathcal{F} \longmapsto \left(\begin{tikzcd} & (1.1.4) \arrow[d, mapsto] \\ y\mathcal{F} \arrow[r, mapsto] & p_L i_R y\mathcal{F} \end{tikzcd} \right)$$

and whose inverse reconstructs each object $\mathcal{F} \in \mathcal{X}$ as the pullback (1.1.3).

This situation is a prototypical instance of Slogan 0.1.1, as well as a special case of Theorem A below: the equivalence (1.1.5) is a macrocosm reconstruction of the noncommutative stack $\mathcal{X}$, and the pullback square (1.1.3) determines a microcosm reconstruction of the quasicoherent sheaf $\mathcal{F} \in \mathcal{X}$.

We have the following fundamental source of recollements.

Example 1.1.2. Suppose we are given a closed-open decomposition of our scheme X as in the diagram

$$\begin{tikzcd} Z \arrow[rr, hook, "i", "\mathsf{closed}"'] \arrow[dr, hook] & & X & U \arrow[l, hook', "j", "\mathsf{open}"'] \\ & X^\wedge_Z \arrow[ur, hook, "\sim"'] & & \end{tikzcd} , \qquad (1.1.6)$$

in which we have additionally included the formal completion $X^\wedge_Z$ of X along Z. Then, we have a recollement

$$\begin{tikzcd} \mathsf{QC}_Z(X) \arrow[r, hook, shift left] \arrow[d, leftrightarrow, "\wr"'] & \mathsf{QC}(X) \arrow[l, shift left] \arrow[l, phantom, "\perp"] \arrow[r, bend left, "j^*"] \arrow[r, bend right] \arrow[dl, shift right, "i^*"'] & \mathsf{QC}(U) \arrow[l, "j_*"] \\ \mathsf{QC}(X^\wedge_Z) \arrow[ur, hook, shift right, "\hat{\imath}_*"'] \arrow[ur, phantom, "\perp"] & & \end{tikzcd} , \qquad (1.1.7)$$

in which

- $\mathsf{QC}_Z(X) := \ker(j^*) \subseteq \mathsf{QC}(X)$ denotes the full subcategory of those quasicoherent sheaves on X that are set-theoretically supported on Z,
- the left vertical equivalence is that between $\mathscr{I}_Z$-torsion and $\mathscr{I}_Z$-complete quasicoherent sheaves of $\mathscr{O}_X$-modules,[20] and
- the triangle commutes.[21]

Warning 1.1.3. In the situation of Example 1.1.2, the full subcategory $\mathsf{QC}_Z(X) \subseteq \mathsf{QC}(X)$ is generated under colimits by the image of the pushforward functor

$$\mathsf{QC}(Z) \xrightarrow{i_*} \mathsf{QC}(X) ,$$

but this latter functor is *not* generally fully faithful.[22]

1.2. **Stratified schemes.** We now generalize the notion of a closed-open decomposition of X. Evidently, the closed-open decomposition (1.1.6) of X is entirely determined by the closed subset $Z \subseteq X$. Let us write $\mathbf{Cls}_X$ for the poset of closed subsets of X ordered by inclusion.

[20] This equivalence is recorded e.g. as [GR14, Proposition 7.1.3]; see also [GM92, DG02]. (Note that it is not generally t-exact, and so is an inherently derived phenomenon.)

[21] The existence of the recollement (1.1.7) is guaranteed by the assumption that X is qcqs: namely, this guarantees that the functor j_* preserves colimits.

[22] On the other hand, it is not hard to recover the closed subset $Z \subseteq X$ from the data of the full subcategory $\mathsf{QC}_Z(X) \subseteq \mathsf{QC}(X)$.

Definition 1.2.1. A ***stratification*** of the scheme X over the poset P is a functor

$$\begin{array}{ccc} \mathsf{P} & \xrightarrow{Z_\bullet} & \mathbf{Cls}_X \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ p & \longmapsto & Z_p \end{array} \tag{1.2.1}$$

satisfying the following conditions:

generation: $X = \bigcup_{p \in \mathsf{P}} Z_p$;

stratification: for any $p, q \in \mathsf{P}$, we have

$$Z_p \cap Z_q = \bigcup_{r \leq p \text{ and } r \leq q} Z_r \ .$$

Example 1.2.2. Suppose that $\mathsf{P} = [1] = \{0 \to 1\}$. Then, a stratification of X over P is equivalent data to that of a closed subset $Z := Z_0 \subseteq X$.

Example 1.2.3. Generalizing Example 1.2.2, suppose that the poset P is in fact a totally ordered set. Then, any functor (1.2.1) satisfies the stratification condition. If P contains a maximal element, then the functor (1.2.1) satisfies the generation condition (and hence defines a stratification of X over P) if and only if the maximal element $X \in \mathbf{Cls}_X$ lies in its image.

Example 1.2.4. Let S be a set, and suppose that $X \to S$ is a morphism to S considered as a discrete scheme (i.e. an S-indexed coproduct of copies of $\mathsf{Spec}(\Bbbk)$). Then, taking preimages defines a stratification

$$S \longrightarrow \mathbf{Cls}_X$$

(where S is considered as a discrete poset).

Example 1.2.5. Suppose that $X = \mathbb{A}^2 = \mathsf{Spec}(\Bbbk[x, y])$ is the affine plane. Choose any $a, b \in \Bbbk^\times$, and consider the three full subposets

$$\begin{tikzcd} & V(x) \arrow[d, hook] \\ V(y) \arrow[r, hook] & \mathbb{A}^2 \end{tikzcd} \quad , \quad \begin{tikzcd} V(x,y) \arrow[r, hook] \arrow[d, hook] & V(x) \arrow[d, hook] \\ V(y) \arrow[r, hook] & \mathbb{A}^2 \end{tikzcd} \quad , \quad \text{and} \quad \begin{tikzcd} V(x,y) \arrow[r, hook] \arrow[d, hook] & V(x) \arrow[d, hook] \\ V(y) \arrow[r, hook] & \mathbb{A}^2 \\ V(x-a, y-b) \arrow[ur, hook] & \end{tikzcd}$$

of $\mathbf{Cls}_{\mathbb{A}^2}$: all three contain $\mathbb{A}^2$, the first contains the two coordinate axes, the second additionally contains the origin $(0, 0)$, and the third additionally contains the point (a, b).[23] The first satisfies the generation condition but not the stratification condition, while the latter two define stratifications of $\mathbb{A}^2$.

Definition 1.2.6. For each element $p \in \mathsf{P}$, the p^{th} ***stratum*** of the stratification (1.2.1) is the locally closed subset

$$X_p := \left(Z_p \Big\backslash \bigcup_{q < p} Z_q \right)$$

of X.

Altogether, the inclusions of the strata of the stratification (1.2.1) assemble into a morphism

$$\coprod_{p \in \mathsf{P}} X_p \longrightarrow X \ . \tag{1.2.2}$$

[23] Here, $V(I) \in \mathbf{Cls}_{\mathbb{A}^2}$ denotes the vanishing locus of an ideal $I \subseteq \Bbbk[x, y]$.

For the stratifications described in Examples 1.2.2, 1.2.4, and 1.2.5, the morphism (1.2.2) defines a bijection on underlying sets. In fact, for any stratification (1.2.1), the morphism (1.2.2) defines an injection on underlying sets: this is a consequence of the stratification condition. However, it does not always define a surjection: for instance, the constant functor

$$\mathbb{N}^{\mathsf{op}} := \{1 \to 2 \to 3 \to \cdots\}^{\mathsf{op}} \xrightarrow{\mathsf{const}_X} \mathbf{Cls}_X \ ,$$

defines a stratification (as a special instance of Example 1.2.3) whose strata are all empty, so that in this case the morphism (1.2.2) is not surjective unless X itself is empty. In fact, it is not hard to see that this counterexample is prototypical, in the sense that the morphism (1.2.2) is guaranteed to be surjective precisely when the poset P is artinian (i.e. every decreasing sequence eventually stabilizes).

Of course, in order to reconstruct X not just as a set but as a scheme, one would need to keep track of not just the strata $\{X_p\}_{p \in \mathsf{P}}$ but also gluing data between them. Theorem A below enacts this idea in the noncommutative setting. In parallel with the commutative situation just described, such reconstruction will depend on certain finiteness properties of the poset P.

1.3. **Stratified noncommutative stacks.** We now introduce our theory of stratified noncommutative stacks, which is closely patterned after the theory of stratified schemes.

Definition 1.3.1. A ***closed noncommutative substack*** of the noncommutative stack $\mathcal{X}$ is a full presentable stable subcategory $\mathcal{Z} \subseteq \mathcal{X}$ whose inclusion extends to a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, bend left] \arrow[r, phantom, bend left=20, "\perp"] \arrow[r, phantom, bend right=20, "\perp"] \arrow[r, hook, dashed, bend right] & \mathcal{X} \arrow[l, dashed] \end{tikzcd}$$

of adjoint functors.[24] We write $\mathbf{Cls}_{\mathcal{X}}$ for the poset of closed noncommutative substacks of $\mathcal{X}$ ordered by inclusion.

Of course, our terminology is motivated by the fact that a closed subset $Z \subseteq X$ determines a closed noncommutative substack $\mathsf{QC}_Z(X) \subseteq \mathsf{QC}(X)$, as indicated in Example 1.1.2. This construction defines a functor

$$\mathbf{Cls}_X \xrightarrow{\mathsf{QC}_{(-)}(X)} \mathbf{Cls}_{\mathsf{QC}(X)} \ .$$

Definition 1.3.2. A ***stratification*** of the noncommutative stack (i.e. presentable stable ∞-category) $\mathcal{X}$ over the poset P is a functor

$$\begin{array}{ccc} \mathsf{P} & \xrightarrow{\mathcal{Z}_\bullet} & \mathbf{Cls}_{\mathcal{X}} \\ \Psi & & \Psi \\ p & \longmapsto & \mathcal{Z}_p \end{array} \tag{1.3.1}$$

satisfying the following conditions:

generation: $\mathcal{X} = \bigcup_{p \in \mathsf{P}} \mathcal{Z}_p$;

stratification: for any $p, q \in \mathsf{P}$, there exists a factorization

$$\begin{tikzcd} \displaystyle\bigcup_{r \leq p \text{ and } r \leq q} \mathcal{Z}_r \arrow[r, hook] & \mathcal{Z}_p \arrow[d] \\ \mathcal{Z}_q \arrow[u, dashed] \arrow[r, hook] & \mathcal{X} \end{tikzcd} .$$

[24]If the right adjoint $\mathcal{X} \to \mathcal{Z}$ admits its own right adjoint, the latter will automatically be fully faithful. (In general, if a functor F has adjoints $F^L \dashv F \dashv F^R$, then F^L is fully faithful if and only if F^R is: this follows from the composite adjunction $FF^L \dashv FF^R$, in which one adjoint is naturally equivalent to the identity functor if and only if the other is.)

Here, the union symbol $\bigcup$ denotes the colimit (i.e. least upper bound) in the poset $\mathbf{Cls}_{\mathcal{X}}$.[25] In this situation, we may also say that $\mathcal{X}$ is P-***stratified***.

Remark 1.3.3. Given a stratification (1.3.1) of $\mathcal{X}$, the commutative square

$$\begin{tikzcd} \displaystyle\bigcup_{r \leq p \text{ and } r \leq q} \mathcal{Z}_r \arrow[r, hook] \arrow[d, hook] & \mathcal{Z}_p \arrow[d, hook] \\ \mathcal{Z}_q \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

of defining fully faithful inclusions is in fact a pullback.[26] Thus, the stratification condition of Definition 1.3.2 is a close cousin of the stratification condition of Definition 1.2.1.

Example 1.3.4. Suppose that $\mathsf{P} = \{a, b\}$ is a two-element set, considered as a discrete poset. A stratification

$$\{a, b\} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_{\mathcal{X}}$$

is the data of a pair of closed noncommutative substacks $\mathcal{Z}_a, \mathcal{Z}_b \in \mathcal{X}$ such that $\mathcal{Z}_a \cup \mathcal{Z}_b = \mathcal{X}$ and such that the composites

$$\mathcal{Z}_a \hookrightarrow \mathcal{X} \longrightarrow \mathcal{Z}_b \qquad \text{and} \qquad \mathcal{Z}_b \hookrightarrow \mathcal{X} \longrightarrow \mathcal{Z}_a$$

are both zero. It follows immediately that we have an adjoint equivalence

$$\begin{tikzcd} \mathcal{Z}_a \times \mathcal{Z}_b \arrow[r, hook, shift left, "i_L \oplus i_L"] \arrow[r, phantom, "\sim"] & \mathcal{X} \arrow[l, shift left, "(y,y)"] \end{tikzcd} \quad ;$$

in other words, a stratification of the noncommutative stack $\mathcal{X}$ over $\{a, b\}$ is nothing other than a decomposition of $\mathcal{X}$ as the product of two closed noncommutative substacks.[27] More generally, for any set S considered as a discrete poset, a stratification of $\mathcal{X}$ over S is the data of a product decomposition $\mathcal{X} \simeq \prod_{s \in S} \mathcal{Z}_s$ by full stable subcategories.[28]

Definition 1.3.5. For each element $p \in \mathsf{P}$, the p^{th} ***stratum*** of the stratification (1.3.1) of $\mathcal{X}$ is the presentable quotient

$$\mathcal{X}_p := \left(\mathcal{Z}_p \Big/ \bigcup_{q < p} \mathcal{Z}_q \right) \quad ,$$

which essentially by definition participates in the recollement

$$\begin{tikzcd} \displaystyle\bigcup_{q<p} \mathcal{Z}_q \arrow[r, hook, bend left, "i_L"] \arrow[r, hook, bend right, "i_R"'] & \mathcal{Z}_p \arrow[l, "y" description] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{X}_p \arrow[l, "\nu" description] \end{tikzcd} \quad .$$

Hence, we obtain a composite adjunction

$$\begin{tikzcd} \Phi_p : \mathcal{X} \arrow[r, shift left, "y"] \arrow[r, phantom, "\perp"] & \mathcal{Z}_p \arrow[l, hook, shift left, "i_R"] \arrow[r, shift left, "p_L"] \arrow[r, phantom, "\perp"] & \mathcal{X}_p : \rho^p \arrow[l, hook, shift left, "\nu"] \end{tikzcd} \quad ,$$

whose left adjoint Φ_p we refer to as the p^{th} ***geometric localization functor*** of the stratification (1.3.1).

[25] In fact, colimits in $\mathbf{Cls}_{\mathcal{X}}$ always exist and are quite straightforward to compute; see Observation 2.3.9.

[26] This follows from Lemma 3.4.5; see Definition 3.1.2.

[27] Conversely, any product decomposition $\mathcal{X} \simeq \mathcal{Z}_a \times \mathcal{Z}_b$ by full stable subcategories is necessarily by closed noncommutative substacks.

[28] This may be compared with Example 1.2.4; note that the functor QC takes disjoint unions to products.

Example 1.3.6. An ordinary stratification (1.2.1) of the scheme X determines a stratification

$$\begin{tikzcd} \mathsf{P} \arrow[r, "Z_\bullet"] & \mathbf{Cls}_X \arrow[r, "\mathsf{QC}_{(-)}(X)"] & \mathbf{Cls}_{\mathsf{QC}(X)} \\ \Cup & & \Cup \\ p \arrow[rr, mapsto] & & \mathsf{QC}_{Z_p}(X) \end{tikzcd} \tag{1.3.2}$$

of its underlying noncommutative stack $\mathsf{QC}(X)$.[29] Given any element $p \in \mathsf{P}$, let us choose a factorization

$$\begin{tikzcd} X_p \arrow[rr, hook, "\text{locally closed}"'] \arrow[dr, hook, dashed, "\text{closed}"'] & & X \\ & U_p \arrow[ur, hook, dashed, "\text{open}"'] & \end{tikzcd} .$$

Then, the p^{th} stratum of the stratification (1.3.2) can be identified as $\mathsf{QC}_{X_p}(U_p) \simeq \mathsf{QC}((U_p)^\wedge_{X_p})$ (recall the equivalence of Example 1.1.2), and thereafter its p^{th} geometric localization functor can be identified as the composite

$$\Phi_p : \mathsf{QC}(X) \xrightarrow{p_L} \mathsf{QC}(U_p) \xrightarrow{y} \mathsf{QC}((U_p)^\wedge_{X_p}) .^{30}$$

In parallel with Example 1.2.2, a stratification of $\mathcal{X}$ over $[1]$ is simply the data of a closed noncommutative substack $\mathcal{Z} := \mathcal{Z}_0 \subseteq \mathcal{X}$. This necessarily extends to a recollement (1.1.1), and indeed the strata of this stratification are simply

$$\mathcal{X}_0 := \mathcal{Z}/0 \simeq \mathcal{Z} \qquad \text{and} \qquad \mathcal{X}_1 := \mathcal{X}/\mathcal{Z} \simeq \mathcal{U} .$$

Moreover, as we have seen in §1.1, the gluing datum necessary for reconstructing $\mathcal{X}$ from these strata is the composite functor

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "p_L i_R"] \arrow[d, equal] & \mathcal{U} \arrow[d, equal] \\ \mathcal{X}_0 \arrow[r, "\Phi_1 \rho^0"'] & \mathcal{X}_1 \end{tikzcd} .^{31} \tag{1.3.3}$$

This suggests the following general construction: given a stratification (1.3.1) of $\mathcal{X}$ over an arbitrary poset P, for each morphism $p \to q$ in P we have an associated ***gluing functor***

$$\Gamma^p_q : \mathcal{X}_p \overset{\rho^p}{\hookrightarrow} \mathcal{X} \xrightarrow{\Phi_q} \mathcal{X}_q$$

[29]Without hypotheses on the scheme X, suppose that the functor $\mathbf{Cls}_X \xrightarrow{\mathsf{QC}_{(-)}(X)} \mathbf{Cls}_{\mathsf{QC}(X)}$ exists, as guaranteed e.g. by the assumption that X is qcqs (recall Footnote 21). Then, the composite functor (1.3.2) automatically satisfies stratification condition. The assumption that X is noetherian guarantees that it also satisfies the generation condition. For an example where the generation condition fails, see Remark 4.6.12.

[30]This identification follows from Lemmas 3.4.5 and 3.2.3(2)(c).

[31]Thus, the recollement (1.1.1) may be thought of as a sort of categorified extension sequence, which is classified by the data of the functor (1.3.3). This analogy will be amplified in Remark 1.5.7.

between the corresponding strata.[32] Given a composable sequence $p \to q \to r$ in P, the associated gluing functors generally do not strictly compose: rather, they fit into a lax-commutative triangle

$$\begin{tikzcd} & \mathcal{X}_q \arrow[dr, "\Gamma^q_r"] \arrow[d, phantom, "\Uparrow"] & \\ \mathcal{X}_p \arrow[ur, "\Gamma^p_q"] \arrow[rr, "\Gamma^p_r"'] & {} & \mathcal{X}_r \end{tikzcd} ,$$

whose natural transformation arises from the unit of the adjunction $\Phi_q \dashv \rho^q$.[33] An elaboration of this observation reveals that the gluing functors assemble into a *left-lax* functor

$$\mathsf{P} \xrightarrow[\mathsf{l.lax}]{\mathscr{G}(\mathcal{X})} \mathsf{Pr_{st}} \tag{1.3.4}$$

to the $(\infty, 2)$-category $\mathsf{Pr_{st}}$ of presentable stable ∞-categories and accessible exact functors between them, which we refer to as the ***gluing diagram*** of the stratification and denote by $\mathscr{G}(\mathcal{X})$ (see Definition 2.5.7).

We can now state our first main theorem, which provides sufficient conditions for the reconstruction of a stratified noncommutative stack $\mathcal{X}$ from its gluing diagram $\mathscr{G}(\mathcal{X})$. As foreshadowed at the end of §1.2, such reconstruction may be obstructed by certain convergence issues, which we precisely codify (see Remark 1.3.9). In order to highlight its recursive structure, we state the theorem succinctly before explaining its terms.

Theorem A (Theorems 6.2.6 and 2.5.14)**.** *Let* P *be a poset.*

(1) **metacosm:** *The gluing diagram functor is the left adjoint in an adjunction*

$$\mathbf{Strat}_{\mathsf{P}} \underset{\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}}{\overset{\mathscr{G}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \quad . \tag{1.3.5}$$

(2) **macrocosm:** *For each* P*-stratified noncommutative stack* $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$*, the unit of the adjunction* (1.3.5) *determines the left adjoint in an adjunction*

$$\mathcal{X} \underset{\mathsf{lim}_{\mathsf{sd(P)}}}{\overset{g}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Glue}(\mathcal{X}) := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) \quad . \tag{1.3.6}$$

(3) **microcosm:** *For each quasicoherent sheaf* $\mathcal{F} \in \mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ *on a* P*-stratified noncommutative stack, the unit of the adjunction* (1.3.6) *is a morphism*

$$\mathcal{F} \longrightarrow \mathsf{glue}(\mathcal{F}) := \mathsf{lim}_{\mathsf{sd(P)}}(g(\mathcal{F})) \tag{1.3.7}$$

[32]When both sources and targets appear in our notation, we put the source as a superscript and the target as a subscript (so e.g. we have $\mathsf{Ar}(\mathcal{C})^{|c} \simeq \mathcal{C}_{c/}$ and $\mathsf{Ar}(\mathcal{C})_{|c} \simeq \mathcal{C}_{/c}$ for any ∞-category $\mathcal{C}$; these conventions are motivated by the fact that $\mathsf{hom}_{\mathcal{C}}(-,-)$ is contravariant in the source and covariant in the target). Moreover, we have chosen to use a subscript in the notation Φ_q and a superscript in the notation ρ^p in order to maintain consistency with the notation Γ^p_q.

[33]For instance, in the situation and notation of Example 1.3.6, the lax-commutative triangle

$$\begin{tikzcd} & & \mathsf{QC}((U_q)^\wedge_{X_q}) \arrow[dr, hook, "\rho^q"] & & \\ & \mathsf{QC}(X) \arrow[ur, "\Phi_q"] & \Uparrow & \mathsf{QC}(X) \arrow[dr, "\Phi_r"] & \\ \mathsf{QC}((U_p)^\wedge_{X_p}) \arrow[ur, hook, "\rho^p"] \arrow[rr, hook, "\rho^p"'] & & \mathsf{QC}(X) \arrow[rr, "\Phi_r"'] & & \mathsf{QC}((U_r)^\wedge_{X_r}) \end{tikzcd}$$

records the difference between push-pull operations either directly from $(U_p)^\wedge_{X_p}$ to $(U_r)^\wedge_{X_r}$ or passing intermediately through $(U_q)^\wedge_{X_q}$.

in $\mathcal{X}$.

(4) **nanocosm:** *For each quasicoherent sheaf* $\mathcal{E} \in \mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ *on a* P*-stratified noncommutative stack, applying* $\underline{\mathsf{hom}}_{\mathcal{X}}(\mathcal{E}, -)$ *to the morphism* (1.3.7) *determines a morphism*

$$\underline{\mathsf{hom}}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}) \longrightarrow \lim_{([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})} \left(\underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)}\mathcal{E}, \Gamma_{\varphi}\Phi_{\varphi(0)}\mathcal{F}) \right) \tag{1.3.8}$$

in $\mathcal{S}\mathsf{p}$.

Moreover, if the poset P *is down-finite,*[34] *then the metacosm adjunction* (1.3.5) *– and hence the macrocosm adjunction* (1.3.6)*, and hence the microcosm morphism* (1.3.7)*, and hence the nanocosm morphism* (1.3.8) *– is an equivalence.*[35]

The various expressions appearing in Theorem A have the following meaning.

metacosm: We write

- $\mathbf{Strat}_{\mathsf{P}}$ for the ∞-category of P-stratified noncommutative stacks,
- $\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})$ for a certain ∞-category whose objects are left-lax left P-modules in $\mathsf{Pr}_{\mathsf{st}}$ (i.e. left-lax functors from P to $\mathsf{Pr}_{\mathsf{st}}$),
- $\mathscr{G}$ for the (*macrocosm*) *gluing diagram* functor (taking a P-stratified noncommutative stack to its gluing diagram), and
- $\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}$ for a certain "parametrized right-lax limit" functor.

We say that $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ is *convergent* if it lies in the image of the right adjoint of the metacosm adjunction (1.3.5), or equivalently if its macrocosm adjunction (1.3.6) is an equivalence.

macrocosm: We refer to $\mathsf{Glue}(\mathcal{X}) := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X}))$ as the *reglued noncommutative stack* of $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$; this is the underlying object of the P-stratified noncommutative stack $\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}(\mathscr{G}(\mathcal{X})) \in \mathbf{Strat}_{\mathsf{P}}$. It may be identified as a full subcategory

$$\mathsf{Glue}(\mathcal{X}) \subseteq \mathsf{Fun}(\mathsf{sd}(\mathsf{P}), \mathcal{X}) \tag{1.3.9}$$

of the ∞-category of functors to $\mathcal{X}$ from the subdivision of P,[36] through which the notation $\lim_{\mathsf{sd}(\mathsf{P})}$ acquires meaning. We write g for the (*microcosm*) *gluing diagram* functor (taking a quasicoherent sheaf to its gluing diagram). For each $p \in \mathsf{P}$, the p^{th} geometric localization functor appears as the factored composite

$$\begin{tikzcd} \mathcal{X} \arrow[r, "g"] \arrow[drr, dashed, bend right, "\Phi_p"'] & \mathsf{Glue}(\mathcal{X}) \arrow[r, "\mathsf{ev}_p"] \arrow[dr, dashed] & \mathcal{X} \\ & & \mathcal{X}_p \arrow[u, hook, "\rho^p"'] \end{tikzcd} .^{37}$$

In particular, the gluing diagram $g(\mathcal{F}) \in \mathsf{Glue}(\mathcal{X})$ indeed consists of the geometric localizations $\{\Phi_p(\mathcal{F}) \in \mathcal{X}_p\}_{p \in \mathsf{P}}$ along with gluing data between them.

[34] A poset P is called *down-finite* if for each element $p \in \mathsf{P}$, its down-closure $({}^{\leq}p) := \{q \in \mathsf{P} : q \leq p\}$ is finite.

[35] Of course, the implied implications are irreversible: respectively, it is possible for (1.3.6), (1.3.7), or (1.3.8) to be an equivalence even if (1.3.5), (1.3.6), or (1.3.7) is not. (On the other hand, fixing some $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$, if for every $\mathcal{F} \in \mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ the microcosm morphism (1.3.7) is an equivalence, then the macrocosm adjunction (1.3.6) is an equivalence. Likewise, fixing some $\mathcal{F} \in \mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$, if for every $\mathcal{E} \in \mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ the nanocosm morphism (1.3.8) is an equivalence, then the microcosm morphism (1.3.7) is an equivalence. See also Remark 1.3.9.)

[36] The *subdivision* of the poset P is the full subcategory $\mathsf{sd}(\mathsf{P}) \subseteq \mathbf{\Delta}_{/\mathsf{P}}$ (which is in fact a poset) on the conservative (or equivalently injective) functors $[n] \to \mathsf{P}$.

[37] More generally, for any $([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$, the composite $\mathcal{X} \xrightarrow{g} \mathsf{Glue}(\mathcal{X}) \xrightarrow{\mathsf{ev}_{\varphi}} \mathcal{X}$ is the composite $\rho^{\varphi(n)}\Phi_{\varphi(n)} \cdots \rho^{\varphi(0)}\Phi_{\varphi(0)}$.

microcosm: We refer to $\mathsf{glue}(\mathcal{F}) := \mathsf{lim}_{\mathsf{sd}(\mathsf{P})}(g(\mathcal{F}))$ as the *reglued quasicoherent sheaf* of $\mathcal{F} \in \mathcal{X}$. We say that $\mathcal{F} \in \mathcal{X}$ is *convergent* if its microcosm morphism (1.3.7) is an equivalence.

nanocosm: We write $\underline{\mathsf{hom}}$ to denoted enriched hom, here meaning the hom-spectra of a stable ∞-category. Moreover, for any $([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$ we write

$$\Gamma_\varphi := \Gamma^{\varphi(n-1)}_{\varphi(n)} \cdots \Gamma^{\varphi(0)}_{\varphi(1)}$$

for brevity.[38]

When the nanocosm morphism (1.3.8) is an equivalence, it may be viewed as affording a description of the "(generalized) elements" of a sheaf (i.e. morphisms into it) entirely in terms of compatible local elements (i.e. morphisms in the strata of the stratification).[39]

Remark 1.3.7. Fix an ∞-category $\mathcal{B} \in \mathsf{Cat}$. Given a functor

$$\mathcal{B} \xrightarrow{F} \mathsf{Cat}$$

to the $(\infty, 2)$-category Cat of ∞-categories, an object of its *limit* may be thought of informally as a system of the following data:

- for each object $b \in \mathcal{B}$, an object $e_b \in F(b)$;
- for each morphism $b_0 \xrightarrow{\gamma} b_1$ in $\mathcal{B}$, an equivalence

$$F(\gamma)(e_{b_0}) \underset{\sim}{\overset{\sigma_\gamma}{\longleftrightarrow}} e_{b_1} \tag{1.3.10}$$

  in $F(b_1)$;
- for each composable pair of morphisms $b_0 \xrightarrow{\gamma} b_1 \xrightarrow{\delta} b_2$ in $\mathcal{B}$, a commutative square

$$\begin{tikzcd} e_{b_2} \arrow[r, leftrightarrow, "\sigma_\delta", "\sim"'] \arrow[d, leftrightarrow, "\sigma_{\delta\gamma}"', "\wr"] & F(\delta)(e_{b_1}) \arrow[d, leftrightarrow, "\wr"', "F(\delta)(\sigma_\gamma)"] \\ F(\delta\gamma)(e_{b_0}) \arrow[r, leftrightarrow, "\sim"'] & F(\delta)(F(\gamma)(e_{b_0})) \end{tikzcd} , \tag{1.3.11}$$

  in which the lower equivalence follows from the fact that F is a functor (and so respects composition of morphisms up to canonical equivalence);
- higher coherence data.

By contrast, to describe an object of its *left-lax limit* or of its *right-lax limit*, we must replace the equivalences (1.3.10) with morphisms

$$F(\gamma)(e_{b_0}) \longrightarrow e_{b_1} \qquad \text{or} \qquad F(\gamma)(e_{b_0}) \longleftarrow e_{b_1} ,$$

respectively.[40] More general definitions apply when the functor F is itself only left- or right-lax, such as the gluing diagram (1.3.4) (whose right-lax limit defines the reglued noncommutative stack $\mathsf{Glue}(\mathcal{X})$); for instance, in the commutative square (1.3.11), the lower morphism will in general no longer be an equivalence.

We provide a comprehensive treatment of these notions in § A. In particular, we unpack the general definition of the right-lax limit of a left-lax left [2]-module (the simplest nontrivial case) in

[38] The nanocosm morphism (1.3.8) is described in detail in Remark 2.6.7.

[39] For instance, via Theorem E this provides an explicit formula for the categorical fixedpoints of a genuine G-spectrum (where G is a finite group) as a finite limit of spectra that are defined in terms of its geometric fixedpoints (see §5.4).

[40] Thus, the limit is a full subcategory of both the left-lax limit and the right-lax limit.

Example A.5.3(1). We also describe the following notions in Example 2.5.16 for a noncommutative stack $\mathcal{X}$ stratified over the poset $\mathsf{P} = [2]$: the gluing diagram $\mathscr{G}(\mathcal{X})$, the reglued noncommutative stack $\mathsf{Glue}(\mathcal{X}) := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X}))$, the gluing diagram functor $\mathcal{X} \xrightarrow{g} \mathsf{Glue}(\mathcal{X})$, and its right adjoint $\mathsf{Glue}(\mathcal{X}) \xrightarrow{\lim_{\mathsf{sd(P)}}} \mathcal{X}$.

Remark 1.3.8. We show as Lemma A.6.5 that the right-lax limit of a left-lax functor

$$\mathsf{P} \xrightarrow[\mathsf{l.lax}]{F} \mathsf{Cat}$$

may be computed as the strict (i.e. ordinary) limit of a certain strict (i.e. ordinary) functor

$$\mathsf{sd(P)} \xrightarrow{\mathfrak{S}(F)} \mathsf{Cat}$$

constructed therefrom:

$$\lim^{\mathsf{r.lax}}\left(\mathsf{P} \xrightarrow[\mathsf{l.lax}]{F} \mathsf{Cat} \right) \simeq \lim\left(\mathsf{sd(P)} \xrightarrow{\mathfrak{S}(F)} \mathsf{Cat}\right) .^{41} \tag{1.3.12}$$

In addition to its technical utility, this result allows for a more uniform perspective on the metacosm adjunction (1.3.5) and the macrocosm adjunction (1.3.6): in both, the right adjoint is computed by taking (strict) limits over $\mathsf{sd(P)}$.

Remark 1.3.9. Theorem A is sharp in the sense that the metacosm adjunction (1.3.5) fails to be an equivalence whenever P is not down-finite.[42] Equivalently (as the forgetful functor $\mathbf{Strat}_{\mathsf{P}} \to \mathsf{Pr}^L_{\mathsf{st}}$ is conservative), when P is not down-finite then there exists a P-stratified noncommutative stack whose macrocosm adjunction (1.3.6) is not an equivalence. Using (both the content and terminology of) Theorems B and D as well as Example 1.6.1, such a P-stratified noncommutative stack may be constructed by choosing an injective functor

$$\mathsf{P}_{\mathbb{Z}} \hookrightarrow \mathsf{P}$$

from the specialization poset of $\mathsf{Spec}(\mathbb{Z})$ and taking the pushforward of the adelic stratification of $\mathsf{Mod}_{\mathbb{Z}}$ along it.[43] It is not hard to see that passing to this new stratification of $\mathsf{Mod}_{\mathbb{Z}}$ yields an equivalent macrocosm adjunction, which is not an equivalence.

Remark 1.3.10. The requirement that P be down-finite is strictly stronger than the requirement that it be artinian. Indeed, the specialization poset of $\mathsf{Spec}(\mathbb{Z})$ (depicted in diagram (1.6.2)) is artinian but not down-finite. In fact, it is not hard to see that the assumption that P be artinian guarantees that the functor

$$\mathcal{X} \xrightarrow{(\Phi_p)_{p\in\mathsf{P}}} \prod_{p\in\mathsf{P}} \mathcal{X}_p \tag{1.3.13}$$

is conservative; this is directly analogous to the guaranteed surjectivity on underlying sets of the morphism (1.2.2) under that same assumption. From this perspective, the further assumption that P be down-finite may be seen as assuring that the gluing data suffice to recover the noncommutative stack structure of $\mathcal{X}$.

Example 1.3.11 (the Goodwillie–Taylor stratification)**.** Goodwillie calculus leads to a stratification over a nonartinian poset in which the functor (1.3.13) generally fails to be conservative, as we now explain.[44] Specifically, we construct a stratification of $\mathcal{X} := \mathsf{Fun}(\mathcal{I}, \mathcal{Y})$, where $\mathcal{Y}$ is any presentable stable ∞-category and $\mathcal{I}$ is any ∞-category that admits finite colimits and has a terminal object.

[41] This equivalence generalizes the identification appearing in the equivalence (1.1.5), which is an instance of the equivalence (1.3.12) in the case that $\mathsf{P} = [1]$.

[42] Hence, convergence is analogous at the metacosm level to the down-finiteness of P.

[43] Indeed, a poset is down-finite precisely when it admits no injective functors from $\mathsf{P}_{\mathbb{Z}}$.

[44] A version of this stratification appears in work of Glasman [Glab, Glaa] (see §0.2.5).

First of all, by [Lur, Theorem 6.1.1.10] (see also [Goo03]), for any $n \geq 0$ the inclusion of the full subcategory of n-excisive functors is the right adjoint in an adjunction

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{I}, \mathcal{Y}) \arrow[r, shift left, "P_n"] \arrow[r, phantom, "\perp"] & \mathsf{Exc}^n(\mathcal{I}, \mathcal{Y}) \arrow[l, hook, shift left] \end{tikzcd} \quad ,$$

whose left adjoint P_n carries a functor to its n-excisive approximation. This inclusion commutes with colimits, and so admits a right adjoint of its own. We trivially extend this to the case that $n = -1$ by declaring that $\mathsf{Exc}^{-1}(\mathcal{I}, \mathcal{Y}) = \{0\} \subseteq \mathsf{Fun}(\mathcal{I}, \mathcal{Y})$, i.e. that only the constant functor at the zero object is (-1)-excisive. Hence, we obtain a stratification

$$\begin{array}{ccc} (\mathbb{Z}_{\geq -1})^{\mathsf{op}} & \longrightarrow & \mathbf{Cls}_{\mathsf{Fun}(\mathcal{I}, \mathcal{Y})} \\ \Psi & & \Psi \\ n & \longmapsto & \mathsf{ker}(P_n) \end{array} \tag{1.3.14}$$

(recall Example 1.2.3). Following the same reasoning as is laid out in Example 4.3.8 (in the case of p-local spectra), we find that the macrocosm adjunction of the stratification (1.3.14) may be identified as the adjunction

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{I}, \mathcal{Y}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{Exc}^{\infty}(\mathcal{I}, \mathcal{Y}) \arrow[l, hook, shift left] \end{tikzcd} \quad ,$$

whose unit morphism at a functor $F \in \mathsf{Fun}(\mathcal{I}, \mathcal{Y})$ is the canonical morphism

$$F \longrightarrow P_{\infty}F := \lim\left(\cdots \longrightarrow P_2F \longrightarrow P_1F \longrightarrow P_0F \longrightarrow P_{-1}F \simeq 0 \right)$$

to the limit of its Goodwillie–Taylor tower, which is not generally an equivalence.

Remark 1.3.12 (filtrations from stratifications)**.** Fix a P-stratified noncommutative stack $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$, and assume for simplicity that P is down-finite.[45] For each $p \in \mathsf{P}$, we have the corresponding recollement (1.1.1) with $\mathcal{Z} = \mathcal{Z}_p$. Writing

$$\mathsf{fil}_p^L := i_L y \qquad \text{and} \qquad \mathsf{fil}_R^p := i_R y \ ,$$

we obtain canonical embeddings with retracts

$$\begin{tikzcd} \mathsf{Fun}(\mathsf{P}, \mathcal{X}) \arrow[r, bend right, "\mathsf{colim}_{\mathsf{P}}"'] & \mathcal{X} \arrow[l, hook', "\mathsf{fil}_{\bullet}^L"'] \arrow[r, hook, "\mathsf{fil}_R^{\bullet}"] & \mathsf{Fun}(\mathsf{P}^{\mathsf{op}}, \mathcal{X}) \arrow[l, bend left, "\mathsf{lim}_{\mathsf{P}^{\mathsf{op}}}"] \end{tikzcd} \quad ;$$

in particular, every object of $\mathcal{X}$ obtains a natural ascending P-filtration as well as a natural descending P^{op}-filtration.[46] In order to proceed, we introduce the composite adjunction

$$\begin{tikzcd} \lambda^p : \mathcal{X}_p \arrow[r, hook, shift left, "\nu"] \arrow[r, phantom, "\perp"] & \mathcal{Z}_p \arrow[l, shift left, "p_R"] \arrow[r, hook, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{X} : \Psi_p \arrow[l, shift left, "y"] \end{tikzcd}$$

for each $p \in \mathsf{P}$, whose right adjoint Ψ_p we refer to as the p^{th} ***reflected geometric localization functor***.[47] Using this notation, for each $p \in \mathsf{P}$ we identify the p^{th} associated graded components of these filtrations as total co/fibers (Definition 7.4.3), namely

$$\mathsf{gr}_p^L := \mathsf{gr}_p(\mathsf{fil}_{\bullet}^L) := \mathsf{tcofib}_{(\leq p)}(\mathsf{fil}_{\bullet}^L) \simeq \lambda^p \Phi_p \qquad \text{and} \qquad \mathsf{gr}_R^p := \mathsf{gr}^{p^{\circ}}(\mathsf{fil}_R^{\bullet}) := \mathsf{tfib}_{(\leq p)^{\mathsf{op}}}(\mathsf{fil}_R^{\bullet}) \simeq \rho^p \Psi_p \ .$$

In fact, we also have canonical filtrations corresponding to upwards-closures (rather than downwards-closures) of elements of P.[48] Namely, for each $p \in \mathsf{P}$, we have the corresponding recollement (1.1.1)

[45] Without hypotheses on P, the filtrations that we define below exist but convergence is more subtle.

[46] We use subscripts for ascending filtrations and superscripts for descending filtrations. (This handedness dictates the corresponding convention for associated graded components as either total cofibers or total fibers (Definition 7.4.3).)

[47] The functors λ^p and Ψ_p play central roles in the theory of reflection (dual to those of ρ^p and Φ_p), which is introduced in §1.10.

[48] These latter filtrations are more natural in the context of stratified topological spaces, where a distinguished role is played by the closures of strata; see Remark 1.8.1.

with $\mathcal{Z} = \bigcup_{q \not\geq p} \mathcal{Z}_q$. Writing

$$\mathsf{fil}^p_L := \nu p_L \qquad \text{and} \qquad \mathsf{fil}^R_p := \nu p_R \ ,$$

we similarly obtain canonical embeddings with retracts

$$\begin{tikzcd} \mathsf{Fun}(\mathsf{P}, \mathcal{X}) \arrow[r, hookleftarrow, "\mathsf{fil}^\bullet_L"] \arrow[r, bend right, "\lim_{\mathsf{P}}"'] & \mathcal{X} \arrow[r, hook, "\mathsf{fil}^R_\bullet"] & \mathsf{Fun}(\mathsf{P}^{\mathsf{op}}, \mathcal{X}) \arrow[l, bend left, "\mathsf{colim}_{\mathsf{P}^{\mathsf{op}}}"] \end{tikzcd} \ .$$

These have the same associated graded components as the above two filtrations: we have

$$\mathsf{gr}^p_L := \mathsf{gr}^p(\mathsf{fil}^\bullet_L) := \mathsf{tfib}_{(\geq p)}(\mathsf{fil}^\bullet_L) \simeq \lambda^p \Phi_p \qquad \text{and} \qquad \mathsf{gr}^R_p := \mathsf{gr}_p(\mathsf{fil}^R_\bullet) := \mathsf{tcofib}_{(\geq p)^{\mathsf{op}}}(\mathsf{fil}^R_\bullet) \simeq \rho^p \Psi_p$$

(i.e. $\mathsf{gr}^p_L \simeq \mathsf{gr}^L_p$ and $\mathsf{gr}^R_p \simeq \mathsf{gr}^p_R$). Moreover, the latter two filtrations can be obtained from the former two by the formulas

$$\mathsf{fil}^p_L \mathcal{F} \simeq \mathsf{cofib}\Big(\mathsf{colim}_{q \in (\not\geq p)} \mathsf{fil}^L_q \mathcal{F} \longrightarrow \mathcal{F}\Big) \qquad \text{and} \qquad \mathsf{fil}^R_p \mathcal{F} \simeq \mathsf{fib}\Big(\mathcal{F} \longrightarrow \lim_{q^\circ \in (\not\geq p)^{\mathsf{op}}} \mathsf{fil}^q_R \mathcal{F}\Big) \ .$$

Remark 1.3.13 (spectral sequences from stratifications)**.** Fix a P-stratified noncommutative stack $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$, and assume that P is down-finite. Suppose that we are additionally given the following data:

- a conservative functor $\mathsf{P} \xrightarrow{d} \mathbb{Z}$;
- an exact functor $\mathcal{X} \xrightarrow{H} \mathcal{V}$ to a stable ∞-category $\mathcal{V}$ equipped with a t-structure (e.g. $\mathcal{X} \xrightarrow{\underline{\mathsf{hom}}_{\mathcal{X}}(\mathcal{E},-)} \mathcal{S}\mathsf{p}$ for some $\mathcal{E} \in \mathcal{X}$).

Then, we obtain a composite

$$\mathcal{X} \xrightarrow{\mathsf{fil}^L_\bullet} \mathsf{Fun}(\mathsf{P}, \mathcal{X}) \xrightarrow{H} \mathsf{Fun}(\mathsf{P}, \mathcal{V}) \xrightarrow{d_!} \mathsf{Fun}(\mathbb{Z}, \mathcal{V}) \ ,$$

i.e. a natural assignment of a filtered object in $\mathcal{V}$ for each object of $\mathcal{X}$. In particular, for each $\mathcal{F} \in \mathcal{X}$ we obtain a spectral sequence (see e.g. [Lur, §1.2.2]), which runs

$$E^1_{s,t} = \bigoplus_{p \in d^{-1}(s)} \pi_{s+t}(H(\mathsf{gr}^L_p \mathcal{F})) \Longrightarrow \pi_{s+t}(H(\mathcal{F})) \ .^{49} \tag{1.3.15}$$

Dually, using $\mathsf{fil}^R_\bullet$ in place of $\mathsf{fil}^L_\bullet$, we obtain a spectral sequence running

$$E^1_{s,t} = \bigoplus_{p \in d^{-1}(-s)} \pi_{s+t}(H(\mathsf{gr}^R_p \mathcal{F})) \Longrightarrow \pi_{s+t}(H(\mathcal{F})) \ . \tag{1.3.16}$$

For the descending filtrations $\mathsf{fil}^\bullet_L$ and $\mathsf{fil}^\bullet_R$, we obtain essentially the same spectral sequences using d_* instead of $d_!$, but with the sums replaced by products.

Remark 1.3.14. More than being convergent, a P-stratified noncommutative stack $\mathcal{X}$ or a quasicoherent sheaf thereon $\mathcal{F} \in \mathcal{X}$ may be ***strict***. In the former case, this is the condition that $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ is convergent and moreover its gluing diagram (1.3.4) is a strict (as opposed to left-lax) functor. In both cases, strictness affords a simplified reconstruction theorem. Moreover, $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ is strict if and only if all of its objects are strict.[50] We study strict objects in §2.7 and strict stratifications in §6.3.

Remark 1.3.15. We establish a number of variations on the metacosm reconstruction of Theorem A(1), which we briefly describe here.

[49] This is guaranteed to converge e.g. if P is finite or if H is colimit-preserving.

[50] So, strictness is analogous at the metacosm level to the condition that the depth of the poset P is at most 1. One may likewise contemplate strictness at the nanocosm level, and (in a sense that is evident from the discussion of §2.7) the object $\mathcal{F} \in \mathcal{X}$ is strict if and only if the pair $(\mathcal{E}, \mathcal{F})$ is strict for all $\mathcal{E} \in \mathcal{X}$.

(1) We establish a theory of stratifications of stable ∞-categories (that are assumed to be idempotent-complete but not necessarily presentable), which we refer to as *stable stratifications*. We provide a metacosm equivalence

$$\mathbf{strat}_{\mathsf{P}} \underset{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}}{\overset{\mathscr{G}}{\underset{\sim}{\rightleftarrows}}} \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}})$$

for stable stratifications as Theorem 7.2.4 (under the assumption that P is finite).

(2) We specialize both metacosm equivalences to *strict* morphisms among (resp. stable) stratifications, which correspond with strict (as opposed to right-lax) morphisms among (the suitable sorts of) left-lax left P-modules: we establish equivalences

$$\mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}} \underset{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}}{\overset{\mathscr{G}}{\underset{\sim}{\rightleftarrows}}} \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \qquad \text{and} \qquad \mathbf{strat}^{\mathsf{strict}}_{\mathsf{P}} \underset{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}}{\overset{\mathscr{G}}{\underset{\sim}{\rightleftarrows}}} \mathsf{LMod}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}})$$

as Theorem 7.3.2 (the former when P is down-finite, the latter when P is finite).

(3) We establish the theory of ***reflection*** for (resp. stable) stratifications, which we describe in §1.10. This affords a dual form of reconstruction, which is desirable for reconstructing constructible sheaves (stratifications of which are discussed in §1.8).

1.4. **Fundamental operations on stratified noncommutative stacks.** The right adjoint in the metacosm adjunction (1.3.5) may be viewed as the inclusion of the full subcategory of *convergent* stratifications. From this point of view, Theorem A (and its sharpness indicated in Remark 1.3.9) may be read as the assertion that all stratifications over P are convergent if and only if P is down-finite.

We view the possibility of nonconvergence not as a bug, but rather as an essential feature. For example, the adelic stratifications guaranteed by Theorem D below are utterly fundamental and must constitute valid examples under any reasonable definition, and yet they do not generally converge. And Example 1.3.11 provides compelling further evidence that nonconvergent stratifications should be considered as a common phenomenon indeed.

Of course, nonconvergent stratifications are not so useful on their own. In order to extract convergent stratifications from nonconvergent ones (and as a key ingredient in the proof of Theorem A), we therefore establish a *pushforward* operation for stratifications. Its utility is illustrated in Example 1.6.1 below, where we show that a certain pushforward of the (nonconvergent) adelic stratification of $\mathsf{Mod}_{\mathbb{Z}}$ gives a (necessarily convergent) stratification over [1] whose microcosm reconstruction theorem (i.e. the pullback square (1.1.3)) recovers the arithmetic fracture square (0.1.1).

In fact, pushforward is but one in a suite of ***fundamental operations*** that we provide for constructing new stratifications from old ones. We indicate their general structure here, and refer the reader to §3.4 for precise definitions and statements.

Theorem B (Observation 3.4.4, Proposition 3.4.9, Proposition 3.4.10, Proposition 3.4.12, and Proposition 3.4.14)**.** *Let* P *be a poset and let* $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ *be a* P*-stratified noncommutative stack.*

(1) **restriction:** *For any down-closed subset* $\mathsf{D} \subseteq \mathsf{P}$*, there is a **restricted stratification** of* $\mathcal{Z}_{\mathsf{D}} := \bigcup_{p \in \mathsf{D}} \mathcal{Z}_p$ *over* D*.*

(2) **pullback:** *For any noncommutative stack* $\widetilde{\mathcal{X}}$ *equipped with a quotient functor* $\widetilde{\mathcal{X}} \to \mathcal{X}$ *by a closed noncommutative substack, there is a **pullback stratification** of* $\widetilde{\mathcal{X}}$ *over* P *(assuming that* P *is nonempty).*

(3) **quotient:** *For any down-closed subset* $\mathsf{D} \subseteq \mathsf{P}$*, there is a **quotient stratification** of* $\mathcal{X}/\mathcal{Z}_{\mathsf{D}}$ *over* $\mathsf{P}\backslash\mathsf{D}$*.*

(4) **pushforward:** *For any functor* $\mathsf{P} \to \mathsf{Q}$ *between posets, there is a **pushforward stratification** of* $\mathcal{X}$ *over* Q*.*

(5) **refinement:** *For any stratification of each stratum $\mathcal{X}_p$ over a poset R_p, there is a* ***refined stratification*** *of $\mathcal{X}$ over the wreath product poset $\mathsf{P} \wr \mathsf{R}_\bullet$.*

Remark 1.4.1. Towards proving Theorem B, in §3.2 we introduce and study the notion of ***alignment*** between closed subcategories. This does not seem to have a direct analog in point-set topology (or even in ∞-topos theory): in the ∞-category of sheaves on a topological space, closed subcategories associated to open subsets are automatically aligned (see §1.8). One manifestation of this idea is that alignment affords excision- and Mayer–Vietoris-type gluing formulas for closed subcategories.

Given a stratification, all of the closed subcategories that it determines (i.e. its values and colimits thereof) are automatically mutually aligned. Our results regarding alignment collectively streamline the arguments that comprise the proof of Theorem B. At the same time, the notion of alignment allows us to obtain generalizations of parts (1) and (3) of Theorem B (see Proposition 3.4.7 (and Remark 3.4.8) for the former).

1.5. **$\mathcal{O}$-monoidal stratifications.** One attractive feature of our definition of a stratification is that it generalizes quite straightforwardly to the case of a presentably $\mathcal{O}$-monoidal stable ∞-category $\mathcal{R}$ (i.e. an $\mathcal{O}$-algebra in the symmetric monoidal ∞-category $(\mathsf{Pr}^L_{\mathsf{st}}, \otimes, \mathcal{S}\mathsf{p})$), as we now describe.

First of all, an *ideal* of $\mathcal{R}$ is a full presentable stable subcategory $\mathcal{I} \subseteq \mathcal{R}$ which is contagious under the $\mathcal{O}$-monoidal structure, and a *closed ideal* is a closed subcategory which is an ideal in a compatible way (Definition 4.2.8). Closed ideals form a full subposet $\mathbf{Idl}_{\mathcal{R}} \subseteq \mathbf{Cls}_{\mathcal{R}}$, and an $\mathcal{O}$***-monoidal stratification*** of $\mathcal{R}$ is simply a stratification that factors through this subposet.

Example 1.5.1. For any closed subset $Z \in \mathbf{Cls}_X$, the corresponding closed subcategory $\mathsf{QC}_Z(X) \in \mathbf{Cls}_{\mathsf{QC}(X)}$ is a closed ideal subcategory.

We have the following macrocosm $\mathcal{O}$-monoidal reconstruction theorem.

Theorem C (Theorem 4.5.1)**.** *Let $\mathcal{O}$ be an ∞-operad satisfying the conditions of Notation 4.1.2(1) (e.g. $\mathbb{E}_n$ for $1 \leq n \leq \infty$), and suppose that $\mathcal{R}$ is a presentably $\mathcal{O}$-monoidal stable ∞-category equipped with an $\mathcal{O}$-monoidal stratification*

$$\mathsf{P} \xrightarrow{\mathcal{I}_\bullet} \mathbf{Idl}_{\mathcal{R}}$$

over a poset P. Then, the strata of the stratification inherit canonical $\mathcal{O}$-monoidal structures, the gluing functors become canonically right-laxly $\mathcal{O}$-monoidal, and these assemble into an $\mathcal{O}$-monoidal gluing diagram $\mathscr{G}^{\otimes}(\mathcal{R})$ that lifts the gluing diagram $\mathscr{G}(\mathcal{R})$, in such a way that we have a canonical identification

$$\begin{tikzcd}
\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \arrow[r, "\mathsf{fgt}"] & \mathsf{Cat} \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}^{\otimes}(\mathcal{R})) =: \mathsf{Glue}^{\otimes}(\mathcal{R}) \arrow[r, mapsto] & \mathsf{Glue}(\mathcal{R}) := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{R}))
\end{tikzcd} .$$

Moreover, the adjunction

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "g"] \arrow[r, phantom, "\perp"] & \mathsf{Glue}(\mathcal{R}) \arrow[l, shift left, "\lim_{\mathsf{sd(P)}}"] \end{tikzcd} \tag{1.5.1}$$

between ∞-categories of Theorem A(2) admits a canonical enhancement to an adjunction

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "g^{\otimes}"] \arrow[r, phantom, "\perp"] & \mathsf{Glue}^{\otimes}(\mathcal{R}) \arrow[l, shift left, "\lim^{\otimes}_{\mathsf{sd(P)}}"] \end{tikzcd} \tag{1.5.2}$$

between $\mathcal{O}$-monoidal ∞-categories, whose left adjoint is $\mathcal{O}$-monoidal and whose right adjoint is right-laxly $\mathcal{O}$-monoidal. In particular, if the adjunction (1.5.1) *is an equivalence between ∞-categories (e.g. as guaranteed by Theorem A in the case that P is down-finite), then the adjunction* (1.5.2) *is an equivalence between $\mathcal{O}$-monoidal ∞-categories.*

Remark 1.5.2. Given two $\mathcal{O}$-monoidal ∞-categories, a *right-laxly $\mathcal{O}$-monoidal functor* between them is a functor between their underlying ∞-categories that preserves the $\mathcal{O}$-monoidal structures up to certain (generally noninvertible) comparison morphisms. For example, a right-laxly monoidal functor

$$(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) \xrightarrow{F} (\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$$

between monoidal ∞-categories involves the data of natural comparison morphisms

$$\mathbb{1}_{\mathcal{D}} \longrightarrow F(\mathbb{1}_{\mathcal{C}}) \qquad \text{and} \qquad F(X) \otimes_{\mathcal{D}} F(Y) \longrightarrow F(X \otimes_{\mathcal{C}} Y) \ .$$

This and related notions are reviewed in §4.1.

Remark 1.5.3. Although we are confident in the existence of a metacosm $\mathcal{O}$-monoidal reconstruction theorem, we state Theorem C at the macrocosm level only.

Remark 1.5.4. It is immediate from Observation 4.2.9 that the fundamental operations described in Theorem B admit direct analogs for $\mathcal{O}$-monoidal stratifications.

Remark 1.5.5. Closed ideals in $\mathcal{R}$ are equivalent data to *central co/augmented idempotent objects* in $\mathcal{R}$ (see Definition 4.2.12 and Proposition 4.2.14). It follows that a morphism $\mathcal{R} \to \mathcal{R}'$ in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Pr}^L_{\mathsf{st}})$ determines a functor

$$\mathbf{Idl}_{\mathcal{R}} \longrightarrow \mathbf{Idl}_{\mathcal{R}'} \ . \tag{1.5.3}$$

Moreover, by Observation 4.3.6, postcomposition with the functor (1.5.3) carries $\mathcal{O}$-monoidal stratifications of $\mathcal{R}$ to $\mathcal{O}$-monoidal stratifications of $\mathcal{R}'$.

Remark 1.5.6. Let $\mathcal{R}$ be a presentably monoidal stable ∞-category equipped with a monoidal stratification

$$\mathsf{P} \xrightarrow{\mathcal{J}_\bullet} \mathbf{Idl}_{\mathcal{R}} \ .$$

For any left $\mathcal{R}$-module $\mathcal{M} \in \mathsf{LMod}_{\mathcal{R}}(\mathsf{Pr}^L_{\mathsf{st}})$, we immediately obtain a stratification

$$\begin{array}{ccc} \mathsf{P} & \longrightarrow & \mathbf{Cls}_{\mathcal{M}} \\ \Psi & & \Psi \\ p & \longmapsto & \mathcal{J}_p \otimes_{\mathcal{R}} \mathcal{M} \simeq \mathsf{LMod}_{\mathbb{1}_{\mathcal{J}_p}}(\mathcal{M}) \end{array}$$

of $\mathcal{M}$ over P (using Notation 4.3.4).[51]

Remark 1.5.7. Our work posits a system of analogies between classical algebra and categorified algebra, which is indicated in Figure 3.[52]

1.6. **Adelic reconstruction.** We now return to our scheme X. Let us write P_X for the specialization poset of its underlying topological space: it has the same underlying set, and its relation is defined so that $x \leq y$ if and only if $x \in \overline{y}$. Then, the closure functor

$$\begin{array}{ccc} \mathsf{P}_X & \xrightarrow{\overline{(-)}} & \mathbf{Cls}_X \\ \Psi & & \Psi \\ x & \longmapsto & \overline{x} \end{array}$$

[51]This appears to be closely related to Elias–Hogancamp's theory of categorical diagonalization [EH].

[52]Colimits categorify addition, $\mathcal{O}$-monoidal structures categorify multiplication, and the distributivity of $\mathcal{O}$-monoidal structures over colimits categorifies the distributivity of multiplication over addition. The analogy between presentable stable ∞-categories and abelian groups is further evinced e.g. by the fact that given compact objects $X, Y \in \mathcal{X}^\omega$, the sequence

$$\langle X \rangle \hookrightarrow \langle X, Y \rangle \twoheadrightarrow \langle Y \rangle$$

(using Notation 2.3.3) is exact if and only if X and Y are "linearly independent", i.e. $\underline{\mathsf{hom}}_{\mathcal{X}}(X, Y) \simeq 0$. (A noncompact object of $\mathcal{X}$ might be thought of as categorifying a nonconvergent infinite sum in an abelian group.)

classical algebra	categorified algebra
abelian group (or spectrum)	presentable stable ∞-category
$\mathcal{O}$-ring (spectrum)	presentably $\mathcal{O}$-monoidal stable ∞-category
filtration	stratification
filtered pieces	$\{\mathcal{Z}_p\}_{p \in \mathsf{P}}$
associated graded pieces	$\{\mathcal{X}_p\}_{p \in \mathsf{P}}$
extension data	gluing diagram

Figure 3. This table lays out a system of analogies between classical algebra and categorified algebra.

defines a stratification of X. Upgrading Example 1.3.6 via Example 1.5.1, we obtain a symmetric monoidal stratification

$$\begin{array}{ccccc} \mathsf{P}_X & \xrightarrow{\overline{(-)}} & \mathbf{Cls}_X & \xrightarrow{\mathsf{QC}_{(-)}(X)} & \mathbf{Idl}_{\mathsf{QC}(X)} \\ \Psi & & & & \Psi \\ x & \longmapsto & & \longrightarrow & \mathsf{QC}_{\overline{x}}(X) \end{array} \tag{1.6.1}$$

of its underlying noncommutative stack $\mathsf{QC}(X)$, which we refer to as its ***adelic stratification***. For each $x \in \mathsf{P}_X$, the x^{th} stratum of this symmetric monoidal stratification is

$$\mathsf{ker}\left(\mathsf{QC}(X^{\wedge}_{\overline{x}}) \longrightarrow \prod_{y<x} \mathsf{QC}(X^{\wedge}_{\overline{y}})\right) .$$ [53]

In general, the poset P_X will not be down-finite, and so the adelic stratification of $\mathsf{QC}(X)$ is not guaranteed to converge. However, writing $d := \dim(X)$ for the dimension of X,[54] we may take the pushforward of the adelic stratification (1.6.1) along the *dimension* functor

$$\mathsf{P}_X \xrightarrow{\dim} [d] \ ;$$

as $[d]$ is finite and hence down-finite, the pushforward symmetric monoidal stratification is guaranteed to converge. Moreover, as the the fibers of the dimension functor are discrete, the strata of the pushforward symmetric monoidal stratification will simply be products of strata of the adelic stratification. We illustrate this maneuver in the following fundamental example.

Example 1.6.1 (the adelic stratification of $\mathbb{Z}$-modules)**.** Suppose that $X = \mathsf{Spec}(\mathbb{Z})$. The specialization poset of this affine scheme (which is the opposite of the poset of prime ideals of $\mathbb{Z}$) is given by

$$\mathsf{P}_{\mathbb{Z}} := \mathsf{P}_{\mathsf{Spec}(\mathbb{Z})} = \left(\begin{array}{cccc} & & (0) & \\ & \nearrow & \uparrow & \nwarrow \\ (2) & (3) & (5) & \cdots \end{array}\right) . \tag{1.6.2}$$

[53] When the subset $({}^{<}x) := (\overline{x} \backslash x) \subseteq X$ is closed, the x^{th} stratum may be identified more simply as $\mathsf{QC}((X \backslash ({}^{<}x))^{\wedge}_{x})$.

[54] Of course, it is here that we use that our scheme X is finite-dimensional.

Then, its adelic stratification

$$
\begin{tikzcd}
\mathsf{P}_{\mathbb{Z}} \arrow[r, "\mathcal{I}_\bullet"] & \mathbf{Idl}_{\mathsf{Mod}_{\mathbb{Z}}} \\[-20pt]
\rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\[-20pt]
\mathfrak{p} \arrow[r, mapsto] & \mathcal{I}_{\mathfrak{p}}
\end{tikzcd}
\tag{1.6.3}
$$

is described by the formulas

$$
\mathcal{I}_{(0)} = \mathsf{Mod}_{\mathbb{Z}} \qquad \text{and} \qquad \mathcal{I}_{(p)} = \mathsf{Mod}_{\mathbb{Z}}^{(p)\text{-torsion}} ,
$$

i.e. it selects the diagram

$$
\begin{tikzcd}
 & & & \mathsf{Mod}_{\mathbb{Z}} & \\
\mathsf{Mod}_{\mathbb{Z}}^{(2)\text{-torsion}} \arrow[urrr, hook] & & \mathsf{Mod}_{\mathbb{Z}}^{(3)\text{-torsion}} \arrow[ur, hook] & \mathsf{Mod}_{\mathbb{Z}}^{(5)\text{-torsion}} \arrow[u, hook] & \cdots \arrow[ul, hook]
\end{tikzcd}
$$

of closed ideal subcategories of $\mathsf{Mod}_{\mathbb{Z}}$.

We now apply Theorem C. We begin by identifying the strata and geometric localization adjunctions as

$$
\mathcal{R} := \mathsf{Mod}_{\mathbb{Z}} \underset{\rho^{(0)} = \mathsf{fgt}}{\overset{\Phi_{(0)} = \mathbb{Q} \otimes_{\mathbb{Z}} (-)}{\underset{\perp}{\rightleftarrows}}} \mathsf{Mod}_{\mathbb{Q}} \simeq \left(\mathsf{Mod}_{\mathbb{Z}} \Big/ \bigcup_{p \text{ prime}} \mathsf{Mod}_{\mathbb{Z}}^{(p)\text{-torsion}} \right) =: \left(\mathcal{I}_{(0)} \Big/ \bigcup_{p \text{ prime}} \mathcal{I}_{(p)} \right) =: \mathcal{R}_{(0)}
$$

and

$$
\begin{tikzcd}
\mathcal{I}_{(p)} := \mathsf{Mod}_{\mathbb{Z}}^{(p)\text{-torsion}} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] \arrow[d, leftrightarrow, "\wr"'] & \mathsf{Mod}_{\mathbb{Z}} =: \mathcal{R} \arrow[l, shift left] \arrow[dl, shift right, "\Phi_{(p)} = \mathbb{Z}_p^\wedge \widehat{\otimes}_{\mathbb{Z}} (-)"'] \arrow[dl, phantom, "\perp"] \\
\mathcal{R}_{(p)} := \mathsf{Mod}_{\mathbb{Z}_p^\wedge}^{(p)\text{-complete}} \arrow[ur, hook, shift right, "\rho^{(p)} = \mathsf{fgt}"'] &
\end{tikzcd}
$$

(recall Example 1.1.2).[55] From here, we see that the symmetric monoidal gluing diagram $\mathscr{G}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}})$ of the adelic stratification (1.6.3) is the diagram

$$
\begin{tikzcd}
 & & & \mathsf{Mod}_{\mathbb{Q}} & \\
\mathsf{Mod}_{\mathbb{Z}_2^\wedge}^{(2)\text{-complete}} \arrow[urrr] & & \mathsf{Mod}_{\mathbb{Z}_3^\wedge}^{(3)\text{-complete}} \arrow[ur] & \mathsf{Mod}_{\mathbb{Z}_5^\wedge}^{(5)\text{-complete}} \arrow[u] & \cdots \arrow[ul]
\end{tikzcd}
$$

of presentably symmetric monoidal stable ∞-categories, in which all gluing functors are given by rationalization.[56] We may now identify the reglued symmetric monoidal ∞-category

$$
\mathsf{Glue}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}}) := \lim^{\mathsf{r.lax}}_{\mathsf{P}_{\mathbb{Z}}} (\mathscr{G}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}}))
$$

as consisting of tuples of data

$$
\left(M_0 \in \mathsf{Mod}_{\mathbb{Q}} \ , \ \left(M_p \in \mathsf{Mod}_{\mathbb{Z}_p^\wedge}^{(p)\text{-complete}} \ , \ \begin{array}{c} M_0 \\ \downarrow \\ \mathbb{Q} \otimes_{\mathbb{Z}} M_p \end{array} \right)_{p \text{ prime}} \right) , \tag{1.6.4}
$$

---

[55] We distinguish between the equivalent ∞-categories $\mathcal{I}_{(p)}$ and $\mathcal{R}_{(p)}$ according to their inclusions into $\mathcal{R}$.

[56] The poset $\mathsf{P}_{\mathbb{Z}}$ has no nondegenerate composite morphisms, and so the functor $\mathscr{G}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}})$ is in fact a strict (instead of left-lax) functor.

equipped with the componentwise symmetric monoidal structure. This brings us to the symmetric monoidal macrocosm adjunction

$$\mathsf{Mod}_{\mathbb{Z}} \underset{\lim^{\otimes}_{\mathsf{sd}(\mathsf{P}_{\mathbb{Z}})}}{\overset{g^{\otimes}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Glue}^{\otimes}(\mathsf{Mod}_{\mathbb{Z}}) \quad , \tag{1.6.5}$$

whose left adjoint $g^{\otimes}$ takes $M \in \mathsf{Mod}_{\mathbb{Z}}$ to the evident tuple (1.6.4) in which $M_0 := \mathbb{Q} \otimes_{\mathbb{Z}} M$ and $M_p := M_p^{\wedge} := \mathbb{Z}_p^{\wedge} \widehat{\otimes}_{\mathbb{Z}} M$ and whose right adjoint takes the tuple (1.6.4) to the evident object

$$\lim \left(\begin{tikzcd} & & & & M_0 \arrow[dlll] \arrow[dll] \arrow[d] \arrow[dr] & & \\ & \mathbb{Q} \otimes_{\mathbb{Z}} M_2 & & \mathbb{Q} \otimes_{\mathbb{Z}} M_3 & \mathbb{Q} \otimes_{\mathbb{Z}} M_5 & \cdots & \\ M_2 \arrow[ur] & & M_3 \arrow[ur] & & M_5 \arrow[u] & & \cdots \arrow[ul] \end{tikzcd} \right) \in \mathsf{Mod}_{\mathbb{Z}} .^{57} \tag{1.6.6}$$

We can now witness the failure of convergence of the adelic stratification (1.6.3). Reorganizing the limit (1.6.6) as the pullback

$$\lim \left(\begin{tikzcd} & M_0 \arrow[d] \\ \prod_{p \text{ prime}} M_p \arrow[r] & \prod_{p \text{ prime}} (\mathbb{Q} \otimes_{\mathbb{Z}} M_p) \end{tikzcd} \right) \in \mathsf{Mod}_{\mathbb{Z}} \ ,$$

we find that the unit of the adjunction (1.6.5) at an object $M \in \mathsf{Mod}_{\mathbb{Z}}$ is a morphism

$$\lim \left(\begin{tikzcd} & \mathbb{Q} \otimes_{\mathbb{Z}} M \arrow[d] \\ \prod_{p \text{ prime}} M_p^{\wedge} \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod_{p \text{ prime}} M_p^{\wedge} \right) \end{tikzcd} \right) \simeq M \longrightarrow \lim \left(\begin{tikzcd} & \mathbb{Q} \otimes_{\mathbb{Z}} M \arrow[d] \\ \prod_{p \text{ prime}} M_p^{\wedge} \arrow[r] & \prod_{p \text{ prime}} (\mathbb{Q} \otimes_{\mathbb{Z}} M_p^{\wedge}) \end{tikzcd} \right) , \tag{1.6.7}$$

in which we have included the equivalence resulting from the arithmetic fracture square (0.1.1) for emphasis. The unit morphism (1.6.7) is not generally an equivalence, because the rationalization functor $\mathbb{Q} \otimes_{\mathbb{Z}} (-)$ does not commute with infinite products.[58] For instance, consider the abelian group

$$M := \bigoplus_{p \text{ prime}} \mathbb{Z}/p \ :$$

for each prime number p we have $M_p^{\wedge} \simeq \mathbb{Z}/p$, and the morphism

$$\mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod_{p \text{ prime}} \mathbb{Z}/p \right) \longrightarrow \prod_{p \text{ prime}} (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p) \simeq 0$$

is not an equivalence. Note that this failure of convergence does not contradict Theorem A, as the poset $\mathsf{P}_{\mathbb{Z}}$ is not down-finite (because the closure of the generic point $(0) \in \mathsf{Spec}(\mathbb{Z})$ is infinite).

---

[57] The right adjoint $\lim^{\otimes}_{\mathsf{sd}(\mathsf{P}_{\mathbb{Z}})}$ of the symmetric monoidal macrocosm adjunction (1.6.5) is only right-laxly (instead of strictly) symmetric monoidal, as the functors $\rho^{(p)}$ are only right-laxly symmetric monoidal.

[58] More precisely, the morphism (1.6.7) between pullbacks arises from a natural transformation between cospans which is an equivalence on the two source terms and is induced by the universal property of the product on the common target term.

In order to rectify this failure of convergence, we apply Theorem B: more precisely, we take the pushforward of the adelic stratification along the dimension functor

$$\mathsf{P}_{\mathbb{Z}} \xrightarrow[(p)\mapsto 0]{(0)\mapsto 1} [1] \ .$$

Recalling Remark 1.5.4, we see that this yields a symmetric monoidal stratification of $\mathsf{Mod}_{\mathbb{Z}}$ over $[1]$,[59] which determines a symmetric monoidal recollement

$$\begin{tikzcd}
\prod\limits_{p \text{ prime}} \mathsf{Mod}_{\mathbb{Z}}^{(p)\text{-torsion}} \arrow[r, hook, shift left] \arrow[d, leftrightarrow, "\wr"'] & \mathsf{Mod}_{\mathbb{Z}} \arrow[l, shift left, "\perp"] \arrow[r, bend left, "\Phi_1=\Phi_{(0)}"] \arrow[r, bend right, "\perp"'] \arrow[dl, shift left, "\Phi_0=\left(\Phi_{(p)}\right)_{p \text{ prime}}"'] & \mathsf{Mod}_{\mathbb{Q}} \arrow[l, hook', "\rho^1=\rho^{(0)}"', "\perp"] \\
\prod\limits_{p \text{ prime}} \mathsf{Mod}_{\mathbb{Z}_p^\wedge}^{(p)\text{-complete}} \arrow[ur, hook, shift left, "\rho^0=\prod_{p \text{ prime}} \rho^{(p)}"', "\perp"] & &
\end{tikzcd} \tag{1.6.8}$$

(in the sense that for $i \in [1]$ the left adjoints Φ_i are symmetric monoidal and their right adjoints ρ^i are right-laxly symmetric monoidal). Combining Theorems A and C, we obtain a macrocosm equivalence

$$\mathsf{Mod}_{\mathbb{Z}} \underset{\lim^{\otimes}_{\mathsf{sd}([1])}}{\overset{g^{\otimes}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \lim{}^{\mathsf{r.lax}} \left(\prod_{p \text{ prime}} \mathsf{Mod}_{\mathbb{Z}_p^\wedge}^{(p)\text{-complete}} \xrightarrow{\Phi_1\rho^0} \mathsf{Mod}_{\mathbb{Q}} \right) \tag{1.6.9}$$

between presentably symmetric monoidal stable ∞-categories. For each $M \in \mathsf{Mod}_{\mathbb{Z}}$, the unit of the adjoint equivalence (1.6.9) recovers the microcosm equivalence

$$M \xrightarrow{\sim} \lim \left(\begin{tikzcd} & \mathbb{Q} \otimes_{\mathbb{Z}} M \arrow[d] \\ \prod\limits_{p \text{ prime}} M_p^\wedge \arrow[r] & \mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod\limits_{p \text{ prime}} M_p^\wedge \right) \end{tikzcd} \right) ,$$

i.e. the arithmetic fracture square (0.1.1).

We generalize the preceding discussion to the setting of tensor-triangular geometry as follows.

Theorem D (Theorem 4.6.11)**.** *Let $\mathcal{R}$ be a presentably symmetric monoidal stable ∞-category, and assume that $\mathcal{R}$ is rigidly-compactly generated (Definition 4.6.3). Then, there is a canonical functor*

$$\mathsf{P}_{\mathcal{R}} \longrightarrow \mathbf{Idl}_{\mathcal{R}} \tag{1.6.10}$$

from the specialization poset $\mathsf{P}_{\mathcal{R}}$ of $\mathsf{Spec}(\mathcal{R}^\omega)$ (i.e. the poset of thick prime ideal subcategories of $\mathcal{R}^\omega$ ordered by inclusion), which is defined in terms of supports. The functor (1.6.10) *satisfies the stratification condition. So, it defines a symmetric monoidal stratification assuming that it also satisfies the generation condition.*

We refer to such a symmetric monoidal stratification (1.6.10) as the ***adelic stratification*** of $\mathcal{R}$. We unpack the adelic stratification of $\mathcal{R} = \mathcal{S}\mathsf{p}$ as Example 4.6.13.

[59]Note that this stratification of $\mathsf{Mod}_{\mathbb{Z}} \simeq \mathsf{QC}(\mathsf{Spec}(\mathbb{Z}))$ does not arise from a stratification of $\mathsf{Spec}(\mathbb{Z})$, as the subset $(\mathsf{Spec}(\mathbb{Z}) \backslash \{(0)\}) \subseteq \mathsf{Spec}(\mathbb{Z})$ is not closed.

Remark 1.6.2. The functor (1.6.10) automatically satisfies the generation condition (and so defines a symmetric monoidal stratification) whenever the topological space $\mathsf{Spec}(\mathcal{R}^\omega)$ has finitely many irreducible components.[60] This holds for example in the case that $\mathsf{Spec}(\mathcal{R}^\omega)$ is noetherian, which also implies that its specialization poset is down-finite.

Remark 1.6.3. Adelic stratifications bring an exciting perspective to tensor-triangular geometry, which seems worthy of further investigation; this is discussed further in Remark 4.6.14.

1.7. **The geometric stratification of genuine G-spectra.** Let G be a compact Lie group. As a matter of notation and perspective, we write $\mathscr{B}G$ for the noncommutative stack whose quasicoherent sheaves are genuine G-spectra:

$$\mathsf{QC}(\mathscr{B}G) := \mathcal{S}\mathsf{p}^{\mathsf{g}G} .$$

We also introduce the following notation.

- We write P_G for the poset of conjugacy classes of closed subgroups of G (ordered by subconjugacy).
- For any element $H \in \mathsf{P}_G$, we write $\mathsf{W}(H) := \mathsf{N}(H)/H$ for its Weyl group (the quotient by it of its normalizer in G).[61]
- We write $\mathcal{S}\mathsf{p}^{\mathsf{h}G} := \mathsf{Fun}(\mathsf{B}G, \mathcal{S}\mathsf{p})$ for the ∞-category of homotopy G-spectra.[62]

Theorem E (Theorem 5.1.27)**.** *The noncommutative stack $\mathscr{B}G$ admits a canonical symmetric monoidal stratification over P_G, with the following features.*

(1) Its stratum corresponding to an element $H \in \mathsf{P}_G$ is the commutative stack $\mathsf{BW}(H)$ (i.e. the presentable stable ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \simeq \mathsf{QC}(\mathsf{BW}(H))$ of homotopy $\mathsf{W}(H)$-spectra).

(2) The geometric localization functors are given by geometric fixedpoints:

$$\mathsf{QC}(\mathscr{B}G) := \mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{\Phi^H} \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \simeq \mathsf{QC}(\mathsf{BW}(H)) .$$ [63]

(3) For any morphism $H \to K$ in P_G, the associated gluing functor

$$\mathsf{QC}(\mathsf{BW}(H)) \simeq \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \xrightarrow{\Gamma^H_K} \mathcal{S}\mathsf{p}^{\mathsf{hW}(K)} \simeq \mathsf{QC}(\mathsf{BW}(K))$$

is given by a version of the Tate construction.[64]

In order to emphasize its relationship with the geometric fixedpoints functors, we refer to the symmetric monoidal stratification of Theorem E as the ***geometric stratification*** of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$.

Remark 1.7.1. In the case that the poset P_G is down-finite, it follows from Theorems E and A that a genuine G-spectrum

$$E \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$$

[60]See Remark 4.6.12 for an example where it fails.

[61]More invariantly, one can also describe $\mathsf{W}(H)$ as the compact Lie group of G-equivariant automorphisms of G/H.

[62]In addition to nicely paralleling the notation $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$, the notation $\mathcal{S}\mathsf{p}^{\mathsf{h}G}$ is consistent: this is the homotopy fixedpoints of the trivial G-action on the ∞-category $\mathcal{S}\mathsf{p}$.

[63]Our notation Φ_p for the p^{th} geometric localization functor, and indeed the terminology itself, are motivated by the example of geometric fixedpoints. However, we use the notation Φ^H instead of Φ_H in order to adhere to standard conventions in equivariant homotopy theory.

[64]In the case that G is abelian, the gluing functor associated to a morphism $H \to K$ in P_G is the *proper* Tate construction

$$\mathcal{S}\mathsf{p}^{\mathsf{h}(G/H)} \xrightarrow{(-)^{\tau(K/H)}} \mathcal{S}\mathsf{p}^{\mathsf{h}(G/K)} ,$$

which quotients by norms from all proper subgroups (rather than just the trivial subgroup, as in the usual Tate construction). When G is not abelian, the corresponding description of the gluing functors is slightly more elaborate (see Remark 5.2.5).

is equivalent data to its geometric fixedpoints spectra

$$\left\{ \Phi^H(E) \in \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \right\}_{H \in \mathsf{P}_G}$$

(considered as homotopy $\mathsf{W}(H)$-spectra) along with gluing data among these; Theorem C guarantees that this equivalence is moreover compatible with symmetric monoidal structures.

However, the poset P_G is down-finite if and only if the compact Lie group G is in fact a finite group. We do not know whether the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ is convergent in the case that G is positive-dimensional, but we see no reason to expect it to be so.[65] In any case, its pushforward to any down-finite poset produces a symmetric monoidal reconstruction theorem for genuine G-spectra. For instance, writing $d := \dim(G)$ we may take its pushforward along the *dimension* functor

$$\mathsf{P}_G \xrightarrow{\dim} [d] \ ;$$

we note that its fibers are down-finite, so in principle this may lead to a fuller understanding of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ in the case that G is positive-dimensional.

Another symmetric monoidal reconstruction theorem resulting from Theorems E and A is unpacked as Example 5.3.11: writing $\mathbb{T}$ for the circle group, the geometric stratification of the noncommutative stack $\mathcal{S}\mathsf{p}^{\mathsf{g}\mathbb{T}}$ of genuine $\mathbb{T}$-spectra over the poset $\mathsf{P}_{\mathbb{T}} \cong (\mathbb{N}^{\mathsf{div}})^{\triangleright}$ (which is not down-finite) restricts to a symmetric monoidal stratification of the noncommutative stack $\mathcal{S}\mathsf{p}^{\mathsf{g}^{<}\mathbb{T}}$ of *proper*-genuine $\mathbb{T}$-spectra over the poset $\mathbb{N}^{\mathsf{div}}$ (which is down-finite). We use the resulting symmetric monoidal reconstruction theorem to study cyclotomic spectra (and their symmetric monoidal structure) in [AMGRd].

Remark 1.7.2. As indicated by our formulation of Theorem E, we view it as providing a sense in which $\mathscr{B}G$ is a "nearly commutative" stack.[66] Indeed, its strata are commutative stacks and its gluing functors are right-laxly symmetric monoidal, just as would be the case for a stratified commutative stack. However, its gluing functors do not appear to be of commutative origin. This is already apparent in the simplest nontrivial case, where $G = \mathsf{C}_p$ is the cyclic group of order p. In this situation, the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{gC}_p}$ amounts to a symmetric monoidal recollement, whose gluing functor is the Tate construction

$$\mathcal{S}\mathsf{p}^{\mathsf{hC}_p} \xrightarrow{(-)^{\mathsf{tC}_p}} \mathcal{S}\mathsf{p} \tag{1.7.1}$$

(as is unpacked further in Example 5.3.4), and there does not appear to be a natural example of a commutative (spectral) stack X equipped with a closed-open decomposition (1.1.6) such that $\mathsf{QC}(X^\wedge_Z) \simeq \mathcal{S}\mathsf{p}^{\mathsf{hC}_p}$, $\mathsf{QC}(U) \simeq \mathcal{S}\mathsf{p}$, and the gluing functor

$$\mathsf{QC}(X^\wedge_Z) \xrightarrow{j^* \hat{\imath}_*} \mathsf{QC}(U)$$

coincides with the Tate construction (1.7.1).

1.8. **Stratified topological spaces and constructible sheaves.** We have discussed how the general theory of stratifications applies in the context of quasicoherent sheaves over a scheme (recall Example 1.3.6). In fact, it applies in other sheaf-theoretic contexts as well, as we now explain.

Let T be a topological space, and suppose that

$$U \underset{\text{open}}{\overset{j}{\hookrightarrow}} T \underset{\text{closed}}{\overset{i}{\hookleftarrow}} Z$$

[65] On the other hand, the poset P_G is always artinian; applying Remark 1.3.10 to the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ recovers the "geometric fixedpoints Whitehead theorem" [Gre, §1.6].

[66] As a nice coincidence, this also gives a second meaning to the terminology "geometric stratification".

is a closed-open decomposition of T (note that the placement is reversed from that of Example 1.1.2). Then, we obtain a recollement

$$\begin{tikzcd} \mathsf{Shv}(U) \arrow[r, bend left, hook, "j_!"] \arrow[r, bend right, hook, "j_*"'] & \mathsf{Shv}(T) \arrow[l, "j^! = j^*"'] \arrow[r, bend left, "i^*"] \arrow[r, bend right, "i^!"'] & \mathsf{Shv}(Z) \arrow[l, hook', "i_* = i_!"'] \end{tikzcd} \tag{1.8.1}$$

among presentable stable ∞-categories of sheaves valued in a presentable stable ∞-category (which we omit from our notation).

This may be upgraded as follows. A ***stratification*** of T over the poset P is a continuous function

$$T \xrightarrow{f} \mathsf{P} \ , \tag{1.8.2}$$

where we consider P as a topological space via the poset topology on its underlying set (in which the closed subsets are precisely the down-closed subsets). This determines a functor

$$\begin{array}{ccc} \mathsf{P}^{\mathsf{op}} & \xrightarrow{U_\bullet} & \mathbf{Open}_T \\ \Psi & & \Psi \\ p^\circ & \longmapsto & U_p := f^{-1}({}^{\geq}p) \end{array}$$

that satisfies the evident analog of Definition 1.2.1: we have $T = \bigcup_{p \in \mathsf{P}} U_p$, and for any $p, q \in \mathsf{P}$ we have $U_p \cap U_q = \bigcup_{p \leq r \text{ and } q \leq r} U_r$. From this we obtain a stratification of $\mathsf{Shv}(T)$ over P^{op}, namely the composite

$$\begin{array}{ccccc} \mathsf{P}^{\mathsf{op}} & \xrightarrow{U_\bullet} & \mathbf{Open}_T & \xrightarrow{\mathsf{Shv}} & \mathbf{Cls}_{\mathsf{Shv}(T)} \\ \Psi & & & & \Psi \\ p^\circ & \longmapsto & & & \mathsf{Shv}(U_p) \end{array} . \tag{1.8.3}$$

For each $p \in \mathsf{P}$, let us write

$$T_p := f^{-1}(p) \overset{\sigma_p}{\hookrightarrow} T$$

for the inclusion of the p^{th} stratum of the stratification (1.8.2) (a locally closed subset). Then, the $(p^\circ)^{\text{th}}$ stratum of the stratification (1.8.3) is $\mathsf{Shv}(T_p)$, and its gluing functor with respect to a morphism $p^\circ \to q^\circ$ in P^{op} is the composite

$$\mathsf{Shv}(T_p) \xhookrightarrow{(\sigma_p)_*} \mathsf{Shv}(T) \xrightarrow{(\sigma_q)^*} \mathsf{Shv}(T_q) \ .$$

Analogous stratifications exist for constructible sheaves. More precisely, the stratification (1.8.3) restricts to stratifications of the full subcategories

$$\mathsf{Shv}^{\mathsf{P}\text{-}\mathsf{cbl}}(T) \subseteq \mathsf{Shv}^{\mathsf{cbl}}(T) \subseteq \mathsf{Shv}(T)$$

of P-constructible sheaves and of constructible sheaves. More generally, for any functor $\mathsf{Q} \to \mathsf{P}$ among posets and any refinement

$$\begin{tikzcd} & \mathsf{Q} \arrow[d] \\ T \arrow[ur, dashed] \arrow[r, "f"'] & \mathsf{P} \end{tikzcd}$$

of the stratification (1.8.2), the stratification (1.8.3) restricts to a P^{op}-stratification of the full subcategory

$$\mathsf{Shv}^{\mathsf{Q}\text{-}\mathsf{cbl}}(T) \subseteq \mathsf{Shv}^{\mathsf{cbl}}(T)$$

of Q-constructible sheaves.[67]

[67]Alternatively, this stratification may be obtained by taking the pushforward (in the sense of Theorem B) of the Q^{op}-stratification of $\mathsf{Shv}^{\mathsf{Q}\text{-}\mathsf{cbl}}(T)$ along the functor $\mathsf{Q}^{\mathsf{op}} \to \mathsf{P}^{\mathsf{op}}$.

Remark 1.8.1. Assume that P^{op} is down-finite, and fix a conservative functor $\mathsf{P} \xrightarrow{d} \mathbb{Z}$ (e.g. the dimension function of strata). Choose any sheaf $\mathcal{F} \in \mathsf{Shv}(T)$, and fix an exact functor $\mathsf{Shv}(T) \xrightarrow{H} \mathcal{V}$ where $\mathcal{V}$ has a t-structure (e.g. cohomology or cohomology with compact support).

(1) The stratification (1.8.3) determines spectral sequences for the cohomology of $\mathcal{F}$ in terms of its cohomologies over strata. Indeed, by Remark 1.3.13, we obtain spectral sequences

$$E^1_{s,t} = \bigoplus_{r \in d^{-1}(-s)} \pi_{s+t}(H((\sigma_r)_!(\sigma_r)^*(\mathcal{F}))) \Longrightarrow \pi_{s+t}(H(\mathcal{F}))$$

and

$$E^1_{s,t} = \bigoplus_{r \in d^{-1}(s)} \pi_{s+t}(H((\sigma_r)_*(\sigma_r)^!(\mathcal{F}))) \Longrightarrow \pi_{s+t}(H(\mathcal{F})) \ .$$[68]

(2) Let us describe the four filtrations of $\mathsf{id}_{\mathsf{Shv}(T)}$ that arise from applying Remark 1.3.12 to the stratification (1.8.3). For each $p \in \mathsf{P}$, let us denote by

$$U_p := f^{-1}({}^{\geq}p) \xhookrightarrow[\text{open}]{j_p} T \xhookleftarrow[\text{closed}]{i_p} f^{-1}({}^{\leq}p) = \overline{T_p} =: Z_p$$

the corresponding open and closed subsets (note that these are *not* generally complements). Then, for any $p \in \mathsf{P}$ we have

$$\mathsf{fil}^L_p \simeq (j_p)_!(j_p)^* \ , \qquad \mathsf{fil}^p_R \simeq (j_p)_*(j_p)^* \ , \qquad \mathsf{fil}^p_L \simeq (i_p)_*(i_p)^* \ , \qquad \text{and} \qquad \mathsf{fil}^R_p \simeq (i_p)_*(i_p)^! \ .$$

(3) Using part (2), we describe the spectral sequences obtained by applying the spectral sequence (1.10.12) discussed in Remark 1.10.5, which arises from categorified Möbius inversion (Example 1.10.4). Applied to the filtration $\mathsf{fil}^{\bullet}_L$, we obtain a spectral sequence

$$E^1_{s,t} = \bigoplus_{r \in d^{-1}(-s) \cap ({}^{\leq}p)} \pi_{s+t}(M^r_p \odot H((i_r)_*(i_r)^*(\mathcal{F}))) \Longrightarrow \pi_{s+t}(H((\sigma_p)_!(\sigma_p)^*(\mathcal{F}))) \ ;$$

taking H to be compactly-supported cohomology, we recover the spectral sequence of [Pet17, Theorem 1.1], which computes compactly-supported cohomology over the p^{th} stratum in terms of those over closures of strata. Next, applied to the filtration $\mathsf{fil}^L_{\bullet}$, we obtain a spectral sequence

$$E^1_{s,t} = \bigoplus_{r \in d^{-1}(-s) \cap ({}^{\leq}p)} \pi_{s+t}(M^r_p \odot H((j_r)_!(j_r)^*(\mathcal{F}))) \Longrightarrow \pi_{s+t}(H((\sigma_p)_!(\sigma_p)^*(\mathcal{F})))$$

with the same abutment but different E^1 page. Finally, applied to the filtrations $\mathsf{fil}^{\bullet}_R$ and $\mathsf{fil}^R_{\bullet}$, we obtain spectral sequences that are Verdier dual to these two (again see Example 1.10.8).

Remark 1.8.2. Under suitable hypotheses, the P^{op}-stratification of $\mathsf{Shv}^{\mathsf{P}\text{-cbl}}(T)$ admits a completely algebraic description; see Example 1.9.1.

Remark 1.8.3. If we consider sheaves valued in a presentably $\mathcal{O}$-monoidal stable ∞-category, the stratification (1.8.3) becomes an $\mathcal{O}$-monoidal stratification.[69]

Remark 1.8.4. The stratification (1.8.3) of sheaves on a topological space generalizes to a stratification of sheaves on an ∞-topos, using the theory of stratified ∞-topoi developed by Barwick–Glasman–Haine [BGH].[70]

[68] Note that these two spectral sequences are related by Verdier duality (see Example 1.10.8).

[69] In particular, the stratification (1.8.3) for an arbitrary target is recovered from the case of $\mathcal{S}\mathsf{p}$ through Remark 1.5.6.

[70] In the case of a presheaf ∞-topos, this may also be recovered as an instance of the stratification (1.9.1) below.

1.9. **Functors to a poset and naive G-spectra.** Let G be a compact Lie group. The ∞-category of genuine G-spectra admits a variant, the ∞-category

$$\mathcal{S}\mathsf{p}^{\mathsf{n}G} := \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathcal{S}\mathsf{p})$$

of ***naive G-spectra***, i.e. of spectral presheaves on the orbit ∞-category of G.[71] Naive G-spectra provide a natural context for computing (generalized) Bredon co/homology, as well as for understanding genuine G-suspension spectra (see e.g. [AMGRa, §**??**]) via the factorization

$$\begin{tikzcd} \mathcal{S}_*^{\mathsf{g}G} \arrow[rr, "\Sigma_G^\infty"] \arrow[dr, "\Sigma^\infty"'] & & \mathcal{S}\mathsf{p}^{\mathsf{g}G} \\ & \mathcal{S}\mathsf{p}^{\mathsf{n}G} \arrow[ur, dashed, "\Psi"'] & \end{tikzcd}$$

(which results from the universal property of stabilization).

The ∞-category of naive G-spectra admits a stratification closely related to the geometric stratification of the ∞-category of genuine G-spectra of Theorem E. In fact, this arises as a special instance of a more general source of stratifications; we return to naive G-spectra in Example 1.9.3.

Fix a presentable stable ∞-category $\mathcal{V}$, as well as an ∞-category $\mathcal{T}$ equipped with a functor

$$\mathcal{T} \longrightarrow \mathsf{P} \ .$$

Then, we obtain a stratification of the presentable stable ∞-category

$$\mathsf{Fun}(\mathcal{T}, \mathcal{V})$$

over the poset P^{op} according to the formula

$$\begin{array}{ccc} \mathsf{P}^{\mathsf{op}} & \longrightarrow & \mathbf{Cls}_{\mathsf{Fun}(\mathcal{T},\mathcal{V})} \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ p^\circ & \longmapsto & \mathsf{Fun}(\mathcal{T}_{\geq p}, \mathcal{V}) \end{array} \ ,^{72} \tag{1.9.1}$$

where we consider

$$\mathsf{Fun}(\mathcal{T}_{\geq p}, \mathcal{V}) \subseteq \mathsf{Fun}(\mathcal{T}, \mathcal{V})$$

as a closed subcategory via left Kan extension (which is simply extension by zero); its right adjoint is restriction, the right adjoint to which is right Kan extension.

The following features of the stratification (1.9.1) are easily verified.

(1) For each $p^\circ \in \mathsf{P}^{\mathsf{op}}$, the $(p^\circ)^{\text{th}}$ stratum of the stratification (1.9.1) is

$$\mathsf{Fun}(\mathcal{T}_p, \mathcal{V}) \ .$$

(2) For any morphism $q^\circ \to p^\circ$ in P^{op}, the corresponding gluing functor of the stratification (1.9.1) is given as follows. First of all, we define the *q^{th} stratum of the link of $\mathcal{T}_p$ in $\mathcal{T}$* as

[71] The terminology "naive" stems from the fact that, whereas the ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ of genuine G-spectra is obtained from the ∞-category $\mathcal{S}_*^{\mathsf{g}G}$ of pointed genuine G-spaces by inverting all representation spheres under the smash product, the ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{n}G}$ of naive G-spectra is obtained from $\mathcal{S}_*^{\mathsf{g}G}$ by inverting merely the spheres with trivial G action under the smash product (i.e. by stabilizing).

[72] Here and throughout, for any subposet $\mathsf{Q} \subseteq \mathsf{P}$ we write $\mathcal{T}_{\mathsf{Q}} := \mathcal{T} \times_{\mathsf{P}} \mathsf{Q}$ for the fiber product; for any element $p \in \mathsf{P}$ we simply write $\mathcal{T}_p := \mathcal{T}_{\{p\}}$.

the limit in the diagram

$$\begin{tikzcd}
\mathsf{Link}_{\mathcal{T}_p}(\mathcal{T})_q \arrow[rr, "t"] \arrow[dd, "s"'] \arrow[dr, hook] & & \mathcal{T}_q \arrow[d, hook] \\
& \mathsf{Ar}(\mathcal{T}) \arrow[r, "t"'] \arrow[d, "s"] & \mathcal{T} \\
\mathcal{T}_p \arrow[r, hook] & \mathcal{T} &
\end{tikzcd} .$$

Then, the corresponding gluing functor is the composite

$$\Gamma^{q^\circ}_{p^\circ} : \mathsf{Fun}(\mathcal{T}_q, \mathcal{V}) \xrightarrow{t^*} \mathsf{Fun}(\mathsf{Link}_{\mathcal{T}_p}(\mathcal{T})_q, \mathcal{V}) \xrightarrow{s_*} \mathsf{Fun}(\mathcal{T}_p, \mathcal{V})$$

of pullback along t followed by right Kan extension along s.

(3) If the functor $\mathcal{T} \to \mathsf{P}$ is an exponentiable fibration, then the stratification (1.9.1) is strict, i.e. the gluing functors strictly compose. Specifically, exponentiability guarantees that links glue: for instance, given any composite $p \to q \to r$ in P, we have an equivalence

$$\mathsf{Link}_{\mathcal{T}_p}(\mathcal{T})_r \simeq \mathsf{Link}_{\mathcal{T}_p}(\mathcal{T})_q \otimes_{\mathcal{T}_q} \mathsf{Link}_{\mathcal{T}_q}(\mathcal{T})_r$$

(expressing the r^{th} stratum of the link of $\mathcal{T}_p$ in $\mathcal{T}$ as a coend over $\mathcal{T}_q$). In this case, the gluing diagram

$$\mathsf{P}^{\mathsf{op}} \xrightarrow{\mathscr{G}(\mathsf{Fun}(\mathcal{T},\mathcal{V}))} \mathsf{Pr}_{\mathsf{st}}$$

is simply the unstraightening of the cartesian fibration

$$\begin{tikzcd}
\mathsf{Fun}^{\mathsf{rel}}_{/\mathsf{P}}(\mathcal{T}, \underline{\mathcal{V}}) \arrow[d] \\
\mathsf{P}
\end{tikzcd} .$$

(4) If $\mathcal{V}$ is presentably $\mathcal{O}$-monoidal, then $\mathsf{Fun}(\mathcal{T}, \mathcal{V})$ is presentably $\mathcal{O}$-monoidal via the pointwise $\mathcal{O}$-monoidal structure, and with respect to this the stratification (1.9.1) is an $\mathcal{O}$-monoidal stratification.

Example 1.9.1. Let us say that a stratified topological space $T \to \mathsf{P}$ is *tamely conical* if the topological space T is paracompact and locally of singular shape and moreover its stratification is conical.[73] In this case, if we take

$$\mathcal{T} := \mathsf{Exit}(T) \longrightarrow \mathsf{P}$$

to be the ***exit-path ∞-category*** of T equipped with its canonical functor to P, then the P^{op}-stratification (1.9.1) recovers that of the ∞-category $\mathsf{Shv}^{\mathsf{P}\text{-}\mathsf{cbl}}(T)$ of P-constructible sheaves on T obtained in §1.8. In this case, for each $p^\circ \in \mathsf{P}^{\mathsf{op}}$, the $(p^\circ)^{\text{th}}$ stratum is the presentable stable ∞-category $\mathsf{Loc}(T_p)$ of local systems on the p^{th} stratum (according to (1)), and the gluing functors are governed by spaces of exiting paths (as described in (2)).

Remark 1.9.2. A converse to Example 1.9.1 is provided by [Hai]: whenever the functor $\mathcal{T} \to \mathsf{P}$ is conservative (i.e. whenever its fibers are ∞-groupoids), there exists a P-stratified topological space $T \to \mathsf{P}$ and an equivalence $\mathcal{T} \simeq \mathsf{Exit}(T)$ in $\mathsf{Cat}_{/\mathsf{P}}$.[74]

Example 1.9.3 (a stratification of naive G-spectra)**.** Taking

$$(\mathcal{T} \longrightarrow \mathsf{P}) := (\mathscr{O}_G^{\mathsf{op}} \longrightarrow \mathsf{P}_G^{\mathsf{op}}) \qquad \text{and} \qquad \mathcal{V} := \mathcal{S}\mathsf{p} ,$$

[73] These are the conditions under which [Lur, Theorem A.9.3] applies.

[74] For instance, this applies to the functor $\mathscr{O}_G^{\mathsf{op}} \to \mathsf{P}_G^{\mathsf{op}}$ considered in Example 1.9.3.

the stratification (1.9.1) specializes to a stratification

$$\begin{array}{ccc} \mathsf{P}_G & \longrightarrow & \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{n}G}} \\ \cup & & \cup \\ H & \longmapsto & \mathsf{Fun}((\mathscr{O}_G^{\mathsf{op}})_{\geq H}, \mathcal{S}\mathsf{p}) \end{array} \tag{1.9.2}$$

of the presentable stable ∞-category of naive G-spectra. The above features of the stratification (1.9.1) bear upon the stratification (1.9.2) as follows.

(1) For each $H \in \mathsf{P}_G$, the H^{th} stratum of the stratification (1.9.2) is

$$\mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} .$$

(2) For any nonidentity morphism $H < K$ in P_G, the corresponding gluing functor of the stratification (1.9.2) is given by pullback followed by right Kan extension along the span

$$\begin{tikzcd} ((G/K)^H)_{\mathsf{h}(\mathsf{W}(K)\times\mathsf{W}(H))} \arrow[d, phantom, "\wr\|"] & \\ \hom_{\mathscr{O}_G}(G/H, G/K)_{\mathsf{h}(\mathsf{W}(K)\times\mathsf{W}(H))} \arrow[d, phantom, "\wr\|"] & \\ \hom_{\mathscr{O}_G^{\mathsf{op}}}((G/K)^\circ, (G/H)^\circ)_{\mathsf{h}(\mathsf{W}(H)\times\mathsf{W}(K))} \arrow[r] \arrow[d] & \mathsf{BW}(H) \\ \mathsf{BW}(K) & \end{tikzcd} .^{75} \tag{1.9.3}$$

In the case that G is abelian, the span (1.9.3) reduces to the span

$$\begin{tikzcd} \mathsf{B}G/H \arrow[r, "\sim"] \arrow[d] & \mathsf{B}G/H \\ \mathsf{B}G/K & \end{tikzcd} ,$$

so that the gluing functor is given by the homotopy (K/H)-fixedpoints functor

$$\mathcal{S}\mathsf{p}^{\mathsf{h}(G/H)} \simeq \mathsf{Fun}(\mathsf{B}(G/H), \mathcal{S}\mathsf{p}) \xrightarrow{(-)^{\mathsf{h}(K/H)}} \mathsf{Fun}(\mathsf{B}(G/K), \mathcal{S}\mathsf{p}) \simeq \mathcal{S}\mathsf{p}^{\mathsf{h}(G/K)} .$$

(3) In the case that G is abelian, the functor

$$\mathscr{O}_G^{\mathsf{op}} \longrightarrow \mathsf{P}_G^{\mathsf{op}}$$

is a right fibration (and in particular an exponentiable fibration). Hence, the stratification (1.9.2) is strict (corresponding to the fact that homotopy fixedpoints strictly compose).

(4) As $\mathcal{S}\mathsf{p}$ is presentably symmetric monoidal, $\mathcal{S}\mathsf{p}^{\mathsf{n}G}$ is presentably symmetric monoidal as well. With respect to this structure, the stratification (1.9.2) is a symmetric monoidal stratification.

Remark 1.9.4. It is not hard to see that the functor

$$\mathcal{S}\mathsf{p}^{\mathsf{n}G} \xrightarrow{\Psi} \mathcal{S}\mathsf{p}^{\mathsf{g}G}$$

[75] At the level of path components, we have an identification

$$\pi_0\left(((G/K)^H)_{\mathsf{h}(\mathsf{W}(K)\times\mathsf{W}(H))}\right) \cong \mathsf{W}(K) \setminus (G/K)^H / \mathsf{W}(H)$$

with the set of double cosets.

defines a morphism in $\mathbf{Strat}_{\mathsf{P}_G}$ (see Definition 6.2.1), where the source is equipped with the stratification (1.9.2) and the target is equipped with the geometric stratification of Theorem E.[76] In fact, considering it as a morphism in $\mathsf{CAlg}(\mathsf{Pr}^L_{\mathsf{st}})$, the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ may be seen as arising from the stratification (1.9.2) of $\mathcal{S}\mathsf{p}^{\mathsf{n}G}$ via Remark 1.5.5.

1.10. **Reflection, Möbius inversion, and Verdier duality.** In §1.8, we established a stratification of ((P-)constructible) sheaves over a P-stratified topological space. Applying Theorem A(2) (in the case that P is down-finite), one obtains a reconstruction theorem for such sheaves that involves $*$-push/pull functors (e.g. the composite $p_L i_R = i^* j_*$ in the recollement (1.8.1)). On the other hand, particularly in the context of constructible sheaves, it is desirable to instead reconstruct sheaves using !-push/pull functors (e.g. the composite $p_R i_L = i^! j_!$ in the recollement (1.8.1)).

We establish a means of passing between these two dual reconstruction patterns, as we describe presently.[77] We refer to this theory as ***reflection***, since in the case that $\mathsf{P} = [1]$ it recovers the theory of reflection functors (see Remark 1.10.3). We use this to give a categorification of the Möbius inversion formula in Example 1.10.4, and we explain a close connection with Verdier duality in Example 1.10.8.

Fix a stratification $\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathcal{X}$. Let us recall its *gluing diagram* from §1.3: this is a left-lax functor

$$\mathsf{P} \xrightarrow[\mathsf{l.lax}]{\mathscr{G}(\mathcal{X})} \mathsf{Pr}_{\mathsf{st}}$$

that carries each morphism $p \to q$ in P to the *gluing functor*

$$\Gamma^p_q : \mathcal{X}_p \overset{\rho^p}{\hookrightarrow} \mathcal{X} \xrightarrow{\Phi_q} \mathcal{X}_q \ ,$$

which is built from the composite *geometric localization* adjunctions

$$\begin{tikzcd} \Phi_p : \mathcal{X} \arrow[r, shift left, "y"] \arrow[r, phantom, "\perp"] & \mathcal{Z}_p \arrow[l, hook, shift left, "i_R"] \arrow[r, shift left, "p_L"] \arrow[r, phantom, "\perp"] & \mathcal{X}_p : \rho^p \arrow[l, hook, shift left, "\nu"] \end{tikzcd}$$

for all $p \in \mathsf{P}$. By contrast, if we instead begin with the composite ***reflected geometric localization*** adjunctions

$$\begin{tikzcd} \lambda^p : \mathcal{X}_p \arrow[r, hook, shift left, "\nu"] \arrow[r, phantom, "\perp"] & \mathcal{Z}_p \arrow[l, shift left, "p_R"] \arrow[r, hook, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{X} : \Psi_p \arrow[l, shift left, "y"] \end{tikzcd}$$

for all $p \in \mathsf{P}$ (first introduced in Remark 1.3.12), we obtain for each morphism $p \to q$ in P the ***reflected gluing functor***

$$\breve{\Gamma}^p_q : \mathcal{X}_p \overset{\lambda^p}{\hookrightarrow} \mathcal{X} \xrightarrow{\Psi_q} \mathcal{X}_q \ ,$$

[76]This may be verified as follows (see Definition 5.1.8 for the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$). It is clear that the i_L inclusions commute. It remains to show that the y projections also commute. For this, let us denote the stratifications by

$$\mathsf{P}_G \xrightarrow{\mathcal{Z}^{\mathsf{n}}_\bullet} \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{n}G}} \qquad \text{and} \qquad \mathsf{P}_G \xrightarrow{\mathcal{Z}^{\mathsf{g}}_\bullet} \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}} \ .$$

Then, we observe that for any $H \in \mathsf{P}_G$ there are conservative factorizations

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{n}G} \arrow[rr, "\mathsf{Res}^G_H"] \arrow[dr, "y"'] & & \mathcal{S}\mathsf{p}^{\mathsf{n}H} \\ & \mathcal{Z}^{\mathsf{n}}_H \arrow[ur, dashed] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[rr, "\mathsf{Res}^G_H"] \arrow[dr, "y"'] & & \mathcal{S}\mathsf{p}^{\mathsf{g}H} \\ & \mathcal{Z}^{\mathsf{g}}_H \arrow[ur, dashed] & \end{tikzcd} .$$

Hence, the fact that the y projections commute follows from the commutativity of the square

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{n}G} \arrow[r, "\Psi"] \arrow[d, "\mathsf{Res}^G_H"'] & \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[d, "\mathsf{Res}^G_H"] \\ \mathcal{S}\mathsf{p}^{\mathsf{n}H} \arrow[r, "\Psi"'] & \mathcal{S}\mathsf{p}^{\mathsf{g}H} \end{tikzcd} .$$

[77]This applies primarily in the case that P is down-finite, but see also Remark 7.4.27.

and these assemble into the ***reflected gluing diagram*** of the stratification: a *right*-lax functor

$$\mathsf{P} \xrightarrow[\mathsf{r.lax}]{\breve{\mathscr{G}}(\mathcal{X})} \mathsf{Pr_{st}} \quad .$$

Theorem F (Corollary 7.4.25)**.** *Let* P *be a down-finite poset.*

(1) **metacosm:** *The reflected gluing diagram functor is an equivalence, as indicated in the canonical commutative diagram*

$$\begin{tikzcd}
 & \prod\limits_{p \in \mathsf{P}} \mathsf{Pr_{st}} & \\
\mathsf{LMod}^{L}_{\mathsf{r.lax.P}}(\mathsf{Pr_{st}}) \arrow[ur, "(\mathsf{ev}_p)_{p\in\mathsf{P}}"] \arrow[r, shift left, "\mathsf{lim}^{\mathsf{l.lax}}_{\mathsf{r.lax.\bullet}}"] \arrow[r, phantom, "\sim"] \arrow[dr, "\mathsf{lim}^{\mathsf{l.lax}}_{\mathsf{r.lax.P}}"'] & \mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}} \arrow[u, "((-)_p)_{p\in\mathsf{P}}"] \arrow[l, shift left, "\breve{\mathscr{G}}"] \arrow[r, shift left, "\mathscr{G}"] \arrow[r, phantom, "\sim"] \arrow[d, "\mathsf{fgt}"] & \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \arrow[ul, "(\mathsf{ev}_p)_{p\in\mathsf{P}}"'] \arrow[l, shift left, "\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}"] \arrow[dl, "\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}"] \\
 & \mathsf{Pr_{st}} &
\end{tikzcd} \quad .^{78} \tag{1.10.1}$$

(2) **macrocosm:** *For each* P*-stratified noncommutative stack* $\mathcal{X} \in \mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}}$*, the equivalence on the left in diagram* (1.10.1) *determines an equivalence*

$$\mathsf{Fun}(\mathsf{sd}(\mathsf{P})^{\mathsf{op}}, \mathcal{X}) \supseteq \mathsf{lim}^{\mathsf{l.lax}}_{\mathsf{r.lax.P}}(\breve{\mathscr{G}}(\mathcal{X})) =: \widetilde{\mathsf{Glue}}(\mathcal{X}) \underset{\breve{g}}{\overset{\mathsf{colim}_{\mathsf{sd(P)^{op}}}}{\underset{\longleftarrow}{\overset{\sim}{\longrightarrow}}}} \mathcal{X} \ . \tag{1.10.2}$$

(3) **microcosm:** *For each quasicoherent sheaf* $\mathcal{F} \in \mathcal{X} \in \mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}}$ *on a* P*-stratified noncommutative stack, the equivalence* (1.10.2) *determines an equivalence*

$$\mathsf{colim}_{\mathsf{sd(P)^{op}}}(\breve{g}(\mathcal{F})) =: \widetilde{\mathsf{glue}}(\mathcal{F}) \xrightarrow{\sim} \mathcal{F} \tag{1.10.3}$$

in $\mathcal{X}$*.*

(4) **nanocosm:** *For each quasicoherent sheaf* $\mathcal{E} \in \mathcal{X} \in \mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}}$ *on a* P*-stratified noncommutative stack, applying* $\underline{\mathsf{hom}}_{\mathcal{X}}(-, \mathcal{E})$ *to the equivalence* (1.10.3) *determines an equivalence*

$$\lim_{([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})} \left(\underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\breve{\Gamma}_{\varphi} \Psi_{\varphi(0)} \mathcal{F}, \Psi_{\varphi(0)} \mathcal{E}) \right) \xleftarrow{\sim} \underline{\mathsf{hom}}_{\mathcal{X}}(\mathcal{F}, \mathcal{E}) \ .^{79}$$

In particular, under the assumption that P is down-finite, Theorems F(2) and A(2) provide dual macrocosm equivalences

$$\mathsf{lim}^{\mathsf{l.lax}}_{\mathsf{r.lax.P}}(\breve{\mathscr{G}}(\mathcal{X})) =: \widetilde{\mathsf{Glue}}(\mathcal{X}) \underset{\sim}{\xleftarrow{\breve{g}}} \mathcal{X} \underset{\sim}{\xrightarrow{g}} \mathsf{Glue}(\mathcal{X}) := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X}))$$

---

[78] Of course, $\mathsf{LMod}^{L}_{\mathsf{r.lax.P}}(\mathsf{Pr_{st}})$ denotes a certain ∞-category whose objects are right-lax functors from P to $\mathsf{Pr_{st}}$, and the notation $\mathsf{lim}^{\mathsf{l.lax}}_{\mathsf{r.lax.\bullet}}$ denotes a certain "parametrized left-lax limit" functor. Here we must restrict to the subcategory $\mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}} \subseteq \mathbf{Strat}_{\mathsf{P}}$ of strict morphisms (as introduced in Remark 1.3.15(2)): there is an implicit laxness in our definition of $\mathbf{Strat}_{\mathsf{P}}$ that is compatible with the gluing diagram functor $\mathscr{G}$ but not with the reflected gluing diagram functor $\breve{\mathscr{G}}$. (Dually, one could instead allow for laxness that is compatible with $\breve{\mathscr{G}}$ but not with $\mathscr{G}$.)

[79] Given an element $([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$, in parallel with the notation $\Gamma_{\varphi} := \Gamma^{\varphi(n-1)}_{\varphi(n)} \cdots \Gamma^{\varphi(0)}_{\varphi(1)}$ we write $\breve{\Gamma}_{\varphi} := \breve{\Gamma}^{\varphi(n-1)}_{\varphi(n)} \cdots \breve{\Gamma}^{\varphi(0)}_{\varphi(1)}$.

in $\mathsf{Pr}_{\mathsf{st}}$ for each P-stratified noncommutative stack $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}^{\mathsf{strict}}$. On the other hand, omitting any reference to stratifications, Theorem F(1) provides a canonical commutative diagram

$$\begin{tikzcd}
& \prod\limits_{p \in \mathsf{P}} \mathsf{Pr}_{\mathsf{st}} & \\
\mathsf{LMod}^{L}_{\mathsf{r.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[ur, "(\mathsf{ev}_p)_{p \in \mathsf{P}}"] \arrow[dr, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.P}}"'] & & \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[ul, "(\mathsf{ev}_p)_{p \in \mathsf{P}}"'] \arrow[ll, "\breve{(-)}"', "\sim"] \arrow[dl, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}"] \\
& \mathsf{Pr}_{\mathsf{st}} &
\end{tikzcd} \tag{1.10.4}$$

for any down-finite poset P.[80] We refer to the equivalence

$$\mathsf{LMod}^{L}_{\mathsf{r.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \underset{\sim}{\xleftarrow{\breve{(-)}}} \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})$$

of diagram (1.10.4) as ***reflection***. In fact, we prove this equivalence more generally for posets whose intervals are finite (see Corollary 7.4.26). Moreover, we give a direct formula for the reflected gluing functors in terms of gluing functors and reversely, as total co/fibers (Definition 7.4.3): for any nonidentity morphism $p < q$ in P we have canonical equivalences

$$\breve{\Gamma}^{p}_{q} \simeq \mathsf{tfib}_{\varphi \in \mathsf{sd}(\mathsf{P})^{|p}_{|q}} \Sigma^{-1} \Gamma_{\varphi} \qquad \text{and} \qquad \Gamma^{p}_{q} \simeq \mathsf{tcofib}_{\varphi^{\circ} \in (\mathsf{sd}(\mathsf{P})^{|p}_{|q})^{\mathsf{op}}} \Sigma \breve{\Gamma}_{\varphi} \tag{1.10.5}$$

in $\mathsf{Fun}(\mathcal{X}_p, \mathcal{X}_q)$ (see Proposition 7.4.5 (and Notation A.6.10)). Note that if $\mathsf{P}_{p//q} \cong [n]$ for some $n \geq 1$, then $\mathsf{sd}(\mathsf{P})^{|p}_{|q}$ is an $(n-1)$-cube; in particular, if $p < q$ admits no nontrivial factorizations then the equivalences (1.10.5) reduce to the equivalent equivalences

$$\breve{\Gamma}^{p}_{q} \simeq \Sigma^{-1} \Gamma^{p}_{q} \qquad \text{and} \qquad \Gamma^{p}_{q} \simeq \Sigma \breve{\Gamma}^{p}_{q} . \tag{1.10.6}$$

Remark 1.10.1. Observe that a closed subcategory

$$\mathcal{Z} \overset{i_L}{\hookrightarrow} \mathcal{X}$$

determines a closed subcategory

$$\mathcal{Z}^{\mathsf{op}} \overset{i_R^{\mathsf{op}}}{\hookrightarrow} \mathcal{X}^{\mathsf{op}} ,$$

which we refer to as its ***reflected closed subcategory***.[81] In concrete terms, Theorem F(1) may be interpreted as saying that given a stratification of $\mathcal{X}$ over a down-finite poset P, passage to reflected closed subcategories determines a stratification of $\mathcal{X}^{\mathsf{op}}$ over P, which we refer to as its ***reflected stratification***: writing $\mathcal{X}^{\mathsf{refl}}$ for $\mathcal{X}^{\mathsf{op}}$ equipped with its reflected stratification, we have an equivalence

$$\mathscr{G}(\mathcal{X}^{\mathsf{refl}}) \simeq \breve{\mathscr{G}}(\mathcal{X})^{\mathsf{op}} .$$ [82]

[80] More systematic notation would allow for the horizontal arrow in diagram (1.10.4) to point in both directions. We have written it in this way in order to maintain consistency, so that for any $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}^{\mathsf{strict}}$ we have a canonical equivalence $\breve{\mathscr{G}(\mathcal{X})} \simeq \breve{\mathscr{G}}(\mathcal{X})$.

[81] Here and throughout this subsection, whenever we mention opposites of presentable stable ∞-categories we are implicitly referring to the theory of *stable* stratifications (as introduced in Remark 1.3.15(1)); we generally omit this distinction from the present discussion in order not to clutter our exposition.

[82] Indeed, our proof of Theorem F (which we establish as Corollary 7.4.25) is based on the analogous result for stable stratifications (Theorem 7.4.11), the main ingredient in the proof of which is the reflected stable stratification (Proposition 7.4.16).

Example 1.10.2. We unpack parts (1) and (2) of Theorem F in the case that $\mathsf{P} = [1]$. First of all, we have identifications

$$\begin{tikzcd} \mathsf{LMod}_{\mathsf{r.lax.}[1]} \arrow[rr, "\sim"] \arrow[dr, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.}[1]}"'] & & \mathsf{coCart}_{[1]} \arrow[dl, "\Gamma"] \\ & \mathsf{Cat} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{LMod}_{\mathsf{l.lax.}[1]} \arrow[rr, "\sim"] \arrow[dr, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[1]}"'] & & \mathsf{Cart}_{[1]^{\mathsf{op}}} \arrow[dl, "\Gamma"] \\ & \mathsf{Cat} & \end{tikzcd} .$$

Let us denote by

$$\begin{tikzcd} \mathsf{LMod}^L_{\mathsf{r.lax.}[1]}(\mathsf{Pr_{st}}) \arrow[r, dashed, "\sim"] \arrow[d, hook] & \mathsf{coCart}^L_{[1]}(\mathsf{Pr_{st}}) \arrow[d, hook] \\ \mathsf{LMod}_{\mathsf{r.lax.}[1]} \arrow[r, "\sim"'] & \mathsf{coCart}_{[1]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{LMod}^L_{\mathsf{l.lax.}[1]}(\mathsf{Pr_{st}}) \arrow[r, dashed, "\sim"] \arrow[d, hook] & \mathsf{Cart}^L_{[1]^{\mathsf{op}}}(\mathsf{Pr_{st}}) \arrow[d, hook] \\ \mathsf{LMod}_{\mathsf{l.lax.}[1]} \arrow[r, "\sim"'] & \mathsf{Cart}_{[1]^{\mathsf{op}}} \end{tikzcd}$$

the indicated corresponding subcategories. Now, for any recollement (1.1.1) we have a canonical equivalence $p_R i_L \simeq \Sigma^{-1} p_L i_R$ (a special case of the equivalences (1.10.6)). It follows that the commutative diagram (1.10.4) specializes to a commutative diagram

$$\begin{tikzcd} & \prod_{p\in[1]} \mathsf{Pr_{st}} & \\ \mathsf{coCart}^L_{[1]}(\mathsf{Pr_{st}}) \arrow[ur, "(\mathsf{ev}_p)_{p\in[1]}"] \arrow[dr, "\Gamma"'] & & \mathsf{Cart}^L_{[1]^{\mathsf{op}}}(\mathsf{Pr_{st}}) \arrow[ul, "(\mathsf{ev}_p)_{p\in[1]^{\mathsf{op}}}"'] \arrow[ll, "\widetilde{(-)}", "\sim"'] \arrow[dl, "\Gamma"] \\ & \mathsf{Pr_{st}} & \end{tikzcd} \tag{1.10.7}$$

in which the equivalence $\widetilde{(-)}$ carries the cartesian unstraightening of a functor $\mathcal{X}_0 \xrightarrow{F} \mathcal{X}_1$ to the cocartesian unstraightening of the functor $\mathcal{X}_0 \xrightarrow{\Sigma^{-1}F} \mathcal{X}_1$. Thereafter, the commutativity of the lower triangle in diagram (1.10.7) records the equivalence

$$\begin{array}{ccc} \lim^{\mathsf{l.lax}}_{\mathsf{r.lax.}[1]}(\mathcal{X}_0 \xrightarrow{\Sigma^{-1}F} \mathcal{X}_1) & \xleftarrow{\sim} & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[1]}(\mathcal{X}_0 \xrightarrow{F} \mathcal{X}_1) \\ \Psi & & \Psi \\ (Z \longmapsto \Sigma^{-1}F(Z) \longrightarrow \mathsf{fib}(\alpha)) & \longleftarrow & (Z \longmapsto F(Z) \xleftarrow{\alpha} U) \end{array} \tag{1.10.8}$$

in $\mathsf{Pr}^L_{\mathsf{st}}$.

Remark 1.10.3. Example 1.10.2 is closely related to the theory of reflection functors [BGP73]. Indeed, we recover [DJW21, Theorem 2.3] as follows. Fix a finite poset Q equipped with a conservative functor $\mathsf{Q} \to [1]$. Additionally fix a functor $\mathsf{Q} \to \mathsf{Cat}$, and let us respectively denote by

$$\mathcal{E}^+ \longrightarrow \mathsf{Q} \qquad \text{and} \qquad \mathcal{E}^- \longrightarrow \mathsf{Q}^{\mathsf{op}}$$

its cocartesian and cartesian unstraightenings. These data determine composite functors

$$\mathcal{E}^+ \longrightarrow \mathsf{Q} \longrightarrow [1] \qquad \text{and} \qquad (\mathcal{E}^-)^{\mathsf{op}} \longrightarrow \mathsf{Q} \longrightarrow [1] .$$

Fix a presentable stable ∞-category $\mathcal{V}$. On the one hand, the functor $\mathcal{E}^+ \to [1]$ determines a stratification of

$$\mathsf{Fun}(\mathcal{E}^+, \mathcal{V})$$

over $[1]^{\mathsf{op}}$ as in §1.9. On the other hand, the functor $(\mathcal{E}^-)^{\mathsf{op}} \to [1]$ similarly determines a stratification of

$$\mathsf{Fun}((\mathcal{E}^-)^{\mathsf{op}}, \mathcal{V}^{\mathsf{op}}) \simeq \mathsf{Fun}(\mathcal{E}^-, \mathcal{V})^{\mathsf{op}}$$

over $[1]^{\mathsf{op}}$, which by Theorem F(1) (as interpreted via Remark 1.10.1) determines a stratification of

$$\mathsf{Fun}(\mathcal{E}^-, \mathcal{V})$$

over $[1]^{\mathsf{op}}$. Unwinding the definitions, we find that the gluing diagram $\mathscr{G}(\mathsf{Fun}(\mathcal{E}^+, \mathcal{V}))$ of the former as well as the reflected gluing diagram $\breve{\mathscr{G}}(\mathsf{Fun}(\mathcal{E}^-, \mathcal{V}))$ of the latter both record the composite functor

$$F : \prod_{q \in \mathsf{Q}_1} \mathsf{Fun}(\mathcal{E}_q, \mathcal{V}) \longrightarrow \prod_{\alpha \in \Gamma(\mathsf{Q}\downarrow[1])} \mathsf{Fun}(\mathcal{E}_{\alpha(0)}, \mathcal{V}) \xrightarrow{\mathcal{F}_\bullet \longmapsto \left(\prod_{\{\alpha : \alpha(0) = p\}} \mathcal{F}_\alpha\right)_{p \in \mathsf{Q}_0}} \prod_{p \in \mathsf{Q}_0} \mathsf{Fun}(\mathcal{E}_p, \mathcal{V}) \ .$$

Hence, applying Theorem F(1) (and the equivalence (1.10.8) of Example 1.10.2 combined with the equivalence $F \simeq \Sigma^{-1} F$ in $\mathsf{Fun}([1], \mathsf{Pr}_{\mathsf{st}})$), we obtain the composite equivalence

$$\mathsf{Fun}(\mathcal{E}^+, \mathcal{V}) \simeq \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.[1]^{\mathsf{op}}}(\mathscr{G}(\mathsf{Fun}(\mathcal{E}^+, \mathcal{V}))) \simeq \lim^{\mathsf{l.lax}}_{\mathsf{r.lax}.[1]^{\mathsf{op}}}(\breve{\mathscr{G}}(\mathsf{Fun}(\mathcal{E}^-, \mathcal{V}))) \simeq \mathsf{Fun}(\mathcal{E}^-, \mathcal{V}) \ .$$

Example 1.10.4 (categorified Möbius inversion)**.** Given a down-finite poset P and a presentable stable ∞-category $\mathcal{V}$, the presentable stable ∞-category

$$\mathcal{X} := \mathsf{Fun}(\mathsf{P}, \mathcal{V})$$

of P-filtered objects in $\mathcal{V}$ admits a stratification

$$\begin{array}{ccc} \mathsf{P} & \longrightarrow & \mathbf{Cls}_{\mathcal{X}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ p & \longmapsto & \mathsf{Fun}((^{\leq}p), \mathcal{V}) \end{array} \ , \tag{1.10.9}$$

where we consider

$$\mathsf{Fun}((^{\leq}p), \mathcal{V}) \subseteq \mathsf{Fun}(\mathsf{P}, \mathcal{V}) =: \mathcal{X}$$

as a closed subcategory via left Kan extension.[83] Unwinding the definitions, for each $p \in \mathsf{P}$ we obtain an identification

$$\begin{tikzcd} \mathcal{X} \arrow[r, "y"] & \mathcal{Z}_p \arrow[l, bend right=50, "i_L"'] \arrow[l, bend left=50, "i_R"] \arrow[r, bend left=50, "p_L"] \arrow[r, bend right=50, "p_R"'] & \mathcal{X}_p \arrow[l, "\nu"'] \end{tikzcd} \qquad \simeq \qquad \begin{tikzcd} \mathsf{Fun}(\mathsf{P}, \mathcal{V}) \arrow[r, "\text{res.}"] & \mathsf{Fun}((^{\leq}p), \mathcal{V}) \arrow[l, bend right=40, "\text{LKE}"'] \arrow[l, bend left=40, "\text{ext. by } 0"] \arrow[r, bend left=40, "\mathsf{tcofib}_{(\leq p)}"] \arrow[r, bend right=40, "\mathsf{ev}_p"'] & \mathcal{V} \arrow[l, "\delta_p"'] \end{tikzcd} \ ,$$

where δ_p denotes the "Dirac delta" functor at $p \in (^{\leq}p)$. In particular, the reflected gluing diagram is the constant locally cartesian fibration

$$\breve{\mathscr{G}}(\mathcal{X}) \simeq \mathcal{V} \times \mathsf{P}^{\mathsf{op}} \xrightarrow{\mathsf{pr}} \mathsf{P}^{\mathsf{op}} \ ,$$

and the p^{th} geometric localization and p^{th} reflected geometric localization functors are respectively the p^{th} associated graded and p^{th} filtered components:

$$\Phi_p(V_\bullet) := p_L y(V_\bullet) \simeq \mathsf{gr}_p(V_\bullet) := \mathsf{tcofib}_{(\leq p)}(V_\bullet) \qquad \text{and} \qquad \Psi_p(V_\bullet) := p_R y(V_\bullet) \simeq \mathsf{fil}_p(V_\bullet) := V_p \ .$$[84]

In this situation, Proposition 7.4.5 yields a ***categorified Möbius inversion formula*** that expresses the p^{th} associated graded component $\mathsf{gr}_p(V_\bullet) \simeq \Phi_p(V_\bullet)$ in terms of the filtered components $\mathsf{fil}_q(V_\bullet) \simeq \Psi_q(V_\bullet) \simeq V_q$ for various $q \in (^{\leq}p)$, as we now explain. First of all, the stratification (1.10.9) endows each object $V_\bullet \in \mathcal{X}$ with a (descending) P^{op}-filtration

$$\mathsf{fil}^\bullet_R(V_\bullet) \in \mathsf{Fun}(\mathsf{P}^{\mathsf{op}}, \mathcal{X})$$

[83]Beware that this is not an instance of the stratification (1.9.1) considered in §1.9.

[84]In what follows, we use either or both of these possible notations, depending on our desired emphasis.

(recall Remark 1.3.12), and for each $r^\circ \in \mathsf{P}^{\mathsf{op}}$ its $(r^\circ)^{\text{th}}$ associated graded component is

$$\mathsf{gr}^r_R(V_\bullet) \simeq \rho^r(\Psi_r(V_\bullet)) \simeq \rho^r(V_r) \simeq \delta_r(V_r) \in \mathcal{X} .$$

Applying the functor $\mathcal{X} \xrightarrow{\Phi_p} \mathcal{X}_p \simeq \mathcal{V}$, we obtain a P^{op}-filtration

$$\mathsf{fil}^\bullet_R(\Phi_p(V_\bullet)) := \Phi_p(\mathsf{fil}^\bullet_R(V_\bullet)) \in \mathsf{Fun}(\mathsf{P}^{\mathsf{op}}, \mathcal{V})$$

of $\Phi_p(V_\bullet) \simeq \mathsf{gr}_p(V_\bullet) \in \mathcal{V}$, whose $(r^\circ)^{\text{th}}$ associated graded component is the object

$$\Phi_p(\mathsf{gr}^r_R(V_\bullet)) \simeq \Phi_p(\rho^r \Psi_r(V_\bullet)) \simeq \Gamma^r_p(V_r) \in \mathcal{V}$$

(which is zero whenever $r \not\leq p$, see Remark 2.5.4). Applying Proposition 7.4.5(2), for any $r \leq p$ we obtain an identification

$$\Gamma^r_p(V_r) \simeq M^r_p \odot V_r \tag{1.10.10}$$

in $\mathcal{V}$, where $M^r_p \in \mathcal{S}^{\mathsf{fin}}_*$ denotes the finite pointed space

$$M^r_p := \begin{cases} S^0 \ , & r = p \\ \Sigma^2|\mathsf{P}_{r//p} \backslash \{r,p\}| \ , & r < p \end{cases} .^{85,86}$$

Note that the reduced Euler characteristic

$$\overline{\chi}(M^r_p) := \chi(\Sigma^\infty M^r_p) \in \mathsf{K}_0(\mathcal{S}\mathsf{p}^{\mathsf{fin}}) \cong \mathbb{Z}$$

is the value $\mu_{\mathsf{P}}(r,p) = \mu_{\mathsf{P}^{\mathsf{op}}}(p^\circ, r^\circ) \in \mathbb{Z}$ of the Möbius function.

Now, assume that $\mathcal{V}$ is compactly generated. In this case, we have two inclusions

$$\mathsf{K}_0(\mathcal{X}^\omega) \underset{i}{\overset{\check{i}}{\rightrightarrows}} \mathsf{hom}_{\mathsf{Set}}(\mathsf{P}^\delta, \mathsf{K}_0(\mathcal{V}^\omega))$$

of abelian groups as the subgroup of finitely-supported functions, given by the two gluing diagrams:

$$\check{i}([V_\bullet])(p) := [\Psi_p(V_\bullet)] = [\mathsf{fil}_p(V_\bullet)] = [V_p] \qquad \text{and} \qquad i([V_\bullet])(p) := [\Phi_p(V_\bullet)] = [\mathsf{gr}_p(V_\bullet)] \ .$$

Now, using the equivalences (1.10.10), we obtain the Möbius inversion formula for P (valued in the abelian group $\mathsf{K}_0(\mathcal{V}^\omega)$):

$$\check{i}([V_\bullet])(p) = \sum_{r \in ({}^{\leq}p)} i([V_\bullet])(r) \qquad \text{and} \qquad i([V_\bullet])(p) = \sum_{r \in ({}^{\leq}p)} \overline{\chi}(M^r_p) \cdot \check{i}([V_\bullet])(r) = \sum_{r \in ({}^{\leq}p)} \mu_{\mathsf{P}}(r,p) \cdot \check{i}([V_\bullet])(r) \ .$$

Remark 1.10.5. In the context of Example 1.10.4, let us fix a conservative functor $\mathsf{P} \xrightarrow{d} \mathbb{Z}$ and an element $p \in \mathsf{P}$, and let us assume that $\mathcal{V}$ is equipped with a t-structure. Then, we obtain two spectral sequences by applying Remark 1.3.13 to the restricted stratification over the poset $({}^{\leq}p)$: taking $H = \Psi_p = \mathsf{fil}_p = (-)_p$ we obtain a spectral sequence

$$E^1_{s,t} = \bigoplus_{r \in d^{-1}(s) \cap ({}^{\leq}p)} \pi_{s+t}(\mathsf{gr}_r(V_\bullet)) \Longrightarrow \pi_{s+t}(V_p) \ , \tag{1.10.11}$$

while taking $H = \Phi_p = \mathsf{gr}_p$ we obtain a spectral sequence

$$E^1_{s,t} = \bigoplus_{r \in d^{-1}(-s) \cap ({}^{\leq}p)} \pi_{s+t}(M^r_p \odot V_r) \Longrightarrow \pi_{s+t}(\mathsf{gr}_p(V_\bullet)) \ .^{87} \tag{1.10.12}$$

[85] In the case that $\mathsf{P}_{r//p} = \{r < p\}$, we have $M^r_p := \Sigma^2(\varnothing) \simeq S^1$.

[86] One could also adopt the convention that $M^r_p := \mathsf{pt}$ in the case that $r \not\leq p$.

[87] The other two spectral sequences that can be constructed in this way (applying (1.3.15) to $H = \Phi_p$ or (1.3.16) to $H = \Psi_p$) collapse immediately.

Note that given any P-stratified noncommutative stack $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ and any exact functor $\mathcal{X} \xrightarrow{H'} \mathcal{V}$, we can apply these spectral sequences to (the value under H' of) any of the four filtrations discussed in Remark 1.3.12.[88]

Example 1.10.6 (filtered objects and chain complexes)**.** Let us specialize Example 1.10.4 to the case that our down-finite poset is $\mathsf{P} = \mathbb{Z}_{\geq 0}$. In this case, the gluing diagram is given by

$$\mathscr{G}(\mathcal{X}) \simeq \left(\begin{tikzcd} \mathcal{V} \arrow[r, "\Sigma", "\sim"'] \arrow[rr, bend left=50, "0", "\Downarrow"'] & \mathcal{V} \arrow[r, "\Sigma", "\sim"'] \arrow[rr, bend right=50, "0"', "\Uparrow"] & \mathcal{V} \arrow[r, "\Sigma", "\sim"'] & \cdots \arrow[r, "\Sigma", "\sim"'] \arrow[rr, bend left=50, "0", "\Downarrow"'] & \mathcal{V} \arrow[r, "\Sigma", "\sim"'] \arrow[rr, bend right=50, "0"', "\Uparrow"] & \mathcal{V} \arrow[r, "\Sigma", "\sim"'] & \cdots \end{tikzcd} \right) \in \mathsf{LMod}^{L}_{\mathsf{l.lax}.\mathbb{Z}_{\geq 0}}(\mathsf{Pr}_{\mathsf{st}}) \ ,$$

and its right-lax limit is the ∞-category

$$\mathsf{Glue}(\mathcal{X}) := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathbb{Z}_{\geq 0}}(\mathscr{G}(\mathcal{X})) \simeq \mathsf{Ch}_{\geq 0}(\mathcal{V})$$

of ***chain complexes*** in $\mathcal{V}$ concentrated in nonnegative degrees.[89] Theorem A(2) grants an equivalence

$$\mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{V}) \simeq \mathsf{Ch}_{\geq 0}(\mathcal{V}) \ ,$$

which is closely related to Lurie's Dold–Kan correspondence for stable ∞-categories; more precisely, it recovers a version of [Lur, Lemma 1.2.2.4].

Warning 1.10.7. As illustrated by Example 1.10.6, reflection does *not* preserve the property of being a strict (as opposed to lax) left P-module.

Example 1.10.8 (Verdier duality)**.** Let T be a locally compact Hausdorff topological space equipped with a stratification $T \to \mathsf{P}$. Assume for simplicity that P is finite, and choose any presentable stable ∞-category $\mathcal{V}$.

(1) Recall that Verdier duality [Lur, Theorem 5.5.5.1] asserts an equivalence

$$\mathsf{Shv}_{\mathcal{V}}(T)^{\mathsf{op}} \underset{\sim}{\overset{\mathbb{D}_T}{\longleftrightarrow}} \mathsf{Shv}_{\mathcal{V}^{\mathsf{op}}}(T) \ . \tag{1.10.13}$$

On the one hand, by §1.8 we have a canonical stratification of

$$\mathsf{Shv}_{\mathcal{V}}(T)$$

over P^{op}, which by Theorem F(1) (as interpreted via Remark 1.10.1) determines a stratification of

$$\mathsf{Shv}_{\mathcal{V}}(T)^{\mathsf{op}}$$

over P^{op}. On the other hand, we similarly have a canonical stratification of

$$\mathsf{Shv}_{\mathcal{V}^{\mathsf{op}}}(T)$$

[88] Note that applying the spectral sequence (1.10.11) in this way simply gives the spectral sequences (1.3.15) and (1.3.16) of Remark 1.3.13.

[89] Informally, an object of $\mathsf{Ch}_{\geq 0}(\mathcal{V})$ may be thought of as a functor $\mathbb{Z}_{\geq 0} \to \mathcal{V}$ equipped with a coherent system of nullhomotopies for its i-fold composites for all $i \geq 2$. (These are equivalent to gapped objects in $\mathcal{V}$ (see [Lur, Definition 1.2.2.2 and Remark 1.2.2.3]).)

over P^{op}. It is not hard to see that the equivalence (1.10.13) respects these P^{op}-stratifications.[90] In particular, it interchanges the induced filtrations of Remark 1.3.12:

$$\mathsf{fil}^{\bullet}_{R} \simeq \mathbb{D}_T \circ \mathsf{fil}^{L}_{\bullet} \circ \mathbb{D}_T \qquad \text{and} \qquad \mathsf{fil}^{R}_{\bullet} \simeq \mathbb{D}_T \circ \mathsf{fil}^{\bullet}_{L} \circ \mathbb{D}_T \ .$$

(2) Suppose that the dualizing complex $\omega_T \in \mathsf{Shv}_{\mathcal{V}}(T)$ is P-constructible. Then, the Verdier duality equivalence (1.10.13) extends to a commutative square

$$\begin{tikzcd} \mathsf{Shv}_{\mathcal{V}}(T)^{\mathsf{op}} \arrow[r, leftrightarrow, "\mathbb{D}_T", "\sim"'] & \mathsf{Shv}_{\mathcal{V}^{\mathsf{op}}}(T) \\ \mathsf{Shv}^{\mathsf{P}\text{-}\mathsf{cbl}}_{\mathcal{V}}(T)^{\mathsf{op}} \arrow[u, hook] \arrow[r, leftrightarrow, dashed, "\sim", "\mathbb{D}^{\mathsf{P}\text{-}\mathsf{cbl}}_T"'] & \mathsf{Shv}^{\mathsf{P}\text{-}\mathsf{cbl}}_{\mathcal{V}^{\mathsf{op}}}(T) \arrow[u, hook] \end{tikzcd} . \tag{1.10.14}$$

The lower two terms in diagram (1.10.14) inherit P^{op}-stratifications from the upper two terms, as in §1.8, such that the entire diagram (1.10.14) respects P^{op}-stratifications.

Remark 1.10.9. In the situation of Example 1.10.8(2), suppose further that $T \to \mathsf{P}$ is tamely conical (as in Example 1.9.1). Then, the lower equivalence of diagram (1.10.14) extends to a commutative square

$$\begin{tikzcd} \mathsf{Shv}^{\mathsf{P}\text{-}\mathsf{cbl}}_{\mathcal{V}}(T)^{\mathsf{op}} \arrow[r, leftrightarrow, "\mathbb{D}^{\mathsf{P}\text{-}\mathsf{cbl}}_T", "\sim"'] \arrow[d, leftrightarrow, "\wr"'] & \mathsf{Shv}^{\mathsf{P}\text{-}\mathsf{cbl}}_{\mathcal{V}^{\mathsf{op}}}(T) \arrow[d, leftrightarrow, "\wr"] \\ \mathsf{Fun}(\mathsf{Exit}(T), \mathcal{V})^{\mathsf{op}} \arrow[r, leftrightarrow, dashed, "\sim"'] & \mathsf{Fun}(\mathsf{Exit}(T), \mathcal{V}^{\mathsf{op}}) \end{tikzcd} \tag{1.10.15}$$

of equivalences. The lower two terms in diagram (1.10.15) inherit P^{op}-stratifications from the functor $\mathsf{Exit}(T) \to \mathsf{P}$ as in Example 1.9.1, and it is not hard to see that the entire diagram (1.10.15) respects P^{op}-stratifications.

1.11. **t-structures.** As we now describe, stratifications give a method for constructing new t-structures from old ones in the spirit of the construction of perverse sheaves [BBD82]; applied to the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ of Theorem E for a finite group G, this technique can also be used to obtain the slice filtration [HY18].

Let $\mathcal{Z}_\bullet$ be a stratification of $\mathcal{X}$ over P. Suppose that each stratum $\mathcal{X}_p$ is endowed with a t-structure. Then, by [Lur, Proposition 1.4.4.11] we obtain a t-structure on $\mathcal{X}$, whose connective objects are precisely those that are taken to connective objects by all geometric localization functors $\mathcal{X} \xrightarrow{\Phi_p} \mathcal{X}_p$, i.e. the composites

$$\mathcal{X} \xrightarrow{y} \mathcal{Z}_p \xrightarrow{p_L} \mathcal{X}_p \ . \tag{1.11.1}$$

Suppose that the functors (1.11.1) are jointly conservative, e.g. as guaranteed by P being artinian (recall Remark 1.3.10). Then, this t-structure becomes particularly computable: we can

[90] This follows from the general fact that Verdier duality is compatible with open embeddings, in the sense that for any open subset $U \overset{j}{\hookrightarrow} T$ we have a commutative diagram

$$\begin{tikzcd} \mathsf{Shv}_{\mathcal{V}}(T)^{\mathsf{op}} \arrow[r, leftrightarrow, "\mathbb{D}_T", "\sim"'] & \mathsf{Shv}_{\mathcal{V}^{\mathsf{op}}}(T) \\ \mathsf{Shv}_{\mathcal{V}}(U)^{\mathsf{op}} \arrow[u, "(j_*)^{\mathsf{op}}"] \arrow[r, leftrightarrow, "\sim", "\mathbb{D}_U"'] & \mathsf{Shv}_{\mathcal{V}^{\mathsf{op}}}(U) \arrow[u, "j_!"'] \end{tikzcd} .$$

also explicitly describe its coconnective objects. Namely, they are precisely those that are taken to coconnective objects by all of the composites

$$\mathcal{X} \xrightarrow{y} \mathcal{Z}_p \xrightarrow{p_R} \mathcal{X}_p \ . \tag{1.11.2}$$

We may see this as follows. Given any down-closed subset $\mathsf{D} \subseteq \mathsf{P}$, let us write

$$\mathcal{Z}_{\mathsf{D}} := \bigcup_{p \in \mathsf{D}} \mathcal{Z}_p \qquad \text{and} \qquad \mathcal{X}_{\mathsf{D}} := \mathcal{X}/\mathcal{Z}_{\mathsf{D}} \ .$$

Then, from Theorem B we obtain

- a restricted stratification of $\mathcal{Z}_{\mathsf{D}}$ over D, whose p^{th} stratum is $\mathcal{X}_p$ for all $p \in \mathsf{D}$, as well as
- a quotient stratification of $\mathcal{X}_{\mathsf{D}}$ over $\mathsf{P} \backslash \mathsf{D}$, whose p^{th} stratum is $\mathcal{X}_p$ for all $p \in \mathsf{P} \backslash \mathsf{D}$.

Hence, $\mathcal{Z}_{\mathsf{D}}$ and $\mathcal{X}_{\mathsf{D}}$ both inherit t-structures, such that in the recollement

$$\begin{tikzcd} \mathcal{Z}_{\mathsf{D}} \arrow[r, hook, bend left, "i_L"] \arrow[r, hook, bend right, "i_R"'] & \mathcal{X} \arrow[l, "y"] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{X}_{\mathsf{D}} \arrow[l, hook', "\nu"] \end{tikzcd} ,$$

the functors y and ν are t-exact, their left adjoints i_L and p_L are right t-exact (i.e. preserve connective objects), and their right adjoints i_R and p_R are left t-exact (i.e. preserve coconnective objects).[91] It follows that the functors (1.11.2) preserve coconnective objects, and the same argument as that for the functors (1.11.1) proves that they too are jointly conservative.

1.12. **Additive and localizing invariants.** We discuss the interaction of stratifications with additive and localizing invariants [BGT13].

Recall that the ind-completion functor on small stable idempotent-complete ∞-categories factors as an equivalence

$$\begin{tikzcd} \mathsf{St}^{\mathsf{idem}} \arrow[rr, "\mathsf{Ind}"] \arrow[dr, dashed, "\sim"'] & & \mathsf{Pr}^L_{\mathsf{st}} \\ & \mathsf{Pr}^{L,\omega}_{\mathsf{st}} \arrow[ur, hook] & \end{tikzcd}$$

onto the subcategory

- whose objects are the compactly generated stable ∞-categories and
- whose morphisms are those functors that preserve both colimits and compact objects.

In fact, for every morphism $\mathcal{C} \xrightarrow{F} \mathcal{D}$ in $\mathsf{St}^{\mathsf{idem}}$, the right adjoint

$$\begin{tikzcd} \mathsf{Ind}(\mathcal{C}) \arrow[r, shift left, "\mathsf{Ind}(F) := F_!"] \arrow[r, phantom, "\perp"] & \mathsf{Ind}(\mathcal{D}) \arrow[l, shift left, dashed, "F^*"] \end{tikzcd}$$

is automatically colimit-preserving, and it preserves compact objects if and only if F itself admits a right adjoint. Hence, the composite functor

$$\begin{tikzcd} \mathsf{St} \arrow[rr, "\mathsf{Ind}"] \arrow[dr, "(-)^{\mathsf{idem}}"'] & & \mathsf{Pr}^L_{\mathsf{st}} \\ & \mathsf{St}^{\mathsf{idem}} \arrow[ur, "\mathsf{Ind}"'] & \end{tikzcd}$$

carries

(1) exact sequences to recollements,

[91] Indeed, y and p_L are right t-exact by definition, while i_L and ν are right t-exact by inspection.

(2) split-exact sequences to recollements in which i_R preserves colimits, and

(3) stable recollements (i.e. recollements among stable ∞-categories (Definition 6.1.8)) to recollements in which i_R preserves both colimits and compact objects.

Fix a stable ∞-category $\mathcal{C} \in \mathsf{St}$. We say that a full stable subcategory of $\mathcal{C}$ is

(1) ***thick*** if it is idempotent-complete (relative to $\mathcal{C}$),

(2) ***split*** if it is thick and its inclusion admits a right adjoint, and

(3) ***closed*** if it is split and the right adjoint to its inclusion admits a further right adjoint.

With the evident notation, we then have a sequence of fully faithful functors

$$\mathbf{cls}_{\mathcal{C}} \hookrightarrow \mathbf{split}_{\mathcal{C}} \hookrightarrow \mathbf{thick}_{\mathcal{C}} \xhookrightarrow{\mathsf{Ind}} \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$$

among posets, and we may define three sorts of stratifications of $\mathcal{C}$ as stratifications of $\mathsf{Ind}(\mathcal{C})$ that factor accordingly.

Remark 1.12.1. A convergent stratification of $\mathsf{Ind}(\mathcal{C})$ gives, in particular, a means of reconstructing its full subcategory $\mathcal{C} \subseteq \mathsf{Ind}(\mathcal{C})$. However, this is somewhat unsatisfying, as it will not generally reconstruct $\mathcal{C}$ in terms of subcategories thereof: neither the geometric localization functors nor the gluing functors for the stratification of $\mathsf{Ind}(\mathcal{C})$ need preserve compact objects. On the other hand, given a stratification

$$\mathsf{P} \longrightarrow \mathbf{cls}_{\mathcal{C}}$$

(as defined just above), the geometric localization functors and gluing functors of the composite stratification

$$\mathsf{P} \longrightarrow \mathbf{cls}_{\mathcal{C}} \xhookrightarrow{\mathsf{Ind}} \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$$

do preserve compact objects. Indeed, these are precisely the stable stratifications introduced in Remark 1.3.15(1) (under the assumption that $\mathcal{C}$ is idempotent-complete), and the metacosm reconstruction theorem indicated there expresses $\mathcal{C}$ entirely in terms of subcategories thereof.

Now, recall that for a presentable stable ∞-category $\mathcal{V}$, a $\mathcal{V}$-valued ***additive*** (resp. ***localizing***) ***invariant*** is a functor

$$\mathsf{St} \longrightarrow \mathcal{V}$$

that

- preserves zero objects and filtered colimits,
- inverts Morita equivalences (i.e. factors through $\mathsf{St} \xrightarrow{(-)^{\mathsf{idem}}} \mathsf{St}^{\mathsf{idem}}$), and
- carries split-exact (resp. exact) sequences to co/fiber sequences;

key examples include algebraic K-theory (the universal additive invariant), nonconnective algebraic K-theory (the universal localizing invariant), and topological Hochschild homology (a localizing invariant). It follows that additive invariants carry

(2) split-exact sequences to split co/fiber sequences and

(3) recollements to doubly-split co/fiber sequences (i.e. co/fiber sequences equipped with two splittings),

while localizing invariants carry

(1) exact sequences to co/fiber sequences.

(2) split-exact sequences to split co/fiber sequences, and

(3) recollements to doubly-split co/fiber sequences.

Putting these observations together, we find that a stratification of $\mathcal{C}$ in each of the senses above determines a corresponding structure on the value at $\mathcal{C}$ of any additive and/or localizing invariant. For instance, given an additive invariant

$$\mathsf{St} \xrightarrow{F} \mathcal{V} \ ,$$

a convergent stratification

$$\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{split}_{\mathcal{C}}$$

determines a direct sum decomposition

$$F(\mathcal{C}) \simeq \bigoplus_{p \in \mathsf{P}} F(\mathcal{C}_p) \ ,$$

where $\mathcal{C}_p$ denotes the p^{th} stratum of the stratification: the stable quotient of $\mathcal{Z}_p$ by $\langle \mathcal{Z}_q \rangle^{\mathsf{thick}}_{q<p}$ (using Notation 7.1.4). Similarly, a stratification

$$\mathsf{P} \longrightarrow \mathbf{thick}_{\mathcal{C}}$$

induces a P-filtration (as studied e.g. in Example 1.10.4) on the value at $\mathcal{C}$ of any localizing invariant (e.g. nonconnective algebraic K-theory).

2. Stratified noncommutative geometry

In this section, we introduce the theory of stratified noncommutative geometry. From here onwards, for simplicity we revert to standard categorical terminology, in particular opting for the term "presentable stable ∞-category" over the term "noncommutative stack" employed in §§0-1.

This section is organized as follows.

§2.1: We collect some notation and terminology regarding posets.

§2.2: We prove the macrocosm reconstruction theorem (Theorem A(2)) for recollements, i.e. stratifications over $[1]$.

§2.3: We study the basic features of closed subcategories (called "closed noncommutative substacks" in §0).

§2.4: We recall the definition of a stratification and related notions.

§2.5: We prove the macrocosm reconstruction theorem (Theorem A(2)) as Theorem 2.5.14. This follows easily from the metacosm reconstruction theorem (Theorem A(1)), which we prove in §6. We also explain the entire theory in the particular case of stratifications over $[2]$ as Example 2.5.16.

§2.6: We explain the microcosm and nanocosm morphisms (over an arbitrary poset).

§2.7: We explain the theory of strict objects in stratified presentable stable ∞-categories.

Local Notation 2.0.1. In this section, we fix a presentable stable ∞-category $\mathcal{X}$ and a poset P.

2.1. Posets.

In this subsection, we collect some basic notation, terminology, and facts regarding posets.

Definition 2.1.1. A ***convex subset*** of P is a full subposet $\mathsf{C} \subseteq \mathsf{P}$ satisfying the condition that if $p, r \in \mathsf{C}$ and $p \leq q \leq r$ in P then also $q \in \mathsf{C}$. We write $\mathsf{Conv}_{\mathsf{P}}$ for the poset of convex subsets of P ordered by inclusion.

Notation 2.1.2. For any element $p \in \mathsf{P}$, we simply write $p \in \mathsf{Conv}_{\mathsf{P}}$ (rather than $\{p\}$) for the corresponding singleton convex subset of P that it defines.

Definition 2.1.3. A ***down-closed subset*** of P is a full subposet $\mathsf{D} \subseteq \mathsf{P}$ satisfying the condition that if $q \in \mathsf{D}$ and $p \leq q$ then also $p \in \mathsf{D}$. We write $\mathsf{Down}_{\mathsf{P}}$ for the poset of down-closed subsets of P ordered by inclusion.

Observation 2.1.4. There is a containment $\mathsf{Down}_{\mathsf{P}} \subseteq \mathsf{Conv}_{\mathsf{P}}$: a down-closed subset of P is automatically convex.

Notation 2.1.5. Choose any $\mathsf{C} \in \mathsf{Conv}_{\mathsf{P}}$.

(1) We write
$$^{\leq}\mathsf{C} := \{p \in \mathsf{P} : p \leq q \text{ for some } q \in \mathsf{C}\} \in \mathsf{Down}_{\mathsf{P}}$$
for the down-closure of C in P.

(2) We write
$$^{<}\mathsf{C} := (^{\leq}\mathsf{C}) \backslash \mathsf{C} \in \mathsf{Down}_{\mathsf{P}}$$
for the down-closed subset of P obtained by removing the elements of C from $^{\leq}\mathsf{C}$.

We also write $^{\not\leq}\mathsf{C} := \mathsf{P} \backslash (^{\leq}\mathsf{C})$ and $^{\not<}\mathsf{C} := \mathsf{P} \backslash (^{<}\mathsf{C})$.

Definition 2.1.6. We say that the poset P is ***down-finite*** if for every $p \in \mathsf{P}$ the subset $(^{\leq}p) \subseteq \mathsf{P}$ is finite.

Definition 2.1.7. We say that the poset P is ***artinian*** if it admits no injective (or equivalently conservative) functors from $\mathbb{N}^{\mathsf{op}}$.

Definition 2.1.8. An ***interval*** in a poset P is a subset of the form $\mathsf{P}_{p//q} \subseteq \mathsf{P}$.

Notation 2.1.9. Given a functor $\mathsf{P} \rightarrow \mathsf{Q}$ between posets, for any subset $\mathsf{S} \subseteq \mathsf{Q}$ we write $\mathsf{P}_{\mathsf{S}} \subseteq \mathsf{P}$ for its preimage.

Observation 2.1.10. For any surjective functor $\mathcal{I} \xrightarrow{G} \mathcal{J}$ among ∞-categories and any functor $\mathcal{J} \xrightarrow{F} \mathsf{P}$, we have a canonical identification $\mathsf{colim}_{\mathcal{I}}(FG) \simeq \mathsf{colim}_{\mathcal{J}}(F)$.[92] We use this fact without further comment.

Remark 2.1.11. Observation 2.1.10 may be articulated informally as the assertion that colimits in posets are all simply unions (taking $\mathcal{I}$ to be a set).

2.2. **Recollements.** In this subsection, we record the (simple and classical) macrocosm reconstruction theorem for recollements (Definition 1.1.1).

Lemma 2.2.1. *Given a recollement* (1.1.1), *the canonical functor*
$$\mathcal{X} \longrightarrow \mathsf{lim}^{\mathsf{r.lax}}\left(\mathcal{Z} \xrightarrow{p_L i_R} \mathcal{U}\right) := \left\{ \left(Z \in \mathcal{Z}\ ,\ U \in \mathcal{U}\ ,\ \begin{array}{c} U \\ \downarrow \\ p_L i_R Z \end{array} \right) \right\}$$
given by the association
$$X \longmapsto (yX \longmapsto p_L i_R yX \longleftarrow p_L X) := \left(yX \in \mathcal{Z}\ ,\ p_L X \in \mathcal{U}\ ,\ \begin{array}{c} p_L X \\ \downarrow \\ p_L i_R yX \end{array} \right)$$
is an equivalence.

[92]In particular, each colimit exists if and only if the other does.

Proof. We claim that this functor has an inverse, given by the association

$$\lim \left(\begin{array}{ccc} & & \nu U \\ & & \downarrow \\ i_R Z & \longrightarrow & \nu p_L i_R y Z \end{array} \right) \longleftarrow \left(\begin{array}{cc} Z \in \mathcal{Z}\ ,\ U \in \mathcal{U}\ , & \begin{array}{c} U \\ \downarrow \\ p_L i_R Z \end{array} \end{array} \right) .$$

Indeed, the composite endofunctor of $\lim^{\mathsf{r.lax}}\left(\mathcal{Z} \xrightarrow{p_L i_R} \mathcal{U}\right)$ is immediately seen to be the identity. To see that the composite endofunctor of $\mathcal{X}$ is also the identity, it suffices to check that for any $X \in \mathcal{X}$ the commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \nu p_L X \arrow[d] \\ i_R y X \arrow[r] & \nu p_L i_R y X \end{tikzcd} \tag{2.2.1}$$

is a pullback square. As a result of the equality $\mathsf{im}(\nu) = \mathsf{ker}(y)$, the fibers of the horizontal morphisms in the commutative square (2.2.1) are equivalent. □

Warning 2.2.2. Recollements play a central role in our work. We generally use the notations of diagram (1.1.1) for the various functors involved (e.g. i_L or p_R), unless there is more pertinent notation in a particular context (such as in our study of genuine G-spectra). For simplicity and readability we do not decorate these symbols further, so that in a single expression (e.g. a composite functor) these various symbols may be referring to *different* recollements – some of which may not even have been explicitly indicated. We hope that the meanings of these functors are always made clear by the context.

2.3. **Closed subcategories.** In this subsection, we study some basic properties of closed subcategories (a.k.a. closed noncommutative substacks).

Definition 2.3.1. For simplicity, here we use the term ***closed subcategory*** of $\mathcal{X}$ in place of the term "closed noncommutative substack" of $\mathcal{X}$ (Definition 1.3.1). We write $\mathbf{Cls}_{\mathcal{X}}$ for the poset of closed subcategories of $\mathcal{X}$ ordered by inclusion.

Example 2.3.2. Given a set $\{K_s \in \mathcal{X}^\omega\}_{s \in S}$ of compact objects of $\mathcal{X}$, the full stable subcategory that they generate under colimits is a closed subcategory of $\mathcal{X}$: the restricted Yoneda embedding commutes with filtered colimits, and hence admits a further right adjoint.

Notation 2.3.3. In the situation of Example 2.3.2, we write

$$\langle K_s \rangle_{s \in S} \in \mathbf{Cls}_{\mathcal{X}}$$

for the closed subcategory of $\mathcal{X}$ generated by the objects $\{K_s \in \mathcal{X}^\omega\}_{s \in S}$.

Notation 2.3.4. Given a full presentable stable subcategory $\mathcal{Z} \subseteq \mathcal{X}$, we write

$$\mathcal{Z}^\perp := \{U \in \mathcal{X} : \underline{\mathsf{hom}}_{\mathcal{X}}(Z, U) \simeq 0 \text{ for all } Z \in \mathcal{Z}\} \subseteq \mathcal{X}$$

for its right-orthogonal subcategory.

Observation 2.3.5. A full presentable stable subcategory $\mathcal{Z} \subseteq \mathcal{X}$ determines a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, shift left, "i"] \arrow[r, phantom, "\perp"] & \mathcal{X} \arrow[l, shift left, "i^R"] \arrow[r, shift left, "j^L"] \arrow[r, phantom, "\perp"] & \mathcal{Z}^\perp \arrow[l, hook, shift left, "j"] \end{tikzcd} \tag{2.3.1}$$

in Cat, in which the functors i and j are the defining fully faithful inclusions and the functor j^L is determined by the formula

$$jj^L \simeq \mathsf{cofib}\left(ii^R \xrightarrow{\varepsilon} \mathsf{id}_{\mathcal{X}}\right) .$$

Moreover, the commutative square

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, "i"] \arrow[d] & \mathcal{X} \arrow[d, "j^L"] \\ 0 \arrow[r, hook] & \mathcal{Z}^\perp \end{tikzcd}$$

is a pushout square in $\mathsf{Pr}^L_{\mathsf{st}}$: given a morphism

$$\mathcal{X} \xrightarrow{F} \mathcal{Y}$$

in $\mathsf{Pr}^L_{\mathsf{st}}$ such that $Fi \simeq 0$, we obtain a colimit-preserving factorization

$$\begin{tikzcd} \mathcal{X} \arrow[r, "j^L"] \arrow[dr, "F"'] & \mathcal{Z}^\perp \arrow[d, dashed, "Fj"] \\ & \mathcal{Y} \end{tikzcd} .$$

Definition 2.3.6. In light of Observation 2.3.5, given a full presentable stable subcategory $\mathcal{Z} \subseteq \mathcal{X}$, we write

$$\mathcal{X}/\mathcal{Z} := \mathcal{Z}^\perp$$

for its right-orthogonal subcategory and refer to it as the ***presentable quotient*** of $\mathcal{X}$ by $\mathcal{Z}$.

Observation 2.3.7. In the special case of Observation 2.3.5 where $\mathcal{Z} \in \mathbf{Cls}_\mathcal{X}$ is a closed subcategory, diagram (2.3.1) (lies in $\mathsf{Pr}^L_{\mathsf{st}}$ and therefore) extends to a recollement (1.1.1) in which

$$i_L := i \ , \qquad y := i^R \ , \qquad \mathcal{U} := \mathcal{Z}^\perp =: \mathcal{X}/\mathcal{Z} \ , \qquad p_L := j^L \ , \qquad \text{and} \qquad \nu := j \ ;$$

the functors p_L and p_R are respectively determined by the formulas

$$\nu p_L \simeq \mathsf{cofib}\left(i_L y \xrightarrow{\varepsilon} \mathsf{id}_\mathcal{X}\right) \qquad \text{and} \qquad \nu p_R \simeq \mathsf{fib}\left(\mathsf{id}_\mathcal{X} \xrightarrow{\eta} i_R y\right) \ .$$

Conversely, any recollement (1.1.1) arises in this way: the functor i_L is the inclusion of a closed subcategory and the functor ν is the inclusion of its right-orthogonal subcategory.

Observation 2.3.8. Inclusions of closed subcategories are stable under composition. Also, if $\mathcal{Z}, \mathcal{Y} \in \mathbf{Cls}_\mathcal{X}$ with $\mathcal{Z} \subseteq \mathcal{Y} \subseteq \mathcal{X}$, then $\mathcal{Z} \in \mathbf{Cls}_\mathcal{Y}$. We use these facts implicitly without further comment.[93]

Observation 2.3.9. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a full stable subcategory that is closed under colimits. Then, $\mathcal{Z}$ is a closed subcategory of $\mathcal{X}$ if and only if its right-orthogonal subcategory $\mathcal{Z}^\perp \subseteq \mathcal{X}$ is also closed under colimits. It follows that for any set $\{\mathcal{Z}_s \in \mathbf{Cls}_\mathcal{X}\}_{s \in S}$ of closed subcategories of $\mathcal{X}$, the full stable subcategory of $\mathcal{X}$ that they generate under colimits is also a closed subcategory of $\mathcal{X}$.[94] We use this fact implicitly without further comment.

Notation 2.3.10. Concordantly with Notation 2.3.3, given a set $\{\mathcal{Z}_s \in \mathbf{Cls}_\mathcal{X}\}_{s \in S}$ of closed subcategories of $\mathcal{X}$, we write

$$\langle \mathcal{Z}_s \rangle_{s \in S} \in \mathbf{Cls}_\mathcal{X}$$

for the closed subcategory of $\mathcal{X}$ that they generate under colimits, i.e. the colimit of the functor $S \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_\mathcal{X}$.[95]

[93] These facts are amplified in §3.2.

[94] To show this, writing $\mathcal{Z} \subseteq \mathcal{X}$ for the full stable subcategory generated under colimits by the subcategories $\{\mathcal{Z}_s\}_{s \in S}$, it suffices to show that $\mathcal{Z}^\perp = \bigcap_{s \in S}((\mathcal{Z}_s)^\perp)$. It is immediate that $\mathcal{Z}^\perp \subseteq \bigcap_{s \in S}((\mathcal{Z}_s)^\perp)$. To verify the inclusion $\mathcal{Z}^\perp \supseteq \bigcap_{s \in S}((\mathcal{Z}_s)^\perp)$, we observe that this intersection of subcategories of $\mathcal{X}$ may be computed as a limit in Pr^R, and therefore its inclusion into $\mathcal{X}$ admits a left adjoint that evidently annihilates all objects of $\mathcal{Z}$.

[95] In §1, this was written as $\bigcup_{s \in S} \mathcal{Z}_s \simeq \mathsf{colim}(S \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_\mathcal{X})$, so as to highlight the analogy with the union of closed subsets of a scheme. Outside of that section, we use the notation $\langle \mathcal{Z}_s \rangle_{s \in S}$ because it is more compact.

Remark 2.3.11. Closed subcategories of presentable stable ∞-category behave much like closed subsets of a topological space, but they are not completely analogous. For instance, increasing unions in the poset $\mathbf{Cls}_{\mathcal{X}}$ commute with the forgetful functor to $\mathsf{Pr}^L_{\mathsf{st}}$, whereas increasing unions in the poset of closed subsets of a topological space do not generally commute with the forgetful functor to topological spaces.

2.4. **Stratifications.** In this subsection, we recall the definitions (originally given in §1.3) of a stratification and of its strata.

Definition 2.4.1. A ***prestratification*** of $\mathcal{X}$ over P is a functor

$$\begin{array}{ccc} \mathsf{P} & \xrightarrow{\mathcal{Z}_\bullet} & \mathbf{Cls}_{\mathcal{X}} \\ \uplus & & \uplus \\ p & \longmapsto & \mathcal{Z}_p \end{array}$$

such that $\mathcal{X} = \langle \mathcal{Z}_p \rangle_{p \in \mathsf{P}}$.

Notation 2.4.2. Given a prestratification $\mathcal{Z}_\bullet$ of $\mathcal{X}$ over P, for any $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}}$ we write

$$\mathcal{Z}_{\mathsf{D}} := \langle \mathcal{Z}_p \rangle_{p \in \mathsf{D}} \in \mathbf{Cls}_{\mathcal{X}} \ .$$

Note that $\mathcal{Z}_{\leq p} = \mathcal{Z}_p$; we use the latter notation for simplicity. Note too that $\mathcal{Z}_{\varnothing} = 0$.

Definition 2.4.3. A prestratification $\mathcal{Z}_\bullet$ of $\mathcal{X}$ over P is a ***stratification*** if it satisfies the following ***stratification condition***: for any $p, q \in \mathsf{P}$, there exists a factorization

$$\begin{tikzcd} \mathcal{Z}_{(\leq p) \cap (\leq q)} \arrow[r, hook, "i_L"] & \mathcal{Z}_p \\ \mathcal{Z}_q \arrow[u, dashed] \arrow[r, hook, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} .$$

Remark 2.4.4. In the stratification condition, the upper functor i_L is a monomorphism (in fact it is the inclusion of a closed subcategory, as indicated by the notation), and so if there exists a factorization then it is unique. Moreover, if the stratification condition holds, then its factorization is necessarily the right adjoint

$$\begin{tikzcd} \mathcal{Z}_{(\leq p) \cap (\leq q)} \arrow[r, hook, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{Z}_q \arrow[l, dashed, shift left, "y"] \end{tikzcd} ;$$

this follows from Lemma 3.1.7.

Observation 2.4.5. The stratification condition is automatic if $p \leq q$ or if $q \leq p$. In particular, in the case that the poset P is totally ordered, every prestratification of $\mathcal{X}$ over P is a stratification.

Definition 2.4.6. Suppose that $\mathcal{Z}_\bullet$ is a prestratification of $\mathcal{X}$ over P, and suppose that $\mathsf{C} \in \mathsf{Conv}_{\mathsf{P}}$.

(1) The C^{th} ***stratum*** of the prestratification is the presentable quotient

$$\mathcal{X}_{\mathsf{C}} := \mathcal{Z}_{\leq \mathsf{C}} / \mathcal{Z}_{< \mathsf{C}} \ .$$

(2) The C^{th} ***geometric localization functor*** is the left adjoint in the composite adjunction

$$\begin{tikzcd} \Phi_{\mathsf{C}} : \mathcal{X} \arrow[r, shift left, "y"] \arrow[r, phantom, "\perp"] & \mathcal{Z}_{\mathsf{C}} \arrow[l, hook, shift left, "i_R"] \arrow[r, shift left, "p_L"] \arrow[r, phantom, "\perp"] & \mathcal{X}_{\mathsf{C}} : \rho^{\mathsf{C}} \arrow[l, hook, shift left, "\nu"] \end{tikzcd} .$$

We also write

$$L_{\mathsf{C}} : \mathcal{X} \xrightarrow{\Phi_{\mathsf{C}}} \mathcal{X}_{\mathsf{C}} \xhookrightarrow{\rho^{\mathsf{C}}} \mathcal{X}$$

for the composite endofunctor, and we write

$$\mathrm{id}_{\mathcal{X}} \xrightarrow{\eta_{\mathsf{C}}} L_{\mathsf{C}}$$

for the unit morphism.

Remark 2.4.7. Considering an element $p \in \mathsf{P}$ as a convex subset of P, Definition 2.4.6 specializes to Definition 1.3.5 of the p^{th} stratum of a stratification.

Remark 2.4.8. For any $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}} \subseteq \mathsf{Conv}_{\mathsf{P}}$, the functor $\mathcal{Z}_{\mathsf{D}} \xrightarrow{p_L} \mathcal{X}_{\mathsf{D}}$ is an equivalence. We use both of these notations, depending on the context: we use the notation $\mathcal{Z}_{\mathsf{D}}$ when we mean to consider this as a subcategory of $\mathcal{X}$ via the inclusion i_L, while we use the notation $\mathcal{X}_{\mathsf{D}}$ when we mean to consider this as a subcategory of $\mathcal{X}$ via the inclusion ρ^{D} (which coincides with i_R in this special case).

2.5. **The macrocosm reconstruction theorem.** This subsection is centered around the macrocosm reconstruction theorem (Theorem 2.5.14), which we prove using the metacosm reconstruction theorem (which is itself proved in §6). We unpack the entire theory in the case that $\mathsf{P} = [2]$ in Example 2.5.16.

Local Notation 2.5.1. In this subsection, we fix a stratification $\mathcal{Z}_\bullet$ of $\mathcal{X}$ over P.

Remark 2.5.2. We use the language of *modules* to discuss certain definitions and constructions. This is explained in detail in §A. In the interest of keeping the main body of this work relatively self-contained, we summarize the essential points here.

- By a left/right module over an ∞-category, we mean a co/cartesian fibration over it, or equivalently a functor from it(s opposite) to Cat.
- These modules become *lax* when our fibrations are only *locally* co/cartesian, which (definitionally) correspond to left/right-lax functors to Cat.
- One can take the strict, left-lax, or right-lax limit of any module (regardless of whether that module is itself strict or left/right-lax).
- The specific construction that is relevant for us here is the right-lax limit of a left-lax module; the precise definition (in our case of interest) is recalled in Remark 2.5.9.

Definition 2.5.3. For any $p, q \in \mathsf{P}$, the corresponding ***gluing functor*** is the composite

$$\Gamma^p_q : \mathcal{X}_p \overset{\rho^p}{\hookrightarrow} \mathcal{X} \xrightarrow{\Phi_q} \mathcal{X}_q \ .$$

Remark 2.5.4. By the stratification condition, the functor $\mathcal{X}_p \xrightarrow{\Gamma^p_q} \mathcal{X}_q$ is zero whenever $p \not\leq q$.

Notation 2.5.5. We define the full subcategory

$$\mathscr{G}(\mathcal{X}) := \{(X, p) \in \mathcal{X} \times \mathsf{P} : X \in \mathcal{X}_p\} \subseteq \mathcal{X} \times \mathsf{P} \ ,$$

which we consider as an object of $\mathsf{Cat}_{/\mathsf{P}}$.

Observation 2.5.6. The functor

$$\mathscr{G}(\mathcal{X}) \longrightarrow \mathsf{P}$$

is a locally cocartesian fibration, whose monodromy functor over each morphism $p \to q$ in P is the gluing functor

$$\mathcal{X}_p \xrightarrow{\Gamma^p_q} \mathcal{X}_q \ .$$

We therefore consider it as defining a left-lax left P-module

$$\mathscr{G}(\mathcal{X}) \in \mathsf{LMod}_{\mathsf{l.lax.P}} := \mathsf{loc.coCart}_{\mathsf{P}} \ .$$

Definition 2.5.7. We refer to the left-lax left P-module

$$\mathscr{G}(\mathcal{X}) \in \mathsf{LMod}_{\mathsf{l.lax.P}}$$

of Observation 2.5.6 as the ***gluing diagram*** of the stratification.

Definition 2.5.8. The ***glued* ∞*-category*** of the stratification is the right-lax limit

$$\mathsf{Glue}(\mathcal{X}) := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) \ .$$

Remark 2.5.9. For the reader's convenience, we unpack the definition of the glued ∞-category $\mathsf{Glue}(\mathcal{X})$. First of all, we write $\mathsf{sd}(\mathsf{P})$ for the *subdivision* of P: the poset of finite nonempty linearly ordered subsets of P (Definition A.4.2). Moreover, the functor

$$\mathsf{sd}(\mathsf{P}) \xrightarrow{\mathsf{max}} \mathsf{P}$$

carrying each subset to its maximal element is a locally cocartesian fibration (Observation A.4.10), with nontrivial cocartesian monodromy functors given by adjoining new maximal elements. Then, by Proposition A.5.2 we have an identification

$$\mathsf{Glue}(\mathcal{X}) := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) \simeq \mathsf{Fun}^{\mathsf{cocart}}_{/\mathsf{P}}(\mathsf{sd}(\mathsf{P}), \mathscr{G}(\mathcal{X})) \ ;$$

that is, the glued ∞-category $\mathsf{Glue}(\mathcal{X})$ is equivalent to that of morphisms

$$\begin{tikzcd} \mathsf{sd}(\mathsf{P}) \arrow[rr, dashed] \arrow[dr, "\mathsf{max}"'] & & \mathscr{G}(\mathcal{X}) \arrow[dl] \\ & \mathsf{P} & \end{tikzcd} \tag{2.5.1}$$

in $\mathsf{loc.coCart}_{\mathsf{P}}$ (i.e. functors over P that preserve cocartesian morphisms thereover).

Observation 2.5.10. We can consider the glued ∞-category as a full subcategory

$$\mathsf{Glue}(\mathcal{X}) \subseteq \mathsf{Fun}(\mathsf{sd}(\mathsf{P}), \mathcal{X})$$

via the composite fully faithful embedding

$$\mathsf{Glue}(\mathcal{X}) := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) \simeq \mathsf{Fun}^{\mathsf{cocart}}_{/\mathsf{P}}(\mathsf{sd}(\mathsf{P}), \mathscr{G}(\mathcal{X})) \overset{\mathsf{f.f.}}{\hookrightarrow} \mathsf{Fun}_{/\mathsf{P}}(\mathsf{sd}(\mathsf{P}), \mathscr{G}(\mathcal{X})) \overset{\mathsf{f.f.}}{\hookrightarrow} \mathsf{Fun}_{/\mathsf{P}}(\mathsf{sd}(\mathsf{P}), \underline{\mathcal{X}}) \simeq \mathsf{Fun}(\mathsf{sd}(\mathsf{P}), \mathcal{X}) \ .$$

Explicitly, its image consists of those functors

$$\mathsf{sd}(\mathsf{P}) \xrightarrow{F} \mathcal{X}$$

satisfying the following conditions.

(1) For every $([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$, we have

$$F(\varphi) \in \rho^{\mathsf{max}(\varphi)}(\mathcal{X}_{\mathsf{max}(\varphi)}) \subseteq \mathcal{X} \ .$$

(2) It carries each morphism in $\mathsf{sd}(\mathsf{P})$ of the form

$$\begin{tikzcd} {[n]} \arrow[rr, hook, "i \mapsto i"] \arrow[dr, hook, "\varphi"'] & & {[n+1]} \arrow[dl, hook', "\psi"] \\ & \mathsf{P} & \end{tikzcd}$$

(which are precisely the cocartesian morphisms with respect to the locally cocartesian fibration $\mathsf{sd}(\mathsf{P}) \xrightarrow{\mathsf{max}} \mathsf{P}$) to a morphism

$$F(\varphi) \longrightarrow F(\psi) \tag{2.5.2}$$

in $\mathcal{X}$ that becomes an equivalence after applying the functor $\mathcal{X} \xrightarrow{\Phi_{\mathsf{max}(\psi)}} \mathcal{X}_{\mathsf{max}(\psi)}$.[96]

We use these facts without further comment.

[96]Assuming condition (1), condition (2) is equivalent to requiring that the morphism (2.5.2) witnesses $F(\psi)$ as the $L_{\mathsf{max}(\psi)}$-localization of $F(\varphi)$.

Notation 2.5.11. We write

$$\mathsf{lim}_{\mathsf{sd(P)}} : \mathsf{Glue}(\mathcal{X}) \hookrightarrow \mathsf{Fun}(\mathsf{sd(P)}, \mathcal{X}) \xrightarrow{\mathsf{lim}_{\mathsf{sd(P)}}} \mathcal{X}$$

for the composite.

Observation 2.5.12. The defining inclusion

$$\underline{\mathcal{X}} \xhookleftarrow{\text{f.f.}} \mathscr{G}(\mathcal{X})$$

is a morphism in $\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{l.lax.P}} := \mathsf{Cat}_{\mathsf{loc.cocart/P}}$ (though not generally in $\mathsf{LMod}_{\mathsf{l.lax.P}} := \mathsf{loc.coCart}_{\mathsf{P}}$). Over each object $p \in \mathsf{P}$, this is the right adjoint in the adjunction

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "\Phi_p"] \arrow[r, phantom, "\perp"] & \mathcal{X}_p \arrow[l, hook', shift left, "\rho^p"] \end{tikzcd} .$$

By Lemma A.3.5, the left adjoints Φ_p assemble into a morphism

$$\mathsf{const}(\mathcal{X}) := \underline{\mathcal{X}} \longrightarrow \mathscr{G}(\mathcal{X})$$

in $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$. Through the definitional adjunction

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, shift left, "\mathsf{const}"] \arrow[r, phantom, "\perp"] & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}} \arrow[l, shift left, "\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}"] \end{tikzcd} ,$$

this corresponds to a functor

$$\mathcal{X} \longrightarrow \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) . \tag{2.5.3}$$

In terms of Observation 2.5.10, the functor (2.5.3) is given by the formula

$$X \longmapsto \Big(([n] \xrightarrow{\varphi} \mathsf{P}) \longmapsto \rho_{\varphi(n)}\Phi_{\varphi(n)} \cdots \rho_{\varphi(0)}\Phi_{\varphi(0)} X \Big) . \tag{2.5.4}$$

Definition 2.5.13. We refer to the functor (2.5.3) as the (***microcosm***) ***gluing diagram functor***, and we denote it by

$$\mathcal{X} \xrightarrow{g} \mathsf{Glue}(\mathcal{X}) := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) .$$

Theorem 2.5.14. *There is a canonical adjunction*

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "g"] \arrow[r, phantom, "\perp"] & \mathsf{Glue}(\mathcal{X}) \arrow[l, shift left, "\mathsf{lim}_{\mathsf{sd(P)}}"] \end{tikzcd} , \tag{2.5.5}$$

which is an equivalence whenever P *is down-finite.*

Proof. By Theorem 6.2.6, the functor $\mathcal{X} \xrightarrow{g} \mathsf{Glue}(\mathcal{X})$ defines a morphism in $\mathsf{Pr}^L_{\mathsf{st}}$ (being the image under the forgetful functor $\mathbf{Strat}_{\mathsf{P}} \to \mathsf{Pr}^L_{\mathsf{st}}$ of the unit of the adjunction (6.2.3)) and is an equivalence whenever P is down-finite. The identification of its right adjoint is contained in the proof of Theorem 6.2.6. ☐

Definition 2.5.15. We say that the stratification of $\mathcal{X}$ over P is ***convergent*** if the adjunction (2.5.5) is an equivalence.

Example 2.5.16. Suppose that $\mathsf{P} = [2]$.

(1) The gluing diagram of the stratification is the lax-commutative triangle

$$\mathscr{G}(\mathcal{X}) = \left(\begin{tikzcd} & \mathcal{X}_1 \arrow[dr, "\Gamma^1_2"] \arrow[d, phantom, "\Uparrow\eta_1"] & \\ \mathcal{X}_0 \arrow[ur, "\Gamma^0_1"] \arrow[rr, "\Gamma^0_2"'] & {} & \mathcal{X}_2 \end{tikzcd} \right) ,$$

in which the natural transformation is the composite

$$\eta_1 : \Gamma^0_2 := \Phi_2\rho^0 \simeq \Phi_2 \,\mathsf{id}_{\mathcal{X}}\, \rho^0 \xrightarrow{\eta_1} \Phi_2\rho^1\Phi_1\rho^0 =: \Gamma^1_2\Gamma^0_1 .$$

(2) An object of the glued ∞-category $\mathsf{Glue}(\mathcal{X})$ amounts to the data of the form

$$\begin{tikzcd}
& X_2 \arrow[rr, "\gamma_2^1"] \arrow[dd, "\gamma_2^0"'] & & \Gamma_2^1(X_1) \arrow[dd, "\Gamma_2^1(\gamma_1^0)"] \\
& & X_1 \arrow[ur, mapsto] \arrow[dd, "\gamma_1^0"'] & \\
& \Gamma_2^0(X_0) \arrow[rr, "\eta_1"'] & & \Gamma_2^1(\Gamma_1^0(X_0)) \\
X_0 \arrow[ur, mapsto] \arrow[rr, mapsto] & & \Gamma_1^0(X_0) \arrow[ur, mapsto] &
\end{tikzcd}\ ,^{97} \tag{2.5.6}$$

where $X_i \in \mathcal{X}_i$ for all $i \in [2]$. One may think of the morphisms γ_j^i as *gluing morphisms* (i.e. 1-cubes) for this object of $\mathsf{Glue}(\mathcal{X})$, and of the commutative square in $\mathcal{X}_2$ as higher-dimensional gluing data, namely a *gluing square* γ_{012}.[98]

(3) Given an object $X \in \mathcal{X}$, its gluing diagram is the object $g(X) = (2.5.6) \in \mathsf{Glue}(\mathcal{X})$ in which

- $X_i = \Phi_i(X)$ for all $0 \leq i \leq 2$,
- the gluing morphism γ_j^i is the unit morphism

$$X_j := \Phi_j(X) \xrightarrow{\eta_i} \Phi_j(\rho^i(\Phi_i(X))) =: \Gamma_j^i(X_i)$$

for all $0 \leq i < j \leq 2$, and

- the commutativity of the gluing square γ_{012} follows from the commutativity of the square

$$\begin{tikzcd}
\Phi_2 \arrow[r, "\eta_1"] \arrow[d, "\eta_0"'] & \Phi_2\rho^1\Phi_1 \arrow[d, "\eta_0"] \\
\Phi_2\rho^0\Phi_0 \arrow[r, "\eta_1"'] & \Phi_2\rho^1\Phi_1\rho^0\Phi_0
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{X}, \mathcal{X}_2)$.

(4) Because $\mathsf{P} = [2]$ is finite and hence down-finite, Theorem 2.5.14 guarantees that each $X \in \mathcal{X}$ is the limit of its gluing diagram: the equivalence

$$X \xrightarrow{\sim} \lim_{\mathsf{sd}(\mathsf{P})}(g(X))$$

[97] The locally cocartesian fibration $\mathsf{sd}([2]) \xrightarrow{\max} [2]$ is illustrated in Figure 9.

[98] The notation $X_j \xrightarrow{\gamma_j^i} \Gamma_j^i(X_i)$ for the gluing morphisms is chosen so as to parallel the notation $\mathcal{X}_i \xrightarrow{\Gamma_j^i} \mathcal{X}_j$ for the gluing functors. More generally, each conservative functor $[n] \xrightarrow{p_\bullet} \mathsf{P}$ determines a gluing n-cube $\gamma_{p_0,\ldots,p_n}$ that is part of the data of an object of the glued ∞-category.

amounts to the limit diagram

$$
\begin{tikzcd}
 & L_2(X) \arrow[rr] \arrow[dd] & & L_2(L_1(X)) \arrow[dd] \\
X \arrow[ur] \arrow[rr] \arrow[dd] & & L_1(X) \arrow[ur] \arrow[dd] & \\
 & L_2(L_0(X)) \arrow[rr] & & L_2(L_1(L_0(X))) \\
L_0(X) \arrow[ur] \arrow[rr] & & L_1(L_0(X)) \arrow[ur] &
\end{tikzcd}
.
$$

2.6. **The microcosm and nanocosm morphisms.** In this subsection, we discuss the microcosm and nanocosm morphisms. In particular, we give a detailed description of the nanocosm morphism in Remark 2.6.7.

Notation 2.6.1. For any $([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$, we write

$$\Gamma_\varphi := \Gamma^{\varphi(n-1)}_{\varphi(n)} \cdots \Gamma^{\varphi(0)}_{\varphi(1)} \qquad \text{and} \qquad L_\varphi := L_{\varphi(n)} \cdots L_{\varphi(0)} \ .$$

Observation 2.6.2. By definition, for any $([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$ the functors Γ_φ and L_φ participate in the commutative diagram

$$
\begin{tikzcd}
\mathcal{X} \arrow[r, "L_\varphi"] \arrow[d, "\Phi_{\varphi(0)}"'] & \mathcal{X} \\
\mathcal{X}_{\varphi(0)} \arrow[r, "\Gamma_\varphi"'] & \mathcal{X}_{\varphi(n)} \arrow[u, hook, "\rho^{\varphi(n)}"']
\end{tikzcd}
.
$$

We use this fact without further comment.

Observation 2.6.3. By definition, the functors L_φ for $\varphi \in \mathsf{sd}(\mathsf{P})$ are the values of a functor

$$\mathsf{sd}(\mathsf{P}) \xrightarrow{L_\bullet} \mathsf{Fun}^{\mathsf{ex}}(\mathcal{X}, \mathcal{X})_{\mathsf{id}_{\mathcal{X}}/} \ , \tag{2.6.1}$$

namely the adjunct of the composite

$$\mathcal{X} \xrightarrow{g} \mathsf{Glue}(\mathcal{X}) \hookrightarrow \mathsf{Fun}(\mathsf{sd}(\mathsf{P}), \mathcal{X})$$

equipped with its coaugmentation given by the unit of the adjunction (2.5.5) of Theorem 2.5.14.

Remark 2.6.4. Using Notation 2.6.1, the formula (2.5.4) for the composite

$$\mathcal{X} \xrightarrow{g} \mathsf{Glue}(\mathcal{X}) := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) \hookrightarrow \mathsf{Fun}(\mathsf{sd}(\mathsf{P}), \mathcal{X})$$

can be expressed more compactly as

$$X \longmapsto (\varphi \longmapsto L_\varphi X) \ .$$

Observation 2.6.5. For each nonidentity morphism $p < q$ in P, the functors $\mathcal{X}_p \xrightarrow{\Gamma_\varphi} \mathcal{X}_q$ for $\varphi \in \mathsf{sd}(\mathsf{P})^{|p}_{|q}$ are the values of a factorization

$$\begin{tikzcd}
\mathsf{sd}(\mathsf{P})^{|p}_{|q} \arrow[rr, dashed, "\Gamma_\bullet"] \arrow[d, hook] & & \mathsf{Fun}^{\mathsf{ex}}(\mathcal{X}_p, \mathcal{X}_q) \arrow[d, hook, "\rho^q \circ -"] \\
\mathsf{sd}(\mathsf{P}) \arrow[r, "L_\bullet"'] & \mathsf{Fun}^{\mathsf{ex}}(\mathcal{X}, \mathcal{X}) \arrow[r, "- \circ \rho^p"'] & \mathsf{Fun}^{\mathsf{ex}}(\mathcal{X}_p, \mathcal{X})
\end{tikzcd} .$$

Definition 2.6.6. Fix any object $X \in \mathcal{X}$.

(1) We define the ***reglued object*** of X to be

$$\mathsf{glue}(X) := \lim_{\mathsf{sd}(\mathsf{P})}(g(X)) \in \mathcal{X} .$$

(2) We define the ***microcosm morphism*** of X to be the unit morphism

$$X \longrightarrow \mathsf{glue}(X)$$

in $\mathcal{X}$ of the adjunction (2.5.5) of Theorem 2.5.14.

(3) For any $Y \in \mathcal{X}$, we define the corresponding ***nanocosm morphism*** to be the composite morphism

$$\underline{\mathsf{hom}}_{\mathcal{X}}(Y, X) \longrightarrow \underline{\mathsf{hom}}_{\mathcal{X}}(Y, \mathsf{glue}(X)) \simeq \lim_{\varphi \in \mathsf{sd}(\mathsf{P})} \underline{\mathsf{hom}}_{\mathcal{X}}(Y, L_\varphi(X)) \simeq \lim_{([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})} \left(\underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)} Y, \Gamma_\varphi \Phi_{\varphi(0)} X) \right)$$

obtained by applying $\underline{\mathsf{hom}}_{\mathcal{X}}(Y, -)$ to the microcosm morphism of X.

Remark 2.6.7. For any objects $X, Y \in \mathcal{X}$, we unpack the nanocosm morphism

$$\underline{\mathsf{hom}}_{\mathcal{X}}(Y, X) \longrightarrow \lim_{([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})} \left(\underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)} Y, \Gamma_\varphi \Phi_{\varphi(0)} X) \right)$$

as follows. First of all, postcomposing with the canonical morphism to the $([n] \xrightarrow{\varphi} \mathsf{P})^{\text{th}}$ constituent of the limit, we obtain the composite

$$\underline{\mathsf{hom}}_{\mathcal{X}}(Y, X) \longrightarrow \underline{\mathsf{hom}}_{\mathcal{X}}(Y, L_\varphi X) \simeq \underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)} Y, \Gamma_\varphi \Phi_{\varphi(0)} X) .$$

The functoriality of the diagram

$$\mathsf{sd}(\mathsf{P}) \xrightarrow{([n] \xrightarrow{\varphi} \mathsf{P}) \longmapsto \underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)} Y, \Gamma_\varphi \Phi_{\varphi(0)} X)} \mathcal{S}\mathsf{p} \tag{2.6.2}$$

may be described informally as follows. Observe that every morphism in $\mathsf{sd}(\mathsf{P})$ factors as a composite of morphisms whose images under the forgetful functor $\mathsf{sd}(\mathsf{P}) \to \mathbf{\Delta}$ are all coface maps $[n] \xrightarrow{\delta^i} [n+1]$ (for some $n \geq 0$ and some $0 \leq i \leq n+1$), so it suffices to describe the functoriality of the diagram (2.6.2) on such morphisms. So, let us fix a morphism

$$\begin{tikzcd}
{[n]} \arrow[rr, hook, "\delta^i"] \arrow[dr, "\varphi"'] & & {[n+1]} \arrow[dl, "\widetilde{\varphi}"] \\
& \mathsf{P} &
\end{tikzcd} \tag{2.6.3}$$

in $\mathsf{sd}(\mathsf{P})$, and describe the morphism

$$\underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)} Y, \Gamma_\varphi \Phi_{\varphi(0)} X) \longrightarrow \underline{\mathsf{hom}}_{\mathcal{X}_{\widetilde{\varphi}(n+1)}}(\Phi_{\widetilde{\varphi}(n+1)} Y, \Gamma_{\widetilde{\varphi}} \Phi_{\widetilde{\varphi}(0)} X) \tag{2.6.4}$$

in $\mathcal{S}\mathsf{p}$ which is the image of the morphism (2.6.3) under the functor (2.6.2).

- If $i = 0$, then $\varphi(n) = \widetilde{\varphi}(n+1)$ and the morphism (2.6.4) is obtained by postcomposition with the morphism

$$\begin{tikzcd} \Gamma_{\varphi}\Phi_{\varphi(0)}X := \Gamma^{\varphi(n-1)}_{\varphi(n)} \cdots \Gamma^{\varphi(0)}_{\varphi(1)}\Phi_{\varphi(0)}X = \Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)} \cdots \Gamma^{\widetilde{\varphi}(1)}_{\widetilde{\varphi}(2)}\Phi_{\widetilde{\varphi}(1)}X \arrow[d, "\eta_{\widetilde{\varphi}(0)}"] \\ \Gamma_{\widetilde{\varphi}}\Phi_{\widetilde{\varphi}(0)}X := \Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)} \cdots \Gamma^{\widetilde{\varphi}(1)}_{\widetilde{\varphi}(2)}\Phi_{\widetilde{\varphi}(1)}L_{\widetilde{\varphi}(0)}X \end{tikzcd}$$

in $\mathcal{X}_{\varphi(n)} = \mathcal{X}_{\widetilde{\varphi}(n+1)}$.

- If $1 \leq i \leq n$, then $\varphi(n) = \widetilde{\varphi}(n+1)$ and the morphism (2.6.4) is obtained by postcomposition with the morphism

$$\begin{tikzcd} \Gamma_{\varphi}\Phi_{\varphi(0)}X := \Gamma^{\varphi(n-1)}_{\varphi(n)} \cdots \Gamma^{\varphi(0)}_{\varphi(1)}\Phi_{\varphi(0)}X = \Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)} \cdots \Gamma^{\widetilde{\varphi}(i-1)}_{\widetilde{\varphi}(i+1)} \cdots \Gamma^{\widetilde{\varphi}(0)}_{\widetilde{\varphi}(1)}\Phi_{\widetilde{\varphi}(0)}X \arrow[d, "\eta_{\widetilde{\varphi}(i)}"] \\ \Gamma_{\widetilde{\varphi}}\Phi_{\widetilde{\varphi}(0)}X := \Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)} \cdots \Gamma^{\widetilde{\varphi}(i)}_{\widetilde{\varphi}(i+1)}\Gamma^{\widetilde{\varphi}(i-1)}_{\widetilde{\varphi}(i)} \cdots \Gamma^{\widetilde{\varphi}(0)}_{\widetilde{\varphi}(1)}\Phi_{\widetilde{\varphi}(0)}X \end{tikzcd}$$

in $\mathcal{X}_{\varphi(n)} = \mathcal{X}_{\widetilde{\varphi}(n+1)}$.

- If $i = n+1$, then the morphism (2.6.4) is the composite

$$\begin{tikzcd} \underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)}Y, \Gamma_{\varphi}\Phi_{\varphi(0)}X) = \underline{\mathsf{hom}}_{\mathcal{X}_{\widetilde{\varphi}(n)}}(\Phi_{\widetilde{\varphi}(n)}Y, \Gamma_{\widetilde{\varphi}|_{[n]}}\Phi_{\widetilde{\varphi}(0)}X) \arrow[d, "\Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)}"] \\ \underline{\mathsf{hom}}_{\mathcal{X}_{\widetilde{\varphi}(n+1)}}(\Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)}\Phi_{\widetilde{\varphi}(n)}Y, \Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)}\Gamma_{\widetilde{\varphi}|_{[n]}}\Phi_{\widetilde{\varphi}(0)}X) \arrow[d, equal] \\ \underline{\mathsf{hom}}_{\mathcal{X}_{\widetilde{\varphi}(n+1)}}(\Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)}\Phi_{\widetilde{\varphi}(n)}Y, \Gamma_{\widetilde{\varphi}}\Phi_{\widetilde{\varphi}(0)}X) \arrow[d, "\eta_{\widetilde{\varphi}(n)}"] \\ \underline{\mathsf{hom}}_{\mathcal{X}_{\widetilde{\varphi}(n+1)}}(\Phi_{\widetilde{\varphi}(n+1)}Y, \Gamma_{\widetilde{\varphi}}\Phi_{\widetilde{\varphi}(0)}X) \end{tikzcd}$$

in which the first morphism is obtained by applying the functor

$$\mathcal{X}_{\widetilde{\varphi}(n)} \xrightarrow{\Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)}} \mathcal{X}_{\widetilde{\varphi}(n+1)}$$

and the second morphism is obtained by precomposing with the morphism

$$\Phi_{\widetilde{\varphi}(n+1)}Y \xrightarrow{\eta_{\widetilde{\varphi}(n)}} \Phi_{\widetilde{\varphi}(n+1)}L_{\widetilde{\varphi}(n)}Y = \Gamma^{\widetilde{\varphi}(n)}_{\widetilde{\varphi}(n+1)}\Phi_{\widetilde{\varphi}(n)}Y \ .$$

2.7. **Strict objects.** In this brief subsection, we lay out the general theory of strict objects.

Local Notation 2.7.1. In this subsection, we fix a stratification $\mathcal{Z}_\bullet$ of $\mathcal{X}$ over P.

Definition 2.7.2.

(1) We say that $X \in \mathcal{X}$ is ***convergent*** if its microcosm morphism
$$X \longrightarrow \mathsf{glue}(X) := \lim_{\mathsf{sd}(\mathsf{P})}(g(X))$$
is an equivalence.

(2) We say that
$$F \in \mathsf{Glue}(\mathcal{X}) \subseteq \mathsf{Fun}(\mathsf{sd}(\mathsf{P}), \mathcal{X})$$
is ***strict*** if it carries every isominmax morphism in $\mathsf{sd}(\mathsf{P})$ (Definition A.4.3) to an equivalence in $\mathcal{X}$.

(3) We say that $X \in \mathcal{X}$ is ***strict*** if it is convergent and moreover its gluing diagram $g(X) \in \mathsf{Glue}(\mathcal{X})$ is strict.

Lemma 2.7.3. *The functor*
$$\begin{array}{ccc} \mathsf{sd}(\mathsf{P}) & \xrightarrow{(\min\to\max)} & \mathsf{TwAr}(\mathsf{P}) \\ \cup & & \cup \\ ([n] \xrightarrow{\varphi} \mathsf{P}) & \longmapsto & (\varphi(0) \to \varphi(n)) \end{array}$$
witnesses $\mathsf{TwAr}(\mathsf{P})$ *as the localization of* $\mathsf{sd}(\mathsf{P})$ *with respect to the isominmax morphisms.*

Proof. Let us write $\mathbf{W} \subseteq \mathsf{sd}(\mathsf{P})$ for the subcategory on the isominmax morphisms, and for any $\mathcal{K} \in \mathsf{Cat}$ let us write $\mathsf{Fun}(\mathcal{K}, \mathsf{sd}(\mathsf{P}))^{\mathbf{W}} \subseteq \mathsf{Fun}(\mathcal{K}, \mathsf{sd}(\mathsf{P}))$ for the subcategory on the natural transformations that are componentwise in $\mathbf{W}$. By [MG19, Theorem 3.8], it suffices to show that for every $n \geq 0$ the evident factorization
$$\begin{tikzcd} \mathsf{Fun}([n], \mathsf{sd}(\mathsf{P})) \arrow[r, "{\mathsf{Fun}([n], (\min\to\max))}"] & \mathsf{Fun}([n], \mathsf{TwAr}(\mathsf{P})) \\ \mathsf{Fun}([n], \mathsf{sd}(\mathsf{P}))^{\mathbf{W}} \arrow[u, hook] \arrow[r, dashed] & \mathsf{hom}_{\mathsf{Cat}}([n], \mathsf{TwAr}(\mathsf{P})) \arrow[u, hook] \end{tikzcd}$$
is an ∞-groupoid completion, which follows from the observation that it admits a fully faithful left adjoint. $\square$

Observation 2.7.4. By Lemma 2.7.3, an object $F \in \mathsf{Glue}(\mathcal{X}) \subseteq \mathsf{Fun}(\mathsf{sd}(\mathsf{P}), \mathcal{X})$ is strict if and only if it admits a factorization
$$\begin{tikzcd} \mathsf{sd}(\mathsf{P}) \arrow[r, "F"] \arrow[d, "(\min\to\max)"'] & \mathcal{X} \\ \mathsf{TwAr}(\mathsf{P}) \arrow[ur, dashed, "F"'] \end{tikzcd}$$
(for which we use the same notation), in which case because localizations are initial we have a canonical equivalence
$$\lim_{\mathsf{sd}(\mathsf{P})}(F) \xleftarrow{\sim} \lim_{\mathsf{TwAr}(\mathsf{P})}(F) .$$
In particular, if $X \in \mathcal{X}$ is strict, then we have a canonical equivalence
$$X \xrightarrow{\sim} \lim_{\mathsf{TwAr}(\mathsf{P})}(g(X))$$
and for any $Y \in \mathcal{X}$ the nanocosm morphism reduces to an equivalence
$$\underline{\mathsf{hom}}_{\mathcal{X}}(Y, X) \xrightarrow{\sim} \lim_{(p\to q)\in\mathsf{TwAr}(\mathsf{P})} \underline{\mathsf{hom}}_{\mathcal{X}_q}(\Phi_q Y, \Gamma^p_q \Phi_p X) .$$

3. Fundamental operations

In this section, we establish our fundamental operations on stratifications. Towards this end, we first study certain fundamental operations on closed subcategories. In particular, we introduce and study the notion of one closed subcategory being *aligned* with another. The notion of alignment allows us to state our fundamental operations on stratifications in greater generality than is done in §1 as Theorem B, while at the same time streamlining their proofs. The assertions of Theorem B are recovered as a consequence of the fact that any two closed subcategories determined by a stratification are mutually aligned (Lemma 3.4.5).

This section is organized as follows.

§3.1: We introduce the notion of alignment and study its basic consequences.

§3.2: We establish a number of fundamental operations on aligned subcategories.

§3.3: We establish excision- and Mayer–Vietoris-type gluing results for closed subcategories in the presence of alignment.

§3.4: We prove our suite of fundamental operations on stratifications.

Local Notation 3.0.1. In this section, we fix a presentable stable ∞-category $\mathcal{X}$.

3.1. Alignment.

In this subsection, we introduce the notion of alignment between closed subcategories and study its basic consequences. We also give an alternative characterization of alignment as Lemma 3.1.7.

Local Notation 3.1.1. In this subsection, we fix two closed subcategories $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$.

Definition 3.1.2. We say that $\mathcal{Z}$ is ***aligned*** with $\mathcal{Y}$ if there exists a factorization

$$\begin{tikzcd} \mathcal{Y} \cap \mathcal{Z} \arrow[r, hook] & \mathcal{Y} \\ \mathcal{Z} \arrow[u, dashed] \arrow[r, hook, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd}$$

through the intersection (with both the intersection and the factorization considered in Cat). To indicate that $\mathcal{Z}$ is aligned with $\mathcal{Y}$, we write either $\mathcal{Z} \rightsquigarrow \mathcal{Y}$ or $\mathcal{Y} \leftsquigarrow \mathcal{Z}$. We say that $\mathcal{Y}$ and $\mathcal{Z}$ are ***mutually aligned*** if $\mathcal{Y}$ is aligned with $\mathcal{Z}$ and $\mathcal{Z}$ is aligned with $\mathcal{Y}$, and in this case we write $\mathcal{Y} \leftrightsquigarrow \mathcal{Z}$. We write

$$\begin{tikzcd} \mathbf{Cls}_{\mathcal{X}}^{\leftrightsquigarrow \mathcal{Y}} \arrow[r, hook] \arrow[d, hook] & \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}} := \{\mathcal{W} \in \mathbf{Cls}_{\mathcal{X}} : \mathcal{W} \rightsquigarrow \mathcal{Y}\} \arrow[d, hook] \\ \{\mathcal{W} \in \mathbf{Cls}_{\mathcal{X}} : \mathcal{W} \leftsquigarrow \mathcal{Y}\} =: \mathbf{Cls}_{\mathcal{X}}^{\leftsquigarrow \mathcal{Y}} \arrow[r, hook] & \mathbf{Cls}_{\mathcal{X}} \end{tikzcd}$$

for the evident pullback diagram among full subposets of $\mathbf{Cls}_{\mathcal{X}}$.

Example 3.1.3. The diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{X} \arrow[l, shift left, "y"] \arrow[r, shift right, "y"'] \arrow[r, phantom, "\perp"] & \mathcal{Z} \arrow[l, hook', shift right, "i_L"'] \\ \mathcal{S}\mathsf{p} \arrow[u, phantom, "\wr\wr"] \arrow[r, hook, shift left, "E \mapsto (0 \to E)"] \arrow[r, phantom, "\perp"] & \mathsf{Fun}([1], \mathcal{S}\mathsf{p}) \arrow[u, phantom, "\wr\wr"] \arrow[l, shift left, "\mathsf{ev}_1"] \arrow[r, shift right, "\mathsf{fib}"'] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p} \arrow[u, phantom, "\wr\wr"] \arrow[l, hook', shift right, "(E \to 0) \leftarrow E"'] \end{tikzcd}$$

depicts the $i_L \dashv y$ adjunctions of two closed subcategories $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$. Note that $\mathcal{Y} \cap \mathcal{Z} = 0$. The composite

$$\mathcal{Z} \xhookrightarrow{i_L} \mathcal{X} \xrightarrow{y} \mathcal{Y}$$

is zero, and so $\mathcal{Z}$ is aligned with $\mathcal{Y}$. On the other hand, the composite

$$\mathcal{Y} \xhookrightarrow{i_L} \mathcal{X} \xrightarrow{y} \mathcal{Z}$$

is given by desuspension, and so $\mathcal{Y}$ is not aligned with $\mathcal{Z}$.

Observation 3.1.4. If either $\mathcal{Y} \subseteq \mathcal{Z}$ or $\mathcal{Y} \supseteq \mathcal{Z}$, then $\mathcal{Y}$ and $\mathcal{Z}$ are mutually aligned.

Observation 3.1.5. The pullback diagram

$$\begin{tikzcd} \mathcal{Y} \cap \mathcal{Z} \arrow[r, hook] \arrow[d, hook] & \mathcal{Y} \arrow[d, hook, "i_L"] \\ \mathcal{Z} \arrow[r, hook, "i_L"'] & \mathcal{X} \end{tikzcd} \tag{3.1.1}$$

lies in $\mathsf{Pr}^L_{\mathsf{st}} \subset \mathsf{Cat}$. We use this fact without further comment.

Local Notation 3.1.6. In this subsection, we use the notation

$$\begin{tikzcd} \mathcal{Y} \cap \mathcal{Z} \arrow[r, hook, "i_{\mathcal{Y}}"] \arrow[d, hook, "i_{\mathcal{Z}}"'] & \mathcal{Y} \arrow[d, hook, "i_L"] \\ \mathcal{Z} \arrow[r, hook, "i_L"'] & \mathcal{X} \end{tikzcd} \tag{3.1.2}$$

for the commutative square (3.1.1) of left adjoints, and we use the notation

$$\begin{tikzcd} \mathcal{Y} \cap \mathcal{Z} & \mathcal{Y} \arrow[l, "i^R_{\mathcal{Y}}"'] \\ \mathcal{Z} \arrow[u, "i^R_{\mathcal{Z}}"] & \mathcal{X} \arrow[l, "y"] \arrow[u, "y"'] \end{tikzcd} \tag{3.1.3}$$

for its corresponding commutative square of right adjoints.

Lemma 3.1.7. *The following are equivalent.*

(1) There exists a factorization

$$\begin{tikzcd} \mathcal{Y} \cap \mathcal{Z} \arrow[r, hook, "i_{\mathcal{Y}}"] & \mathcal{Y} \\ \mathcal{Z} \arrow[u, dashed, "y'"] \arrow[r, hook, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} ,$$

i.e. $\mathcal{Z}$ is aligned with $\mathcal{Y}$.

(2) The morphism

$$i_{\mathcal{Y}} i^R_{\mathcal{Y}} y i_L \xrightarrow{\varepsilon} y i_L \tag{3.1.4}$$

in $\mathsf{Fun}(\mathcal{Z}, \mathcal{Y})$ is an equivalence.

(3) The morphism

$$y i_L i_{\mathcal{Z}} i^R_{\mathcal{Z}} \xrightarrow{\varepsilon} y i_L \tag{3.1.5}$$

in $\mathsf{Fun}(\mathcal{Z}, \mathcal{Y})$ is an equivalence.

(4) The lax-commutative square

$$\begin{tikzcd} \mathcal{Y} \cap \mathcal{Z} \arrow[r, hook, "i_{\mathcal{Y}}"] & \mathcal{Y} \\ \mathcal{Z} \arrow[u, "i^R_{\mathcal{Z}}"] \arrow[r, hook, "i_L"'] \arrow[ur, phantom, "\Searrow"] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} \tag{3.1.6}$$

determined by either commutative square (3.1.2) *or* (3.1.3) *commutes.*

Moreover, if these equivalent conditions are satisfied, then the factorization y' admits canonical identifications

$$i_{\mathcal{Y}}^R y i_L \simeq y' \simeq i_{\mathcal{Z}}^R \ .$$

Proof. We begin by proving the diagram of implications

$$\begin{tikzcd} (1) \arrow[r, Rightarrow] & (2) \arrow[d, Leftrightarrow] \\ (3) \arrow[r, Leftrightarrow] & (4) \arrow[ul, Rightarrow] \end{tikzcd} .$$

- Given a factorization y', we obtain an identification

$$\begin{tikzcd} i_{\mathcal{Y}} i_{\mathcal{Y}}^R y i_L \arrow[r, "\varepsilon"] \arrow[d, phantom, "\wr"] & y i_L \arrow[d, phantom, "\wr"] \\ i_{\mathcal{Y}} i_{\mathcal{Y}}^R i_{\mathcal{Y}} y' \arrow[r, "\sim", "\varepsilon"'] & i_{\mathcal{Y}} y' \end{tikzcd}$$

  among morphisms in $\mathsf{Fun}(\mathcal{Z}, \mathcal{Y})$. This proves that (1) $\Rightarrow$ (2).

- Trivially, (4) $\Rightarrow$ (1).

- Considering the lax-commutative square (3.1.6) as being determined by the commutative square (3.1.3), its natural transformation is the composite $i_{\mathcal{Y}} i_{\mathcal{Z}}^R \underset{\sim}{\xrightarrow{\eta}} i_{\mathcal{Y}} i_{\mathcal{Z}}^R y i_L \simeq i_{\mathcal{Y}} i_{\mathcal{Y}}^R y i_L \xrightarrow{\varepsilon} y i_L$. This proves that (2) $\Leftrightarrow$ (4).

- Considering the lax-commutative square (3.1.6) as being determined by the commutative square (3.1.2), its natural transformation is the composite $i_{\mathcal{Y}} i_{\mathcal{Z}}^R \underset{\sim}{\xrightarrow{\eta}} y i_L i_{\mathcal{Y}} i_{\mathcal{Z}}^R \simeq y i_L i_{\mathcal{Z}} i_{\mathcal{Z}}^R \xrightarrow{\varepsilon} y i_L$. This proves that (3) $\Leftrightarrow$ (4).

We now conclude by observing that if (2) holds then setting $y' := i_{\mathcal{Y}}^R y i_L$ defines a factorization. $\square$

3.2. **Fundamental operations on aligned subcategories.** In this subsection we undertake a deeper analysis of alignment, particularly regarding its interactions with colimits and intersections in $\mathbf{Cls}_{\mathcal{X}}$ as well as its with quotients of $\mathcal{X}$ by closed subcategories.

Local Notation 3.2.1. In this subsection, given two closed subcategories $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$ we continue to use the notation $i_{\mathcal{Y}}$, $i_{\mathcal{Z}}$, $i_{\mathcal{Y}}^R$, and $i_{\mathcal{Z}}^R$ of Local Notation 3.1.6.

Lemma 3.2.2. *For any closed subcategory $\mathcal{Y} \in \mathbf{Cls}_{\mathcal{X}}$, all four functors in the commutative square*

$$\begin{tikzcd} \mathbf{Cls}_{\mathcal{X}}^{\leftrightsquigarrow \mathcal{Y}} \arrow[r, hook] \arrow[d, hook] & \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}} \arrow[d, hook] \\ \mathbf{Cls}_{\mathcal{X}}^{\leftsquigarrow \mathcal{Y}} \arrow[r, hook] & \mathbf{Cls}_{\mathcal{X}} \end{tikzcd} \tag{3.2.1}$$

preserve colimits.

Proof. Since the commutative square (3.2.1) is a pullback among full subposets of $\mathbf{Cls}_{\mathcal{X}}$, it suffices to check that its right vertical functor and its lower horizontal functor both preserve colimits. We address each of these in turn.

Suppose first that we are given any $\{\mathcal{Z}_s \in \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}}\}_{s \in S}$, and let us write $\mathcal{Z} = \langle \mathcal{Z}_s \rangle_{s \in S} \in \mathbf{Cls}_{\mathcal{X}}$. For each $s \in S$, by assumption we have a factorization

$$\begin{tikzcd} \mathcal{Y} \cap \mathcal{Z}_s \arrow[r, hook] & \mathcal{Y} \cap \mathcal{Z} \arrow[r, hook] & \mathcal{Y} \\ \mathcal{Z}_s \arrow[u, dashed] \arrow[r, hook, "i_L"'] & \mathcal{Z} \arrow[r, hook, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} . \tag{3.2.2}$$

Because all solid functors in the diagram (3.2.2) preserve colimits, we find that $\mathcal{Z}$ is aligned with $\mathcal{Y}$, i.e. that $\mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}}$.

Suppose now that we are given any $\{\mathcal{Z}_s \in \mathbf{Cls}_{\mathcal{X}}^{\leftrightsquigarrow \mathcal{Y}}\}_{s \in S}$, and let us write $\mathcal{Z} = \langle \mathcal{Z}_s \rangle_{s\in S} \in \mathbf{Cls}_{\mathcal{X}}$. For an arbitrary element $s \in S$, consider the diagram

$$\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z}_s \arrow[r, hook, "i_{\mathcal{Z}_s}"] \arrow[dr, phantom, "\Searrow"] & \mathcal{Z}_s \\
\mathcal{Y} \cap \mathcal{Z} \arrow[u, "i^R"] \arrow[r, hook, "i_{\mathcal{Z}}"] \arrow[dr, phantom, "\Searrow"] & \mathcal{Z} \arrow[u, "y"'] \\
\mathcal{Y} \arrow[u, "i_{\mathcal{Y}}^R"] \arrow[r, hook, "i_L"'] & \mathcal{X} \arrow[u, "y"']
\end{tikzcd} , \tag{3.2.3}$$

in which the functor i^R is the evident right adjoint. By Lemma 3.1.7, to show that $\mathcal{Y}$ is aligned with $\mathcal{Z}$ it suffices to show that the lower natural transformation in diagram (3.2.3) is an equivalence. Also by Lemma 3.1.7, because $\mathcal{Y}$ is aligned with $\mathcal{Z}_s$, the composite natural transformation

$$i_{\mathcal{Z}_s} i^R i_{\mathcal{Y}}^R \longrightarrow y y i_L \tag{3.2.4}$$

in diagram (3.2.3) is an equivalence. This implies that the upper natural transformation in diagram (3.2.3) is also an equivalence, as it is given by the composite

$$i_{\mathcal{Z}_s} i^R \xrightarrow[\sim]{\eta} i_{\mathcal{Z}_s} i^R i_{\mathcal{Y}}^R i_{\mathcal{Y}} \xrightarrow[\sim]{(3.2.4)} y y i_L i_{\mathcal{Y}} \simeq y y i_L i_{\mathcal{Z}} \simeq y i_{\mathcal{Z}} \ .$$

So, the lower natural transformation in diagram (3.2.3) is indeed an equivalence, because the functors $\{\mathcal{Z} \xrightarrow{y} \mathcal{Z}_s\}_{s\in S}$ are jointly conservative. $\square$

Lemma 3.2.3. *Let $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$ be closed subcategories, and suppose that $\mathcal{Z}$ is aligned with $\mathcal{Y}$.*

(1) The functor $i_{\mathcal{Z}}$ is the inclusion of $\mathcal{Y} \cap \mathcal{Z}$ as a closed subcategory of $\mathcal{Z}$.

(2) Consider the resulting commutative diagram

$$\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z} \arrow[r, hook, "i_{\mathcal{Y}}"] \arrow[d, hook, "i_{\mathcal{Z}}"'] & \mathcal{Y} \arrow[d, hook, "i_L"] \\
\mathcal{Z} \arrow[r, hook, "i_L"] \arrow[d, "p_L"'] & \mathcal{X} \arrow[d, "p_L"] \\
\mathcal{Z}/(\mathcal{Y} \cap \mathcal{Z}) \arrow[r, dashed, "i"'] & \mathcal{X}/\mathcal{Y}
\end{tikzcd} \tag{3.2.5}$$

in $\mathsf{Pr}^L_{\mathsf{st}}$, in which i is the canonical morphism between presentable quotients.

(a) The functor i is the fully faithful inclusion of $\mathcal{Z}/(\mathcal{Y}\cap\mathcal{Z})$ as a closed subcategory of $\mathcal{X}/\mathcal{Y}$.

(b) The lax-commutative square

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, hook, "i_L"] \arrow[dr, phantom, "\Searrow"] & \mathcal{X} \\
\mathcal{Z}/(\mathcal{Z} \cap \mathcal{Y}) \arrow[u, hook, "\nu"] \arrow[r, "i"'] & \mathcal{X}/\mathcal{Y} \arrow[u, hook, "\nu"']
\end{tikzcd} \tag{3.2.6}$$

determined by the lower commutative square in diagram (3.2.5) *commutes.*

(c) Suppose further that $\mathcal{Y}$ *is aligned with* $\mathcal{Z}$*. Then, the lax-commutative square*

$$\begin{tikzcd} \mathcal{Z} \arrow[d, "p_L"'] & \mathcal{X} \arrow[l, "y"'] \arrow[d, "p_L"] \arrow[dl, phantom, "\Searrow"] \\ \mathcal{Z}/(\mathcal{Y}\cap\mathcal{Z}) & \mathcal{X}/\mathcal{Y} \arrow[l, "i^R"] \end{tikzcd} \tag{3.2.7}$$

determined by the lower commutative square in diagram (3.2.5) *commutes.*

Proof. We begin by proving part (1). Because $i_{\mathcal{Z}}$ is fully faithful, it remains to show that its right adjoint $i_{\mathcal{Z}}^R$ preserves colimits. For this, because $i_{\mathcal{Y}}$ is fully faithful and colimit-preserving, it suffices to show that the composite $i_{\mathcal{Y}} i_{\mathcal{Z}}^R$ preserves colimits, which follows from the equivalence $i_{\mathcal{Y}} i_{\mathcal{Z}}^R \simeq y i_L$ guaranteed by Lemma 3.1.7.

We now prove part (2)(b). By definition, the natural transformation in the lax-commutative square (3.2.6) is the composite $i_L\nu \xrightarrow{\eta} \nu p_L i_L \nu \simeq \nu i p_L \nu \underset{\sim}{\xrightarrow{\varepsilon}} \nu i$. To show that it is an equivalence is therefore equivalent to showing that the composite functor

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, "i_L"] & \mathcal{X} \\ \mathcal{Z}/(\mathcal{Y}\cap\mathcal{Z}) \arrow[u, hook, "\nu"] & \end{tikzcd}$$

lands in the image of the functor

$$\begin{tikzcd} \mathcal{X} \\ \mathcal{X}/\mathcal{Y} \arrow[u, hook, "\nu"'] \end{tikzcd} .$$

This is equivalent to showing that the composite functor

$$\begin{tikzcd} & \mathcal{Y} \\ \mathcal{Z} \arrow[r, hook, "i_L"] & \mathcal{X} \arrow[u, "y"'] \\ \mathcal{Z}/(\mathcal{Y}\cap\mathcal{Z}) \arrow[u, hook, "\nu"] & \end{tikzcd}$$

is zero. This follows from the commutativity of the diagram

$$\begin{tikzcd} \mathcal{Y}\cap\mathcal{Z} \arrow[r, hook, "i_{\mathcal{Y}}"] & \mathcal{Y} \\ \mathcal{Z} \arrow[u, "i_{\mathcal{Z}}^R"] \arrow[r, hook, "i_L"] & \mathcal{X} \arrow[u, "y"'] \\ \mathcal{Z}/(\mathcal{Y}\cap\mathcal{Z}) \arrow[u, hook, "\nu"] & \end{tikzcd}$$

guaranteed by Lemma 3.1.7, because its left vertical composite is zero. So indeed, the lax-commutative square (3.2.6) is commutative.

We now prove part (2)(a). By part (2)(b), the functor i is fully faithful. Note too that by definition i is colimit-preserving. Passing to right adjoints in the lower commutative square in diagram (3.2.5), we obtain a commutative square

$$\begin{tikzcd} \mathcal{Z} & \mathcal{X} \arrow[l, "y"'] \\ \mathcal{Z}/(\mathcal{Y}\cap\mathcal{Z}) \arrow[u, hook, "\nu"] & \mathcal{X}/\mathcal{Y} \arrow[l, "i^R"] \arrow[u, hook, "\nu"'] \end{tikzcd} ,$$

which implies that i^R is colimit-preserving. So indeed, i is the inclusion of a closed subcategory.

We now conclude by proving part (2)(c). By Lemma 3.1.7 (with the roles of $\mathcal{Y}$ and $\mathcal{Z}$ reversed), we have a commutative diagram

$$\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z} \arrow[d, hook, "i_{\mathcal{Z}}"'] & \mathcal{Y} \arrow[l, "i_{\mathcal{Y}}^R"'] \arrow[d, hook, "i_L"] \\
\mathcal{Z} \arrow[d, "p_L"'] & \mathcal{X} \arrow[l, "y"'] \arrow[d, "p_L"] \\
\mathcal{Z}/(\mathcal{Y} \cap \mathcal{Z}) & \mathcal{X}/\mathcal{Y} \arrow[l, dashed, "j"]
\end{tikzcd} \tag{3.2.8}$$

in $\mathsf{Pr}^L_{\mathsf{st}}$, in which j is the canonical morphism between presentable quotients. Hence, j fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[d, "p_L"'] & \mathcal{X} \arrow[l, "y"'] \\
\mathcal{Z}/(\mathcal{Y} \cap \mathcal{Z}) & \mathcal{X}/\mathcal{Y} \arrow[l, "j"] \arrow[u, hook, "\nu"']
\end{tikzcd} .$$

On the other hand, note the commutative square

$$\begin{tikzcd}
\mathcal{Z} & \mathcal{X} \arrow[l, "y"'] \\
\mathcal{Z}/(\mathcal{Y} \cap \mathcal{Z}) \arrow[u, hook, "\nu"] & \mathcal{X}/\mathcal{Y} \arrow[l, "j"] \arrow[u, hook, "\nu"']
\end{tikzcd}$$

obtained by passing to right adjoints in the lower commutative square of diagram (3.2.5). Using this, we obtain the identification $j \simeq p_L y \nu \simeq p_L \nu i^R \simeq i_R$. Thereafter, we see that indeed the lax-commutative square (3.2.7) is precisely the lower commutative square in diagram (3.2.8). $\square$

Observation 3.2.4. Let $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$ be closed subcategories, and suppose that $\mathcal{Z}$ is aligned with $\mathcal{Y}$. By Lemma 3.2.3(1), we have $(\mathcal{Y} \cap \mathcal{Z}) \in \mathbf{Cls}_{\mathcal{Z}}$. It follows that $(\mathcal{Y} \cap \mathcal{Z}) \in \mathbf{Cls}_{\mathcal{X}}$, and thereafter that $(\mathcal{Y} \cap \mathcal{Z}) \in \mathbf{Cls}_{\mathcal{Y}}$. In other words, all four functors in the pullback diagram

$$\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z} \arrow[r, hook, "i_{\mathcal{Y}}"] \arrow[d, hook, "i_{\mathcal{Z}}"'] & \mathcal{Y} \arrow[d, hook, "i_L"] \\
\mathcal{Z} \arrow[r, hook, "i_L"'] & \mathcal{X}
\end{tikzcd} \tag{3.2.9}$$

are inclusions of closed subcategories.

Lemma 3.2.5. *Let $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$ be closed subcategories, and suppose that $\mathcal{Z}$ is aligned with $\mathcal{Y}$. Then, the commutative square*

$$\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z} \arrow[r, hook, "i_R"] \arrow[d, hook, "i_R"'] & \mathcal{Y} \arrow[d, hook, "i_R"] \\
\mathcal{Z} \arrow[r, hook, "i_R"'] & \mathcal{X}
\end{tikzcd} \tag{3.2.10}$$

in Cat *obtained by taking right adjoints twice in the commutative square* (3.2.9) *in* Cat *(which is possible by Observation 3.2.4) is a pullback square.*

Proof. By Lemma 3.1.7, the square

$$\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z} \arrow[r, hook, "i_L"] & \mathcal{Y} \\
\mathcal{Z} \arrow[r, hook, "i_L"'] \arrow[u, "y"] & \mathcal{X} \arrow[u, "y"']
\end{tikzcd}$$

commutes, which implies that the square

$$
\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z} \arrow[d, hook, "i_R"'] & \mathcal{Y} \arrow[l, "y"'] \arrow[d, hook, "i_R"] \\
\mathcal{Z} & \mathcal{X} \arrow[l, "y"]
\end{tikzcd}
\tag{3.2.11}
$$

commutes by passing to right adjoints. Now, consider the solid commutative diagram

$$
\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z} \arrow[dr, shift left] \arrow[drr, hook, bend left, "i_R"] \arrow[ddr, hook, bend right, "i_R"'] & & \\
& \mathcal{Y} \cap_R \mathcal{Z} \arrow[r, hook, "j_{\mathcal{Y}}"'] \arrow[d, hook, "j_{\mathcal{Z}}"] \arrow[ul, dashed, shift left] & \mathcal{Y} \arrow[d, hook, "i_R"] \\
& \mathcal{Z} \arrow[r, hook, "i_R"'] & \mathcal{X}
\end{tikzcd}
\tag{3.2.12}
$$

in which $\mathcal{Y} \cap_R \mathcal{Z}$ denotes the pullback in Cat. Because both functors to $\mathcal{Z}$ in diagram (3.2.12) are fully faithful, it suffices to show that there exists the dashed factorization of $j_{\mathcal{Z}}$. This follows from the sequence of equivalences

$$
j_{\mathcal{Z}} \simeq y i_R j_{\mathcal{Z}} \simeq y i_R j_{\mathcal{Y}} \simeq i_R y j_{\mathcal{Y}} \ ,
$$

in which the last equivalence follows from the commutativity of the square (3.2.11). □

Remark 3.2.6. By Observation 3.2.4, two closed subcategories $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$ are mutually aligned if and only if the diagram

$$
\begin{tikzcd}
\mathcal{Y} \cap \mathcal{Z} \arrow[r, hook, "i_L"] \arrow[d, hook, "i_L"'] & \mathcal{Y} \arrow[d, hook, "i_L"] \\
\mathcal{Z} \arrow[r, hook, "i_L"'] & \mathcal{X}
\end{tikzcd}
$$

defines a stratification of $\mathcal{X}$ over $[1] \times [1]$.

Remark 3.2.7. Given closed subcategories $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$, the most important consequence of $\mathcal{Z}$ being aligned with $\mathcal{Y}$ is that the image of the composite

$$
\mathcal{Z} \overset{i_L}{\hookrightarrow} \mathcal{X} \xrightarrow{p_L} \mathcal{X}/\mathcal{Y}
$$

is a closed subcategory, as guaranteed by Lemma 3.2.3(2)(a). This need not hold if $\mathcal{Z}$ is not aligned with $\mathcal{Y}$. We may see this as follows.

Let us take $\mathcal{X} := \mathsf{Fun}(\mathcal{I}, \mathcal{S}\mathsf{p})$, where $\mathcal{I}$ denotes the category generated by the quiver

$$
\begin{tikzcd}
u \arrow[r, bend left, "a"] & v \arrow[l, bend left, "b"]
\end{tikzcd} \quad ,
$$

i.e. the pushout

$$
\mathcal{I} := \mathsf{colim} \left(\begin{tikzcd}
\mathsf{pt} \sqcup \mathsf{pt} \arrow[r, "{(0,1)}"] \arrow[d, "{(1,0)}"'] & {[1]} \\
{[1]} &
\end{tikzcd} \right) .
$$

Consider the full subcategories

$$
\mathcal{Y} := \{E_\bullet \in \mathcal{X} : E_u \simeq 0\} \subseteq \mathcal{X} \supseteq \{E_\bullet \in \mathcal{X} : E_v \simeq 0\} =: \mathcal{Z} \ .
$$

They are clearly closed under colimits. Moreover, via the identifications

$$
\mathcal{Y} \underset{\sim}{\xrightarrow{\mathsf{ev}_v}} \mathcal{S}\mathsf{p} \underset{\sim}{\xleftarrow{\mathsf{ev}_u}} \mathcal{Z} \ ,
$$

their inclusions' right adjoints are given by the formulas

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{X} \arrow[l, dashed, shift left, "y"] \arrow[r, dashed, shift right, "y"'] \arrow[r, phantom, "\perp"] & \mathcal{Z} \arrow[l, hook', shift right, "i_L"'] \\ \mathrm{fib}(E_b) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & E_\bullet \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[l, mapsto] \arrow[r, mapsto] & \mathrm{fib}(E_a) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd} ,$$

which preserve colimits so that $\mathcal{Y}$ and $\mathcal{Z}$ are indeed closed subcategories of $\mathcal{X}$ (which justifies the notations i_L and y). Note that $\mathcal{Y} \cap \mathcal{Z} \simeq 0$. On the other hand, the composite functors

$$\mathcal{Y} \xhookrightarrow{i_L} \mathcal{X} \xrightarrow{y} \mathcal{Z} \qquad \text{and} \qquad \mathcal{Y} \xleftarrow{y} \mathcal{X} \xhookleftarrow{i_L} \mathcal{Z}$$

may both be identified with desuspension, and in particular are equivalences. So, $\mathcal{Y}$ is not aligned with $\mathcal{Z}$ and $\mathcal{Z}$ is not aligned with $\mathcal{Y}$.

Now, observe the identification

$$\begin{tikzcd} \mathcal{X} \arrow[d, phantom, "\rotatebox{-90}{$:=$}"] & & \mathcal{X}/\mathcal{Y} \arrow[ll, hook', "\nu"'] \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] \\ \mathsf{Fun}(\mathcal{I}, \mathcal{S}\mathsf{p}) & & \{E_\bullet \in \mathcal{X} : E_b \text{ is an equivalence}\} \arrow[ll, hook'] \end{tikzcd} ,$$

and thereafter the identification of its left adjoint $\mathcal{X} \xrightarrow{p_L} \mathcal{X}/\mathcal{Y}$ as the assignment

$$\left(\begin{tikzcd} E_u \arrow[r, bend left, "E_a"] & E_v \arrow[l, bend left, "E_b"] \end{tikzcd} \right) \longmapsto \mathsf{cofib}\left(\left(\begin{tikzcd} 0 \arrow[r, bend left] & \mathsf{fib}(E_b) \arrow[l, bend left] \end{tikzcd} \right) \longrightarrow \left(\begin{tikzcd} E_u \arrow[r, bend left, "E_a"] & E_v \arrow[l, bend left, "E_b"] \end{tikzcd} \right) \right) \simeq \left(\begin{tikzcd} E_u \arrow[r, bend left, "E_b E_a"] & E_u \arrow[l, bend left, "\sim", "\mathsf{id}"'] \end{tikzcd} \right) .$$

Hence, the composite

$$\mathcal{S}\mathsf{p} \underset{\sim}{\xleftarrow{\mathsf{ev}_u}} \mathcal{Z} \xhookrightarrow{i_L} \mathcal{X} \xrightarrow{p_L} \mathcal{X}/\mathcal{Y}$$

is given by the assignment

$$E \longmapsto \left(\begin{tikzcd} E \arrow[r, bend left, "0"] & E \arrow[l, bend left, "\sim", "\mathsf{id}"'] \end{tikzcd} \right) ,$$

and so its image is not even closed under colimits – nor does it define a fully faithful functor from $\mathcal{Z}/(\mathcal{Y} \cap \mathcal{Z}) \simeq \mathcal{Z}/0 \simeq \mathcal{Z}$ to $\mathcal{X}/\mathcal{Y}$.

Remark 3.2.8. In Lemma 3.2.3(2)(c), the lax-commutative square (3.2.7) need not commute if $\mathcal{Y}$ is not aligned with $\mathcal{Z}$. Indeed, in the situation of Example 3.1.3, it may be identified with the canonical lax-commutative square

$$\begin{tikzcd} \mathcal{S}\mathsf{p} \arrow[d, "\mathsf{id}"', "\wr"] \arrow[dr, phantom, "\Searrow"] & \mathsf{Fun}([1], \mathcal{S}\mathsf{p}) \arrow[l, "\mathsf{fib}"'] \arrow[d, "\mathsf{ev}_0"] \\ \mathcal{S}\mathsf{p} & \mathcal{S}\mathsf{p} \arrow[l, "\sim", "\mathsf{id}"'] \end{tikzcd} .$$

Proposition 3.2.9. *For any closed subcategory $\mathcal{Y} \in \mathbf{Cls}_{\mathcal{X}}$, taking the image or preimage (in Cat) of a closed subcategory along either functor in the composite*

$$\mathcal{Y} \xhookrightarrow{i_L} \mathcal{X} \xrightarrow{p_L} \mathcal{X}/\mathcal{Y}$$

yields a closed subcategory, and these constructions define adjunctions

$$\begin{tikzcd}
& (\mathbf{Cls}_{\mathcal{X}})_{/\mathcal{Y}} \arrow[d, hook] & \\
\mathbf{Cls}_{\mathcal{Y}} \arrow[ur, dashed, "\sim"] \arrow[r, hook, shift left=1.5, "i_L"] \arrow[r, phantom, "\perp"] & \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}} \arrow[l, shift left=1.5, "i_L^{-1}"] \arrow[r, shift left=1.5, "p_L"] \arrow[r, phantom, "\perp"] & \mathbf{Cls}_{\mathcal{X}/\mathcal{Y}} \arrow[l, hook', shift left=1.5, "p_L^{-1}"] \arrow[dl, dashed, "\sim"] \\
& (\mathbf{Cls}_{\mathcal{X}})_{\mathcal{Y}/} \arrow[u, hook] &
\end{tikzcd}$$

with fully faithful images as indicated.

Proof. It is immediate that i_L and p_L^{-1} respectively carry closed subcategories of $\mathcal{Y}$ and $\mathcal{X}/\mathcal{Y}$ to closed subcategories of $\mathcal{X}$, which are aligned with $\mathcal{Y}$ by Observation 3.1.4. Moreover, for any $\mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}}$, we have $i_L^{-1}(\mathcal{Z}) = (\mathcal{Y} \cap \mathcal{Z}) \in \mathbf{Cls}_{\mathcal{Y}}$ by Observation 3.2.4 and $p_L(\mathcal{Z}) = \mathcal{Z}/(\mathcal{Y} \cap \mathcal{Z}) \in \mathbf{Cls}_{\mathcal{X}/\mathcal{Y}}$ by Lemma 3.2.3(2)(a). The asserted co/reflective adjunctions among posets, as well as the identifications of the resulting fully faithful images, are now immediate. $\square$

Lemma 3.2.10. *Fix any closed subcategory $\mathcal{Y} \in \mathbf{Cls}_{\mathcal{X}}$.*

(1) The functor $\mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}} \xrightarrow{i_L^{-1}} \mathbf{Cls}_{\mathcal{Y}}$ preserves colimits.

(2) The functor $\mathbf{Cls}_{\mathcal{X}/\mathcal{Y}} \xrightarrow{p_L^{-1}} \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}}$ preserves nonempty colimits.

Proof. We first prove part (1). Let $\{\mathcal{Z}_s \in \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}}\}_{s \in S}$ be a set of closed subcategories of $\mathcal{X}$ that are aligned with $\mathcal{Y}$. We have an evident inclusion $\left\langle i_L^{-1}(\mathcal{Z}_s)\right\rangle_{s \in S} \subseteq i_L^{-1}(\langle \mathcal{Z}_s \rangle_{s \in S})$. On the other hand, because $\mathcal{X} \xrightarrow{y} \mathcal{Y}$ preserves colimits, we also have an inclusion $i_L^{-1}(\langle \mathcal{Z}_s \rangle_{s \in S}) \subseteq \left\langle i_L^{-1}(\mathcal{Z}_s)\right\rangle_{s \in S}$.

We now prove part (2). Let now $\{\mathcal{Z}_s \in \mathbf{Cls}_{\mathcal{X}/\mathcal{Y}}\}_{s \in S}$ be a nonempty set of closed subcategories of $\mathcal{X}/\mathcal{Y}$. We have an evident inclusion $\left\langle p_L^{-1}(\mathcal{Z}_s)\right\rangle_{s \in S} \subseteq p_L^{-1}(\langle \mathcal{Z}_s \rangle_{s \in S})$. On the other hand, for any $X \in p_L^{-1}(\langle \mathcal{Z}_s \rangle_{s \in S})$, consider the co/fiber sequence $i_L y X \to X \to \nu p_L X$. Because $i_L y X \in p_L^{-1}(0) \subseteq \left\langle p_L^{-1}(\mathcal{Z}_s)\right\rangle_{s \in S}$ (using that S is nonempty), to show that $X \in \left\langle p_L^{-1}(\mathcal{Z}_s)\right\rangle_{s \in S}$ it suffices to show that $\nu p_L X \in \left\langle p_L^{-1}(\mathcal{Z}_s)\right\rangle_{s \in S}$, which follows from the fact that $\mathcal{X}/\mathcal{Y} \overset{\nu}{\hookrightarrow} \mathcal{X}$ preserves colimits. $\square$

Observation 3.2.11. Fix any closed subcategory $\mathcal{Y} \in \mathbf{Cls}_{\mathcal{X}}$ and any $\mathcal{W} \in \mathbf{Cls}_{\mathcal{Y}} \subseteq \mathbf{Cls}_{\mathcal{X}}$. Then, by the equivalence (1) $\Leftrightarrow$ (2) of Lemma 3.1.7 there exists a factorization

$$\begin{tikzcd}
\mathbf{Cls}_{\mathcal{Y}} & \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}} \arrow[l, "i_L^{-1}"'] \\
\mathbf{Cls}_{\mathcal{Y}}^{\rightsquigarrow \mathcal{W}} \arrow[u, hook] & \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}} \cap \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{W}} \arrow[u, hook] \arrow[l, dashed]
\end{tikzcd} \quad ;$$

that is, if $\mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$ is aligned with both $\mathcal{Y}$ and $\mathcal{W}$ then $i_L^{-1}(\mathcal{Z}) := \mathcal{Y} \cap \mathcal{Z}$ is aligned with $\mathcal{W}$.

3.3. **Gluing aligned subcategories.** In this brief subsection, we establish gluing formulas for closed subcategories of $\mathcal{X}$ in the presence of alignment. More precisely, one may view Lemma 3.3.4 (which merely requires alignment) as an excision principle and Lemma 3.3.5 (which requires mutual alignment) as a Mayer–Vietoris principle.[99]

[99]Recall that closed subcategories of a presentable stable ∞-category correspond to open subsets of a topological space, as indicated in §1.8.

Local Notation 3.3.1. In this subsection, we fix closed subcategories $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$.

Remark 3.3.2. In this subsection, we implicitly use Observation 3.2.4 (that if $\mathcal{Z}$ is aligned with $\mathcal{Y}$ then $\mathcal{Y} \cap \mathcal{Z}$ is a closed subcategory of both $\mathcal{Y}$ and $\mathcal{Z}$).

Local Notation 3.3.3. Given co/reflective localizations

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{X} \arrow[l, shift left, "G"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{X} \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, hook, shift left, "G'"] \end{tikzcd}$$

of $\mathcal{X}$, we write

$$C_{\mathcal{C}} := FG \xrightarrow{\varepsilon_{\mathcal{C}}} \mathsf{id}_{\mathcal{X}} \qquad \text{and} \qquad \mathsf{id}_{\mathcal{X}} \xrightarrow{\eta_{\mathcal{C}}} G'F' =: L_{\mathcal{C}'}$$

for the corresponding co/monads on $\mathcal{X}$ and their co/unit maps.[100] In particular, given a closed subcategory $\mathcal{Y} \in \mathbf{Cls}_{\mathcal{X}}$ we obtain the endofunctors

$$C_{\mathcal{Y}} := i_L y \ , \qquad L_{\mathcal{Y}} := i_R y \ , \qquad L_{\mathcal{X}/\mathcal{Y}} := \nu p_L \ , \qquad \text{and} \qquad C_{\mathcal{X}/\mathcal{Y}} := \nu p_R$$

of $\mathcal{X}$.

Lemma 3.3.4. *Suppose that $\mathcal{Z}$ is aligned with $\mathcal{Y}$.*

(1) There is a canonical identification

$$C_{\langle \mathcal{Y}, \mathcal{Z} \rangle} \simeq \mathsf{cofib}(\Sigma^{-1} C_{\mathcal{Z}} L_{\mathcal{X}/\mathcal{Y}} \longrightarrow \Sigma^{-1} L_{\mathcal{X}/\mathcal{Y}} \longrightarrow C_{\mathcal{Y}}) \ .$$

(2) There is a canonical identification

$$L_{\langle \mathcal{Y}, \mathcal{Z} \rangle} \simeq \mathsf{fib}(L_{\mathcal{Z}} \longrightarrow \Sigma C_{\mathcal{X}/\mathcal{Z}} \longrightarrow \Sigma L_{\mathcal{Y}} C_{\mathcal{X}/\mathcal{Z}}) \ .$$

Proof. We begin with part (1). For this, consider the morphism

$$\begin{tikzcd} \Sigma^{-1} C_{\mathcal{Z}} L_{\mathcal{X}/\mathcal{Y}} \arrow[r] \arrow[d, "\Sigma^{-1}\varepsilon_{\mathcal{Z}} L_{\mathcal{X}/\mathcal{Y}}"'] & C_{\mathcal{Y}} \arrow[r] \arrow[d, equal] & \mathsf{cofib} \arrow[d] \\ \Sigma^{-1} L_{\mathcal{X}/\mathcal{Y}} \arrow[r] & C_{\mathcal{Y}} \arrow[r, "\varepsilon_{\mathcal{Y}}"'] & \mathsf{id}_{\mathcal{X}} \end{tikzcd} \tag{3.3.1}$$

of cofiber sequences in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{X}, \mathcal{X})$, where we simply write cofib for the indicated cofiber. It suffices to show that the right vertical morphism in diagram (3.3.1) becomes an equivalence after applying $C_{\mathcal{Y}}$ and $C_{\mathcal{Z}}$. It is clear that it becomes an equivalence after applying $C_{\mathcal{Z}}$. To see that it becomes an equivalence after applying $C_{\mathcal{Y}}$, it suffices to observe the containment

$$\mathsf{ker}(C_{\mathcal{Y}}) \subseteq \mathsf{ker}(C_{\mathcal{Y}} C_{\mathcal{Z}})$$

resulting from the fact that $\mathcal{Z}$ is aligned with $\mathcal{Y}$.

We now turn to part (2). For this, consider the morphism

$$\begin{tikzcd} \mathsf{id}_{\mathcal{X}} \arrow[r, "\eta_{\mathcal{Z}}"] \arrow[d] & L_{\mathcal{Z}} \arrow[r] \arrow[d, equal] & \Sigma C_{\mathcal{X}/\mathcal{Z}} \arrow[d, "\Sigma \eta_{\mathcal{Y}} C_{\mathcal{X}/\mathcal{Z}}"] \\ \mathsf{fib} \arrow[r] & L_{\mathcal{Z}} \arrow[r] & \Sigma L_{\mathcal{Y}} C_{\mathcal{X}/\mathcal{Z}} \end{tikzcd} \tag{3.3.2}$$

of cofiber sequences in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{X}, \mathcal{X})$, where we simply write fib for the indicated fiber. It suffices to show that the left vertical morphism in diagram (3.3.2) becomes an equivalence after applying $L_{\mathcal{Y}}$ and $L_{\mathcal{Z}}$. It is clear that it becomes an equivalence after applying $L_{\mathcal{Y}}$. To see that it becomes an equivalence after applying $L_{\mathcal{Z}}$, it suffices to observe the containment

$$\mathsf{ker}(L_{\mathcal{Z}}) \subseteq \mathsf{ker}(L_{\mathcal{Z}} L_{\mathcal{Y}})$$

resulting from the fact that $\mathcal{Z}$ is aligned with $\mathcal{Y}$. □

Lemma 3.3.5. *Suppose that $\mathcal{Y}$ and $\mathcal{Z}$ are mutually aligned.*

[100] So, in the notation of Definition 2.4.6 we simply write $L_{\mathsf{C}} := L_{\mathcal{X}_{\mathsf{C}}}$.

(1) The commutative square

$$\begin{tikzcd} C_{\mathcal{Y}\cap\mathcal{Z}} \arrow[r] \arrow[d] & C_{\mathcal{Y}} \arrow[d] \\ C_{\mathcal{Z}} \arrow[r] & C_{\langle \mathcal{Y},\mathcal{Z}\rangle} \end{tikzcd} \tag{3.3.3}$$

in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{X},\mathcal{X})$ *is a pushout.*

(2) The commutative square

$$\begin{tikzcd} L_{\mathcal{Y}\cap\mathcal{Z}} & L_{\mathcal{Y}} \arrow[l] \\ L_{\mathcal{Z}} \arrow[u] & L_{\langle \mathcal{Y},\mathcal{Z}\rangle} \arrow[l] \arrow[u] \end{tikzcd} \tag{3.3.4}$$

in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{X},\mathcal{X})$ *is a pullback.*

Proof. We begin with part (1). It suffices to show that the square (3.3.3) becomes a pushout after applying $C_{\mathcal{Y}}$ and $C_{\mathcal{Z}}$.

- Applying $C_{\mathcal{Y}}$ to the square (3.3.3), we see that both vertical morphisms become equivalences, the right by inspection and the left because $\mathcal{Z}$ is aligned with $\mathcal{Y}$.
- Applying $C_{\mathcal{Z}}$ to the square (3.3.3), we see that both horizontal morphisms become equivalences, the lower by inspection and the upper because $\mathcal{Y}$ is aligned with $\mathcal{Z}$.

So the square (3.3.3) is indeed a pushout.

We now turn to part (2). It suffices to show that the square (3.3.4) becomes a pullback after applying $L_{\mathcal{Y}}$ and $L_{\mathcal{Z}}$.

- Applying $L_{\mathcal{Y}}$ to the square (3.3.4), we see that both vertical morphisms become equivalences, the right by inspection and the left because $\mathcal{Y}$ is aligned with $\mathcal{Z}$.
- Applying $L_{\mathcal{Z}}$ to the square (3.3.4), we see that both horizontal morphisms become equivalences, the lower by inspection and the upper because $\mathcal{Z}$ is aligned with $\mathcal{Y}$.

So the square (3.3.4) is indeed a pullback. □

3.4. **Fundamental operations on stratifications.** In this subsection, we record our fundamental operations on stratifications. For ease of navigation, it is organized into subsubsections.

Local Notation 3.4.1. In this subsection, we fix a poset P, a stratification $\mathcal{Z}_\bullet$ of $\mathcal{X}$ over P, down-closed subsets $\mathsf{D},\mathsf{E} \in \mathsf{Down}_{\mathsf{P}}$, and a closed subcategory $\mathcal{Y} \in \mathbf{Cls}_{\mathcal{X}}$.

Definition 3.4.2. We respectively say that the stratification $\mathcal{Z}_\bullet$ is ***aligned*** or ***mutually aligned*** with $\mathcal{Y}$ if each of its values $\mathcal{Z}_p$ is so.

Definition 3.4.3. We name the key outputs of this subsection as follows.

(1) Proposition 3.4.7 provides a ***restricted stratification*** of $\mathcal{Y}$ over P (under the assumption that $\mathcal{Z}_\bullet$ is mutually aligned with $\mathcal{Y}$).

(2) Given a functor $\widetilde{\mathcal{X}} \to \mathcal{X}$ that is the quotient by a closed subcategory, Proposition 3.4.9 provides a ***pullback stratification*** of $\widetilde{\mathcal{X}}$ over P (under the assumption that P is nonempty).

(3) Proposition 3.4.10 provides a ***quotient stratification*** of $\mathcal{X}/\mathcal{Y}$ over P (under the assumption that $\mathcal{Z}_\bullet$ is aligned with $\mathcal{Y}$).

(4) Given a functor $\mathsf{P} \to \mathsf{Q}$ between posets, Proposition 3.4.12 provides a ***pushforward stratification*** of $\mathcal{X}$ over Q.

(5) Given a stratification of each stratum $\mathcal{X}_p$ over a poset R_p, Proposition 3.4.14 provides a ***refined stratification*** of $\mathcal{X}$ over the *wreath product* poset $\mathsf{P} \wr \mathsf{R}_\bullet$.

3.4.1. *Preliminary results.*

Observation 3.4.4. The evident factorization

$$\begin{tikzcd} \mathsf{P} \arrow[r, "\mathcal{Z}_\bullet"] & \mathbf{Cls}_{\mathcal{X}} \\ \mathsf{D} \arrow[u, hook] \arrow[r, dashed, "\mathcal{Z}_\bullet"'] & \mathbf{Cls}_{\mathcal{Z}_{\mathsf{D}}} \arrow[u, hook] \end{tikzcd}$$

is a stratification of $\mathcal{Z}_{\mathsf{D}}$ over D, whose p^{th} stratum is $\mathcal{X}_p$ for every $p \in \mathsf{D} \subseteq \mathsf{P}$.

Lemma 3.4.5. *The closed subcategories $\mathcal{Z}_{\mathsf{D}}, \mathcal{Z}_{\mathsf{E}} \in \mathbf{Cls}_{\mathcal{X}}$ are mutually aligned and $(\mathcal{Z}_{\mathsf{D}} \cap \mathcal{Z}_{\mathsf{E}}) = \mathcal{Z}_{\mathsf{D} \cap \mathsf{E}}$.*

Proof. We first show that the lax-commutative square

$$\begin{tikzcd} \mathcal{Z}_{\mathsf{D}\cap\mathsf{E}} \arrow[r, hook, "i_L"] & \mathcal{Z}_{\mathsf{D}} \\ \mathcal{Z}_{\mathsf{E}} \arrow[u, "y"] \arrow[r, hook, "i_L"'] \arrow[ur, phantom, "\Searrow"] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} \tag{3.4.1}$$

determined by the commutative square

$$\begin{tikzcd} \mathcal{Z}_{\mathsf{D}\cap\mathsf{E}} \arrow[r, hook, "i_L"] \arrow[d, hook, "i_L"'] & \mathcal{Z}_{\mathsf{D}} \arrow[d, hook, "i_L"] \\ \mathcal{Z}_{\mathsf{E}} \arrow[r, hook, "i_L"'] & \mathcal{X} \end{tikzcd} \tag{3.4.2}$$

commutes. By an identical argument to that proving the equivalence (1) $\Leftrightarrow$ (4) of Lemma 3.1.7, it suffices to show that there exists a factorization

$$\begin{tikzcd} \mathcal{Z}_{\mathsf{D}\cap\mathsf{E}} \arrow[r, hook, "i_L"] & \mathcal{Z}_{\mathsf{D}} \\ \mathcal{Z}_{\mathsf{E}} \arrow[u, dashed] \arrow[r, hook, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} . \tag{3.4.3}$$

In the special case that $\mathsf{D} = ({}^{\leq}p)$ and $\mathsf{E} = ({}^{\leq}q)$, this is precisely the stratification condition. In order to prove the general case, we first prove the intermediate case that $\mathsf{E} \in \mathsf{Down}_{\mathsf{P}}$ is arbitrary but $\mathsf{D} = ({}^{\leq}p)$ for some $p \in \mathsf{P}$. Then, for each $q \in \mathsf{E}$, we have a factorization

$$\begin{tikzcd} \mathcal{Z}_{({}^{\leq}p)\cap({}^{\leq}q)} \arrow[r, hook, "i_L"] & \mathcal{Z}_{({}^{\leq}p)\cap\mathsf{E}} \arrow[r, hook, "i_L"] & \mathcal{Z}_p \\ \mathcal{Z}_q \arrow[u, dashed] \arrow[r, hook, "i_L"'] & \mathcal{Z}_{\mathsf{E}} \arrow[r, hook, "i_L"'] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} \tag{3.4.4}$$

by the stratification condition. So, the intermediate case follows from the facts that $\mathcal{Z}_{\mathsf{E}} := \langle \mathcal{Z}_q \rangle_{q \in \mathsf{E}}$ and that all solid morphisms in diagram (3.4.4) preserve colimits. Passing to the general case, for each $p \in \mathsf{D}$ let us extend the lax-commutative square (3.4.1) to a diagram

$$\begin{tikzcd} \mathcal{Z}_{({}^{\leq}p)\cap\mathsf{E}} \arrow[r, hook, "i_L"] & \mathcal{Z}_p \\ \mathcal{Z}_{\mathsf{D}\cap\mathsf{E}} \arrow[u, "y"] \arrow[r, hook, "i_L"] & \mathcal{Z}_{\mathsf{D}} \arrow[u, "y"'] \\ \mathcal{Z}_{\mathsf{E}} \arrow[u, "y"] \arrow[r, hook, "i_L"'] \arrow[ur, phantom, "\Searrow"] & \mathcal{X} \arrow[u, "y"'] \end{tikzcd} , \tag{3.4.5}$$

in which the upper (commutative) square is obtained by applying the intermediate case to the restricted stratification of $\mathcal{Z}_{\mathsf{D}}$ over D of Observation 3.4.4 (replacing $\mathsf{D}, \mathsf{E} \in \mathsf{Down}_{\mathsf{P}}$ respectively with $({}^{\leq}p), (\mathsf{D} \cap \mathsf{E}) \in \mathsf{Down}_{\mathsf{D}}$). Note too that the intermediate case is precisely the assertion that the composite lax-commutative rectangle of diagram (3.4.5) is in fact commutative. So, the lax-commutative square (3.4.1) must be commutative because the functors $\{\mathcal{Z}_{\mathsf{D}} \xrightarrow{y} \mathcal{Z}_p\}_{p \in \mathsf{D}}$ are jointly conservative.

Now, the commutativity of the square (3.4.2) implies that $\mathcal{Z}_{\mathsf{D} \cap \mathsf{E}} \subseteq (\mathcal{Z}_{\mathsf{D}} \cap \mathcal{Z}_{\mathsf{E}})$. On the other hand, the existence of the factorization (3.4.3) implies that $(\mathcal{Z}_{\mathsf{D}} \cap \mathcal{Z}_{\mathsf{E}}) \subseteq \mathcal{Z}_{\mathsf{D} \cap \mathsf{E}}$, as any object of $(\mathcal{Z}_{\mathsf{D}} \cap \mathcal{Z}_{\mathsf{E}})$ must lie in the image of the composite $\mathcal{Z}_{\mathsf{E}} \xhookrightarrow{i_L} \mathcal{X} \xrightarrow{y} \mathcal{Z}_{\mathsf{D}}$. So indeed, $(\mathcal{Z}_{\mathsf{D}} \cap \mathcal{Z}_{\mathsf{E}}) = \mathcal{Z}_{\mathsf{D} \cap \mathsf{E}}$. Hence, the factorization (3.4.3) witnesses $\mathcal{Z}_{\mathsf{E}}$ as being aligned with $\mathcal{Z}_{\mathsf{D}}$. That $\mathcal{Z}_{\mathsf{D}}$ is aligned with $\mathcal{Z}_{\mathsf{E}}$ follows by reversing the roles of D and E. $\square$

Remark 3.4.6. Evidently, a prestratification $\mathsf{P} \xrightarrow{\mathcal{Z}'_\bullet} \mathbf{Cls}_{\mathcal{X}'}$ satisfies the stratification condition if for all $p, q \in \mathsf{P}$ we have that $\mathcal{Z}'_q \rightsquigarrow \mathcal{Z}'_p$ and $(\mathcal{Z}'_p \cap \mathcal{Z}'_q) = \mathcal{Z}'_{({}^{\leq}p) \cap ({}^{\leq}q)}$. Lemma 3.4.5 provides a converse.

3.4.2. *Restricted stratifications.*

Proposition 3.4.7. *Suppose that the stratification $\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_{\mathcal{X}}$ is mutually aligned with $\mathcal{Y} \in \mathbf{Cls}_{\mathcal{X}}$.*

(1) The composite functor

$$\begin{tikzcd} \mathsf{P} \arrow[r, "\mathcal{Z}_\bullet"] & \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}} \arrow[r, "i_L^{-1}"] & \mathbf{Cls}_{\mathcal{Y}} \\[-20pt] \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\[-20pt] p \arrow[rr, mapsto] & & i_L^{-1}(\mathcal{Z}_p) \end{tikzcd} \tag{3.4.6}$$

is a stratification of $\mathcal{Y}$ over P.

(2) For any $p \in \mathsf{P}$, the p^{th} stratum of the stratification (3.4.6) is $i_L^{-1}(\mathcal{X}_p)$.

Proof. We begin with part (1). By Lemma 3.2.10(1), the composite functor (3.4.6) is a prestratification. So, it remains to verify the stratification condition. Choose any $p, q \in \mathsf{P}$, and consider the diagram

$$\begin{tikzcd} & \mathcal{Z}_{({}^{\leq}p) \cap ({}^{\leq}q)} \arrow[rr, hook, "i_L"] & & \mathcal{Z}_p \\ i_L^{-1}(\mathcal{Z}_{({}^{\leq}p) \cap ({}^{\leq}q)}) \arrow[ur, hook, "i_L"] \arrow[rr, hook, "i_L"] & & i_L^{-1}(\mathcal{Z}_p) \arrow[ur, hook, "i_L"] & \\ & \mathcal{Z}_q \arrow[uu, dashed] \arrow[rr, hook, "i_L"] & & \mathcal{X} \arrow[uu, "y"'] \\ i_L^{-1}(\mathcal{Z}_q) \arrow[uu, dashed] \arrow[ur, hook, "i_L"] \arrow[rr, hook, "i_L"'] & & i_L^{-1}(\mathcal{X}) = \mathcal{Y} \arrow[uu, "y"] \arrow[ur, hook, "i_L"] & \end{tikzcd}$$

in which the upper and lower squares commute by definition of i_L^{-1} and the right square commutes because $\mathcal{Y}$ is aligned with $\mathcal{Z}_p$. The back factorization exists because $\mathcal{Z}_\bullet$ is a stratification, and hence the front factorization exists because the upper square is a pullback. So, the stratification condition follows from the identification

$$i_L^{-1}(\mathcal{Z}_{({}^{\leq}p) \cap ({}^{\leq}q)}) := i_L^{-1}(\langle \mathcal{Z}_r \rangle_{r \in ({}^{\leq}p) \cap ({}^{\leq}q)}) \simeq \left\langle i_L^{-1}(\mathcal{Z}_r) \right\rangle_{r \in ({}^{\leq}p) \cap ({}^{\leq}q)}$$

resulting from Lemma 3.2.10(1).

We now turn to part (2). Note that $\mathcal{Y}$ is aligned with $\mathcal{Z}_{<p}$ by Lemma 3.2.2. By Observation 3.2.11, it follows that $i_L^{-1}(\mathcal{Z}_p)$ is also aligned with $\mathcal{Z}_{<p}$. Using this and Lemma 3.2.10(1), we identify the

p^{th} stratum of the stratification (3.4.6) as

$$\frac{i_L^{-1}(\mathcal{Z}_p)}{i_L^{-1}(\mathcal{Z}_{<p})} \simeq \mathsf{ker}(i_L^{-1}(\mathcal{Z}_p) \xrightarrow{y} i_L^{-1}(\mathcal{Z}_{<p})) \simeq \mathsf{ker}(i_L^{-1}(\mathcal{Z}_p) \xrightarrow{y} i_L^{-1}(\mathcal{Z}_{<p}) \xhookrightarrow{i_L} \mathcal{Z}_{<p})$$
$$\simeq \mathsf{ker}(i_L^{-1}(\mathcal{Z}_p) \xhookrightarrow{i_L} \mathcal{Z}_p \xrightarrow{y} \mathcal{Z}_{<p}) \simeq i_L^{-1}(\mathcal{X}_p) \ ,$$

as desired. □

Remark 3.4.8. Taking $\mathcal{Y} = \mathcal{Z}_{\mathsf{D}}$ in Proposition 3.4.7, we obtain a stratification of $\mathcal{Z}_{\mathsf{D}}$ over P, whose restriction to D is the stratification of $\mathcal{Z}_{\mathsf{D}}$ over D of Observation 3.4.4.[101]

3.4.3. *Pullback stratifications.*

Proposition 3.4.9. *Let $\widetilde{\mathcal{X}}$ be a presentable stable ∞-category. Suppose that $\widetilde{\mathcal{X}} \xrightarrow{\pi} \mathcal{X}$ is the quotient by a closed subcategory (i.e. the functor p_L in a recollement), and suppose further that P is nonempty.*

(1) The composite functor

$$\begin{array}{ccccc} \mathsf{P} & \xrightarrow{\mathcal{Z}_\bullet} & \mathbf{Cls}_{\mathcal{X}} & \xrightarrow{\pi^{-1}} & \mathbf{Cls}_{\widetilde{\mathcal{X}}} \\ \rotatebox{90}{$\in$} & & & & \rotatebox{90}{$\in$} \\ p & \longmapsto & & & \pi^{-1}(\mathcal{Z}_p) \end{array} \tag{3.4.7}$$

is a stratification of $\widetilde{\mathcal{X}}$ over P.

(2) For any $p \in \mathsf{P}$, the p^{th} stratum of the stratification (3.4.7) is $\mathcal{X}_p$ if $({}^{<}p) \neq \varnothing$ and is $\pi^{-1}(\mathcal{X}_p)$ if $({}^{<}p) = \varnothing$.

Proof. We begin with part (1). Because P is nonempty, the functor (3.4.7) is a prestratification by Lemma 3.2.10(2). So, it remains to verify the stratification condition. Choose any $p, q \in \mathsf{P}$, and consider the diagram

$$\begin{tikzcd}
& \mathcal{Z}_{(\leq p)\cap(\leq q)} \arrow[rr, hook, "i_L"] & & \mathcal{Z}_p \\
\pi^{-1}(\mathcal{Z}_{(\leq p)\cap(\leq q)}) \arrow[ur] \arrow[rr, hook, "i_L"] & & \pi^{-1}(\mathcal{Z}_p) \arrow[ur] & \\
& \mathcal{Z}_q \arrow[uu, dashed] \arrow[rr, hook, "i_L"] & & \mathcal{X} \arrow[uu, "y"'] \\
\pi^{-1}(\mathcal{Z}_q) \arrow[uu, dashed] \arrow[ur] \arrow[rr, hook, "i_L"'] & & \pi^{-1}(\mathcal{X}) = \widetilde{\mathcal{X}} \arrow[uu, "y"] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

in which the upper and lower squares commute by definition of π^{-1} and the right square commutes by Lemma 3.2.3(2)(c) and Observation 3.1.4. The back factorization exists because $\mathcal{Z}_\bullet$ is a stratification, and hence the front factorization exists because the upper square is a pullback. So, the stratification condition follows from the identification

$$\pi^{-1}(\mathcal{Z}_{(\leq p)\cap(\leq q)}) := \pi^{-1}(\langle \mathcal{Z}_r \rangle_{r \in (\leq p)\cap(\leq q)}) \simeq \left\langle \pi^{-1}(\mathcal{Z}_r) \right\rangle_{r \in (\leq p)\cap(\leq q)}$$

resulting from Lemma 3.2.10(2).

---

[101]In general, if the stratification $\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_{\mathcal{X}}$ has the property that $\mathcal{Z}_p = \mathcal{Z}_{(\leq p)\cap \mathsf{D}}$ for every $p \in \mathsf{P}$, then its restriction $\mathsf{D} \hookrightarrow \mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_{\mathcal{X}}$ is evidently also a stratification.

We now turn to part (2). In the case that $({}^{<}p) \neq \varnothing$, using Lemma 3.2.10(2) and Proposition 3.2.9 we identify the p^{th} stratum of the stratification (3.4.7) as

$$\frac{\pi^{-1}(\mathcal{Z}_p)}{\langle \pi^{-1}(\mathcal{Z}_{p'}) \rangle_{p'<p}} \simeq \frac{\pi^{-1}(\mathcal{Z}_p)}{\pi^{-1}(\langle \mathcal{Z}_{p'} \rangle_{p'<p})} =: \frac{\pi^{-1}(\mathcal{Z}_p)}{\pi^{-1}(\mathcal{Z}_{<p})} \simeq \frac{\pi^{-1}(\mathcal{Z}_p)/\pi^{-1}(0)}{\pi^{-1}(\mathcal{Z}_{<p})/\pi^{-1}(0)} \simeq \frac{\mathcal{Z}_p}{\mathcal{Z}_{<p}} =: \mathcal{X}_p \ ,$$

as desired. In the case that $({}^{<}p) = \varnothing$, we identify the p^{th} stratum of the stratification (3.4.7) as

$$\frac{\pi^{-1}(\mathcal{Z}_p)}{\langle \pi^{-1}(\mathcal{Z}_{p'}) \rangle_{p'<p}} = \frac{\pi^{-1}(\mathcal{Z}_p)}{0} = \pi^{-1}(\mathcal{Z}_p) \simeq \pi^{-1}(\mathcal{X}_p) \ ,$$

as desired. □

3.4.4. *Quotient stratifications.*

Proposition 3.4.10. *Suppose that the stratification* $\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_{\mathcal{X}}$ *is aligned with* $\mathcal{Y} \in \mathbf{Cls}_{\mathcal{X}}$.

(1) The composite functor

$$\begin{tikzcd} \mathsf{P} \arrow[r, "\mathcal{Z}_\bullet"] & \mathbf{Cls}_{\mathcal{X}}^{\rightsquigarrow \mathcal{Y}} \arrow[r, "p_L"] & \mathbf{Cls}_{\mathcal{X}/\mathcal{Y}} \\ \Psi & & \Psi \\ p \arrow[rr, mapsto] & & p_L(\mathcal{Z}_p) \end{tikzcd} \tag{3.4.8}$$

is a stratification of $\mathcal{X}/\mathcal{Y}$ *over* P.

(2) Suppose further that $\mathcal{Y}$ *is aligned with the stratification* $\mathcal{Z}_\bullet$. *For any* $p \in \mathsf{P}$, *the subcategory* $i_L^{-1}(\mathcal{X}_p) \subseteq \mathcal{X}_p$ *is closed and the* p^{th} *stratum of the stratification* (3.4.8) *is* $\mathcal{X}_p / i_L^{-1}(\mathcal{X}_p)$.

Proof. Over the course of the proof, for clarity we write $\mathcal{Y} \xrightarrow{\iota} \mathcal{X} \xrightarrow{\pi} \mathcal{X}/\mathcal{Y}$ for the canonical functors.

We begin with part (1). The functor (3.4.8) is a prestratification by Proposition 3.2.9 and the fact that $\pi(\mathcal{X}) = \mathcal{X}/\mathcal{Y}$. It remains to check the stratification condition. For any $p, q \in \mathsf{P}$, we have the solid commutative diagram

$$\begin{tikzcd} & \pi(\mathcal{Z}_{(\leq p)\cap(\leq q)}) \arrow[rr, hook, "i_L"] & & \pi(\mathcal{Z}_p) \\ \mathcal{Z}_{(\leq p)\cap(\leq q)} \arrow[ur, "p_L"] \arrow[rr, hook, "i_L"] & & \mathcal{Z}_p \arrow[ur, "p_L"'] & \\ & \pi(\mathcal{Z}_q) \arrow[uu, dashed] \arrow[rr, hook, "i_L"] & & \pi(\mathcal{X}) = \mathcal{X}/\mathcal{Y} \arrow[uu, "y"'] \\ \mathcal{Z}_q \arrow[uu, dashed] \arrow[ur, "p_L"] \arrow[rr, hook, "i_L"'] & & \mathcal{X} \arrow[uu, "y"] \arrow[ur, "p_L"'] & \end{tikzcd} ,$$

in which the bottom and top squares commute by the functoriality of presentable quotients and the right square commutes by Lemma 3.2.3(2)(c). The front factorization exists because $\mathcal{Z}_\bullet$ is a stratification, and hence the back factorization exists because the functor $\mathcal{Z}_q \xrightarrow{p_L} \pi(\mathcal{Z}_q)$ is surjective. So, the stratification condition follows from the identification

$$\pi(\mathcal{Z}_{(\leq p)\cap(\leq q)}) := \pi(\langle \mathcal{Z}_r \rangle_{r \in (\leq p)\cap(\leq q)}) \simeq \langle \pi^{-1}(\mathcal{Z}_r) \rangle_{r \in (\leq p)\cap(\leq q)}$$

resulting from Proposition 3.2.9.

We now proceed to part (2). First of all, $\mathcal{Y}$ is aligned with $\mathcal{Z}_{<p}$ by Lemma 3.2.2, and thereafter $\mathcal{Y} \cap \mathcal{Z}_p$ is aligned with $\mathcal{Z}_{<p}$ by Observation 3.2.11. Hence, the fact that $\iota^{-1}(\mathcal{X}_p) \in \mathbf{Cls}_{\mathcal{X}_p}$ follows from Lemma 3.2.3(2)(a) along with the observation that

$$\frac{\mathcal{Y} \cap \mathcal{Z}_p}{\mathcal{Y} \cap \mathcal{Z}_{<p}} \simeq \mathcal{Y} \cap \mathcal{X}_p =: \iota^{-1}(\mathcal{X}_p) \ .$$

Using Proposition 3.2.9, we now identify the p^{th} stratum of the stratification (3.4.8) as

$$\frac{\pi(\mathcal{Z}_p)}{\langle \pi(\mathcal{Z}_{p'}) \rangle_{p'<p}} \simeq \frac{\pi(\mathcal{Z}_p)}{\pi(\mathcal{Z}_{<p})} \simeq \frac{\mathcal{Z}_p/(\mathcal{Y} \cap \mathcal{Z}_p)}{\mathcal{Z}_{<p}/(\mathcal{Y} \cap \mathcal{Z}_{<p})} \simeq \frac{\mathcal{Z}_p/\mathcal{Z}_{<p}}{(\mathcal{Y} \cap \mathcal{Z}_p)/(\mathcal{Y} \cap \mathcal{Z}_{<p})} \simeq \frac{\mathcal{X}_p}{\mathcal{Y} \cap \mathcal{X}_p} =: \frac{\mathcal{X}_p}{\iota^{-1}(\mathcal{X}_p)} ,$$

as desired. ☐

Observation 3.4.11. Taking $\mathcal{Y} = \mathcal{Z}_{\mathsf{D}}$ in Proposition 3.4.10, we obtain a stratification of $\mathcal{X}/\mathcal{Z}_{\mathsf{D}} =: \mathcal{X}_{\mathsf{P}\backslash\mathsf{D}}$ over P, whose p^{th} stratum is 0 whenever $p \in \mathsf{D}$ and is $\mathcal{X}_p$ whenever $p \notin \mathsf{D}$ (because in this case $((^{\leq}p) \cap \mathsf{D}) \subseteq (^{<}p)$ (and using Lemma 3.4.5)). Evidently, the restriction to $(\mathsf{P}\backslash\mathsf{D}) \subseteq \mathsf{P}$ is also a stratification of $\mathcal{X}_{\mathsf{P}\backslash\mathsf{D}}$.[102] And in fact, the entire gluing diagram of $\mathcal{X}/\mathcal{Z}_{\mathsf{D}}$ with respect to this latter stratification is the restriction of that of $\mathcal{X}$, in the sense that we have a pullback diagram

$$\begin{tikzcd}
\mathscr{G}(\mathcal{X}/\mathcal{Z}_{\mathsf{D}}) \arrow[r, hook] \arrow[d] & \mathscr{G}(\mathcal{X}) \arrow[d] \\
\mathsf{P}\backslash\mathsf{D} \arrow[r, hook] & \mathsf{P}
\end{tikzcd} .$$

This follows from the existence of a factorization

$$\begin{tikzcd}
\mathscr{G}(\mathcal{X}/\mathcal{Z}_{\mathsf{D}}) \arrow[rr, dashed] \arrow[d, hook] & & \mathscr{G}(\mathcal{X}) \arrow[d, hook] \\
(\mathcal{X}/\mathcal{Z}_{\mathsf{D}}) \times (\mathsf{P}\backslash\mathsf{D}) \arrow[r, hook, "\nu \times \mathrm{id}_{\mathsf{P}\backslash\mathsf{D}}"'] & \mathcal{X} \times (\mathsf{P}\backslash\mathsf{D}) \arrow[r, hook] & \mathcal{X} \times \mathsf{P}
\end{tikzcd} ,$$

which itself results from Lemma 3.2.3(2)(c) (which applies by Lemma 3.4.5) by passing to right adjoints in the commutative square (3.2.7).

3.4.5. *Pushforward stratifications.*

Proposition 3.4.12. *Suppose that $\mathsf{P} \to \mathsf{Q}$ is any functor between posets.*

(1) The functor

$$\begin{array}{ccc}
\mathsf{Q} & \longrightarrow & \mathbf{Cls}_{\mathcal{X}} \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
q & \longmapsto & \mathcal{Z}_q := \mathcal{Z}_{\mathsf{P}_{\leq q}}
\end{array} \tag{3.4.9}$$

defines a stratification of $\mathcal{X}$ over Q.

(2) For any $q \in \mathsf{Q}$, the q^{th} stratum of the stratification (3.4.9) *is $\mathcal{X}_{\mathsf{P}_q}$.*

Proof. We begin with part (1). Since $\mathcal{X} = \langle \mathcal{Z}_p \rangle_{p \in \mathsf{P}}$, then also $\mathcal{X} = \langle \mathcal{Z}_q \rangle_{q \in \mathsf{Q}}$. So, it remains to check the stratification condition. For any $q, r \in \mathsf{Q}$, we must show that there is a factorization

$$\begin{tikzcd}
\mathcal{Z}_{\mathsf{P}_{(\leq q) \cap (\leq r)}} \arrow[r, hook, "i_L"] & \mathcal{Z}_{\mathsf{P}_{\leq q}} \\
\mathcal{Z}_{\mathsf{P}_{\leq r}} \arrow[u, dashed] \arrow[r, hook, "i_L"'] & \mathcal{X} \arrow[u, "y"']
\end{tikzcd}$$

This follows from Lemma 3.4.5 by taking $\mathsf{D} = \mathsf{P}_{\leq q}$ and $\mathsf{E} = \mathsf{P}_{\leq r}$ and noting that $\mathsf{P}_{\leq q} \cap \mathsf{P}_{\leq r} = \mathsf{P}_{(\leq q) \cap (\leq r)}$.

[102]In general, if the stratification $\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_{\mathcal{X}}$ has the property that $\mathcal{Z}_p = 0$ for all $p \in \mathsf{D}$, then its restriction $(\mathsf{P}\backslash\mathsf{D}) \hookrightarrow \mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_{\mathcal{X}}$ is also a stratification.

We now proceed to part (2). We write $\mathcal{Z} := \mathcal{Z}_{\mathsf{P}_{\leq q}}$ for simplicity, and we apply Observation 3.4.4 (taking $\mathsf{D} = \mathsf{P}_{\leq q}$) to pass to the restricted stratification

$$\mathsf{P}_{\leq q} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{Cls}_{\mathcal{Z}}$$

of $\mathcal{Z}$ over $\mathsf{P}_{\leq q}$ with the same strata. Writing

$$\mathsf{Span} := \left\{ \begin{tikzcd} s \arrow[r] \arrow[d] & t \\ u & \end{tikzcd} \right\}$$

for the walking span, we define a functor $\mathsf{P}_{\leq q} \xrightarrow{\pi} \mathsf{Span}$ between posets according to the prescriptions

$$\pi^{-1}(s) = ({}^{<}\mathsf{P}_q) \ , \qquad \pi^{-1}(t) = ({}^{\leq}\mathsf{P}_q \backslash {}^{<}\mathsf{P}_q) \ , \qquad \text{and} \qquad \pi^{-1}(u) = (\mathsf{P}_{<q} \backslash {}^{<}\mathsf{P}_q) \ .$$

By part (1), we obtain a stratification of $\mathcal{Z}$ over Span. Thereafter, applying Observation 3.4.11 (and Proposition 3.4.10) with $\mathsf{D} = \{s \to u\} \in \mathsf{Down}_{\mathsf{Span}}$, we obtain a quotient stratification

$$\begin{array}{ccc} \{t\} & \longrightarrow & \mathbf{Cls}_{\mathcal{Z}/\mathcal{Z}_u} \\ \Psi & & \Psi \\ t & \longmapsto & \mathcal{Z}_t/\mathcal{Z}_s \end{array}$$

over the one-element poset (since $({}^{\leq}t) \cap \{s \to u\} = \{s\}$). In particular, we find that

$$\mathcal{X}_q := \mathcal{Z}_q/\mathcal{Z}_{<q} := \mathcal{Z}_{\mathsf{P}_{\leq q}}/\mathcal{Z}_{\mathsf{P}_{<q}} =: \mathcal{Z}/\mathcal{Z}_u \simeq \mathcal{Z}_t/\mathcal{Z}_s =: \mathcal{Z}_{{}^{\leq}\mathsf{P}_q}/\mathcal{Z}_{{}^{<}\mathsf{P}_q} =: \mathcal{X}_{\mathsf{P}_q} \ ,$$

as desired. $\square$

3.4.6. *Refined stratifications.*

Definition 3.4.13. Given a functor

$$\iota_0\mathsf{P} \xrightarrow{\mathsf{R}_\bullet} \mathsf{Poset} \ ,$$

we define the ***wreath product*** of P with $\mathsf{R}_\bullet$ to be the poset $\mathsf{P} \wr \mathsf{R}_\bullet$ whose objects are pairs (p, r) where $p \in \mathsf{P}$ and $r \in \mathsf{R}_p$ equipped with the lexicographic ordering: $(p, r) \leq (p', r')$ in $\mathsf{P} \wr \mathsf{R}_\bullet$ if and only if either $p < p'$ in P or else $p = p'$ in P and $r \leq r'$ in R_p. This comes equipped with a canonical functor

$$\begin{array}{ccc} \mathsf{P} \wr \mathsf{R}_\bullet & \longrightarrow & \mathsf{P} \\ \Psi & & \Psi \\ (p, r) & \longmapsto & p \end{array} .$$

Proposition 3.4.14. *Choose any functor* $\iota_0\mathsf{P} \xrightarrow{\mathsf{R}_\bullet} \mathsf{Poset}$ *and, for each* $p \in \mathsf{P}$*, a stratification*

$$\begin{array}{ccc} \mathsf{R}_p & \xrightarrow{(\mathcal{Y}_p)_\bullet} & \mathbf{Cls}_{\mathcal{X}_p} \\ \Psi & & \Psi \\ r & \longmapsto & (\mathcal{Y}_p)_r \end{array} . \tag{3.4.10}$$

(1) *The functor*

$$\begin{array}{ccc} \mathsf{P} \wr \mathsf{R}_\bullet & \xrightarrow{\widetilde{\mathcal{Z}}_\bullet} & \mathbf{Cls}_{\mathcal{X}} \\ \Psi & & \Psi \\ (p, r) & \longmapsto & \widetilde{\mathcal{Z}}_{(p,r)} := p_L^{-1}((\mathcal{Y}_p)_r) \end{array} \tag{3.4.11}$$

defines a stratification of $\mathcal{X}$ *over* $\mathsf{P} \wr \mathsf{R}_\bullet$.

(2) *For any* $(p, r) \in \mathsf{P} \wr \mathsf{R}_\bullet$*, the* $(p, r)^{th}$ *stratum of the stratification* (3.4.11) *is* $(\mathcal{X}_p)_r$.

Proof. We begin with part (1).

We first verify that the functor (3.4.11) defines a prestratification. For this, consider any $p \in \mathsf{P}$. If $\mathsf{R}_p = \varnothing$, then it must be the case that $\mathcal{X}_p = 0$ and so $\mathcal{Z}_p = \mathcal{Z}_{<p}$. Otherwise, we have $\mathcal{Z}_p = \left\langle \widetilde{\mathcal{Z}}_{(p,r)} \right\rangle_{r \in \mathsf{R}_p}$ by Lemma 3.2.10(2). Hence, we find that

$$\mathcal{X} = \langle \mathcal{Z}_p \rangle_{p \in \mathsf{P}} = \langle \mathcal{Z}_p \rangle_{\{p \in \mathsf{P} : \mathsf{R}_p \neq \varnothing\}} = \left\langle \left\langle \widetilde{\mathcal{Z}}_{(p,r)} \right\rangle_{r \in \mathsf{R}_p} \right\rangle_{\{p \in \mathsf{P} : \mathsf{R}_p \neq \varnothing\}} = \left\langle \widetilde{\mathcal{Z}}_{(p,r)} \right\rangle_{(p,r) \in \mathsf{P} \wr \mathsf{R}_\bullet} .$$

We note here that the same argument shows that for any $\mathsf{D} \in \mathsf{Down}_\mathsf{P}$ we have an identification

$$\widetilde{\mathcal{Z}}_{(\mathsf{P} \wr \mathsf{R}_\bullet)_\mathsf{D}} = \mathcal{Z}_\mathsf{D} \tag{3.4.12}$$

in $\mathbf{Cls}_\mathcal{X}$.

We now verify the stratification condition. By Observation 2.4.5, it suffices to verify it for incomparable pairs of elements of $\mathsf{P} \wr \mathsf{R}_\bullet$. There are two types of such pairs: pairs (p,r) and (q,s) where p and q are incomparable in P, and pairs (p,r) and (p,s) where r and s are incomparable in R_p. We address these two cases in turn.

- Choose elements $(p,r),(q,s) \in \mathsf{P} \wr \mathsf{R}_\bullet$ such that p and q are incomparable in P. Note the equality
$$({}^{\leq}(p,r)) \cap ({}^{\leq}(q,s)) = (\mathsf{P} \wr \mathsf{R}_\bullet)_{({}^{\leq}p) \cap ({}^{\leq}q)}$$
in $\mathsf{Down}_{\mathsf{P} \wr \mathsf{R}_\bullet}$. Hence, we obtain a diagram
$$\begin{tikzcd}
\widetilde{\mathcal{Z}}_{({}^{\leq}(p,r)) \cap ({}^{\leq}(q,s))} \arrow[r, phantom, "="] & \mathcal{Z}_{({}^{\leq}p) \cap ({}^{\leq}q)} \arrow[r, hook, "i_L"] & \widetilde{\mathcal{Z}}_{(p,r)} \arrow[r, hook, "i_L"] & \mathcal{Z}_p \\
& \widetilde{\mathcal{Z}}_{(q,s)} \arrow[r, hook, "i_L"'] & \mathcal{Z}_q \arrow[r, hook, "i_L"'] \arrow[ul, dashed] & \mathcal{X} \arrow[u, "y"']
\end{tikzcd} ,$$
in which the identification is (3.4.12) with $\mathsf{D} = ({}^{\leq}p) \cap ({}^{\leq}q)$ and the factorization is guaranteed by the stratification condition for the stratification of $\mathcal{X}$ over P.

- Given elements $(p,r),(p,s) \in \mathsf{P} \wr \mathsf{R}_\bullet$ such that r and s are incomparable in R_p, the factorization
$$\begin{tikzcd}
\widetilde{\mathcal{Z}}_{({}^{\leq}(p,r)) \cap ({}^{\leq}(p,s))} \arrow[rr, hook, "i_L"] & & \widetilde{\mathcal{Z}}_{(p,r)} \\
& & \mathcal{Z}_p \arrow[u, "y"'] \\
\widetilde{\mathcal{Z}}_{(p,s)} \arrow[r, hook, "i_L"'] \arrow[uu, dashed] & \mathcal{Z}_p \arrow[r, hook, "i_L"'] \arrow[ur, "\mathrm{id}"] & \mathcal{X} \arrow[u, "y"']
\end{tikzcd}$$
follows from Proposition 3.4.9.

We now proceed to part (2). In light of the equalities

$${}^{\leq}(p,r) = \{(p,r') \in \mathsf{P} \wr \mathsf{R}_\bullet : r' \leq r\} \cup \{(p',r') \in \mathsf{P} \wr \mathsf{R}_\bullet : p' < p\} =: (p, ({}^{\leq}r)) \cup (\mathsf{P} \wr \mathsf{R}_\bullet)_{<p}$$

and

$${}^{<}(p,r) = \{(p,r') \in \mathsf{P} \wr \mathsf{R}_\bullet : r' < r\} \cup \{(p',r') \in \mathsf{P} \wr \mathsf{R}_\bullet : p' < p\} =: (p, ({}^{<}r)) \cup (\mathsf{P} \wr \mathsf{R}_\bullet)_{<p}$$

in $\mathsf{Down}_{\mathsf{P} \wr \mathsf{R}_\bullet}$, we find that

$$\mathcal{X}_{(p,r)} := \frac{\widetilde{\mathcal{Z}}_{(p,r)}}{\widetilde{\mathcal{Z}}_{<(p,r)}} \simeq \frac{\widetilde{\mathcal{Z}}_{(p,r)} / \widetilde{\mathcal{Z}}_{(\mathsf{P} \wr \mathsf{R}_\bullet)_{<p}}}{\widetilde{\mathcal{Z}}_{<(p,r)} / \widetilde{\mathcal{Z}}_{(\mathsf{P} \wr \mathsf{R}_\bullet)_{<p}}} = \frac{\widetilde{\mathcal{Z}}_{(p,r)} / \mathcal{Z}_{<p}}{\widetilde{\mathcal{Z}}_{<(p,r)} / \mathcal{Z}_{<p}} \simeq \frac{(\mathcal{Y}_p)_r}{(\mathcal{Y}_p)_{<r}} =: (\mathcal{X}_p)_r ,$$

as desired, using the identification (3.4.12) with $\mathsf{D} = ({}^{<}p)$. $\square$

4. The $\mathcal{O}$-monoidal reconstruction theorem

In this section, we upgrade our macrocosm reconstruction theorem (Theorem A(2)) to one that accounts for operadic structures (Theorem C). We also establish the adelic stratification (Theorem D), which is a symmetric monoidal stratification of a presentably symmetric monoidal stable ∞-category (satisfying mild finiteness hypotheses) over the specialization poset of its Balmer spectrum.

This section is organized as follows.

- §4.1: We fix an ∞-operad $\mathcal{O}$ (satisfying mild conditions) and recall the notions of $\mathcal{O}$-monoidal ∞-categories and laxly $\mathcal{O}$-monoidal functors.
- §4.2: We study the appropriate notion of an ideal subcategory of a presentably $\mathcal{O}$-monoidal stable ∞-category.
- §4.3: We define $\mathcal{O}$-monoidal stratifications of a presentably $\mathcal{O}$-monoidal stable ∞-category. We unpack the chromatic stratification of $\mathcal{S}\mathsf{p}$ in Example 4.3.8, which organizes the fundamental objects of chromatic homotopy theory.
- §4.4: We define the ∞-category that contains the $\mathcal{O}$-monoidal gluing diagram of an $\mathcal{O}$-monoidal stratification.
- §4.5: We prove Theorem C as Theorem 4.5.1.
- §4.6: We recall the basic notions of tensor-triangular geometry and then prove Theorem D as Theorem 4.6.11. We discuss the adelic stratification of $\mathcal{S}\mathsf{p}$ in Example 4.6.13. We explain how symmetric monoidal stratifications contribute to the theory of tensor-triangular geometry in Remark 4.6.14.

4.1. Preliminaries on $\mathcal{O}$-monoidal ∞-categories.

In this subsection, we fix an ∞-operad $\mathcal{O}$ satisfying mild conditions and recall the notions of $\mathcal{O}$-monoidal ∞-categories and laxly $\mathcal{O}$-monoidal functors.

Remark 4.1.1. We are primarily interested in symmetric monoidal ∞-categories. Indeed, the reader will not lose much by simply reading every instance of the ∞-operad "$\mathcal{O}$" as "Comm" (a.k.a. "$\mathbb{E}_\infty$", a.k.a. Fin_*), every instance of "$\mathcal{O}$-monoidal" as "symmetric monoidal", and so on. However, we work in this greater generality because it requires almost no extra effort and yet encompasses other situations of potential interest, notably ($\mathbb{E}_1$-)monoidal, braided (i.e. $\mathbb{E}_2$-)monoidal, and more generally $\mathbb{E}_n$-monoidal ∞-categories for any $1 \leq n \leq \infty$ (e.g. recall Remark 1.5.6).

Notation 4.1.2.

(1) We fix an ∞-operad

$$\mathcal{O} ,$$

which we assume

(a) to be unital,

(b) to be reduced (i.e. to have a contractible ∞-category of colors), and

(c) to have a nonempty space of binary operations.

We write

$$(\mathcal{O}^{\otimes} \downarrow \mathsf{Fin}_*) \in \mathsf{Cat}_{/\mathsf{Fin}_*}$$

for its defining object.

(2) Justified by the fact that the functor $\mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ restricts as an equivalence on underlying ∞-groupoids (by the assumption that $\mathcal{O}$ is reduced), we notationally identify objects of $\mathcal{O}^{\otimes}$ with their images in Fin_*; for any $n \geq 0$ we write $\underline{n} := \{1, \ldots, n\} \in \mathsf{Fin}$ and $\underline{n}_+ := \{1, \ldots, n\}_+ \in \mathsf{Fin}_*$.

(3) For any $n \geq 0$, we write

$$\begin{tikzcd} \mathcal{O}(n) \arrow[r, hook] \arrow[d] & \mathsf{hom}_{\mathcal{O}^{\otimes}}(\underline{n}_+, \underline{1}_+) \arrow[d] \\ \mathsf{pt} \arrow[r, hook] & \mathsf{hom}_{\mathsf{Fin}_*}(\underline{n}_+, \underline{1}_+) \end{tikzcd}$$

for the fiber over the unique active morphism, the space of n-ary operations in $\mathcal{O}$.

(4) We write

$$\mathcal{O}^{\otimes}_{\mathsf{cls}} \subseteq \mathcal{O}^{\otimes}$$

for the subcategory of *closed* (a.k.a. *inert*) morphisms.

Remark 4.1.3. A few comments regarding assumptions on the ∞-operad $\mathcal{O}$ are in order.

(1) All three assumptions of Notation 4.1.2(1) are motivated by examples and by a desire for simplicity of exposition; we expect that our results go through (mutatis mutandis) in greater generality.

(2) It follows from assumption (b) of Notation 4.1.2(1) that $\mathcal{O}$ is the underlying ∞-operad of an ordinary (i.e. single-colored) operad in topological spaces or simplicial sets.

(3) Assumption (c) of Notation 4.1.2(1) is primarily useful in that it allows us to simplify our notation, e.g. in Observation 4.2.9, Remark 4.3.5, and Observation 4.3.6. However, it also serves to guarantee that the unique morphism $\mathbb{E}_0 \to \mathcal{O}$ from the initial reduced unital ∞-operad is not an equivalence; this is convenient, as a number of our results do not hold as stated in this degenerate case.

(4) The additional assumption that $\mathcal{O}$ is *quadratic* (i.e. that for all $n \geq 2$ every n-ary operation is ((possibly only noncanonically) equivalent to) an iterated composite of binary operations) would allow us to very slightly simplify certain conditions in §4.2 (from quantifying over all $n \geq 2$ to quantifying merely over $n = 2$).

Definition 4.1.4.

(1) An $\mathcal{O}$-***monoidal*** ∞-***category*** is a reduced Segal functor

$$\mathcal{O}^{\otimes} \xrightarrow{\mathcal{C}^{\otimes}} \mathsf{Cat} .$$

We also write

$$(\mathcal{C}^{\otimes} \downarrow \mathcal{O}^{\otimes}) \in \mathsf{coCart}_{\mathcal{O}^{\otimes}}$$

for the cocartesian fibration that such a functor classifies, and we write

$$\mathcal{C} := \mathcal{C}^{\otimes}(\underline{1}_+) \in \mathsf{Cat}$$

for its underlying ∞-category. These assemble into the full subcategory

$$\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \subseteq \mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{Cat}) ,$$

whose morphisms we refer to as $\mathcal{O}$-***monoidal functors***.

(2) We define the ∞-category whose objects are $\mathcal{O}$-monoidal ∞-categories and whose morphisms are ***right-laxly*** $\mathcal{O}$***-monoidal functors*** to be the indicated image in the diagram

$$
\begin{tikzcd}
\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \arrow[r, hook, two heads, dashed] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{Alg}^{\mathsf{r.lax}}_{\mathcal{O}}(\mathsf{Cat}) \arrow[d, hook, "\mathsf{f.f.}"] & \\
\mathsf{coCart}_{\mathcal{O}^{\otimes}} \arrow[r, hook, two heads] & \mathsf{Cat}^{\mathsf{cls}}_{\mathsf{cocart}/\mathcal{O}^{\otimes}} \arrow[r, hook, two heads] \arrow[d] & \mathsf{Cat}_{\mathsf{cocart}/\mathcal{O}^{\otimes}} \arrow[d] \\
& \mathsf{coCart}_{\mathcal{O}^{\otimes}_{\mathsf{cls}}} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{cocart}/\mathcal{O}^{\otimes}_{\mathsf{cls}}}
\end{tikzcd}
$$

whose lower right square is a pullback.

(3) We define the ∞-category whose objects are $\mathcal{O}$-monoidal ∞-categories and whose morphisms are ***left-laxly*** $\mathcal{O}$***-monoidal functors*** to be the indicated image in the diagram

$$
\begin{tikzcd}
\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \arrow[r, hook, two heads, dashed] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{Alg}^{\mathsf{l.lax}}_{\mathcal{O}}(\mathsf{Cat}) \arrow[d, hook, "\mathsf{f.f.}"] & \\
\mathsf{Cart}_{(\mathcal{O}^{\otimes})^{\mathsf{op}}} \arrow[r, hook, two heads] & \mathsf{Cat}^{\mathsf{cls}}_{\mathsf{cart}/(\mathcal{O}^{\otimes})^{\mathsf{op}}} \arrow[r, hook, two heads] \arrow[d] & \mathsf{Cat}_{\mathsf{cart}/(\mathcal{O}^{\otimes})^{\mathsf{op}}} \arrow[d] \\
& \mathsf{Cart}_{(\mathcal{O}^{\otimes}_{\mathsf{cls}})^{\mathsf{op}}} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{cart}/(\mathcal{O}^{\otimes}_{\mathsf{cls}})^{\mathsf{op}}}
\end{tikzcd}
$$

whose lower right square is a pullback.

Notation 4.1.5. For each $n \geq 0$, we write

$$
\begin{array}{ccc}
\mathcal{O}(n) \times \mathcal{C}^{\times n} & \longrightarrow & \mathcal{C} \\
\Uparrow\!\!\!\!\cup \;\; \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\big(\mu, (X_i)_{i \in \underline{n}}\big) & \longmapsto & \bigotimes_{\mu} (X_i)_{i \in \underline{n}}
\end{array}
$$

for the value of an n-ary operation $\mu \in \mathcal{O}(n)$ on an n-tuple $(X_i)_{i \in \underline{n}} \in \mathcal{C}^{\times n}$ of objects of $\mathcal{C}$.

Remark 4.1.6. For each $n \geq 0$, each $\mu \in \mathcal{O}(n)$, and each $(X_i)_{i \in \underline{n}} \in \mathcal{C}^{\times n}$, a right-laxly $\mathcal{O}$-monoidal functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ determines a natural comparison morphism

$$
\bigotimes_{\mu}^{\mathcal{D}} (F(X_i))_{i \in \underline{n}} \longrightarrow F\left(\bigotimes_{\mu}^{\mathcal{C}} (X_i)_{i \in \underline{n}} \right)
$$

in $\mathcal{D}$.[103] In fact, directly from the definitions, a right-laxly $\mathcal{O}$-monoidal functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ determines a functor $\mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{F} \mathsf{Alg}_{\mathcal{O}}(\mathcal{D})$ on ∞-categories of $\mathcal{O}$-algebras. Dually, a left-laxly $\mathcal{O}$-monoidal functor determines comparison morphisms in the opposite direction, and determines a functor on ∞-categories of $\mathcal{O}$-coalgebras.

Observation 4.1.7. It follows from Lemma A.3.5 that given an adjunction $F \dashv G$ between the underlying ∞-categories of $\mathcal{O}$-monoidal ∞-categories, the following two types of data are equivalent:

- the additional structure on the left adjoint F of a left-laxly $\mathcal{O}$-monoidal functor;
- the additional structure on the right adjoint G of a right-laxly $\mathcal{O}$-monoidal functor.[104]

It follows in particular e.g. that the right adjoint of an $\mathcal{O}$-monoidal functor is canonically right-laxly $\mathcal{O}$-monoidal. We will use these facts without further comment.

4.2. **Ideals in presentably $\mathcal{O}$-monoidal ∞-categories.** In this subsection, we study the appropriate notion of an ideal subcategory of a presentably $\mathcal{O}$-monoidal stable ∞-category. We also show as Proposition 4.2.14 that these are equivalent data to certain idempotent objects.

Local Notation 4.2.1. For the remainder of this section, we fix a presentably $\mathcal{O}$-monoidal stable ∞-category $\mathcal{R}$: that is, $\mathcal{R}$ is a presentable stable ∞-category equipped with the structure of an $\mathcal{O}$-monoidal ∞-category such that for all $n \geq 2$ and all $\mu \in \mathcal{O}(n)$ the functor $\mathcal{R}^{\times n} \xrightarrow{\otimes_\mu} \mathcal{R}$ commutes with colimits separately in each variable.

Notation 4.2.2. We write $\mathbb{1}_{\mathcal{R}} \in \mathcal{R}$ for the $\mathcal{O}$-monoidal unit object of $\mathcal{R}$.

Remark 4.2.3. The object $\mathbb{1}_{\mathcal{R}} \in \mathcal{R}$ is the unit with respect to all possible monoidal products in $\mathcal{R}$: for any $n \geq 1$, for any $\mu \in \mathcal{O}(n)$, and for any $X \in \mathcal{R}$, there is a canonical equivalence

$$\bigotimes_{\mu}(X, \mathbb{1}_{\mathcal{R}}, \ldots, \mathbb{1}_{\mathcal{R}}) \xrightarrow{\sim} X$$

(where there are $(n-1)$ copies of $\mathbb{1}_{\mathcal{R}}$), and similarly where X is put in a different slot from the first.

Notation 4.2.4. We simply write $\otimes := \otimes_\mu$ in any situation where this notation is canonically unambiguous, such as throughout Observation 4.2.10. (This unambiguity will then be an implicit assertion.)

Definition 4.2.5. A full presentable stable subcategory $\mathcal{I} \subseteq \mathcal{R}$ is called an ***ideal*** if it is contagious under the $\mathcal{O}$-monoidal structure, i.e. for all $n \geq 2$ and all $\mu \in \mathcal{O}(n)$ there exists a factorization

$$\begin{tikzcd} \mathcal{I} \times \mathcal{R}^{\times(n-1)} \arrow[rr, "\otimes_\mu"] \arrow[dr, dashed] & & \mathcal{R} \\ & \mathcal{I} \arrow[ur, hook, "\text{f.f.}"'] & \end{tikzcd} .$$

Notation 4.2.6. Given a set $\{K_s \in \mathcal{R}\}_{s \in S}$ of objects, we write $\langle K_s \rangle^{\otimes}_{s \in S}$ for the ideal that they generate. Likewise, given a subcategory $\mathcal{D} \subseteq \mathcal{R}$, we write $\langle \mathcal{D} \rangle^{\otimes} \subseteq \mathcal{R}$ for the ideal that it generates.

Observation 4.2.7. Suppose that $\mathcal{I} \subseteq \mathcal{R}$ is an ideal that is also a closed subcategory. Then, $\mathcal{I}$ inherits an $\mathcal{O}$-monoidal structure with unit object $\mathbb{1}_{\mathcal{I}} := y(\mathbb{1}_{\mathcal{R}}) \in \mathcal{I}$, such that in the adjunction

$$\begin{tikzcd} \mathcal{I} \arrow[r, hook, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{R} \arrow[l, shift left, "y"] \end{tikzcd} \tag{4.2.1}$$

[103]In the case that $n = 0$, by assumption the space $\mathcal{O}(0)$ is contractible, and the comparison morphism determined by its unique point is a morphism

$$\mathbb{1}_{\mathcal{D}} \longrightarrow F(\mathbb{1}_{\mathcal{C}}) .$$

[104]Indeed, this fact motivates our choice of handedness in parts (2) and (3) of Definition 4.1.4: we take concordance with the handedness of the adjoint as more fundamental than concordance with the handedness of the fibrations.

the left adjoint i_L is left-laxly $\mathcal{O}$-monoidal and nonunitally $\mathcal{O}$-monoidal, i.e. it preserves tensor products up to natural equivalence but the unit only up to a morphism

$$i_L(\mathbb{1}_{\mathcal{I}}) := i_L(y(\mathbb{1}_{\mathcal{R}})) \xrightarrow{\varepsilon_{\mathbb{1}_{\mathcal{R}}}} \mathbb{1}_{\mathcal{R}} \ . \tag{4.2.2}$$

It follows that the right adjoint y is right-laxly $\mathcal{O}$-monoidal.

Definition 4.2.8. An ideal $\mathcal{I} \subseteq \mathcal{R}$ which is also a closed subcategory is called a ***closed ideal*** if the right adjoint y in the adjunction (4.2.1) is $\mathcal{O}$-monoidal. We write

$$\mathbf{Idl}_{\mathcal{R}} \subseteq \mathbf{Cls}_{\mathcal{R}}$$

for the full subposet consisting of the closed ideals.

Observation 4.2.9. Because the $\mathcal{O}$-monoidal structure on $\mathcal{R}$ commutes with colimits separately in each variable, the full subposet $\mathbf{Idl}_{\mathcal{R}} \subseteq \mathbf{Cls}_{\mathcal{R}}$ is stable under colimits.

Observation 4.2.10. Suppose that $\mathcal{I} \subseteq \mathcal{R}$ is a closed ideal, and consider the recollement

$$\begin{tikzcd} \mathcal{I} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathcal{R} \arrow[l, "y"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{R}/\mathcal{I} \arrow[l, hook', "\nu"'] \end{tikzcd} \ .$$

It is straightforward to verify the following facts, which we will use without further comment.

(1) The object $i_L(\mathbb{1}_{\mathcal{I}}) \in \mathcal{R}$ is an idempotent $\mathcal{O}$-coalgebra object with counit morphism (4.2.2). Moreover, tensoring with this counit morphism implements the colocalization $i_L \dashv y$: for any $X \in \mathcal{R}$, the diagram

$$\begin{tikzcd} i_L(\mathbb{1}_{\mathcal{I}}) \otimes X \arrow[r, "\varepsilon_{\mathbb{1}_{\mathcal{R}}} \otimes \mathrm{id}_X"] \arrow[d, phantom, "\wr"] & \mathbb{1}_{\mathcal{R}} \otimes X \arrow[d, phantom, "\wr"] \\ i_L(\mathbb{1}_{\mathcal{I}} \otimes y(X)) \arrow[d, phantom, "\wr"] & X \\ i_L y(X) \arrow[ur, "\varepsilon_X"'] & \end{tikzcd}$$

canonically commutes.[105]

(2) There is a canonical $\mathcal{O}$-monoidal structure on $\mathcal{R}/\mathcal{I}$, such that

(a) the unit object is $\mathbb{1}_{\mathcal{R}/\mathcal{I}} := p_L(\mathbb{1}_{\mathcal{R}}) \in \mathcal{R}/\mathcal{I}$,

(b) the functor p_L is $\mathcal{O}$-monoidal, and

(c) the functor ν is right-laxly $\mathcal{O}$-monoidal and nonunitally $\mathcal{O}$-monoidal, i.e. it preserves tensor products up to natural equivalence but the unit only up to a morphism

$$\mathbb{1}_{\mathcal{R}} \xrightarrow{\eta_{\mathbb{1}_{\mathcal{R}}}} \nu(p_L(\mathbb{1}_{\mathcal{R}})) =: \nu(\mathbb{1}_{\mathcal{R}/\mathcal{I}}) \ . \tag{4.2.3}$$

Hence, the object $\nu(\mathbb{1}_{\mathcal{R}/\mathcal{I}}) \in \mathcal{R}$ is an idempotent $\mathcal{O}$-algebra object with unit morphism (4.2.3). Moreover, tensoring with this unit morphism implements the localization $p_L \dashv \nu$:

[105] That is, for every $\mu \in \mathcal{O}(2)$, the functor $i_L(\mathbb{1}_{\mathcal{I}}) \otimes_\mu (-)$ is canonically equivalent to the composite $i_L y$ (recall Notation 4.2.4).

for any $X \in \mathcal{R}$, the diagram

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{R}} \otimes X \arrow[r, "\eta_{\mathbb{1}_{\mathcal{R}}} \otimes \mathsf{id}_X"] \arrow[d, phantom, "\wr"] & \nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}}) \otimes X \arrow[d, phantom, "\wr"] \\
X \arrow[dr, "\eta_X"'] & \nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}} \otimes p_L(X)) \arrow[d, phantom, "\wr"] \\
 & \nu(p_L(X))
\end{tikzcd}$$

canonically commutes.[106]

Remark 4.2.11.

(1) An idempotent $\mathcal{O}$-coalgebra object in $\mathcal{R}$ is equivalently an object

$$(C \xrightarrow{\varepsilon} \mathbb{1}_{\mathcal{R}}) \in \mathcal{R}_{/\,\mathbb{1}_{\mathcal{R}}} \tag{4.2.4}$$

such that for all $n \geq 2$ and all $\mu \in \mathcal{O}(n)$ the morphism

$$\bigotimes_{\mu}(C, \ldots, C) \xrightarrow{\bigotimes_{\mu}(\varepsilon, \mathsf{id}_C, \ldots, \mathsf{id}_C)} \bigotimes_{\mu}(\mathbb{1}_{\mathcal{R}}, C, \ldots, C) \tag{4.2.5}$$

is an equivalence.

(2) An idempotent $\mathcal{O}$-algebra object in $\mathcal{R}$ is equivalently an object

$$(\mathbb{1}_{\mathcal{R}} \xrightarrow{\eta} A) \in \mathcal{R}_{\mathbb{1}_{\mathcal{R}}\,/} \tag{4.2.6}$$

such that for all $n \geq 2$ and all $\mu \in \mathcal{O}(n)$ the morphism

$$\bigotimes_{\mu}(\mathbb{1}_{\mathcal{R}}, A, \ldots, A) \xrightarrow{\bigotimes_{\mu}(\eta, \mathsf{id}_A, \ldots, \mathsf{id}_A)} \bigotimes_{\mu}(A, \ldots, A) \tag{4.2.7}$$

is an equivalence.

Definition 4.2.12.

(1) An ***augmented idempotent*** in $\mathcal{R}$ is an object (4.2.4) such that for all $n \geq 2$ and all $\mu \in \mathcal{O}(n)$ the morphism (4.2.5) is an equivalence.[107] We say that it is ***central*** if for all $n \geq 3$, all $\mu \in \mathcal{O}(n)$, and all $X_1, \ldots, X_{n-2} \in \mathcal{R}$, the morphism

$$\bigotimes_{\mu}(C, C, X_1, \ldots, X_{n-2}) \xrightarrow{\bigotimes_{\mu}(\varepsilon, \mathsf{id}_C, \mathsf{id}_{X_1}, \ldots, \mathsf{id}_{X_{n-2}})} \bigotimes_{\mu}(\mathbb{1}_{\mathcal{R}}, C, X_1 \ldots, X_{n-2})$$

is an equivalence. We write

$$\mathsf{ZAugIdem}_{\mathcal{R}} \subseteq \mathcal{R}_{/\,\mathbb{1}_{\mathcal{R}}}$$

for the full subcategory on the central augmented idempotents.

(2) A ***coaugmented idempotent*** in $\mathcal{R}$ is an object (4.2.6) such that for all $n \geq 2$ and all $\mu \in \mathcal{O}(n)$ the morphism (4.2.7) is an equivalence.[108] We say that it is ***central*** if for all $n \geq 3$, all $\mu \in \mathcal{O}(n)$, and all $X_1, \ldots, X_{n-2} \in \mathcal{R}$, the morphism

$$\bigotimes_{\mu}(\mathbb{1}_{\mathcal{R}}, A, X_1, \ldots, X_{n-2}) \xrightarrow{\bigotimes_{\mu}(\eta, \mathsf{id}_A, \mathsf{id}_{X_1}, \ldots, \mathsf{id}_{X_{n-2}})} \bigotimes_{\mu}(A, A, X_1 \ldots, X_{n-2})$$

[106] In particular, $\mathcal{R}/\mathcal{J} \overset{\nu}{\hookrightarrow} \mathcal{R}$ is also the inclusion of an ideal (which is not generally a closed ideal).
[107] So, an augmented idempotent is equivalently an idempotent $\mathcal{O}$-coalgebra by Remark 4.2.11(1).
[108] So, a coaugmented idempotent is equivalently an idempotent $\mathcal{O}$-algebra by Remark 4.2.11(2).

is an equivalence. We write

$$\mathsf{ZcoAugIdem}_{\mathcal{R}} \subseteq \mathcal{R}_{\mathbb{1}_{\mathcal{R}}/}$$

for the full subcategory on the central coaugmented idempotents.

Observation 4.2.13. In the case that $\mathcal{O}$ is quadratic, it suffices to verify centrality for ternary operations. For instance, if $\mathcal{O} = \mathbb{E}_1$, an augmented idempotent $C \in \mathsf{ZAugIdem}_{\mathcal{R}}$ is central if and only if for every $X \in \mathcal{R}$ the morphisms

$$C \otimes X \simeq C \otimes X \otimes \mathbb{1}_{\mathcal{R}} \xleftarrow{\mathsf{id}_C \otimes \mathsf{id}_X \otimes \varepsilon} C \otimes X \otimes C \xrightarrow{\varepsilon \otimes \mathsf{id}_X \otimes \mathsf{id}_C} \mathbb{1}_{\mathcal{R}} \otimes X \otimes C \simeq X \otimes C$$

are equivalences, while a coaugmented idempotent $A \in \mathsf{ZcoAugIdem}_{\mathcal{R}}$ is central if and only if for every $X \in \mathcal{R}$ the morphisms

$$A \otimes X \simeq A \otimes X \otimes \mathbb{1}_{\mathcal{R}} \xrightarrow{\mathsf{id}_A \otimes \mathsf{id}_X \otimes \eta} A \otimes X \otimes A \xleftarrow{\eta \otimes \mathsf{id}_X \otimes \mathsf{id}_A} \mathbb{1}_{\mathcal{R}} \otimes X \otimes A \simeq X \otimes A$$

are equivalences. If additionally $\mathcal{O}(2)$ is connected (e.g. if $\mathcal{O} = \mathbb{E}_n$ for any $2 \leq n \leq \infty$), then the condition of centrality is vacuous: every co/augmented idempotent is automatically central.

Proposition 4.2.14.

(1) The full subcategories

$$\mathsf{ZAugIdem}_{\mathcal{R}} \subseteq \mathcal{R}_{/\mathbb{1}_{\mathcal{R}}} \qquad \textit{and} \qquad \mathsf{ZcoAugIdem}_{\mathcal{R}} \subseteq \mathcal{R}_{\mathbb{1}_{\mathcal{R}}/}$$

are posets.

(2) There is a canonical commutative diagram

$$\begin{tikzcd}
 & \mathbf{Idl}_{\mathcal{R}} \arrow[dl, shift left, "\mathcal{J} \longmapsto i_L(\mathbb{1}_{\mathcal{J}})", "\sim"'] \arrow[dr, shift left, "\mathcal{J} \longmapsto \nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}})", "\sim"'] & \\
\mathsf{ZAugIdem}_{\mathcal{R}} \arrow[ur, shift left, "C \longmapsto \langle C \rangle^{\oplus}"] \arrow[rr, shift left, "\mathsf{cofib}(\varepsilon)", "\sim"'] & & \mathsf{ZcoAugIdem}_{\mathcal{R}} \arrow[ul, shift left, "\langle \mathsf{fib}(\eta) \rangle^{\oplus}"] \arrow[ll, shift left, "\mathsf{fib}(\eta)"]
\end{tikzcd} \tag{4.2.8}$$

of equivalences.

(3) Given a central augmented idempotent $C \in \mathsf{ZAugIdem}_{\mathcal{R}}$, for any $\tau \in \mathcal{O}(2)$ we have an identification

$$\langle C \rangle^{\otimes} = \mathcal{I}_{C,\tau} := \left\{ X \in \mathcal{R} : \text{the morphism } C \otimes_\tau X \xrightarrow{\varepsilon \otimes_\tau \mathsf{id}_X} \mathbb{1}_{\mathcal{R}} \otimes_\tau X \simeq X \text{ is an equivalence} \right\} ,$$

and we may identify the right adjoint to its inclusion as

$$\begin{tikzcd} \mathcal{I}_{C,\tau} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{R} \arrow[l, dashed, shift left, "C \otimes_\tau (-)"] \end{tikzcd}$$

with counit $C \otimes_\tau (-) \xrightarrow{\varepsilon \otimes_\tau \mathsf{id}} \mathbb{1}_{\mathcal{R}} \otimes_\tau (-) \simeq \mathsf{id}_{\mathcal{R}}$.

(4) Given a central coaugmented idempotent $A \in \mathsf{ZcoAugIdem}_{\mathcal{R}}$, for any $\tau \in \mathcal{O}(2)$ we have an identification

$$\mathcal{R}/\langle \mathsf{fib}(\eta) \rangle^{\otimes} = \mathcal{R}/\mathcal{I}_{\mathsf{fib}(\eta),\tau} = \left\{ X \in \mathcal{R} : \text{the morphism } X \simeq \mathbb{1}_{\mathcal{R}} \otimes_\tau X \xrightarrow{\eta \otimes_\tau \mathsf{id}_X} A \otimes_\tau X \text{ is an equivalence} \right\} ,$$

and we may identify the left adjoint to its inclusion as

$$\begin{tikzcd} \mathcal{R} \arrow[r, dashed, shift left, "A \otimes_\tau (-)"] \arrow[r, phantom, "\perp"] & \mathcal{R}/\mathcal{I}_{\mathsf{fib}(\eta),\tau} \arrow[l, hook, shift left] \end{tikzcd}$$

with unit $\mathsf{id}_{\mathcal{R}} \simeq \mathbb{1}_{\mathcal{R}} \otimes_\tau (-) \xrightarrow{\eta \otimes_\tau \mathsf{id}} A \otimes_\tau (-)$.

Proof. We fix arbitrary $C \in \mathsf{ZAugIdem}_{\mathcal{R}}$ and $\tau \in \mathcal{O}(2)$, to which we will refer throughout the proof.

We begin by proving part (3), and then use it implicitly through the remainder of the proof. We first verify that $\mathcal{I}_{C,\tau} \subseteq \mathcal{R}$ is an ideal. It is clearly a full presentable stable subcategory. Now, for any $n \geq 2$, any $\mu \in \mathcal{O}(n)$, any $X \in \mathcal{I}$, and any $Y_1, \ldots, Y_{n-1} \in \mathcal{R}$, we may factor the morphism

$$C \otimes_\tau \bigotimes_\mu (X, Y_1, \ldots, Y_{n-1}) \longrightarrow \mathbb{1}_{\mathcal{R}} \otimes_\tau \bigotimes_\mu (X, Y_1, \ldots, Y_{n-1}) \simeq \bigotimes_\mu (X, Y_1, \ldots, Y_{n-1})$$

as the sequence of equivalences

$$\begin{aligned} C \otimes_\tau \bigotimes_\mu (X, Y_1, \ldots, Y_{n-1}) &\xleftarrow{\sim} C \otimes_\tau \bigotimes_\mu (C \otimes_\tau X, Y_1, \ldots, Y_{n-1}) && (4.2.9) \\ &\xrightarrow{\sim} \mathbb{1}_{\mathcal{R}} \otimes_\tau \bigotimes_\mu (C \otimes_\tau X, Y_1, \ldots, Y_{n-1}) && (4.2.10) \\ &\xrightarrow{\sim} \bigotimes_\mu (C \otimes_\tau X, Y_1, \ldots, Y_{n-1}) \\ &\xrightarrow{\sim} \bigotimes_\mu (X, Y_1, \ldots, Y_{n-1}) && (4.2.11) \end{aligned}$$

in which equivalences (4.2.9) and (4.2.11) use that $X \in \mathcal{I}_{C,\tau}$ and equivalence (4.2.10) uses the centrality of C. So indeed, the subcategory $\mathcal{I}_{C,\tau} \subseteq \mathcal{R}$ is an ideal. Now, we have $C \in \mathcal{I}_{C,\tau}$ because C is an augmented idempotent, so we obtain the containment $\langle C \rangle^{\otimes} \subseteq \mathcal{I}_{C,\tau}$. On the other hand, clearly $(C \otimes_\tau X) \in \langle C \rangle^{\otimes}$ for any $X \in \mathcal{R}$, which implies that $\langle C \rangle^{\otimes} \supseteq \mathcal{I}_{C,\tau}$. This proves the asserted equality $\langle C \rangle^{\otimes} = \mathcal{I}_{C,\tau}$. To verify that the right adjoint to its inclusion is as asserted, we observe that for any $X \in \mathcal{I}_{C,\tau}$ and any $Y \in \mathcal{R}$, we have $(C \otimes_\tau Y) \in \langle C \rangle^{\otimes} = \mathcal{I}_{C,\tau}$ and moreover we have the commutative diagram

$$\begin{tikzcd} \mathsf{hom}_{\mathcal{I}_{C,\tau}}(X, C \otimes_\tau Y) := \mathsf{hom}_{\mathcal{R}}(X, C \otimes_\tau Y) \arrow[r] \arrow[d, "\wr"] & \mathsf{hom}_{\mathcal{R}}(X, Y) \arrow[dl, "C \otimes_\tau (-)"'] \arrow[d, "\wr"] \\ \mathsf{hom}_{\mathcal{R}}(C \otimes_\tau X, C \otimes_\tau Y) \arrow[r] & \mathsf{hom}_{\mathcal{R}}(C \otimes_\tau X, Y) \end{tikzcd},$$

which implies that its upper morphism is an equivalence. This completes the proof of part (3).

We now verify that the ideal $\mathcal{I}_{C,\tau} \subseteq \mathcal{R}$ is in fact a closed ideal. First of all, it is a closed subcategory because the right adjoint $\mathcal{R} \xrightarrow{C \otimes_\tau (-)} \mathcal{I}_{C,\tau}$ preserves colimits. So, it remains to verify that this right adjoint is $\mathcal{O}$-monoidal. Clearly $\mathbb{1}_{\mathcal{I}_{C,\tau}} \simeq C$, and hence this right adjoint preserves unit objects. We now observe that for any $n \geq 2$, any $\mu \in \mathcal{O}(n)$, and any $Y_1, \ldots, Y_n \in \mathcal{R}$, we may factor the canonical morphism

$$\bigotimes_\mu (C \otimes_\tau Y_i)_{i \in \underline{n}} \longrightarrow C \otimes_\tau \left(\bigotimes_\mu (Y_i)_{i \in \underline{n}} \right)$$

as the sequence of equivalences

$$\bigotimes_\mu (C \otimes_\tau Y_i)_{i \in \underline{n}} \xleftarrow{\sim} C \otimes_\tau \bigotimes_\mu (C \otimes_\tau Y_i)_{i \in \underline{n}} \xrightarrow{\sim} C \otimes_\tau \bigotimes_\mu (Y_i)_{i \in \underline{n}}$$

using the centrality of C. So indeed, $\mathcal{I}_{C,\tau} \subseteq \mathcal{R}$ is a closed ideal.

We now verify that the association $C \mapsto \mathcal{I}_{C,\tau}$ defines a functor

$$\mathsf{ZAugIdem}_{\mathcal{R}} \xrightarrow{\mathcal{I}_{(-),\tau}} \mathbf{Idl}_{\mathcal{R}} \ :$$

given a morphism

$$
\begin{tikzcd}
C \arrow[rr, "\alpha"] \arrow[dr, "\varepsilon_C"'] & & C' \arrow[dl, "\varepsilon_{C'}"] \\
& \mathbb{1}_{\mathcal{R}} &
\end{tikzcd}
\tag{4.2.12}
$$

in $\mathsf{ZAugIdem}_{\mathcal{R}}$, we must verify the inclusion $\mathcal{I}_{C,\tau} \subseteq \mathcal{I}_{C',\tau}$. Using the equality $\mathcal{I}_{C,\tau} = \langle C \rangle^{\otimes}$, it suffices to verify that $C \in \mathcal{I}_{C',\tau}$. For this, we apply the functor $C \otimes_\tau (-)$ to the commutative triangle (4.2.12), which yields a retraction diagram

$$
\begin{tikzcd}
C \otimes_\tau C \arrow[rr, "\mathsf{id}_C \otimes_\tau \alpha"] \arrow[dr, "\sim", "\mathsf{id}_C \otimes_\tau \varepsilon_C"'] & & C \otimes_\tau C' \arrow[dl, "\mathsf{id}_C \otimes_\tau \varepsilon_{C'}"] \\
& C \otimes_\tau \mathbb{1}_{\mathcal{R}} &
\end{tikzcd}
,
$$

which proves the claim since $(C \otimes_\tau C') \in \mathcal{I}_{C',\tau}$ and $\mathcal{I}_{C',\tau} \subseteq \mathcal{R}$ is closed under retracts.

We now prove that the subcategory $\mathsf{ZAugIdem}_{\mathcal{R}} \subseteq \mathcal{R}_{/\mathbb{1}_{\mathcal{R}}}$ is a poset, i.e. the first half of part (1). Suppose there exists a morphism $C \to C'$ in $\mathsf{ZAugIdem}_{\mathcal{R}}$. As we have just seen, this implies that $C \in \mathcal{I}_{C',\tau} \subseteq \mathcal{R}$. Hence, we find that

$$
\mathsf{hom}_{\mathsf{ZAugIdem}_{\mathcal{R}}}(C, C') := \mathsf{hom}_{\mathcal{R}_{/\mathbb{1}_{\mathcal{R}}}}(C, C') \simeq \mathsf{hom}_{(\mathcal{I}_{C',\tau})_{/C'}}(C, C') \simeq \mathsf{pt} ,
$$

as desired.

Now, given any closed ideal $\mathcal{I} \in \mathbf{Idl}_{\mathcal{R}}$, it is clear that $i_L(\mathbb{1}_{\mathcal{I}}) \in \mathcal{R}_{/\mathbb{1}_{\mathcal{R}}}$ is a central augmented idempotent, and moreover that a morphism $\mathcal{I} \subseteq \mathcal{I}'$ in $\mathbf{Idl}_{\mathcal{R}}$ determines a morphism $i_L(\mathbb{1}_{\mathcal{I}}) \to i_L(\mathbb{1}_{\mathcal{I}'})$ in $\mathcal{R}_{\mathbb{1}_{\mathcal{R}}}$: in other words, the association $\mathcal{I} \mapsto i_L(\mathbb{1}_{\mathcal{I}})$ defines a functor

$$
\mathsf{ZAugIdem}_{\mathcal{R}} \xleftarrow{i_L(\mathbb{1}_{(-)})} \mathbf{Idl}_{\mathcal{R}} .
$$

From here, we immediately obtain the mutually inverse equivalences on the left in diagram (4.2.8). It is straightforward to verify the horizontal mutually inverse equivalences in diagram (4.2.8). Part (2) immediately follows, as do part (4) and the second half of part (1). $\square$

Corollary 4.2.15. *Assume that $\mathcal{O}$ is quadratic and that $\mathcal{O}(2)$ is connected (e.g. $\mathcal{O} = \mathbb{E}_n$ for $2 \leq n \leq \infty$). Let $\mathcal{R} \xrightarrow{F} \mathcal{R}'$ be a morphism in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Pr}^L_{\mathsf{st}})$, i.e. an $\mathcal{O}$-monoidal left adjoint functor between presentably $\mathcal{O}$-monoidal stable ∞-categories. Then for any closed ideal $\mathcal{I} \in \mathbf{Idl}_{\mathcal{R}}$, the ideal*

$$
\mathcal{I}' := \langle F(\mathcal{I}) \rangle^{\otimes} \subseteq \mathcal{R}'
$$

is a closed ideal of $\mathcal{R}'$, and moreover

$$
i_L(\mathbb{1}_{\mathcal{I}'}) \simeq F(i_L(\mathbb{1}_{\mathcal{I}})) \in \mathsf{ZAugIdem}_{\mathcal{R}'} \qquad \textit{and} \qquad \nu(\mathbb{1}_{\mathcal{R}'/\mathcal{I}'}) \simeq F(\nu(\mathbb{1}_{\mathcal{R}/\mathcal{I}})) \in \mathsf{ZcoAugIdem}_{\mathcal{R}'} .
$$

Proof. Because F is $\mathcal{O}$-monoidal, it preserves co/augmented idempotents. Moreover, by Observation 4.2.13, our assumptions on $\mathcal{O}$ imply that the condition of centrality is vacuous, so that we obtain factorizations

$$
\begin{tikzcd}
\mathcal{R}_{/\mathbb{1}_{\mathcal{R}}} \arrow[r, "F"] & \mathcal{R}'_{/\mathbb{1}_{\mathcal{R}'}} \\
\mathsf{ZAugIdem}_{\mathcal{R}} \arrow[u, hook] \arrow[r, dashed, "F"'] & \mathsf{ZAugIdem}_{\mathcal{R}'} \arrow[u, hook]
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathcal{R}_{\mathbb{1}_{\mathcal{R}}/} \arrow[r, "F"] & \mathcal{R}'_{\mathbb{1}_{\mathcal{R}'}/} \\
\mathsf{ZcoAugIdem}_{\mathcal{R}} \arrow[u, hook] \arrow[r, dashed, "F"'] & \mathsf{ZcoAugIdem}_{\mathcal{R}'} \arrow[u, hook]
\end{tikzcd}
.
$$

Now, using Proposition 4.2.14(2), we find that

$$
\mathcal{R}' \supseteq \mathcal{I}' := \langle F(\mathcal{I}) \rangle^{\otimes} = \left\langle F(\langle i_L(\mathbb{1}_{\mathcal{I}}) \rangle^{\otimes}) \right\rangle^{\otimes} = \langle F(i_L(\mathbb{1}_{\mathcal{I}})) \rangle^{\otimes}
$$

is indeed a closed ideal with $i_L(\mathbb{1}_{\mathcal{J}'}) \simeq F(i_L(\mathbb{1}_{\mathcal{J}}))$. Using this, we compute that

$$\nu(\mathbb{1}_{\mathcal{R}'/\mathcal{J}'}) \simeq \mathsf{cofib}(i_L(\mathbb{1}_{\mathcal{J}'}) \xrightarrow{\varepsilon} \mathbb{1}_{\mathcal{R}'}) \simeq \mathsf{cofib}(F(i_L(\mathbb{1}_{\mathcal{J}}) \xrightarrow{\varepsilon} \mathbb{1}_{\mathcal{R}})) \simeq F(\mathsf{cofib}(i_L(\mathbb{1}_{\mathcal{J}}) \xrightarrow{\varepsilon} \mathbb{1}_{\mathcal{R}})) \simeq F(\nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}})) ,$$

as desired. □

Remark 4.2.16. Let us assume for simplicity that $\mathcal{O} = \mathsf{Comm}$, and let us simply write $\underline{\mathsf{hom}}_{\mathcal{R}}(-,-)$ for the internal hom bifunctor of $\mathcal{R}$. Then, in light of Observation 4.2.10(1) we may identify the composite adjoints

$$\begin{tikzcd} i_L(\mathbb{1}_{\mathcal{J}}) \otimes (-) : \mathcal{R} \arrow[r, shift left, "y"] \arrow[r, phantom, "\perp"] & \mathcal{J} \arrow[l, hook, shift left, "i_R"] \arrow[r, hook, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{R} : \underline{\mathsf{hom}}_{\mathcal{R}}(i_L(\mathbb{1}_{\mathcal{J}}), -) \arrow[l, shift left, "y"] \end{tikzcd} .$$

If $i_L(\mathbb{1}_{\mathcal{J}}) \in \mathcal{R}$ is dualizable, then the composite right adjoint admits a further identification

$$i_R y \simeq \underline{\mathsf{hom}}_{\mathcal{R}}(i_L(\mathbb{1}_{\mathcal{J}}), -) \simeq i_L(\mathbb{1}_{\mathcal{J}})^{\vee} \otimes (-) ,$$

in which case it itself admits a further right adjoint. Because y is a left adjoint and i_R is fully faithful, this is the case if and only if i_R itself admits a further right adjoint. Likewise, in light of Observation 4.2.10(2) we may identify the composite adjoints

$$\begin{tikzcd} \nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}}) \otimes (-) : \mathcal{R} \arrow[r, shift left, "p_L"] \arrow[r, phantom, "\perp"] & \mathcal{R}/\mathcal{J} \arrow[l, hook, shift left, "\nu"] \arrow[r, hook, shift left, "\nu"] \arrow[r, phantom, "\perp"] & \mathcal{R} : \underline{\mathsf{hom}}_{\mathcal{R}}(\nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}}), -) \arrow[l, shift left, "p_R"] \end{tikzcd} .$$

Now, the dualizability of $\nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}})$ implies the further identification

$$\nu p_R \simeq \underline{\mathsf{hom}}_{\mathcal{R}}(\nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}}), -) \simeq \nu(\mathbb{1}_{\mathcal{R}/\mathcal{J}})^{\vee} \otimes (-) ,$$

which implies that this composite right adjoint itself admits a further right adjoint. Because ν is a fully faithful left adjoint, this is the case if and only if p_R admits a further right adjoint.[109] See e.g. [BDS16] for more on these considerations.

4.3. **$\mathcal{O}$-monoidal stratifications.** In this subsection, we define $\mathcal{O}$-monoidal stratifications and study their basic properties. We also discuss the chromatic stratification of $\mathcal{S}\mathsf{p}$ (Example 4.3.8).

Local Notation 4.3.1. For the remainder of this section, we fix a poset P.

Definition 4.3.2. A prestratification of $\mathcal{R}$ over P is an $\mathcal{O}$***-monoidal prestratification*** if it admits a factorization

$$\begin{tikzcd} \mathsf{P} \arrow[rr] \arrow[dr, dashed] & & \mathbf{Cls}_{\mathcal{R}} \\ & \mathbf{Idl}_{\mathcal{R}} \arrow[ur, hook, "\text{f.f.}"'] & \end{tikzcd} .$$

An $\mathcal{O}$-monoidal prestratification is an $\mathcal{O}$***-monoidal stratification*** if its underlying prestratification is a stratification.

Observation 4.3.3. Suppose that

$$\mathsf{P} \xrightarrow{\mathcal{J}_\bullet} \mathbf{Idl}_{\mathcal{R}}$$

is an $\mathcal{O}$-monoidal prestratification. By Observation 4.2.9, for any $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}}$ we have

$$\mathcal{J}_{\mathsf{D}} := \langle \mathcal{J}_p \rangle_{p \in \mathsf{D}} \in \mathbf{Idl}_{\mathcal{R}} \subseteq \mathbf{Cls}_{\mathcal{R}} .$$

Notation 4.3.4. In the setting of Observation 4.3.3, we write

$$\mathbb{1}_{\mathcal{J}_{\mathsf{D}}} := y(\mathbb{1}_{\mathcal{R}}) \in \mathcal{J}_{\mathsf{D}}$$

for the $\mathcal{O}$-monoidal unit object of $\mathcal{J}_{\mathsf{D}}$.

[109] It is not hard to see that i_R admits a further right adjoint if and only if p_R does.

Remark 4.3.5. Suppose that

$$\mathsf{P} \xrightarrow{\mathcal{J}_\bullet} \mathbf{Idl}_{\mathcal{R}}$$

is an $\mathcal{O}$-monoidal stratification. Then, for any $p \leq q$ in P we have an equivalence

$$i_L(\mathbb{1}_{\mathcal{J}_p}) \otimes i_L(\mathbb{1}_{\mathcal{J}_q}) \xrightarrow{\sim} i_L(\mathbb{1}_{\mathcal{J}_p}) .$$

More generally, for any $\mathsf{D} \to \mathsf{E}$ in $\mathsf{Down}_{\mathsf{P}}$ we have an equivalence

$$i_L(\mathbb{1}_{\mathcal{J}_{\mathsf{D}}}) \otimes i_L(\mathbb{1}_{\mathcal{J}_{\mathsf{E}}}) \xrightarrow{\sim} i_L(\mathbb{1}_{\mathcal{J}_{\mathsf{D}}}) .$$

Conversely, with the evident notation, there exists an (automatically unique) extension

$$\begin{tikzcd} \iota_0\mathsf{P} \arrow[r, "\mathcal{J}_\bullet"] \arrow[d, hook] & \mathbf{Idl}_{\mathcal{R}} \\ \mathsf{P} \arrow[ur, dashed] & \end{tikzcd}$$

if and only if for any $p \leq q$ in P the canonical morphism

$$i_L(\mathbb{1}_{\mathcal{J}_p}) \otimes i_L(\mathbb{1}_{\mathcal{J}_q}) \longrightarrow i_L(\mathbb{1}_{\mathcal{J}_p})$$

is an equivalence.

Observation 4.3.6. An $\mathcal{O}$-monoidal prestratification

$$\mathsf{P} \xrightarrow{\mathcal{J}_\bullet} \mathbf{Idl}_{\mathcal{R}}$$

is a(n automatically $\mathcal{O}$-monoidal) stratification if and only if for any $p, q \in \mathsf{P}$ the canonical morphism

$$i_L(\mathbb{1}_{\mathcal{J}_p}) \otimes i_L(\mathbb{1}_{\mathcal{J}_q}) \otimes i_L\left(\mathbb{1}_{\mathcal{J}_{(\leq p)\cap(\leq q)}}\right) \longrightarrow i_L(\mathbb{1}_{\mathcal{J}_p}) \otimes i_L(\mathbb{1}_{\mathcal{J}_q})$$

is an equivalence.

Observation 4.3.7. Suppose that

$$\mathsf{P} \xrightarrow{\mathcal{J}_\bullet} \mathbf{Idl}_{\mathcal{R}}$$

is an $\mathcal{O}$-monoidal prestratification. For each $\mathsf{Q} \in \mathsf{Down}_{\mathsf{P}}$, this restricts to an $\mathcal{O}$-monoidal prestratification

$$\mathsf{Q} \xrightarrow{\mathcal{J}_\bullet} \mathbf{Idl}_{\mathcal{J}_{\mathsf{Q}}} .$$

Hence, for every $p \in \mathsf{P}$ the geometric localization functor

$$\Phi_p : \mathcal{R} \xrightarrow{y} \mathcal{J}_p \xrightarrow{p_L} \mathcal{J}_p/\mathcal{J}_{<p} =: \mathcal{R}_p$$

is $\mathcal{O}$-monoidal. It follows from the composite adjunction

$$\begin{tikzcd} \Phi_p : \mathcal{R} \arrow[r, shift left, "y"] \arrow[r, phantom, "\perp"] & \mathcal{J}_p \arrow[l, shift left, "i_R"] \arrow[r, shift left, "p_L"] \arrow[r, phantom, "\perp"] & \mathcal{R}_p : \rho^p \arrow[l, shift left, "\nu"] \end{tikzcd}$$

that its right adjoint ρ^p is right-laxly $\mathcal{O}$-monoidal. So for every $p \leq q$, the gluing functor

$$\Gamma^p_q : \mathcal{R}_p \overset{\rho^p}{\hookrightarrow} \mathcal{R} \xrightarrow{\Phi_q} \mathcal{R}_q$$

is right-laxly $\mathcal{O}$-monoidal.

Example 4.3.8 (the chromatic stratification of spectra)**.** Consider the presentably symmetric monoidal stable ∞-category $\mathcal{R} = \mathcal{S}\mathsf{p}$ of spectra. We introduce the following notation.

- We write $n \in \mathbb{N}$ for an arbitrary (finite, positive) natural number.
- We respectively write $K_p(n)$ and $E_{p,n}$ for the n^{th} Morava K- and E-theory spectra at the prime p. By convention, we also set $K_p(0) = E_{p,0} = \mathbb{Q}$.

$$
\mathsf{P}_{\mathcal{S}\mathsf{p}} = \left(\begin{tikzcd}[column sep=small]
 & & (0) & \\
((2),1) \arrow[urr] & ((3),1) \arrow[ur] & ((5),1) \arrow[u] & \cdots \arrow[ul] \\
((2),2) \arrow[u] & ((3),2) \arrow[u] & ((5),2) \arrow[u] & \cdots \\
((2),3) \arrow[u] & ((3),3) \arrow[u] & ((5),3) \arrow[u] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \cdots \\
((2),\infty) \arrow[u] & ((3),\infty) \arrow[u] & ((5),\infty) \arrow[u] & \cdots
\end{tikzcd} \right)
$$

FIGURE 4. The poset $\mathsf{P}_{\mathcal{S}\mathsf{p}}$ is the union of the totally ordered sets $\{\{((p),\infty) \to \cdots \to ((p),2) \to ((p),1) \to (0)\}\}_{p \text{ prime}}$ over their common maximal element (0).

- For any $E \in \mathcal{S}\mathsf{p}$, we respectively write L_E and $A_E := \mathsf{fib}(\mathsf{id}_{\mathcal{S}\mathsf{p}} \to L_E)$ for E-localization and E-acyclification. We simply write $L_{(p)}$ for p-localization, $L_{p,n}$ for $E_{p,n}$-localization, and $L_{p,\infty}$ for $(\bigoplus_{0 \leq n < \infty} E_{p,n})$-localization, and similarly for the corresponding acyclifications.[110]

Then, the ***chromatic stratification*** is a symmetric monoidal stratification of $\mathcal{S}\mathsf{p}$ over the poset $\mathsf{P}_{\mathcal{S}\mathsf{p}}$ described in Figure 4:[111] namely, it is the functor

$$
\begin{array}{ccc}
\mathsf{P}_{\mathcal{S}\mathsf{p}} & \xrightarrow{\mathcal{I}_\bullet} & \mathbf{Idl}_{\mathcal{S}\mathsf{p}} \\
\Psi & & \Psi \\
\mathfrak{p} & \longmapsto & \mathcal{I}_{\mathfrak{p}}
\end{array}
\tag{4.3.1}
$$

defined by the assignments

$$
\mathcal{I}_{(0)} = \mathcal{S}\mathsf{p} \ , \qquad \mathcal{I}_{((p),n)} = A_{p,n-1} L_{(p)} \mathcal{S}\mathsf{p} \ , \qquad \text{and} \qquad \mathcal{I}_{((p),\infty)} = A_{p,\infty} L_{(p)} \mathcal{S}\mathsf{p} \ .
$$

So, the minimal strata are equivalent to dissonant p-local spectra, while the remaining strata and their geometric localization adjunctions may be identified as

$$
\begin{tikzcd}
\mathcal{S}\mathsf{p} \arrow[r, shift left, "\Phi_{(0)} = \mathbb{Q} \otimes_{\mathbb{S}} (-)"] \arrow[r, phantom, "\perp"] & \mathsf{Mod}_{\mathbb{Q}} \simeq \mathcal{S}\mathsf{p}_{(0)} \arrow[l, shift left, hook, "\rho^{(0)}"]
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathcal{S}\mathsf{p} \arrow[r, shift left, "\Phi_{((p),n)} = L_{K_p(n)}"] \arrow[r, phantom, "\perp"] & L_{K_p(n)} \mathcal{S}\mathsf{p} \simeq \mathcal{S}\mathsf{p}_{((p),n)} \arrow[l, shift left, hook, "\rho^{((p),n)}"]
\end{tikzcd}
$$

(simply by verifying that their kernels are respectively $A_{\mathbb{Q}} \mathcal{S}\mathsf{p}$ and $A_{K_p(n)} \mathcal{S}\mathsf{p}$). The poset $\mathsf{P}_{\mathcal{S}\mathsf{p}}$ is not down-finite, and indeed the chromatic stratification (4.3.1) fails to converge for essentially the same reasons that the adelic stratification of $\mathsf{Mod}_{\mathbb{Z}}$ fails to converge as illustrated in Example 1.6.1.

Of course, the failure of the poset $\mathsf{P}_{\mathcal{S}\mathsf{p}}$ to be down-finite is not simply due to the infinitude of the primes, but also to its failure to be artinian. Let us therefore study the chromatic stratification of $L_{(p)} \mathcal{S}\mathsf{p}$, a quotient stratification (in the sense of Proposition 3.4.10) of the chromatic stratification (4.3.1) of $\mathcal{S}\mathsf{p}$: namely, writing

$$
\mathsf{P}_{\mathcal{S}\mathsf{p}} \supset \mathsf{P}_{L_{(p)} \mathcal{S}\mathsf{p}} = \{((p),\infty) \longrightarrow \cdots \longrightarrow ((p),3) \longrightarrow ((p),2) \longrightarrow ((p),1) \longrightarrow (0)\}
$$

---

[110] We use this notation because it is standard, but note that it mildly conflicts with that of Definition 2.4.6.

[111] The adelic stratification of $\mathcal{S}\mathsf{p}$ is also defined over the poset $\mathsf{P}_{\mathcal{S}\mathsf{p}}$, but it is slightly different (see Example 4.6.13).

and employing the identification $\mathsf{P}_{L_{(p)}\mathcal{S}\mathsf{p}} \cong (\mathbb{N}^{\triangleleft\triangleright})^{\mathsf{op}} = \{\infty \to \cdots \to 2 \to 1 \to 0\}$ for notational simplicity, this is the functor

$$\begin{array}{ccc} \mathsf{P}_{L_{(p)}\mathcal{S}\mathsf{p}} & \xrightarrow{\mathcal{J}_\bullet} & \mathbf{Idl}_{L_{(p)}\mathcal{S}\mathsf{p}} \\ \uplus & & \uplus \\ \mathfrak{p} & \longmapsto & \mathcal{J}_{\mathfrak{p}} \end{array} \tag{4.3.2}$$

given by

$$\mathcal{J}_0 = L_{(p)}\mathcal{S}\mathsf{p} \ , \qquad \mathcal{J}_n = \mathcal{I}_{((p),n)} = A_{p,n-1}L_{(p)}\mathcal{S}\mathsf{p} \ , \qquad \text{and} \qquad \mathcal{J}_\infty = \mathcal{I}_{((p),\infty)} = A_{p,\infty}L_{(p)}\mathcal{S}\mathsf{p} \ .$$

In order to understand the behavior of the chromatic stratification (4.3.2) of $L_{(p)}\mathcal{S}\mathsf{p}$, we pass further to its quotient stratification over

$$[n]^{\mathsf{op}} = \{n \to \cdots \to 0\} \cong (\mathbb{N}^{\triangleleft\triangleright})^{\mathsf{op}} \backslash ({}^{\leq}(n+1)) \ :$$

this provides a (necessarily convergent) stratification of

$$\mathcal{J}_0/\mathcal{J}_{n+1} := L_{(p)}\mathcal{S}\mathsf{p}/A_{p,n}L_{(p)}\mathcal{S}\mathsf{p} \simeq L_{p,n}\mathcal{S}\mathsf{p}$$

over $[n]^{\mathsf{op}}$, whose microcosm reconstruction theorem recovers the n-dimensional fracture cube of [ACB22] (recall Example 2.5.16). Hence, the chromatic stratification (4.3.2) fails to converge as a result of the difference between harmonic localization and chromatic completion [Bar16].

4.4. **$\mathcal{O}$-algebra objects in $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}$.** In this subsection, we define the ∞-category that contains the $\mathcal{O}$-monoidal gluing diagram of an $\mathcal{O}$-monoidal stratification.

Local Notation 4.4.1. In this subsection, we fix an ∞-category $\mathcal{B}$.

Observation 4.4.2. In the composite

$$\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} \hookrightarrow \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}} \xrightarrow{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}} \mathsf{Cat} \ ,$$

all three ∞-categories admit finite products and both functors preserve them.

Notation 4.4.3. We write

$$\mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}) \subseteq \mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}})$$

for the full subcategory on the reduced Segal objects.

Observation 4.4.4. In light of Observation 4.4.2, we obtain a canonical lift

$$\begin{tikzcd} \mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}) \arrow[r, dashed, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}"] \arrow[d, "\mathsf{fgt}"'] & \mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \arrow[d, "\mathsf{fgt}"] \\ \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}} \arrow[r, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}"'] & \mathsf{Cat} \end{tikzcd}$$

as the restriction to reduced Segal objects of the value of the functor $\mathsf{Fun}(\mathcal{O}^{\otimes}, -)$ on the finite-product-preserving functor $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}} \xrightarrow{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}} \mathsf{Cat}$.

Observation 4.4.5. There is a canonical equivalence

$$\iota_0 \mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}) \simeq \mathsf{hom}_{2\mathsf{Cat}}(\mathsf{l.lax}(\mathcal{B}), \mathsf{Alg}^{\mathsf{r.lax}}_{\mathcal{O}}(\mathsf{Cat})) \tag{4.4.1}$$

of spaces. Indeed, by Theorem B.4.1 we have an equivalence

$$\mathsf{hom}_{\mathsf{Cat}}(\mathcal{O}^{\otimes}, \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}) := \mathsf{hom}_{2\mathsf{Cat}}(\mathcal{O}^{\otimes}, 2\mathsf{Cat}_{\mathsf{1cart}/\mathsf{l.lax}(\mathcal{B})^{\mathsf{1op}}}) \simeq \mathsf{hom}_{2\mathsf{Cat}}(\mathsf{l.lax}(\mathcal{B}), \mathsf{Cat}_{\mathsf{cocart}/\mathcal{O}^{\otimes}})$$

of spaces, through which the equivalence (4.4.1) can be obtained as an equivalence of subspaces.[112]

[112]On the other hand, the ∞-categories $\mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}})$ and $\mathsf{hom}_{2\mathsf{Cat}}(\mathsf{l.lax}(\mathcal{B}), \mathsf{Alg}^{\mathsf{r.lax}}_{\mathcal{O}}(\mathsf{Cat}))$ are not equivalent; this can already be seen in the case that $\mathcal{B} = \mathsf{pt}$.

4.5. **The $\mathcal{O}$-monoidal reconstruction theorem.** In this subsection, we prove our $\mathcal{O}$-monoidal reconstruction theorem.

Theorem 4.5.1. *Let $\mathcal{R}$ be a presentably $\mathcal{O}$-monoidal stable ∞-category, let P be a poset, and let*

$$\mathsf{P} \xrightarrow{\mathcal{I}_\bullet} \mathbf{Idl}_{\mathcal{R}}$$

be an $\mathcal{O}$-monoidal stratification.

(1) *There is a canonical lift*

$$\begin{tikzcd} \mathsf{P} \arrow[r, dashed, "\mathscr{G}^{\otimes}(\mathcal{R})", "\mathsf{l.lax}"'] \arrow[dr, "\mathsf{l.lax}", "\mathscr{G}(\mathcal{R})"'] & \mathsf{Alg}^{\mathsf{r.lax}}_{\mathcal{O}}(\mathsf{Cat}) \arrow[d, "\mathsf{fgt}"] \\ & \mathsf{Cat} \end{tikzcd}$$

of the gluing diagram of the underlying stratification of $\mathcal{R}$ to an $\mathcal{O}$-monoidal gluing diagram.

(2) *There is a canonical morphism*

$$\mathcal{R} \xrightarrow{g^{\otimes}} \mathsf{Glue}^{\otimes}(\mathcal{R}) := \lim{}^{\mathsf{r.lax}} \left(\mathsf{P} \xrightarrow[\mathsf{l.lax}]{\mathscr{G}^{\otimes}(\mathcal{R})} \mathsf{Alg}^{\mathsf{r.lax}}_{\mathcal{O}}(\mathsf{Cat}) \right) \tag{4.5.1}$$

in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat})$ whose image under the functor $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \xrightarrow{\mathsf{fgt}} \mathsf{Cat}$ is the morphism

$$\mathcal{R} \xrightarrow{g} \mathsf{Glue}(\mathcal{R}) := \lim{}^{\mathsf{r.lax}} \left(\mathsf{P} \xrightarrow[\mathsf{l.lax}]{\mathscr{G}(\mathcal{R})} \mathsf{Cat} \right) \tag{4.5.2}$$

in Cat, so that the adjunction

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "g"] \arrow[r, phantom, "\perp"] & \mathsf{Glue}(\mathcal{R}) \arrow[l, shift left, "\lim_{\mathsf{sd}(\mathsf{P})}"] \end{tikzcd}$$

between ∞-categories admits a canonical enhancement to an adjunction

$$\begin{tikzcd} \mathcal{R} \arrow[r, shift left, "g^{\otimes}"] \arrow[r, phantom, "\perp"] & \mathsf{Glue}^{\otimes}(\mathcal{R}) \arrow[l, shift left, "\lim^{\otimes}_{\mathsf{sd}(\mathsf{P})}"] \end{tikzcd}$$

between $\mathcal{O}$-monoidal ∞-categories whose left adjoint is $\mathcal{O}$-monoidal and whose right adjoint is right-laxly $\mathcal{O}$-monoidal.

In particular, if the morphism (4.5.2) *is an equivalence then the morphism* (4.5.1) *is also an equivalence.*

Proof. We begin with part (1). By Observation 4.4.5, it is equivalent to enhance the gluing diagram

$$\mathscr{G}(\mathcal{R}) \in \mathsf{LMod}_{\mathsf{l.lax.P}} \subseteq \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$$

to an $\mathcal{O}$-monoidal gluing diagram

$$\mathscr{G}^{\otimes}(\mathcal{R}) \in \mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}) \subseteq \mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}) .$$

We use Lemma A.3.4 to construct this as a locally cocartesian fibration over $\mathcal{O}^{\otimes} \times \mathsf{P}$, which we define to be the full subcategory

$$\begin{tikzcd} \mathscr{G}^{\otimes}(\mathcal{R}) \arrow[r, hook, "\mathsf{f.f.}"] \arrow[d] & \mathcal{R}^{\otimes} \times \mathsf{P} \arrow[dl] \\ \mathcal{O}^{\otimes} \times \mathsf{P} \end{tikzcd}$$

on the objects

$$\{((X_1, \ldots, X_n), p) \in \mathcal{R}^{\otimes} \times \mathsf{P} : X_i \in \mathcal{R}_p \subseteq \mathcal{R} \text{ for all } i\} .$$[113]

[113]Because we are working over both P and $\mathcal{O}^{\otimes} \times \mathsf{P}$ in this proof, we avoid the potentially ambiguous notation $\underline{\mathcal{R}^{\otimes}}$ for $\mathcal{R}^{\otimes} \times \mathsf{P}$.

We first observe that this is indeed a locally cocartesian fibration over $\mathcal{O}^{\otimes} \times \mathsf{P}$: over a morphism $(S_+, p) \xrightarrow{\widetilde{\alpha}} (T_+, q)$ in $\mathcal{O}^{\otimes} \times \mathsf{P}$ lying over a morphism $S_+ \xrightarrow{\alpha} T_+$ in $\mathcal{O}^{\otimes}$, for any $X \in (\mathcal{R}_p)^{\times S}$ and $Y \in (\mathcal{R}_q)^{\times T}$ we have the string of equivalences

$$\begin{aligned}
\mathsf{hom}^{\widetilde{\alpha}}_{\mathcal{R}^{\otimes} \times \mathsf{P}}(((\rho^p)^{\times S}(X), p), ((\rho^q)^{\times T}(Y), q)) &\simeq \mathsf{hom}^{\alpha}_{\mathcal{R}^{\otimes}}((\rho^p)^{\times S}(X), (\rho^q)^{\times T}(Y)) && (4.5.3)\\
&\simeq \mathsf{hom}_{\mathcal{R}^{\times T}}(\alpha_*(\rho^p)^{\times S}(X), (\rho^q)^{\times T}(Y)) && (4.5.4)\\
&\simeq \mathsf{hom}_{(\mathcal{R}_q)^{\times T}}((\Phi_q)^{\times T}\alpha_*(\rho^p)^{\times S}(X), Y) && (4.5.5)\\
&\simeq \mathsf{hom}_{(\mathcal{R}_q)^{\times T}}(\alpha_*(\Phi_q)^{\times S}(\rho^p)^{\times S}(X), Y) && (4.5.6)\\
&=: \mathsf{hom}_{(\mathcal{R}_q)^{\times T}}(\alpha_*(\Phi_q\rho^p)^{\times S}(X), Y)\\
&=: \mathsf{hom}_{(\mathcal{R}_q)^{\times T}}(\alpha_*(\Gamma^p_q)^{\times S}(X), Y) \ ,
\end{aligned}$$

in which

- equivalence (4.5.3) follows from the fact that P is a poset,
- equivalence (4.5.4) follows from the fact that $\mathcal{R}^{\otimes} \to \mathcal{O}^{\otimes}$ is a locally cocartesian fibration,
- equivalence (4.5.5) follows from the adjunction $\Phi_q \dashv \rho^q$ (or really the adjunction $(\Phi_q)^{\times T} \dashv (\rho^q)^{\times T}$), and
- equivalence (4.5.6) follows from the fact that Φ_q is $\mathcal{O}$-monoidal.

This identification of the cocartesian monodromy in the locally cocartesian fibration $\mathscr{G}^{\otimes}(\mathcal{R}) \downarrow (\mathcal{O}^{\otimes} \times \mathsf{P})$ immediately implies condition (2) of Lemma A.3.4, and for its condition (1) we observe that for each $p \in \mathsf{P}$ the pullback to $\mathcal{O}^{\otimes} \times \{p\}$ is the cocartesian fibration $(\mathcal{R}_p)^{\otimes} \downarrow \mathcal{O}^{\otimes}$. Thus, we have indeed constructed a functor

$$\mathcal{O}^{\otimes} \xrightarrow{\mathscr{G}^{\otimes}(\mathcal{R})} \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}} \ .$$

This is moreover a reduced Segal functor, which evaluates on each object $\underline{n}_+ \in \mathcal{O}^{\otimes}$ as the locally cocartesian fibration $\mathscr{G}(\mathcal{R})^{\times_{\mathsf{P}} n} \downarrow \mathsf{P}$ (the n-fold fiber product with itself of the locally cocartesian fibration $\mathscr{G}(\mathcal{R}) \downarrow \mathsf{P}$ (recall Observation 4.4.2)). In other words, it defines an object

$$\mathscr{G}^{\otimes}(\mathcal{R}) \in \mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}})$$

lifting the object $\mathscr{G}(\mathcal{R}) \in \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$, as desired.

We now proceed to part (2).

We begin by constructing a morphism

$$\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{R}^{\otimes} \times \mathsf{P}) \longrightarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}^{\otimes}(\mathcal{R})) \ . \qquad (4.5.7)$$

By definition, we have a morphism

$$\mathcal{R}^{\otimes} \times \mathsf{P} \longleftarrow \mathscr{G}^{\otimes}(\mathcal{R})$$

in $\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{l.lax.}(\mathcal{O}^{\otimes} \times \mathsf{P})}$, which on each fiber is a right adjoint: over the object $(S_+, p) \in \mathsf{Fin}_* \times \mathsf{P}$ it is the product right adjoint

$$\mathcal{R}^{\times S} \xleftarrow{(\rho^p)^{\times S}} (\mathcal{R}_p)^{\times S} \ .$$

By Lemma A.3.5, the fiberwise left adjoints

$$\mathcal{R}^{\times S} \xrightarrow{(\Phi_p)^{\times S}} (\mathcal{R}_p)^{\times S}$$

therefore assemble into a morphism

$$\mathcal{R}^{\otimes} \times \mathsf{P} \longrightarrow \mathscr{G}^{\otimes}(\mathcal{R}) \qquad (4.5.8)$$

in $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\mathcal{O}^{\otimes}\times\mathsf{P})}$. We consider this morphism as a point in the lower right space in the diagram

$$\begin{tikzcd}
\mathsf{hom}_{\mathsf{Cat}}([1],\mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}})) \arrow[d, hook] \\
\mathsf{hom}_{\mathsf{Cat}}([1],\mathsf{Fun}(\mathcal{O}^{\otimes},\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}})) \arrow[d, phantom, "\wr"] \\
\mathsf{hom}_{\mathsf{Cat}}([1]\times\mathcal{O}^{\otimes},\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}) \arrow[d, hook] \\
\iota_0\mathsf{loc.coCart}_{[1]\times\mathcal{O}^{\otimes}\times\mathsf{P}} & \mathsf{hom}_{\mathsf{Cat}}([1],\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\mathcal{O}^{\otimes}\times\mathsf{P})}) \ni (4.5.8) \arrow[l, hook']
\end{tikzcd} \tag{4.5.9}$$

of spaces, in which the upper vertical inclusion is definitional and the other two inclusions follow from Lemma A.3.4. As such, we aim to show that the point (4.5.8) lies in the upper left space of diagram (4.5.9). So, let us consider its image

$$\left(\begin{array}{c} \mathcal{E} \\ \downarrow \\ [1]\times\mathcal{O}^{\otimes}\times\mathsf{P} \end{array}\right) \in \iota_0\mathsf{loc.coCart}_{[1]\times\mathcal{O}^{\otimes}\times\mathsf{P}} \ . \tag{4.5.10}$$

To first show that the point (4.5.10) factors through the lower vertical inclusion in diagram (4.5.9), we verify conditions (1) and (2) of Lemma A.3.4 in turn.

(1) For each $p \in \mathsf{P}$, the pullback

$$\begin{tikzcd}
\mathcal{E}_{|[1]\times\mathcal{O}^{\otimes}\times\{p\}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
{[1]\times\mathcal{O}^{\otimes}} \arrow[r, "(\mathsf{id}_{[1]\times\mathcal{O}^{\otimes}},\mathsf{const}_p)"'] & {[1]\times\mathcal{O}^{\otimes}\times\mathsf{P}}
\end{tikzcd}$$

is indeed a cocartesian fibration: it is classified by the morphism $\mathcal{R} \xrightarrow{\Phi_p} \mathcal{R}_p$ in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat})$.

(2) Any pair of a morphism $(i,S_+) \xrightarrow{(\leq,\alpha)} (j,T_+)$ in $[1]\times\mathcal{O}^{\otimes}$ and a morphism $p \leq q$ in P determines a functor $[2] \to [1]\times\mathcal{O}^{\otimes}\times\mathsf{P}$ classifying the commutative triangle

$$\begin{tikzcd}
(i,S_+,p) \arrow[d] \arrow[dr] & \\
(i,S_+,q) \arrow[r] & (j,T_+,q)
\end{tikzcd} ,$$

and we must show that the resulting pullback

$$\begin{tikzcd}
\mathcal{E}_{|[2]} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
{[2]} \arrow[r] & {[1]\times\mathcal{O}^{\otimes}\times\mathsf{P}}
\end{tikzcd} \tag{4.5.11}$$

defines a cocartesian fibration over $[2]$. By what we have already seen, this holds when $i = j$ (because both of the locally cocartesian fibrations $(\mathcal{R}^{\otimes} \times \mathsf{P}) \downarrow (\mathcal{O}^{\otimes} \times \mathsf{P})$ and $\mathscr{G}^{\otimes}(\mathcal{R}) \downarrow (\mathcal{O}^{\otimes} \times \mathsf{P})$ satisfy condition (2)). In the remaining case where $i = 0$ and $j = 1$, the pullback (4.5.11) is the cocartesian fibration over $[2]$ classifying the commutative triangle

$$\begin{tikzcd}
\mathcal{R}^{\times S} \arrow[d, "\mathsf{id}_{\mathcal{R}^{\times S}}"'] \arrow[dr] & \\
\mathcal{R}^{\times S} \arrow[r] & (\mathcal{R}_p)^{\times T}
\end{tikzcd}$$

in Cat in which both rightward functors coincide with the diagonal composite in the commutative square

$$\begin{tikzcd}
\mathcal{R}^{\times S} \arrow[r, "\alpha_*"] \arrow[d, "(\Phi_p)^{\times S}"'] & \mathcal{R}^{\times T} \arrow[d, "(\Phi_p)^{\times T}"] \\
(\mathcal{R}_p)^{\times S} \arrow[r, "\alpha_*"'] & (\mathcal{R}_p)^{\times T}
\end{tikzcd}$$

in Cat (which commutes because $\mathcal{R} \xrightarrow{\Phi_p} \mathcal{R}_p$ is $\mathcal{O}$-monoidal).

Hence, the point (4.5.10) does indeed factor through the lower vertical inclusion in diagram (4.5.9). Thereafter, considered as a point in $\mathsf{hom}_{\mathsf{Cat}}([1], \mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}))$, i.e. as a morphism in $\mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}})$, its source and target evidently both lie in the full subcategory $\mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}) \subseteq \mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}})$: its source is the composite

$$\mathcal{O}^{\otimes} \xrightarrow{\mathcal{R}^{\otimes}} \mathsf{Cat} \xrightarrow{-\times\mathsf{P}} \mathsf{coCart}_{\mathsf{P}} \hookrightarrow \mathsf{loc.coCart}_{\mathsf{P}} =: \mathsf{LMod}_{\mathsf{l.lax.P}} \hookrightarrow \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}} , \tag{4.5.12}$$

while its target is the object $\mathscr{G}^{\otimes}(\mathcal{R})$. Therefore, the point (4.5.10) lies in the uppermost space of diagram (4.5.9), as we aimed to show. Hence, we may take its postcomposition

$$[1] \longrightarrow \mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}) \xrightarrow{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}} \mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) ,$$

which provides the desired morphism (4.5.7).

Now, as observed above, the object

$$(\mathcal{R}^{\otimes} \times \mathsf{P}) \in \mathsf{Alg}_{\mathcal{O}}(\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}) \subseteq \mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}})$$

factors as the composite (4.5.12), so that we may identify the source $\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{R}^{\otimes} \times \mathsf{P})$ of the morphism (4.5.7) in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \subseteq \mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{Cat})$ in simple terms: it is the composite

$$\mathcal{O}^{\otimes} \xrightarrow{\mathcal{R}^{\otimes}} \mathsf{Cat} \xrightarrow{-\times\mathsf{P}^{\mathsf{op}}} \mathsf{Cart}_{\mathsf{P}^{\mathsf{op}}} \xrightarrow{\Gamma} \mathsf{Cat} ,$$

which classifies the ∞-category $\mathsf{Fun}(\mathsf{P}^{\mathsf{op}}, \mathcal{R})$ equipped with its pointwise $\mathcal{O}$-monoidal structure. This receives a canonical morphism

$$\mathcal{R} \longrightarrow \mathsf{Fun}(\mathsf{P}^{\mathsf{op}}, \mathcal{R})$$

in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat})$.[114] So, we obtain a composite comparison morphism

$$\mathcal{R} \longrightarrow \mathsf{Fun}(\mathsf{P}^{\mathsf{op}}, \mathcal{R}) \simeq \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{R}^{\otimes} \times \mathsf{P}) \xrightarrow{(4.5.7)} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}^{\otimes}(\mathcal{R}))$$

in $\mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat})$. Moreover, by construction, upon applying the forgetful functor

$$\mathsf{fgt} : \mathsf{Alg}_{\mathcal{O}}(\mathsf{Cat}) \xhookrightarrow{\mathsf{f.f.}} \mathsf{Fun}(\mathcal{O}^{\otimes}, \mathsf{Cat}) \xrightarrow{\mathsf{ev}_{1_+}} \mathsf{Cat}$$

we recover the morphism

$$\mathcal{R} \xrightarrow{g} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{R})) ,$$

as desired. $\square$

[114] This canonical morphism factors through the strict limit $\lim_{\mathsf{l.lax.P}}(\mathcal{R}^{\otimes} \times \mathsf{P})$, which can be similarly identified with $\mathsf{Fun}(|\mathsf{P}|, \mathcal{R}) \simeq \mathsf{Fun}(|\mathsf{P}^{\mathsf{op}}|, \mathcal{R})$ equipped with its pointwise $\mathcal{O}$-monoidal structure.

Remark 4.5.2. It is not possible to prove Theorem 4.5.1 directly from Theorem 2.5.14, because a presentably $\mathcal{O}$-monoidal ∞-category is not defined by a diagram in $\mathsf{Pr}^L_{\mathsf{st}}$ (as the tensor product functors are required to be multi-cocontinuous rather than cocontinuous).

4.6. **Symmetric monoidal stratifications and tensor-triangular geometry.** In this subsection, we construct the adelic stratification of a presentably symmetric monoidal stable ∞-category (satisfying mild finiteness hypotheses) as Theorem 4.6.11. This is based in the theory of tensor-triangular geometry, which we begin by reviewing; we refer the reader to the survey [Ste18] for more background on this topic, which highlights the interaction between the small and presentable settings. We unpack the adelic stratification of $\mathcal{S}\mathsf{p}$ in Example 4.6.13, and we explain how symmetric monoidal stratifications contribute to the theory of tensor-triangular geometry in Remark 4.6.14.

Observation 4.6.1. The homotopy category of a stable ∞-category is canonically triangulated, and (presentably) symmetric monoidal structures descend to (resp. exact and coproduct-preserving) symmetric monoidal structures. Through this, one can largely apply results concerning triangulated categories to stable ∞-categories without any modification; for instance, the condition of an object being zero can be checked in the homotopy category, and the projection to the homotopy category preserves co/products (indeed, this is true for any ∞-category). We use this fact without further comment.

Local Notation 4.6.2. For the remainder of this section, we specialize Local Notation 4.2.1 to further assume that $\mathcal{O} = \mathsf{Comm}$, i.e. that $\mathcal{R}$ is a presentably symmetric monoidal stable ∞-category. We assume moreover that $\mathcal{R}$ is compactly generated, and that its full subcategory $\mathcal{R}^\omega$ of compact objects inherits a symmetric monoidal structure (i.e. the unit object is compact and the tensor product of compact objects is again compact).

Definition 4.6.3. We say that $\mathcal{R}$ is ***rigidly-compactly generated*** if (in addition to the hypotheses of Local Notation 4.6.2) its full subcategory of dualizable (a.k.a. rigid) objects is precisely $\mathcal{R}^\omega \subseteq \mathcal{R}$.

Definition 4.6.4. A full proper stable subcategory $\mathfrak{p} \subsetneq \mathcal{R}^\omega$ is called a ***thick prime ideal*** if

- it is idempotent-complete,
- it is contagious under the symmetric monoidal structure, and
- for all $X, Y \in \mathcal{R}^\omega$, if $X \otimes Y \in \mathfrak{p}$ then $X \in \mathfrak{p}$ or $Y \in \mathfrak{p}$.

We write $\mathsf{P}_\mathcal{R}$ for the poset of thick prime ideal subcategories of $\mathcal{R}^\omega$ ordered by inclusion.

Definition 4.6.5. The ***Balmer spectrum*** of $\mathcal{R}^\omega$ is the topological space $\mathsf{Spec}(\mathcal{R}^\omega) \in \mathsf{Top}$ defined as follows. First of all, the underlying set of $\mathsf{Spec}(\mathcal{R}^\omega)$ is that of thick prime ideals in $\mathcal{R}^\omega$. Then, for any object $X \in \mathcal{R}^\omega$, we define its ***support*** to be the subset

$$\mathsf{supp}(X) := \{\mathfrak{p} \in \mathsf{Spec}(\mathcal{R}^\omega) : X \notin \mathfrak{p}\} .$$

Finally, the topology on $\mathsf{Spec}(\mathcal{R}^\omega)$ is obtained by declaring that the subsets $\{\mathsf{supp}(X) \subseteq \mathsf{Spec}(\mathcal{R}^\omega)\}_{X \in \mathcal{R}^\omega}$ are closed.[115]

Remark 4.6.6. The specialization poset of the topological space $\mathsf{Spec}(\mathcal{R}^\omega) \in \mathsf{Top}$ is precisely $\mathsf{P}_\mathcal{R}$: the membership $\mathfrak{p} \in \overline{\{\mathfrak{q}\}}$ is equivalent to the containment $\mathfrak{p} \subseteq \mathfrak{q}$. So, we may consider the support of an object $X \in \mathcal{R}^\omega$ either as a closed subset of $\mathsf{Spec}(\mathcal{R}^\omega)$ or as a down-closed subset of $\mathsf{P}_\mathcal{R}$.

Remark 4.6.7. Let X be a qcqs scheme. Thomason proved that there is a canonical isomorphism

$$\mathsf{Spec}(\mathsf{Perf}(X)) \cong X \tag{4.6.1}$$

[115] In fact, these subsets form a basis, so that every closed subset is of the form

$$\bigcap_{s \in S} \mathsf{supp}(X_s) = \{\mathfrak{p} \in \mathsf{Spec}(\mathcal{R}^\omega) : \{X_s\}_{s \in S} \cap \mathfrak{p} = \varnothing\}$$

for some set $\{X_s \in \mathcal{R}^\omega\}_{s \in S}$ of objects of $\mathcal{R}^\omega$.

of topological spaces [Tho97, Theorem 3.15],[116] with the correspondence being given by the support of perfect complexes. Thereafter, Balmer upgraded the topological space $\mathsf{Spec}(\mathcal{R}^\omega)$ to a ringed topological space [Bal05, Definition 6.1], in such a way that the isomorphism (4.6.1) naturally upgrades to one of ringed topological spaces (and therefore one of schemes) [Bal05, Theorem 6.3].

Notation 4.6.8. For each $\mathfrak{p} \in \mathsf{P}_\mathcal{R}$, we define the full subcategories

$$\mathcal{I}^\omega_\mathfrak{p} := \{X \in \mathcal{R}^\omega : \mathsf{supp}(X) \subseteq ({}^{\leq}\mathfrak{p})\} \subseteq \mathcal{R}^\omega \qquad \text{and} \qquad \mathcal{I}_\mathfrak{p} := \mathsf{Ind}(\mathcal{I}^\omega_\mathfrak{p}) = \langle \mathcal{I}^\omega_\mathfrak{p} \rangle \subseteq \mathcal{R} \ .$$

Observation 4.6.9. For each $\mathfrak{p} \in \mathsf{P}_\mathcal{R}$, the subcategory $\mathcal{I}_\mathfrak{p} \subseteq \mathcal{R}$ is obviously closed (recall Example 2.3.2), and in fact it is a closed ideal by [HPS97, Theorem 3.3.3] (which is an abstraction of [Mil92, Corollary 8]). As the assignment $\mathfrak{p} \mapsto \mathcal{I}_\mathfrak{p}$ is order-preserving, we therefore obtain a functor

$$\begin{array}{ccc} \mathsf{P}_\mathcal{R} & \xrightarrow{\mathcal{I}_\bullet} & \mathbf{Idl}_\mathcal{R} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ \mathfrak{p} & \longmapsto & \mathcal{I}_\mathfrak{p} \end{array} \ . \tag{4.6.2}$$

Definition 4.6.10. Whenever the functor (4.6.2) is a symmetric monoidal stratification, we refer to it as the ***adelic stratification*** of $\mathcal{R}$ over $\mathsf{P}_\mathcal{R}$.

Theorem 4.6.11. *Suppose that $\mathcal{R}$ is a rigidly-compactly generated presentably symmetric monoidal stable ∞-category, and suppose that $\mathcal{R} = \langle \mathcal{I}_\mathfrak{p} \rangle_{\mathfrak{p} \in \mathsf{P}_\mathcal{R}}$. Then, the functor* (4.6.2) *defines a symmetric monoidal stratification of $\mathcal{R}$ over $\mathsf{P}_\mathcal{R}$.*

Proof. By assumption, the functor (4.6.2) is a symmetric monoidal prestratification. Note that if $X, Y \in \mathcal{R}^\omega$ then

$$\mathsf{supp}(X \otimes Y) \subseteq \mathsf{supp}(X) \cap \mathsf{supp}(Y)$$

by definition of a thick prime ideal. Since the symmetric monoidal structure commutes with colimits separately in each variable, the stratification condition follows from Observation 4.3.6. □

Remark 4.6.12. We indicate an example in which the condition that $\mathcal{R} = \langle \mathcal{I}_\mathfrak{p} \rangle_{\mathfrak{p} \in \mathsf{P}_\mathcal{R}}$ appearing in Theorem 4.6.11 fails to hold.[117] Let S be a countably infinite set, let S^+ denote its one-point compactification, let $R := \mathsf{hom}_{\mathsf{Top}}(S^+, \mathbb{F}_2)$ denote the commutative ring of continuous $\mathbb{F}_2$-valued functions on S^+, and let $\mathcal{R} := \mathsf{QC}(\mathsf{Spec}(R))$. Then there are canonical homeomorphisms $S^+ \cong \mathsf{Spec}(R) \cong \mathsf{Spec}(\mathcal{R}^\omega)$, and in particular the specialization poset $\mathsf{P}_\mathcal{R}$ is discrete (as R has Krull dimension 0). However, the functor

$$\mathcal{R} \xrightarrow{(y)_{\mathfrak{p} \in \mathsf{P}_\mathcal{R}}} \prod_{\mathfrak{p} \in \mathsf{P}_\mathcal{R}} \mathcal{I}_\mathfrak{p}$$

is not an equivalence.

Example 4.6.13 (the adelic stratification of spectra)**.** The adelic stratification of $\mathcal{S}\mathsf{p}$ is quite similar to its chromatic stratification (Example 4.3.8): namely, it is the functor

$$\begin{array}{ccc} \mathsf{P}_{\mathcal{S}\mathsf{p}} & \xrightarrow{\mathcal{I}^f_\bullet} & \mathbf{Idl}_{\mathcal{S}\mathsf{p}} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ \mathfrak{p} & \longmapsto & \mathcal{I}^f_\mathfrak{p} \end{array}$$

defined by the assignments

$$\mathcal{I}^f_{(0)} = \mathcal{S}\mathsf{p} \ , \qquad \mathcal{I}^f_{((p),n)} = A^f_{p,n-1} L^f_{(p)} \mathcal{S}\mathsf{p} \ , \qquad \text{and} \qquad \mathcal{I}^f_{((p),\infty)} = 0 \ ,$$

where

[116] See also [Nee92] for an affine version of this result, which originates in [Hop87].

[117] We thank Scott Balchin for pointing out this example to us.

- we identify the poset of primes in the Balmer spectrum as $\mathsf{P}_{\mathcal{S}\mathsf{p}}$ (depicted in Figure 4) by [Bal10, Corollary 9.5] (see also [HS98]),
- we use the superscript f to denote the finite localization/acyclification functors [Mil92, Definition 3], and
- the identifications of the minimal strata as zero follows from the fact that finite spectra are harmonic [Rav84, Corollary 4.5].

Note that the telescope conjecture asserts that the morphisms $L^f_{p,n-1} \to L_{p,n-1}$ (or equivalently the morphisms $A^f_{p,n-1} \to A_{p,n-1}$) are equivalences.

Remark 4.6.14. We view the theory of symmetric monoidal stratifications as an important complement to the study of tensor-triangular geometry, for the following two reasons.

(1) While the Balmer spectrum has a universal property [Bal05, Theorem 3.2] it can be quite difficult to compute. By contrast, our general theory of (symmetric monoidal) stratifications is substantially more flexible.

   (a) For instance, in addition to its rather subtle stratification indicated in Example 4.3.8, the ∞-category $\mathcal{S}\mathsf{p}$ of spectra admits an "arithmetic" stratification over $\mathsf{P}_{\mathsf{Mod}_{\mathbb{Z}}}$, which behaves just as that of $\mathsf{Mod}_{\mathbb{Z}}$ itself as described in Example 1.6.1.

   (b) Likewise, as we prove in Theorem 5.1.27, for a compact Lie group G, the ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ of genuine G-spectra admits a relatively straightforward stratification over the poset P_G of closed subgroups of G; compare this with the computations of its Balmer spectrum [BS17, BHN+19, BGH20].

   This flexibility allows for the systematic study of tensor-triangulated categories that is compatible with, but not bound to, their Balmer spectra; and it is of course further augmented by the fundamental operations for (symmetric monoidal (recall Remark 1.5.4)) stratifications developed in §3.4.

(2) Our theory of symmetric monoidal stratifications appears to provide a compelling framework for studying the "presheaf of triangulated categories" that serves as motivation throughout the literature on tensor-triangular geometry (originating with [Bal02]), enhancing as it does the presheaf of commutative rings introduced in [Bal05, Definition 6.1]. In this vein, we view our symmetric monoidal reconstruction theorem (Theorem 4.5.1) as encoding a form of descent for this (pre?)sheaf. In particular, we expect that our theory straightforwardly recovers the reconstruction results of e.g. [Bal07, BF07].[118]

5. The geometric stratification of genuine G-spectra

In this section, we prove our symmetric monoidal stratification of genuine G-spectra (Theorem E). This gives a reconstruction theorem for genuine G-spectra when G is a finite group, which we unpack in a number of examples.

Local Notation 5.0.1. In this section, we write G for an arbitrary compact Lie group, and we write H for an arbitrary closed subgroup of G.

This section is organized as follows.

§5.1: We set our conventions regarding genuine G-spectra and prove Theorem E as Theorem 5.1.27.

[118]Of course, this notion of descent is necessarily ∞-categorical, and cannot be carried through at the level of homotopy categories. In particular, we expect that such recovery would repair the failure of uniqueness of gluings that arises in [BF07], which appears to come of working with homotopy categories instead of ∞-categories.

§5.2: We study the gluing functors of the geometric stratification of genuine G-spectra, which are versions of the Tate construction.

§5.3: We unpack our reconstruction theorem for genuine G-spectra in the cases where $G \in \{\mathsf{C}_p, \mathsf{C}_{p^2}, \mathsf{C}_{pq}, \mathsf{S}_3\}$ (for p and q distinct primes). We also discuss the geometric stratification of genuine $\mathbb{T}$-spectra and the resulting reconstruction theorem for proper-genuine $\mathbb{T}$-spectra.

§5.4: We specialize our nanocosm reconstruction theorem to give a formula for the categorical H-fixedpoints of genuine G-spectra (when G is finite).

5.1. **The geometric stratification of genuine G-spectra.** In this subsection, we establish the symmetric monoidal stratification of genuine G-spectra as Theorem 5.1.27. We begin by laying out our notation and recalling the facts that we need; for further background on genuine G-spectra, we refer the reader to [LMSM86, May96, MM02].

Notation 5.1.1.

(1) We write
$$\mathcal{S}^{\mathsf{g}G}$$
for the ∞-category of *genuine G-spaces.*

(2) We write
$$\mathscr{O}_G \subseteq \mathcal{S}^{\mathsf{g}G}$$
for the *orbit ∞-category* of G, the full subcategory on those objects of the form G/H.

(3) We write P_G for the poset of conjugacy classes of closed subgroups of G ordered by subconjugacy (the posetification (i.e. homwise (-1)-truncation) of $\mathscr{O}_G$).

(4) We write
$$\mathcal{S}\mathsf{p}^{\mathsf{g}G}$$
for the ∞-category of *genuine G-spectra*, i.e. the stable ∞-category of spectral presheaves on $\mathscr{O}_G$ with the representation spheres inverted under the symmetric monoidal structure.

(5) We write
$$\begin{tikzcd} \mathcal{S}^{\mathsf{g}G}_* \arrow[r, shift left, "\Sigma^\infty_G"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[l, shift left, "\Omega^\infty_G"] \end{tikzcd}$$
for the adjoint functors of (*genuine G-*)*suspension spectrum* and (*pointed genuine G-*)*infinite loopspace.*

(6) We write
$$\mathcal{S}\mathsf{p}^{\mathsf{h}G} := \mathsf{Fun}(\mathsf{B}G, \mathcal{S}\mathsf{p})$$
for the ∞-category of *homotopy G-spectra.*

(7) We write
$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, shift left, "U_G"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p}^{\mathsf{h}G} \arrow[l, shift left, hook, "\beta_G"] \end{tikzcd}$$
for the adjunction – a reflective localization – whose left adjoint is the forgetful functor and whose right adjoint is the *Borel-complete genuine G-spectrum* functor.[119] We may also omit the subscripts, simply writing $U \dashv \beta$ instead of $U_G \dashv \beta_G$.

[119]That is, β_G is the inclusion of the full subcategory of *Borel-complete* genuine G-spectra, i.e. those objects $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ such that the canonical map $E^H \to E^{\mathsf{h}H}$ (from genuine H-fixedpoints to homotopy H-fixedpoints) is an equivalence for all closed subgroups $H \leq G$.

Warning 5.1.2. Notation 5.1.1(3) introduces a mild clash: given closed subgroups H and K of G, we may write $H \leq K$ when H is subconjugate to K but not necessarily actually contained in it. On the other hand, in such situations we generally assume (without real loss of generality) that H is in fact contained in K. To emphasize that we truly mean containment, we use the notation $\subseteq$.

Remark 5.1.3. We will often refer to the set $\{G/H \in \mathscr{O}_G\}_{H \in \mathsf{P}_G}$ (and variants thereof). This may appear to be ill-defined, as the objects of P_G are only conjugacy classes of subgroups of G. However, a conjugation relation $H' = gHg^{-1}$ determines an equivalence $G/H' \simeq G/H$. Thus, this notation is effectively unambiguous.

Notation 5.1.4. We respectively write

$$\mathsf{N}(H) := \mathsf{N}_G(H) \qquad \text{and} \qquad \mathsf{W}(H) := \mathsf{W}_G(H) := \mathsf{N}(H)/H$$

for the normalizer and Weyl group of the closed subgroup $H \leq G$.

Observation 5.1.5. We record the following facts, which we use without further comment.

(1) The set $\{G/H \in \mathscr{O}_G \subseteq \mathcal{S}^{\mathsf{g}G}\}_{H \in \mathsf{P}_G}$ of orbits compactly generates $\mathcal{S}^{\mathsf{g}G}$: by Elmendorf's theorem, the restricted Yoneda functor is an equivalence

$$\mathcal{S}^{\mathsf{g}G} \xrightarrow{\sim} \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathcal{S}) .$$

Under this identification, the genuine H-fixedpoints functor $(-)^H$ corresponds to evaluation at the object $(G/H)^\circ \in \mathscr{O}_G^{\mathsf{op}}$.

(2) The set $\{\Sigma^\infty_G(G/H)_+ \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}\}_{H \in \mathsf{P}_G}$ of suspension spectra of orbits compactly generates $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$.

(3) The ∞-categories $\mathcal{S}^{\mathsf{g}G}$ and $\mathcal{S}^{\mathsf{g}G}_*$ are both presentably symmetric monoidal, with their respective cartesian product (denoted $\times$) and smash product (denoted $\wedge$) defined pointwise: that is, these symmetric monoidal structures commute with taking genuine fixedpoints.

(4) The ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ is presentably symmetric monoidal via the smash product (denoted $\otimes$).

(5) The genuine G-suspension spectrum functor

$$\mathcal{S}^{\mathsf{g}G}_* \xrightarrow{\Sigma^\infty_G} \mathcal{S}\mathsf{p}^{\mathsf{g}G}$$

is symmetric monoidal.

(6) The Weyl group $\mathsf{W}(H)$ is (the underlying ∞-group of) the compact Lie group of G-equivariant automorphisms of G/H.

(7) Given a normal closed subgroup $H \in \mathsf{P}_G$, the categorical H-fixedpoints functor fits into a commutative square

$$\begin{tikzcd}
\mathcal{S}^{\mathsf{g}(G/H)}_* & \mathcal{S}^{\mathsf{g}G}_* \arrow[l, "(-)^H"'] \\
\mathcal{S}\mathsf{p}^{\mathsf{g}(G/H)} \arrow[u, "\Omega^\infty_{G/H}"] & \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[l, "(-)^H"] \arrow[u, "\Omega^\infty_G"']
\end{tikzcd}$$

that is obtained by passing to right adjoints in the commutative square

$$\begin{tikzcd}
\mathcal{S}_*^{\mathsf{g}(G/H)} \arrow[r, "\mathsf{Res}_G^{G/H}"] \arrow[d, "\Sigma^\infty_{G/H}"'] & \mathcal{S}_*^{\mathsf{g}G} \arrow[d, "\Sigma^\infty_G"] \\
\mathcal{S}\mathsf{p}^{\mathsf{g}(G/H)} \arrow[r, "\mathsf{Res}_G^{G/H}"'] & \mathcal{S}\mathsf{p}^{\mathsf{g}G}
\end{tikzcd}$$

in $\mathsf{CAlg}(\mathsf{Pr}^L)$ (which itself is deduced from the universal property of genuine G/H-spectra). For an arbitrary closed subgroup $H \in \mathsf{P}_G$, the categorical H-fixedpoints functor is the composite

$$(-)^H : \mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{\mathsf{Res}^G_{\mathsf{N}(H)}} \mathcal{S}\mathsf{p}^{\mathsf{gN}(H)} \xrightarrow{(-)^H} \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} ~.$$

(8) Categorical fixedpoints compose: if $K \leq H \leq \mathsf{N}_G(K) \leq G$ then the triangle

$$\begin{tikzcd}
\mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "(-)^K"] \arrow[dr, "(-)^H"'] & \mathcal{S}\mathsf{p}^{\mathsf{gW}_G(K)} \arrow[d, "(-)^{H/K}"] \\
& \mathcal{S}\mathsf{p}^{\mathsf{gW}_G(H)}
\end{tikzcd}$$

commutes.[120]

(9) At the level of underlying homotopy $\mathsf{W}(H)$-spectra, categorical H-fixedpoints are corepresented by $\Sigma^\infty_G(G/H)_+$: the diagram

$$\begin{tikzcd}
\mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "(-)^H"] \arrow[dr, "\underline{\mathsf{hom}}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}}(\Sigma^\infty_G(G/H)_+,-)"'] & \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \arrow[d, "U"] \\
& \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)}
\end{tikzcd}$$

canonically commutes.

Notation 5.1.6. We often simply write

$$(-)^H : \mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{(-)^H} \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \xrightarrow{U} \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)}$$

for the composite.[121] Our meaning will always be clear from context.

Notation 5.1.7. We denote by $\odot$ the action on $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ of $\mathcal{S}_*^{\mathsf{g}G}$. So by definition, for any $X \in \mathcal{S}_*^{\mathsf{g}G}$ and $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ we have

$$X \odot E \simeq \Sigma^\infty_G X \otimes E \in \mathcal{S}\mathsf{p}^{\mathsf{g}G} ~.$$

Definition 5.1.8. The ***geometric prestratification*** of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ over P_G is the functor

$$\begin{array}{ccc}
\mathsf{P}_G & \xrightarrow{\mathcal{S}\mathsf{p}^{\mathsf{g}G}_{\leq \bullet}} & \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}} \\
\Cup & & \Cup \\
H & \longmapsto & \mathcal{S}\mathsf{p}^{\mathsf{g}G}_{\leq H} := \langle \Sigma^\infty_G(G/K)_+ \rangle_{K \leq H}
\end{array}$$

[120]Note the canonical isomorphism $\mathsf{W}_{\mathsf{W}_G(K)}(H/K) \cong \mathsf{W}_G(H)$.

[121]This is in contrast with our conventions for geometric fixedpoints appearing in Definition 5.1.20.

sending an element $H \in \mathsf{P}_G$ to the closed subcategory generated by the set

$$\{\Sigma^\infty_G (G/K)_+ \in (\mathcal{S}\mathsf{p}^{\mathsf{g}G})^\omega\}_{K \leq H}$$

of compact objects (recall Example 2.3.2).

Definition 5.1.9. A ***family*** is an element of the poset $\mathsf{Down}_{\mathsf{P}_G}$, i.e. a set of closed subgroups of G that is closed under subconjugacy. To align with standard notation, we denote an arbitrary family by $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$, and given an element $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}_G}$ we also write $\mathscr{F}_\mathsf{D} := \mathsf{D}$.

Local Notation 5.1.10. In this subsection, in the course of proving that the geometric prestratification is in fact a symmetric monoidal stratification, we may write

$$\mathcal{I}_H := \mathcal{S}\mathsf{p}^{\mathsf{g}G}_{\leq H} \ ,$$

for brevity. Similarly, for any family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$, we may write

$$\mathcal{I}_\mathscr{F} := \langle \mathcal{I}_K \rangle_{K \in \mathscr{F}} \simeq \langle \Sigma^\infty_G (G/K)_+ \rangle_{K \in \mathscr{F}} \ .$$

Notation 5.1.11. For any family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$, we write $\mathsf{E}\mathscr{F} \in \mathcal{S}^{\mathsf{g}G}$ for the genuine G-space characterized by the fact that

$$(\mathsf{E}\mathscr{F})^H \simeq \begin{cases} \mathsf{pt} \ , & H \in \mathscr{F} \\ \varnothing \ , & H \notin \mathscr{F} \end{cases} .$$ [122]

Definition 5.1.12. For any family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$, the corresponding ***isotropy separation sequence*** is the cofiber sequence

$$\mathsf{E}\mathscr{F}_+ \longrightarrow S^0 \longrightarrow \widetilde{\mathsf{E}}\mathscr{F} \qquad (5.1.1)$$

in $\mathcal{S}^{\mathsf{g}G}_*$, where the first morphism is obtained by applying the functor $\mathcal{S}^{\mathsf{g}G} \xrightarrow{(-)_+} \mathcal{S}^{\mathsf{g}G}_*$ to the unique morphism $\mathsf{E}\mathscr{F} \to \mathsf{pt}$ in $\mathcal{S}^{\mathsf{g}G}$.

Observation 5.1.13. Applying the genuine H-fixedpoints functor $(-)^H$ to the isotropy separation sequence (5.1.1), we obtain the cofiber sequence

$$\left(\mathsf{E}\mathscr{F}_+ \longrightarrow S^0 \longrightarrow \widetilde{\mathsf{E}}\mathscr{F}\right)^H \simeq \begin{cases} S^0 \xrightarrow{\sim} S^0 \longrightarrow \mathsf{pt} \ , & H \in \mathscr{F} \\ \mathsf{pt} \longrightarrow S^0 \xrightarrow{\sim} S^0 \ , & H \notin \mathscr{F} \end{cases}$$

in $\mathcal{S}_*$. Extending Definition 4.2.12 and Observation 4.2.13 to the unstable setting in the evident way, we find that the objects

$$(\mathsf{E}\mathscr{F}_+ \longrightarrow S^0) \in (\mathcal{S}^{\mathsf{g}G}_*)_{/S^0} \qquad \text{and} \qquad (\Sigma^\infty_G \mathsf{E}\mathscr{F}_+ \longrightarrow \Sigma^\infty_G S^0 \simeq \mathbb{S}) \in (\mathcal{S}\mathsf{p}^{\mathsf{g}G})_{/\mathbb{S}}$$

are central augmented idempotents and that the objects

$$(S^0 \longrightarrow \widetilde{\mathsf{E}}\mathscr{F}) \in (\mathcal{S}^{\mathsf{g}G}_*)_{S^0/} \qquad \text{and} \qquad (\mathbb{S} \simeq \Sigma^\infty_G S^0 \longrightarrow \Sigma^\infty_G \widetilde{\mathsf{E}}\mathscr{F}) \in (\mathcal{S}\mathsf{p}^{\mathsf{g}G})_{\mathbb{S}/}$$

are central coaugmented idempotents. We use these facts without further comment.

[122] Said differently, $(\mathsf{E}\mathscr{F} \downarrow \mathscr{O}_G) \in \mathsf{RFib}(\mathscr{O}_G) \simeq \mathsf{Fun}(\mathscr{O}_G^{\mathsf{op}}, \mathcal{S}) \simeq \mathcal{S}^{\mathsf{g}G}$ fits into a pullback square

$$\begin{tikzcd} \mathsf{E}\mathscr{F} \arrow[r, hook, "\mathsf{f.f.}"] \arrow[d] & \mathscr{O}_G \arrow[d] \\ \mathscr{F} \arrow[r, hook, "\mathsf{f.f.}"'] & \mathsf{P}_G \end{tikzcd} .$$

Observation 5.1.14. For any family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$, the counit of the adjunction

$$\begin{tikzcd} \mathcal{I}_{\mathscr{F}} \arrow[r, hook, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[l, shift left, "y"] \end{tikzcd}$$

at an object $X \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ is the morphism

$$\mathsf{E}\mathscr{F}_+ \odot X \longrightarrow S^0 \odot X \simeq X \ . \tag{5.1.2}$$

In particular, the full subcategory $i_L(\mathcal{I}_{\mathscr{F}}) \subseteq \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ consists of those objects $X \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ such that the counit morphism (5.1.2) is an equivalence.

Observation 5.1.15. It follows from Observation 5.1.14 that for any family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$, the closed subcategory $\mathcal{I}_{\mathscr{F}} \subseteq \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ is a closed ideal subcategory (as anticipated by the notation), with symmetric monoidal unit object $\Sigma^\infty_G \mathsf{E}\mathscr{F}_+ \simeq i_L(y(\mathbb{1}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}}))$. In particular, there exists a factorization

$$\begin{tikzcd} \mathsf{P} \arrow[rr, "\mathcal{I}_\bullet"] \arrow[dr, dashed] & & \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}} \\ & \mathbf{Idl}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}} \arrow[ur, hook, "\text{f.f.}"'] & \end{tikzcd} :$$

the geometric prestratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ is a symmetric monoidal prestratification.

Observation 5.1.16. It follows from Observation 5.1.14 that the unit of the adjunction

$$\begin{tikzcd} \mathcal{I}_H \arrow[r, shift left, "p_L"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p}^{\mathsf{g}G}_H \arrow[l, hook, shift left, "\nu"] \end{tikzcd}$$

at an object $X \in i_L(\mathcal{I}_H) \subseteq \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ is the morphism

$$X \simeq S^0 \odot X \longrightarrow \widetilde{\mathsf{E}}\mathscr{F}_{<H} \odot X \ .$$

Hence, the full subcategory $i_L(\nu(\mathcal{S}\mathsf{p}^{\mathsf{g}G}_H)) \subseteq \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ consists of those objects $X \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ such that in the canonical commutative square

$$\begin{tikzcd} (\mathsf{E}\mathscr{F}_{\leq H})_+ \odot X \arrow[r] \arrow[d] & X \arrow[d] \\ ((\mathsf{E}\mathscr{F}_{\leq H})_+ \wedge \widetilde{\mathsf{E}}\mathscr{F}_{<H}) \odot X \arrow[r] & \widetilde{\mathsf{E}}\mathscr{F}_{<H} \odot X \end{tikzcd} \tag{5.1.3}$$

the upper and right morphisms are equivalences. In turn, this is the case if and only if the square (5.1.3) consists entirely of equivalences.

Notation 5.1.17. For brevity, we write

$$\mathsf{E}\delta_H := ((\mathsf{E}\mathscr{F}_{\leq H})_+ \wedge \widetilde{\mathsf{E}}\mathscr{F}_{<H}) \in \mathcal{S}^{\mathsf{g}G}_* \ .$$

This notation is motivated by the Dirac delta function: this pointed genuine G-space is characterized by the fact that

$$(\mathsf{E}\delta_H)^K \simeq \begin{cases} S^0 \ , & K = H \\ \mathsf{pt} \ , & K \neq H \end{cases} \ .$$

Observation 5.1.18. The object $\mathsf{E}\delta_H \in \mathcal{S}^{\mathsf{g}G}_*$ is idempotent with respect to the smash product.

Notation 5.1.19. We define the family

$$(\not\geq H) := \{K \in \mathsf{P}_G : K \not\geq H\} \in \mathsf{Down}_{\mathsf{P}_G} \ .$$

Definition 5.1.20. The ***geometric H-fixedpoints*** functor

$$\mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{\Phi^H_{\mathsf{g}}} \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)}$$

is defined by the formula

$$\Phi^H_{\mathsf{g}}(X) := (\widetilde{\mathsf{E}}\mathscr{F}_{\ngeq H} \odot X)^H \ .$$

We will be primarily interested in the composite

$$\Phi^H : \mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{\Phi^H_{\mathsf{g}}} \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \xrightarrow{U} \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \ ,$$

which we refer to by the same name.

Remark 5.1.21. For any normal closed subgroup $H \leq G$, there is a canonical commutative diagram

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "p_L"] \arrow[dr, "\Phi^H_{\mathsf{g}}"'] & \mathcal{S}\mathsf{p}^{\mathsf{g}G}/\mathcal{I}_{\mathscr{F}_{\ngeq H}} \arrow[d, leftrightarrow, "\wr"] \\ & \mathcal{S}\mathsf{p}^{\mathsf{g}(G/H)} \end{tikzcd} \quad . \tag{5.1.4}$$

Recall from Proposition 3.4.10 and Observation 3.4.11 that the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ over P_G determines a quotient stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}/\mathcal{I}_{\mathscr{F}_{\ngeq H}}$ over $\mathsf{P}_G \backslash ({}^{\ngeq} H)$. Under the equivalence in diagram (5.1.4), this corresponds to the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}(G/H)}$ over $\mathsf{P}_{G/H}$ (recall the third isomorphism theorem).

Observation 5.1.22. One may also define the functor Φ^H (but not the functor Φ^H_{g}) using the family $({}^{<}H) \in \mathsf{Down}_{\mathsf{P}_G}$, in fact using any family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$ that does not contain H and such that moreover $H \in \mathsf{P}_G \backslash \mathscr{F}$ is a minimal element. Namely, for any such family we have a canonical equivalence

$$\Phi^H(-) \simeq (\widetilde{\mathsf{E}}\mathscr{F} \odot (-))^H$$

in $\mathsf{Fun}(\mathcal{S}\mathsf{p}^{\mathsf{g}G}, \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)})$.

To explain this, observe that $({}^{<}H) \in \mathsf{Down}_{\mathsf{P}_G}$ is the initial such family, so that for any such family $\mathscr{F} \in \mathsf{Down}_{\mathsf{P}_G}$ we have a canonical morphism

$$\widetilde{\mathsf{E}}\mathscr{F}_{<H} \longrightarrow \widetilde{\mathsf{E}}\mathscr{F} \tag{5.1.5}$$

in $\mathcal{S}^{\mathsf{g}G}_*$ determined by the inclusion $({}^{<}H) \subseteq \mathscr{F}$. Then, we claim that for any $X \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$, the composite functor

$$\mathcal{S}^{\mathsf{g}G}_* \xrightarrow{\Sigma^\infty_G} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{(-)\otimes X} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{(-)^H} \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \xrightarrow{U} \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \tag{5.1.6}$$

carries the morphism (5.1.5) to an equivalence (although its truncation ending at $\mathcal{S}\mathsf{p}^{\mathsf{gW}(H)}$ does not generally do so). Indeed, this follows from the fact that in the commutative diagram

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "(-)^H"] \arrow[d, "\mathsf{Res}^G_H"'] & \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \arrow[r, "U"] & \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \arrow[d, "\mathsf{fgt}"] \\ \mathcal{S}\mathsf{p}^{\mathsf{g}H} \arrow[rr, "(-)^H"'] & & \mathcal{S}\mathsf{p} \end{tikzcd} \ ,$$

the left vertical functor is symmetric monoidal and carries the morphism Σ^∞_G(5.1.5) to an equivalence while the right vertical functor is conservative. Hence, the composite (5.1.6) carries the span

$$\widetilde{\mathsf{E}}\mathscr{F}_{\ngeq H} \longleftarrow \widetilde{\mathsf{E}}\mathscr{F}_{<H} \longrightarrow \widetilde{\mathsf{E}}\mathscr{F}$$

in $\mathcal{S}^{\mathsf{g}G}_*$ to a span of equivalences.

Observation 5.1.23. Geometric fixedpoints functors compose: if $K \leq H \leq \mathsf{N}_G(K) \leq G$ then the triangle

$$\begin{tikzcd}
\mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "\Phi^K_{\mathrm{g}}"] \arrow[dr, "\Phi^H_{\mathrm{g}}"'] & \mathcal{S}\mathsf{p}^{\mathsf{gW}_G(K)} \arrow[d, "\Phi^{H/K}_{\mathrm{g}}"] \\
& \mathcal{S}\mathsf{p}^{\mathsf{gW}_G(H)}
\end{tikzcd}$$

commutes. We use this fact without further comment.

Observation 5.1.24. The geometric H-fixedpoints functor

$$\mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{\Phi^H} \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)}$$

is symmetric monoidal.

Observation 5.1.25. The geometric H-fixedpoints functor fits into a canonical commutative diagram

$$\begin{tikzcd}
\mathcal{S}^{\mathsf{g}G}_{*} \arrow[r, "\Sigma^\infty_G"] \arrow[d, "(-)^H"'] & \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[d, "\Phi^H"] \\
\mathcal{S}^{\mathsf{hW}(H)}_{*} \arrow[r, "\Sigma^\infty"'] & \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)}
\end{tikzcd} .$$

Observation 5.1.26. There is a unique nonzero morphism

$$\mathsf{E}\delta_H \longrightarrow \widetilde{\mathsf{E}}\mathscr{F}_{\ngeq H}$$

in $\mathcal{S}^{\mathsf{g}G}_{*}$, and it becomes an equivalence

$$\mathsf{E}\delta_H \simeq \mathsf{E}\delta_H \wedge \mathsf{E}\delta_H \xrightarrow{\sim} \widetilde{\mathsf{E}}\mathscr{F}_{\ngeq H} \wedge \mathsf{E}\delta_H$$

upon smashing it with its source.

Theorem 5.1.27. *The geometric prestratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ over P_G is a symmetric monoidal stratification. Moreover,*

(1) its H^{th} stratum is the ∞-category

$$\mathcal{S}\mathsf{p}^{\mathsf{g}G}_H \simeq \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)}$$

of homotopy $\mathsf{W}(H)$-spectra, and

(2) its H^{th} geometric localization functor is the geometric H-fixedpoints functor

$$\mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{\Phi^H} \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \simeq \mathcal{S}\mathsf{p}^{\mathsf{g}G}_H .$$

Proof. Applying Observation 5.1.15, we see that it suffices to show that the geometric prestratification is a stratification. For this, we first verify the two asserted identifications for the geometric prestratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ over P_G, and then we use these identifications to verify that it is indeed a stratification.

Towards verifying the two identifications, for any $X \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$, referring to the functors in the diagram

$$\begin{tikzcd}
\mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, shift right, "y"'] \arrow[r, phantom, "\perp"] & \mathcal{J}_H \arrow[l, hook', shift right, "i_L"'] \arrow[r, shift left, "p_L"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p}^{\mathsf{g}G}_H \arrow[l, hook', shift left, "\nu"]
\end{tikzcd}$$

we compute in $\mathcal{S}\mathsf{p}^{\mathsf{hW}(H)}$ that

$$(i_L \nu p_L y(X))^H \simeq (\mathsf{E}\delta_H \odot X)^H \tag{5.1.7}$$

$$
\begin{aligned}
&\simeq ((\widetilde{\mathsf{E}}\mathscr{F}_{\not\geq H} \wedge \mathsf{E}\delta_H) \odot X)^H && (5.1.8)\\
&\simeq (\widetilde{\mathsf{E}}\mathscr{F}_{\not\geq H} \odot (\mathsf{E}\delta_H \odot X))^H \\
&:= \Phi^H(\mathsf{E}\delta_H \odot X) \\
&\simeq \Phi^H(\Sigma^\infty_G(\mathsf{E}\delta_H)) \otimes \Phi^H(X) && (5.1.9)\\
&\simeq \Sigma^\infty((\mathsf{E}\delta_H)^H) \otimes \Phi^H(X) && (5.1.10)\\
&\simeq \Phi^H(X) \ , && (5.1.11)
\end{aligned}
$$

where

- equivalence (5.1.7) follows from Observation 5.1.16,
- equivalence (5.1.8) follows from Observation 5.1.26,
- equivalence (5.1.9) follows from Observation 5.1.24,
- equivalence (5.1.10) follows from Observation 5.1.25, and
- equivalence (5.1.11) follows from the equivalence $(\mathsf{E}\delta_H)^H \simeq S^0$ in $\mathcal{S}_*^{\mathsf{hW}(H)}$.

Now, to verify part (1), we begin by observing via the recollement

$$
\begin{tikzcd}
\mathcal{J}_{<H} \arrow[r, hook, bend left=50, "i_L"] \arrow[r, hook, bend right=50, "i_R"'] & \mathcal{J}_H \arrow[l, "y"'] \arrow[l, phantom, shift left=3, "\perp"] \arrow[l, phantom, shift right=3, "\perp"] \arrow[r, bend left=50, "p_L"] \arrow[r, bend right=50, "p_R"'] & \mathcal{S}\mathsf{p}^{\mathsf{g}G}_H \arrow[l, hook', "\nu"'] \arrow[l, phantom, shift left=3, "\perp"] \arrow[l, phantom, shift right=3, "\perp"]
\end{tikzcd}
$$

that the object $p_L(\Sigma^\infty_G(G/H)_+) \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}_H$ is a compact generator, so that it suffices to verify that the composite morphism

$$
\Sigma^\infty_+\mathsf{W}(H) \simeq \Sigma^\infty \mathsf{end}_{\mathcal{S}^{\mathsf{g}G}_*}((G/H)_+) \xrightarrow{\Sigma^\infty_G} \underline{\mathsf{end}}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}}(\Sigma^\infty_G(G/H)_+) \xrightarrow[\sim]{y} \underline{\mathsf{end}}_{\mathcal{J}_H}(\Sigma^\infty_G(G/H)_+) \xrightarrow{p_L} \underline{\mathsf{end}}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}_H}(p_L(\Sigma^\infty_G(G/H)_+)) \tag{5.1.12}
$$

of ring spectra is an equivalence. For this, by adjunction we compute that

$$
\begin{aligned}
\underline{\mathsf{end}}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}_H}(p_L(\Sigma^\infty_G(G/H)_+)) &:= \underline{\mathsf{hom}}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}_H}(p_L(\Sigma^\infty_G(G/H)_+), p_L(\Sigma^\infty_G(G/H)_+)) \\
&\simeq \underline{\mathsf{hom}}_{\mathcal{J}_H}(\Sigma^\infty_G(G/H)_+, \nu p_L(\Sigma^\infty_G(G/H)_+)) \\
&\simeq \underline{\mathsf{hom}}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}}(\Sigma^\infty_G(G/H)_+, i_L\nu p_L(\Sigma^\infty_G(G/H)_+)) \\
&\simeq (i_L\nu p_L(\Sigma^\infty_G(G/H)_+))^H \\
&\simeq \Phi^H(\Sigma^\infty_G(G/H)_+) && (5.1.13)\\
&\simeq \Sigma^\infty(((G/H)_+)^H) && (5.1.14)\\
&\simeq \Sigma^\infty(\mathsf{hom}_{\mathcal{S}^{\mathsf{g}G}}(G/H, G/H)_+) \\
&\simeq \Sigma^\infty(\mathsf{W}(H)_+) \ ,
\end{aligned}
$$

where equivalence (5.1.13) follows from the equivalences (5.1.7)-(5.1.11) and equivalence (5.1.14) follows from Observation 5.1.25. This string of equivalences of spectra evidently underlies the composite morphism (5.1.12) of ring spectra, which proves part (1). To verify part (2), we compute for any $X \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ that

$$
\begin{aligned}
\underline{\mathsf{hom}}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}_H}(p_L(\Sigma^\infty_G(G/H)_+), p_L y(X)) &\simeq \underline{\mathsf{hom}}_{\mathcal{J}_H}(\Sigma^\infty_G(G/H)_+, \nu p_L y(X)) \\
&\simeq \underline{\mathsf{hom}}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}}(\Sigma^\infty_G(G/H)_+, i_L \nu p_L y(X)) \\
&\simeq (i_L \nu p_L y(X))^H \\
&\simeq \Phi^H(X) \ , && (5.1.15)
\end{aligned}
$$

where equivalence (5.1.15) follows from equivalences (5.1.7)-(5.1.11).

We now verify that the geometric prestratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ over P_G is indeed a stratification. Using Observations 4.3.6 and 5.1.14, it suffices to observe that for any $\mathsf{D}, \mathsf{D}' \in \mathsf{Down}_{\mathsf{P}_G}$ the morphism

$$(\mathsf{E}\mathscr{F}_{\mathsf{D}})_+ \wedge (\mathsf{E}\mathscr{F}_{\mathsf{D}'})_+ \wedge (\mathsf{E}\mathscr{F}_{\mathsf{D}\cap\mathsf{D}'})_+ \longrightarrow (\mathsf{E}\mathscr{F}_{\mathsf{D}})_+ \wedge (\mathsf{E}\mathscr{F}_{\mathsf{D}'})_+$$

in $\mathcal{S}_*^{\mathsf{g}G}$ is an equivalence. □

5.2. **The proper Tate construction.** In this brief subsection we discuss the gluing functors of the geometric stratification of genuine G-spectra, which are versions of the Tate construction.

Observation 5.2.1. By definition, the H^{th} geometric localization functor of the geometric stratification of genuine G-spectra is the left adjoint in the composite adjunction

$$\begin{tikzcd} \Phi^H : \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, shift left=1.5, "\mathsf{Res}^G_{\mathsf{N}(H)}"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p}^{\mathsf{gN}(H)} \arrow[l, shift left=1.5, "\mathsf{coInd}^G_{\mathsf{N}(H)}"] \arrow[r, shift left=1.5, "\Phi^H_{\mathsf{g}}"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \arrow[l, hook', shift left=1.5, "\rho^H_{\mathsf{g}}"] \arrow[r, shift left=1.5, "U_{\mathsf{W}(H)}"] \arrow[r, phantom, "\perp"] & \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} : \rho^H \arrow[l, hook', shift left=1.5, "\beta_{\mathsf{W}(H)}"] \end{tikzcd} .$$

It follows that for any $H \leq K$ in P_G, the gluing functor Γ^H_K is the composite

$$\begin{tikzcd} & \mathcal{S}\mathsf{p}^{\mathsf{gN}(K)} \arrow[r, "\Phi^K_{\mathsf{g}}"] & \mathcal{S}\mathsf{p}^{\mathsf{gW}(K)} \arrow[r, "U_{\mathsf{W}(K)}"] & \mathcal{S}\mathsf{p}^{\mathsf{hW}(K)} \\ \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[ur, "\mathsf{Res}^G_{\mathsf{N}(K)}"] & & & \\ & \mathcal{S}\mathsf{p}^{\mathsf{gN}(H)} \arrow[ul, "\mathsf{coInd}^G_{\mathsf{N}(H)}"] & \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \arrow[l, hook', "\rho^H_{\mathsf{g}}"] & \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \arrow[l, hook', "\beta_{\mathsf{W}(H)}"] \arrow[uu, dashed, "\Gamma^H_K"'] \end{tikzcd} . \tag{5.2.1}$$

When H and K are both normal subgroups of G (which is automatic when G is abelian), then the composite (5.2.1) reduces to the composite

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{g}(G/K)} \arrow[r, "U_{G/K}"] & \mathcal{S}\mathsf{p}^{\mathsf{h}(G/K)} \\ \mathcal{S}\mathsf{p}^{\mathsf{g}(G/H)} \arrow[u, "\Phi^{K/H}_{\mathsf{g}}"] & \mathcal{S}\mathsf{p}^{\mathsf{h}(G/H)} \arrow[l, hook', "\beta_{G/H}"] \arrow[u, dashed, "\Gamma^H_K"'] \end{tikzcd} .$$

Observation 5.2.2. The subcomposite

$$\mathcal{S}\mathsf{p}^{\mathsf{gN}(H)} \xrightarrow{\mathsf{coInd}^G_{\mathsf{N}(H)}} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{\mathsf{Res}^G_{\mathsf{N}(K)}} \mathcal{S}\mathsf{p}^{\mathsf{gN}(K)} \xrightarrow{\Phi^K_{\mathsf{g}}} \mathcal{S}\mathsf{p}^{\mathsf{gW}(K)}$$

of the composite (5.2.1) is zero whenever $\mathsf{N}(H) \ngeq K$.

Definition 5.2.3. We define the ***proper H-Tate construction*** to be the composite functor

$$\begin{tikzcd} (-)^{\tau H} : \mathcal{S}\mathsf{p}^{\mathsf{h}G} \arrow[r, hook, "\beta"] & \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "\Phi^H_{\mathsf{g}}"] \arrow[rr, bend right, "\Phi^H"'] & \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \arrow[r, "U"] & \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \end{tikzcd} .$$

Remark 5.2.4. We make Definition 5.2.3 here in the interest of self-containment, but in fact the proper H-Tate construction

$$\mathcal{S}\mathsf{p}^{\mathsf{h}G} \xrightarrow{(-)^{\tau H}} \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)}$$

admits a description making no reference to genuine equivariant homotopy theory, at least assuming that G is finite. Namely, we prove as [AMGRa, Proposition **??**] that it is given by quotienting by norms from all proper subgroups of H: it is the lower composite in the left Kan extension diagram

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{h}G} \arrow[r, "(-)^{\mathsf{h}H}"] \arrow[d, "p"'] \arrow[dr, phantom, "\not\Downarrow"] & \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \\ \mathcal{S}\mathsf{p}^{\mathsf{h}G}/^{\mathsf{St}}\mathcal{I} \arrow[ur, dashed] & \end{tikzcd} ,$$

where p denotes the projection to the stable quotient by the thick ideal subcategory $\mathcal{I} \subseteq \mathcal{S}\mathsf{p}^{\mathsf{h}G}$ generated by the objects $\{\Sigma^\infty(G/K)_+ \in \mathcal{S}\mathsf{p}^{\mathsf{h}G}\}_{K<H}$. In particular, when $G = H = \mathsf{C}_p$ for a prime p, this recovers the ordinary Tate construction

$$(-)^{\tau\mathsf{C}_p} \simeq (-)^{\mathsf{tC}_p} := \mathsf{cofib}\left((-)_{\mathsf{hC}_p} \xrightarrow{\mathsf{Nm}_{\mathsf{C}_p}} (-)^{\mathsf{hC}_p} \right) .$$

Remark 5.2.5. Assuming that G is finite, as [AMGRa, Theorem **??**] we identify the gluing functor

$$\mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \xrightarrow{\Gamma_K^H} \mathcal{S}\mathsf{p}^{\mathsf{hW}(K)}$$

for any $H \leq K$ in P_G: writing

$$\tilde{C}(H,K) := \{g \in G : H \subseteq gKg^{-1} \subseteq \mathsf{N}(H)\} ,$$

it is given by the formula

$$E \longmapsto \bigoplus_{[g] \in \mathsf{N}(H)\backslash \tilde{C}(H,K)/\mathsf{N}(K)} \mathsf{Ind}^{\mathsf{W}(K)}_{(\mathsf{N}(H)\cap(\mathsf{N}(gKg^{-1})))/(gKg^{-1})} E^{\tau(gKg^{-1})/H} .$$

(In particular, by Remark 5.2.4 this description also makes no reference to genuine equivariant homotopy theory.)

Observation 5.2.6. We record here the following arithmetic facts surrounding the proper Tate construction, which we use in §5.3.

(1) By [NS18, Lemma II.6.7], if G is a finite group whose order is not a prime power, then the proper Tate construction $(-)^{\tau G}$ vanishes.

(2) By [NS18, Lemma I.2.8], if $E \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_p}$ and p acts invertibly on $\pi_n E$ for all $n \in \mathbb{Z}$, then $E^{\tau\mathsf{C}_p} \simeq 0$.

(3) By the Segal conjecture [Lin80, AGM85] combined with [NS18, Theorem I.3.1], for any $E \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_p}$ the spectrum $E^{\tau\mathsf{C}_p} \in \mathcal{S}\mathsf{p}$ is p-complete.

Warning 5.2.7. In [AMGRc], for brevity we omit the word "proper" from the terminology "proper Tate construction".

5.3. **Examples of reconstruction of genuine G-spectra.** In this subsection, we give a number of examples of reconstruction (via Theorem 2.5.14) that follow from the geometric stratification of genuine G-spectra (Theorem 5.1.27). It is straightforward but notationally cumbersome to describe the symmetric monoidal structures (which result from Theorem 4.5.1), and so we omit them from the present discussion.

Notation 5.3.1. In this subsection, in the interest of uniformity, even in the case that G is the trivial group we may include the forgetful functor

$$\mathcal{S}\mathsf{p}^{\mathsf{g}G} \xrightarrow{U} \mathcal{S}\mathsf{p}^{\mathsf{h}G}$$

in our notation.

Remark 5.3.2. In this subsection, we continue to distinguish between the two geometric H-fixedpoints functors appearing in the commutative diagram

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "\Phi^H_{\mathsf{g}}"] \arrow[dr, "\Phi^H"'] & \mathcal{S}\mathsf{p}^{\mathsf{gW}(H)} \arrow[d, "U"] \\ & \mathcal{S}\mathsf{p}^{\mathsf{hW}(H)} \end{tikzcd} ,$$

as introduced in Definition 5.1.20. The H^{th} geometric localization functor for the geometric stratification of genuine G-spectra is the functor Φ^H, but we also use its identification as the composite $U\Phi^H_{\mathsf{g}}$ in order to describe the structure maps in the right-lax limit (as first indicated in Remark 1.3.7), which are given by the unit maps of various adjunctions of the form $U \dashv \beta$.

Notation 5.3.3. We write

$$\mathsf{P}_G \xrightarrow[\mathsf{l.lax}]{\mathcal{S}\mathsf{p}^{\mathsf{hW}_G(\bullet)}} \mathsf{Pr}_{\mathsf{st}}$$

for the gluing diagram of the geometric stratification of genuine G-spectra.

Example 5.3.4 (genuine C_p-spectra)**.** Let C_p denote the cyclic group of order p, where p is a prime. Its poset of conjugacy classes of closed subgroups is

$$\mathsf{P}_{\mathsf{C}_p} = \{e \longrightarrow \mathsf{C}_p\} \ .$$

Theorems 5.1.27 and 2.5.14 provide an equivalence

$$\mathcal{S}\mathsf{p}^{\mathsf{gC}_p} \xrightarrow[\sim]{g} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}_{\mathsf{C}_p}} \left(\mathcal{S}\mathsf{p}^{\mathsf{hW}_{\mathsf{C}_p}(\bullet)}\right) := \lim^{\mathsf{r.lax}}\left(\mathcal{S}\mathsf{p}^{\mathsf{hC}_p} \xrightarrow{(-)^{\tau\mathsf{C}_p}} \mathcal{S}\mathsf{p}\right) := \left\{\left(E_0 \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_p}\ ,\ E_1 \in \mathcal{S}\mathsf{p}\ ,\ \begin{matrix} E_1 \\ \downarrow \\ (E_0)^{\tau\mathsf{C}_p}\end{matrix}\right)\right\} . \tag{5.3.1}$$

Via the equivalence (5.3.1), a genuine C_p-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{gC}_p}$ is specified by the data of

- its underlying homotopy C_p-spectrum
$$E_0 := UE \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_p} \ ,$$
- its geometric C_p-fixedpoints spectrum
$$E_1 := \Phi^{\mathsf{C}_p} E := U\Phi^{\mathsf{C}_p}_{\mathsf{g}} E \in \mathcal{S}\mathsf{p} \ ,$$
and
- the gluing data of a comparison map
$$U\Phi^{\mathsf{C}_p}_{\mathsf{g}}(E \longrightarrow \beta UE) =: \left(E_1 \longrightarrow (E_0)^{\tau\mathsf{C}_p}\right)$$
from E_1 to the C_p-Tate construction of E_0 (recall Remark 5.2.4).

In other words, we have a recollement

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{hC}_p} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "\beta"'] & \mathcal{S}\mathsf{p}^{\mathsf{gC}_p} \arrow[l, "U"'] \arrow[r, bend left, "\Phi^{\mathsf{C}_p}"] \arrow[r, bend right, "p_R"'] & \mathcal{S}\mathsf{p} \arrow[l, "\rho^{\mathsf{C}_p}"'] \end{tikzcd} . \tag{5.3.2}$$

Remark 5.3.5. It is not hard to see the Wirthmüller isomorphism $\mathsf{Ind}^{\mathsf{C}_p}_e \simeq \mathsf{coInd}^{\mathsf{C}_p}_e$ within the context of Example 5.3.4. Indeed, writing $\mathsf{pt} \simeq \mathsf{B}e \xrightarrow{\iota} \mathsf{BC}_p$ for the canonical basepoint, the adjoint functors $\mathsf{Ind}_e^{\mathsf{C}_p} \dashv \mathsf{Res}_e^{\mathsf{C}_p} \dashv \mathsf{coInd}_e^{\mathsf{C}_p}$ are obtained as the horizontal composites in the diagram

$$\begin{tikzcd} \mathcal{S}\mathsf{p} \arrow[r, bend left, "\iota_!"] \arrow[r, bend right, "\iota_*"'] & \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{C}_p} \arrow[l, "\iota^*"'] \arrow[r, hook, bend left, "i_L"] \arrow[r, hook, bend right, "\beta"'] & \mathcal{S}\mathsf{p}^{\mathsf{g}\mathsf{C}_p} \arrow[l, "U"'] \end{tikzcd} .$$

Note that for any $E \in \mathcal{S}\mathsf{p}$ we have

$$\iota_!(E) \simeq \coprod_{\mathsf{C}_p/e} E \simeq \bigoplus_{\mathsf{C}_p/e} E \simeq \prod_{\mathsf{C}_p/e} E \simeq \iota_*(E) \ :$$

both adjoints to the forgetful functor ι^* are given by inducing up from e to C_p. On the other hand, in the recollement (5.3.2), we see that for any $E \in \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{C}_p}$ we have

$$i_L(E) = (E \longmapsto E^{\mathsf{\tau}\mathsf{C}_p} \longleftarrow 0) \qquad \text{and} \qquad \beta(E) = (E \longmapsto E^{\mathsf{\tau}\mathsf{C}_p} \xleftarrow{\sim} E^{\mathsf{\tau}\mathsf{C}_p})$$

(via the identification of Lemma 2.2.1). Hence, the equivalence

$$\mathsf{Ind}_e^{\mathsf{C}_p} := i_L \iota_! \simeq \beta \iota_* =: \mathsf{coInd}_e^{\mathsf{C}_p}$$

follows from the fact that the C_p-Tate construction vanishes on any homotopy C_p-spectra that are induced from the proper subgroup $e < \mathsf{C}_p$.

Example 5.3.6 (genuine C_{p^2}-spectra)**.** Let C_{p^2} denote the cyclic group of order p^2, where p is a prime. Its poset of conjugacy classes of closed subgroups is

$$\mathsf{P}_{\mathsf{C}_{p^2}} = \left\{ \begin{tikzcd} & \mathsf{C}_p \arrow[dr] & \\ e \arrow[ur] \arrow[rr] & & \mathsf{C}_{p^2} \end{tikzcd} \right\} .$$

Theorems 5.1.27 and 2.5.14 provide an equivalence

$$\mathcal{S}\mathsf{p}^{\mathsf{g}\mathsf{C}_{p^2}} \xrightarrow[\sim]{g} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathsf{P}_{\mathsf{C}_{p^2}}} \left(\mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{W}_{\mathsf{C}_{p^2}}(\bullet)} \right) := \lim^{\mathsf{r.lax}} \left(\begin{tikzcd} & \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{C}_p} \arrow[dr, "(-)^{\tau \mathsf{C}_p}"] & \\ \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{C}_{p^2}} \arrow[ur, "(-)^{\tau \mathsf{C}_p}"] \arrow[rr, "(-)^{\tau \mathsf{C}_{p^2}}"'] \arrow[rr, phantom, bend left=30, "\Uparrow"] & & \mathcal{S}\mathsf{p} \end{tikzcd} \right) . \tag{5.3.3}$$

Via the equivalence (5.3.3), a genuine C_{p^2}-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}\mathsf{C}_{p^2}}$ is specified by the following data, which is precisely that of an object of this right-lax limit.[123]

- First of all, it determines the objects

$$E_0 := UE \in \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{C}_{p^2}} , \qquad E_1 := \Phi^{\mathsf{C}_p} E := U\Phi_{\mathsf{g}}^{\mathsf{C}_p} E \in \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{C}_p} , \qquad \text{and} \qquad E_2 := \Phi^{\mathsf{C}_{p^2}} E := U\Phi_{\mathsf{g}}^{\mathsf{C}_{p^2}} E \in \mathcal{S}\mathsf{p} ,$$

  the homotopy-equivariant spectra underlying the genuine-equivariant spectra which are its geometric fixedpoints with respect to the various subgroups of C_{p^2}.

- Thereafter, the unit maps of various adjunctions of the form $U \dashv \beta$ yield

[123]Right-lax limits of left-lax left [2]-modules are described in Example A.5.3(1). See also Example 2.5.16 for a discussion of macrocosm and microcosm reconstruction for an arbitrary stratification over [2].

– a map
$$U\Phi_{\mathsf{g}}^{\mathsf{C}_p}\left(E \longrightarrow \beta U E\right) =: \left(E_1 \longrightarrow \left(U\Phi_{\mathsf{g}}^{\mathsf{C}_p}\beta\right) E_0\right) =: \left(E_1 \longrightarrow (E_0)^{\tau\mathsf{C}_p}\right) \tag{5.3.4}$$
in $\mathcal{S}\mathsf{p}^{\mathsf{hC}_p}$,

– a map
$$U\Phi_{\mathsf{g}}^{\mathsf{C}_{p^2}}\left(E \longrightarrow \beta U E\right) =: \left(E_2 \longrightarrow \left(U\Phi_{\mathsf{g}}^{\mathsf{C}_{p^2}}\beta\right) E_0\right) =: \left(E_2 \longrightarrow (E_0)^{\tau\mathsf{C}_{p^2}}\right) \tag{5.3.5}$$
in $\mathcal{S}\mathsf{p}$, and

– a map
$$U\Phi_{\mathsf{g}}^{\mathsf{C}_p}\left(\Phi_{\mathsf{g}}^{\mathsf{C}_p}E \longrightarrow \beta U\Phi_{\mathsf{g}}^{\mathsf{C}_p}E\right) \simeq \left(U\Phi_{\mathsf{g}}^{\mathsf{C}_{p^2}}E \longrightarrow \left(U\Phi_{\mathsf{g}}^{\mathsf{C}_p}\beta\right)\left(U\Phi_{\mathsf{g}}^{\mathsf{C}_p}E\right)\right) =: \left(E_2 \longrightarrow (E_1)^{\tau\mathsf{C}_p}\right) \tag{5.3.6}$$
in $\mathcal{S}\mathsf{p}$.

- Finally, these maps fit into a commutative square
$$\begin{tikzcd} E_2 \arrow[r, "(5.3.6)"] \arrow[d, "(5.3.5)"'] & (E_1)^{\tau\mathsf{C}_p} \arrow[d, "(5.3.4)^{\tau\mathsf{C}_p}"] \\ (E_0)^{\tau\mathsf{C}_{p^2}} \arrow[r] & \left((E_0)^{\tau\mathsf{C}_p}\right)^{\tau\mathsf{C}_p} \end{tikzcd} \tag{5.3.7}$$
in $\mathcal{S}\mathsf{p}$, as a consequence of the commutativity of the diagram
$$\begin{tikzcd} \left(U\Phi_{\mathsf{g}}^{\mathsf{C}_p}\right)\left(\Phi_{\mathsf{g}}^{\mathsf{C}_p}\right) \arrow[r] \arrow[d] & \left(U\Phi_{\mathsf{g}}^{\mathsf{C}_p}\right)\beta U\left(\Phi_{\mathsf{g}}^{\mathsf{C}_p}\right) \arrow[d] \\ \left(U\Phi_{\mathsf{g}}^{\mathsf{C}_p}\right)\left(\Phi_{\mathsf{g}}^{\mathsf{C}_p}\right)U\beta \arrow[r] & \left(U\Phi_{\mathsf{g}}^{\mathsf{C}_p}\right)\beta U\left(\Phi_{\mathsf{g}}^{\mathsf{C}_p}\right)\beta U \end{tikzcd}$$
in $\mathsf{Fun}(\mathcal{S}\mathsf{p}^{\mathsf{gC}_{p^2}}, \mathcal{S}\mathsf{p})$ and the canonical equivalence $\Phi_{\mathsf{g}}^{\mathsf{C}_{p^2}} \simeq \Phi_{\mathsf{g}}^{\mathsf{C}_p}\Phi_{\mathsf{g}}^{\mathsf{C}_p}$.

Indeed, the lower morphism in the commutative square (5.3.7) is precisely the component at $E_0 \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_{p^2}}$ of the natural transformation in the lax-commutative triangle appearing in equivalence (5.3.3).

Remark 5.3.7. Note that we have an equivalence
$$(-)^{\tau\mathsf{C}_{p^2}} \simeq \left((-)^{\mathsf{hC}_p}\right)^{\tau\mathsf{C}_p}$$
in $\mathsf{Fun}(\mathcal{S}\mathsf{p}^{\mathsf{hC}_{p^2}}, \mathcal{S}\mathsf{p})$ (see e.g. [AMGRa, Lemma **??**]). Using this, we can apply results of Nikolaus–Scholze to identify certain genuine C_{p^2}-spectra $E \in \mathcal{S}\mathsf{p}^{\mathsf{gC}_{p^2}}$ as strict (Definition 2.7.2(3)). Strictness amounts to the assertion that the underlying homotopy C_{p^2}-spectrum $E_0 := \Phi^e E \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_{p^2}}$ satisfies the condition that the morphism
$$(E_0)^{\tau\mathsf{C}_{p^2}} \longrightarrow \left((E_0)^{\tau\mathsf{C}_p}\right)^{\tau\mathsf{C}_p}$$
is an equivalence (so that the morphism (5.3.5) is uniquely determined by the morphisms (5.3.4) and (5.3.6)). Namely, this condition is guaranteed to hold assuming that the underlying spectrum $E_0 \in \mathcal{S}\mathsf{p}$

- is bounded below by [NS18, Lemma I.2.1], or alternatively

- admits a $\mathbb{Z}$-module structure by [NS18, Footnote 9] (see also [NS18, Lemma I.2.7]).

In fact, similar arguments can be applied to give a simplified description of the strict objects of $\mathcal{S}\mathsf{p}^{\mathsf{gC}_{p^n}}$, as described in [NS18, Remark II.4.8] (see [Sha, §4]).[124] Indeed, [AMGRa, Theorem **??**] applies them to give a simplified description of $\mathbb{Z}$-linear genuine C_{p^n}-spectra (a.k.a. derived Mackey functors).

Example 5.3.8 (genuine C_{pq}-spectra)**.** Let $\mathsf{C}_{pq} \cong \mathsf{C}_p \times \mathsf{C}_q$ denote the cyclic group of order pq, where p and q are distinct primes. Its poset of conjugacy classes of closed subgroups is

$$\mathsf{P}_{\mathsf{C}_{pq}} = \left\{ \begin{tikzcd} e \arrow[r] \arrow[d] & \mathsf{C}_p \arrow[d] \\ \mathsf{C}_q \arrow[r] & \mathsf{C}_{pq} \end{tikzcd} \right\} .$$

Theorems 5.1.27 and 2.5.14 provide an equivalence

$$\mathcal{S}\mathsf{p}^{\mathsf{gC}_{pq}} \xrightarrow[\sim]{g} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}_{\mathsf{C}_{pq}}} \left(\mathcal{S}\mathsf{p}^{\mathsf{hW}_{\mathsf{C}_{pq}}(\bullet)} \right) := \lim^{\mathsf{r.lax}} \left(\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{hC}_{pq}} \arrow[r, "(-)^{\tau\mathsf{C}_p}"] \arrow[d, "(-)^{\tau\mathsf{C}_q}"'] \arrow[dr, "(-)^{\tau\mathsf{C}_{pq}}" description] & \mathcal{S}\mathsf{p}^{\mathsf{hC}_q} \arrow[d, "(-)^{\tau\mathsf{C}_q}"] \arrow[dl, phantom, "\Nearrow" near end] \\ \mathcal{S}\mathsf{p}^{\mathsf{hC}_p} \arrow[r, "(-)^{\tau\mathsf{C}_p}"'] \arrow[ur, phantom, "\Swarrow" near end] & \mathcal{S}\mathsf{p} \end{tikzcd} \right) . \tag{5.3.8}$$

By Observation 5.2.6, all three functors $\mathcal{S}\mathsf{p}^{\mathsf{hC}_{pq}} \to \mathcal{S}\mathsf{p}$ appearing in the lax-commutative diagram in equivalence (5.3.8) are zero (the two composite functors by parts (2) and (3), the direct functor by part (1)). It follows that via the equivalence (5.3.8), a genuine C_{pq}-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{gC}_{pq}}$ is completely specified by the data of

- the objects

$$E_{00} := UE \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_{pq}} \qquad E_{01} := \Phi^{\mathsf{C}_p} E \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_q}$$

$$E_{10} := \Phi^{\mathsf{C}_q} E \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_p} \qquad E_{11} := \Phi^{\mathsf{C}_{pq}} E \in \mathcal{S}\mathsf{p}$$

  and

- the structure maps

$$\begin{tikzcd} & (E_{00})^{\tau\mathsf{C}_p} & E_{01} \arrow[l] \\ (E_{00})^{\tau\mathsf{C}_q} & & E_{01}^{\tau\mathsf{C}_q} \\ E_{10} \arrow[u] & E_{10}^{\tau\mathsf{C}_p} & E_{11} \arrow[l] \arrow[u] \end{tikzcd} .$$ [125]

[124] Note that bounded below objects of $\mathcal{S}\mathsf{p}^{\mathsf{gC}_{p^n}}$ need not be strict (assuming $n \geq 3$).

[125] These data are organized so as to reflect their positions within the diagram appearing in the equivalence (5.3.8).

Remark 5.3.9. Example 5.3.8 makes manifest the equivalence

$$\mathcal{S}\mathsf{p}^{\mathsf{gC}_p} \otimes \mathcal{S}\mathsf{p}^{\mathsf{gC}_q} \xrightarrow{\sim} \mathcal{S}\mathsf{p}^{\mathsf{gC}_{pq}}$$

(where the tensor product is taken in $\mathsf{Pr}^L_{\mathsf{st}}$). In other words, a genuine C_{pq}-spectrum is equivalent data to a genuine C_p-object in genuine C_q-spectra (and vice versa).[126]

Example 5.3.10 (genuine S_3-spectra)**.** Let S_3 denote the symmetric group on three letters. Its poset of conjugacy classes of closed subgroups is

$$\mathsf{P}_{\mathsf{S}_3} = \left\{ \begin{tikzcd} e \arrow[r] \arrow[d] & \mathsf{C}_2 \arrow[d] \\ \mathsf{C}_3 \arrow[r] & \mathsf{S}_3 \end{tikzcd} \right\} ,$$

where $\mathsf{C}_3 = \mathsf{A}_3$ denotes the alternating group (the normal subgroup of sign-preserving symmetries) and C_2 denotes the equivalence class of the three (non-normal) order-two subgroups generated by the three transpositions. Theorems 5.1.27 and 2.5.14 provide an equivalence

$$\mathcal{S}\mathsf{p}^{\mathsf{gS}_3} \xrightarrow[\sim]{g} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}_{\mathsf{S}_3}} \left(\mathcal{S}\mathsf{p}^{\mathsf{hW}_{\mathsf{S}_3}(\bullet)} \right) := \lim^{\mathsf{r.lax}} \left(\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{hS}_3} \arrow[r, "(-)^{\tau \mathsf{C}_2}"] \arrow[d, "(-)^{\tau \mathsf{C}_3}"'] \arrow[dr, "(-)^{\tau \mathsf{S}_3}" description] & \mathcal{S}\mathsf{p} \arrow[d] \arrow[dl, phantom, shift left=3, "\Nearrow"] \\ \mathcal{S}\mathsf{p}^{\mathsf{hC}_2} \arrow[r, "(-)^{\tau \mathsf{C}_2}"'] \arrow[ur, phantom, shift right=3, "\Swarrow"] & \mathcal{S}\mathsf{p} \end{tikzcd} \right) . \tag{5.3.9}$$

By Observation 5.2.6, the functors $(-)^{\tau \mathsf{S}_3}$ and $((-)^{\tau \mathsf{C}_3})^{\tau \mathsf{C}_2}$ are zero (the former by part (1), the latter by parts (2) and (3)). Moreover, by Observation 5.2.2, the gluing functor corresponding to the relation $\mathsf{C}_2 \to \mathsf{S}_3$ is also zero. Therefore, via the equivalence (5.3.9), a genuine S_3-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{gS}_3}$ is completely specified by the data of

- the objects

$$E_{00} := UE \in \mathcal{S}\mathsf{p}^{\mathsf{hS}_3} \qquad E_{01} := \Phi^{\mathsf{C}_2} E \in \mathcal{S}\mathsf{p}$$

$$E_{10} := \Phi^{\mathsf{C}_3} E \in \mathcal{S}\mathsf{p}^{\mathsf{hC}_2} \qquad E_{11} := \Phi^{\mathsf{S}_3} E \in \mathcal{S}\mathsf{p}$$

  and

- the structure maps

$$\begin{tikzcd} & (E_{00})^{\tau \mathsf{C}_2} & E_{01} \arrow[l] \\ (E_{00})^{\tau \mathsf{C}_3} & & \\ E_{10} \arrow[u] & (E_{10})^{\tau \mathsf{C}_2} & E_{11} \arrow[l] \end{tikzcd} .$$

[126] We refer the reader to the brief discussion of [AMGRa, §§??-??] for further support regarding these assertions.

Example 5.3.11 (genuine and proper-genuine $\mathbb{T}$-spectra)**.** Let $\mathbb{T}$ denote the circle group. Its poset of conjugacy classes of closed subgroups admits an identification

$$\mathsf{P}_{\mathbb{T}} \cong (\mathbb{N}^{\mathsf{div}})^{\triangleright}$$

as the right cone on the poset of natural numbers ordered by divisibility (under which the subgroup $\mathsf{C}_n \leq \mathbb{T}$ corresponds to the element $n \in \mathbb{N}^{\mathsf{div}} \subseteq (\mathbb{N}^{\mathsf{div}})^{\triangleright}$), which we use implicitly for notational convenience. The gluing diagram

$$(\mathbb{N}^{\mathsf{div}})^{\triangleright} \xrightarrow[\mathsf{l.lax}]{\mathcal{S}\mathsf{p}^{\mathsf{hW}_{\mathbb{T}}(\bullet)}} \mathsf{Pr}_{\mathsf{st}}$$

of the geometric stratification of genuine $\mathbb{T}$-spectra may be depicted as

$$\begin{tikzcd}
& \mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_2)} \arrow[r] \arrow[dr] & \mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_4)} & \cdots \arrow[ddr] & \\
\mathcal{S}\mathsf{p}^{\mathsf{h}\mathbb{T}} \arrow[ur] \arrow[urr] \arrow[rr] \arrow[dr] \arrow[ddr] & & \mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_6)} & \cdots \arrow[dr] & \vdots \\
& \mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_3)} \cdots \arrow[ur] & & \cdots \arrow[r] & \mathcal{S}\mathsf{p} \\
& \cdots & & \cdots \arrow[ur] &
\end{tikzcd}$$

- its values are described by the assignments

$$r \longmapsto \mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_r)} \xleftrightarrow[\sim]{(\mathbb{T}/\mathsf{C}_r)\cong\mathbb{T}} \mathcal{S}\mathsf{p}^{\mathsf{h}\mathbb{T}} \qquad \text{and} \qquad \infty \longmapsto \mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathbb{T})} \simeq \mathcal{S}\mathsf{p} ,$$

- it assigns to morphisms $r \to rs$ and $r \to \infty$ the horizontal functors in the diagrams

$$\begin{tikzcd}
\mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_r)} \arrow[r, "(-)^{\tau(\mathsf{C}_{rs}/\mathsf{C}_r)}"] \arrow[d, "(\mathbb{T}/\mathsf{C}_r)\cong\mathbb{T}"', "\wr"] & \mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_{rs})} \arrow[d, "\wr"', "(\mathbb{T}/\mathsf{C}_{rs})\cong(\mathbb{T}/\mathsf{C}_s)"] \\
\mathcal{S}\mathsf{p}^{\mathsf{h}\mathbb{T}} \arrow[r, "(-)^{\tau\mathsf{C}_s}"] \arrow[dr, dashed, "(-)^{\tau\mathsf{C}_s}"'] & \mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_s)} \arrow[d, "\wr"', "(\mathbb{T}/\mathsf{C}_s)\cong\mathbb{T}"] \\
& \mathcal{S}\mathsf{p}^{\mathsf{h}\mathbb{T}}
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\mathcal{S}\mathsf{p}^{\mathsf{h}(\mathbb{T}/\mathsf{C}_r)} \arrow[r, "(-)^{\tau(\mathbb{T}/\mathsf{C}_r)}"] \arrow[d, "(\mathbb{T}/\mathsf{C}_r)\cong\mathbb{T}"', "\wr"] & \mathcal{S}\mathsf{p} \\
\mathcal{S}\mathsf{p}^{\mathsf{h}\mathbb{T}} \arrow[ur, "(-)^{\tau\mathbb{T}}"'] &
\end{tikzcd}$$

  (in which the notation for the dashed functor is mildly abusive), and

- we have suppressed the natural transformations

$$(-)^{\tau\mathsf{C}_{rs}} \longrightarrow \left((-)^{\tau\mathsf{C}_r}\right)^{\tau\mathsf{C}_s}$$

  (not to mention higher coherences) for typographical ease.

Theorems 5.1.27 and 2.5.14 provide an adjunction

$$\mathcal{S}\mathsf{p}^{\mathsf{g}\mathbb{T}} \underset{\lim_{\mathsf{sd}((\mathbb{N}^{\mathsf{div}})^{\triangleright})}}{\overset{g}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\mathbb{N}^{\mathsf{div}})^{\triangleright}} \left(\mathcal{S}\mathsf{p}^{\mathsf{hW}_{\mathbb{T}}(\bullet)}\right) \quad . \tag{5.3.10}$$

However, Theorem 2.5.14 does not guarantee that the adjunction (5.3.10) is an equivalence, because the poset $(\mathbb{N}^{\mathsf{div}})^{\triangleright}$ is not down-finite (recall Remark 1.7.1). On the other hand, there is evidently a restricted stratification

$$\begin{tikzcd} (\mathbb{N}^{\mathsf{div}})^{\triangleright} \arrow[r, "\mathcal{S}\mathsf{p}^{\mathsf{g}\mathbb{T}}_{\leq \bullet}"] & \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{g}\mathbb{T}}} \\ \mathbb{N}^{\mathsf{div}} \arrow[u, hook] \arrow[r, dashed] & \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{g}<\mathbb{T}}} \arrow[u, hook] \end{tikzcd} \tag{5.3.11}$$

of the presentable stable ∞-category $\mathcal{S}\mathsf{p}^{\mathsf{g}^{<\mathbb{T}}}$ of *proper*-genuine $\mathbb{T}$-spectra (recall Observation 3.4.4).[127] As the poset $\mathbb{N}^{\mathsf{div}}$ is down-finite, Theorem 2.5.14 provides an equivalence

$$\mathcal{S}\mathsf{p}^{\mathsf{g}^{<\mathbb{T}}} \xrightarrow[\sim]{g} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathbb{N}^{\mathsf{div}}} \left(\mathcal{S}\mathsf{p}^{\mathsf{hW}_{\mathbb{T}}(\bullet)} \right) .$$

5.4. **Categorical fixedpoints via stratifications.** In this subsection, we describe categorical fixedpoints of genuine G-spectra as well as restriction and transfer morphisms among them in terms of the geometric stratification.

Local Notation 5.4.1. In this subsection, we assume that the group G is finite.

Observation 5.4.2. The poset P_G is finite, and hence the geometric stratification of $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$ over it converges by Theorem 2.5.14. We use this fact without further comment.

Observation 5.4.3. Given a genuine G-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$, using the nanocosm reconstruction of Theorem A(4) (recall Remark 2.6.7), we may identify its categorical H-fixedpoints via the equivalences

$$\begin{aligned} E^H &\simeq \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{g}G}}(\Sigma^\infty_G (G/H)_+, E) \\ &\simeq \lim_{([n] \xrightarrow{\varphi} \mathsf{P}_G) \in \mathsf{sd}(\mathsf{P}_G)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Phi^{\varphi(n)}(\Sigma^\infty_G (G/H)_+), \Gamma_\varphi \Phi^{\varphi(0)} E) \\ &\simeq \lim_{([n] \xrightarrow{\varphi} \mathsf{P}_G) \in \mathsf{sd}(\mathsf{P}_G)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Sigma^\infty ((G/H)^{\varphi(n)})_+, \Gamma_\varphi \Phi^{\varphi(0)} E) && (5.4.1) \\ &\simeq \lim_{([n] \xrightarrow{\varphi} (^{\leq} H)) \in \mathsf{sd}(^{\leq} H)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Sigma^\infty ((G/H)^{\varphi(n)})_+, \Gamma_\varphi \Phi^{\varphi(0)} E) && (5.4.2) \end{aligned}$$

in $\mathcal{S}\mathsf{p}^{\mathsf{hW}(H)}$, in which

- equivalence (5.4.1) follows from Observation 5.1.25 and
- equivalence (5.4.2) follows from the facts
  - that the functor
  $$\mathsf{sd}(^{\leq} H) \longrightarrow \mathsf{sd}(\mathsf{P}_G)$$
  is a fully faithful right fibration and
  - that for any $K \not\leq H$ in P_G we have
  $$\Sigma^\infty ((G/H)^K)_+ \simeq \Sigma^\infty (\varnothing)_+ \simeq 0 .$$

Example 5.4.4 (categorical e-fixedpoints)**.** Suppose that $H = e \leq G$ is the trivial subgroup. Then, for any genuine G-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$, the composite equivalence of Observation 5.4.3 reduces to an equivalence

$$E^e \simeq \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{h}G}}(\Sigma^\infty ((G/e)^e)_+, UE) \simeq \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{h}G}}(\Sigma^\infty (G/e)_+, UE) \simeq UE$$

in $\mathcal{S}\mathsf{p}^{\mathsf{h}G}$.

[127] The right vertical functor of diagram (5.3.11) arises from the fact that we may identify proper-genuine $\mathbb{T}$-spectra as the closed subcategory $\mathcal{S}\mathsf{p}^{\mathsf{g}^{<\mathbb{T}}} \in \mathbf{Cls}_{\mathcal{S}\mathsf{p}^{\mathsf{g}\mathbb{T}}}$ consisting of those objects $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}\mathbb{T}}$ such that the canonical morphism $E^{\mathbb{T}} \to E^{\mathsf{h}\mathbb{T}}$ is an equivalence.

Example 5.4.5 (categorical C_p-fixedpoints)**.** Suppose that $H = G = \mathsf{C}_p$ (and recall Example 5.3.4). For any genuine C_p-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}\mathsf{C}_p}$, the composite equivalence of Observation 5.4.3 reduces to an equivalence

$$E^{\mathsf{C}_p} \simeq \lim \left(\begin{tikzcd} & \mathsf{hom}_{\mathcal{S}\mathsf{p}}(\Sigma^\infty((\mathsf{C}_p/\mathsf{C}_p)^{\mathsf{C}_p})_+, \Phi^{\mathsf{C}_p}E) \arrow[d] \\ \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{C}_p}}(\Sigma^\infty((\mathsf{C}_p/\mathsf{C}_p)^e)_+, UE) \arrow[r] & \mathsf{hom}_{\mathcal{S}\mathsf{p}}(\Sigma^\infty((\mathsf{C}_p/\mathsf{C}_p)^{\mathsf{C}_p})_+, (UE)^{\tau\mathsf{C}_p}) \end{tikzcd} \right) \simeq \lim \left(\begin{tikzcd} & \Phi^{\mathsf{C}_p}E \arrow[d] \\ (UE)^{\mathsf{h}\mathsf{C}_p} \arrow[r] & (UE)^{\tau\mathsf{C}_p} \end{tikzcd} \right)$$

in $\mathcal{S}\mathsf{p}$.

Example 5.4.6 (categorical C_{p^2}-fixedpoints)**.** Suppose that $H = G = \mathsf{C}_{p^2}$ (and recall Example 5.3.6). For any genuine C_{p^2}-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}\mathsf{C}_{p^2}}$, the composite equivalence of Observation 5.4.3 yields a limit diagram

$$\begin{tikzcd} & \Phi^{\mathsf{C}_{p^2}}E \arrow[rr] \arrow[dd] & & (\Phi^{\mathsf{C}_p}E)^{\tau\mathsf{C}_p} \arrow[dd] \\ E^{\mathsf{C}_{p^2}} \arrow[ur] \arrow[rr] \arrow[dd] & & (\Phi^{\mathsf{C}_p}E)^{\mathsf{h}\mathsf{C}_p} \arrow[ur] \arrow[dd] & \\ & (UE)^{\tau\mathsf{C}_{p^2}} \arrow[rr] & & ((UE)^{\tau\mathsf{C}_p})^{\tau\mathsf{C}_p} \\ (UE)^{\mathsf{h}\mathsf{C}_{p^2}} \arrow[ur] \arrow[rr] & & ((UE)^{\tau\mathsf{C}_p})^{\mathsf{h}\mathsf{C}_p} \arrow[ur] & \end{tikzcd}$$

in $\mathcal{S}\mathsf{p}$.

Local Notation 5.4.7. For the remainder of this subsection, we fix a subgroup $K \subseteq H$ of the chosen subgroup $H \subseteq G$.

Definition 5.4.8. The ***relative Weyl group*** of the nested pair $K \subseteq H$ of subgroups of G is

$$\mathsf{W}(K \subseteq H) := \mathsf{W}_G(K \subseteq H) := \frac{\mathsf{N}_G(K) \cap \mathsf{N}_G(H)}{K} \ ,$$

the quotient by K of the intersection of the normalizers of K and H in G.[128] By definition, this comes equipped with homomorphisms

$$\begin{tikzcd} \mathsf{W}(K) \arrow[d, phantom, "\rotatebox{90}{$\coloneqq$}"] & \mathsf{W}(K \subseteq H) \arrow[l] \arrow[rr] \arrow[d, phantom, "\rotatebox{90}{$\coloneqq$}"] & & \mathsf{W}(H) \arrow[d, phantom, "\rotatebox{90}{$\coloneqq$}"] \\ \dfrac{\mathsf{N}(K)}{K} & \dfrac{\mathsf{N}(K) \cap \mathsf{N}(H)}{K} \arrow[l] \arrow[r] & \dfrac{\mathsf{N}(K) \cap \mathsf{N}(H)}{\mathsf{N}(K) \cap H} \arrow[r] & \dfrac{\mathsf{N}(H)}{H} \end{tikzcd} .$$

Observation 5.4.9. Restriction defines a natural transformation

$$\begin{tikzcd} \mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "(-)^H"] \arrow[d, "(-)^K"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{W}_G(H)} \arrow[d] \\ \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{W}_G(K)} \arrow[r] & \mathcal{S}\mathsf{p}^{\mathsf{h}\mathsf{W}_G(K \subseteq H)} \end{tikzcd} ,$$

which is corepresented by the morphism

$$\Sigma^\infty_G(G/K \longrightarrow G/H)_+$$

[128]More invariantly, one can also describe $\mathsf{W}(K \subseteq H)$ as the group of automorphisms of the object $(G/K \to G/H) \in \mathsf{Ar}(\mathscr{O}_G) \subseteq \mathsf{Ar}(\mathcal{S}^{\mathsf{g}G})$.

in $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$. In terms of nanocosm reconstruction, for any genuine G-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ it may be expressed as the composite

$$\begin{aligned}
E^H &\simeq \lim_{([n] \xrightarrow{\varphi} ({}^{\leq}H)) \in \mathsf{sd}({}^{\leq}H)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Sigma^\infty((G/H)^{\varphi(n)})_+, \Gamma_\varphi \Phi^{\varphi(0)} E) && (5.4.3) \\
&\longrightarrow \lim_{([n] \xrightarrow{\varphi} ({}^{\leq}K)) \in \mathsf{sd}({}^{\leq}K)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Sigma^\infty((G/H)^{\varphi(n)})_+, \Gamma_\varphi \Phi^{\varphi(0)} E) && (5.4.4) \\
&\longrightarrow \lim_{([n] \xrightarrow{\varphi} ({}^{\leq}K)) \in \mathsf{sd}({}^{\leq}K)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Sigma^\infty((G/K)^{\varphi(n)})_+, \Gamma_\varphi \Phi^{\varphi(0)} E) && (5.4.5) \\
&\simeq E^K \ , && (5.4.6)
\end{aligned}$$

where

- the equivalences (5.4.3) and (5.4.6) follow from Observation 5.4.3,
- the morphism (5.4.4) is that on limits induced by the functor
$$\mathsf{sd}({}^{\leq}K) \longrightarrow \mathsf{sd}({}^{\leq}H) \ ,$$
  and
- the morphism (5.4.5) is that on limits determined by a morphism in $\mathsf{Fun}(\mathsf{sd}({}^{\leq}K), \mathcal{S}\mathsf{p})$ whose component at an object $([n] \xrightarrow{\varphi} ({}^{\leq}K)) \in \mathsf{sd}({}^{\leq}K)$ is precomposition with the morphism
$$\Sigma^\infty((G/K \longrightarrow G/H)^{\varphi(n)})_+$$
  in $\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}$.

Observation 5.4.10. Transfer defines a natural transformation

$$\begin{tikzcd}
\mathcal{S}\mathsf{p}^{\mathsf{g}G} \arrow[r, "(-)^H"] \arrow[d, "(-)^K"'] \arrow[dr, phantom, "\nearrow"] & \mathcal{S}\mathsf{p}^{\mathsf{hW}_G(H)} \arrow[d] \\
\mathcal{S}\mathsf{p}^{\mathsf{hW}_G(K)} \arrow[r] & \mathcal{S}\mathsf{p}^{\mathsf{hW}_G(K \subseteq H)}
\end{tikzcd} ,$$

which is corepresented by a morphism

$$\Sigma^\infty_G(G/H)_+ \longrightarrow \Sigma^\infty_G(G/K)_+ \qquad (5.4.7)$$

in $\mathcal{S}\mathsf{p}^{\mathsf{g}G}$.[129] In terms of nanocosm reconstruction, for any genuine G-spectrum $E \in \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ it may be expressed as the composite

$$\begin{aligned}
E^K &\simeq \lim_{([n] \xrightarrow{\varphi} ({}^{\leq}K)) \in \mathsf{sd}({}^{\leq}K)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Sigma^\infty((G/K)^{\varphi(n)})_+, \Gamma_\varphi \Phi^{\varphi(0)} E) && (5.4.8) \\
&\simeq \lim_{([n] \xrightarrow{\varphi} ({}^{\leq}H)) \in \mathsf{sd}({}^{\leq}H)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Sigma^\infty((G/K)^{\varphi(n)})_+, \Gamma_\varphi \Phi^{\varphi(0)} E) && (5.4.9) \\
&\longrightarrow \lim_{([n] \xrightarrow{\varphi} ({}^{\leq}H)) \in \mathsf{sd}({}^{\leq}H)} \mathsf{hom}_{\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}}(\Sigma^\infty((G/H)^{\varphi(n)})_+, \Gamma_\varphi \Phi^{\varphi(0)} E) && (5.4.10) \\
&\simeq E^H \ , && (5.4.11)
\end{aligned}$$

where

- the equivalences (5.4.8), (5.4.9), and (5.4.11) follow from Observation 5.4.3, and

---

[129] The morphism (5.4.7) may be obtained by applying the functor $\mathcal{S}\mathsf{p}^{\mathsf{g}H} \xrightarrow{\mathsf{Ind}_H^G} \mathcal{S}\mathsf{p}^{\mathsf{g}G}$ to the morphism

$$\Sigma^\infty_H(H/H)_+ \longrightarrow \Sigma^\infty_H(H/K)_+ \simeq \mathsf{coInd}_K^H(\Sigma^\infty_K(K/K)_+)$$

corresponding to the identity morphism

$$\mathsf{Res}_K^H(\Sigma^\infty_H(H/H)_+) \longrightarrow \Sigma^\infty_K(K/K)_+$$

in $\mathcal{S}\mathsf{p}^{\mathsf{g}K}$.

- the morphism (5.4.10) is that on limits determined by a morphism in $\mathsf{Fun}(\mathsf{sd}({}^{\leq}H), \mathcal{S}\mathsf{p})$ whose component at an object $([n] \xrightarrow{\varphi} ({}^{\leq}H)) \in \mathsf{sd}({}^{\leq}H)$ is precomposition with the morphism

$$\Sigma^{\infty}((G/H)^{\varphi(n)})_{+} \simeq \Phi^{\varphi(n)}(\Sigma^{\infty}_{G}(G/H)_{+}) \xrightarrow{\Phi^{\varphi(n)}(5.4.7)} \Phi^{\varphi(n)}(\Sigma^{\infty}_{G}(G/K)_{+}) \simeq \Sigma^{\infty}((G/K)^{\varphi(n)})_{+}$$

in $\mathcal{S}\mathsf{p}^{\mathsf{hW}(\varphi(n))}$ (using Observation 5.1.25).

6. The metacosm reconstruction theorem

In this section, we prove the metacosm reconstruction theorem (Theorem A(1)), which easily implies the macrocosm reconstruction theorem (Theorem A(2)) as proved in §2. It is organized as follows.

§6.1: We establish a canonical stratification of certain right-lax limits.

§6.2: We prove Theorem A(1) as Theorem 6.2.6. Recall that this is an adjunction, which is an equivalence when the poset is down-finite. Its left adjoint takes a stratified presentable stable ∞-category to its gluing diagram; its right adjoint is essentially constructed in §6.1.

§6.3: We explain the theory of strict stratifications.

Local Notation 6.0.1. In this section, we fix a poset P.

6.1. Stratifications of right-lax limits.

In this subsection we prove the omnibus Proposition 6.1.6, which establishes a canonical stratification of certain right-lax limits as well as a number of its essential properties.

Definition 6.1.1. A ***presentable stable left-lax left*** P***-module*** is a left-lax left P-module whose fibers are presentable stable ∞-categories and whose monodromy functors are exact and accessible. These assemble into a subcategory

$$\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \subseteq \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$$

whose morphisms are those morphisms in $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$ that are fiberwise left adjoints.

Local Notation 6.1.2. In this subsection, we fix a presentable stable left-lax left P-module

$$(\mathcal{E} \downarrow \mathsf{P}) \in \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \ .$$

For any morphism $p \to q$ in P we write

$$\mathcal{E}_p \xrightarrow{\Gamma^p_q} \mathcal{E}_q$$

for its corresponding cocartesian monodromy functor.

Local Notation 6.1.3. In this subsection, we write

$$\mathcal{X} := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \in \mathsf{Cat} \ ,$$

and for any subposet $\mathsf{Q} \subseteq \mathsf{P}$ we write

$$\mathcal{X} := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \xrightarrow{\Phi_{\mathsf{Q}}} \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.Q}}(\mathcal{E})$$

for the restriction functor.

Observation 6.1.4. It follows from Lemma A.6.5 that $\mathcal{X}$ is accessible.

Observation 6.1.5. It follows from Lemma A.6.5 that $\mathcal{X}$ is stable. We use this fact without further comment.

Proposition 6.1.6.

(1) The ∞-category $\mathcal{X} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})$ is cocomplete, and hence presentable by Observation 6.1.4.

(2) The functor

$$\mathcal{X} \xrightarrow{(\Phi_p)_{p \in \mathsf{P}}} \prod_{p \in \mathsf{P}} \mathcal{E}_p$$

is conservative.

(3) For any subposet $\mathsf{Q} \subseteq \mathsf{P}$, the restriction functor

$$\mathcal{X} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \xrightarrow{\Phi_{\mathsf{Q}}} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.Q}}(\mathcal{E})$$

preserves colimits, and hence admits a right adjoint by part (1).

(4) Choose any $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}}$.

(a) The restriction functor

$$\mathcal{X} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \xrightarrow{y} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathcal{E}) =: \mathcal{Z}_{\mathsf{D}}$$

admits not only a right adjoint i_R as guaranteed by part (3) but also a fully faithful left adjoint i_L, whose image consists of those objects $X \in \mathcal{X}$ such that $\Phi_q(X) \simeq 0$ for all $q \in \mathsf{P} \backslash \mathsf{D}$. In particular, for any $p \in \mathsf{P}$, we may consider

$$\mathcal{Z}_p := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E}) \tag{6.1.1}$$

as a closed subcategory of $\mathcal{X}$ via i_L.

(b) The right adjoint ν to the restriction functor

$$\mathcal{X} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \xrightarrow{p_L} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\mathsf{P} \backslash \mathsf{D})}(\mathcal{E})$$

guaranteed by part (3) is fully faithful, and its image consists of those objects $X \in \mathcal{X}$ such that $\Phi_q(X) \simeq 0$ for all $q \in \mathsf{D}$.

(5) The closed subcategories (6.1.1) assemble into a stratification

$$\begin{array}{ccl} \mathsf{P} & \xrightarrow{\mathcal{Z}_\bullet} & \mathbf{Cls}_{\mathcal{X}} := \mathbf{Cls}_{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})} \\ \Psi & & \Psi \\ p & \longmapsto & \mathcal{Z}_p \ := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E}) \end{array} . \tag{6.1.2}$$

Moreover, our existing notation is consistent with this stratification in the following ways.

(a) For any $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}}$, we have

$$\mathcal{Z}_{\mathsf{D}} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathcal{E}) \simeq \left\langle \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E}) \right\rangle_{p \in \mathsf{D}} =: \langle \mathcal{Z}_p \rangle_{p \in \mathsf{D}} .$$

(b) For any $\mathsf{C} \in \mathsf{Conv}_{\mathsf{P}}$, the C^{th} stratum of the stratification (6.1.2) is

$$\mathcal{X}_{\mathsf{C}} := \mathcal{Z}_{\leq \mathsf{C}} / \mathcal{Z}_{< \mathsf{C}} \simeq \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.C}}(\mathcal{E}) ,$$

and its C^{th} geometric localization functor

$$\mathcal{X} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \xrightarrow{\Phi_{\mathsf{C}}} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.C}}(\mathcal{E}) \simeq \mathcal{X}_{\mathsf{C}}$$

is the restriction functor.

(c) *For any $p < q$ in P, the lax-commutative square*

$$\begin{tikzcd} \mathcal{E}_p \arrow[r, hook, "\rho^p"] \arrow[d, equal] \arrow[dr, phantom, "\Swarrow"] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) =: \mathcal{X} \arrow[d, "\Phi_{\{p<q\}}"] \\ \mathcal{E}_p \arrow[r, hook, "\rho^p"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\{p<q\}}(\mathcal{E}) \end{tikzcd} \tag{6.1.3}$$

determined by the commutative square

$$\begin{tikzcd} \mathcal{E}_p \arrow[d, equal] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) =: \mathcal{X} \arrow[l, "\Phi_p"'] \arrow[d, "\Phi_{\{p<q\}}"] \\ \mathcal{E}_p & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\{p<q\}}(\mathcal{E}) \arrow[l, "\Phi_p"] \end{tikzcd} \tag{6.1.4}$$

commutes. In particular, for every morphism $p \to q$ in P, there is a canonical identification

$$\begin{tikzcd} \mathcal{X}_p \arrow[r, "\Gamma^p_q"] \arrow[d, phantom, "\wr"] & \mathcal{X}_q \arrow[d, phantom, "\wr"] \\ \mathcal{E}_p \arrow[r, "\Gamma^p_q"'] & \mathcal{E}_q \end{tikzcd}$$

between the corresponding gluing functor for $\mathcal{X}$ (with respect to the stratification (6.1.2)*) and the corresponding monodromy functor of $\mathcal{E}$.*

Warning 6.1.7. In the statement and proof of Proposition 6.1.6, we use notation corresponding to recollements (such as i_L, y, etc.) even before those recollements have been established.

Definition 6.1.8. A ***stable recollement*** is a diagram (1.1.1) among stable ∞-categories such that there are equalities (1.1.2). (In particular, we use the same notation for the functors involved in a stable recollement as we do for those involved in a recollement.)

Remark 6.1.9. A recollement in the sense of Definition 1.1.1 is simply a stable recollement among presentable stable ∞-categories.

Observation 6.1.10. Lemma 2.2.1 applies not just to recollements but to stable recollements: neither the statement nor the proof relies on presentability in any way. We will use this fact without further comment.

Lemma 6.1.11. *Proposition 6.1.6 holds when $\mathsf{P} = [1]$.*

Proof. It is immediate that we have a stable recollement

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, bend left, hook, "i_L"] \arrow[r, bend right, hook, "i_R"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{X} \arrow[l, "y"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{E}_1 \arrow[l, hook', "\nu"'] \end{tikzcd} \tag{6.1.5}$$

in which, writing

$$(E_0 \longmapsto \Gamma^0_1(E_0) \xleftarrow{\gamma} E_1) \tag{6.1.6}$$

for an arbitrary object of $\mathcal{X}$ (where $E_i \in \mathcal{E}_i$ for $i \in [1]$),

- the three functors with source $\mathcal{X}$ are defined by the formulas
$$y(6.1.6) := E_0 \ , \qquad p_L(6.1.6) := E_1 \ , \qquad \text{and} \qquad p_R(6.1.6) := \mathsf{fib}(\gamma) \ ,$$
- the two functors with source $\mathcal{E}_0$ are defined by the formulas
$$i_L(E) := (E \longmapsto \Gamma^0_1(E) \longleftarrow 0) \qquad \text{and} \qquad i_R(E) := (E \longmapsto \Gamma^0_1(E) \xleftarrow{\sim} \Gamma^0_1(E)) \ ,$$
and

- the one functor with source $\mathcal{E}_1$ is defined by the formula

$$\nu(E) := (0 \longmapsto 0 \longleftarrow E) \ .$$

In particular, we have an evident identification $\Gamma_1^0 \simeq p_L i_R$. Moreover, applying Lemma 2.2.1 to the stable recollement (6.1.5), it is straightforward to verify that any functor $\mathcal{J} \xrightarrow{F} \mathcal{X}$ has a colimit

$$\Big(\mathsf{colim}_{\mathcal{J}}(yF) \longmapsto \Gamma_1^0(\mathsf{colim}_{\mathcal{J}}(yF)) \simeq p_L i_R(\mathsf{colim}_{\mathcal{J}}(yF)) \longleftarrow \mathsf{colim}_{\mathcal{J}}(p_L i_R y F) \xleftarrow{\eta_{y \dashv i_R}} \mathsf{colim}_{\mathcal{J}}(p_L F)\Big) \ ,$$

so that $\mathcal{X}$ is cocomplete. The remaining claims are now evident. $\square$

Lemma 6.1.12. *Proposition 6.1.6 holds when* $\mathsf{P} = [n] \in \mathbf{\Delta}$ *(for any* $n \geq 0$*).*

Proof. The claim is immediate if $n = 0$, and if $n = 1$ this is the content of Lemma 6.1.11. So suppose that $n \geq 2$. Let us write $\mathcal{Y} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\{1 < \cdots < n\}}(\mathcal{E})$.

Consider the functor $[n] \xrightarrow{\pi} [1]$ characterized by the fact that $\pi^{-1}(0) = \{0\}$. In light of Lemma A.6.5, using the composability of right Kan extensions with respect to the composite $\mathsf{sd}([n]) \xrightarrow{\mathsf{sd}(\pi)} \mathsf{sd}([1]) \to \mathsf{pt}$, we obtain a pullback square

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathsf{Fun}([1], \mathcal{Y}) \arrow[d, "t"] \\
\mathcal{E}_0 \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

in which the left vertical functor and the composite $\mathcal{X} \to \mathsf{Fun}([1], \mathcal{Y}) \xrightarrow{s} \mathcal{Y}$ are the canonical restriction functors. This immediately yields a stable recollement

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathcal{X} \arrow[l, "y" description] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{Y} \arrow[l, "\nu" description]
\end{tikzcd} \ , \tag{6.1.7}$$

in which the functors y and p_L are the canonical restriction functors. Note moreover that $\mathcal{E}_0$ is presentable by assumption, $\mathcal{Y}$ is presentable by induction, and the composite functor $p_L i_R$ is accessible in light of Observation 6.1.4. So, it follows from Lemma 6.1.11 that $\mathcal{X}$ is presentable: that is, we have proved part (1).

Using the recollement (6.1.7) and Lemma 2.2.1, we see by induction that the functor

$$\mathcal{X} \xrightarrow{(\Phi_i)_{i \in [n]}} \prod_{i \in [n]} \mathcal{E}_i$$

is conservative and preserves colimits; in particular, we have proved part (2). Since any subposet $\mathsf{Q} \subseteq [n]$ whose inclusion is not an isomorphism is of the form $\mathsf{Q} \cong \coprod_{j=1}^{k} [i_j]$ where $i_j < n$ for all j, we then also see by induction (with respect to parts (2) and (3)) that the restriction functor

$$\mathcal{X} \xrightarrow{\Phi_{\mathsf{Q}}} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.Q}}(\mathcal{E})$$

preserves colimits. So, we have proved part (3).

We now turn to part (4). If $\mathsf{D} = \varnothing$ then part (4) is trivial, while if $\mathsf{D} = \{0\}$ then part (4) follows from the recollement (6.1.7) (and part (2) applied to $\mathcal{Y}$). So, we may assume that $\mathsf{D} = ({}^{\leq} i) = [i]$ where $1 \leq i \leq n$. Noting the factorization

$$\mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([n] \backslash [i])}(\mathcal{E})$$

of the restriction functor, we find that part (4)(b) follows from induction and the recollement (6.1.7). So it remains to prove part (4)(a). For this, we introduce the notation

$$\mathcal{W}_i := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\{1 < \cdots < i\}}(\mathcal{E})$$

and make the following observations.

- By induction, we have $\mathcal{W}_i \in \mathbf{Cls}_{\mathcal{Y}}$.
- Replacing $[n]$ with $[i]$, the recollement (6.1.7) becomes an analogous recollement

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, hook, bend left, "i_L"] \arrow[r, hook, bend right, "i_R"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{Z}_i \arrow[l, "y" description] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{W}_i \arrow[l, hook, "\nu" description] \end{tikzcd} . \tag{6.1.8}$$

- The diagram

$$\begin{tikzcd} \mathcal{E}_0 \arrow[d, equal] & \mathcal{X} \arrow[l, "y"'] \arrow[r, "p_L"] \arrow[d, "y"'] & \mathcal{Y} \arrow[d, "y"] \\ \mathcal{E}_0 & \mathcal{Z}_i \arrow[l, "y"] \arrow[r, "p_L"'] & \mathcal{W}_i \end{tikzcd} \tag{6.1.9}$$

  among restriction functors commutes.
- The fully faithful inclusion i_R of recollement (6.1.7) (resp. (6.1.8)) has image consisting of those objects $X \in \mathcal{X}$ (resp. $X \in \mathcal{Z}_i$) such that for all $j \in \{1 < \cdots < n\}$ (resp. $j \in \{1 < \cdots < i\}$) the structure morphism $\Phi_j(X) \to \Gamma^0_j(\Phi_0(X))$ is an equivalence. It follows that the lax-commutative square

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, hook, "i_R"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow"] & \mathcal{X} \arrow[d, "y"] \\ \mathcal{E}_0 \arrow[r, hook, "i_R"'] & \mathcal{Z}_i \end{tikzcd}$$

  determined by the left commutative square in diagram (6.1.9) commutes.

Using these observations and applying Lemma 2.2.1 to the recollements (6.1.7) and (6.1.8), we find that the restriction functor $\mathcal{X} \xrightarrow{y} \mathcal{Z}_i$ is described by the formula

$$\begin{array}{ccc} \mathcal{Z}_i \simeq \lim^{\mathsf{r.lax}}\left(\mathcal{E}_0 \xrightarrow{p_L i_R} \mathcal{W}_i\right) & \xleftarrow{\quad y \quad} & \lim^{\mathsf{r.lax}}\left(\mathcal{E}_0 \xrightarrow{p_L i_R} \mathcal{Y}\right) \simeq \mathcal{X} \\ \Uparrow & & \Uparrow \\ (E \longmapsto p_L i_R(E) \longleftarrow y(Y)) & \longleftarrow\!\!\shortmid & (E \longmapsto p_L i_R(E) \longleftarrow Y) \end{array} ,$$

so that it admits a left adjoint described by the formula

$$\begin{array}{ccc} \mathcal{Z}_i \simeq \lim^{\mathsf{r.lax}}\left(\mathcal{E}_0 \xrightarrow{p_L i_R} \mathcal{W}_i\right) & \xrightarrow{\quad i_L \quad} & \lim^{\mathsf{r.lax}}\left(\mathcal{E}_0 \xrightarrow{p_L i_R} \mathcal{Y}\right) \simeq \mathcal{X} \\ \Uparrow & & \Uparrow \\ (E \longmapsto p_L i_R(E) \longleftarrow W) & \longmapsto & (E \longmapsto p_L i_R(E) \longleftarrow i_L(W)) \end{array} ,$$

which by induction is fully faithful and has image as desired.

We now conclude with part (5). Observe that the closed subcategories

$$\left\{\mathcal{Z}_i := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq i)}(\mathcal{E}) \in \mathbf{Cls}_{\mathcal{X}}\right\}_{i \in [n]}$$

evidently assemble into a functor $[n] \xrightarrow{(6.1.2)} \mathbf{Cls}_{\mathcal{X}}$, which is clearly a prestratification and hence is a stratification by Observation 2.4.5. Moreover, assertion (5)(a) is trivial, and assertion (5)(b) follows from part (4)(b) (applied to D instead of P). To prove part (5)(c), in light of the commutative diagram

$$\begin{tikzcd}
\mathcal{E}_p \arrow[rr, hook, "\rho^p"] \arrow[dr, hook, "\rho^p"'] & & \mathcal{X} \arrow[rr, "\Phi_q"] \arrow[dr, "\Phi_{[n]_{p//q}}"] & & \mathcal{E}_q \\
& \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.[n]_{p//q}}(\mathcal{E}) \arrow[ur, hook, "\rho^{[n]_{p//q}}"] \arrow[rr, "\mathsf{id}"'] & & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.[n]_{p//q}}(\mathcal{E}) \arrow[ur, "\Phi_q"'] &
\end{tikzcd} ,$$

we see that it suffices to assume that $p = 0$ and $q = n$. Moreover, applying part (2) of Lemma 6.1.11, we see that it suffices to prove that the natural transformation of diagram (6.1.3) becomes an equivalence upon postcomposition with the functor

$$\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\{0<n\}}(\mathcal{E}) \xrightarrow{\Phi_n} \mathcal{E}_n \quad :$$

that is, that the natural transformation in the diagram

$$\begin{tikzcd}
& \mathcal{Z}_{n-1} \arrow[d, hook, "\rho^{[n-1]}"] \\
\mathcal{E}_0 \arrow[ur, hook, "\rho^0"] \arrow[r, hook, "\rho^0"] \arrow[d, hook, "\rho^0"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{X} \arrow[d, "\Phi_n"] \\
\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\{0<n\}}(\mathcal{E}) \arrow[r, "\Phi_n"'] & \mathcal{E}_n
\end{tikzcd}$$

is an equivalence. By Lemma A.6.5, every object of $\mathcal{Z}_{n-1} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.[n-1]}(\mathcal{E})$ is the limit of a diagram indexed by the finite poset $\mathsf{sd}([n-1])$; by our inductive hypothesis, for the image $\rho^0(X) \in \mathcal{Z}_{n-1}$ of any object $X \in \mathcal{E}_0$, this diagram is equivalent to its right Kan extension from the full subposet $\mathsf{sd}_0([n-1]) \subseteq \mathsf{sd}([n-1])$ on those objects $([i] \hookrightarrow [n-1]) \in \mathsf{sd}([n-1])$ whose image contains $0 \in [n-1]$. Note that this finite limit is preserved by the composite $\mathcal{Z}_{n-1} \xhookrightarrow{\rho^{[n-1]}} \mathcal{X} \xrightarrow{\Phi_n} \mathcal{E}_n$ of exact functors. Because $(\{0\} \hookrightarrow [n-1]) \in \mathsf{sd}_0([n-1])$ is an initial object, it follows that the composite functor $\mathcal{E}_0 \xhookrightarrow{\rho^0} \mathcal{X} \xrightarrow{\Phi_n} \mathcal{E}_n$ is canonically equivalent to the monodromy functor $\mathcal{E}_0 \to \mathcal{E}_n$, which proves the claim. $\square$

Proof of Proposition 6.1.6. Observe the equivalence

$$\mathcal{X} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \simeq \lim_{([n]\downarrow \mathsf{P})^\circ \in (\mathbf{\Delta}_{/\mathsf{P}})^{\mathsf{op}}} \left(\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.[n]}(\mathcal{E}) \right) . \tag{6.1.10}$$

It follows from Lemma 6.1.12 that the functor $(\mathbf{\Delta}_{/\mathsf{P}})^{\mathsf{op}} \xrightarrow{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\bullet}(\mathcal{E})} \mathsf{Cat}$ factors through the subcategory $\mathsf{Pr}^L_{\mathsf{st}} \subset \mathsf{Cat}$: each ∞-category $\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.[n]}(\mathcal{E})$ is presentable by its part (1), and for each morphism $[m] \to [n]$ in $\mathbf{\Delta}_{/\mathsf{P}}$ the corresponding restriction functor $\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.[m]}(\mathcal{E}) \leftarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.[n]}(\mathcal{E})$ preserves colimits by its parts (2) and (3). Hence, the identification (6.1.10) shows that $\mathcal{X}$ is presentable; that is, we have proved part (1). Using part (2) of Lemma 6.1.12, equivalence (6.1.10) also proves part (2). Thereafter, the evident functoriality of equivalence (6.1.10) in the variable P proves part (3).

We now prove part (4)(a). Given our fixed element $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}}$, observe the adjunction

$$\begin{tikzcd}
\mathbf{\Delta}_{/\mathsf{D}} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathbf{\Delta}_{/\mathsf{P}} \arrow[l, shift left, "(-)\cap\mathsf{D}"]
\end{tikzcd}$$

in which the right adjoint is given by intersection with $\mathsf{D} \subseteq \mathsf{P}$; thereafter, observe its opposite adjunction

$$\begin{tikzcd}
(\mathbf{\Delta}_{/\mathsf{P}})^{\mathsf{op}} \arrow[r, shift left, "((-)\cap\mathsf{D})^{\mathsf{op}}"] \arrow[r, phantom, "\perp"] & (\mathbf{\Delta}_{/\mathsf{D}})^{\mathsf{op}} \arrow[l, hook, shift left]
\end{tikzcd} . \tag{6.1.11}$$

Using the unit of the adjunction (6.1.11), we obtain a morphism

$$\begin{tikzcd}
(\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}} \arrow[rr, "\mathsf{id}"] \arrow[dr, "((-)\cap\mathsf{D})^{\mathsf{op}}"'] & \arrow[d, phantom, "\Downarrow"] & (\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}} \arrow[r, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}(\mathcal{E})"] & \mathsf{Cat} \\
& (\boldsymbol{\Delta}_{/\mathsf{D}})^{\mathsf{op}} \arrow[ur, hook] & &
\end{tikzcd} \tag{6.1.12}$$

in $\mathsf{Fun}((\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}}, \mathsf{Cat})$, which upon taking limits over $(\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}}$ yields a morphism

$$\lim_{([n]\downarrow\mathsf{P})^\circ\in(\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}}}\left(\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[n]}(\mathcal{E})\right) \longrightarrow \lim_{([n]\downarrow\mathsf{P})^\circ\in(\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}}}\left(\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([n]\cap\mathsf{D})}(\mathcal{E})\right) \ . \tag{6.1.13}$$

On the one hand, the source of the morphism (6.1.13) is identified as $\mathcal{X}$ via equivalence (6.1.10). On the other hand, because the functor $(\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}} \xrightarrow{((-)\cap\mathsf{D})^{\mathsf{op}}} (\boldsymbol{\Delta}_{/\mathsf{D}})^{\mathsf{op}}$ is initial (being a left adjoint), we may identify the target of the morphism (6.1.13) as

$$\lim_{([n]\downarrow\mathsf{P})^\circ\in(\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}}}\left(\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([n]\cap\mathsf{D})}(\mathcal{E})\right) \simeq \lim_{([n]\downarrow\mathsf{D})^\circ\in(\boldsymbol{\Delta}_{/\mathsf{D}})^{\mathsf{op}}}\left(\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[n]}(\mathcal{E})\right) \simeq \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathcal{E}) \ .$$

Hence, the morphism (6.1.13) is the restriction morphism

$$\mathcal{X} := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \xrightarrow{y} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathcal{E}) \ .$$

We now make the following observations regarding the morphism (6.1.12) in $\mathsf{Fun}((\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}}, \mathsf{Cat})$.

- For each object $([n]\downarrow\mathsf{P})^\circ \in (\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}}$, the component of the morphism (6.1.12) is the restriction functor
$$\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[n]}(\mathcal{E}) \xrightarrow{y} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([n]\cap\mathsf{D})}(\mathcal{E}) \ . \tag{6.1.14}$$
By part (4)(a) of Lemma 6.1.12, the functor (6.1.14) admits a fully faithful left adjoint i_L, whose image consists of those objects $X \in \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[n]}(\mathcal{E})$ such that $\Phi_q(X) \simeq 0$ for all $q \in ([n]\cap(\mathsf{P}\backslash\mathsf{D}))$.

- For each morphism $([m]\downarrow\mathsf{P})^\circ \to ([n]\downarrow\mathsf{P})^\circ$ in $(\boldsymbol{\Delta}_{/\mathsf{P}})^{\mathsf{op}}$, i.e. for each commutative triangle
$$\begin{tikzcd}
{[m]} \arrow[dr] & & {[n]} \arrow[ll] \arrow[dl] \\
& \mathsf{P} &
\end{tikzcd} ,$$
the component of the morphism (6.1.12) is the commutative square
$$\begin{tikzcd}
\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[m]}(\mathcal{E}) \arrow[r, "y"] \arrow[d] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([m]\cap\mathsf{D})}(\mathcal{E}) \arrow[d] \\
\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[n]}(\mathcal{E}) \arrow[r, "y"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([n]\cap\mathsf{D})}(\mathcal{E})
\end{tikzcd} \tag{6.1.15}$$
of restriction functors. Moreover, the lax-commutative square
$$\begin{tikzcd}
\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[m]}(\mathcal{E}) \arrow[d] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([m]\cap\mathsf{D})}(\mathcal{E}) \arrow[l, hook', "i_L"'] \arrow[d] \arrow[dl, phantom, "\Leftarrow"] \\
\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[n]}(\mathcal{E}) & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([n]\cap\mathsf{D})}(\mathcal{E}) \arrow[l, hook', "i_L"]
\end{tikzcd}$$
determined by the commutative square (6.1.15) is in fact commutative as a result of our characterization of both functors i_L.

Hence, we find that the morphism (6.1.12) in $\mathsf{Fun}((\mathbf{\Delta}_{/\mathsf{P}})^{\mathsf{op}}, \mathsf{Cat})$ admits a left adjoint

$$\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\bullet\cap\mathsf{D})}(\mathcal{E}) \longrightarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}(\mathcal{E}) \tag{6.1.16}$$

whose components are fully faithful. Therefore, upon taking limits over $(\mathbf{\Delta}_{/\mathsf{P}})^{\mathsf{op}}$, we obtain a fully faithful left adjoint

$$\begin{tikzcd} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathcal{E}) \arrow[r, hook, dashed, shift left, "i_L"] \arrow[r, phantom, "\perp"] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[l, shift left, "y"] \end{tikzcd} .$$

In order to characterize its image, we note that by construction, for any $([n] \downarrow \mathsf{P})^{\circ} \in (\mathbf{\Delta}_{/\mathsf{P}})^{\mathsf{op}}$ we have a commutative square

$$\begin{tikzcd} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathcal{E}) \arrow[r, hook, "i_L"] \arrow[d, "y"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[d, "y"] \\ \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}([n]\cap\mathsf{D})}(\mathcal{E}) \arrow[r, hook, "i_L"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}[n]}(\mathcal{E}) \end{tikzcd} . \tag{6.1.17}$$

Taking $n = 0$, the commutative square (6.1.17) immediately implies that for any $X \in \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathcal{E})$ and any $q \in \mathsf{P}\backslash\mathsf{D}$ we have $\Phi_q(i_L(X)) \simeq 0$. On the other hand, given an object $X \in \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})$ such that $\Phi_q(X) \simeq 0$ whenever $q \in \mathsf{P}\backslash\mathsf{D}$, again using the commutative square (6.1.17) with $n = 0$, by part (2) we see that the counit morphism $i_L y X \to X$ is an equivalence. So, we have proved part (4)(a).

Part (4)(b) follows from an essentially identical argument to part (4)(a).

We now conclude with part (5). We first observe that the closed subcategories

$$\left\{ \mathcal{Z}_p := \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E}) \in \mathbf{Cls}_{\mathcal{X}} \right\}_{p \in \mathsf{P}}$$

evidently assemble into a functor $\mathsf{P} \xrightarrow{(6.1.2)} \mathbf{Cls}_{\mathcal{X}}$, which is a prestratification by part (2) and satisfies the stratification condition as a result of part (4)(a) (applied to both P and $({}^{\leq}p)$). Moreover, assertion (5)(a) follows from part (2) (applied to D instead of P), and assertion (5)(b) follows from part (4)(b) (applied to D instead of P). To prove part (5)(c), writing $\mathsf{C} \in \mathsf{Conv}_{\mathsf{P}}$ for the convex hull of the subset $\{p, q\} \subseteq \mathsf{P}$ (i.e. the full subposet on those elements $r \in \mathsf{P}$ such that there exist morphisms $p \to r \to q$), in light of the commutative diagram

$$\begin{tikzcd} \mathcal{E}_p \arrow[rr, hook, "\rho^p"] \arrow[dr, hook, "\rho^p"'] & & \mathcal{X} \arrow[rr, "\Phi_q"] \arrow[dr, "\Phi_{\mathsf{C}}"] & & \mathcal{E}_q \\ & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.C}}(\mathcal{E}) \arrow[ur, hook, "\rho^{\mathsf{C}}"] \arrow[rr, "\mathrm{id}"'] & & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.C}}(\mathcal{E}) \arrow[ur, "\Phi_q"'] & \end{tikzcd} ,$$

we see that it suffices to assume that $p \in \mathsf{P}$ is initial and $q \in \mathsf{P}$ is terminal. Now, consider the full subposet $\mathsf{sd}_{p,q}(\mathsf{P}) \subseteq \mathsf{sd}(\mathsf{P})$ consisting of those objects $([i] \hookrightarrow \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$ that contain both elements p and q in their image, and consider the morphisms

$$\mathsf{const}_{\mathcal{E}_p} \longleftarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}(\mathcal{E}) \tag{6.1.18}$$

in $\mathsf{Fun}(\mathsf{sd}_{p,q}(\mathsf{P})^{\mathsf{op}}, \mathsf{Cat})$ whose components are given by restriction. Because the inclusion $\mathsf{sd}_{p,q}(\mathsf{P}) \subseteq \mathsf{sd}(\mathsf{P})$ is final (so that its opposite is initial) and moreover $\mathsf{sd}_{p,q}(\mathsf{P})$ has contractible ∞-groupoid completion (as it has an initial object), applying the functor $\lim_{\mathsf{sd}_{p,q}(\mathsf{P})^{\mathsf{op}}}$ to the morphism (6.1.18) yields the morphism

$$\mathcal{E}_p \xleftarrow{\Phi_p} \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})$$

in Cat. On the other hand, by (5)(c) of Lemma 6.1.12, the morphism (6.1.18) admits a right adjoint

$$\mathsf{const}_{\mathcal{E}_p} \longrightarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}(\mathcal{E}) \tag{6.1.19}$$

in $\mathsf{Fun}(\mathsf{sd}_{p,q}(\mathsf{P})^{\mathsf{op}}, \mathsf{Cat})$. The component at the object $([1] \xrightarrow{\{p<q\}} \mathsf{P})^{\circ} \in \mathsf{sd}_{p,q}(\mathsf{P})^{\mathsf{op}}$ of the limiting cone of the morphism (6.1.18) is the commutative square (6.1.4), and so the component at that same object of the limiting cone of the morphism (6.1.19) is the desired commutative square (6.1.3). $\square$

6.2. **The metacosm reconstruction theorem.** In this subsection, we prove the metacosm reconstruction theorem as Theorem 6.2.6.

Definition 6.2.1. Let $\mathcal{X}$ and $\mathcal{X}'$ be P-stratified presentable stable ∞-categories. We define a morphism between them to be a left adjoint functor $\mathcal{X} \to \mathcal{X}'$ satisfying the condition that for every $p \in \mathsf{P}$ there exist (necessarily unique) factorizations

$$\begin{tikzcd} \mathcal{X} \arrow[r] & \mathcal{X}' \\ \mathcal{Z}_p \arrow[u, hook, "i_L"] \arrow[r, dashed] & \mathcal{Z}'_p \arrow[u, hook, "i_L"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "y"'] & \mathcal{X}' \arrow[d, "y"] \\ \mathcal{Z}_p \arrow[r, dashed] & \mathcal{Z}'_p \end{tikzcd} .$$

In this way, we obtain an ∞-category

$$\mathbf{Strat}_{\mathsf{P}}$$

that we refer to as that of P***-stratified presentable stable*** ∞***-categories***.

Observation 6.2.2. The forgetful functor $\mathbf{Strat}_{\mathsf{P}} \to \mathsf{Pr}^L_{\mathsf{st}}$ is conservative.

Notation 6.2.3. For any $(\mathcal{E} \downarrow \mathsf{P}) \in \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})$, we write

$$\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}(\mathcal{E}) \in \mathbf{Strat}_{\mathsf{P}}$$

for the P-stratified presentable stable ∞-category $\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})$ of Proposition 6.1.6.

Observation 6.2.4. Given a morphism

$$\mathcal{E} \longrightarrow \mathcal{E}' \tag{6.2.1}$$

in $\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})$, the induced functor

$$\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \longrightarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}') \tag{6.2.2}$$

lies in $\mathbf{Strat}_{\mathsf{P}}$: in other words, we may upgrade Notation 6.2.3 to a functor

$$\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \xrightarrow{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}} \mathbf{Strat}_{\mathsf{P}} .$$

Indeed, the functor (6.2.2) preserves colimits by parts (2) and (3) of Proposition 6.1.6, it obviously commutes with the restriction functors y, and it commutes with their left adjoints i_L by Proposition 6.1.6(4)(a). We use this fact without further comment.

Observation 6.2.5. For any P-stratified presentable stable ∞-category $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$, its gluing diagram

$$\mathscr{G}(\mathcal{X}) \in \mathsf{LMod}_{\mathsf{l.lax.P}}$$

is in fact a presentable stable left-lax left P-module. We use this fact without further comment.

Theorem 6.2.6. *There is a canonical adjunction*

$$\begin{tikzcd} \mathbf{Strat}_{\mathsf{P}} \arrow[r, shift left, "\mathscr{G}"] \arrow[r, phantom, "\perp"] & \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[l, shift left, hook, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}"] \end{tikzcd} \tag{6.2.3}$$

whose right adjoint is fully faithful, which is an equivalence whenever P *is down-finite.*

Proof. Fix arbitrary objects $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ and $(\mathcal{E} \downarrow \mathsf{P}) \in \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})$. The adjunction (6.2.3) may be extracted from the commutative diagram

$$\begin{tikzcd}[column sep=small] \hom_{\mathsf{Cat}}(\mathcal{X}, \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})) \arrow[r, leftrightarrow, "\sim"] & \hom_{\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}}(\underline{\mathcal{X}}, \mathcal{E}) & & \\ \hom_{\mathsf{Pr}^L_{\mathsf{st}}}(\mathcal{X}, \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow, "\sim"] & \hom_{\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})}(\underline{\mathcal{X}}, \mathcal{E}) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow, "\sim"] & \hom_{\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})}(\mathcal{E}, \underline{\mathcal{X}}) & \\ \hom_{\mathbf{Strat}_{\mathsf{P}}}(\mathcal{X}, \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}(\mathcal{E})) \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, leftrightarrow, "\sim"] & & \hom_{\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})}(\mathcal{E}, \mathscr{G}(\mathcal{X})) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, leftrightarrow, "\sim"] & \hom_{\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})}(\mathscr{G}(\mathcal{X}), \mathcal{E}) \end{tikzcd} \tag{6.2.4}$$

in $\mathcal{S}$ that we explain presently.

- The equivalence in the top row of diagram (6.2.4) follows from the adjunction $\mathsf{const} \dashv \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$.

- The notation $\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ has the evident meaning, analogous to the notation $\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ introduced in Definition 6.1.1.[130]

- By parts (2) and (3) of Proposition 6.1.6, a functor $\mathcal{X} \to \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})$ preserves colimits if and only if for every $p \in \mathsf{P}$ the composite functor $\mathcal{X} \to \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \to \mathcal{E}_p$ preserves colimits. Hence, in diagram (6.2.4) the equivalence in the top row factors as the left equivalence in the middle row.

- The right equivalences in the middle and bottom rows of diagram (6.2.4) follow directly from Lemma A.3.5: over each object $p \in \mathsf{P}$, these equivalences are obtained by passage between adjoints.

- In diagram (6.2.4), we deduce the factorization of the composite equivalence in the middle row as the left equivalence in the bottom row as follows.

  - Given a morphism $\mathcal{X} \to \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}(\mathcal{E})$ in $\mathbf{Strat}_{\mathsf{P}}$, it is immediate that for each $p \in \mathsf{P}$ there exists a factorization

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "\Phi_p"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[r] & \mathcal{E}_p \\ \mathcal{X}_p \arrow[urr, dashed, bend right] & & \end{tikzcd}$$

    that is necessarily a left adjoint. This proves the rightwards factorization.

  - Suppose we are given a morphism $\underline{\mathcal{X}} \leftarrow \mathcal{E}$ in $\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ that admits a factorization

$$\begin{tikzcd} \underline{\mathcal{X}} & \mathcal{E} \arrow[l] \arrow[dl, dashed] \\ \mathscr{G}(\mathcal{X}) \arrow[u, hook, "\mathsf{f.f.}"] & \end{tikzcd} . \tag{6.2.5}$$

    Fix any $p \in \mathsf{P}$, and note that the existence of the factorization (6.2.5) implies (and in fact is equivalent to) the existence for every $q \in \mathsf{P}$ of a factorization

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "\Phi_q"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[d, "\Phi_q"] \\ \mathcal{X}_q \arrow[r, dashed] & \mathcal{E}_q \end{tikzcd} . \tag{6.2.6}$$

    We make the following observations.

[130] It is also explained in Notation 7.0.3.

∗ In the diagram

$$\begin{tikzcd}
\prod_{q \not\leq p} \mathcal{X}_q \arrow[r] & \prod_{q \not\leq p} \mathcal{E}_q \\
\mathcal{X} \arrow[u, "(\Phi_q)_{q\not\leq p}"] \arrow[r] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[u, "(\Phi_q)_{q\not\leq p}"'] \\
\mathcal{Z}_p \arrow[u, hook, "i_L"] \arrow[r, dashed] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E}) \arrow[u, hook, "i_L"']
\end{tikzcd} ,$$

the upper square commutes as a result of the factorizations (6.2.6) and the left vertical composite is zero as a result of the stratification condition. Because the right vertical composite is a fiber sequence by Proposition 6.1.6(4)(a), we obtain the indicated factorization.

∗ The existence of a factorization

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d, "y"'] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[d, "y"] \\
\mathcal{Z}_p \arrow[r, dashed] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E})
\end{tikzcd}$$

is equivalent to the assertion that if an object $X \in \mathcal{X}$ is in the kernel of the functor $\mathcal{X} \xrightarrow{y} \mathcal{Z}_p$ then it is sent to zero under the composite $\mathcal{X} \to \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \xrightarrow{y} \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E})$. This latter assertion follows from the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d, "y"'] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[d, "y"] \\
\mathcal{Z}_p \arrow[d, "(\Phi_q)_{q\leq p}"'] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E}) \arrow[d, "(\Phi_q)_{q\leq p}"] \\
\prod_{q \leq p} \mathcal{X}_q \arrow[r] & \prod_{q \leq p} \mathcal{E}_q
\end{tikzcd} ,$$

which commutes on account of the factorizations (6.2.6) and in which the lower right vertical functor is conservative by Proposition 6.1.6(2) (applied to the poset $({}^{\leq}p)$).

It follows that our chosen morphism $\underline{\mathcal{X}} \leftarrow \mathcal{E}$ in $\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})$ corresponds to a morphism not just in $\mathsf{Pr}^L_{\mathsf{st}}$ but in $\mathbf{Strat}_{\mathsf{P}}$: i.e., it proves the leftwards factorization.

We now prove that the counit of the adjunction (6.2.3) is an equivalence. Unwinding the equivalences of diagram (6.2.4), we see that the counit is given by the following sequence of operations.

- Begin with the counit morphism

$$\underline{\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})} \longrightarrow \mathcal{E} \tag{6.2.7}$$

in $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$ of the adjunction $\mathsf{const} \dashv \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$, which lies in $\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})$: over each $p \in \mathsf{P}$ it restricts as the left adjoint

$$\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \xrightarrow{\Phi_p} \mathcal{E}_p \ .$$

- Use Lemma A.3.5 to pass to the corresponding morphism
$$\underline{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})} \longleftarrow \mathcal{E} \tag{6.2.8}$$
in $\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ to the morphism (6.2.7) in $\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$, which restricts over each $p \in \mathsf{P}$ as the right adjoint
$$\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \overset{\rho^p}{\hookleftarrow} \mathcal{E}_p \ .$$
- Observe the factorization of the morphism (6.2.8) in $\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ as
$$\begin{tikzcd} \underline{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E})} & \mathcal{E} \arrow[l] \arrow[dl, dashed] \\ \mathscr{G}(\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}(\mathcal{E})) \arrow[u, hook] & \end{tikzcd} . \tag{6.2.9}$$
- Use Lemma A.3.5 to pass to the corresponding morphism
$$\mathscr{G}(\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}(\mathcal{E})) \longrightarrow \mathcal{E} \tag{6.2.10}$$
in $\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ to the factorization of diagram (6.2.9) in $\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$.

Evidently, the factorization of diagram (6.2.9) restricts as an equivalence over each $p \in \mathsf{P}$. In fact, it is an equivalence by Proposition 6.1.6(5)(c). Hence the counit (6.2.10) is also an equivalence.

We now study the unit of the adjunction (6.2.3). In order to verify that its component at the object $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ to be an equivalence, by Observation 6.2.2 it suffices to show that the underlying morphism
$$\mathcal{X} \longrightarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) \tag{6.2.11}$$
in $\mathsf{Pr}^L_{\mathsf{st}}$ is an equivalence. Unwinding the equivalences of diagram (6.2.4), we see that the morphism (6.2.11) is the composite
$$\mathcal{X} \longrightarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{X}) \longrightarrow \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X}))$$
in which the first functor is the unit of the adjunction $\mathsf{const} \dashv \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$ and the second morphism is obtained by applying Lemma A.3.5 to the defining morphism $\underline{\mathcal{X}} \hookleftarrow \mathscr{G}(\mathcal{X})$ in $\mathsf{LMod}^{\mathsf{l.lax},R}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ (which restricts over each $p \in \mathsf{P}$ as the right adjoint $\mathcal{X} \overset{\rho^p}{\hookleftarrow} \mathcal{X}_p$) and then applying the functor $\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}$.

We now prove that the morphism (6.2.11) is an equivalence under the assumption that P is finite. We proceed by induction on the number of elements of P, the base case where $\mathsf{P} = \varnothing$ being trivial. So, choose any maximal element $\infty \in \mathsf{P}$, and write $\mathsf{P}' := \mathsf{P}\backslash\{\infty\} \in \mathsf{Down_P}$ for its complement. This defines a functor $\mathsf{P} \xrightarrow{\pi} [1]$ with $\pi^{-1}(0) = \mathsf{P}'$ and $\pi^{-1}(1) = \{\infty\}$. Taking pushforwards of stratifications along π via Proposition 3.4.12 allows us to consider the morphism (6.2.11) as lying in $\mathbf{Strat}_{[1]}$. To show that the morphism (6.2.11) is an equivalence, by Observation 2.3.7 and Lemma 2.2.1 it suffices to show that the lax-commutative square
$$\begin{tikzcd} \mathcal{X}_{\mathsf{P}'} \arrow[r, hook, "i_R"] \arrow[d, "\wr"'] & \mathcal{X} \arrow[r, "p_L"] \arrow[d, phantom, "\swarrow"] & \mathcal{X}_\infty \arrow[d, "\wr"] \\ \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P'}}(\mathscr{G}(\mathcal{X})) \arrow[r, hook, "i_R"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) \arrow[r, "p_L"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\{\infty\}}(\mathscr{G}(\mathcal{X})) \end{tikzcd} \tag{6.2.12}$$
(whose left vertical morphism is an equivalence by induction) commutes. By Lemma A.6.5, every object of $\mathcal{X}_{\mathsf{P}'}$ is a limit indexed over the finite poset $\mathsf{sd}(\mathsf{P}')$ of objects lying in the images of the fully faithful inclusions $\mathcal{X}_p \overset{\rho^p}{\hookrightarrow} \mathcal{X}_{\mathsf{P}'}$ for elements $p \in \mathsf{P}'$; because all functors in the diagram (6.2.12)

are exact, it suffices to show that its natural transformation is an equivalence when restricted along each such fully faithful inclusion. After this restriction, the source is precisely the gluing functor

$$\Gamma^p_\infty : \mathcal{X}_p \xhookrightarrow{\rho^p} \mathcal{X} \xrightarrow{\Phi_\infty} \mathcal{X}_\infty \ ;$$

by Proposition 6.1.6(5)(c) the target is (canonically equivalent to) the gluing functor Γ^p_∞ as well, and unwinding the construction of the morphism (6.2.11) we see that the natural transformation in diagram (6.2.12) is indeed an equivalence. So when P is finite the morphism (6.2.11) is indeed an equivalence.

We now prove that the morphism (6.2.11) is an equivalence under the assumption that P is down-finite. Let us write $\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}} \subseteq \mathsf{Down}_{\mathsf{P}}$ for the full subposet on the finite down-closed subsets of P. Consider the composite

$$\mathsf{const}_{\mathcal{X}} \longrightarrow \mathcal{Z}_\bullet \longrightarrow \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}(\mathscr{G}(\mathcal{X})) \tag{6.2.13}$$

in $\mathsf{Fun}((\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}})^{\mathsf{op}}, \mathsf{Pr}^L_{\mathsf{st}})$, in which

- the functor $\mathcal{Z}_\bullet$ takes a morphism $\mathsf{D}_0^\circ \to \mathsf{D}_1^\circ$ in $(\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}})^{\mathsf{op}}$ corresponding to a morphism $\mathsf{D}_0 \leftarrow \mathsf{D}_1$ in $\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}}$ to the functor $\mathcal{Z}_{\mathsf{D}_0} \xrightarrow{y} \mathcal{Z}_{\mathsf{D}_1}$,
- the functor $\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}(\mathscr{G}(\mathcal{X}))$ takes a morphism $\mathsf{D}_0^\circ \to \mathsf{D}_1^\circ$ in $(\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}})^{\mathsf{op}}$ corresponding to a morphism $\mathsf{D}_0 \leftarrow \mathsf{D}_1$ in $\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}}$ to the restriction functor

  $$\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.D}_0}(\mathscr{G}(\mathcal{X})) \longrightarrow \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.D}_1}(\mathscr{G}(\mathcal{X}))$$

  (recall that this lies in $\mathsf{Pr}^L_{\mathsf{st}}$ by parts (1) and (3) of Proposition 6.1.6),
- the component at $\mathsf{D}^\circ \in (\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}})^{\mathsf{op}}$ of the first morphism is the functor $\mathcal{X} \xrightarrow{y} \mathcal{Z}_{\mathsf{D}}$, and
- the component at $\mathsf{D}^\circ \in (\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}})^{\mathsf{op}}$ of the second morphism is the functor

  $$\mathcal{Z}_{\mathsf{D}} \longrightarrow \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathscr{G}(\mathcal{X}))$$

  obtained as the instance of the functor (6.2.11) in the case of the restricted stratification of Observation 3.4.4.

Applying the functor $\mathsf{lim}_{(\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}})^{\mathsf{op}}}$ to the composite (6.2.13), we obtain the upper composite in the commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[rrd, "(6.2.11)"'] & \mathsf{lim}_{\mathsf{D}^\circ \in (\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}})^{\mathsf{op}}} \mathcal{Z}_{\mathsf{D}} \arrow[r] & \mathsf{lim}_{\mathsf{D}^\circ \in (\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}})^{\mathsf{op}}} \left(\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.D}}(\mathscr{G}(\mathcal{X})) \right) \\ & & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{G}(\mathcal{X})) \arrow[u] \end{tikzcd} \tag{6.2.14}$$

in $\mathsf{Pr}^L_{\mathsf{st}}$. Because P is down-finite, the canonical morphism

$$\mathsf{colim}\left(\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}} \xrightarrow{\mathsf{fgt}} \mathsf{Cat}\right) \xrightarrow{\sim} \mathsf{P}$$

in Cat is an equivalence. This implies that in diagram (6.2.14), the upper left horizontal morphism is an equivalence (by definition of a prestratification) and also the right vertical morphism is an equivalence (note that $\mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}}$ is filtered). Meanwhile, because each $\mathsf{D} \in \mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}}$ is finite, the second morphism in the composite (6.2.13) is an equivalence, which implies that the upper right horizontal morphism in diagram (6.2.14) is an equivalence. So the morphism (6.2.11) is an equivalence. □

6.3. **Strict stratifications.** In this brief subsection, we lay out the general theory of strict stratifications.

Definition 6.3.1.

(1) We say that $\mathscr{F} \in \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ is ***strict*** if it lies in the full subcategory

$$\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{P}}(\mathsf{Pr_{st}}) \subseteq \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \ .$$

(2) We say that $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ is ***strict*** if it is convergent (Definition 2.5.15) and moreover its gluing diagram $\mathscr{G}(\mathcal{X}) \in \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$ is strict.

Observation 6.3.2. Note that $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ is strict if and only if it is convergent and its gluing functors strictly compose, i.e. for every composable sequence $p \to q \to r$ in P the morphism

$$\Gamma^p_r \xrightarrow{\eta_q} \Gamma^q_r \Gamma^p_q$$

in $\mathsf{Fun}(\mathcal{X}_p, \mathcal{X}_r)$ is an equivalence. It follows that $\mathcal{X}$ is strict if and only if every object $X \in \mathcal{X}$ is strict (Definition 2.7.2).

Remark 6.3.3. Choose any $\mathscr{F} \in \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$. Considering it as an object $\mathscr{F} \in \mathsf{LMod}_{\mathsf{l.lax.P}}$, through Lemma A.6.5 we obtain an object $\mathfrak{S}(\mathscr{F}) \in \mathsf{LMod}_{\mathsf{sd(P)}}$ and an equivalence

$$\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathscr{F}) \simeq \lim_{\mathsf{sd(P)}}(\mathfrak{S}(\mathscr{F})) \ .$$

However, in contrast with Observation 2.7.4, the strictness of $\mathscr{F}$ is *not* equivalent to the existence of a factorization

$$\begin{tikzcd} \mathsf{sd(P)} \arrow[r, "\mathfrak{S}(\mathscr{F})"] \arrow[d, "(\mathsf{min}\to\mathsf{max})"'] & \mathsf{Cat} \\ \mathsf{TwAr(P)} \arrow[ur, dashed] \end{tikzcd} \ . \tag{6.3.1}$$

This distinction is already visible when $\mathsf{P} = [2]$, in which case the factorization (6.3.1) exists if and only if the ∞-category $\mathscr{F}_2$ is an ∞-groupoid.

Observation 6.3.4. The commutative triangle

$$\begin{tikzcd} \mathsf{sd(P)} \arrow[rr, "(\mathsf{min}\to\mathsf{max})"] \arrow[dr, "\mathsf{max}"'] & & \mathsf{TwAr(P)} \arrow[dl, "t"] \\ & \mathsf{P} \end{tikzcd} \tag{6.3.2}$$

defines a morphism in $\mathsf{loc.coCart_P}$, and moreover $\mathsf{TwAr(P)} \in \mathsf{coCart_P} \subseteq \mathsf{loc.coCart_P}$. Moreover, by Lemma 2.7.3 (recall Definition A.4.3), the functor

$$\mathsf{sd(P)} \xrightarrow{(\mathsf{min}\to\mathsf{max})} \mathsf{TwAr(P)}$$

is precisely the localization at the comparison morphisms in the locally cocartesian fibration $\mathsf{sd(P)} \xrightarrow{\mathsf{max}} \mathsf{P}$ as well as their locally cocartesian pushforwards. It follows that the morphism (6.3.2) is the initial morphism from $\mathsf{sd(P)} \in \mathsf{loc.coCart_P}$ to an object of the full subcategory $\mathsf{coCart_P} \subseteq \mathsf{loc.coCart_P}$.

Observation 6.3.5. By Observation 6.3.4, for any $\mathcal{E} \in \mathsf{coCart_P}$ we have an equivalence

$$\lim^{\mathsf{r.lax}}_{\mathsf{P}}(\mathcal{E}) := \mathsf{Fun}^{\mathsf{cocart}}_{/\mathsf{P}}(\mathsf{sd(P)}, \mathcal{E}) \underset{\sim}{\xleftarrow{(\mathsf{min}\to\mathsf{max})^*}} \mathsf{Fun}^{\mathsf{cocart}}_{/\mathsf{P}}(\mathsf{TwAr(P)}, \mathcal{E}) =: \Gamma_{\mathsf{P^{op}}}\left(\mathcal{E}^{\overset{\mathsf{cocart}}{\vee}}\right) \ .$$

In particular, if $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ is strict, then taking $\mathcal{E} = \mathscr{G}(\mathcal{X})$ gives a canonical equivalence

$$\mathcal{X} \xrightarrow{\sim} \Gamma_{\mathsf{P^{op}}}\left(\mathscr{G}(\mathcal{X})^{\overset{\mathsf{cocart}}{\vee}}\right) \ .$$

7. Variations on the metacosm reconstruction theorem

In this section, we provide three variations on metacosm reconstruction (Theorem A(1), proved as Theorem 6.2.6). It is organized as follows.

§7.1: We recall some preliminary notions regarding various sorts of subcategories of an idempotent-complete stable ∞-category that is not necessarily presentable.

§7.2: We extend our theory of stratifications to the case of idempotent-complete stable ∞-categories that are not necessarily presentable; for disambiguation, we refer to these as *stable stratifications*. We establish metacosm reconstruction for stable stratifications over finite posets as Theorem 7.2.4.

§7.3: Our definitions of morphisms in the ∞-categories of (resp. stable) stratifications require commutativity for i_L and y. We show as Theorem 7.3.2 that additionally requiring commutativity for i_R corresponds to strict (as opposed to possibly right-lax) morphisms between left-lax left modules over our poset. We refer to such morphisms between stratifications as *strict*.

§7.4: We establish the theory of *reflection* (as discussed in §1.10) for (resp. stable) stratifications over a finite poset: this is a dual form of reconstruction, which is functorial for strict morphisms between stratifications. We begin by establishing reflection for stable stratifications (which are the more natural context for reflection) as Theorem 7.4.11. Using this, we establish reflection for stratifications (i.e. Theorem F) as Corollary 7.4.25.[131] We also give formulas expressing the gluing functors and reflected gluing functors in terms of each other as Proposition 7.4.5.

Local Notation 7.0.1. In this section, we fix a poset P and an idempotent-complete stable ∞-category $\mathcal{C}$.

Remark 7.0.2. It is straightforward to treat the more general case of stable ∞-categories that are not necessarily idempotent-complete. We restrict to idempotent-complete stable ∞-categories merely to ease our language (e.g. so that we can recover $\mathcal{C} \simeq \mathsf{Ind}(\mathcal{C})^\omega \subseteq \mathsf{Ind}(\mathcal{C})$ as the compact objects of its ind-completion).

Notation 7.0.3. We extend the notation $\mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}})$ of Definition 6.1.1 to a systematic notational scheme for the various ∞-categories of lax left P-modules that appear in this section.

- The subscript on LMod indicates the handedness of the lax left P-modules that we consider.
- The parenthetical indicates the restrictions placed both on the fibers and monodromy functors of objects as well as on the fiberwise behavior of morphisms. (Those that arise are $\mathsf{St}^{\mathsf{idem}}$, $\mathsf{Pr}^{L,\omega}_{\mathsf{st}}$, $\mathsf{Pr}^{L}_{\mathsf{st}}$, and $\mathsf{Pr}_{\mathsf{st}}$.)
- A superscript $\mathsf{l.lax}$ or $\mathsf{r.lax}$ on LMod indicates the handedness of the laxness that we allow for the morphisms. (The absence of either of these indicates that we require strictly P-equivariant morphisms.)
- A superscript L on LMod indicates that morphisms are additionally required to be fiberwise left adjoints. (This will only arise in the case that the parenthetical is $\mathsf{Pr}_{\mathsf{st}}$.)

7.1. **Closed, split, and thick subcategories.**

In this subsection, we recall some preliminary notions regarding various sorts of subcategories of an idempotent-complete stable ∞-category that is not necessarily presentable.

[131] More precisely, we prove part (1) of Theorem F; parts (2)-(4) follow trivially therefrom. (We have stated Theorem F in four parts primarily to highlight the parallel with Theorem A.)

Definition 7.1.1. A full stable subcategory $\mathcal{Z} \subseteq \mathcal{C}$ is called

(1) ***thick*** if it is idempotent-complete,

(2) ***split*** if it is thick and its inclusion extends to a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, dashed, shift left] \end{tikzcd} ,$$

and

(3) ***closed*** if it is thick and its inclusion extends to a diagram

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook, bend left] \arrow[r, phantom, "\perp", shift left=2] \arrow[r, phantom, "\perp", shift right=2] & \mathcal{C} \arrow[l, dashed] \arrow[l, dashed, hook, bend left] \end{tikzcd} .$$

These various sorts of full stable subcategories of $\mathcal{C}$ assemble into posets ordered by inclusion, which we respectively denote by

$$\mathbf{thick}_{\mathcal{C}} , \qquad \mathbf{split}_{\mathcal{C}} , \qquad \text{and} \qquad \mathbf{cls}_{\mathcal{C}} .$$

Observation 7.1.2. If $\mathcal{C}$ is presentable, then there is a canonical equivalence

$$\mathbf{Cls}_{\mathcal{C}} \simeq \mathbf{cls}_{\mathcal{C}} .$$[132]

Observation 7.1.3. We record a number of basic facts surrounding Definition 7.1.1, which we thereafter use without further comment.[133]

(1) There are fully faithful inclusions

$$\mathbf{cls}_{\mathcal{C}} \hookrightarrow \mathbf{split}_{\mathcal{C}} \hookrightarrow \mathbf{thick}_{\mathcal{C}} .$$

(2) The poset $\mathbf{thick}_{\mathcal{C}}$ has all colimits.

(3) Ind-completion defines a fully faithful colimit-preserving functor

$$\mathbf{thick}_{\mathcal{C}} \xhookrightarrow{\mathsf{Ind}} \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})} ,$$

whose image consists of those closed subcategories $\mathcal{Z} \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$ that are compactly generated.[134]

(4) The image of the composite functor

$$\mathbf{split}_{\mathcal{C}} \hookrightarrow \mathbf{thick}_{\mathcal{C}} \xhookrightarrow{\mathsf{Ind}} \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$$

consists of those closed subcategories $\mathcal{Z} \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$ such that the functor

$$\mathcal{Z} \xhookrightarrow{i_R} \mathsf{Ind}(\mathcal{C})$$

preserves colimits, or equivalently such that the composite functor

$$\mathsf{Ind}(\mathcal{C}) \xrightarrow{y} \mathcal{Z} \xhookrightarrow{i_R} \mathsf{Ind}(\mathcal{C})$$

preserves colimits.[135]

[132]It follows that the terminology of Definition 7.1.1(3) is unambiguous.

[133]Many of these facts have already been discussed in §1.12.

[134]The inclusion $\mathcal{Z} \xhookrightarrow{i_L} \mathsf{Ind}(\mathcal{C})$ automatically preserves compact objects, as its right adjoint $\mathsf{Ind}(\mathcal{C}) \xrightarrow{y} \mathcal{Z}$ preserves colimits.

[135]This implies that $\mathcal{Z}$ is compactly generated, with compact objects the image of the composite $\mathcal{C} \simeq \mathsf{Ind}(\mathcal{C})^{\omega} \hookrightarrow \mathsf{Ind}(\mathcal{C}) \xrightarrow{y} \mathcal{Z}$.

(5) The image of the composite functor

$$\mathbf{cls}_{\mathcal{C}} \longrightarrow \mathbf{split}_{\mathcal{C}} \longrightarrow \mathbf{thick}_{\mathcal{C}} \xhookrightarrow{\mathsf{Ind}} \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$$

consists of those closed subcategories $\mathcal{Z} \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$ such that the functor

$$\mathcal{Z} \xhookrightarrow{i_R} \mathsf{Ind}(\mathcal{C})$$

preserves colimits and compact objects, or equivalently such that the composite functor

$$\mathsf{Ind}(\mathcal{C}) \xrightarrow{y} \mathcal{Z} \xhookrightarrow{i_R} \mathsf{Ind}(\mathcal{C})$$

preserves colimits and compact objects.

Notation 7.1.4. Given a set $\{\mathcal{Z}_s \in \mathbf{thick}_{\mathcal{C}}\}_{s\in S}$ of thick subcategories of $\mathcal{C}$, we write

$$\langle \mathcal{Z}_s \rangle^{\mathsf{thick}}_{s\in S} \in \mathbf{thick}_{\mathcal{C}}$$

for the thick subcategory that they generate, i.e. the colimit of the functor $S \xrightarrow{\mathcal{Z}_\bullet} \mathbf{thick}_{\mathcal{C}}$.

Notation 7.1.5. Given a thick subcategory $\mathcal{Z} \in \mathbf{thick}_{\mathcal{C}}$, we write

$$\mathcal{C}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z} \in \mathsf{St}^{\mathsf{idem}}$$

for the idempotent-complete stable quotient of $\mathcal{C}$ by $\mathcal{Z}$, i.e. the cofiber of the inclusion in $\mathsf{St}^{\mathsf{idem}}$.

Remark 7.1.6. Concretely, the idempotent-complete stable quotient of $\mathcal{C}$ by a thick subcategory $\mathcal{Z} \in \mathbf{thick}_{\mathcal{C}}$ may be realized as the full subcategory

$$\mathcal{C}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z} \simeq (\mathsf{Ind}(\mathcal{C})/\mathsf{Ind}(\mathcal{Z}))^{\omega} \subseteq \mathsf{Ind}(\mathcal{C})/\mathsf{Ind}(\mathcal{Z})$$

of compact objects of the corresponding presentable quotient.[136] On the other hand, the idempotent-complete stable quotient of $\mathcal{C}$ by a split subcategory $\mathcal{Z} \in \mathbf{split}_{\mathcal{C}}$ may be realized more simply as $\mathsf{ker}(\mathcal{C} \to \mathcal{Z})$.[137]

Observation 7.1.7. If $\mathcal{C}$ is presentable and $\mathcal{Z} \subseteq \mathcal{C}$ is a full presentable stable subcategory, then the idempotent-complete stable quotient and the presentable quotient of $\mathcal{C}$ by $\mathcal{Z}$ coincide: the canonical morphism

$$\mathcal{C}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z} \longrightarrow \mathcal{C}/\mathcal{Z}$$

is an equivalence. Indeed, the presentable quotient satisfies the universal property of the stable quotient: given any stable ∞-category $\mathcal{D}$ and any exact functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ such that $Fi_L \simeq 0$, the morphism

$$F \longrightarrow \nu p_L F$$

is an equivalence (because for each $X \in \mathcal{C}$ the cofiber sequence $i_L y X \to X \to \nu p_L X$ is carried by F to a cofiber sequence). We use this fact without further comment.

Observation 7.1.8. The functor

$$\mathsf{St}^{\mathsf{idem}} \xrightarrow{\mathsf{Ind}} \mathsf{Pr}^L_{\mathsf{st}}$$

preserves colimits. In particular, given a thick subcategory $\mathcal{Z} \in \mathbf{thick}_{\mathcal{C}}$ we have an equivalence

$$\mathsf{Ind}(\mathcal{C}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z}) \simeq \mathsf{Ind}(\mathcal{C})/\mathsf{Ind}(\mathcal{Z}) \ .$$

We use this fact without further comment.

[136] By contrast, the stable quotient $\mathcal{C}/^{\mathsf{St}}\mathcal{Z}$ (i.e. the cofiber of the inclusion in St) may be realized as the image of the composite

$$\mathcal{C} \simeq \mathsf{Ind}(\mathcal{C})^{\omega} \hookrightarrow \mathsf{Ind}(\mathcal{C}) \xrightarrow{p_L} \mathsf{Ind}(\mathcal{C})/\mathsf{Ind}(\mathcal{Z}) \ ;$$

its idempotent-completion recovers $\mathcal{C}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z}$.

[137] In particular, in this case the canonical morphism $\mathcal{C}/^{\mathsf{St}}\mathcal{Z} \to \mathcal{C}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z}$ is an equivalence.

Observation 7.1.9. The inclusion of a closed subcategory $\mathcal{Z} \in \mathbf{cls}_{\mathcal{C}}$ extends to a stable recollement

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hookrightarrow, bend left, "i_L"] \arrow[r, hookrightarrow, bend right, "i_R"'] & \mathcal{C} \arrow[l, "y"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{C}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z} \arrow[l, hookrightarrow, "\nu"'] \end{tikzcd} \quad . \tag{7.1.1}$$

We use this fact without further comment.

7.2. **Stratifications of stable ∞-categories.** In this subsection, we extend our theory of stratifications to the case of idempotent-complete stable ∞-categories that are not necessarily presentable; we refer to these as *stable stratifications*. We establish metacosm reconstruction for stable stratifications over finite posets as Theorem 7.2.4. We state this result as quickly as possible; much of the rest of the subsection is devoted to its proof. Although we define stable stratifications in terms of stratifications of ind-completions, we also characterize them in a way that does not make reference to ind-completions as Proposition 7.2.11.

Definition 7.2.1. A ***stable stratification*** of $\mathcal{C}$ over P is a functor

$$\begin{array}{ccc} \mathsf{P} & \xrightarrow{\mathcal{Z}_\bullet} & \mathbf{cls}_{\mathcal{C}} \\ \cup & & \cup \\ p & \longmapsto & \mathcal{Z}_p \end{array}$$

such that the composite functor

$$\begin{array}{ccccc} \mathsf{P} & \xrightarrow{\mathcal{Z}_\bullet} & \mathbf{cls}_{\mathcal{C}} & \xrightarrow{\mathsf{Ind}} & \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})} \\ \cup & & & & \cup \\ p & & \longmapsto & & \mathsf{Ind}(\mathcal{Z}_p) \end{array}$$

is a stratification. In this situation, we may also say that $\mathcal{C}$ is ***stably*** P***-stratified***.

Definition 7.2.2. We define the ∞-category

$$\mathbf{strat}_{\mathsf{P}}$$

of ***stably*** P***-stratified idempotent-complete stable*** ∞***-categories*** analogously to the ∞-category $\mathbf{Strat}_{\mathsf{P}}$ of Definition 6.2.1: its objects are stably P-stratified idempotent-complete stable ∞-categories, and its morphisms are those exact functors that commute with both the i_L inclusions and the y projections.

Observation 7.2.3. Ind-completion defines a faithful functor

$$\mathbf{strat}_{\mathsf{P}} \overset{\mathsf{Ind}}{\hookrightarrow} \mathbf{Strat}_{\mathsf{P}} \ .$$

Explicitly, an object $\mathcal{X} \in \mathbf{Strat}_{\mathsf{P}}$ is in its image precisely when its underlying presentable stable ∞-category $\mathcal{X} \in \mathsf{Pr}^L_{\mathsf{st}}$ is compactly generated and moreover there exists a factorization

$$\begin{tikzcd} \mathsf{P} \arrow[r] \arrow[dr, dashed] & \mathbf{Cls}_{\mathcal{X}} \\ & \mathbf{cls}_{\mathcal{X}^\omega} \arrow[u, hookrightarrow, "\mathsf{Ind}"'] \end{tikzcd}$$

of its defining functor, and a morphism $\mathcal{X} \to \mathcal{X}'$ in $\mathbf{Strat}_{\mathsf{P}}$ between objects in its image lies in its image precisely when its underlying morphism $\mathcal{X} \to \mathcal{X}'$ in $\mathsf{Pr}^L_{\mathsf{st}}$ preserves compact objects (i.e. lies in the subcategory $\mathsf{Pr}^{L,\omega}_{\mathsf{st}} \subseteq \mathsf{Pr}^L_{\mathsf{st}}$).

Theorem 7.2.4. *Assume that* P *is finite. Then, the metacosm adjunction* (6.2.3) *restricts to an equivalence*

$$\mathbf{strat}_{\mathsf{P}} \underset{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}}{\overset{\mathscr{G}}{\underset{\sim}{\rightleftarrows}}} \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \quad . \tag{7.2.1}$$

Definition 7.2.5. We say that two thick subcategories $\mathcal{Y}, \mathcal{Z} \in \mathbf{thick}_{\mathcal{C}}$ are (resp. ***mutually***) ***aligned*** if the two closed subcategories $\mathsf{Ind}(\mathcal{Y}), \mathsf{Ind}(\mathcal{Z}) \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$ are (resp. mutually) aligned.

Remark 7.2.6. In the case that $\mathcal{C}$ is presentable and $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{C}} \simeq \mathbf{cls}_{\mathcal{C}} \subseteq \mathbf{thick}_{\mathcal{C}}$, it is not hard to see that Definition 7.2.5 coincides with Definition 3.1.2.

Definition 7.2.7. We respectively say that a closed subcategory of $\mathsf{Ind}(\mathcal{C})$ is ***compact-thick***, ***compact-split***, or ***compact-closed*** if it is the ind-completion of a thick, split, or closed subcategory of $\mathcal{C}$.

Lemma 7.2.8. *Let $\mathcal{Y}, \mathcal{Z} \in \mathbf{cls}_{\mathcal{C}}$ be closed subcategories, and suppose that $\mathcal{Z}$ is aligned with $\mathcal{Y}$. Then, the thick subcategory $\langle \mathcal{Y}, \mathcal{Z} \rangle^{\mathsf{thick}} \subseteq \mathcal{C}$ generated by $\mathcal{Y}$ and $\mathcal{Z}$ is a closed subcategory.*

Proof. Note the identification

$$\mathsf{Ind}(\langle \mathcal{Y}, \mathcal{Z} \rangle^{\mathsf{thick}}) = \langle \mathsf{Ind}(\mathcal{Y}), \mathsf{Ind}(\mathcal{Z}) \rangle \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})} .$$

Now, by Lemma 3.3.4(1) (and the fact that $\mathsf{Ind}(\mathcal{Z})$ is aligned with $\mathsf{Ind}(\mathcal{Y})$), we have that the composite

$$\mathsf{Ind}(\mathcal{C}) \xrightarrow{y} \langle \mathsf{Ind}(\mathcal{Y}), \mathsf{Ind}(\mathcal{Z}) \rangle \xhookrightarrow{i_R} \mathsf{Ind}(\mathcal{C})$$

preserves colimits and compact objects, which proves the claim. $\square$

Corollary 7.2.9. *Fix a stable stratification $\mathcal{Z}_\bullet$ of $\mathcal{C}$ over P. For every finite down-closed subset $\mathsf{D} \in \mathsf{Down}^{\mathsf{fin}}_{\mathsf{P}}$, the thick subcategory*

$$\langle \mathcal{Z}_p \rangle^{\mathsf{thick}}_{p \in \mathsf{D}} \subseteq \mathcal{C}$$

generated by the corresponding closed subcategories is itself a closed subcategory.

Proof. This follows by applying Lemmas 7.2.8 and 3.4.5 inductively. $\square$

Proof of Theorem 7.2.4. Under the assumption that P is finite (and hence down-finite), the metacosm adjunction (6.2.3) is an equivalence by Theorem 6.2.6. It therefore suffices to prove that there exist factorizations

$$\begin{tikzcd} \mathbf{Strat}_{\mathsf{P}} \arrow[r, "\mathscr{G}", "\sim"'] & \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \\ \mathbf{strat}_{\mathsf{P}} \arrow[u, hook, "\mathsf{Ind}"] \arrow[r, dashed] & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \arrow[u, hook, "\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{Ind})"'] \end{tikzcd} \tag{7.2.2}$$

and

$$\begin{tikzcd} \mathbf{Strat}_{\mathsf{P}} & \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[l, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}"', "\sim"] \\ \mathbf{strat}_{\mathsf{P}} \arrow[u, hook, "\mathsf{Ind}"] & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \arrow[u, hook, "\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{Ind})"'] \arrow[l, dashed] \end{tikzcd} . \tag{7.2.3}$$

We first prove that factorization (7.2.2) exists.[138] Fix a stable stratification

$$\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{cls}_{\mathcal{C}} ,$$

and consider the composite stratification

$$\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{cls}_{\mathcal{C}} \xrightarrow{\mathsf{Ind}} \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})} .$$

Because P is finite, every $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}}$ is finite. Hence, by Corollary 7.2.9, for every $\mathsf{D} \in \mathsf{Down}_{\mathsf{P}}$ the closed subcategory

$$\mathsf{Ind}(\mathcal{Z}_\bullet)_{\mathsf{D}} := \langle \mathsf{Ind}(\mathcal{Z}_p) \rangle_{p \in \mathsf{D}} = \mathsf{Ind}(\langle \mathcal{Z}_p \rangle^{\mathsf{thick}}_{p \in \mathsf{D}}) \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$$

[138] In fact, for this factorization to exist it suffices that P merely be down-finite.

is compact-closed. It follows that for every $p \in \mathsf{P}$, all functors in the recollement

$$\begin{tikzcd} \mathsf{Ind}(\mathcal{Z}_\bullet)_{<p} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathsf{Ind}(\mathcal{Z}_\bullet)_p \arrow[l, "y"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathsf{Ind}(\mathcal{C})_p \arrow[l, hook', "\nu"'] \end{tikzcd}$$

preserve colimits and compact objects, and hence all functors in the composite adjunction

$$\Phi_p : \mathsf{Ind}(\mathcal{C}) \underset{i_R}{\overset{y}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Ind}(\mathcal{Z}_\bullet)_p \underset{\nu}{\overset{p_L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Ind}(\mathcal{C})_p : \rho^p$$

preserve colimits and compact objects. This implies that factorization (7.2.2) exists on objects, and thereafter it is straightforward to see that it exists on morphisms as well.

We now prove that factorization (7.2.3) exists. To avoid unnecessary notation involving ind-completions, we simply begin with an object

$$(\mathcal{E} \downarrow \mathsf{P}) \in \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{Pr}^{L,\omega}_{\mathsf{st}}) \ ,$$

and prove that its image under the composite

$$\begin{tikzcd} \mathbf{Strat}_{\mathsf{P}} & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[l, "\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}"', "\sim"] \\ & \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}^{L,\omega}_{\mathsf{st}}) \arrow[u, hook] \end{tikzcd}$$

lies in the image of the inclusion

$$\mathbf{strat}_{\mathsf{P}} \xhookrightarrow{\mathsf{Ind}} \mathbf{Strat}_{\mathsf{P}} \ ,$$

as described in Observation 7.2.3. To further simplify our notation, we write

$$\mathcal{X} := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \ ,$$

and for any $p \in \mathsf{P}$ and any $\mathsf{C} \in \mathsf{Conv}_{\mathsf{P}}$ we write

$$\mathcal{Z}_p := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}(\leq p)}(\mathcal{E}) \qquad \text{and} \qquad \mathcal{X}_{\mathsf{C}} := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.C}}(\mathcal{E}) \ ,$$

as justified by Proposition 6.1.6(5); in particular, we have $\mathcal{X}_p = \mathcal{E}_p$.

As a preliminary observation, we note that for every $p, q \in \mathsf{P}$ the composite functor

$$\mathcal{X}_p \xhookrightarrow{\rho^p} \mathcal{X} \xrightarrow{\Phi_q} \mathcal{X}_q \tag{7.2.4}$$

is the monodromy functor $\mathcal{E}_p \xrightarrow{\mathcal{E}_{p \leq q}} \mathcal{E}_q$ if $p \leq q$ (by Proposition 6.1.6(5)(c)) and is zero if $p \not\leq q$; in particular, in either case it preserves colimits and compact objects.

Now, fix any $p \in \mathsf{P}$. The functor

$$\mathcal{X} \xrightarrow{(\Phi_q)_{q \in \mathsf{P}}} \prod_{q \in \mathsf{P}} \mathcal{X}_q$$

is conservative (because P is finite) and preserves colimits. Combining this with the fact that for all $q \in \mathsf{P}$ the composite functor (7.2.4) preserves colimits, it follows that in the adjunction

$$\mathcal{X} \underset{\rho^p}{\overset{\Phi_p}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{X}_p$$

the right adjoint ρ^p preserves colimits, which implies that the left adjoint Φ_p preserves compact objects.

We now claim that the converse also holds: if an object $X \in \mathcal{X}$ satisfies the condition that $\Phi_p(X) \in \mathcal{X}_p$ is compact for all $p \in \mathsf{P}$, then it is compact. To see this, for any filtered diagram $\mathcal{J} \xrightarrow{Y_\bullet} \mathcal{X}$ we compute that

$$\underline{\mathsf{hom}}_{\mathcal{X}}(X, \mathsf{colim}_{\mathcal{J}}(Y_\bullet)) \simeq \mathsf{lim}_{([n] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})} \left(\underline{\mathsf{hom}}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)}(X), \Gamma_\varphi \Phi_{\varphi(0)}(\mathsf{colim}_{\mathcal{J}}(Y_\bullet))) \right) \tag{7.2.5}$$

$$\begin{aligned}
&\simeq \lim_{([n]\xrightarrow{\varphi}\mathsf{P})\in\mathsf{sd}(\mathsf{P})}\left(\underline{\hom}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)}(X),\mathrm{colim}_{\mathcal{J}}(\Gamma_\varphi\Phi_{\varphi(0)}(Y_\bullet)))\right) && (7.2.6)\\
&\simeq \lim_{([n]\xrightarrow{\varphi}\mathsf{P})\in\mathsf{sd}(\mathsf{P})}\left(\mathrm{colim}_{\mathcal{J}}\left(\underline{\hom}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)}(X),\Gamma_\varphi\Phi_{\varphi(0)}(Y_\bullet))\right)\right) && (7.2.7)\\
&\simeq \mathrm{colim}_{\mathcal{J}}\left(\lim_{([n]\xrightarrow{\varphi}\mathsf{P})\in\mathsf{sd}(\mathsf{P})}\left(\underline{\hom}_{\mathcal{X}_{\varphi(n)}}(\Phi_{\varphi(n)}(X),\Gamma_\varphi\Phi_{\varphi(0)}(Y_\bullet))\right)\right) && (7.2.8)\\
&\simeq \mathrm{colim}_{\mathcal{J}}\left(\underline{\hom}_{\mathcal{X}}(X,Y_\bullet)\right)\ , && (7.2.9)
\end{aligned}$$

where

- equivalences (7.2.5) and (7.2.9) use nanocosm reconstruction (recall Remark 2.6.7),
- equivalence (7.2.6) follows from the fact that the composite

$$\Gamma_\varphi\Phi_{\varphi(0)} := \Gamma^{\varphi(n-1)}_{\varphi(n)}\cdots\Gamma^{\varphi(0)}_{\varphi(1)}\Phi_{\varphi(0)} := \Phi_{\varphi(n)}\rho^{\varphi(n-1)}\cdots\Phi_{\varphi(1)}\rho^{\varphi(0)}\Phi_{\varphi(0)}$$

  preserves colimits,
- equivalence (7.2.7) uses the assumption that $\Phi_p(X)\in\mathcal{X}_p$ is compact for all $p\in\mathsf{P}$, and
- equivalence (7.2.8) uses the facts that $\mathcal{J}$ is filtered and $\mathsf{sd}(\mathsf{P})$ is finite (because P is finite).

So in fact, an object $X\in\mathcal{X}$ is compact if and only if the object $\Phi_p(X)\in\mathcal{X}_p$ is compact for all $p\in\mathsf{P}$. It now follows that for every $p\in\mathsf{P}$ the functor ρ^p preserves compact objects, because for every $q\in\mathsf{P}$ the functor (7.2.4) preserves compact objects.

We now verify that $\mathcal{X}$ is compactly generated. We proceed by induction on the cardinality of P, the base case where $\mathsf{P}=\varnothing$ being trivial. So, assume that $\mathsf{P}\neq\varnothing$, choose any minimal element $-\infty\in\mathsf{P}$, and write $\mathsf{P}':=\mathsf{P}\backslash\{-\infty\}\in\mathsf{Conv}_{\mathsf{P}}$ for its complement. This defines a functor $\mathsf{P}\xrightarrow{\pi}[1]$ with $\pi^{-1}(0)=\{-\infty\}$ and $\pi^{-1}(1)=\mathsf{P}'$. Taking pushforwards of stratifications along π via Proposition 3.4.12 yields a recollement

$$\begin{tikzcd}
\mathcal{X}_{-\infty} \arrow[r, bend left, "i_L"] \arrow[r, bend right, "i_R"'] & \mathcal{X} \arrow[l, "y"'] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{X}_{\mathsf{P}'} \arrow[l, "\nu"']
\end{tikzcd}$$

in which $\mathcal{X}_{-\infty}=\mathcal{E}_{-\infty}$ is compactly generated by assumption and $\mathcal{X}_{\mathsf{P}'}=\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}'}(\mathcal{E})$ is compactly generated by induction. Moreover, by Proposition 6.1.6(4) the functors i_L and ν are both given by extension by 0, and so preserve compact objects by our above characterization of compact objects in $\mathcal{X}$ (applied also to $\mathcal{X}_{\mathsf{P}'}$). Since i_L and ν also both preserve colimits, we find that the smallest cocomplete full subcategory of $\mathcal{X}$ containing its compact objects is in fact all of $\mathcal{X}$, i.e. that $\mathcal{X}$ is compactly generated.

We now show that for every $p\in\mathsf{P}$ the closed subcategory $\mathcal{Z}_p\in\mathbf{Cls}_{\mathcal{X}}$ is compact-closed. For this, we use the restricted stratification of $\mathcal{Z}_p$ over $({}^{\leq}p)$ of Observation 3.4.4, which converges since the poset $({}^{\leq}p)$ is finite. For each $q\in({}^{\leq}p)$, we use the notation

$$\begin{tikzcd}
\mathcal{Z}_p \arrow[r, shift left, "\widetilde{\Phi}_q"] \arrow[r, phantom, "\perp"] & \mathcal{X}_q \arrow[l, shift left, hook, "\widetilde{\rho}^q"]
\end{tikzcd}$$

for the corresponding geometric localization adjunction, and we write

$$\widetilde{L}_q := \widetilde{\rho}^q\widetilde{\Phi}_q$$

for the corresponding idempotent endofunctor of $\mathcal{Z}_p$. Consider the commutative diagram

$$\begin{tikzcd}
 & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}(^{\leq}p)}(\mathscr{G}(\mathcal{Z}_p)) \arrow[d, hook] & \\
\mathcal{Z}_p \arrow[ur, "\mathfrak{g}", "\sim"'] \arrow[r, "g'"] \arrow[dr, "\mathsf{id}_{\mathcal{Z}_p}"', "\sim"] & \mathsf{Fun}(\mathsf{sd}(^{\leq}p), \mathcal{Z}_p) \arrow[r, hook, "\mathsf{Fun}(\mathsf{sd}(^{\leq}p), i_R)"] \arrow[d, "\lim_{\mathsf{sd}(^{\leq}p)}"] & \mathsf{Fun}(\mathsf{sd}(^{\leq}p), \mathcal{X}) \arrow[d, "\lim_{\mathsf{sd}(^{\leq}p)}"] \\
 & \mathcal{Z}_p \arrow[r, hook, "i_R"'] & \mathcal{X}
\end{tikzcd} \tag{7.2.10}$$

in Cat in which

- the functor g' is described by the formula

$$\begin{array}{ccc}
\mathcal{Z}_p & \xrightarrow{\quad g' \quad} & \mathsf{Fun}(\mathsf{sd}(^{\leq}p), \mathcal{Z}_p) \\
\cup & & \cup \\
X & \longmapsto & \left(\left([n] \xrightarrow{\varphi} (^{\leq}p) \right) \longmapsto \widetilde{L}_{\varphi}(X) \right)
\end{array}$$

  where we write
$$\widetilde{L}_{\varphi} := \widetilde{L}_{\varphi(n)} \cdots \widetilde{L}_{\varphi(0)}$$
  for brevity (recall Observation 2.5.12 (and Remark 2.6.4)),

- the commutativity of the left two triangles follow from macrocosm reconstruction (Theorem 2.5.14) for the stratification of $\mathcal{Z}_p$ over $(^{\leq}p)$, and

- the square commutes because $\mathsf{sd}(^{\leq}p)$ is finite (since $(^{\leq}p)$ is finite) and i_R is exact.

Observe that the functor
$$\mathsf{Fun}(\mathsf{sd}(^{\leq}p), \mathcal{X}) \xrightarrow{\lim_{\mathsf{sd}(^{\leq}p)}} \mathcal{X}$$
carries pointwise colimits to colimits and carries pointwise compact objects to compact objects (both using the facts that $\mathcal{X}$ is stable and that $\mathsf{sd}(^{\leq}p)$ is finite). So, using the commutativity of diagram (7.2.10), to show that the functor
$$\mathcal{Z}_p \overset{i_R}{\hookrightarrow} \mathcal{X}$$
preserves colimits and compact objects, it suffices to show that the composite
$$\mathcal{Z}_p \xrightarrow{g'} \mathsf{Fun}(\mathsf{sd}(^{\leq}p), \mathcal{Z}_p) \xhookrightarrow{\mathsf{Fun}(\mathsf{sd}(^{\leq}p), i_R)} \mathsf{Fun}(\mathsf{sd}(^{\leq}p), \mathcal{X})$$
carries colimits to pointwise colimits and carries compact objects to pointwise compact objects, or equivalently that for every $([n] \xrightarrow{\varphi} (^{\leq}p)) \in \mathsf{sd}(^{\leq}p)$ the composite
$$\mathcal{Z}_p \xrightarrow{\widetilde{L}_{\varphi}} \mathcal{Z}_p \overset{i_R}{\hookrightarrow} \mathcal{X}$$
preserves colimits and compact objects. For this, it suffices to show that the composite
$$\mathcal{Z}_p \xrightarrow{\widetilde{L}_{\varphi}} \mathcal{Z}_p \overset{i_R}{\hookrightarrow} \mathcal{X} \xrightarrow{(\Phi_q)_{q \in \mathsf{P}}} \prod_{q \in \mathsf{P}} \mathcal{X}_q$$
carries colimits to pointwise colimits and carries compact objects to pointwise compact objects – the former because the functor $(\Phi_q)_{q \in \mathsf{P}}$ is conservative (because P is finite) and preserves colimits,

the latter by our above characterization of compact objects in $\mathcal{X}$. Equivalently, it suffices to show that for every $q \in \mathsf{P}$ the composite

$$\mathcal{Z}_p \xrightarrow{\widetilde{L}_\varphi} \mathcal{Z}_p \overset{i_R}{\hookrightarrow} \mathcal{X} \xrightarrow{\Phi_q} \mathcal{X}_q \tag{7.2.11}$$

preserves colimits and compact objects. To see this, first observe the factorization of the composite (7.2.11) according to the commutative diagram

$$\begin{tikzcd} \mathcal{Z}_p \arrow[r, "\widetilde{L}_\varphi"] \arrow[d, "\widetilde{L}_{\varphi_{|[n-1]}}"'] & \mathcal{Z}_p \arrow[r, hook, "i_R"] & \mathcal{X} \arrow[r, "\Phi_q"] & \mathcal{X}_q \\ \mathcal{Z}_p \arrow[r, "\widetilde{\Phi}_{\varphi(n)}"'] & \mathcal{X}_{\varphi(n)} \arrow[u, hook, "\widetilde{\rho}^{\varphi(n)}"] \arrow[ur, hook, "\rho^{\varphi(n)}"'] & & \end{tikzcd},$$

in which the square commutes by definition and the commutativity of the triangle follows from the commutativity of the triangle

$$\begin{tikzcd} \mathcal{Z}_p \arrow[r, hook, "i_R"] & \mathcal{X} \\ \mathcal{Z}_{\varphi(n)} \arrow[u, hook, "i_R"] \arrow[ur, hook, "i_R"'] & \end{tikzcd}.$$

Now, because the composite functor (7.2.4) (replacing p with $\varphi(n)$) preserves colimits and compact objects, it follows that the functor

$$\mathcal{Z}_p \overset{i_R}{\hookrightarrow} \mathcal{X}$$

preserves colimits and compact objects, i.e. the closed subcategory $\mathcal{Z}_p \in \mathbf{Cls}_{\mathcal{X}}$ is indeed compact-closed.

We have shown that the factorization (7.2.3) exists on objects, and thereafter it is straightforward to see that it exists on morphisms as well. $\square$

Remark 7.2.10. One may interpret our proof of Theorem 7.2.4 as establishing the commutativity of the functor

$$\mathsf{St}^{\mathsf{idem}} \xrightarrow{\mathsf{Ind}} \mathsf{Pr}_{\mathsf{st}}$$

with certain right-lax limits.

Proposition 7.2.11. *Choose any functor*

$$\begin{array}{ccc} \mathsf{P} & \xrightarrow{\mathcal{Z}_\bullet} & \mathbf{cls}_{\mathcal{C}} \\ \Psi & & \Psi \\ p & \longmapsto & \mathcal{Z}_p \end{array},$$

and consider the composite functor

$$\begin{array}{ccccc} \mathsf{P} & \xrightarrow{\mathcal{Z}_\bullet} & \mathbf{cls}_{\mathcal{C}} & \xrightarrow{\mathsf{Ind}} & \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})} \\ \Psi & & & & \Psi \\ p & & \longmapsto & & \mathsf{Ind}(\mathcal{Z}_p) \end{array}.$$

(1) *The composite $\mathsf{Ind}(\mathcal{Z}_\bullet)$ is a prestratification if and only if $\langle \mathcal{Z}_p \rangle^{\mathsf{thick}}_{p \in \mathsf{P}} = \mathcal{C}$.*

(2) *The composite $\mathsf{Ind}(\mathcal{Z}_\bullet)$ satisfies the stratification condition if and only if for every $p, q \in \mathsf{P}$,*

(a) *the thick subcategory*

$$\mathcal{Z}_{(\leq p) \cap (\leq q)} := \langle \mathcal{Z}_r \rangle^{\mathsf{thick}}_{r \in (\leq p) \cap (\leq q)} \subseteq \mathcal{C}$$

is in fact a closed subcategory, and

(b) there exists a factorization

$$\begin{tikzcd} \mathcal{Z}_{(\leq p)\cap(\leq q)} \arrow[r, hook, "i_L"] & \mathcal{Z}_p \\ \mathcal{Z}_q \arrow[u, dashed] \arrow[r, hook, "i_L"'] & \mathcal{C} \arrow[u, "y"'] \end{tikzcd} .$$

Lemma 7.2.12. *Let $\mathcal{Y}, \mathcal{Z} \in \mathbf{cls}_{\mathcal{C}}$ be closed subcategories, and suppose that $\mathcal{Z}$ is aligned with $\mathcal{Y}$. Then, the intersection $(\mathcal{Y} \cap \mathcal{Z}) \subseteq \mathcal{C}$ is a closed subcategory, and moreover we have an identification*

$$\mathsf{Ind}(\mathcal{Y} \cap \mathcal{Z}) = (\mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z})) \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})} .$$

Proof. Consider the pullback square

$$\begin{tikzcd} \mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z}) \arrow[r, hook, "i_{\mathsf{Ind}(\mathcal{Y})}"] \arrow[d, hook, "i_{\mathsf{Ind}(\mathcal{Z})}"'] & \mathsf{Ind}(\mathcal{Y}) \arrow[d, hook, "i_L = \mathsf{Ind}(i_L)"] \\ \mathsf{Ind}(\mathcal{Z}) \arrow[r, hook, "i_L = \mathsf{Ind}(i_L)"'] & \mathsf{Ind}(\mathcal{C}) \end{tikzcd} \tag{7.2.12}$$

in $\mathsf{Pr}^L_{\mathsf{st}}$. By Observation 3.2.4 (and the fact that $\mathsf{Ind}(\mathcal{Z})$ is aligned with $\mathsf{Ind}(\mathcal{Y})$), we have that $(\mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z})) \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$. Thereafter, by Lemma 3.2.5, the commutative square

$$\begin{tikzcd} \mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z}) \arrow[r, hook, "i_R"] \arrow[d, hook, "i_R"'] & \mathsf{Ind}(\mathcal{Y}) \arrow[d, hook, "i_R = \mathsf{Ind}(i_R)"] \\ \mathsf{Ind}(\mathcal{Z}) \arrow[r, hook, "i_R = \mathsf{Ind}(i_R)"'] & \mathsf{Ind}(\mathcal{C}) \end{tikzcd} \tag{7.2.13}$$

in Cat obtained by taking right adjoints twice in the commutative square (7.2.12) is a pullback square. In particular, the identifications $i_R = \mathsf{Ind}(i_R)$ in the pullback square (7.2.13) imply that it lies in $\mathsf{Pr}^L_{\mathsf{st}}$. It follows that $(\mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z})) \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$ is compact-closed. Now, using the fact that $(\mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z})) \xhookrightarrow{i_L} \mathsf{Ind}(\mathcal{C})$ preserves compact objects (because its right adjoint preserves colimits), we obtain the composite identification

$$\mathbf{cls}_{\mathcal{C}} \ni (\mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z}))^{\omega} = ((\mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z})) \cap \mathsf{Ind}(\mathcal{C})^{\omega}) = (\mathsf{Ind}(\mathcal{Y}) \cap \mathsf{Ind}(\mathcal{Z}) \cap \mathcal{C}) = (\mathcal{Y} \cap \mathcal{Z}) ,$$

which proves both assertions. □

Proof of Proposition 7.2.11. Part (1) is clear. So, we proceed to part (2). First of all, it is clear that for any $p, q \in \mathsf{P}$, if the functor $\mathcal{Z}_\bullet$ satisfies conditions (a) and (b) then the composite $\mathsf{Ind}(\mathcal{Z}_\bullet)$ satisfies the stratification condition (using the fact that the functor $\mathbf{thick}_{\mathcal{C}} \xhookrightarrow{\mathsf{Ind}} \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$ commutes with colimits). Conversely, suppose that the composite $\mathsf{Ind}(\mathcal{Z}_\bullet)$ satisfies the stratification condition. Then, by Lemma 3.4.5 we have that the closed subcategories $\mathsf{Ind}(\mathcal{Z}_p), \mathsf{Ind}(\mathcal{Z}_q) \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$ are mutually aligned and moreover

$$\mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})} \ni (\mathsf{Ind}(\mathcal{Z}_p) \cap \mathsf{Ind}(\mathcal{Z}_q)) = \langle \mathsf{Ind}(\mathcal{Z}_r) \rangle_{r \in (\leq p)\cap(\leq q)} = \mathsf{Ind}(\langle \mathcal{Z}_r \rangle^{\mathsf{thick}}_{r \in (\leq p)\cap(\leq q)}) =: \mathsf{Ind}(\mathcal{Z}_{(\leq p)\cap(\leq q)}) . \tag{7.2.14}$$

In particular, the closed subcategories $\mathcal{Z}_p, \mathcal{Z}_q \in \mathbf{cls}_{\mathcal{C}}$ are mutually aligned, which by Lemma 7.2.12 implies that

$$\mathsf{Ind}(\mathcal{Z}_p \cap \mathcal{Z}_q) = (\mathsf{Ind}(\mathcal{Z}_p) \cap \mathsf{Ind}(\mathcal{Z}_q)) \in \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})} . \tag{7.2.15}$$

As the functor $\mathbf{thick}_{\mathcal{C}} \xhookrightarrow{\mathsf{Ind}} \mathbf{Cls}_{\mathsf{Ind}(\mathcal{C})}$ is fully faithful, the identifications (7.2.14) and (7.2.15) along with Lemma 7.2.12 now imply that we have an identification

$$\mathbf{cls}_{\mathcal{C}} \ni (\mathcal{Z}_p \cap \mathcal{Z}_q) = \mathsf{Ind}(\mathcal{Z}_{(\leq p)\cap(\leq q)})^{\omega} ,$$

i.e. the functor $\mathcal{Z}_\bullet$ satisfies condition (a). Now, by Lemma 3.1.7 the square

$$\begin{tikzcd}
\mathsf{Ind}(\mathcal{Z}_p) \cap \mathsf{Ind}(\mathcal{Z}_q) \arrow[r, hook, "i_L"] & \mathsf{Ind}(\mathcal{Z}_p) \\
\mathsf{Ind}(\mathcal{Z}_q) \arrow[u, "y"] \arrow[r, hook, "i_L"'] & \mathsf{Ind}(\mathcal{C}) \arrow[u, "y"']
\end{tikzcd}$$

commutes, and by Lemma 7.2.12 all four of its functors preserve compact objects (because their right adjoints preserve colimits). Hence, again using identification (7.2.14) we see that the functor $\mathcal{Z}_\bullet$ satisfies condition (b). $\square$

7.3. **Strict morphisms among stratifications.** In this brief subsection, we introduce strict morphisms among (resp. stable) stratifications and show as Theorem 7.3.2 that they correspond through metacosm reconstruction to strict (as opposed to possibly right-lax) morphisms between left-lax left P-modules.

Definition 7.3.1. We say that a morphism $\mathcal{X} \to \mathcal{X}'$ in $\mathbf{Strat}_\mathsf{P}$ or in $\mathbf{strat}_\mathsf{P}$ is ***strict*** if for every $p \in \mathsf{P}$ there exists a (necessarily unique) factorization

$$\begin{tikzcd}
\mathcal{X} \arrow[r] & \mathcal{X}' \\
\mathcal{Z}_p \arrow[u, hook, "i_R"] \arrow[r, dashed] & \mathcal{Z}'_p \arrow[u, hook, "i_R"']
\end{tikzcd}$$

(with the evident notation). We denote by

$$\mathbf{Strat}_\mathsf{P}^{\mathsf{strict}} \subseteq \mathbf{Strat}_\mathsf{P} \qquad \text{and} \qquad \mathbf{strat}_\mathsf{P}^{\mathsf{strict}} \subseteq \mathbf{strat}_\mathsf{P}$$

the respective subcategories on the strict morphisms.

Theorem 7.3.2.

(1) Assume that P *is down-finite. Then, the metacosm equivalence* (6.2.3) *of Theorem 6.2.6 restricts to an equivalence*

$$\mathbf{Strat}_\mathsf{P}^{\mathsf{strict}} \underset{\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}}{\overset{\mathscr{G}}{\underset{\sim}{\rightleftarrows}}} \mathsf{LMod}^L_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \quad .$$

(2) Assume that P *is finite. Then, the metacosm equivalence* (7.2.1) *of Theorem 7.2.4 restricts to an equivalence*

$$\mathbf{strat}_\mathsf{P}^{\mathsf{strict}} \underset{\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}}{\overset{\mathscr{G}}{\underset{\sim}{\rightleftarrows}}} \mathsf{LMod}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \quad .$$

Proof. We begin with part (1). On the one hand, it is clear that there exists a factorization

$$\begin{tikzcd}
\mathbf{Strat}_\mathsf{P} \arrow[r, "\mathscr{G}", "\sim"'] & \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \\
\mathbf{Strat}_\mathsf{P}^{\mathsf{strict}} \arrow[u, hook] \arrow[r, dashed] & \mathsf{LMod}^L_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \arrow[u, hook]
\end{tikzcd} \quad .$$

So, it remains to show that there exists a factorization

$$\begin{tikzcd}
\mathbf{Strat}_\mathsf{P} & \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \arrow[l, "\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}"', "\sim"] \\
\mathbf{Strat}_\mathsf{P}^{\mathsf{strict}} \arrow[u, hook] & \mathsf{LMod}^L_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}}) \arrow[u, hook] \arrow[l, dashed]
\end{tikzcd} \quad .$$

Given a morphism $\mathcal{E} \to \mathcal{E}'$ in $\mathsf{LMod}^L_{\mathsf{l.lax.P}}(\mathsf{Pr_{st}})$, for each $p \in \mathsf{P}$ we obtain a commutative square

$$\begin{tikzcd}
\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[r] \arrow[d, "y"'] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}') \arrow[d, "y"] \\
\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}({}^{\leq}p)}(\mathcal{E}) \arrow[r] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}({}^{\leq}p)}(\mathcal{E}')
\end{tikzcd} , \tag{7.3.1}$$

and it suffices to show that the corresponding lax-commutative square

$$\begin{tikzcd}
\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[r] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}') \\
\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}({}^{\leq}p)}(\mathcal{E}) \arrow[r] \arrow[u, hook, "i_R"] \arrow[ur, phantom, "\Nwarrow"] & \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}({}^{\leq}p)}(\mathcal{E}') \arrow[u, hook, "i_R"']
\end{tikzcd} \tag{7.3.2}$$

commutes. For this, we use the restricted stratifications of $\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}({}^{\leq}p)}(\mathcal{E})$ and $\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}({}^{\leq}p)}(\mathcal{E}')$ over $({}^{\leq}p)$ of Observation 3.4.4, which converge since the poset $({}^{\leq}p)$ is finite (because P is down-finite). To simplify our notation, we write

$$\mathcal{X} := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \qquad \text{and} \qquad \mathcal{X}' := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}') \ ,$$

and for any $p \in \mathsf{P}$ we write

$$\mathcal{Z}_p := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \ , \qquad \mathcal{Z}'_p := \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}') \ , \qquad \mathcal{X}_p := \mathcal{E}_p \ , \qquad \text{and} \qquad \mathcal{X}'_p := \mathcal{E}'_p \ ,$$

as justified by Proposition 6.1.6(5). Moreover, for each $q \in ({}^{\leq}p)$, we use the notation

$$\begin{tikzcd}
\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}({}^{\leq}p)}(\mathcal{E}) \arrow[r, shift left, "\widetilde{\Phi}_q"] \arrow[r, phantom, "\perp"] & \mathcal{E}_q \arrow[l, hook, shift left, "\widetilde{\rho}^q"]
\end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}
\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}({}^{\leq}p)}(\mathcal{E}') \arrow[r, shift left, "\widetilde{\Phi}_q"] \arrow[r, phantom, "\perp"] & \mathcal{E}'_q \arrow[l, hook, shift left, "\widetilde{\rho}^q"]
\end{tikzcd}$$

for the corresponding geometric localization adjunctions. Now, for each $q \in ({}^{\leq}p)$, we may extend the commutative square (7.3.1) to a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d, "y"] \arrow[dd, bend right=60, "\Phi_q"'] & \mathcal{X}' \arrow[d, "y"'] \arrow[dd, bend left=60, "\Phi_q"] \\
\mathcal{Z}_p \arrow[r] \arrow[d, "\widetilde{\Phi}_q"] & \mathcal{Z}'_p \arrow[d, "\widetilde{\Phi}_q"'] \\
\mathcal{X}_q \arrow[r] & \mathcal{X}'_q
\end{tikzcd} ,$$

which determines the lower two squares in the lax-commutative diagram

$$\begin{tikzcd}
\prod\limits_{p \in \mathsf{P}} \mathcal{X}_p \arrow[r] & \prod\limits_{p \in \mathsf{P}} \mathcal{X}'_p \\
\mathcal{X} \arrow[r] \arrow[u, "(\Phi_p)_{p \in \mathsf{P}}"] & \mathcal{X}' \arrow[u, "(\Phi_p)_{p \in \mathsf{P}}"'] \\
\mathcal{Z}_p \arrow[r] \arrow[u, hook, "i_R"] \arrow[ur, phantom, "\Nwarrow"] & \mathcal{Z}'_p \arrow[u, hook, "i_R"'] \\
\mathcal{X}_q \arrow[r] \arrow[u, hook, "\widetilde{\rho}^q"] \arrow[uu, hook, bend left=60, "\rho^q"] \arrow[ur, phantom, "\Nwarrow"] & \mathcal{X}'_q \arrow[u, hook, "\widetilde{\rho}^q"'] \arrow[uu, hook, bend right=60, "\rho^q"']
\end{tikzcd} \tag{7.3.3}$$

whose middle square is (7.3.2). Because the upper two vertical functors in diagram (7.3.3) are conservative, its composite natural transformation is an equivalence by Proposition 6.1.6(5)(c) (and the fact that $\Phi_p \rho^q \simeq 0$ whenever $q \not\leq p$). Meanwhile, the same argument (applied to the poset $({}^{\leq}p)$) shows that the lower natural transformation in diagram (7.3.3) is also an equivalence. So, the upper natural transformation in diagram (7.3.3) is an equivalence on every object in the image of $\widetilde{\rho}^q$. Now, microcosm reconstruction for $\mathcal{Z}_p$ (Theorem A(3)) implies that each of its objects is a limit over $\mathsf{sd}({}^{\leq}p)$ of objects in the image of $\widetilde{\rho}^q$ for various $q \in ({}^{\leq}p)$ (using the fact that $\mathsf{sd}({}^{\leq}p)$ is finite (because $({}^{\leq}p)$ is finite because P is down-finite)). So, because $\mathsf{sd}({}^{\leq}p)$ is finite, the upper natural transformation in diagram (7.3.3) is indeed an equivalence; in other words, the lax-commutative square (7.3.2) commutes.

Now, part (2) follows from part (1) and the fact that the commutative squares

$$\begin{tikzcd} \mathbf{strat}^{\mathsf{strict}}_{\mathsf{P}} \arrow[r, hook] \arrow[d, hook] & \mathbf{strat}_{\mathsf{P}} \arrow[d, hook, "\mathsf{Ind}"] \\ \mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}} \arrow[r, hook] & \mathbf{Strat}_{\mathsf{P}} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathsf{LMod}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \arrow[r, hook] \arrow[d, hook] & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \arrow[d, hook, "\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathsf{Ind})"] \\ \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[r, hook] & \mathsf{LMod}^{\mathsf{r.lax},L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \end{tikzcd}$$

are both pullbacks. □

7.4. **Reflection.** In this section, we establish the theory of *reflection* for (resp. stable) stratifications. We establish reflection for stable stratifications over a finite poset as Theorem 7.4.11; using this, we establish reflection for stratifications over a down-finite poset as Corollary 7.4.25.

The key input to the proof of Theorem 7.4.11 is the fact that a stable stratification of $\mathcal{C}$ over a finite poset determines a *reflected* stable stratification of $\mathcal{C}^{\mathsf{op}}$ over the same poset, which we prove as Proposition 7.4.16. Another important input to the proof of Corollary 7.4.25 (in addition to Theorem 7.4.11) is the fact that a stratification of a presentable stable ∞-category may also be considered as a stable stratification thereof, which we prove as Proposition 7.4.21.[139]

We also give a direct formula for the gluing functors in terms of the reflected gluing functors and reversely, as total co/fibers; this is recorded as Proposition 7.4.5.

Local Notation 7.4.1. In this subsection we assume that our poset P is finite, except in Corollaries 7.4.25 and 7.4.26 and Remark 7.4.27 (which apply under strictly weaker hypotheses on P). Moreover, we fix a stable stratification $\mathcal{Z}_\bullet$ of $\mathcal{C}$ over P.

Definition 7.4.2.

(1) For any $p \in \mathsf{P}$, we write

$$\begin{tikzcd} \mathcal{Z}_{<p} \arrow[r, hook, bend left=50, "i_L"] \arrow[r, hook, bend right=50, "i_R"'] & \mathcal{Z}_p \arrow[l, "y" description] \arrow[r, bend left=50, "p_L"] \arrow[r, bend right=50, "p_R"'] & \mathcal{Z}_p/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z}_{<p} =: \mathcal{C}_p \arrow[l, hook', "\nu" description] \end{tikzcd}$$

for the idempotent-complete stable quotient participating in the indicated recollement guaranteed by Corollary 7.2.9, which we refer to as the p^{th} ***stratum*** of the stable stratification.

(2) For any $p \in \mathsf{P}$, we write

$$\Phi_p : \mathcal{C} \underset{i_R}{\overset{y}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{Z}_p \underset{\nu}{\overset{p_L}{\underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{C}_p : \rho^p \qquad \text{and} \qquad \lambda^p : \mathcal{C}_p \underset{p_R}{\overset{\nu}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}}}} \mathcal{Z}_p \underset{y}{\overset{i_L}{\underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}}}} \mathcal{C} : \Psi_p$$

for the indicated composite adjoint functors. We respectively refer to the functors Φ_p and Ψ_p as the corresponding ***geometric localization functor*** and ***reflected geometric localization functor***.

(3) For any morphism $p \to q$ in P, we write

$$\Gamma^p_q : \mathcal{C}_p \xhookrightarrow{\rho^p} \mathcal{C} \xrightarrow{\Phi_q} \mathcal{C}_q \qquad \text{and} \qquad \breve{\Gamma}^p_q : \mathcal{C}_p \xhookrightarrow{\lambda^p} \mathcal{C} \xrightarrow{\Psi_q} \mathcal{C}_q$$

[139]This may be contrasted with the fact that stable stratifications are *definitionally* related to stratifications (although recall Proposition 7.2.11).

for the indicated composite functors, which we respectively refer to as the corresponding ***gluing functor*** and ***reflected gluing functor***.

Definition 7.4.3. Fix a stable ∞-category $\mathcal{D} \in \mathsf{St}$ and an ∞-category $\mathcal{J} \in \mathsf{Cat}$.

(1) Suppose that $\mathcal{J}$ admits a terminal object $t \in \mathcal{J}$, and write $\mathcal{J}_0 := \mathcal{J} \backslash \{t\}$. Then, for any functor $\mathcal{J} \xrightarrow{F} \mathcal{D}$, we define its ***total cofiber*** to be

$$\mathsf{tcofib}(F) := \mathsf{cofib}\left(\mathsf{colim}_{\mathcal{J}_0}(F) \longrightarrow F(t)\right) \in \mathcal{D} \ .$$

(2) Suppose that $\mathcal{J}$ admits an initial object $i \in \mathcal{J}$, and write $\mathcal{J}_0 := \mathcal{J} \backslash \{i\}$. Then, for any functor $\mathcal{J} \xrightarrow{F} \mathcal{D}$, we define its ***total fiber*** to be

$$\mathsf{tfib}(F) := \mathsf{fib}\left(F(i) \longrightarrow \mathsf{lim}_{\mathcal{J}_0}(F)\right) \in \mathcal{D} \ .$$

Remark 7.4.4. Here are two alternative descriptions of the total cofiber functor (using the notation of Definition 7.4.3(1)).

(1) It is the left adjoint

$$\begin{tikzcd} \mathsf{Fun}(\mathcal{J}, \mathcal{D}) \arrow[r, dashed, shift left, "\mathsf{tcofib}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "\delta_t"] \end{tikzcd}$$

to the "Dirac delta at t" functor (i.e. extension by zero over $\mathcal{J}_0$).

(2) It is the composite

$$\mathsf{Fun}(\mathcal{J}, \mathcal{D}) \longrightarrow \mathsf{Fun}(\mathcal{J}_+, \mathcal{D}) \xrightarrow{\mathsf{colim}_{\mathcal{J}_+}} \mathcal{D} \ ,$$

where we write $\mathcal{J}_+ := \mathcal{J} \coprod_{\mathcal{J}_0} \mathcal{J}_0^{\triangleright}$ and the first functor is extension by zero over the cone point of $\mathcal{J}_0^{\triangleright}$.

Of course, the total fiber functor admits dual descriptions.

Proposition 7.4.5. *Fix a nonidentity morphism $p < q$ in P.*

(1) There is a canonical equivalence

$$\breve{\Gamma}^p_q \simeq \mathsf{tfib}\left(\mathsf{sd}(\mathsf{P})|^p_{|q} \xrightarrow{\Sigma^{-1}\Gamma_\bullet} \mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}_p, \mathcal{C}_q)\right)$$

in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}_p, \mathcal{C}_q)$, where the functor $\Gamma_\bullet$ is given by Observation 2.6.5.

(2) There is a canonical equivalence

$$\Gamma^p_q \simeq \mathsf{tcofib}\left(\left(\mathsf{sd}(\mathsf{P})|^p_{|q}\right)^{\mathsf{op}} \xrightarrow{\Sigma\breve{\Gamma}_\bullet} \mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}_p, \mathcal{C}_q)\right)$$

in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}_p, \mathcal{C}_q)$, where the functor $\breve{\Gamma}_\bullet$ is given by Observation 2.6.5 and Proposition 7.4.16.[140]

Lemma 7.4.6. *Given a stable recollement* (7.1.1)*, there is a canonical equivalence*

$$p_R i_L \simeq \Sigma^{-1} p_L i_R \ ;$$

that is, Proposition 7.4.5 holds when $\mathsf{P} = [1]$.

[140]Proposition 7.4.16 does not rely on the present result in any way.

Proof. Consider the commutative diagram

$$\begin{tikzcd} \nu p_R i_L y \arrow[r] \arrow[d] & i_L y \arrow[r] \arrow[d] & 0 \arrow[d] \\ \nu p_R \arrow[r] \arrow[d] & \mathsf{id}_{\mathcal{C}} \arrow[r] \arrow[d] & \nu p_L \arrow[d] \\ 0 \arrow[r] & i_R y \arrow[r] & \nu p_L i_R y \end{tikzcd}$$

in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}, \mathcal{C})$ (in which all morphisms are co/units or zero). Note that the lower-right square is a pullback by (the proof of) Lemma 2.2.1, while the upper-left square is a pushout by an identical argument (or by appealing to Proposition 7.4.16). On the other hand, the lower-left and upper-right squares are clearly both co/fiber sequences. So, the outer square is a pullback, which proves the claim (by precomposing with either i_L or i_R and postcomposing with either p_L or p_R). □

Proof of Proposition 7.4.5. Part (2) follows from part (1) by applying Proposition 7.4.16, so it suffices to verify the latter.

We reduce to the case that $p \in \mathsf{P}$ is initial and $q \in \mathsf{P}$ is terminal, using Observation 7.2.3 in order to apply our previous results regarding stratifications of presentable stable ∞-categories, as follows. First of all, by passing to the restricted stratification of $\mathcal{Z}_q$ over $({}^{\leq}q) \in \mathsf{Down}_{\mathsf{P}}$ (Observation 3.4.4), we may clearly assume that $q \in \mathsf{P}$ is terminal. From here, we claim that passing to the quotient stratification of $\mathcal{C}/\mathcal{Z}_{\not\geq p}$ over $({}^{\geq}p) = \mathsf{P} \backslash ({}^{\not\geq}p)$ (Observation 3.4.11) allows us to assume that $p \in \mathsf{P}$ is initial. On the one hand, the fact that the gluing diagram of $\mathcal{C}$ over P restricts to that of $\mathcal{C}/\mathcal{Z}_{\not\geq p}$ over $\mathsf{P}\backslash({}^{\not\geq}p)$ (as explained in Observation 3.4.11) implies that the diagram

$$\begin{tikzcd} \mathsf{sd}(\mathsf{P})^{|p}_{|q} \arrow[rr, "\Gamma_\bullet"] & & \mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}_p, \mathcal{C}_q) \arrow[d, leftrightarrow, "\wr"] \\ \mathsf{sd}(\mathsf{P}\backslash({}^{\not\geq}p))^{|p}_{|q} \arrow[u, "\wr"] \arrow[r, "\Gamma_\bullet"'] & \mathsf{Fun}^{\mathsf{ex}}((\mathcal{C}/\mathcal{Z}_{\not\geq p})_p, (\mathcal{C}/\mathcal{Z}_{\not\geq p})_q) \end{tikzcd}$$

commutes. On the other hand, the diagram

$$\begin{tikzcd} \mathcal{Z}_p \arrow[r, hook, "i_L"] & \mathcal{C} \arrow[r, "p_R"] & \mathcal{C}_q \\ \mathcal{C}_p \arrow[u, hook, "\nu"] \arrow[r, hook, "i_L"'] & \mathcal{C}/\mathcal{Z}_{\not\geq p} \arrow[u, hook, "\nu"] \arrow[r, "p_R"'] & (\mathcal{C}/\mathcal{Z}_{\not\geq p})_q \arrow[u, "\wr"'] \end{tikzcd}$$

commutes: the left square commutes by Lemma 3.2.3(2)(b) (which applies by Lemma 3.4.5), while the right square commutes by passing to left adjoints (recall Proposition 3.4.10). It follows that we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{C}_p \arrow[r, "\breve{\Gamma}^p_q"] \arrow[d, phantom, "\wr|"] & \mathcal{C}_q \arrow[d, phantom, "\wr|"] \\ (\mathcal{C}/\mathcal{Z}_{\not\geq p})_p \arrow[r, "\breve{\Gamma}^p_q"'] & (\mathcal{C}/\mathcal{Z}_{\not\geq p})_q \end{tikzcd} .$$

So we may indeed assume that $p \in \mathsf{P}$ is initial.

To simplify our notation, we write $\mathsf{P}' := \mathsf{P}\backslash\{q\} = ({}^{<}q) \in \mathsf{Down}_{\mathsf{P}}$. Observe that we obtain a stable recollement

$$\begin{tikzcd} \mathcal{C}_{\mathsf{P}'} \arrow[r, hook, bend left, "i_L"] \arrow[r, hook, bend right, "i_R"'] & \mathcal{C} \arrow[l, "y" description] \arrow[r, bend left, "p_L"] \arrow[r, bend right, "p_R"'] & \mathcal{C}_q \arrow[l, hook', "\nu" description] \end{tikzcd} . \tag{7.4.1}$$

To finish the proof, we work within the context of the commutative diagram

$$
\begin{tikzcd}
\mathcal{C}_p \arrow[d, hook, "i_L"'] \arrow[rr, bend left, "\Sigma\breve{\Gamma}^p_q"] & & & \\
\mathcal{C}_{\mathsf{P}'} \arrow[r, hook, "i_R"] \arrow[d, "L_\bullet"'] \arrow[rr, bend left, "\Sigma p_R i_L"] & \mathcal{C} \arrow[r, "p_L"] & \mathcal{C}_q & \\
\mathsf{Fun}(\mathsf{sd}(\mathsf{P}'), \mathcal{C}_{\mathsf{P}'}) \arrow[r, hook, "i_R"'] & \mathsf{Fun}(\mathsf{sd}(\mathsf{P}'), \mathcal{C}) \arrow[r, "p_L"'] & \mathsf{Fun}(\mathsf{sd}(\mathsf{P}'), \mathcal{C}_q) \arrow[u, "\mathsf{lim}_{\mathsf{sd}(\mathsf{P}')}"] \arrow[r, "\pi_*"'] & \mathsf{Fun}(\mathsf{sd}([1]), \mathcal{C}_q) \arrow[ul, "\mathsf{lim}_{\mathsf{sd}([1])}"']
\end{tikzcd}
\tag{7.4.2}
$$

in St, in which

- the upper left (curved) triangle commutes by definition of $\breve{\Gamma}^p_q$,
- the middle (flattened) triangle commutes by applying Lemma 7.4.6 to the stable recollement (7.4.1),
- the lower left vertical functor $L_\bullet$ is the functor (2.6.1) of Observation 2.6.3,
- the lower left rectangle commutes due to the equivalence $\mathsf{lim}_{\mathsf{sd}(\mathsf{P}')} \circ L_\bullet \simeq \mathsf{id}_{\mathcal{C}_{\mathsf{P}'}}$ that follows from Theorem 2.5.14 (using that $\mathsf{sd}(\mathsf{P}')$ is finite),
- the functor $\mathsf{sd}(\mathsf{P}') \xrightarrow{\pi} \mathsf{sd}([1])$ to the walking cospan is given by the prescriptions

$$\pi^{-1}(0) = \{([0] \xrightarrow{p} \mathsf{P}')\} , \qquad \pi^{-1}(01) = \mathsf{sd}(\mathsf{P}')^{|p} \backslash ([0] \xrightarrow{p} \mathsf{P}') , \qquad \text{and} \qquad \pi^{-1}(1) = \mathsf{sd}(\mathsf{P}') \backslash \mathsf{sd}(\mathsf{P}')^{|p} ,$$

  and
- the lower right triangle commutes because right Kan extensions compose.

We claim that the composite exact functor $\mathcal{C}_p \to \mathsf{Fun}(\mathsf{sd}([1]), \mathcal{C}_q)$ in diagram (7.4.2) selects the evident cospan

$$
\begin{tikzcd}
 & 0 \arrow[d] \\
\Gamma^p_q \arrow[r] & \mathsf{lim}_{\mathsf{sd}(\mathsf{P})^{|p}_{|q} \backslash ([1] \xrightarrow{p<q} \mathsf{P})} (\Gamma_\bullet)
\end{tikzcd}
\tag{7.4.3}
$$

in $\mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}_p, \mathcal{C}_q)$. To see this, observe first that the functor $\mathsf{sd}(\mathsf{P}') \xrightarrow{\pi} \mathsf{sd}([1])$ is a cartesian fibration: over the morphism $0 \to 01$ this follows from the fact that the object $([0] \xrightarrow{p} \mathsf{P}') \in \mathsf{sd}(\mathsf{P}')^{|p}$ is initial, while its cartesian monodromy over the morphism $1 \to 01$ is given by removing the element $p \in \mathsf{P}'$ from each object. Therefore, the right Kan extension π_* is computed by fiberwise limit. From here, it suffices to make the following two observations.

- The composite functor $\mathcal{C}_p \xhookrightarrow{i_L} \mathcal{C}_{\mathsf{P}'} \xrightarrow{L_\bullet} \mathsf{Fun}(\mathsf{sd}(\mathsf{P}'), \mathcal{C}_{\mathsf{P}'})$ takes values in the full subcategory of functors that restrict to zero on $\mathsf{sd}(\mathsf{P}') \backslash \mathsf{sd}(\mathsf{P}')^{|p}$. This implies that the composite functor $\mathcal{C}_p \to \mathsf{Fun}(\mathsf{sd}([1]), \mathcal{C}_q) \xrightarrow{\mathsf{ev}_1} \mathcal{C}_q$ is indeed zero.
- For each object $([n] \xrightarrow{\varphi} \mathsf{P}') \in \mathsf{sd}(\mathsf{P}')^{|p}$, the composite functor

$$\mathcal{C}_p \xhookrightarrow{i_L} \mathcal{C}_{\mathsf{P}'} \xrightarrow{L_\bullet} \mathsf{Fun}(\mathsf{sd}(\mathsf{P}'), \mathcal{C}_{\mathsf{P}'}) \xrightarrow{\mathsf{ev}_\varphi} \mathcal{C}_{\mathsf{P}'}$$

is canonically equivalent to the composite

$$\mathcal{C}_p = \mathcal{C}_{\mathsf{min}(\varphi)} \xrightarrow{\Gamma_\varphi} \mathcal{C}_{\mathsf{max}(\varphi)} \xhookrightarrow{\rho^{\mathsf{max}(\varphi)}} \mathcal{C}_{\mathsf{P}'} \ .$$

This explains the identification of the composite functor $\mathcal{C}_p \to \mathsf{Fun}(\mathsf{sd}([1]), \mathcal{C}_q)$ at the objects $0, 01 \in \mathsf{sd}([1])$.

So, the claim follows from the fact that the limit of the cospan (7.4.3) is by definition the total fiber of the functor $\mathsf{sd}(\mathsf{P})|^p_{|q} \xrightarrow{\Gamma_\bullet} \mathsf{Fun}^{\mathsf{ex}}(\mathcal{C}_p, \mathcal{C}_q)$. □

Remark 7.4.7. We now simultaneously introduce the gluing diagram and reflected gluing diagram of our stable stratification $\mathcal{Z}_\bullet$ of $\mathcal{C}$ over P. The former was already implicitly defined in Theorem 7.2.4, but we nevertheless spell it out here for clarity and in order to highlight the comparison.[141]

Notation 7.4.8.

(1) We define the full subcategory

$$\mathscr{G}(\mathcal{C}) := \{(X, p) \in \mathcal{C} \times \mathsf{P} : X \in \rho^p(\mathcal{C}_p)\} \subseteq \mathcal{C} \times \mathsf{P} \ ,$$

which we consider as an object of $\mathsf{Cat}_{/\mathsf{P}}$.

(2) We define the full subcategory

$$\breve{\mathscr{G}}(\mathcal{C}) := \{(X, p^\circ) \in \mathcal{C} \times \mathsf{P}^{\mathsf{op}} : X \in \lambda^p(\mathcal{C}_p)\} \subseteq \mathcal{C} \times \mathsf{P}^{\mathsf{op}} \ ,$$

which we consider as an object of $\mathsf{Cat}_{/\mathsf{P}^{\mathsf{op}}}$.

Observation 7.4.9.

(1) The functor

$$\mathscr{G}(\mathcal{C}) \longrightarrow \mathsf{P}$$

is a locally cocartesian fibration, whose monodromy functor over each morphism $p \to q$ in P is the functor

$$\mathcal{C}_p \xrightarrow{\Gamma^p_q} \mathcal{C}_q \ .$$

Moreover, its fibers are idempotent-complete and stable and its monodromy functors are exact. We therefore consider it as defining an object

$$\mathscr{G}(\mathcal{C}) \in \mathsf{LMod}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \subseteq \mathsf{LMod}_{\mathsf{l.lax.P}} := \mathsf{loc.coCart}_{\mathsf{P}} \ .$$

(2) The functor

$$\breve{\mathscr{G}}(\mathcal{C}) \longrightarrow \mathsf{P}^{\mathsf{op}}$$

is a locally cartesian fibration, whose monodromy functor over each morphism $p^\circ \leftarrow q^\circ$ in P^{op} is the functor

$$\mathcal{C}_p \xrightarrow{\breve{\Gamma}^p_q} \mathcal{C}_q \ .$$

Moreover, its fibers are idempotent-complete and stable and its monodromy functors are exact. We therefore consider it as defining an object

$$\breve{\mathscr{G}}(\mathcal{C}) \in \mathsf{LMod}_{\mathsf{r.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \subseteq \mathsf{LMod}_{\mathsf{r.lax.P}} := \mathsf{RMod}_{\mathsf{r.lax.P}^{\mathsf{op}}} := \mathsf{loc.Cart}_{\mathsf{P}^{\mathsf{op}}} \ .$$

[141]Inspecting Definition 2.5.7 and Theorem 7.2.4, it is clear that Definition 7.4.10(1) coincides with the gluing diagram as implicitly defined in Theorem 7.2.4.

Definition 7.4.10.

(1) We refer to the object

$$\mathscr{G}(\mathcal{C}) \in \mathsf{LMod}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}})$$

as the ***gluing diagram*** of the stratification.

(2) We refer to the object

$$\check{\mathscr{G}}(\mathcal{C}) \in \mathsf{LMod}_{\mathsf{r.lax.P}}(\mathsf{St}^{\mathsf{idem}})$$

as the ***reflected gluing diagram*** of the stratification.

Theorem 7.4.11. *There is a canonical commutative diagram*

$$\begin{tikzcd}
 & \prod\limits_{p \in \mathsf{P}} \mathsf{St}^{\mathsf{idem}} & \\
\mathsf{LMod}_{\mathsf{r.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \arrow[ur, "(\mathsf{ev}_p)_{p\in\mathsf{P}}"] \arrow[dr, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.P}}"'] \arrow[r, shift left, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.}\bullet}", "\sim"'] & \mathbf{strat}^{\mathsf{strict}}_{\mathsf{P}} \arrow[u, "((-)_p)_{p\in\mathsf{P}}"'] \arrow[d, "\mathsf{fgt}"] \arrow[l, shift left, "\check{\mathscr{G}}"] \arrow[r, shift left, "\mathscr{G}", "\sim"'] & \mathsf{LMod}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \arrow[ul, "(\mathsf{ev}_p)_{p\in\mathsf{P}}"'] \arrow[dl, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}"] \arrow[l, shift left, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.}\bullet}"] \\
 & \mathsf{St}^{\mathsf{idem}} &
\end{tikzcd} .$$

Remark 7.4.12. In what follows, we use the notation $(-)^{\mathsf{refl}}$ to denote opposite ∞-categories that are considered in some nonstandard ("reflected") way. (We explain both usages of this notation as they arise; see Notation 7.4.15 and Definition 7.4.18.) We continue to use the notation $(-)^{\mathsf{op}}$ to denote the opposite ∞-category considered in its own right.

Definition 7.4.13. The ***reflected closed subcategory*** (or simply ***reflection***) of a closed subcategory $\mathcal{Z} \in \mathbf{cls}_{\mathcal{C}}$ of $\mathcal{C}$ is the closed subcategory

$$\mathcal{Z}^{\mathsf{op}} \xhookrightarrow{i_R^{\mathsf{op}}} \mathcal{C}^{\mathsf{op}}$$

of $\mathcal{C}^{\mathsf{op}}$.[142]

Observation 7.4.14. Passage to reflected closed subcategories determines an equivalence

$$\begin{tikzcd}
\mathbf{cls}_{\mathcal{C}} \arrow[r, "(-)^{\mathsf{refl}}", "\sim"'] & \mathbf{cls}_{\mathcal{C}^{\mathsf{op}}} \\[-20pt]
\cup\!\!\!\!\!\shortmid & \cup\!\!\!\!\!\shortmid \\[-20pt]
\left(\mathcal{Z} \xhookrightarrow{i_L} \mathcal{C}\right) \arrow[r, mapsto] & \left(\mathcal{Z}^{\mathsf{op}} \xhookrightarrow{i_R^{\mathsf{op}}} \mathcal{C}^{\mathsf{op}}\right)
\end{tikzcd} ,$$

which when applied twice yields the identity functor

$$\mathsf{id}_{\mathbf{cls}_{\mathcal{C}}} : \mathbf{cls}_{\mathcal{C}} \xrightarrow[\sim]{(-)^{\mathsf{refl}}} \mathbf{cls}_{\mathcal{C}^{\mathsf{op}}} \xrightarrow[\sim]{(-)^{\mathsf{refl}}} \mathbf{cls}_{(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}}} \simeq \mathbf{cls}_{\mathcal{C}} .$$

We use these facts without further comment.

[142] The right adjoint of i_R^{op} is y^{op}, and the right adjoint of y^{op} is i_L^{op}; see Notation 7.4.15.

Notation 7.4.15. As indicated in Observation 7.4.14, given a closed subcategory $\mathcal{Z} \in \mathbf{cls}_{\mathcal{C}}$ we write $\mathcal{Z}^{\mathsf{refl}} \in \mathbf{cls}_{\mathcal{C}^{\mathsf{op}}}$ for its reflection. Moreover, we write

$$\begin{tikzcd}[column sep=large]
\mathcal{Z}^{\mathsf{refl}} \arrow[r, hook, bend left=50, "i_L^{\mathsf{refl}}" name=A] \arrow[r, hook, bend right=50, "i_R^{\mathsf{refl}}"' name=B] & \mathcal{C}^{\mathsf{op}} \arrow[l, "y^{\mathsf{refl}}" description] \arrow[r, bend left=50, "p_L^{\mathsf{refl}}"] \arrow[r, bend right=50, "p_R^{\mathsf{refl}}"'] & \mathcal{C}^{\mathsf{op}}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z}^{\mathsf{refl}} \arrow[l, hook', "\nu^{\mathsf{refl}}" description]
\end{tikzcd}
\quad := \quad
\begin{tikzcd}[column sep=large]
\mathcal{Z}^{\mathsf{op}} \arrow[r, hook, bend left=50, "i_R^{\mathsf{op}}"] \arrow[r, hook, bend right=50, "i_L^{\mathsf{op}}"'] & \mathcal{C}^{\mathsf{op}} \arrow[l, "y^{\mathsf{op}}" description] \arrow[r, bend left=50, "p_R^{\mathsf{op}}"] \arrow[r, bend right=50, "p_L^{\mathsf{op}}"'] & (\mathcal{C}/^{\mathsf{St}^{\mathsf{idem}}}\mathcal{Z})^{\mathsf{op}} \arrow[l, hook', "\nu^{\mathsf{op}}" description]
\end{tikzcd}$$

for the functors in the stable recollement that is opposite to the stable recollement (7.1.1).[143]

Proposition 7.4.16. *The composite*

$$\mathcal{Z}_\bullet^{\mathsf{refl}} : \mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{cls}_{\mathcal{C}} \xrightarrow[\sim]{(-)^{\mathsf{refl}}} \mathbf{cls}_{\mathcal{C}^{\mathsf{op}}}$$

is a stable stratification of $\mathcal{C}^{\mathsf{op}}$ *over* P*.*

Observation 7.4.17. Passage to opposites defines an equivalence

$$\begin{array}{ccc}
\mathbf{thick}_{\mathcal{C}} & \xleftarrow[\sim]{(-)^{\mathsf{op}}} \!\!\!\! \rightarrow & \mathbf{thick}_{\mathcal{C}^{\mathsf{op}}} \\
\Psi & & \Psi \\
(\mathcal{Z} \subseteq \mathcal{C}) & \longmapsto & (\mathcal{Z}^{\mathsf{op}} \subseteq \mathcal{C}^{\mathsf{op}})
\end{array} .$$

In particular, it preserves colimits, so that given a set $\{\mathcal{Y}_s \in \mathbf{thick}_{\mathcal{C}}\}_{s \in S}$ of thick subcategories of $\mathcal{C}$, we have an identification

$$\langle \mathcal{Y}_s^{\mathsf{op}} \rangle_{s \in S}^{\mathsf{thick}} = \left(\langle \mathcal{Y}_s \rangle_{s \in S}^{\mathsf{thick}} \right)^{\mathsf{op}} \in \mathbf{thick}_{\mathcal{C}^{\mathsf{op}}} .$$

We use this fact without further comment.

Proof of Proposition 7.4.16. We apply the criteria of Proposition 7.2.11.

We begin with condition (1) of Proposition 7.2.11. Observe first that

$$\langle i_R(\mathcal{Z}_p) \rangle_{p \in \mathsf{P}}^{\mathsf{thick}} \supseteq \langle \rho^p(\mathcal{C}_p) \rangle_{p \in \mathsf{P}}^{\mathsf{thick}} = \mathcal{C} ,$$

where the equality is guaranteed by Theorem 7.2.4 (and the fact that P is finite). Since $\mathcal{C} \in \mathbf{thick}_{\mathcal{C}}$ is terminal, this implies the equality

$$\langle i_R(\mathcal{Z}_p) \rangle_{p \in \mathsf{P}}^{\mathsf{thick}} = \mathcal{C} \in \mathbf{thick}_{\mathcal{C}} , \tag{7.4.4}$$

which implies the equality

$$\left\langle \mathcal{Z}_p^{\mathsf{refl}} \right\rangle_{p \in \mathsf{P}}^{\mathsf{thick}} := \left\langle i_L^{\mathsf{refl}}(\mathcal{Z}_p^{\mathsf{refl}}) \right\rangle_{p \in \mathsf{P}}^{\mathsf{thick}} := \left\langle i_R^{\mathsf{op}}(\mathcal{Z}_p^{\mathsf{op}}) \right\rangle_{p \in \mathsf{P}}^{\mathsf{thick}} = \langle i_R(\mathcal{Z}_p)^{\mathsf{op}} \rangle_{p \in \mathsf{P}}^{\mathsf{thick}} = \left(\langle i_R(\mathcal{Z}_p) \rangle_{p \in \mathsf{P}}^{\mathsf{thick}} \right)^{\mathsf{op}} = \mathcal{C}^{\mathsf{op}} \in \mathbf{thick}_{\mathcal{C}^{\mathsf{op}}} .$$

Before turning to condition (2) of Proposition 7.2.11, we make some preliminary deductions.

Fix any $p, q \in \mathsf{P}$. First of all, applying Proposition 7.2.11(2)(a) to the stable stratification $\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{cls}_{\mathcal{C}}$, we find that the thick subcategory

$$\mathcal{Z}_{({}^{\leq}p) \cap ({}^{\leq}q)} := \langle \mathcal{Z}_r \rangle_{r \in ({}^{\leq}p) \cap ({}^{\leq}q)}^{\mathsf{thick}} \in \mathbf{thick}_{\mathcal{C}}$$

is a closed subcategory. Thereafter, by Observation 3.4.4 it is clear that the evident factorization

$$\begin{tikzcd}
\mathsf{P} \arrow[r, "\mathcal{Z}_\bullet"] & \mathbf{cls}_{\mathcal{C}} \\
({}^{\leq}p) \cap ({}^{\leq}q) \arrow[u, hook] \arrow[r, dashed, "\mathcal{Z}_\bullet"'] & \mathbf{cls}_{\mathcal{Z}_{({}^{\leq}p) \cap ({}^{\leq}q)}} \arrow[u, hook]
\end{tikzcd} \tag{7.4.5}$$

[143] Here, we implicitly use the evident fact that the involution $\mathsf{St}^{\mathsf{idem}} \xrightarrow[\sim]{(-)^{\mathsf{op}}} \mathsf{St}^{\mathsf{idem}}$ preserves stable recollements.

defines a stable stratification of $\mathcal{Z}_{(\le p)\cap(\le q)}$ over $({}^{\le}p)\cap({}^{\le}q)$. For any $r \in ({}^{\le}p)\cap({}^{\le}q)$, let us denote by

$$\mathcal{Z}_r \overset{\widetilde{i_R}}{\hookrightarrow} \mathcal{Z}_{(\le p)\cap(\le r)}$$

the corresponding i_R inclusion, so that we have a commutative triangle

$$\begin{tikzcd} \mathcal{Z}_r \arrow[r, hook, "\widetilde{i_R}"] \arrow[dr, hook, "i_R"'] & \mathcal{Z}_{(\le p)\cap(\le q)} \arrow[d, hook, "i_R"] \\ & \mathcal{C} \end{tikzcd} . \tag{7.4.6}$$

Then, the equality (7.4.4) applied to the factorization (7.4.5) becomes an equality

$$\left\langle \widetilde{i_R}(\mathcal{Z}_r) \right\rangle^{\mathsf{thick}}_{r \in ({}^{\le}p)\cap({}^{\le}q)} = \mathcal{Z}_{(\le p)\cap(\le q)} \in \mathbf{thick}_{\mathcal{Z}_{(\le p)\cap(\le q)}} ,$$

which by the commutativity of the triangle (7.4.6) yields an equality

$$\langle i_R(\mathcal{Z}_r)\rangle^{\mathsf{thick}}_{r \in ({}^{\le}p)\cap({}^{\le}q)} = i_R(\mathcal{Z}_{(\le p)\cap(\le q)}) \in \mathbf{thick}_{\mathcal{C}} .$$

This implies the composite equality

$$\begin{aligned} \langle \mathcal{Z}^{\mathsf{refl}}_r \rangle^{\mathsf{thick}}_{r \in ({}^{\le}p)\cap({}^{\le}q)} &:= \langle i^{\mathsf{refl}}_L(\mathcal{Z}^{\mathsf{refl}}_r)\rangle^{\mathsf{thick}}_{r \in ({}^{\le}p)\cap({}^{\le}q)} := \langle i^{\mathsf{op}}_R(\mathcal{Z}^{\mathsf{op}}_r)\rangle^{\mathsf{thick}}_{r \in ({}^{\le}p)\cap({}^{\le}q)} \\ &= \langle i_R(\mathcal{Z}_r)^{\mathsf{op}}\rangle^{\mathsf{thick}}_{r \in ({}^{\le}p)\cap({}^{\le}q)} = \left(\langle i_R(\mathcal{Z}_r)\rangle^{\mathsf{thick}}_{r \in ({}^{\le}p)\cap({}^{\le}q)} \right)^{\mathsf{op}} \\ &= i_R(\mathcal{Z}_{(\le p)\cap(\le q)})^{\mathsf{op}} \in \mathbf{thick}_{\mathcal{C}^{\mathsf{op}}} , \end{aligned}$$

which we record for readability as the equality

$$\langle \mathcal{Z}^{\mathsf{refl}}_r \rangle^{\mathsf{thick}}_{r \in ({}^{\le}p)\cap({}^{\le}q)} = i_R(\mathcal{Z}_{(\le p)\cap(\le q)})^{\mathsf{op}} \in \mathbf{thick}_{\mathcal{C}^{\mathsf{op}}} . \tag{7.4.7}$$

We now turn to condition (2) of Proposition 7.2.11. Applying Proposition 7.2.11(2)(a) to the stable stratification $\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{cls}_{\mathcal{C}}$, we immediately find that equality (7.4.7) implies part (a) of condition (2) of Proposition 7.2.11. Then, applying Proposition 7.2.11(2)(b) to the stable stratification $\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet} \mathbf{cls}_{\mathcal{C}}$ with the roles of p and q reversed and invoking Lemma 3.1.7, we obtain a commutative square

$$\begin{tikzcd} \mathcal{Z}_{(\le p)\cap(\le q)} \arrow[d, hook, "i_L"'] & \mathcal{Z}_p \arrow[l, "y"'] \arrow[d, hook, "i_L"] \\ \mathcal{Z}_q & \mathcal{C} \arrow[l, "y"] \end{tikzcd} ,$$

which upon passing to right adjoints yields a commutative square

$$\begin{tikzcd} \mathcal{Z}_{(\le p)\cap(\le q)} \arrow[r, hook, "i_R"] & \mathcal{Z}_p \\ \mathcal{Z}_q \arrow[u, "y"] \arrow[r, hook, "i_R"'] & \mathcal{C} \arrow[u, "y"'] \end{tikzcd} ,$$

which upon passing to opposites and applying equality (7.4.7) yields a commutative square

$$\begin{tikzcd} \langle \mathcal{Z}^{\mathsf{refl}}_r \rangle_{r \in ({}^{\le}p)\cap({}^{\le}q)} \arrow[r, hook, "i^{\mathsf{refl}}_L"] & \mathcal{Z}^{\mathsf{refl}}_p \\ \mathcal{Z}^{\mathsf{refl}}_q \arrow[u, "y^{\mathsf{refl}}"] \arrow[r, hook, "i^{\mathsf{refl}}_L"'] & \mathcal{C}^{\mathsf{op}} \arrow[u, "y^{\mathsf{refl}}"'] \end{tikzcd} ,$$

which verifies part (b) of condition (2) of Proposition 7.2.11. $\square$

Definition 7.4.18. We refer to the stable stratification $\mathcal{Z}_\bullet^{\mathsf{refl}}$ of $\mathcal{C}^{\mathsf{op}}$ over P of Proposition 7.4.16 as the ***reflected stable stratification*** (or simply the ***reflection***) of the stable stratification $\mathcal{Z}_\bullet$ of $\mathcal{C}$ over P, and we denote it by

$$\mathcal{C}^{\mathsf{refl}} := \left(\mathsf{P} \xrightarrow{\mathcal{Z}_\bullet^{\mathsf{refl}}} \mathbf{cls}_{\mathcal{C}^{\mathsf{op}}} \right) \in \mathbf{strat}_{\mathsf{P}}^{\mathsf{strict}} .$$

Observation 7.4.19. Passage to reflected stable stratifications determines an involution

$$\begin{tikzcd} \mathbf{strat}_{\mathsf{P}}^{\mathsf{strict}} \arrow[r, leftrightarrow, "(-)^{\mathsf{refl}}", "\sim"'] & \mathbf{strat}_{\mathsf{P}}^{\mathsf{strict}} \\ \mathcal{C} \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & \mathcal{C}^{\mathsf{refl}} \arrow[u, phantom, "\Cup"] \end{tikzcd} .$$

Observation 7.4.20.

(1) There is a canonical equivalence

$$\begin{tikzcd} \mathsf{LMod}_{\mathsf{l.lax.P}} := \mathsf{loc.coCart}_{\mathsf{P}} \arrow[r, leftrightarrow, "(-)^{\mathsf{op}}", "\sim"'] & \mathsf{loc.Cart}_{\mathsf{P}^{\mathsf{op}}} =: \mathsf{RMod}_{\mathsf{r.lax.P}^{\mathsf{op}}} =: \mathsf{LMod}_{\mathsf{r.lax.P}} \\ (\mathcal{E} \downarrow \mathsf{P}) \arrow[u, phantom, "\Cup"] \arrow[r, mapsto] & (\mathcal{E} \downarrow \mathsf{P})^{\mathsf{op}} := (\mathcal{E}^{\mathsf{op}} \downarrow \mathsf{P}^{\mathsf{op}}) \arrow[u, phantom, "\Cup"] \end{tikzcd} . \tag{7.4.8}$$

(2) The equivalence (7.4.8) of part (1) restricts to an equivalence

$$\begin{tikzcd} \mathsf{LMod}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \arrow[r, leftrightarrow, dashed, "(-)^{\mathsf{op}}", "\sim"'] \arrow[d, hook] & \mathsf{LMod}_{\mathsf{r.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \arrow[d, hook] \\ \mathsf{LMod}_{\mathsf{l.lax.P}} \arrow[r, leftrightarrow, "\sim", "(-)^{\mathsf{op}}"'] & \mathsf{LMod}_{\mathsf{r.lax.P}} \end{tikzcd} .$$

(3) In view of the identification

$$\left(\mathsf{sd}(\mathsf{P}) \xrightarrow{\mathsf{max}} \mathsf{P} \right)^{\mathsf{op}} \simeq \left(\mathsf{sd}(\mathsf{P}^{\mathsf{op}})^{\mathsf{op}} \xrightarrow{\mathsf{min}} \mathsf{P}^{\mathsf{op}} \right) ,$$

the equivalence (7.4.8) of part (1) participates in a commutative square

$$\begin{tikzcd} \mathsf{LMod}_{\mathsf{l.lax.P}} \arrow[r, leftrightarrow, "(-)^{\mathsf{op}}", "\sim"'] \arrow[d, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}"'] & \mathsf{LMod}_{\mathsf{r.lax.P}} \arrow[d, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.P}}"] \\ \mathsf{Cat} \arrow[r, leftrightarrow, "\sim", "(-)^{\mathsf{op}}"'] & \mathsf{Cat} \end{tikzcd} .$$[144]

Proof of Theorem 7.4.11. First of all, it follows immediately from the definitions that the upper two triangles commute. Next, the inverse equivalences on the right are precisely the content of Theorem 7.3.2(2), which also implies the commutativity of the lower right triangle. Thereafter, unwinding its definition, we see that the construction $\breve{\mathscr{G}}$ is precisely the composite functor

$$\breve{\mathscr{G}} : \mathbf{strat}_{\mathsf{P}}^{\mathsf{strict}} \xrightarrow[\sim]{(-)^{\mathsf{refl}}} \mathbf{strat}_{\mathsf{P}}^{\mathsf{strict}} \xrightarrow[\sim]{\mathscr{G}} \mathsf{LMod}_{\mathsf{l.lax.P}}(\mathsf{St}^{\mathsf{idem}}) \xrightarrow[\sim]{(-)^{\mathsf{op}}} \mathsf{LMod}_{\mathsf{r.lax.P}}(\mathsf{St}^{\mathsf{idem}})$$

(as asserted by Theorem 7.4.11), in which the three functors are respectively equivalences by Observation 7.4.19, Theorem 7.3.2(2), and Observation 7.4.20(2). This implies that the functor $\breve{\mathscr{G}}$ is indeed an equivalence. Combining these three results with Observation 7.4.20(3) justifies the notation $\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.}\bullet}$ for its inverse (referring to the evident analog of Notation 6.2.3), and in particular implies the commutativity of the lower left triangle. $\square$

[144] This may be seen as resulting from the fact that the equivalence (7.4.8) between ∞-categories enhances to an equivalence $\mathsf{LMod}_{\mathsf{l.lax.P}} \simeq (\mathsf{LMod}_{\mathsf{r.lax.P}})^{2\mathsf{op}}$ between $(\infty, 2)$-categories.

Proposition 7.4.21. *Fix a presentable stable ∞-category $\mathcal{X}$. For any stratification*

$$\mathsf{P} \longrightarrow \mathbf{Cls}_{\mathcal{X}}$$

of $\mathcal{X}$ over P, its postcomposition

$$\mathsf{P} \longrightarrow \mathbf{Cls}_{\mathcal{X}} \xrightarrow{\sim} \mathbf{cls}_{\mathcal{X}}$$

with the equivalence of Observation 7.1.2 is a stable stratification.

Proof. Choose any closed subcategories $\mathcal{Y}, \mathcal{Z} \in \mathbf{Cls}_{\mathcal{X}}$ such that $\mathcal{Z}$ is aligned with $\mathcal{Y}$. By Lemma 3.3.4(1), the colocalization $i_L y$ into the closed subcategory $\langle \mathcal{Y}, \mathcal{Z} \rangle \in \mathbf{Cls}_{\mathcal{X}}$ is the cofiber of a morphism from an object of $\mathcal{Z} \subseteq \mathcal{X}$ to an object of $\mathcal{Y} \subseteq \mathcal{X}$. This implies that the inclusion

$$\langle \mathcal{Y}, \mathcal{Z} \rangle^{\mathsf{thick}} \subseteq \langle \mathcal{Y}, \mathcal{Z} \rangle$$

in $\mathbf{thick}_{\mathcal{X}}$ is an equality. Hence, the claim follows from Lemma 3.4.5 and Proposition 7.2.11 (and the fact that P is finite). □

Observation 7.4.22. Considering a stratification of a presentable stable ∞-category as a stable stratification via Proposition 7.4.21 does not change its gluing diagram: Definitions 2.5.7 and 7.4.10(1) are compatible.

Notation 7.4.23. We use a hat in order to emphasize that we are referring to a huge ∞-category whose objects are possibly large.

Observation 7.4.24. By Proposition 7.4.21, we have inclusions

$$\begin{tikzcd}
\mathbf{Strat}_{\mathsf{P}} \arrow[r, hook, dashed] & \widehat{\mathbf{strat}}_{\mathsf{P}} \\
\mathbf{Strat}_{\mathsf{P}}^{\mathsf{strict}} \arrow[u, hook] \arrow[r, hook, dashed] & \widehat{\mathbf{strat}}_{\mathsf{P}}^{\mathsf{strict}} \arrow[u, hook]
\end{tikzcd} .$$

Corollary 7.4.25. *Assume that the poset P is down-finite. Then, there is a canonical commutative diagram*

$$\begin{tikzcd}
 & \prod_{p \in \mathsf{P}} \mathsf{Pr}_{\mathsf{st}} & \\
\mathsf{LMod}^{L}_{\mathsf{r.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[ur, "(\mathsf{ev}_p)_{p\in\mathsf{P}}"] \arrow[r, shift left, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.\bullet}}", "\sim"'] \arrow[dr, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.P}}"'] & \mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}} \arrow[u, "((-)_p)_{p\in\mathsf{P}}"] \arrow[l, shift left, "\breve{\mathscr{G}}"] \arrow[r, shift left, "\mathscr{G}", "\sim"'] \arrow[d, "\mathsf{fgt}"] & \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[ul, "(\mathsf{ev}_p)_{p\in\mathsf{P}}"'] \arrow[l, shift left, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}"] \arrow[dl, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}"] \\
 & \mathsf{Pr}_{\mathsf{st}} &
\end{tikzcd} .[145]$$

Proof. We first address the case that P is finite. By interpreting Theorem 7.4.11 in a larger universe and appealing to Observation 7.4.22, it suffices to verify the image factorizations

$$\begin{tikzcd}
\mathsf{LMod}_{\mathsf{r.lax.P}}(\widehat{\mathsf{St}^{\mathsf{idem}}}) & \widehat{\mathbf{strat}}^{\mathsf{strict}}_{\mathsf{P}} \arrow[l, "\breve{\mathscr{G}}"', "\sim"] \arrow[r, "\mathscr{G}", "\sim"'] & \mathsf{LMod}_{\mathsf{l.lax.P}}(\widehat{\mathsf{St}^{\mathsf{idem}}}) \\
\mathsf{LMod}^{L}_{\mathsf{r.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[u, hook] & \mathbf{Strat}^{\mathsf{strict}}_{\mathsf{P}} \arrow[u, hook] \arrow[l, dashed, "\sim"] \arrow[r, dashed, "\sim"'] & \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[u, hook]
\end{tikzcd} \tag{7.4.9}$$

[145]By $\breve{\mathscr{G}}$ here we refer to the evident analog of Definition 7.4.10(2).

of the indicated composites, where the middle vertical inclusion is that of Observation 7.4.24. The lower right equivalence in diagram (7.4.9) follows from Theorem 7.3.2(1). To conclude, we observe the outer commutative rectangle in diagram (7.4.9): the upper composite equivalence is the identity on fibers, and the conditions of accessibility of monodromy functors coincide by Proposition 7.4.5.

We now turn to the case that P is merely down-finite. Writing $\mathsf{Down}_{\mathsf{P}}^{\mathsf{fin}} \subseteq \mathsf{Down}_{\mathsf{P}}$ for the full subposet on the finite down-closed subposets of P, observe that $\mathsf{P} \simeq \mathsf{colim}_{\mathsf{D} \in \mathsf{Down}_{\mathsf{P}}^{\mathsf{fin}}}(\mathsf{D})$. Now, for an arbitrary finite poset Q, we have just argued that we have a commutative diagram

$$\begin{tikzcd}
 & \prod\limits_{q \in \mathsf{Q}} \mathsf{Pr}_{\mathsf{st}} & \\
\mathsf{LMod}^{L}_{\mathsf{r.lax.Q}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[ur, "(\mathsf{ev}_q)_{q\in\mathsf{Q}}"] \arrow[r, shift left, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.\bullet}}"] \arrow[r, phantom, "\sim"] & \mathbf{Strat}^{\mathsf{strict}}_{\mathsf{Q}} \arrow[u, "((-)_q)_{q\in\mathsf{Q}}"'] \arrow[l, shift left, "\breve{\mathscr{G}}"] \arrow[r, shift left, "\mathscr{G}"] \arrow[r, phantom, "\sim"] & \mathsf{LMod}^{L}_{\mathsf{l.lax.Q}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[ul, "(\mathsf{ev}_q)_{q\in\mathsf{Q}}"'] \arrow[l, shift left, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.\bullet}}"]
\end{tikzcd} .$$

Moreover, this diagram is clearly contravariantly functorial as we vary Q over the category of finite posets and inclusions of down-closed subposets. From here, it is not hard to see that we obtain the desired diagram for P by passing to cofiltered limits over $(\mathsf{Down}_{\mathsf{P}}^{\mathsf{fin}})^{\mathsf{op}}$. $\square$

Corollary 7.4.26. *Let* P *be an arbitrary poset whose intervals are finite. Then, there is a canonical commutative diagram*

$$\begin{tikzcd}
 & \prod\limits_{p \in \mathsf{P}} \mathsf{Pr}_{\mathsf{st}} & \\
\mathsf{LMod}^{L}_{\mathsf{r.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[ur, "(\mathsf{ev}_p)_{p\in\mathsf{P}}"] \arrow[rr, leftrightarrow, "\sim", "\widetilde{(-)}"'] & & \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[ul, "(\mathsf{ev}_p)_{p\in\mathsf{P}}"']
\end{tikzcd} .$$

Moreover, on monodromy functors, the equivalence $\widetilde{(-)}$ *acts as described in Proposition 7.4.5.*

Proof. This diagram is clearly contravariantly functorial as we vary P over the category of finite posets and inclusions of finite convex subposets. So, the claim follows from Corollary 7.4.25 by passing to cofiltered limits over the poset $(\mathsf{Conv}_{\mathsf{P}}^{\mathsf{fin}})^{\mathsf{op}}$, the opposite of the poset of finite convex subposets of P. $\square$

Remark 7.4.27. In the situation of Corollary 7.4.26, if the poset P is not down-finite then the equivalence does not necessarily commute with the lax limit functors: rather, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{LMod}^{L}_{\mathsf{r.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[rr, leftrightarrow, "\widetilde{(-)}", "\sim"'] \arrow[dr, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax.P}}"'] & \Rightarrow & \mathsf{LMod}^{L}_{\mathsf{l.lax.P}}(\mathsf{Pr}_{\mathsf{st}}) \arrow[dl, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}"] \\
 & \mathsf{Pr}_{\mathsf{st}} &
\end{tikzcd} ,$$

in which the components of the natural transformation are left adjoints. For example, let us take $\mathsf{P} = \mathbb{Z}$ and fix a presentable stable ∞-category $\mathcal{V} \in \mathsf{Pr}_{\mathsf{st}}$. Then, taking the constant diagram

$$\underline{\mathcal{V}} \in \mathsf{LMod}^{L}_{\mathsf{r.lax.}\mathbb{Z}}(\mathsf{Pr}_{\mathsf{st}})$$

we obtain an adjunction

$$\lim^{\mathsf{l.lax}}_{\mathsf{r.lax}.\mathbb{Z}}(\underline{\mathcal{V}}) \simeq \mathsf{Fun}(\mathbb{Z}, \mathcal{V}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Ch}(\mathcal{V}) \simeq \lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathbb{Z}}(\check{\underline{\mathcal{V}}}) \quad ,$$

in which the right adjoint is fully faithful with image the subcategory of *complete* filtered objects, i.e. those whose limit is zero (compare with Example 1.10.6).

Appendix A. Actions and limits, strict and lax

In this section, we provide definitions of strict, left-lax, and right-lax modules over ∞-categories: in effect, functors of the corresponding sort into the $(\infty, 2)$-category Cat.[146] We also provide definitions of strict, left-lax, and right-lax functors among them (and in particular, limits thereof); perhaps surprisingly, these various notions are actually well-defined in all nine cases. Moreover, we record a number of fundamental results regarding these notions.

Local Notation A.0.1. Throughout this section, we fix a base ∞-category $\mathcal{B}$.

This section is organized as follows.

- §A.1: We introduce all of the notions of $\mathcal{B}$-modules and most of the notions of equivariant functors.
- §A.2: We introduce the more straightforward sorts of limits.
- §A.3: We introduce the remaining sorts of equivariant functors (and in particular the remaining sorts of limits), using the theory of $(\infty, 2)$-categories developed in §B: namely, those in which the handedness of the laxness of the $\mathcal{B}$-modules disagrees with that of the equivariant functors among them.
- §A.4: We study the subdivision $\mathsf{sd}(\mathcal{B}) \in \mathsf{Cat}$.
- §A.5: We give an alternative and more explicit description of the right-lax limit of a left-lax left $\mathcal{B}$-module using $\mathsf{sd}(\mathcal{B})$.
- §A.6: Given a poset P, we provide a useful alternative description of the right-lax limit of a left-lax left P-module as the strict limit of a strict left $\mathsf{sd}(\mathsf{P})$-module.

Remark A.0.2. In §§A.1 and A.2 we give a comprehensive account of the theory, explaining all possible handednesses and how they relate. However, thereafter we specialize in order to streamline our discussion.

Remark A.0.3. We omit essentially all mention of lax *colimits*, as we will have no explicit need for them. On the other hand, they will certainly be present: for example, the left-lax colimit of a functor $\mathcal{B} \to \mathsf{Cat}$ is nothing other than the total ∞-category $\mathcal{E}$ of the cocartesian fibration $\mathcal{E} \downarrow \mathcal{B}$ that it classifies.

Remark A.0.4. The lax $\mathcal{B}$-modules and lax equivariant functors that we study are all strictly unital (in the sense that the corresponding functors to Cat strictly respect identity morphisms).[147] This stands in contrast with the laxly $\mathcal{O}$-monoidal functors between $\mathcal{O}$-monoidal ∞-categories that arise in §4: as described in Remark 4.1.6, we do not require those to be strictly unital (in the sense that we do not require them to strictly respect the unit objects of $\mathcal{O}$-monoidal structures).

A.1. **Strict and lax actions.**

In this subsection, we introduce all of the notions of $\mathcal{B}$-modules and most of the notions of equivariant functors. We begin with an omnibus definition, which the remainder of the subsection is dedicated to discussing.

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[146] The terminology "module" is inspired by ordinary group actions: for instance, a left (resp. right) G-module in an ∞-category $\mathcal{C}$ is the data of a functor $\mathsf{B}G \to \mathcal{C}$ (resp. $\mathsf{B}G^{\mathsf{op}} \to \mathcal{C}$).

[147] Of course, more general definitions exist (see §B.1).

Definition A.1.1. In Figure 5, various ∞-categories of $\mathcal{B}$***-modules*** depicted on the left side are defined as indicated on the right side. The objects in the ∞-categories in the upper left diagram are (various sorts of) ***left*** $\mathcal{B}$-modules, while the objects in the ∞-categories in the lower left diagram are (various sorts of) ***right*** $\mathcal{B}$-modules. In both diagrams on the left side, we refer

- to the objects
  - in the middle rows as (***strict***) $\mathcal{B}$-modules,
  - in the top rows as ***left-lax*** $\mathcal{B}$-modules, and
  - in the bottom rows as ***right-lax*** $\mathcal{B}$-modules,

  and
- to the morphisms
  - in the middle columns as (***strictly***) ***equivariant***,
  - in the left columns as ***left-lax equivariant***, and
  - in the right columns as ***right-lax equivariant***.

So in our notation, laxness of the actions is indicated by a subscript (placed before ".$\mathcal{B}$"), while laxness of the morphisms is indicated by a superscript.

Remark A.1.2. We give definitions in §A.3 that extend the diagrams of Figure 5 to full 3×3 grids.

Example A.1.3. Let us unwind the definitions of the ∞-categories

$$\mathsf{LMod}_{\mathcal{B}} \ , \qquad \mathsf{LMod}_{\mathcal{B}}^{\mathsf{l.lax}} \ , \qquad \mathsf{RMod}_{\mathcal{B}} \ , \qquad \text{and} \qquad \mathsf{RMod}_{\mathcal{B}}^{\mathsf{r.lax}}$$

in the simplest nontrivial case, namely when $\mathcal{B} = [1]$.

(1) Let $\mathcal{E} \downarrow [1]$ and $\mathcal{F} \downarrow [1]$ be cocartesian fibrations, the unstraightenings of functors

$$[1] \xrightarrow{\left\langle \mathcal{E}_0 \xrightarrow{E} \mathcal{E}_1 \right\rangle} \mathsf{Cat}$$

and

$$[1] \xrightarrow{\left\langle \mathcal{F}_0 \xrightarrow{F} \mathcal{F}_1 \right\rangle} \mathsf{Cat} \ ,$$

respectively. Then, let us consider a left-lax equivariant functor

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\alpha"] \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & {[1]} & \end{tikzcd} .$$

Given a cocartesian morphism $e \to E(e)$ in $\mathcal{E}$ with $e \in \mathcal{E}_0$ and $E(e) \in \mathcal{E}_1$, the functor α takes it to some not-necessarily-cocartesian morphism $\alpha(e) \to \alpha(E(e))$ in $\mathcal{F}$ with $\alpha(e) \in \mathcal{F}_0$ and $\alpha(E(e)) \in \mathcal{F}_1$. This admits a unique factorization

$$\begin{tikzcd} \alpha(e) \arrow[r, dashed] \arrow[dr] & F(\alpha(e)) \arrow[d] \\ & \alpha(E(e)) \end{tikzcd}$$

$$
\begin{tikzcd}
\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}} & \mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} \arrow[l, hook, two heads] & \\
\mathsf{LMod}^{\mathsf{l.lax}}_{\mathcal{B}} \arrow[u, hook, "\mathsf{f.f.}"] & \mathsf{LMod}_{\mathcal{B}} \arrow[l, hook, two heads] \arrow[r, hook, two heads] \arrow[u, hook, "\mathsf{f.f.}"'] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathcal{B}} \arrow[d, hook, "\mathsf{f.f.}"] \\
& \mathsf{LMod}_{\mathsf{r.lax}.\mathcal{B}} \arrow[r, hook, two heads] & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}}
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
\mathsf{Cat}_{\mathsf{loc.cocart}/\mathcal{B}} & \mathsf{loc.coCart}_{\mathcal{B}} \arrow[l, hook, two heads] & & \\
\mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}} \arrow[u, hook, "\mathsf{f.f.}"] & \mathsf{coCart}_{\mathcal{B}} \arrow[l, hook, two heads] \arrow[u, hook, "\mathsf{f.f.}"'] \arrow[dr, phantom, "\simeq" sloped] & & \\
& & \mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}} \arrow[r, hook, two heads] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{Cat}_{\mathsf{cart}/\mathcal{B}^{\mathsf{op}}} \arrow[d, hook, "\mathsf{f.f.}"] \\
& & \mathsf{loc.Cart}_{\mathcal{B}^{\mathsf{op}}} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{loc.cart}/\mathcal{B}^{\mathsf{op}}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{RMod}^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}} & \mathsf{RMod}_{\mathsf{l.lax}.\mathcal{B}} \arrow[l, hook, two heads] & \\
\mathsf{RMod}^{\mathsf{l.lax}}_{\mathcal{B}} \arrow[u, hook, "\mathsf{f.f.}"] & \mathsf{RMod}_{\mathcal{B}} \arrow[l, hook, two heads] \arrow[r, hook, two heads] \arrow[u, hook, "\mathsf{f.f.}"'] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{RMod}^{\mathsf{r.lax}}_{\mathcal{B}} \arrow[d, hook, "\mathsf{f.f.}"] \\
& \mathsf{RMod}_{\mathsf{r.lax}.\mathcal{B}} \arrow[r, hook, two heads] & \mathsf{RMod}^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}}
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
\mathsf{Cat}_{\mathsf{loc.cocart}/\mathcal{B}^{\mathsf{op}}} & \mathsf{loc.coCart}_{\mathcal{B}^{\mathsf{op}}} \arrow[l, hook, two heads] & & \\
\mathsf{Cat}_{\mathsf{cocart}/\mathcal{B}^{\mathsf{op}}} \arrow[u, hook, "\mathsf{f.f.}"] & \mathsf{coCart}_{\mathcal{B}^{\mathsf{op}}} \arrow[l, hook, two heads] \arrow[u, hook, "\mathsf{f.f.}"'] \arrow[dr, phantom, "\simeq" sloped] & & \\
& & \mathsf{Cart}_{\mathcal{B}} \arrow[r, hook, two heads] \arrow[d, hook, "\mathsf{f.f.}"'] & \mathsf{Cat}_{\mathsf{cart}/\mathcal{B}} \arrow[d, hook, "\mathsf{f.f.}"] \\
& & \mathsf{loc.Cart}_{\mathcal{B}} \arrow[r, hook, two heads] & \mathsf{Cat}_{\mathsf{loc.cart}/\mathcal{B}}
\end{tikzcd}
$$

Figure 5. The commutative diagrams of monomorphisms among ∞-categories on the left are defined to be those on the right.

as a cocartesian morphism followed by a fiber morphism. This operation is functorial in $e \in \mathcal{E}_0$, which implies that our left-lax equivariant functor amounts to the data of a lax-commutative square

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, "E"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{E}_1 \arrow[d, "\alpha_1"] \\ \mathcal{F}_0 \arrow[r, "F"'] & \mathcal{F}_1 \end{tikzcd} .$$

To say that the left-lax equivariant functor is actually strictly equivariant is equivalently to say that this square actually commutes, i.e. that the natural transformation is a natural equivalence.

(2) Dually, let $\mathcal{E} \downarrow [1]$ and $\mathcal{F} \downarrow [1]$ be cartesian fibrations, the unstraightenings of functors

$$[1]^{\mathsf{op}} \xrightarrow{\left\langle \mathcal{E}_{0^\circ} \xleftarrow{E} \mathcal{E}_{1^\circ} \right\rangle} \mathsf{Cat}$$

and

$$[1]^{\mathsf{op}} \xrightarrow{\left\langle \mathcal{F}_{0^\circ} \xleftarrow{F} \mathcal{F}_{1^\circ} \right\rangle} \mathsf{Cat} ,$$

respectively. Then, a right-lax equivariant functor

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\alpha"] \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & {[1]} & \end{tikzcd}$$

likewise amounts to the data of a lax-commutative square

$$\begin{tikzcd} \mathcal{E}_{0^\circ} \arrow[d, "\alpha_{0^\circ}"'] \arrow[dr, phantom, "\Searrow"] & \mathcal{E}_{1^\circ} \arrow[l, "E"'] \arrow[d, "\alpha_{1^\circ}"] \\ \mathcal{F}_{0^\circ} & \mathcal{F}_{1^\circ} \arrow[l, "F"] \end{tikzcd} .$$

To say that the right-lax equivariant functor is actually strictly equivariant is equivalently to say that this square actually commutes, i.e. that the natural transformation is a natural equivalence.

Example A.1.4. Let us unwind the definitions of the ∞-categories

$$\mathsf{LMod}^{\mathsf{l.lax}}_{\mathcal{B}} , \qquad \mathsf{LMod}^{\mathsf{r.lax}}_{\mathcal{B}} , \qquad \mathsf{RMod}^{\mathsf{r.lax}}_{\mathcal{B}} , \qquad \text{and} \qquad \mathsf{RMod}^{\mathsf{l.lax}}_{\mathcal{B}}$$

in the simple but illustrative case that $\mathcal{B} = \mathsf{B}G$ for a group or monoid G. Choose any two objects

$$\mathcal{E}, \mathcal{F} \in \mathsf{Cat}_{(\mathsf{co})\mathsf{cart}/\mathsf{B}G^{(\mathsf{op})}} ,$$

with the two choices of whether or not to include the parenthesized bits made independently. These are classified by left or right G-actions on the fibers $\mathcal{E}_0$ and $\mathcal{F}_0$ over the basepoint of $\mathsf{B}G^{(\mathsf{op})}$ – right if the choices coincide, left if they do not – and morphisms between them are left-lax equivariant in the case of "cocart" and right-lax equivariant in the case of "cart". In all four cases, a morphism

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\alpha"] \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & \mathsf{B}G^{(\mathsf{op})} & \end{tikzcd}$$

is the data of a functor

$$\mathcal{E}_0 \xrightarrow{\alpha_0} \mathcal{F}_0$$

on underlying ∞-categories equipped with certain natural transformations indexed over all $g \in G$, as recorded in Figure 6. Moreover, these must be equipped with compatibility data with respect to

$\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{B}G}$	$g \cdot \alpha_0(-) \longrightarrow \alpha_0(g \cdot -)$
$\mathsf{RMod}^{\mathsf{r.lax}}_{\mathsf{B}G}$	$\alpha_0(- \cdot g) \longrightarrow \alpha_0(-) \cdot g$
$\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{B}G}$	$\alpha_0(g \cdot -) \longrightarrow g \cdot \alpha_0(-)$
$\mathsf{RMod}^{\mathsf{l.lax}}_{\mathsf{B}G}$	$\alpha_0(- \cdot g) \longrightarrow \alpha_0(-) \cdot g$

FIGURE 6. Given two ∞-categories equipped with (strict) left or right G-actions, defining a left- or right-lax equivariant functor between them amounts to defining a functor on underlying ∞-categories along with compatible lax structure maps indexed by $g \in G$, as indicated.

the multiplication in G: for example, in the case of $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{B}G}$, for all $g, h \in G$ the diagram

$$\begin{tikzcd} \alpha_0(ghe) \arrow[rr] \arrow[dr] & & gh\alpha_0(e) \\ & g\alpha_0(he) \arrow[ur] & \end{tikzcd}$$

must commute, naturally in $e \in \mathcal{E}_0$.

Example A.1.5. Let us unwind the definitions of the ∞-categories

$$\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}} \qquad \text{and} \qquad \mathsf{RMod}^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}}$$

in the simplest nontrivial case, namely when $\mathcal{B} = [2]$.

(1) (a) Let $\mathcal{E} \downarrow [2]$ be a locally cocartesian fibration; let us write $\mathcal{E}_i$ for its fibers (for $i \in [2]$) and E_{ij} for its cocartesian monodromy functors (for $0 \leq i < j \leq 2$). An object $e \in \mathcal{E}_0$ determines a pair of composable locally cocartesian morphisms $e \to E_{01}(e) \to E_{12}(E_{01}(e))$ with $E_{01}(e) \in \mathcal{E}_1$ and $E_{12}(E_{01}(e)) \in \mathcal{E}_2$. Their composite is a not-necessarily-locally-cocartesian morphism, which admits a unique factorization

$$\begin{tikzcd} e \arrow[r, dashed] \arrow[dr] & E_{02}(e) \arrow[d] \\ & E_{12}(E_{01}(e)) \end{tikzcd}$$

as a locally cocartesian morphism followed by a fiber morphism. This operation is functorial in $e \in \mathcal{E}_0$, which implies that our left-lax left $[2]$-module amounts to the data of a lax-commutative triangle

$$\begin{tikzcd} & \mathcal{E}_1 \arrow[dr, "E_{12}"] \arrow[d, phantom, "\Uparrow"] & \\ \mathcal{E}_0 \arrow[ur, "E_{01}"] \arrow[rr, "E_{02}"'] & {} & \mathcal{E}_2 \end{tikzcd} .$$

This should be thought as the unstraightening of a *left-lax* functor

$$[2] \xrightarrow{\mathsf{l.lax}} \mathsf{Cat}$$

of $(\infty, 2)$-categories.

(b) Let $\mathcal{E} \downarrow [2]$ and $\mathcal{F} \downarrow [2]$ be locally cocartesian fibrations, and let us continue to use notation as in part (a) for both $\mathcal{E}$ and $\mathcal{F}$. Then, a left-lax equivariant functor

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "\alpha"] \arrow[dr] & & \mathcal{F} \arrow[dl] \\
& {[2]} &
\end{tikzcd}$$

amounts to the data of left-lax equivariant functors over the three nonidentity morphisms in [2] (as described in Example A.1.3(1)), along with an equivalence between the composite 2-morphisms

$$\begin{tikzcd}
& \mathcal{E}_1 \arrow[dr, "E_{12}"] \arrow[dd, "\alpha_1"] & \\
\mathcal{E}_0 \arrow[ur, "E_{01}"] \arrow[dd, "\alpha_0"'] \arrow[r, phantom, "\Nearrow"] & & \mathcal{E}_2 \arrow[dd, "\alpha_2"] \arrow[l, phantom, "\Nearrow"] \\
& \mathcal{F}_1 \arrow[dr, "F_{12}"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{F}_0 \arrow[ur, "F_{01}"] \arrow[rr, "F_{02}"'] & {} & \mathcal{F}_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathcal{E}_1 \arrow[dr, "E_{12}"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{E}_0 \arrow[ur, "E_{01}"] \arrow[rr, "E_{02}"'] \arrow[dd, "\alpha_0"'] \arrow[ddrr, phantom, "\Nearrow"] & {} & \mathcal{E}_2 \arrow[dd, "\alpha_2"] \\
& & \\
\mathcal{F}_0 \arrow[rr, "F_{02}"'] & & \mathcal{F}_2
\end{tikzcd}$$

(i.e. a 3-morphism filling in the triangular prism).

(2) (a) Dually, let $\mathcal{E} \downarrow [2]$ be a locally cartesian fibration; let us write $\mathcal{E}_{i^\circ}$ for its fibers (for $i \in [2]$) and $E_{j^\circ i^\circ}$ for its cartesian monodromy functors (for $0 \leq i < j \leq 2$). Then, this right-lax right [2]-module amounts to the data of a lax-commutative triangle

$$\begin{tikzcd}
& \mathcal{E}_{1^\circ} \arrow[dl, "E_{1^\circ 0^\circ}"'] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{E}_{0^\circ} & {} & \mathcal{E}_{2^\circ} \arrow[ul, "E_{2^\circ 1^\circ}"'] \arrow[ll, "E_{2^\circ 0^\circ}"]
\end{tikzcd} .$$

This should be thought as the unstraightening of a *right-lax* functor

$$[2]^{\mathsf{op}} \xrightarrow{\mathsf{r.lax}} \mathsf{Cat}$$

of $(\infty, 2)$-categories.

(b) Let $\mathcal{E} \downarrow [2]$ and $\mathcal{F} \downarrow [2]$ be locally cartesian fibrations, and let us continue to use notation as in part (a) for both $\mathcal{E}$ and $\mathcal{F}$. Then, a right-lax equivariant functor

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, "\alpha"] \arrow[dr] & & \mathcal{F} \arrow[dl] \\
& {[2]} &
\end{tikzcd}
$$

amounts to the data of right-lax equivariant functors over the three nonidentity morphisms in [2] (as described in Example A.1.3(2)), along with an equivalence between the composite 2-morphisms

$$
\begin{tikzcd}
& \mathcal{E}_{1^\circ} \arrow[dl, "E_{1^\circ 0^\circ}"'] \arrow[dd, "\alpha_{1^\circ}"] & \\
\mathcal{E}_{0^\circ} \arrow[dd, "\alpha_{0^\circ}"'] \arrow[r, phantom, "\Searrow"] & & \mathcal{E}_{2^\circ} \arrow[ul, "E_{2^\circ 1^\circ}"'] \arrow[dd, "\alpha_{2^\circ}"] \arrow[l, phantom, "\Searrow"] \\
& \mathcal{F}_{1^\circ} \arrow[dl, "F_{1^\circ 0^\circ}"'] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{F}_{0^\circ} & & \mathcal{F}_{2^\circ} \arrow[ul, "F_{2^\circ 1^\circ}"'] \arrow[ll, "F_{2^\circ 0^\circ}"]
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& \mathcal{E}_{1^\circ} \arrow[dl, "E_{1^\circ 0^\circ}"'] \arrow[d, phantom, "\Downarrow"] & \\
\mathcal{E}_{0^\circ} \arrow[dd, "\alpha_{0^\circ}"'] & {} & \mathcal{E}_{2^\circ} \arrow[ul, "E_{2^\circ 1^\circ}"'] \arrow[ll, "E_{2^\circ 0^\circ}"] \arrow[dd, "\alpha_{2^\circ}"] \\
& \Searrow & \\
\mathcal{F}_{0^\circ} & & \mathcal{F}_{2^\circ} \arrow[ll, "F_{2^\circ 0^\circ}"]
\end{tikzcd}
$$

(i.e. a 3-morphism filling in the triangular prism).

Example A.1.6. Let us unwind the definitions of the ∞-categories

$$
\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} , \qquad \mathsf{RMod}_{\mathsf{r.lax}.\mathcal{B}} , \qquad \mathsf{LMod}_{\mathsf{r.lax}.\mathcal{B}} , \qquad \text{and} \qquad \mathsf{RMod}_{\mathsf{l.lax}.\mathcal{B}}
$$

in the simple but illustrative case that $\mathcal{B} = \mathsf{B}G$ for a group or monoid G. Choose an object

$$
\mathcal{E} \in \mathsf{Cat}_{\mathsf{loc.(co)cart}/\mathsf{B}G^{(\mathsf{op})}} ,
$$

with the two choices of whether or not to include the parenthesized bits made independently. Write $\mathcal{E}_0$ for the fiber over the basepoint of $\mathsf{B}G^{(\mathsf{op})}$, the underlying ∞-category. Then, this is the data of an endofunctor $(g \cdot -)$ or $(- \cdot g)$ of $\mathcal{E}_0$ for each $g \in G$, along with compatible natural transformations, as recorded in Figure 7. Of course, these must also be compatible with iterated multiplication in G.

Observation A.1.7. Consider a colimit

$$
\mathcal{B} \simeq \mathsf{colim}_{i \in \mathcal{J}}(\mathcal{B}_i) \tag{A.1.1}
$$

in Cat. It is clear that the canonical functor

$$
\mathsf{coCart}_{\mathcal{B}} \longrightarrow \mathsf{lim}_{i^\circ \in \mathcal{J}^{\mathsf{op}}}(\mathsf{coCart}_{\mathcal{B}_i})
$$

is an equivalence (of $(\infty, 2)$-categories). On the other hand, the canonical functor

$$
\mathsf{loc.coCart}_{\mathcal{B}} \longrightarrow \mathsf{lim}_{i^\circ \in \mathcal{J}^{\mathsf{op}}}(\mathsf{loc.coCart}_{\mathcal{B}_i}) \tag{A.1.2}
$$

$\mathsf{LMod}_{\mathsf{l.lax.B}G}$	$(gh \cdot -) \longrightarrow g \cdot (h \cdot -)$
$\mathsf{RMod}_{\mathsf{r.lax.B}G}$	$(- \cdot g) \cdot h \longrightarrow (- \cdot gh)$
$\mathsf{LMod}_{\mathsf{r.lax.B}G}$	$g \cdot (h \cdot -) \longrightarrow (gh \cdot -)$
$\mathsf{RMod}_{\mathsf{l.lax.B}G}$	$(- \cdot gh) \longrightarrow (- \cdot g) \cdot h$

FIGURE 7. Equipping an ∞-category with a left- or right-lax left or right G-action amounts to defining endofunctors indexed by $g \in G$, equipped with lax structure maps corresponding to multiplication in G, as indicated.

is not generally an equivalence. However, the functor (A.1.2) is an equivalence under the condition that the colimit (A.1.1), considered in complete Segal spaces, is in fact a colimit in simplicial spaces.

A.2. **Strict and lax limits.** In this subsection, we introduce the more straightforward sorts of limits. We begin with an omnibus definition, which the remainder of the subsection is dedicated to discussing.

Definition A.2.1. In Figure 8, we define various ***limit*** functors on various ∞-categories of $\mathcal{B}$-modules. Our notation is largely concordant with that of Definition A.1.1; we indicate the handedness of the original module in the subscript by writing $\mathcal{B}$ for left modules and $\mathcal{B}^{\mathsf{op}}$ for right modules.[148] We refer to a limit functor according to its superscript (which is more relevant anyways), e.g. we refer to $\lim^{\mathsf{l.lax}}_{\mathsf{l.lax.}\mathcal{B}^{\mathsf{op}}}$ as the ***left-lax limit*** functor. We also write e.g.

$$\lim^{\mathsf{l.lax}}_{\mathcal{B}} : \mathsf{LMod}_{\mathcal{B}} \hookrightarrow \mathsf{LMod}_{\mathsf{l.lax.}\mathcal{B}} \xrightarrow{\lim^{\mathsf{l.lax}}_{\mathsf{l.lax.}\mathcal{B}}} \mathsf{Cat}$$

for the composite functor, which carries each strict left $\mathcal{B}$-module to its left-lax limit.

Example A.2.2. Let us unwind the definitions of the functors in the diagrams

$$\begin{tikzcd} \mathsf{LMod}_{\mathcal{B}} \arrow[r, bend left, "\lim_{\mathcal{B}}"] \arrow[r, bend right, "\lim^{\mathsf{l.lax}}_{\mathcal{B}}"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{Cat} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{RMod}_{\mathcal{B}} \arrow[r, bend left, "\lim_{\mathcal{B}^{\mathsf{op}}}"] \arrow[r, bend right, "\lim^{\mathsf{r.lax}}_{\mathcal{B}^{\mathsf{op}}}"'] \arrow[r, phantom, "\Downarrow"] & \mathsf{Cat} \end{tikzcd}$$

in the simple but illustrative case that $\mathcal{B} = \mathsf{B}G$ for a group or monoid G.

(1) Suppose that
$$(\mathcal{E} \downarrow \mathsf{B}G) \in \mathsf{coCart}_{\mathsf{B}G} =: \mathsf{LMod}_{\mathsf{B}G}$$
is classified by a left G-action on $\mathcal{E}_0$.

(a) An object of the strict limit is given by an object $e \in \mathcal{E}_0$ equipped with equivalences
$$g \cdot e \xrightarrow{\sim} e$$
for all $g \in G$ that are compatible with the multiplication in G.

(b) An object of the left-lax limit is given by an object $e \in \mathcal{E}_0$ equipped with morphisms
$$g \cdot e \longrightarrow e$$
for all $g \in G$ that are compatible with the multiplication in G.

[148]This coincides with the corresponding notation for G-actions: the limit of a left (resp. right) G-action is a limit over $\mathsf{B}G$ (resp. over $\mathsf{B}G^{\mathsf{op}}$).

$$
\begin{tikzcd}
\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}} \arrow[ddr, bend left, "\lim^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}}"] & \\
\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} \arrow[u, hook, two heads] \arrow[dr, "\lim_{\mathsf{l.lax}.\mathcal{B}}"] \arrow[r, phantom, "\Nearrow"] & \\
\mathsf{LMod}_{\mathcal{B}} \arrow[u, hook, "\mathrm{f.f.}"] \arrow[d, hook, "\mathrm{f.f.}"'] \arrow[r, dashed, "\lim_{\mathcal{B}}"] & \mathsf{Cat} \\
\mathsf{LMod}_{\mathsf{r.lax}.\mathcal{B}} \arrow[d, hook, two heads] \arrow[ur, "\lim_{\mathsf{r.lax}.\mathcal{B}}"'] \arrow[r, phantom, "\Searrow"] & \\
\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}} \arrow[uur, bend right, "\lim^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}}"'] &
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
\mathsf{Cat}_{\mathsf{loc.cocart}/\mathcal{B}} \arrow[dddr, bend left, "\Gamma"] & \\
\mathsf{loc.coCart}_{\mathcal{B}} \arrow[u, hook, two heads] \arrow[ddr, "\Gamma^{\mathsf{cocart}}"] \arrow[r, phantom, "\Nearrow"] & \\
\mathsf{coCart}_{\mathcal{B}} \arrow[u, hook, "\mathrm{f.f.}"] \arrow[dd, phantom, "\wr"] & \\
 & \mathsf{Cat} \\
\mathsf{Cart}_{\mathcal{B}^{\mathsf{op}}} \arrow[d, hook, "\mathrm{f.f.}"'] & \\
\mathsf{loc.Cart}_{\mathcal{B}^{\mathsf{op}}} \arrow[d, hook, two heads] \arrow[uur, "\Gamma^{\mathsf{cart}}"'] \arrow[r, phantom, "\Searrow"] & \\
\mathsf{Cat}_{\mathsf{loc.cart}/\mathcal{B}^{\mathsf{op}}} \arrow[uuur, bend right, "\Gamma"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{RMod}^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}} \arrow[ddr, bend left, "\lim^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}^{\mathsf{op}}}"] & \\
\mathsf{RMod}_{\mathsf{l.lax}.\mathcal{B}} \arrow[u, hook, two heads] \arrow[dr, "\lim_{\mathsf{l.lax}.\mathcal{B}^{\mathsf{op}}}"] \arrow[r, phantom, "\Nearrow"] & \\
\mathsf{RMod}_{\mathcal{B}} \arrow[u, hook, "\mathrm{f.f.}"] \arrow[d, hook, "\mathrm{f.f.}"'] \arrow[r, dashed, "\lim_{\mathcal{B}^{\mathsf{op}}}"] & \mathsf{Cat} \\
\mathsf{RMod}_{\mathsf{r.lax}.\mathcal{B}} \arrow[d, hook, two heads] \arrow[ur, "\lim_{\mathsf{r.lax}.\mathcal{B}^{\mathsf{op}}}"'] \arrow[r, phantom, "\Searrow"] & \\
\mathsf{RMod}^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}} \arrow[uur, bend right, "\lim^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}^{\mathsf{op}}}"'] &
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
\mathsf{Cat}_{\mathsf{loc.cocart}/\mathcal{B}^{\mathsf{op}}} \arrow[dddr, bend left, "\Gamma"] & \\
\mathsf{loc.coCart}_{\mathcal{B}^{\mathsf{op}}} \arrow[u, hook, two heads] \arrow[ddr, "\Gamma^{\mathsf{cocart}}"] \arrow[r, phantom, "\Nearrow"] & \\
\mathsf{coCart}_{\mathcal{B}^{\mathsf{op}}} \arrow[u, hook, "\mathrm{f.f.}"] \arrow[dd, phantom, "\wr"] & \\
 & \mathsf{Cat} \\
\mathsf{Cart}_{\mathcal{B}} \arrow[d, hook, "\mathrm{f.f.}"'] & \\
\mathsf{loc.Cart}_{\mathcal{B}} \arrow[d, hook, two heads] \arrow[uur, "\Gamma^{\mathsf{cart}}"'] \arrow[r, phantom, "\Searrow"] & \\
\mathsf{Cat}_{\mathsf{loc.cart}/\mathcal{B}} \arrow[uuur, bend right, "\Gamma"'] &
\end{tikzcd}
\quad .
$$

FIGURE 8. The rightwards functors to Cat on the left are defined to be those on the right (except that each dashed functor may be defined as either adjacent composite: the inner triangles all commute).

(2) Suppose that

$$(\mathcal{E} \downarrow \mathsf{B}G) \in \mathsf{Cart}_{\mathsf{B}G} =: \mathsf{RMod}_{\mathsf{B}G}$$

is classified by a right G-action on $\mathcal{E}_0$.

(a) An object of the strict limit is given by an object $e \in \mathcal{E}_0$ equipped with equivalences

$$e \xrightarrow{\sim} e \cdot g$$

for all $g \in G$ that are compatible with the multiplication in G.

(b) An object of the left-lax limit is given by an object $e \in \mathcal{E}_0$ equipped with morphisms

$$e \longrightarrow e \cdot g$$

for all $g \in G$ that are compatible with the multiplication in G.

Example A.2.3. Let us unwind the definitions of the functors

$$\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}} \xrightarrow{\lim^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}}} \mathsf{Cat} \qquad \text{and} \qquad \mathsf{RMod}^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}} \xrightarrow{\lim^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}}} \mathsf{Cat}$$

in the simplest nontrivial case, namely when $\mathcal{B} = [2]$.

(1) Let $\mathcal{E} \downarrow [2]$ be a locally cocartesian fibration, and let us employ the notation of Example A.1.5(1)(a). Then, an object of the left-lax limit of this left-lax left $[2]$-module is given by the data of

- objects $e_i \in \mathcal{E}_i$ (for $0 \leq i \leq 2$),
- morphisms

$$E_{ij}(e_i) \xrightarrow{\varepsilon_{ij}} e_j$$

(for $0 \leq i < j \leq 2$), and

- a commutative square

$$\begin{tikzcd}
E_{02}(e_0) \arrow[rr, "\varepsilon_{02}"] \arrow[d] & & e_2 \\
E_{12}(E_{01}(e_0)) \arrow[rr, "E_{12}(\varepsilon_{01})"'] & & E_{12}(e_1) \arrow[u, "\varepsilon_{12}"']
\end{tikzcd}$$

in $\mathcal{E}_2$, where the morphism on the left is the canonical one (recall Example A.1.5(1)(a)).

Note that the structure map ε_{02} is *canonically* determined by the structure maps ε_{01} and ε_{12}.

(2) Let $\mathcal{E} \downarrow [2]$ be a locally cartesian fibration, and let us employ the notation of Example A.1.5(2)(a). Then, an object of the right-lax limit of this right-lax right $[2]$-module is given by the data of

- objects $e_{i^\circ} \in \mathcal{E}_{i^\circ}$ (for $0 \leq i \leq 2$),
- morphisms

$$e_{i^\circ} \xrightarrow{\varepsilon_{j^\circ i^\circ}} E_{j^\circ i^\circ}(e_{j^\circ})$$

(for $0 \leq i < j \leq 2$), and

- a commutative square

$$\begin{tikzcd}
e_{0^\circ} \arrow[r, "\varepsilon_{2^\circ 0^\circ}"] \arrow[d, "\varepsilon_{1^\circ 0^\circ}"'] & E_{2^\circ 0^\circ}(e_{2^\circ}) \\
E_{1^\circ 0^\circ}(e_{1^\circ}) \arrow[r, "E_{1^\circ 0^\circ}(\varepsilon_{2^\circ 1^\circ})"'] & E_{1^\circ 0^\circ}(E_{2^\circ 1^\circ}(e_{2^\circ})) \arrow[u]
\end{tikzcd}$$

in $\mathcal{E}_{0^\circ}$, where the morphism on the right is the canonical one (recall Example A.1.5(2)(a)).

Note that the structure map $\varepsilon_{2^\circ 0^\circ}$ is likewise *canonically* determined by the structure maps $\varepsilon_{2^\circ 1^\circ}$ and $\varepsilon_{1^\circ 0^\circ}$.

A.3. **Lax actions and lax limits with mixed handedness.** In this subsection, we introduce lax morphisms among lax modules (and in particular lax limits) of mixed handedness, using the theory of $(\infty, 2)$-categories developed in §B.[149] We also record some key results here: an $(\infty, 1)$-categorical description of such morphisms (Lemma A.3.4), as well as two results that relate such morphisms with those in the ∞-categories appearing in §A.1 via passage to adjoints (Lemmas A.3.5 and A.3.6). To streamline our discussion, we restrict our attention to *left* $\mathcal{B}$-modules; the case of right $\mathcal{B}$-modules is obtained by replacing $\mathcal{B}$ with $\mathcal{B}^{\mathsf{op}}$.

Observation A.3.1. By Observation B.4.2, we can describe the ∞-categories of lax modules as well as their lax limit functors described in §§A.1 and A.2 in terms of $(\infty, 2)$-categories according to the identifications

$$\begin{tikzcd}
\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} \arrow[rr, hook, two heads] \arrow[dr, "\lim_{\mathsf{l.lax}.\mathcal{B}}"'] & & \mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}} \arrow[dl, "\lim^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}}"] & \simeq & \mathsf{1coCart}_{\mathsf{l.lax}(\mathcal{B})} \arrow[rr, hook, two heads] \arrow[dr, "\lim_{\mathsf{l.lax}(\mathcal{B})}"'] & & \mathsf{2Cat}_{\mathsf{1cocart}/\mathsf{l.lax}(\mathcal{B})} \arrow[dl, "\lim^{\mathsf{l.lax}}_{\mathsf{l.lax}(\mathcal{B})}"] \\
& \mathsf{Cat} & & & & \mathsf{Cat} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{LMod}_{\mathsf{r.lax}.\mathcal{B}} \arrow[rr, hook, two heads] \arrow[dr, "\lim_{\mathsf{r.lax}.\mathcal{B}}"'] & & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}} \arrow[dl, "\lim^{\mathsf{r.lax}}_{\mathsf{r.lax}.\mathcal{B}}"] & \simeq & \mathsf{1Cart}_{\mathsf{r.lax}(\mathcal{B})^{\mathsf{1op}}} \arrow[rr, hook, two heads] \arrow[dr, "\lim_{\mathsf{r.lax}(\mathcal{B})}"'] & & \mathsf{2Cat}_{\mathsf{1cart}/\mathsf{r.lax}(\mathcal{B})^{\mathsf{1op}}} \arrow[dl, "\lim^{\mathsf{r.lax}}_{\mathsf{r.lax}(\mathcal{B})}"] \\
& \mathsf{Cat} & & & & \mathsf{Cat} &
\end{tikzcd}.$$

Definition A.3.2. We define ***right-lax equivariant*** morphisms between left-lax left $\mathcal{B}$-modules and right-lax limits thereof as well as ***left-lax equivariant*** morphisms between right-lax left $\mathcal{B}$-modules and left-lax limits thereof according to the diagrams

$$\begin{tikzcd}
\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}} := \mathsf{2Cat}_{\mathsf{1cart}/\mathsf{l.lax}(\mathcal{B})^{\mathsf{1op}}} \arrow[d, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}"', shift right=6] \arrow[d, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}(\mathcal{B})}", shift left=6] \\
\mathsf{Cat}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{r.lax}.\mathcal{B}} := \mathsf{2Cat}_{\mathsf{1cocart}/\mathsf{r.lax}(\mathcal{B})} \arrow[d, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax}.\mathcal{B}}"', shift right=6] \arrow[d, "\lim^{\mathsf{l.lax}}_{\mathsf{r.lax}(\mathcal{B})}", shift left=6] \\
\mathsf{Cat}
\end{tikzcd}.$$

Remark A.3.3. We give a purely $(\infty, 1)$-categorical description of the right-lax limit of a left-lax left $\mathcal{B}$-module in §A.5 (see Proposition A.5.2). In the main body of the paper, we take this as an alternate definition. In the case that $\mathcal{B}$ is a poset, we also give another description in §A.6 as a strict limit over its subdivision category.

[149] Here we freely refer to some basic $(\infty, 2)$-categorical notions (such as left- and right-laxification and 1-co/cartesian fibrations) described there.

Lemma A.3.4. *For any ∞-category $\mathcal{C}$, the datum of a functor*

$$\mathcal{C} \longrightarrow \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}$$

is equivalent to the datum of a locally cocartesian fibration $\mathcal{E} \downarrow (\mathcal{C} \times \mathcal{B})$ satisfying the following two conditions.

(1) *For every object $b \in \mathcal{B}$, the base change $\mathcal{E}_b \to \mathcal{C}$ is a (strict) cocartesian fibration.*

(2) *For any pair of morphisms $c \to c'$ in $\mathcal{C}$ and $b \to b'$ in $\mathcal{B}$, the pullback along the functor $[2] \to \mathcal{C} \times \mathcal{B}$ selecting the commutative triangle*

$$\begin{tikzcd} (c,b) \arrow[d] \arrow[dr] & \\ (c,b') \arrow[r] & (c',b') \end{tikzcd}$$

is a (strict) cocartesian fibration.

Proof. Using Theorem B.4.1, we have the identification

$$\mathsf{hom}_{\iota_1 2\mathsf{Cat}}(\mathcal{C}, \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}) := \mathsf{hom}_{\iota_1 2\mathsf{Cat}}(\mathcal{C}, 2\mathsf{Cat}_{1\mathsf{cart}/\mathsf{l.lax}(\mathcal{B})^{1\mathsf{op}}}) \simeq \mathsf{hom}_{\iota_1 2\mathsf{Cat}}(\mathsf{l.lax}(\mathcal{B}), 2\mathsf{Cat}_{1\mathsf{cocart}/\mathcal{C}}) \ .$$

The result now follows from Theorem B.4.3. □

Lemma A.3.5. *The datum of a morphism $\mathcal{E}_0 \leftarrow \mathcal{E}_1$ in $\mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{l.lax}.\mathcal{B}}$ whose restriction to each $b \in \mathcal{B}$ is a right adjoint is equivalent to the datum of a morphism $\mathcal{E}_0 \to \mathcal{E}_1$ in $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}$ whose restriction to each $b \in \mathcal{B}$ is a left adjoint, with the equivalence given fiberwise by passing to adjoints.*

Proof. This follows from Lemma B.5.7 by taking $\mathcal{C} = \mathsf{l.lax}(\mathcal{B})$. □

Lemma A.3.6. *A morphism $\mathcal{E}_0 \leftarrow \mathcal{E}_1$ in $\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}}$ whose restriction to each $b \in \mathcal{B}$ is a right adjoint becomes a right adjoint in $\mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}$, i.e. there exists a (necessarily unique) extension*

$$\begin{tikzcd} {[1]^{\mathsf{op}}} \arrow[r] \arrow[d, hook] & \mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} \arrow[d, hook, two heads] \\ \mathsf{Adj} \arrow[r, dashed] & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}} \end{tikzcd} .$$

Proof. This follows from Lemma B.5.8 by taking $\mathcal{C} = \mathsf{l.lax}(\mathcal{B})$. □

A.4. **Subdivisions.** In this subsection, we study subdivisions of ∞-categories.

Local Notation A.4.1. In this subsection, we fix a poset $\mathsf{P} \in \mathsf{Poset}$.

Definition A.4.2. The ***subdivision*** of P is the full subcategory

$$\mathsf{sd}(\mathsf{P}) \subseteq \mathbf{\Delta}_{/\mathsf{P}} := \mathbf{\Delta} \times_{\mathsf{Cat}} \mathsf{Cat}_{/\mathsf{P}}$$

on the conservative (or equivalently injective) functors $[n] \to \mathsf{P}$.

Definition A.4.3. A morphism $[m] \xrightarrow{\alpha} [n]$ in $\mathbf{\Delta}$ is called ***isomin*** if $\alpha(0) = 0$, ***isomax*** if $\alpha(m) = n$, and ***isominmax*** if it is both isomin and isomax. We use the same terminology for morphisms in $\mathsf{sd}(\mathsf{P})$ according to their images under the forgetful functor $\mathsf{sd}(\mathsf{P}) \to \mathbf{\Delta}$.

Remark A.4.4. The ∞-category $\mathsf{sd}(\mathsf{P})$ is in fact a poset, namely the full subposet of the power set $\mathscr{P}(\mathsf{P})$ (ordered by inclusion) on those subsets of P which are nonempty and totally ordered.

Example A.4.5. For each $[n] \in \mathbf{\Delta} \subseteq \mathsf{Poset}$, we have an identification

$$\mathsf{sd}([n]) \simeq \mathscr{P}_{\neq \varnothing}([n])$$

of its subdivision with its power set with its initial element removed, which is a punctured $(n+1)$-cube.

Observation A.4.6. The defining fully faithful inclusion $\mathsf{sd}(\mathsf{P}) \subseteq \mathbf{\Delta}_{/\mathsf{P}}$ admits a left adjoint

$$\begin{tikzcd} \mathbf{\Delta}_{/\mathsf{P}} \arrow[r, dashed, shift left, "\mathsf{im}"] \arrow[r, phantom, "\perp"] & \mathsf{sd}(\mathsf{P}) \arrow[l, hook, shift left] \end{tikzcd} ,$$

which takes each functor $[n] \xrightarrow{\varphi} \mathsf{P}$ to the second factor in its epi-mono factorization

$$\begin{tikzcd} {[n]} \arrow[rr, "\varphi"] \arrow[dr, dashed, two heads] & & \mathsf{P} \\ & \mathsf{im}(\varphi) \arrow[ur, dashed, hook, "\text{f.f.}"'] & \end{tikzcd} .$$

Observation A.4.7. Subdivisions of posets assemble into a functor

$$\mathsf{Poset} \xrightarrow{\mathsf{sd}} \mathsf{Cat}$$

(whose unstraightening is a full subcategory of that of the functor $\mathsf{Poset} \xrightarrow{\mathbf{\Delta}_{/(-)}} \mathsf{Cat}$).

Lemma A.4.8. *The commutative triangle*

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, hook, "\text{f.f.}"] \arrow[d, hook, "\text{f.f.}"'] & \mathsf{Poset} \arrow[r, "\mathsf{sd}"] & \mathsf{Cat} \\ \mathsf{Poset} \arrow[urr, bend right, "\mathsf{sd}"'] & & \end{tikzcd}$$

is a left Kan extension diagram.

Proof. We must show that the canonical functor

$$\mathsf{colim}\left(\mathbf{\Delta}_{/\mathsf{P}} \xrightarrow{\mathsf{fgt}} \mathbf{\Delta} \hookrightarrow \mathsf{Poset} \xrightarrow{\mathsf{sd}} \mathsf{Cat}\right) \longrightarrow \mathsf{sd}(\mathsf{P})$$

is an equivalence. By Observation A.4.6, the functor $\mathsf{sd}(\mathsf{P}) \hookrightarrow \mathbf{\Delta}_{/\mathsf{P}}$ is final. So, it is equivalent to show that the functor

$$\mathsf{colim}\left(\mathsf{sd}(\mathsf{P}) \hookrightarrow \mathbf{\Delta}_{/\mathsf{P}} \xrightarrow{\mathsf{fgt}} \mathbf{\Delta} \hookrightarrow \mathsf{Poset} \xrightarrow{\mathsf{sd}} \mathsf{Cat}\right) \longrightarrow \mathsf{sd}(\mathsf{P}) \tag{A.4.1}$$

is an equivalence. Consider the composite

$$\mathsf{sd}(\mathsf{P}) \hookrightarrow \mathbf{\Delta}_{/\mathsf{P}} \xrightarrow{\mathsf{fgt}} \mathbf{\Delta} \hookrightarrow \mathsf{Poset} \xrightarrow{\mathsf{sd}} \mathsf{Cat} \tag{A.4.2}$$

(whose colimit is the source of the functor (A.4.1)). It is not hard to see that the unstraightening of the composite (A.4.2) is the cocartesian fibration

$$\mathsf{Ar}(\mathsf{sd}(\mathsf{P})) \xrightarrow{t} \mathsf{sd}(\mathsf{P}) , \tag{A.4.3}$$

and that the composite

$$\mathsf{Ar}(\mathsf{sd}(\mathsf{P})) \longrightarrow \mathsf{colim}(\text{A.4.2}) \xrightarrow{(\text{A.4.1})} \mathsf{sd}(\mathsf{P}) \tag{A.4.4}$$

is precisely the functor $\mathsf{Ar}(\mathsf{sd}(\mathsf{P})) \xrightarrow{s} \mathsf{sd}(\mathsf{P})$, where the first functor in the composite (A.4.4) is the localization of $\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))$ with respect to the cocartesian morphisms in the cocartesian fibration (A.4.3). These cocartesian morphisms are precisely the morphisms that are sent to equivalences by the functor $\mathsf{Ar}(\mathsf{sd}(\mathsf{P})) \xrightarrow{s} \mathsf{sd}(\mathsf{P})$. But this functor is itself a localization (because it admits a fully faithful left adjoint), which shows that the functor (A.4.1) is an equivalence. □

Definition A.4.9. Justified by Lemma A.4.8, we define the ***subdivision*** endofunctor on Cat as the left Kan extension

$$\begin{tikzcd} \mathbf{\Delta} \arrow[r, "\mathsf{sd}"] \arrow[d, hook, "\text{f.f.}"'] & \mathsf{Cat} \\ \mathsf{Cat} \arrow[ur, dashed, "\mathsf{sd}"'] & \end{tikzcd} .$$

Observation A.4.10. For any $[n] \in \mathbf{\Delta}$, there are evident functors

$$\mathsf{sd}([n]) \xrightarrow{\mathsf{max}} [n] \qquad \text{and} \qquad \mathsf{sd}([n])^{\mathsf{op}} \xrightarrow{\mathsf{min}} [n] \ ,$$

which respectively take a nonempty subset of $[n]$ to its maximal or minimal element. These are respectively a locally cocartesian fibration and a locally cartesian fibration: in both cases the monodromy functors are given by union, as illustrated in Figure 9. By functoriality of left Kan extension,

$$\begin{tikzcd}
 & 2 \arrow[rr] \arrow[dd] & & 12 \arrow[dd] & & \\
 & & 1 \arrow[ur] \arrow[dd] & & & \mathsf{sd}([2]) \arrow[dddd, "\mathsf{max}"] \\
 & 02 \arrow[rr] & & 012 & & \\
0 \arrow[ur] \arrow[rr] & & 01 \arrow[ur] & & & \\
 & & & 2 & & \\
0 \arrow[urrr] \arrow[rr] & & 1 \arrow[ur] & & & [2]
\end{tikzcd} \tag{A.4.5}$$

$$\begin{tikzcd}
 & {[1]} \arrow[dr, "{(\mathrm{id},\mathrm{const}_1)}"] & \\
{[0]} \arrow[ur, "1"] \arrow[rr, "{(1,0)}"'] \arrow[rr, phantom, shift left=4ex, "\Uparrow"] & & {[1] \times [1]}
\end{tikzcd} \tag{A.4.6}$$

FIGURE 9. The functor $\mathsf{sd}([2]) \xrightarrow{\mathsf{max}} [2]$ is a locally cocartesian fibration, as illustrated in diagram (A.4.5); its unstraightening is illustrated in diagram (A.4.6). Note that the functor $\mathsf{sd}([1]) \xrightarrow{\mathsf{max}} [1]$ can also be seen in diagram (A.4.5) in three different ways, corresponding to the three nonidentity morphisms in [2].

these induce augmentations

$$\mathsf{sd} \xrightarrow{\mathsf{max}} \mathsf{id} \qquad \text{and} \qquad \mathsf{sd}^{\mathsf{op}} \xrightarrow{\mathsf{min}} \mathsf{id}$$

in $\mathsf{Fun}(\mathsf{Cat}, \mathsf{Cat})$, whose components are respectively locally cocartesian fibrations and locally cartesian fibrations.[150] We often use these facts without further comment.

A.5. **Lax limits with mixed handedness via subdivisions.** In this subsection, we give an alternative description of left-lax limits of right-lax left $\mathcal{B}$-modules in terms of the subdivision of $\mathcal{B}$ (Proposition A.5.2). In the main body of the paper, we use this description as an alternate definition.

Notation A.5.1. Given two objects

$$(\mathcal{E} \downarrow \mathcal{B}), (\mathcal{F} \downarrow \mathcal{B}) \in \mathsf{Cat}_{/\mathcal{B}} \ ,$$

[150] It is not hard to see that the colimit defining $\mathsf{sd}(\mathcal{B})$, when translated to complete Segal spaces, can actually be computed in simplicial spaces; thereafter, it is straightforward to check that the functor $\mathsf{sd}(\mathcal{B}) \xrightarrow{\mathsf{max}} \mathcal{B}$ is indeed a locally cocartesian fibration. An identical argument proves that $\mathsf{sd}(\mathcal{B})^{\mathsf{op}} \xrightarrow{\mathsf{min}} \mathcal{B}$ is a locally cartesian fibration.

we write

$$\mathsf{Fun}^{\mathsf{(co)cart}}_{/\mathcal{B}}(\mathcal{E},\mathcal{F}) \subseteq \mathsf{Fun}_{/\mathcal{B}}(\mathcal{E},\mathcal{F})$$

for the full subcategory on those functors which take all (co)cartesian morphisms over $\mathcal{B}$ in $\mathcal{E}$ to (co)cartesian morphisms over $\mathcal{B}$ in $\mathcal{F}$.

Proposition A.5.2. *Right-lax limits of left-lax left $\mathcal{B}$-modules are corepresented by the object $(\mathsf{sd}(\mathcal{B}) \downarrow \mathcal{B}) \in \mathsf{loc.coCart}_{\mathcal{B}} =: \mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}}$. In other words, there is a canonical commutative diagram*

$$\begin{tikzcd}
\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} \arrow[r, hook, two heads] \arrow[d, phantom, "\rotatebox{90}{$\coloneqq$}"] & \mathsf{LMod}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}} \arrow[dr, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}"] & \\
\mathsf{loc.coCart}_{\mathcal{B}} \arrow[rr, "\mathsf{Fun}^{\mathsf{cocart}}_{/\mathcal{B}}(\mathsf{sd}(\mathcal{B}),-)"'] & & \mathsf{Cat}
\end{tikzcd}.$$

Proof. This is Theorem B.6.2. □

Example A.5.3. Using Proposition A.5.2, let us unwind the definitions of the functors

$$\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} \xrightarrow{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}} \mathsf{Cat} \qquad \text{and} \qquad \mathsf{RMod}_{\mathsf{r.lax}.\mathcal{B}} \xrightarrow{\lim^{\mathsf{l.lax}}_{\mathsf{r.lax}.\mathcal{B}^{\mathsf{op}}}} \mathsf{Cat}$$

in the simplest nontrivial case, namely when $\mathcal{B} = [2]$.

(1) Let $\mathcal{E} \downarrow [2]$ be a locally cocartesian fibration, and let us employ the notation of Example A.1.5(1)(a). Then, an object of the right-lax limit of this left-lax left [2]-module is given by the data of

- objects $e_i \in \mathcal{E}_i$ (for $0 \leq i \leq 2$),
- morphisms
$$e_j \xrightarrow{\varepsilon_{ij}} E_{ij}(e_i)$$
(for $0 \leq i < j \leq 2$), and
- a commutative square
$$\begin{tikzcd}
e_2 \arrow[r, "\varepsilon_{01}"] \arrow[d, "\varepsilon_{02}"'] & E_{12}(e_1) \arrow[d, "E_{12}(\varepsilon_{01})"] \\
E_{02}(e_0) \arrow[r] & E_{12}(E_{01}(e_0))
\end{tikzcd}$$
in $\mathcal{E}_2$, where the lower morphism is the canonical one (recall Example A.1.5(1)(a)).

Note that the structure map ε_{02} is *not* generally determined by the structure maps ε_{01} and ε_{12} (in contrast with Example A.2.3(1)).

(2) Let $\mathcal{E} \downarrow [2]$ be a locally cartesian fibration, and let us employ the notation of Example A.1.5(2)(a). Then, an object of the left-lax limit of this right-lax right [2]-module is given by the data of

- objects $e_{i^\circ} \in \mathcal{E}_{i^\circ}$ (for $0 \leq i \leq 2$),
- morphisms
$$E_{j^\circ i^\circ}(e_{j^\circ}) \xrightarrow{\varepsilon_{j^\circ i^\circ}} e_{i^\circ}$$
(for $0 \leq i < j \leq 2$), and

- a commutative square

$$\begin{tikzcd} E_{1^\circ 0^\circ}(E_{2^\circ 1^\circ}(e_{2^\circ})) \arrow[r] \arrow[d, "E_{1^\circ 0^\circ}(\varepsilon_{2^\circ 1^\circ})"'] & E_{2^\circ 0^\circ}(e_{2^\circ}) \arrow[d, "\varepsilon_{2^\circ 0^\circ}"] \\ E_{1^\circ 0^\circ}(e_{1^\circ}) \arrow[r, "\varepsilon_{1^\circ 0^\circ}"'] & e_{0^\circ} \end{tikzcd}$$

  in $\mathcal{E}_{0^\circ}$, , where the upper morphism is the canonical one (recall Example A.1.5(2)(a)).

Note that the structure map $\varepsilon_{2^\circ 0^\circ}$ is likewise *not* generally determined by the structure maps $\varepsilon_{2^\circ 1^\circ}$ and $\varepsilon_{1^\circ 0^\circ}$ (again in contrast with Example A.2.3(2)).

Remark A.5.4. It is because we are taking e.g. the *right*-lax limit of a *left*-lax module that we end up with the perhaps unfamiliar compatibility conditions of the commutative squares in Example A.5.3. Comparing with Example A.2.3, we see that the analogous compatibility condition for e.g. the left-lax limit of a left-lax module as a section of a locally cocartesian fibration is simply that the section preserves composition of morphisms – which is of course built into the very definition of a functor.

Example A.5.5. Consider the projection from the product

$$\underline{\mathcal{G}} := \mathcal{G} \times \mathcal{B} \xrightarrow{\mathsf{pr}} \mathcal{B} \tag{A.5.1}$$

as an object of $\mathsf{LMod}_{\mathcal{B}}$. Note that the localization of the category $\mathsf{sd}([n])$ with respect to the locally cocartesian morphisms for the locally cocartesian fibration

$$\mathsf{sd}([n]) \xrightarrow{\mathsf{max}} [n]$$

is $[n]^{\mathsf{op}}$; it follows that the localization of the ∞-category $\mathsf{sd}(\mathcal{B})$ with respect to the cocartesian morphisms for the locally cocartesian fibration

$$\mathsf{sd}(\mathcal{B}) \xrightarrow{\mathsf{max}} \mathcal{B}$$

is simply $\mathcal{B}^{\mathsf{op}}$. Hence, we find that

$$\mathsf{lim}^{\mathsf{r.lax}}_{\mathcal{B}}(\underline{\mathcal{G}}) := \mathsf{Fun}^{\mathsf{cocart}}_{/\mathcal{B}}\left(\mathsf{sd}(\mathcal{B}), \underline{\mathcal{G}}\right) := \mathsf{Fun}^{\mathsf{cocart}}_{/\mathcal{B}}\left(\mathsf{sd}(\mathcal{B}), \mathcal{G} \times \mathcal{B}\right) \simeq \mathsf{Fun}\left(\mathcal{B}^{\mathsf{op}}, \mathcal{G}\right) .$$

Dually, considering (A.5.1) $\in \mathsf{RMod}_{\mathcal{B}}$, we have that

$$\mathsf{lim}^{\mathsf{l.lax}}_{\mathcal{B}^{\mathsf{op}}}(\underline{\mathcal{G}}) \simeq \mathsf{Fun}(\mathcal{B}, \mathcal{G}) .$$

Observation A.5.6. Given a $\mathcal{B}$-module of any sort, there are canonical fully faithful inclusions from its strict limit to its various lax limits. In terms of Proposition A.5.2, the canonical morphism

$$\mathsf{lim}_{\mathsf{l.lax}.\mathcal{B}}(-) \hookrightarrow \mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax}.\mathcal{B}}(-)$$

in $\mathsf{Fun}(\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}}, \mathsf{Cat})$ is corepresented by the epimorphism (in fact localization) $\mathcal{B} \xleftarrow{\mathsf{max}} \mathsf{sd}(\mathcal{B})$ in $\mathsf{LMod}_{\mathsf{l.lax}.\mathcal{B}} := \mathsf{loc.coCart}_{\mathcal{B}}$.

A.6. **An alternative description of right-lax limits of left-lax modules over posets.** In this subsection, we provide a useful alternative description of right-lax limits of left-lax modules in the special case that the base ∞-category is a poset.

Local Notation A.6.1. In this subsection, we fix a poset P.

Definition A.6.2. For any object $\varphi \in \mathsf{sd}(\mathsf{P})$, its ***isomax undercategory*** is the fiber

$$\begin{tikzcd} \mathsf{sd}(\mathsf{P})_{\varphi/\mathsf{isomax}} \arrow[r, hook, "\mathsf{f.f.}"] \arrow[dd] & \mathsf{sd}(\mathsf{P})_{\varphi/} \arrow[d, "t"] \\ & \mathsf{sd}(\mathsf{P}) \arrow[d, "\mathsf{max}"] \\ \{\mathsf{max}(\varphi)\} \arrow[r, hook, "\mathsf{f.f.}"'] & \mathsf{P} \end{tikzcd} ,$$

i.e. the poset of isomax morphisms in $\mathsf{sd}(\mathsf{P})$ with source φ.

Observation A.6.3. There is a factorization system on $\mathbf{\Delta}$, whose left factor consists of isomax morphisms and whose right factor consists of isomin morphisms that are moreover consecutive inclusions: it takes a morphism $[m] \xrightarrow{\alpha} [n]$ in $\mathbf{\Delta}$ to the factorization

$$\begin{tikzcd} {[m]} \arrow[rr, "\alpha"] \arrow[dr, dashed, "\alpha_L"'] & & {[n]} \\ & {[n]_{/\alpha(m)}} \arrow[ur, dashed, "\alpha_R"'] & \end{tikzcd} .$$

It lifts to a factorization system on $\mathbf{\Delta}_{/\mathsf{P}}$, which restricts to a factorization system on $\mathsf{sd}(\mathsf{P}) \subseteq \mathbf{\Delta}_{/\mathsf{P}}$.

Notation A.6.4. Given a locally cocartesian fibration $\mathcal{E} \downarrow \mathsf{P}$ and an object $([m] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$, we write

$$\mathcal{E}_\varphi : \mathcal{E}_{\varphi(0)} \xrightarrow{\mathcal{E}_{\varphi(\{0<1\})}} \mathcal{E}_{\varphi(1)} \xrightarrow{\mathcal{E}_{\varphi(\{1<2\})}} \cdots \xrightarrow{\mathcal{E}_{\varphi(\{(m-1)<m\})}} \mathcal{E}_{\varphi(m)}$$

for the composite of locally cocartesian monodromy functors.

Lemma A.6.5. *There is a commutative diagram*

$$\begin{tikzcd} \mathsf{LMod}_{\mathsf{l.lax.P}} \arrow[rr, "\lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}"] \arrow[dr, dashed, "\mathfrak{S}"'] & & \mathsf{Cat} \\ & \mathsf{LMod}_{\mathsf{sd}(\mathsf{P})} \arrow[ur, "\lim_{\mathsf{sd}(\mathsf{P})}"'] & \end{tikzcd} .$$

Given a left-lax left P*-module* $(\mathcal{E} \downarrow \mathsf{P}) \in \mathsf{LMod}_{\mathsf{l.lax.P}}$*, the left* $\mathsf{sd}(\mathsf{P})$*-module* $(\mathfrak{S}(\mathcal{E}) \downarrow \mathsf{sd}(\mathsf{P})) \in \mathsf{LMod}_{\mathsf{sd}(\mathsf{P})}$ *has the following properties.*

- *Its fiber over an object* $([m] \xrightarrow{\varphi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$ *is the* ∞*-category*

$$\mathfrak{S}(\mathcal{E})_\varphi := \mathsf{Fun}\left(\mathsf{sd}(\mathsf{P})_{\varphi/\mathsf{isomax}}, \mathcal{E}_{\mathsf{max}(\varphi)}\right) .$$

- *Over a morphism*

$$\begin{tikzcd} {[m]} \arrow[rr, hook, "\alpha"] \arrow[dr, "\varphi"'] & & {[n]} \arrow[dl, "\psi"] \\ & \mathsf{P} & \end{tikzcd} \tag{A.6.1}$$

in $\mathsf{sd}(\mathsf{P})$*, its cocartesian monodromy functor*

$$\mathsf{Fun}\left(\mathsf{sd}(\mathsf{P})_{\varphi/\mathsf{isomax}}, \mathcal{E}_{\mathsf{max}(\varphi)}\right) =: \mathfrak{S}(\mathcal{E})_\varphi \xrightarrow{\mathfrak{S}(\mathcal{E})_\alpha} \mathfrak{S}(\mathcal{E})_\psi := \mathsf{Fun}\left(\mathsf{sd}(\mathsf{P})_{\psi/\mathsf{isomax}}, \mathcal{E}_{\mathsf{max}(\psi)}\right) \tag{A.6.2}$$

evaluates on a functor

$$\mathsf{sd}(\mathsf{P})_{\varphi/\mathsf{isomax}} \xrightarrow{F} \mathcal{E}_{\mathsf{max}(\varphi)} \tag{A.6.3}$$

as a functor that evaluates as

$$\begin{array}{ccc} \mathsf{sd}(\mathsf{P})_{\psi/\mathsf{isomax}} & \xrightarrow{\mathfrak{S}(\mathcal{E})_\alpha(F)} & \mathcal{E}_{\mathsf{max}(\psi)} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ (\psi \xrightarrow{\beta} \omega) & \longmapsto & \mathcal{E}_{\omega'}(F((\beta\alpha)_L)) \end{array} ,$$

where writing $([k] \xrightarrow{\omega} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$ *we denote by*

$$\omega' := \left([k]_{\mathsf{max}((\beta\alpha)_R)/} \hookrightarrow [k] \overset{\omega}{\hookrightarrow} \mathsf{P}\right) \in \mathsf{sd}(\mathsf{P})$$

the restriction of ω *to the undercategory of* $\mathsf{max}((\beta\alpha)_R) \in [k]$.

Example A.6.6. Suppose that $\mathsf{P} = [1]$, so that

$$\mathsf{sd}(\mathsf{P}) = \mathsf{sd}([1]) = \left(\begin{tikzcd} & 1 \arrow[d] \\ 0 \arrow[r] & 01 \end{tikzcd}\right) .$$

Given a (locally) cocartesian fibration $\mathcal{E} \downarrow [1]$ classified by a diagram $\mathcal{E}_0 \xrightarrow{F} \mathcal{E}_1$, the functor $\mathsf{sd}([1]) \xrightarrow{\mathfrak{S}(\mathcal{E})} \mathsf{Cat}$ selects the diagram

$$\begin{tikzcd} & \mathsf{Fun}([1], \mathcal{E}_1) \arrow[d, "t"] \\ \mathcal{E}_0 \arrow[r, "F"'] & \mathcal{E}_1 \end{tikzcd} ,$$

whose limit is indeed

$$\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax}.[1]}(\mathcal{E}) \simeq \mathsf{lim}^{\mathsf{r.lax}}_{[1]}(\mathcal{E}) := \Gamma\left(\left(\begin{array}{c} \mathcal{E} \\ \downarrow \\ [1] \end{array}\right)^{\overset{\mathsf{cocart}}{\vee}}\right) .$$

Example A.6.7. Suppose that $\mathsf{P} = [2]$, so that $\mathsf{sd}(\mathsf{P}) = \mathsf{sd}([2])$ is as depicted in diagram (A.4.5) of Figure 9. Given a locally cocartesian fibration $\mathcal{E} \downarrow [2]$ selecting a lax-commutative triangle

$$\begin{tikzcd} & \mathcal{E}_1 \arrow[dr, "G"] & \\ \mathcal{E}_0 \arrow[ur, "F"] \arrow[rr, "H"'] & \arrow[u, phantom, "\eta\Uparrow"] & \mathcal{E}_2 \end{tikzcd} ,$$

the functor $\mathsf{sd}([2]) \xrightarrow{\mathfrak{S}(\mathcal{E})} \mathsf{Cat}$ selects the diagram

$$
\begin{tikzcd}
 & \mathsf{Fun}([1] \times [1], \mathcal{E}_2) \arrow[rr, "(\mathsf{id}, \mathsf{const}_1)^*"] \arrow[dd, "(\mathsf{const}_1, \mathsf{id})^*"'] & & \mathsf{Fun}([1], \mathcal{E}_2) \arrow[dd, "t"] \\
 & & \mathsf{Fun}([1], \mathcal{E}_1) \arrow[ur, "\mathcal{G}"] \arrow[dd, "t"'] & \\
 & \mathsf{Fun}([1], \mathcal{E}_2) \arrow[rr, "t"'] & & \mathcal{E}_2 \\
\mathcal{E}_0 \arrow[ur, "\eta"] \arrow[rr, "F"'] & & \mathcal{E}_1 \arrow[ur, "\mathcal{G}"'] &
\end{tikzcd} ,
$$

whose limit is indeed $\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.}[2]}(\mathcal{E})$ (as described in Example A.5.3(1)).

Remark A.6.8. By construction, for any inclusion $\mathsf{D} \hookrightarrow \mathsf{P}$ of a down-closed subset we have a commutative square

$$
\begin{tikzcd}
\mathsf{LMod}_{\mathsf{l.lax.P}} \arrow[r, "\mathfrak{S}"] \arrow[d] & \mathsf{LMod}_{\mathsf{sd(P)}} \arrow[d] \\
\mathsf{LMod}_{\mathsf{l.lax.D}} \arrow[r, "\mathfrak{S}"'] & \mathsf{LMod}_{\mathsf{sd(D)}}
\end{tikzcd}
$$

(in which the horizontal functors are those of Lemma A.6.5 and the vertical functors are restriction). For more general functors between posets, the functoriality of the functor $\mathfrak{S}$ is more subtle.

Remark A.6.9. Given a left-lax left P-module $(\mathcal{E} \downarrow \mathsf{P}) \in \mathsf{LMod}_{\mathsf{l.lax.P}}$, it is a rather intricate matter to give a complete description of the cocartesian monodromy functors in the left $\mathsf{sd(P)}$-module $(\mathfrak{S}(\mathcal{E}) \downarrow \mathsf{sd(P)}) \in \mathsf{LMod}_{\mathsf{sd(P)}}$. Such a description can be extracted from the proof of Lemma A.6.5, but we omit it since it is not necessary for our purposes here.

Notation A.6.10. For any $p, q \in \mathsf{P}$, we write

$$
\begin{tikzcd}
\mathsf{sd(P)}^{|p} \arrow[r] \arrow[d] & \mathsf{sd(P)} \arrow[d, "\mathsf{min}"] \\
\mathsf{pt} \arrow[r, "p^\circ"'] & \mathsf{P}^{\mathsf{op}}
\end{tikzcd} ,
\qquad
\begin{tikzcd}
\mathsf{sd(P)}_{|q} \arrow[r] \arrow[d] & \mathsf{sd(P)} \arrow[d, "\mathsf{max}"] \\
\mathsf{pt} \arrow[r, "q"'] & \mathsf{P}
\end{tikzcd} ,
\qquad \text{and} \qquad
\begin{tikzcd}
\mathsf{sd(P)}^{|p}_{|q} \arrow[r] \arrow[d] & \mathsf{sd(P)} \arrow[d, "(\mathsf{min}, \mathsf{max})"] \\
\mathsf{pt} \arrow[r, "(p^\circ, q)"'] & \mathsf{P}^{\mathsf{op}} \times \mathsf{P}
\end{tikzcd}
$$

for the indicated pullbacks.

Remark A.6.11. Notation A.6.10 is chosen so as to be suggestive e.g. of the pullback

$$
\begin{tikzcd}
\mathsf{TwAr}(\mathcal{C})^{|c}_{|d} \arrow[r] \arrow[d] & \mathsf{TwAr}(\mathcal{C}) \arrow[d, "(s,t)"] \\
\mathsf{pt} \arrow[r, "(c^\circ, d)"'] & \mathcal{C}^{\mathsf{op}} \times \mathcal{C}
\end{tikzcd}
$$

(for $c, d \in \mathcal{C} \in \mathsf{Cat}$); indeed, the locally cocartesian fibration $\mathsf{sd}(\mathsf{P}) \xrightarrow{(\min,\max)} \mathsf{P}^{\mathsf{op}} \times \mathsf{P}$ may be thought of as the unstraightening of the composite

$$\mathsf{P}^{\mathsf{op}} \times \mathsf{P} \xrightarrow{\mathsf{l.lax}} \mathsf{l.lax}(\mathsf{P}^{\mathsf{op}}) \times \mathsf{l.lax}(\mathsf{P}) \simeq \mathsf{l.lax}(\mathsf{P})^{1\mathsf{op}} \times \mathsf{l.lax}(\mathsf{P}) \xrightarrow{\underline{\mathsf{hom}}_{\mathsf{l.lax}(\mathsf{P})}} \mathsf{Cat} \quad .$$

Observation A.6.12. For any $p \in \mathsf{P}$, the composite functor

$$\max : \mathsf{sd}(\mathsf{P})^{|p} \longrightarrow \mathsf{sd}(\mathsf{P}) \xrightarrow{\max} \mathsf{P}$$

is a locally cocartesian fibration, whose locally cocartesian morphisms are precisely those that map to locally cocartesian morphisms in the locally cocartesian fibration $\mathsf{sd}(\mathsf{P}) \xrightarrow{\max} \mathsf{P}$.

Lemma A.6.13. *Fix any $\mathcal{C} \in \mathsf{Cat}$ and $p \in \mathsf{P}$.*

(1) The composite functor

$$\mathcal{C} \times \mathsf{sd}(\mathsf{P})^{|p} \xrightarrow{\mathsf{pr}} \mathsf{sd}(\mathsf{P})^{|p} \xrightarrow{\max} \mathsf{P}$$

is a locally cocartesian fibration, whose locally cocartesian morphisms are those that project to equivalences in $\mathcal{C}$ and to locally cocartesian morphisms in $\mathsf{sd}(\mathsf{P})^{|p}$ with respect to the locally cocartesian fibration $\mathsf{sd}(\mathsf{P})^{|p} \xrightarrow{\max} \mathsf{P}$.

(2) The morphism

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "{\left(\mathsf{id}_{\mathcal{C}}, \mathsf{const}_{([0] \xrightarrow{p} \mathsf{P})}\right)}"] \arrow[dr, "\mathsf{const}_p"'] & & \mathcal{C} \times \mathsf{sd}(\mathsf{P})^{|p} \arrow[dl, "\max \circ \mathsf{pr}"] \\
& \mathsf{P} &
\end{tikzcd}$$

in $\mathsf{Cat}_{/\mathsf{P}}$ witnesses its target as the free locally cocartesian fibration on its source: for any object $(\mathcal{E} \downarrow \mathsf{P}) \in \mathsf{loc.coCart}_{\mathsf{P}}$, restriction defines an equivalence

$$\underline{\mathsf{hom}}_{\mathsf{Cat}_{/\mathsf{P}}}(\mathcal{C}, \mathcal{E}) \xleftarrow{\sim} \underline{\mathsf{hom}}_{\mathsf{loc.coCart}_{\mathsf{P}}}(\mathcal{C} \times \mathsf{sd}(\mathsf{P})^{|p}, \mathcal{E}) \ .$$

Observation A.6.14. Fix an ∞-category $\mathcal{B} \in \mathsf{Cat}$, and write

$$\mathcal{B} \overset{i}{\hookrightarrow} \mathcal{B}^{\triangleright} \xrightarrow{p} \mathsf{pt}^{\triangleright} = [1]$$

for the evident functors. Then, we have a canonical pullback square

$$\begin{tikzcd}
\mathsf{loc.coCart}_{\mathcal{B}^{\triangleright}} \arrow[r, "i^*"] \arrow[d, "p \circ (-)"'] & \mathsf{loc.coCart}_{\mathcal{B}} \arrow[d, "\mathsf{fgt}"] \\
\mathsf{coCart}_{[1]} \arrow[r, "0^*"'] & \mathsf{Cat}
\end{tikzcd} \tag{A.6.4}$$

among $(\infty, 2)$-categories. (In particular, a locally cocartesian fibration $\mathcal{E} \to \mathcal{B}^{\triangleright}$ is equivalent data to a locally cocartesian fibration $\mathcal{E}_{|\mathcal{B}} \to \mathcal{B}$ along with a functor $\mathcal{E}_{|\mathcal{B}} \to \mathcal{E}_{\infty}$.) To see this, observe first the right adjoint

$$\begin{tikzcd}
\mathsf{Cat}_{/[1]} \arrow[r, shift left, "0^*"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, shift left, dashed, "(-)^{\triangleright}"]
\end{tikzcd} \quad .$$

This implies that we have a pullback square

$$\begin{tikzcd}
\mathsf{Cat}_{/\mathcal{B}^{\triangleright}} \arrow[r, "i^*"] \arrow[d, "p \circ (-)"'] & \mathsf{Cat}_{/\mathcal{B}} \arrow[d, "\mathsf{fgt}"] \\
\mathsf{Cat}_{/[1]} \arrow[r, "0^*"'] & \mathsf{Cat}
\end{tikzcd}$$

among $(\infty, 2)$-categories, which it is easy to check restricts to give the pullback square (A.6.4).

Proof of Lemma A.6.13. Part (1) is clear.

We first prove part (2) in the case that $\mathsf{P} = [n]$ and $p = 0$, by induction on n. The case that $n = 0$ is a tautology, so we may assume that $n \geq 1$. For the inductive step, note the evident equivalence

$$\mathsf{sd}([n-1])^{|0} \times [1] \xrightarrow{\sim} \mathsf{sd}([n])^{|0}$$

in $\mathsf{Cat}_{/[1]}$ that selects the natural transformation carrying an object $([m] \hookrightarrow [n-1]) \in \mathsf{sd}([n-1])^{|0}$ to the morphism

$$\begin{tikzcd}
{[m]} \arrow[rr, hook] \arrow[dr, hook] & & {[m]^{\triangleright}} \arrow[dl, hook] \\
& {[n-1]^{\triangleright}} &
\end{tikzcd}$$

in $\mathsf{sd}([n])^{|0}$. The inductive step then follows by Observation A.6.14.

We now prove part (2) in the general case. We may clearly assume that $p \in \mathsf{P}$ is initial. Now, observe that the colimit

$$\mathsf{colim}_{([n]\downarrow \mathsf{P}) \in \mathsf{sd}(\mathsf{P})^{|p}} [n] \xrightarrow{\sim} \mathsf{P}$$

can be computed in simplicial spaces (not merely complete Segal spaces), so that we may apply Observation A.1.7 to obtain an equivalence

$$\mathsf{loc.coCart}_{\mathsf{P}} \xrightarrow{\sim} \mathsf{lim}_{([n]\downarrow \mathsf{P})^{\circ} \in (\mathsf{sd}(\mathsf{P})^{|p})^{\mathsf{op}}} \mathsf{loc.coCart}_{[n]} \ .$$

Now, observe that for every object $([n] \downarrow \mathsf{P}) \in \mathsf{sd}(\mathsf{P})^{|p}$ we have a pullback diagram

$$\begin{tikzcd}
\mathcal{C} \times \mathsf{sd}([n])^{|0} \arrow[r] \arrow[d, "\mathsf{max} \circ \mathsf{pr}"'] & \mathcal{C} \times \mathsf{sd}(\mathsf{P})^{|p} \arrow[d, "\mathsf{max} \circ \mathsf{pr}"] \\
{[n]} \arrow[r] & \mathsf{P}
\end{tikzcd} .$$

Hence, the claim follows from the case that $\mathsf{P} = [n]$. □

Proof of Lemma A.6.5. For any object $(\mathcal{K} \downarrow \mathsf{sd}(\mathsf{P})) \in \mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})}$, observe that the horizontal composite in the diagram

$$\begin{tikzcd}
\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\mathcal{K}} \arrow[r] \arrow[d] & \mathsf{Ar}(\mathsf{sd}(\mathsf{P})) \arrow[r, "t"] \arrow[d, "s"] & \mathsf{sd}(\mathsf{P}) \arrow[r, "\mathsf{max}"] & \mathsf{P} \\
\mathcal{K} \arrow[r] & \mathsf{sd}(\mathsf{P}) & &
\end{tikzcd}$$

(in which the square is a pullback) is a locally cocartesian fibration: the composite of the first two horizontal functors is a cocartesian fibration (because $\mathsf{Ar}(\mathsf{sd}(\mathsf{P})) \xrightarrow{(s,t)} \mathsf{sd}(\mathsf{P}) \times \mathsf{sd}(\mathsf{P})$ is a bifibration), and the functor $\mathsf{sd}(\mathsf{P}) \xrightarrow{\mathsf{max}} \mathsf{P}$ is a locally cocartesian fibration. Moreover, the locally cocartesian morphisms in $(\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\mathcal{K}} \downarrow \mathsf{P}) \in \mathsf{loc.coCart}_{\mathsf{P}}$ are those that project to an equivalence in $\mathcal{K}$ and to a locally cocartesian morphism in $\mathsf{sd}(\mathsf{P})$ with respect to the locally cocartesian fibration $\mathsf{sd}(\mathsf{P}) \xrightarrow{\mathsf{max}} \mathsf{P}$. It follows that we have a factorization

$$\begin{tikzcd}
\mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})} \arrow[r, "\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\bullet}"] \arrow[dr, dashed] & \mathsf{Cat}_{/\mathsf{P}} \\
& \mathsf{loc.coCart}_{\mathsf{P}} \arrow[u, hook]
\end{tikzcd} . \tag{A.6.5}$$

Now, consider the functor

$$\mathsf{LMod}_{\mathsf{l.lax.P}} := \mathsf{loc.coCart}_{\mathsf{P}} \xrightarrow{\mathfrak{S}} \mathsf{Fun}\left(\left(\mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})}\right)^{\mathsf{op}}, \mathcal{S}\right)$$

defined by the formula

$$\begin{array}{ccc} \left(\mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})}\right)^{\mathsf{op}} & \xrightarrow{\qquad \mathfrak{S}(\mathcal{E}) \qquad} & \mathcal{S} \\ \Psi & & \Psi \\ (\mathcal{K} \longrightarrow \mathsf{sd}(\mathsf{P})) & \longmapsto & \mathsf{hom}_{\mathsf{loc.coCart}_{\mathsf{P}}}\left(\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\mathcal{K}}, \mathcal{E}\right) \end{array}$$

We observe that the factorization in diagram (A.6.5) preserves colimits: indeed, it is the left adjoint in the composite adjunction

$$\begin{tikzcd} \mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})} \arrow[r, shift left, "\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\bullet}"] \arrow[r, phantom, "\perp"] & \mathsf{coCart}_{/\mathsf{sd}(\mathsf{P})} \arrow[l, shift left] \arrow[r, shift left, "\mathsf{max}\circ -"] \arrow[r, phantom, "\perp"] & \mathsf{loc.coCart}_{/\mathsf{P}} \arrow[l, shift left] \end{tikzcd} .^{151}$$

So by the presentability of $\mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})}$, for any $\mathcal{E} \in \mathsf{loc.coCart}_{\mathsf{P}}$ the presheaf $\mathfrak{S}(\mathcal{E}) \in \mathsf{Fun}\left(\left(\mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})}\right)^{\mathsf{op}}, \mathcal{S}\right)$ is representable. Therefore, we have the uppermost factorization in the diagram

$$\begin{tikzcd} \mathsf{LMod}_{\mathsf{l.lax.P}} := \mathsf{loc.coCart}_{\mathsf{P}} \arrow[r, "\mathfrak{S}"] \arrow[dr, dashed] \arrow[ddr, dashed] \arrow[dddr, dashed] & \mathsf{Fun}\left(\left(\mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})}\right)^{\mathsf{op}}, \mathcal{S}\right) \\ & \mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})} \arrow[u, hook, "\mathsf{f.f.}"'] \\ & \mathsf{Cat}_{\mathsf{cocart}/\mathsf{sd}(\mathsf{P})} =: \mathsf{LMod}^{\mathsf{l.lax}}_{\mathsf{sd}(\mathsf{P})} \arrow[u, hook, "\mathsf{f.f.}"'] \\ & \mathsf{coCart}_{\mathsf{sd}(\mathsf{P})} =: \mathsf{LMod}_{\mathsf{sd}(\mathsf{P})} \arrow[u, two heads] \end{tikzcd} . \tag{A.6.6}$$

In order to complete the proof, we will demonstrate the two further factorizations indicated in diagram (A.6.6), deducing the asserted properties of $(\mathfrak{S}(\mathcal{E}) \downarrow \mathsf{sd}(\mathsf{P})) \in \mathsf{LMod}_{\mathsf{sd}(\mathsf{P})}$ along the way, and then prove a natural equivalence $\mathsf{lim}^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \simeq \mathsf{lim}_{\mathsf{sd}(\mathsf{P})}(\mathfrak{S}(\mathcal{E}))$.

For any object $\varphi \in \mathsf{sd}(\mathsf{P})$, observe the evident identification

$$\begin{tikzcd} \mathsf{sd}(\mathsf{P})_{\varphi/} \arrow[rr, "(\varphi \xrightarrow{\alpha} \psi) \longmapsto (\alpha_L, \psi')", "\sim"'] \arrow[dr, "t"'] & & \mathsf{sd}(\mathsf{P})_{\varphi/\mathsf{isomax}} \times \mathsf{sd}(\mathsf{P})^{|\mathsf{max}(\varphi)} \arrow[dl] \\ & \mathsf{sd}(\mathsf{P}) & \end{tikzcd}$$

in $\mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})}$, where

- (as in the statement of the result) writing $([k] \xrightarrow{\psi} \mathsf{P}) \in \mathsf{sd}(\mathsf{P})$ we denote by
$$\psi' := \left([k]_{\mathsf{max}(\alpha_R)/} \hookrightarrow [k] \xhookrightarrow{\psi} \mathsf{P} \right) \in \mathsf{sd}(\mathsf{P})$$
the restriction of ψ to the undercategory of $\mathsf{max}(\alpha_R) \in [k]$, and
- the lower right morphism is given by concatenation.

By Lemma A.6.13, we may then identify the fiber of $(\mathfrak{S}(\mathcal{E}) \downarrow \mathsf{sd}(\mathsf{P})) \in \mathsf{Cat}_{/\mathsf{sd}(\mathsf{P})}$ over $\varphi \in \mathsf{sd}(\mathsf{P})$ as

$$\mathfrak{S}(\mathcal{E})_{\varphi} := \underline{\mathsf{hom}}_{\mathsf{loc.coCart}_{\mathsf{P}}}\left(\mathsf{sd}(\mathsf{P})_{\varphi/}, \mathcal{E}\right) \xrightarrow{\sim} \underline{\mathsf{hom}}_{\mathsf{Cat}_{/\mathsf{P}}}\left(\mathsf{sd}(\mathsf{P})_{\varphi/\mathsf{isomax}}, \mathcal{E}\right) \simeq \mathsf{Fun}\left(\mathsf{sd}(\mathsf{P})_{\varphi/\mathsf{isomax}}, \mathcal{E}_{\mathsf{max}(\varphi)}\right) , \tag{A.6.7}$$

as asserted.

[151] For any locally cocartesian fibration $\mathcal{F} \xrightarrow{\pi} \mathcal{B}$, the functor $\mathsf{coCart}_{\mathcal{F}} \xrightarrow{\pi\circ -} \mathsf{loc.coCart}_{\mathcal{B}}$ preserves colimits and therefore admits a right adjoint; this follows from the fact that the forgetful functor $\mathsf{coCart}_{\mathcal{F}} \to \mathsf{Cat}_{/\mathcal{F}}$ preserves colimits and that any cocone in $\mathsf{loc.coCart}_{\mathcal{B}} \subseteq \mathsf{Cat}_{/\mathcal{B}}$ that is a colimit diagram in $\mathsf{Cat}_{/\mathcal{B}}$ is also a colimit diagram in $\mathsf{loc.coCart}_{\mathcal{B}}$.

Consider the diagram

$$\begin{tikzcd}
& \mathsf{sd}(\mathsf{P})_{\psi/} \simeq \mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\{1\}} \arrow[dr, hook, "\text{f.f.}"] \arrow[dd] & & & & \\
\mathsf{sd}(\mathsf{P})_{\varphi/} \simeq \mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\{0\}} \arrow[rr, hook, "\text{f.f.}"] \arrow[dd] & & \mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|[1]} \arrow[r] \arrow[dd] & \mathsf{Ar}(\mathsf{sd}(\mathsf{P})) \arrow[r, "t"] \arrow[dd, "s"] & \mathsf{sd}(\mathsf{P}) \arrow[r, "\mathsf{max}"] & \mathsf{P} \\
& \{1\} \arrow[dr, hook, "\text{f.f.}"] & & & & \\
\{0\} \arrow[rr, hook, "\text{f.f.}"'] & & {[1]} \arrow[r, "(\text{A.6.1})"'] & \mathsf{sd}(\mathsf{P}) & &
\end{tikzcd} \quad , \tag{A.6.8}$$

in which all three quadrilaterals are pullbacks. All four vertical functors in diagram (A.6.8) are cartesian fibrations; in particular, the cartesian fibration $\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|[1]} \downarrow [1]$ is classified by the diagram

$$\mathsf{sd}(\mathsf{P})_{\varphi/} \xleftarrow{\alpha^*} \mathsf{sd}(\mathsf{P})_{\psi/} \ .$$

Now, observe the existence of a right adjoint

$$\begin{tikzcd}
\mathsf{sd}(\mathsf{P})_{\varphi/} \simeq \mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\{0\}} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|[1]} \arrow[l, dashed, shift left]
\end{tikzcd}$$

in $\mathsf{loc.coCart}_{\mathsf{P}}$. This determines an adjunction

$$\begin{tikzcd}
\Gamma_{\{0\}}(\mathfrak{S}(\mathcal{E})) := \underline{\mathsf{hom}}_{\mathsf{loc.coCart}_{\mathsf{P}}}(\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|\{0\}}, \mathcal{E}) \arrow[r, hook, dashed, shift left] \arrow[r, phantom, "\perp"] & \underline{\mathsf{hom}}_{\mathsf{loc.coCart}_{\mathsf{P}}}(\mathsf{Ar}(\mathsf{sd}(\mathsf{P}))^{|[1]}, \mathcal{E}) =: \Gamma_{[1]}(\mathfrak{S}(\mathcal{E})) \arrow[l, shift left]
\end{tikzcd} \quad ,$$

which implies that the functor $\mathfrak{S}(\mathcal{E}) \downarrow \mathsf{sd}(\mathsf{P})$ is a locally cocartesian fibration, whose locally cocartesian monodromy functor over the morphism (A.6.1) in $\mathsf{sd}(\mathsf{P})$ is the functor

$$\underline{\mathsf{hom}}_{\mathsf{loc.coCart}_{\mathsf{P}}}\left(\mathsf{sd}(\mathsf{P})_{\varphi/}, \mathcal{E}\right) \xrightarrow{(\alpha^*)^*} \underline{\mathsf{hom}}_{\mathsf{loc.coCart}_{\mathsf{P}}}\left(\mathsf{sd}(\mathsf{P})_{\psi/}, \mathcal{E}\right) \ .$$

This description immediately implies that the functor $\mathfrak{S}(\mathcal{E}) \downarrow \mathsf{sd}(\mathsf{P})$ is a cocartesian fibration, which gives the middle factorization in diagram (A.6.6); from here, the lower factorization in diagram (A.6.6) is immediate as well. On the other hand, given a functor (A.6.3) and writing $\widetilde{F} \in \underline{\mathsf{hom}}_{\mathsf{loc.coCart}_{\mathsf{P}}}\left(\mathsf{sd}(\mathsf{P})_{\varphi/}, \mathcal{E}\right)$ for its corresponding object under the equivalence (A.6.7), we see that its image under the cocartesian monodromy functor (A.6.2) is given by the factorization

$$\begin{tikzcd}
& \mathsf{sd}(\mathsf{P})_{\varphi/\mathsf{isomax}} \arrow[r, "F"] \arrow[d, hook] & \mathcal{E}_{\mathsf{max}(\varphi)} \arrow[d, hook] \\
\mathsf{sd}(\mathsf{P})_{\psi/} \arrow[r, "\alpha^*"] & \mathsf{sd}(\mathsf{P})_{\varphi/} \arrow[r, "\widetilde{F}"'] & \mathcal{E} \\
\mathsf{sd}(\mathsf{P})_{\psi/\mathsf{isomax}} \arrow[u, hook] \arrow[rr, dashed] & & \mathcal{E}_{\mathsf{max}(\psi)} \arrow[u, hook]
\end{tikzcd} \quad ,$$

which implies the asserted description of the cocartesian monodromy functor (A.6.2).

Lastly, observe the reflective localization adjunction

$$\begin{tikzcd}
\mathsf{Ar}(\mathsf{sd}(\mathsf{P})) \arrow[rr, shift left, "t"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\mathsf{max} \circ t"'] & & \mathsf{sd}(\mathsf{P}) \arrow[ll, hook, dashed, shift left] \arrow[dl, "\mathsf{max}"] \\
& \mathsf{P} &
\end{tikzcd}$$

in $\mathsf{loc.coCart_P}$. From this, we immediately deduce a commutative diagram

$$\begin{tikzcd} \Gamma_{\mathsf{sd(P)}}(\mathfrak{S}(\mathcal{E})) \arrow[r, leftrightarrow, "\sim"] & \underline{\mathsf{hom}}_{\mathsf{loc.coCart_P}}(\mathsf{Ar}(\mathsf{sd(P)}), \mathcal{E}) \\ \Gamma^{\mathsf{cocart}}_{\mathsf{sd(P)}}(\mathfrak{S}(\mathcal{E})) \arrow[u, phantom, "\cup\!\!|"] \arrow[r, leftrightarrow, "\sim"] & \underline{\mathsf{hom}}_{\mathsf{loc.coCart_P}}(\mathsf{sd(P)}, \mathcal{E}) \arrow[u, phantom, "\cup\!\!|"] \\ \lim_{\mathsf{sd(P)}}(\mathfrak{S}(\mathcal{E})) \arrow[u, phantom, "\wr\wr"] \arrow[r, leftrightarrow, "\sim"'] & \lim^{\mathsf{r.lax}}_{\mathsf{l.lax.P}}(\mathcal{E}) \arrow[u, phantom, "\wr\wr"] \end{tikzcd}$$

that is natural in $\mathcal{E}$. □

Appendix B. Some $(\infty, 2)$-category theory

In this section, we establish some aspects of $(\infty, 2)$-category theory. (In the main body of the paper, we only refer to applications thereof that are recorded in §A.)

A large part of §§B.1-B.5 is adapted from [GR17, Appendix A]. As explained in [GR17, Chapter 10, §0.4], some of the results there rely on results whose proofs do not appear in the literature at the time of writing. Here, we give a logically complete account of the material that we use; in particular, this section does not logically depend on [GR17] in any way. Nevertheless, we provide references where appropriate.

This section is organized as follows.

§B.1: We introduce some basic notions in $(\infty, 2)$-category theory.

§B.2: We define various notions of fibrations among $(\infty, 2)$-categories.

§B.3: We give (a lax version of) un/straightening for $(\infty, 2)$-categories.

§B.4: We study parametrized versions of un/straightening for $(\infty, 2)$-categories.

§B.5: We study adjunctions in $(\infty, 2)$-categories, and prove parametrized versions of the mate correspondence.

§B.6: We define lax limits in Cat over $(\infty, 2)$-categories, and give an alternative $(\infty, 1)$-categorical description in the case that the base is the left-laxification of an $(\infty, 1)$-category.

B.1. **Basic notions in $(\infty, 2)$-category theory.** In this subsection, we discuss (strict and) lax versions of functors and natural transformations among $(\infty, 2)$-categories. Relatedly, we discuss various laxifications of an $(\infty, 2)$-category, and we give an explicit identification in one important case (Proposition B.1.24). We also introduce the class of thin $(\infty, 2)$-categories (Definition B.1.13), for which homotopy-coherence data is vacuous (see Observation B.1.14) – analogously to the class of posets among $(\infty, 1)$-categories.

Definition B.1.1 ([Bar05])**.** An $(\infty, 2)$-***category*** is a complete Segal ∞-category whose 0^{th} ∞-category is an ∞-groupoid.[152] These assemble into a full subcategory

$$\iota_1 2\mathsf{Cat} \subset \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat}) \ ,$$

whose morphisms we refer to as ***functors*** (or occasionally ***strict functors*** in order to contrast with the notions introduced in Definition B.1.7). We consider ∞-categories (which we may refer to

[152] Said differently, it is a two-fold complete Segal space.

as $(\infty,1)$-categories for emphasis) as forming a full subcategory of $\iota_1 2\mathsf{Cat}$ according to the pullback

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, hook, "\text{f.f.}"] \arrow[d, hook, "\text{f.f.}"'] & \iota_1 2\mathsf{Cat} \arrow[d, hook, "\text{f.f.}"] \\ \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S}) \arrow[r, hook, "\text{f.f.}"'] & \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat}) \end{tikzcd} .$$

We refer to an $(\infty,2)$-category as a ***2-category*** if its hom-∞-categories are ordinary categories.[153]

Notation B.1.2. Given an $(\infty,2)$-category $\mathcal{C} \in \iota_1 2\mathsf{Cat}$ and objects $c, c' \in \mathcal{C}$, we write $\mathsf{hom}_{\mathcal{C}}(c,c') \in \mathsf{Cat}$ for the ∞-category of morphisms from c to c'.

Notation B.1.3. The ∞-category $\iota_1 2\mathsf{Cat}$ is cartesian closed [Rez10, BSP21], and we denote its internal hom by $\mathsf{Fun}(-,-)$. By [GH15, Hau15], it follows that $\iota_1 2\mathsf{Cat}$ is the underlying $(\infty,1)$-category of an $(\infty,3)$-category. We write $2\mathsf{Cat}$ for its underlying $(\infty,2)$-category.[154]

Definition B.1.4. Given an $(\infty,2)$-category $\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathcal{C}} \mathsf{Cat}$, its ***1-opposite*** and ***2-opposite*** are the $(\infty,2)$-categories

$$\mathcal{C}^{1\mathsf{op}} : \mathbf{\Delta}^{\mathsf{op}} \xrightarrow[\sim]{\mathsf{rev}} \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathcal{C}} \mathsf{Cat} \qquad \text{and} \qquad \mathcal{C}^{2\mathsf{op}} : \mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathcal{C}} \mathsf{Cat} \xrightarrow[\sim]{(-)^{\mathsf{op}}} \mathsf{Cat}$$

given respectively by pre- and postcomposing with the indicated involutions.[155] These two operations define commuting involutions of $\iota_1 2\mathsf{Cat}$, and we write

$$(-)^{1\&2\mathsf{op}} := ((-)^{1\mathsf{op}})^{2\mathsf{op}} \simeq ((-)^{2\mathsf{op}})^{1\mathsf{op}}$$

for their composite.

Notation B.1.5. We denote by

$$(-)^{\oint} : \iota_1 2\mathsf{Cat} \xhookrightarrow{\text{f.f.}} \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Cat}) \xrightarrow{\sim} \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$$

the composite functor carrying an $(\infty,2)$-category to its corresponding cocartesian fibration over $\mathbf{\Delta}^{\mathsf{op}}$.

Definition B.1.6. A morphism in $\mathbf{\Delta}$ is called ***convex*** if its image is convex.[156] A morphism in $\mathbf{\Delta}$ is called ***inert*** if it is convex and injective (or equivalently conservative). We use the same terms for corresponding morphisms in $\mathbf{\Delta}^{\mathsf{op}}$.

Definition B.1.7 ([GR17, Chapter 10, §3.1.3])**.** Given $(\infty,2)$-categories $\mathcal{C}, \mathcal{D} \in \iota_1 2\mathsf{Cat}$, a ***non-unital right-lax functor*** from $\mathcal{C}$ to $\mathcal{D}$ is a morphism

$$\begin{tikzcd} \mathcal{C}^{\oint} \arrow[rr] \arrow[dr] & & \mathcal{D}^{\oint} \arrow[dl] \\ & \mathbf{\Delta}^{\mathsf{op}} & \end{tikzcd}$$

in $\mathsf{Cat}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}}$ that preserves inert-cocartesian morphisms. It is called a (***unital***) ***right-lax functor*** if it preserves convex-cocartesian morphisms. These respectively define the morphisms in ∞-categories that we denote by

$$\iota_1 2\mathsf{Cat}_{\mathsf{n.u.r.lax}} \qquad \text{and} \qquad \iota_1 2\mathsf{Cat}_{\mathsf{r.lax}} ,$$

so that we have monomorphisms

$$\iota_1 2\mathsf{Cat} \hookrightarrow \iota_1 2\mathsf{Cat}_{\mathsf{r.lax}} \hookrightarrow \iota_1 2\mathsf{Cat}_{\mathsf{n.u.r.lax}} .$$

[153] Note that these are most naturally modeled by the classical notion of a bicategory.

[154] That is, we do not make any use of the full $(\infty,3)$-category of $(\infty,2)$-categories. In particular, we only ever consider $(\infty,2)$-categorical laxness in $2\mathsf{Cat}$, i.e. laxness in 1-morphisms but not 2-morphisms.

[155] Here, rev denotes the involution given by reversing linear orders.

[156] In [GR17], such morphisms are referred to as "idle".

We define a (***non-unital*** or ***unital***) ***left-lax functor*** from $\mathcal{C}$ to $\mathcal{D}$ to be a (respectively non-unital or unital) right-lax functor from $\mathcal{C}^{1\mathsf{op}}$ to $\mathcal{D}^{1\mathsf{op}}$, and we use the evident corresponding notation. Given $(\infty,2)$-categories $\mathcal{C},\mathcal{D} \in 2\mathsf{Cat}$, we write

$$\mathcal{C} \rightsquigarrow \mathcal{D}$$

to denote a (possibly) lax functor (be it non-unital or unital, right- or left-lax).[157]

Remark B.1.8. One can similarly define left-lax functors in terms of cartesian fibrations over $\mathbf{\Delta}$. We systematically privilege right-lax functors in our treatment here (so that for instance we do not introduce a cartesian variant of Notation B.1.5).

Remark B.1.9. Informally, a lax functor is one that only laxly respects composition of 1-morphisms. More specifically, given a pair of composable 1-morphisms $c_0 \xrightarrow{\varphi} c_1 \xrightarrow{\psi} c_2$, a right-lax functor F determines a 2-morphism $F(\psi)\circ F(\varphi) \to F(\psi\circ\varphi)$ while a left-lax functor G determines a 2-morphism $G(\psi\circ\varphi) \to G(\psi)\circ G(\varphi)$. A lax functor is strict precisely when these 2-morphisms are all invertible.

Remark B.1.10. Our primary interest will be in *unital* lax functors. While non-unital lax functors are of independent interest, for our purposes they serve as an auxiliary notion (see Definitions B.1.15 and B.1.19).

Definition B.1.11 ([GR17, Chapter 10, §3.2.7])**.** Fix $(\infty,2)$-categories $\mathcal{C},\mathcal{D} \in 2\mathsf{Cat}$.

(1) A ***right-lax natural transformation*** between right-lax functors from $\mathcal{C}$ to $\mathcal{D}$ is a right-lax functor

$$\mathcal{C} \times [1] \rightsquigarrow \mathcal{D} \tag{B.1.1}$$

that is strict on pairs of composable 1-morphisms of the form $(c_0,0) \xrightarrow{(\varphi,\mathsf{id}_0)} (c_1,0) \xrightarrow{(\mathsf{id}_{c_1},\iota)} (c_1,1)$.[158]

(2) A ***left-lax natural transformation*** between right-lax functors from $\mathcal{C}$ to $\mathcal{D}$ is a right-lax functor (B.1.1) that is strict on pairs of composable 1-morphisms of the form $(c_0,0) \xrightarrow{(\mathsf{id}_{c_0},\iota)} (c_0,1) \xrightarrow{(\varphi,\mathsf{id}_1)} (c_1,1)$.

Dually, a ***right-*** (resp. ***left-***)***lax natural transformation*** between left-lax functors from $\mathcal{C}$ to $\mathcal{D}$ is a right- (resp. left-)lax natural transformation between the corresponding right-lax functors from $\mathcal{C}^{1\mathsf{op}}$ to $\mathcal{D}^{1\mathsf{op}}$.[159] As in Definition B.1.7, given (right- or left-lax) functors F and G, we simply write

$$F \rightsquigarrow G$$

to denote a lax natural transformation between them (regardless of its handedness).

Remark B.1.12. Given left- or right-lax functors F and G from $\mathcal{C}$ to $\mathcal{D}$, a right-lax natural transformation from F to G specifies the data, for each 1-morphism $c_0 \to c_1$ in $\mathcal{C}$, of a lax-commutative square

$$\begin{tikzcd} F(c_0) \arrow[r] \arrow[d] \arrow[dr] & F(c_1) \arrow[d] \arrow[dl, phantom, "\Nearrow" near start] \\ G(c_0) \arrow[r] & G(c_1) \end{tikzcd} \quad =: \quad \begin{tikzcd} F(c_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & F(c_1) \arrow[d] \\ G(c_0) \arrow[r] & G(c_1) \end{tikzcd} \quad := \quad \begin{tikzcd} F(c_0) \arrow[r] \arrow[d] \arrow[dr] & F(c_1) \arrow[d] \\ G(c_0) \arrow[r] \arrow[ur, phantom, "\Nearrow" near start] & G(c_1) \end{tikzcd}$$

in $\mathcal{D}$ – the square on the left (resp. right) applying in the case that F and G are left-lax (resp. right-lax).

[157] Outside of §B, we explicitly label our arrows according to the handedness of the laxness.
[158] That is, its precomposition with the corresponding functor $[2] \to \mathcal{C} \times [1]$ defines a strict functor $[2] \to \mathcal{D}$.
[159] Evidently, these can also be expressed as left-lax functors $\mathcal{C} \times [1] \rightsquigarrow \mathcal{D}$.

Definition B.1.13. An $(\infty,2)$-category is called a ***thin 2-category*** (or simply ***thin***) if its hom-∞-categories lie in $\mathsf{Poset} \subset \mathsf{Cat}$ and its endomorphism ∞-categories are all equivalent to $\mathsf{pt} \in \mathsf{Cat}$.[160]

Observation B.1.14. We collect the following apparent facts about thin 2-categories, which we use without further reference.

(1) Thin 2-categories form a full subcategory of the $(\infty,2)$-category (in fact strict 2-category) of strict 2-categories. Under this identification, non-unital right-lax functors correspond to lax functors, while unital right-lax functors correspond to normal lax functors. Moreover, right-lax natural transformations correspond to lax natural transformations.

(2) Given a thin 2-category $\mathcal{D}$ and an $(\infty,2)$-category $\mathcal{C}$, a (possibly left- or right-lax) functor from $\mathcal{C}$ to $\mathcal{D}$ is uniquely determined by its values on 1-morphisms, i.e. by the morphism of sets

$$\pi_0 \mathsf{hom}_{\iota_1 2\mathsf{Cat}}([1], \mathcal{C}) \longrightarrow \mathsf{hom}_{\iota_1 2\mathsf{Cat}}([1], \mathcal{D}) \ .$$

Hence, given (possibly left- or right-lax) functors from $\mathcal{C}$ to $\mathcal{D}$, a (possibly left- or right-lax) natural transformation between them is uniquely determined by its values on objects, i.e. by the morphism of sets

$$\pi_0 \iota_0 \mathcal{C} \longrightarrow \mathsf{hom}_{\iota_1 2\mathsf{Cat}}([1], \mathcal{D}) \ .$$

(3) Given a functor $\mathcal{C} \to \mathcal{D}$ in $2\mathsf{Cat}$ such that $\mathcal{D}$ is thin, for every object $d \in \mathcal{D}$ the functor $\mathcal{C}_d \to \mathcal{C}$ is fully faithful.

(4) Given a thin 2-category $\mathcal{D}$, the forgetful functor

$$2\mathsf{Cat}_{/\mathcal{D}} \longrightarrow 2\mathsf{Cat}$$

is 1-full (i.e. it is fully faithful on hom-∞-categories).[161]

Definition B.1.15 ([GR17, Chapter 11, §A.1])**.** Given an object $(\mathcal{E} \downarrow \boldsymbol{\Delta}^{\mathsf{op}}) \in \mathsf{coCart}_{\boldsymbol{\Delta}^{\mathsf{op}}}$, we write

$$\begin{tikzcd} \mathsf{F}^{\mathsf{act}}(\mathcal{E}) := \mathcal{E} \underset{\boldsymbol{\Delta}^{\mathsf{op}}}{\times} \mathsf{Ar}^{\mathsf{act}}(\boldsymbol{\Delta}^{\mathsf{op}}) \arrow[r] \arrow[d] & \mathsf{Ar}^{\mathsf{act}}(\boldsymbol{\Delta}^{\mathsf{op}}) \arrow[r, "t"] \arrow[d, "s"] & \boldsymbol{\Delta}^{\mathsf{op}} \\ \mathcal{E} \arrow[r] & \boldsymbol{\Delta}^{\mathsf{op}} & \end{tikzcd} \tag{B.1.2}$$

for the indicated fiber product (where $\mathsf{Ar}^{\mathsf{act}}(\boldsymbol{\Delta}^{\mathsf{op}}) \subset \mathsf{Ar}(\boldsymbol{\Delta}^{\mathsf{op}})$ denotes the full subcategory on the active morphisms). Noting that the functor $\mathsf{Ar}^{\mathsf{act}}(\boldsymbol{\Delta}^{\mathsf{op}}) \xrightarrow{t} \boldsymbol{\Delta}^{\mathsf{op}}$ is a cocartesian fibration, we find that the horizontal composite of diagram (B.1.2) defines a functor

$$\mathsf{coCart}_{\boldsymbol{\Delta}^{\mathsf{op}}} \xrightarrow{\mathsf{F}^{\mathsf{act}}} \mathsf{coCart}_{\boldsymbol{\Delta}^{\mathsf{op}}} \ .$$

In particular, given an $(\infty,2)$-category $\mathcal{C} \in 2\mathsf{Cat}$, it is straightforward to see that $\mathsf{F}^{\mathsf{act}}(\mathcal{C}^{\oint}) \in \mathsf{coCart}_{\boldsymbol{\Delta}^{\mathsf{op}}}$ defines an $(\infty,2)$-category, which we denote by $\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}) \in 2\mathsf{Cat}$ (so that $\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint} \simeq \mathsf{F}^{\mathsf{act}}(\mathcal{C}^{\oint})$) and refer to as its ***non-unital right-laxification***. Altogether, this defines a functor

$$\iota_1 2\mathsf{Cat} \xrightarrow{\mathsf{r.lax}^{\mathsf{n.u.}}(-)} \iota_1 2\mathsf{Cat} \ .$$

[160] This notion is strictly stronger than that of gauntness, which merely requires that all *invertible* k-morphisms are identities. It is also strictly stronger than the requirement that every k-morphism has a contractible space of endomorphisms.

[161] Said differently, given $\mathcal{C}_0, \mathcal{C}_1 \in 2\mathsf{Cat}_{/\mathcal{D}}$ it is merely a condition for a morphism $\mathcal{C}_0 \to \mathcal{C}_1$ in $2\mathsf{Cat}$ to lie in $2\mathsf{Cat}_{/\mathcal{D}}$.

Observation B.1.16. The functor $\mathsf{F}^{\mathsf{act}}$ has the following universal property. First of all, the functors $s, t \in \mathsf{hom}_{\mathsf{Cat}}(\mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}), \mathbf{\Delta}^{\mathsf{op}})$ admit a common section $\mathbf{\Delta}^{\mathsf{op}} \xrightarrow{\mathsf{id}_{(-)}} \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}})$, which induces for any $\mathcal{E} \in \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ a natural morphism

$$\mathcal{E} \longrightarrow \mathsf{F}^{\mathsf{act}}(\mathcal{E})$$

in $\mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$. Then, by [AMGRb, Proposition 2.18], for any $\mathcal{F} \in \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$, restriction therealong defines a monomorphism

$$\mathsf{hom}_{\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}}(\mathsf{F}^{\mathsf{act}}(\mathcal{E}), \mathcal{F}) \hookrightarrow \mathsf{hom}_{\mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}}(\mathcal{E}, \mathcal{F})$$

in $\mathcal{S}$ whose image consists of those morphisms that preserve cocartesian lifts of inert morphisms in $\mathbf{\Delta}^{\mathsf{op}}$.

Observation B.1.17 ([GR17, Chapter 11, Theorem A.1.5])**.** By Observation B.1.16, non-unital right-laxification defines a left adjoint

$$\begin{tikzcd} \iota_1 2\mathsf{Cat}_{\mathsf{n.u.r.lax}} \arrow[r, dashed, shift left, "\mathsf{r.lax}^{\mathsf{n.u.}}(-)"] \arrow[r, phantom, "\perp"] & \iota_1 2\mathsf{Cat} \arrow[l, hook, shift left] \end{tikzcd}$$

to the inclusion; in particular, for any $(\infty, 2)$-category $\mathcal{C} \in 2\mathsf{Cat}$, we have a universal non-unital right-lax functor

$$\mathcal{C} \rightsquigarrow \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}) \quad .$$

We use this fact without further comment.

Observation B.1.18. Noting the identification

$$\mathsf{r.lax}^{\mathsf{n.u.}}(\mathsf{pt})^{\oint} \simeq \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \xrightarrow{t} \mathbf{\Delta}^{\mathsf{op}} ,$$

we see that $\mathsf{r.lax}^{\mathsf{n.u.}}(\mathsf{pt})$ has a single object $* := ([0]^\circ \xrightarrow{\sim} [0]^\circ) \in \mathsf{r.lax}^{\mathsf{n.u.}}(\mathsf{pt})$ as well as a distinguished 2-morphism

$$\begin{tikzcd} * \arrow[r, bend left, "\mathsf{id}_* = ([0]^\circ \to [1]^\circ)"] \arrow[r, bend right, "e_* := ([1]^\circ \xrightarrow{\sim} [1]^\circ)"'] \arrow[r, phantom, "\Downarrow"] & * \end{tikzcd} .$$

By the functoriality of $\mathsf{r.lax}^{\mathsf{n.u.}}(-)$, for any object $c \in \mathcal{C} \in 2\mathsf{Cat}$ we obtain a canonical 2-morphism $\mathsf{id}_c \to e_c$ in $\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})$.

Definition B.1.19. We refer to the 2-morphism $\mathsf{id}_c \to e_c$ in $\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})$ of Observation B.1.18 as the ***quasi-unit 2-morphism*** corresponding to the object $c \in \mathcal{C}$. Inverting these determines an $(\infty, 2)$-category

$$\mathsf{r.lax}(\mathcal{C}) \in 2\mathsf{Cat} ,$$

which we refer to as the (***unital***) ***right-laxification*** of $\mathcal{C}$. This construction defines an endofunctor

$$\iota_1 2\mathsf{Cat} \xrightarrow{\mathsf{r.lax}(-)} \iota_1 2\mathsf{Cat}$$

equipped with a natural epimorphism from $\mathsf{r.lax}^{\mathsf{n.u.}}(-)$.

Observation B.1.20. For any $\mathcal{C}, \mathcal{D} \in 2\mathsf{Cat}$, pullback along the involution $\mathbf{\Delta}^{\mathsf{op}} \underset{\sim}{\xrightarrow{\mathsf{rev}}} \mathbf{\Delta}^{\mathsf{op}}$ defines equivalences

$$\mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{n.u.r.lax}}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{n.u.r.lax}}}(\mathcal{C}^{1\mathsf{op}}, \mathcal{D}^{1\mathsf{op}}) \qquad \text{and} \qquad \mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{C}^{1\mathsf{op}}, \mathcal{D}^{1\mathsf{op}})$$

in $\mathcal{S}$. Therefore, we have canonical equivalences

$$\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}^{1\mathsf{op}}) \simeq \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{1\mathsf{op}} \qquad \text{and} \qquad \mathsf{r.lax}(\mathcal{C}^{1\mathsf{op}}) \simeq \mathsf{r.lax}(\mathcal{C})^{1\mathsf{op}} .$$

Observation B.1.21. In the case that $\mathcal{C} \in \mathsf{Cat} \subset 2\mathsf{Cat}$ is an ∞-category, we have a natural equivalence

$$\mathsf{colim}_{([n]\downarrow\mathcal{C})\in\mathbf{\Delta}_{/\mathcal{C}}} \mathsf{r.lax}([n]) \xrightarrow{\sim} \mathsf{r.lax}(\mathcal{C}) \ ;$$

by the universal property of $\mathsf{r.lax}(\mathcal{C}) \in \iota_1 2\mathsf{Cat}$, this follows from the equivalence

$$\mathsf{colim}_{([n]\downarrow\mathcal{C})\in\mathbf{\Delta}_{/\mathcal{C}}} [n]^{\oint} \xrightarrow{\sim} \mathcal{C}^{\oint}$$

in which the colimit can be computed either in $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ or in $\mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$.

Observation B.1.22. The composite

$$\mathcal{C} \rightsquigarrow \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}) \longrightarrow \mathsf{r.lax}(\mathcal{C})$$

is a (unital) right-lax functor. Moreover, it is the universal right-lax functor from $\mathcal{C}$. We use this fact without further comment.

Definition B.1.23. Given an $(\infty,2)$-category $\mathcal{C} \in 2\mathsf{Cat}$, we define its ***non-unital left-laxification*** and its (***unital***) ***left-laxification*** respectively as the $(\infty,2)$-categories

$$\mathsf{l.lax}^{\mathsf{n.u.}}(\mathcal{C}) := \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}^{2\mathsf{op}})^{2\mathsf{op}} \qquad \text{and} \qquad \mathsf{l.lax}(\mathcal{C}) := \mathsf{r.lax}(\mathcal{C}^{2\mathsf{op}})^{2\mathsf{op}} \ .$$

Proposition B.1.24. *The left-laxification $\mathsf{l.lax}([n]) \in 2\mathsf{Cat}$ is the thin 2-category that is characterized as follows: its objects are those of $[n]$, and for any $i, j \in [n]$ the poset $\mathsf{hom}_{\mathsf{l.lax}([n])}(i,j)$ is that of strictly increasing sequences*

$$i < k_1 < \cdots < k_l < j$$

in $[n]$ (for some $l \geq 0$) from i to j (ordered by inclusion), with composition given by concatenation.[162]

Proof. We establish the corresponding description of $\mathsf{r.lax}([n]) \simeq \mathsf{l.lax}([n]^{2\mathsf{op}})^{2\mathsf{op}} \simeq \mathsf{l.lax}([n])^{2\mathsf{op}}$.

We begin by noting the identification

$$[n]^{\oint} \simeq \left((\mathbf{\Delta}_{/[n]})^{\mathsf{op}} \xrightarrow{\mathsf{fgt}} \mathbf{\Delta}^{\mathsf{op}} \right)$$

in $\mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$. For $i, j \in [n]$ with $i \leq j$, let us write $[i,j] := [n]_{i//j} \in \mathbf{\Delta}$ for the corresponding closed interval. Using this notation, $\mathsf{r.lax}^{\mathsf{n.u.}}([n]) \in 2\mathsf{Cat}$ can be characterized as follows: its objects are those of $[n]$, and for $i \leq j$ we have

$$\mathsf{hom}_{\mathsf{r.lax}^{\mathsf{n.u.}}([n])}(i,j) \simeq \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}})_{|[i,j]^{\circ}} \ ,$$

with composition given by concatenation (and for $i > j$ we have $\mathsf{hom}_{\mathsf{r.lax}^{\mathsf{n.u.}}([n])}(i,j) = \varnothing$).[163]

Now, let us define the further pullback

$$\begin{tikzcd}
{}'\mathsf{r.lax}([n])^{\oint} \arrow[r, hook] \arrow[d] & \mathsf{r.lax}^{\mathsf{n.u.}}([n])^{\oint} \arrow[r] \arrow[d] & \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \arrow[r, "t"] \arrow[d, "s"] & \mathbf{\Delta}^{\mathsf{op}} \\
(\mathbf{\Delta}^{\mathsf{inj}}_{/[n]})^{\mathsf{op}} \arrow[r, hook] & (\mathbf{\Delta}_{/[n]})^{\mathsf{op}} \arrow[r] & \mathbf{\Delta}^{\mathsf{op}} & \\
& {[n]^{\oint}} \arrow[u, phantom, "\wr"] & &
\end{tikzcd} \tag{B.1.3}$$

in Cat, where we write $\mathbf{\Delta}^{\mathsf{inj}}_{/[n]} := (\mathbf{\Delta}^{\mathsf{inj}})_{/[n]}$ and we consider ${}'\mathsf{r.lax}([n])^{\oint} \in \mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$ via the upper horizontal composite. We claim that ${}'\mathsf{r.lax}([n])^{\oint} \simeq \mathsf{r.lax}([n])^{\oint}$, which will prove the desired result.

We first note that ${}'\mathsf{r.lax}([n])^{\oint} \in \mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$ lies in the image of the monomorphism $\iota_1 2\mathsf{Cat} \xhookrightarrow{(-)^{\oint}} \mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$; we write ${}'\mathsf{r.lax}([n]) \in 2\mathsf{Cat}$ for the corresponding $(\infty,2)$-category. Moreover, in diagram

[162] This $(\infty,2)$-category can be presented as the simplicially-enriched category $\mathfrak{C}(\Delta^n)$ (where $\mathfrak{C}$ denotes the left adjoint of the homotopy-coherent nerve functor to simplicial sets), but thought of as enriched in ∞-categories (via the Joyal model structure) rather than in spaces (via the Kan–Quillen model structure).

[163] In other words, $\mathsf{hom}_{\mathsf{r.lax}^{\mathsf{n.u.}}([n])}(i,j)$ has objects the nondecreasing sequences in $[n]$ from i to j.

(B.1.3), the upper left horizontal functor is fully faithful (because the lower left horizontal functor is) and admits a right adjoint

$$
\begin{tikzcd}
{'\mathsf{r.lax}([n])^{\oint}} \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathsf{r.lax}^{\mathsf{n.u.}}([n])^{\oint} \arrow[l, dashed, shift left, "q"]
\end{tikzcd}
$$

in $\mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$.[164] In particular, q is a localization (considered in $\mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$ or in Cat). Moreover, it is clear that q defines a morphism in $\mathsf{coCart}_{/\mathbf{\Delta}^{\mathsf{op}}}$ and therefore a functor $'\mathsf{r.lax}([n]) \xleftarrow{\tilde{q}} \mathsf{r.lax}^{\mathsf{n.u.}}([n])$ in $2\mathsf{Cat}$. Hence, by the Segal condition, it follows that $\tilde{q}$ is a localization at certain 2-morphisms; and unwinding the definitions, we see that these are generated under (horizontal) composition by the quasi-unit 2-morphisms of the objects of $[n]$.[165] □

B.2. **Fibrations.** In this subsection, we introduce several notions of fibrations among $(\infty,2)$-categories. These will feature in our study of un/straightening in §B.3.

Local Notation B.2.1. Throughout this subsection, we fix a functor $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ between $(\infty,2)$-categories.

Definition B.2.2 ([GR17, Chapter 11, Definition 1.1.2])**.** We say that a 1-morphism $e_0 \to e_1$ in $\mathcal{E}$ is ***cartesian*** (with respect to π), or π***-cartesian***, if for all $e \in \mathcal{E}$ the commutative square

$$
\begin{tikzcd}
\mathsf{hom}_{\mathcal{E}}(e,e_0) \arrow[r] \arrow[d] & \mathsf{hom}_{\mathcal{E}}(e,e_1) \arrow[d] \\
\mathsf{hom}_{\mathcal{C}}(\pi(e),\pi(e_0)) \arrow[r] & \mathsf{hom}_{\mathcal{C}}(\pi(e),\pi(e_1))
\end{tikzcd}
\tag{B.2.1}
$$

in Cat is a pullback. We then say that $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a ***2-cartesian fibration*** if the following conditions are satisfied.

(1) For every object $e \in \mathcal{E}$ and 1-morphism $c \to \pi(e)$ in $\mathcal{C}$ there exists a cartesian 1-morphism in $\mathcal{E}$ covering it with target e.

(2) For all $e_0, e_1 \in \mathcal{E}$, the morphism

$$\mathsf{hom}_{\mathcal{E}}(e_0,e_1) \longrightarrow \mathsf{hom}_{\mathcal{C}}(\pi(e_0),\pi(e_1))$$

in Cat is a cocartesian fibration.

(3) For all $e_0, e_1, e_2 \in \mathcal{E}$, in the commutative square

$$
\begin{tikzcd}
\mathsf{hom}_{\mathcal{E}}(e_0,e_1) \times \mathsf{hom}_{\mathcal{E}}(e_1,e_2) \arrow[r] \arrow[d] & \mathsf{hom}_{\mathcal{E}}(e_0,e_2) \arrow[d] \\
\mathsf{hom}_{\mathcal{C}}(p(e_0),p(e_1)) \times \mathsf{hom}_{\mathcal{C}}(p(e_1),p(e_2)) \arrow[r] & \mathsf{hom}_{\mathcal{C}}(p(e_0),p(e_2))
\end{tikzcd}
$$

in Cat, the upper horizontal functor preserves cocartesian morphisms with respect to the vertical functors (which are cocartesian fibrations by condition (2)).

We say that $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a ***homwise cocartesian fibration*** if condition (2) is satisfied, and a ***strict homwise cocartesian fibration*** if additionally condition (3) is satisfied. If $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a homwise

[164] On objects, this right adjoint is given by taking images of morphisms to $[n] \in \mathbf{\Delta}$.

[165] Given an object $i \in [n]$, its corresponding quasi-unit 2-morphism corresponds to the diagram

$$
\begin{tikzcd}
{[1]} \arrow[rr] \arrow[dr] & & {[0]} \arrow[dl, "i"] \\
& {[n]} &
\end{tikzcd},
$$

considered as a morphism in $\mathbf{\Delta}_{/[n]}$.

cocartesian fibration, we refer to the 2-morphisms in $\mathcal{E}$ that define cocartesian 1-morphisms in a hom-∞-category $\mathsf{hom}_{\mathcal{E}}(e_0, e_1)$ as ***cocartesian 2-morphisms*** (with respect to π).

We write

$$\mathsf{2Cat}_{\mathsf{2cart}/\mathcal{C}} \subseteq \mathsf{2Cat}_{/\mathcal{C}}$$

for the 1-full subcategory on the 2-cartesian fibrations, whose 1-morphisms are those that preserve cocartesian 2-morphisms. Moreover, we write

$$\mathsf{2Cart}_{\mathcal{C}} \subseteq \mathsf{2Cat}_{\mathsf{2cart}/\mathcal{C}}$$

for the 1-full subcategory on the same objects, whose 1-morphisms are those that additionally preserve cartesian 1-morphisms.

A ***1-cartesian fibration*** is a 2-cartesian fibration whose fibers are $(\infty, 1)$-categories. We write

$$\mathsf{1Cart}_{\mathcal{C}} \subseteq \mathsf{2Cart}_{\mathcal{C}} \qquad \text{and} \qquad \mathsf{2Cart}_{\mathsf{1cart}/\mathcal{C}} \subseteq \mathsf{2Cat}_{\mathsf{2cart}/\mathcal{C}}$$

for the full subcategories on the 1-cartesian fibrations.

Dually, we say that $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a ***2-cocartesian fibration*** if $\mathcal{E}^{1\&2\mathsf{op}} \xrightarrow{\pi^{1\&2\mathsf{op}}} \mathcal{C}^{1\&2\mathsf{op}}$ is a 2-cartesian fibration. We use the evident notation and terminology for the corresponding related notions.

Observation B.2.3. The functor $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ in $\mathsf{2Cat}$ is a strict homwise cocartesian fibration if and only if the corresponding functor $\mathcal{E}^{\oint} \xrightarrow{\pi^{\oint}} \mathcal{C}^{\oint}$ in Cat is a cocartesian fibration.[166]

Definition B.2.4 ([GR17, Chapter 11, Definition 3.1.2])**.** We say that a 1-morphism $e_0 \to e_1$ in $\mathcal{E}$ classified by a functor $[1] \xrightarrow{\varphi} \mathcal{E}$ is ***locally cartesian*** (with respect to π), or ***locally π-cartesian***, if it defines a cartesian 1-morphism with respect to the pullback $(\pi\varphi)^*\mathcal{E} \to [1]$. We then say that $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a ***locally 2-cartesian fibration*** if for every object $e \in \mathcal{E}$ and 1-morphism $c \to \pi(e)$ in $\mathcal{C}$ there exists a locally cartesian 1-morphism in $\mathcal{E}$ covering it with target e and moreover π is a strict homwise cocartesian fibration.

We employ the evident variants of the remaining notation and terminology of Definition B.2.2, e.g. the 1-full subcategories

$$\mathsf{loc.2Cart}_{\mathcal{C}} \subseteq \mathsf{2Cat}_{\mathsf{loc.2cart}/\mathcal{C}} \subseteq \mathsf{2Cat}_{/\mathcal{C}}$$

and the notion of a ***locally 2-cocartesian fibration*** are defined similarly.

Lemma B.2.5. *Suppose that $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a locally 2-cartesian fibration. Then, it is a 2-cartesian fibration if and only if its locally cartesian 1-morphisms are closed under composition.*

Proof. Clearly, in a 2-cartesian fibration the cartesian 1-morphisms are closed under composition. Conversely, suppose that the locally cartesian 1-morphisms in $\mathcal{E}$ are closed under composition. Then, each locally cartesian 1-morphism $e_0 \to e_1$ in $\mathcal{E}$ is in fact a cartesian 1-morphism. Indeed, since $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a strict homwise cocartesian fibration, the commutative square (B.2.1) is a pullback if and only if it induces an equivalence on fibers, which follows from the assumption that locally cartesian 1-morphisms are closed under composition. □

Observation B.2.6. It follows from Observation B.2.3 that we can pull back locally 2-cartesian fibrations along right-lax functors: given a diagram

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "\mathsf{loc.2cart}"] \\ \mathcal{D} \arrow[r, rightsquigarrow, "F"'] & \mathcal{C} \end{tikzcd}$$

[166]For the forwards implication, π being a homwise cocartesian fibration implies that $\pi^{\oint}$ is a locally cocartesian fibration, and thereafter its strictness guarantees composability of the locally cocartesian morphisms.

(in which F is a right-lax functor), we obtain a locally 2-cartesian fibration $F^*\mathcal{E} \to \mathcal{D}$ via the pullback square

$$\begin{tikzcd} (F^*\mathcal{E})^{\oint} \arrow[r] \arrow[d] & \mathcal{E}^{\oint} \arrow[d] \\ \mathcal{D}^{\oint} \arrow[r] & \mathcal{C}^{\oint} \end{tikzcd}$$

in $\mathsf{Cat}_{/\mathbf{\Delta}^{\mathsf{op}}}$ (in fact in $\mathsf{Cat}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}}$).[167] Evidently, if F is a strict functor then this construction coincides with ordinary pullback therealong.

Observation B.2.7. We collect the following apparent facts about (locally) 2-cartesian fibrations.

(1) Suppose that the functor $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a homwise cocartesian fibration and moreover all its fibers are $(\infty,1)$-categories. Then it is automatically a *strict* homwise cocartesian fibration.

(2) The functor $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ in $\mathsf{2Cat}$ is a (locally) 1-cartesian fibration if and only if the functor $\iota_1\mathcal{E} \xrightarrow{\iota_1(\pi)} \iota_1\mathcal{C}$ in Cat is a (resp. locally) cartesian fibration and moreover for all $e_0, e_1 \in \mathcal{E}$ the functor $\mathsf{hom}_{\mathcal{E}}(e_0, e_1) \to \mathsf{hom}_{\mathcal{C}}(\pi(e_0), \pi(e_1))$ is a left fibration.

(3) Suppose that $\mathcal{C} \in \mathsf{2Cat}$ is an $(\infty,1)$-category. Then, the functor $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is automatically a strict homwise cocartesian fibration. Hence, the inclusions $\mathsf{2Cat}_{\mathsf{2cart}/\mathcal{C}} \subseteq \mathsf{2Cat}_{\mathsf{loc.2cart}/\mathcal{C}} \subseteq \mathsf{2Cat}_{/\mathcal{C}}$ are fully faithful. In particular, $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$ is a locally 2-cartesian fibration if and only if its pullback along every functor $[1] \to \mathcal{C}$ defines a 2-cartesian fibration over $[1]$.

(4) A morphism in $\mathsf{loc.2Cart}_{\mathcal{C}}$ is an equivalence if and only if it's an equivalence on fibers (because the latter condition implies that it is both surjective and fully faithful).

B.3. **Un/straightening.** In this subsection, we consider variants of the Grothendieck construction for $(\infty,2)$-categories. Its main result is Theorem B.3.5, which establishes the Grothendieck construction for lax functors as an equivalence of ∞-categories. We later enhance it to an equivalence of $(\infty,2)$-categories (see Observation B.4.2).

Definition B.3.1. We refer to the equivalences of Theorem B.3.4 as ***un/straightening***, and to the equivalence (B.3.1) of Theorem B.3.5 as ***lax un/straightening***.

Remark B.3.2. Theorem B.3.4 appears as [GR17, Chapter 11, Theorem-Construction 1.1.8(b)], while Theorem B.3.5 is a slight variant of [GR17, Chapter 11, Theorem-Construction 3.2.2] (with essentially the same proof).

Local Notation B.3.3. In this subsection, we fix an $(\infty,2)$-category $\mathcal{C} \in \mathsf{2Cat}$.

Theorem B.3.4. *There are canonical equivalences*

$$\mathsf{Fun}(\mathcal{C}, \mathsf{2Cat}) \simeq \mathsf{2coCart}_{\mathcal{C}} \qquad \text{and} \qquad \mathsf{Fun}(\mathcal{C}^{\mathsf{1op}}, \mathsf{2Cat}) \simeq \mathsf{2Cart}_{\mathcal{C}}$$

among $(\infty,2)$*-categories.*

Proof. This is a special case of [Nui, Theorem 6.21].[168] □

Theorem B.3.5. *Pullback (in the sense of Observation B.2.6) along the universal right-lax functor*

$$\mathcal{C} \rightsquigarrow \mathsf{r.lax}(\mathcal{C})$$

[167] The Segal condition for $(F^*\mathcal{E})^{\oint}$ follows from the fact that it can be checked over the subcategory of inert morphisms in $\mathbf{\Delta}^{\mathsf{op}}$, and the completeness condition therefor follows from the fact that (unital) right-lax functors preserve equivalences. Thereafter, the conditions of Definition B.2.4 for the functor $F^*\mathcal{E} \to \mathcal{D}$ follow from the facts that the functor $(F^*\mathcal{E})^{\oint} \to \mathcal{D}^{\oint}$ is a cocartesian fibration and that for every functor $[1] \to \mathcal{D}$ the composite $[1]^{\oint} \to \mathcal{D}^{\oint} \to \mathcal{C}^{\oint}$ defines a strict (as opposed to right-lax) functor $[1] \to \mathcal{C}$.

[168] Note that our definition of $\mathsf{2Cat}$ agrees with that of [Nui] by [Nui, Remark 4.21].

determines an equivalence

$$\iota_1\mathsf{loc.2Cart}_{\mathcal{C}} \xleftarrow{\sim} \iota_1\mathsf{2Cart}_{\mathsf{r.lax}(\mathcal{C})}\ .$$

In particular, there is an equivalence

$$\iota_1\mathsf{loc.2Cart}_{\mathcal{C}} \simeq \iota_1\mathsf{Fun}(\mathsf{r.lax}(\mathcal{C}^{1\mathsf{op}}), \mathsf{2Cat})\ . \tag{B.3.1}$$

Proof. The second statement follows from the first using Observation B.1.20 and un/straightening.

To prove the first statement, we will construct an inverse to the pullback functor as the ultimate factorization in a diagram

$$\begin{tikzcd}
\iota_1\mathsf{loc.2Cart}_{\mathcal{C}} \arrow[d, dashed, "\Phi"'] \arrow[dr, dashed, "\Phi^{n.u.}"'] \arrow[drr, dashed, "\Phi^{n.u.}"] \arrow[drrr, dashed, bend left, "\Phi^{n.u.}(-)^{\oint}"] & & & \\
\iota_1\mathsf{2Cart}_{\mathsf{r.lax}(\mathcal{C})} \arrow[r, hook] & \iota_1\mathsf{2Cart}_{\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})} \arrow[r, hook] & \iota_1\mathsf{2Cat}_{/\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})} \arrow[r, hook, "(-)^{\oint}"'] & \mathsf{Cat}_{/\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint}}
\end{tikzcd}\ .$$

Let $S_{m,n}$ denote the thin 2-category

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \cdots \arrow[r] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \cdots \arrow[r] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[d] \\
\vdots \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \vdots \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \ddots \arrow[dr, phantom, "\Swarrow"] & \vdots \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet
\end{tikzcd}$$

with m vertical 1-morphisms in each column and n horizontal 1-morphisms in each row. These assemble into a bicosimplicial thin 2-category $\mathbf{\Delta} \times \mathbf{\Delta} \xrightarrow{S_{\bullet,\bullet}} \iota_1\mathsf{2Cat}$, and we write

$$\mathsf{Sq} := \hom_{\iota_1\mathsf{2Cat}}(S_{\bullet,\bullet}, -) : \iota_1\mathsf{2Cat} \longrightarrow \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}} \times \mathbf{\Delta}^{\mathsf{op}}, \mathcal{S})$$

for the functor that it corepresents. By [HORR, Corollary 4.4.2] (see also [Col]), this functor lands in the full subcategory of double ∞-categories: simplicial objects in $\mathsf{Cat} \subseteq \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S})$ satisfying the Segal and completeness conditions.[169] Similarly, we write

$$\iota_1\mathsf{2Cat} \xrightarrow{\mathsf{Sq}(-)^{\oint}} \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$$

for the cocartesian unstraightening of the functor carrying each object $[n]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$ to the functor $\iota_1\mathsf{2Cat} \xrightarrow{\mathsf{Sq}(-)_{\bullet,n}} \mathsf{Cat} \subset \mathsf{Fun}(\mathbf{\Delta}^{\mathsf{op}}, \mathcal{S})$. Note that $(-)^{\oint} \subseteq \mathsf{Sq}(-)^{\oint}$ is a 1-full subcategory: it contains the same objects, and in the fiber over $[n]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$ its 1-morphisms are those functors $S_{1,n} \to (-)$ that carry the vertical 1-morphisms to equivalences.

To proceed, let us note that any morphism $I \xrightarrow{\varphi} J$ in $\mathbf{\Delta}^{\mathsf{act}}$ admits a canonical factorization

$$\begin{tikzcd}
I \arrow[rr, "\varphi"] \arrow[dr, dashed, "i \mapsto i"'] & & J \\
& K \arrow[ur, dashed, "j \mapsto j", "i \mapsto \varphi(i)"'] &
\end{tikzcd} \tag{B.3.2}$$

in $\mathbf{\Delta}^{\mathsf{act}}$, where $K := (I \sqcup J)/\{\mathsf{max}(I) \sim \mathsf{max}(J)\}$ with ordering characterized by the requirement that for all $i \in I$ and $j \in J$ we have $i < j$ if and only if $\varphi(i) \leq j$. This determines a monomorphism $\mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \hookrightarrow \mathsf{Fun}([2], \mathbf{\Delta}^{\mathsf{op}})$, which is adjoint to natural transformations

$$t \longleftarrow \mu \longleftarrow s$$

in $\mathsf{Fun}(\mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}), \mathbf{\Delta}^{\mathsf{op}})$ (whose components at $(I \xrightarrow{\varphi} J)^\circ \in \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}})$ correspond to the diagram (B.3.2) in $\mathbf{\Delta}^{\mathsf{act}}$).

[169] So, a double ∞-category is an $(\infty, 2)$-category if and only if its 0^{th} ∞-category is an ∞-groupoid.

Additionally, let us observe that for any functor $\mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \xrightarrow{\chi} \mathbf{\Delta}^{\mathsf{op}}$ we obtain an endofunctor

$$\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}} \xrightarrow{\mathsf{F}^{\mathsf{act}}_{\chi}} \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$$

given by taking $\mathcal{E} \downarrow \mathbf{\Delta}^{\mathsf{op}}$ to the fiber product

$$\begin{tikzcd}
\mathsf{F}^{\mathsf{act}}_{\chi}(\mathcal{E}) \arrow[r] \arrow[d] & \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \arrow[r, "t"] \arrow[d, "\chi"] & \mathbf{\Delta}^{\mathsf{op}} \\
\mathcal{E} \arrow[r] & \mathbf{\Delta}^{\mathsf{op}}
\end{tikzcd}$$

considered in $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ by the horizontal composite (compare with Definition B.1.15); in particular, $\mathsf{F}^{\mathsf{act}} := \mathsf{F}^{\mathsf{act}}_{s}$. This defines a functor $\mathsf{Fun}(\mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}), \mathbf{\Delta}^{\mathsf{op}}) \rightarrow \mathsf{Fun}(\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}, \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}})$.

Now, given an object $(\mathcal{E} \downarrow \mathcal{C}) \in \mathsf{loc.2Cart}_{\mathcal{C}}$, we define the 1-full subcategory

$$\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint} \subseteq \mathsf{F}^{\mathsf{act}}_{\mu}(\mathsf{Sq}(\mathcal{E})^{\oint})$$

as follows, using the notation of diagram (B.3.2) throughout. First of all, an object of $\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint}$ is given by $I \rightarrow K \xrightarrow{e_\bullet} \mathcal{E}$ such that for every $k \in K \backslash \{\mathsf{max}(K)\}$, the morphism $e_k \rightarrow e_{k+1}$ is sent to an equivalence in $\mathcal{C}$ if $k \in I$ and is locally cartesian over $\mathcal{C}$ if $k \in J$. Then, a morphism in $\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint}$ from $I' \rightarrow K' \xrightarrow{e'_\bullet} \mathcal{E}$ to $I \rightarrow K \xrightarrow{e_\bullet} \mathcal{E}$ is given by a morphism

$$\begin{tikzcd}
I' \arrow[r] & K' \arrow[r] & J' \\
I \arrow[r] \arrow[u] & K \arrow[r] \arrow[u, "\beta"] & J \arrow[u]
\end{tikzcd}$$

in $\mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \subset \mathsf{Fun}([2], \mathbf{\Delta}^{\mathsf{op}})$ along with a diagram

$$\begin{tikzcd}
e'_{\beta(\mathsf{min}(K))} \arrow[r] \arrow[d, "\gamma_{\mathsf{min}(K)}"'] \arrow[dr, phantom, "\swarrow \eta_{\mathsf{min}(K)}"] & e'_{\beta(\mathsf{min}(K)+1)} \arrow[r] \arrow[d, "\gamma_{\mathsf{min}(K)+1}"'] \arrow[dr, phantom, "\swarrow \eta_{\mathsf{min}(K)+1}"] & \cdots \arrow[r] \arrow[dr, phantom, "\swarrow \eta_{\mathsf{max}(K)-1}"] & e'_{\beta(\mathsf{max}(K))} \arrow[d, "\gamma_{\mathsf{max}(K)}"] \\
e_{\mathsf{min}(K)} \arrow[r] & e_{\mathsf{min}(K)+1} \arrow[r] & \cdots \arrow[r] & e_{\mathsf{max}(K)}
\end{tikzcd} \tag{B.3.3}$$

in $\mathcal{E}$,[170] such that

- for every $k \in K$, the 1-morphism γ_k in $\mathcal{E}$ is sent to an equivalence in $\mathcal{C}$;
- for every $i \in I \subseteq K$, the 1-morphism γ_i in $\mathcal{E}$ is an equivalence; and
- for every $i \in (I \backslash \{\mathsf{max}(I)\}) \subseteq (K \backslash \{\mathsf{max}(K)\})$, the 2-morphism η_i in $\mathcal{E}$ is sent to an equivalence in $\mathcal{C}$;
- for every $j \in (J \backslash \{\mathsf{max}(J)\}) \subseteq (K \backslash \{\mathsf{max}(K)\})$, the 2-morphism η_j in $\mathcal{E}$ is a cocartesian 2-morphism over $\mathcal{C}$.[171]

---

[170] Beware that the upper row in diagram (B.3.3) may not define an object of $\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint}$, although for every $k \in (I \backslash \{\mathsf{max}(I)\}) \subseteq (K \backslash \{\mathsf{max}(K)\})$ the 1-morphism $e'_{\beta(k)} \rightarrow e'_{\beta(k+1)}$ is sent to an equivalence in $\mathcal{C}$.

[171] As the notation suggests, we will soon verify that this construction defines an $(\infty, 2)$-category $\Phi^{\mathsf{n.u.}}(\mathcal{E}) \in \mathsf{2Cat}$. Informally, its objects are those of $\mathcal{E}$, its 1-morphisms are strings of 1-morphisms $e_0 \rightarrow \cdots \rightarrow e_m$ in $\mathcal{E}$ with $m \geq 1$ such that the 1-morphism $e_0 \rightarrow e_1$ in $\mathcal{E}$ is sent to an equivalence in $\mathcal{C}$ and the 1-morphisms $e_i \rightarrow e_{i+1}$ in $\mathcal{E}$ are locally cartesian over $\mathcal{C}$ for all $1 \leq i < m$, and a typical 2-morphism is given by a diagram

$$\begin{tikzcd}
e'_0 \arrow[r] \arrow[d, "\gamma_0"', "\wr"] \arrow[dr, phantom, "\swarrow \eta_0"] & e'_1 \arrow[r] \arrow[d, "\gamma_1"] & e'_2 \arrow[r] \arrow[drr, phantom, "\swarrow \eta_1 \eta_2"] & e'_3 \arrow[r] & e'_4 \arrow[r] \arrow[d, "\gamma_3"'] \arrow[drr, phantom, "\swarrow \eta_3"] & e'_5 \arrow[r] & e'_6 \arrow[rr] \arrow[d, "\gamma_4"] \arrow[drr, phantom, "\swarrow \eta_4 \eta_5"] & & e'_7 \arrow[d, "\wr"', "\gamma_6"] \\
e_0 \arrow[r] & e_1 \arrow[r] & e_2 \arrow[rr] & & e_3 \arrow[rr] & & e_4 \arrow[r] & e_5 \arrow[r] & e_6
\end{tikzcd}$$

Observe that by the definition of $\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint}$, we have a factorization

$$\begin{tikzcd}
\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint} \arrow[rrr, hook] \arrow[d, dashed] & & & \mathsf{F}^{\mathsf{act}}_{\mu}(\mathsf{Sq}(\mathcal{E}^{\oint})) \arrow[d] \\
\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint} \simeq \mathsf{F}^{\mathsf{act}}(\mathcal{C}^{\oint}) \arrow[r, hook] & \mathsf{F}^{\mathsf{act}}(\mathsf{Sq}(\mathcal{C})^{\oint}) \arrow[rr, hook, "s \longrightarrow \mu"'] & & \mathsf{F}^{\mathsf{act}}_{\mu}(\mathsf{Sq}(\mathcal{C})^{\oint})
\end{tikzcd},$$

where the lower right horizontal functor (induced by the natural transformation $s \to \mu$) is 1-full since $\mathsf{Sq}(\mathcal{C})$ is a double ∞-category (in particular it is complete). This factorization $\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint} \to \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint}$ is a cocartesian fibration: a morphism in $\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint}$ is locally cocartesian over $\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint}$ when all of its constituent 2-morphisms η_k (as in diagram (B.3.3)) are cocartesian,[172] and clearly such morphisms are closed under composition. Hence, we obtain a functor

$$\iota_1\mathsf{loc.2Cart}_{\mathcal{C}} \xrightarrow{\Phi^{\mathsf{n.u.}}(-)^{\oint}} \mathsf{coCart}_{\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint}} .$$

Moreover, the composite cocartesian fibration

$$\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint} \longrightarrow \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint} \longrightarrow \mathbf{\Delta}^{\mathsf{op}}$$

defines an $(\infty, 2)$-category.[173] So altogether, we obtain a functor

$$\iota_1\mathsf{loc.2Cart}_{\mathcal{C}} \xrightarrow{\Phi^{\mathsf{n.u.}}} \mathsf{2Cat}_{/\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})} .$$

We now verify that $\Phi^{\mathsf{n.u.}}$ factors through $\mathsf{2Cart}_{\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})} \subseteq \mathsf{2Cat}_{/\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})}$. Indeed, by Observation B.2.3, the functor $\Phi^{\mathsf{n.u.}}(\mathcal{E}) \to \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})$ is a strict homwise cocartesian fibration, and its cartesian 1-morphisms are those 1-morphisms in $\Phi^{\mathsf{n.u.}}(\mathcal{E})$ in which all constituent 1-morphisms in $\mathcal{E}$ are locally cartesian (i.e. the first 1-morphism is an equivalence).

We now verify that in fact $\Phi^{\mathsf{n.u.}}$ factors further through $\mathsf{2Cart}_{\mathsf{r.lax}(\mathcal{C})} \subseteq \mathsf{2Cart}_{\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})}$, a subcategory via un/straightening. Because the construction of $\Phi^{\mathsf{n.u.}}$ commutes with pullback in the variable $\mathcal{C} \in \iota_1\mathsf{2Cat}^{\mathsf{op}}$, it suffices to check the case that $\mathcal{C} = \mathsf{pt}$. And in this case we have $\Phi^{\mathsf{n.u.}}(\mathcal{E}) \simeq \mathcal{E} \times \mathsf{r.lax}^{\mathsf{n.u.}}(\mathsf{pt})$, where the projection $\Phi^{\mathsf{n.u.}}(\mathcal{E}) \to \mathcal{E}$ is given by the factorization

$$\begin{tikzcd}
\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint} \arrow[r, hook] \arrow[d, dashed] & \mathsf{F}^{\mathsf{act}}_{\mu}(\mathsf{Sq}(\mathcal{E})^{\oint}) \arrow[r, "\mu \longrightarrow t"] & \mathsf{F}^{\mathsf{act}}_{t}(\mathsf{Sq}(\mathcal{E})^{\oint}) \simeq \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \underset{\mathbf{\Delta}^{\mathsf{op}}}{\times} \mathsf{Sq}(\mathcal{E})^{\oint} \arrow[d, "\mathsf{pr}"] \\
\mathcal{E}^{\oint} \arrow[rr, hook] & & \mathsf{Sq}(\mathcal{E})^{\oint}
\end{tikzcd} \tag{B.3.4}$$

in $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$. So indeed, we obtain our desired functor Φ.

To conclude, we show that Φ is indeed inverse to the functor given by pullback along the universal right-lax functor, which we denote here by $\mathcal{C} \overset{\theta}{\rightsquigarrow} \mathsf{r.lax}(\mathcal{C})$.

---

in $\mathcal{E}$ in which the 2-morphism η_0 is sent to an equivalence in $\mathcal{C}$ and all remaining 2-morphisms in the diagram are required to be cocartesian over $\mathcal{C}$.

[172] Such lifts always exist: first consider cocartesian lifts of the 2-morphisms in $\mathcal{C}$, and then enforce locally cartesianness of the relevant 1-morphisms (starting at the end and working backwards).

[173] The Segal and completeness conditions follow from those for $\mathsf{Sq}(\mathcal{E})^{\oint} \in \mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$, and we have $\Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint}_0 \simeq \iota_0(\mathcal{E})$.

We first verify the equivalence $\theta^*\Phi \simeq \mathsf{id}_{\iota_1 \mathsf{loc.2Cart}_{\mathcal{C}}}$. For this, given any $\mathcal{E} \in \iota_1\mathsf{loc.2Cart}_{\mathcal{C}}$, consider the commutative diagram

$$\begin{tikzcd}
 & & \Phi^{\mathsf{n.u.}}(\mathcal{E})^{\oint} \arrow[dl] \arrow[d, hook, "\text{1-full}"] & \mathcal{E}^{\oint} \arrow[d, hook, "\text{1-full}"] & \\
 & & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathsf{Sq}(\mathcal{E})^{\oint} \arrow[d] \\
\mathcal{C}^{\oint} \arrow[r, hook] \arrow[rrr, bend right, "\mathsf{id}_{\mathcal{C}^{\oint}}"'] & \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint} \arrow[r, hook, "s \longrightarrow \mu"] \arrow[d, phantom, "\Downarrow"] & \mathsf{F}^{\mathsf{act}}_{\mu}(\mathcal{C}^{\oint}) \arrow[r, "\mathsf{pr}"] & \mathcal{C}^{\oint} \arrow[r, hook] & \mathsf{Sq}(\mathcal{C})^{\oint} \\
 & {} & & &
\end{tikzcd} \tag{B.3.5}$$

in Cat, in which both squares are pullbacks and the 2-morphism arises from the commutative diagram

$$\begin{tikzcd}
\mathbf{\Delta}^{\mathsf{op}} \arrow[r, "\mathsf{id}_{(-)}"] & \mathsf{Ar}^{\mathsf{act}}(\mathbf{\Delta}^{\mathsf{op}}) \arrow[r, bend left, "\mu"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{\Delta}^{\mathsf{op}}
\end{tikzcd}$$

in Cat. It follows that we obtain the cospan in the diagram

$$\begin{tikzcd}
 & \mathcal{E}^{\oint} \arrow[d, hook, "\text{1-full}"] \\
\theta^*\Phi(\mathcal{E})^{\oint} \arrow[r] \arrow[ur, dashed] & \mathcal{F}
\end{tikzcd}, \tag{B.3.6}$$

in which the lower horizontal functor arises from the 2-morphism of diagram (B.3.5). Unraveling the definitions, we obtain the factorization in diagram (B.3.6), and further observe that it is an equivalence.[174]

We now verify the equivalence $\Phi\theta^* \simeq \mathsf{id}_{\iota_1 \mathsf{2Cart}_{\mathsf{r.lax}(\mathcal{C})}}$. For this, fix an object $\mathcal{E} \in \iota_1\mathsf{2Cart}_{\mathsf{r.lax}(\mathcal{C})}$, and let us write $\mathcal{E}^{\mathsf{n.u.}} \in \iota_1\mathsf{2Cart}_{\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})}$ for its pullback. Observe the diagram

$$\begin{tikzcd}
 & & (\mathcal{E}^{\mathsf{n.u.}})^{\oint} \arrow[d] \\
\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint} \arrow[r] \arrow[rr, bend right, "\mathsf{id}_{\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint}}"'] & \mathcal{C}^{\oint} \arrow[r] \arrow[d, phantom, "\Uparrow"] & \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint} \\
 & {} &
\end{tikzcd}$$

[174]The $(\infty, 2)$-category $\theta^*\Phi(\mathcal{E})$ can be informally described as follows: its objects are those of $\mathcal{E}$, a 1-morphism from e_0 to e_1 is a pair of 1-morphisms $e_0 \to e \to e_1$ in $\mathcal{E}$ such that $e_0 \to e$ is sent to an equivalence in $\mathcal{C}$ and $e \to e_1$ is locally cartesian over $\mathcal{C}$, and a 2-morphism from $e_0 \to e' \to e_1$ to $e_0 \to e \to e_1$ is given by a diagram

$$\begin{tikzcd}
e_0 \arrow[r] \arrow[d, "\parallel"'] \arrow[dr, phantom, "\Swarrow \eta_0"] & e' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow \eta_1"] & e_1 \arrow[d, "\parallel"] \\
e_0 \arrow[r] & e \arrow[r] & e_1
\end{tikzcd}$$

in $\mathcal{E}$ in which the 2-morphism η_0 is sent to an equivalence in $\mathcal{C}$ and the 2-morphism η_1 is cocartesian over $\mathcal{C}$. In these terms, the corresponding functor $\theta^*\Phi(\mathcal{E}) \to \mathcal{E}$ is given by composition of 1- and 2-morphisms (and this is clearly both surjective and fully faithful).

in Cat, in which the left horizontal functor lies in $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ and corresponds to $\mathsf{id}_{\mathcal{C}}$ and the 2-morphism is the unit of an adjunction in $\mathsf{Cat}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}}$.[175] Since $(\mathcal{E}^{\mathsf{n.u.}})^{\oint} \to \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint}$ is a cocartesian fibration (recall Observation B.2.3), we obtain a commutative square

$$\begin{tikzcd}
\mathcal{E}^{\mathsf{n.u.}} \arrow[r] \arrow[d] & \theta^*\mathcal{E} \arrow[d] \\
\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

in $2\mathsf{Cat}$. Applying $\mathsf{Sq}(-)^{\oint}$, the composite $\mathcal{C}^{\oint} \to \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint} \to \mathsf{Sq}(\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}))^{\oint}$ gives rise to a morphism

$$\mathsf{Sq}(\mathcal{E}^{\mathsf{n.u.}})^{\oint} \underset{\mathsf{Sq}(\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}))^{\oint}}{\times} \mathcal{C}^{\oint} \longrightarrow \mathsf{Sq}(\theta^*\mathcal{E})^{\oint} \underset{\mathsf{Sq}(\mathcal{C})^{\oint}}{\times} \mathcal{C}^{\oint} \tag{B.3.7}$$

in $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$,[176] which is an equivalence because it is so on fibers over each $[n]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$. From here, we observe the factorization

$$\begin{tikzcd}
\Phi^{\mathsf{n.u.}}(\theta^*\mathcal{E})^{\oint} \arrow[r, hook] \arrow[d, dashed, "\alpha^{\mathsf{n.u.}}"'] & \mathsf{F}^{\mathsf{act}}_{\mu}\left(\mathsf{Sq}(\theta^*\mathcal{E})^{\oint} \underset{\mathsf{Sq}(\mathcal{C})^{\oint}}{\times} \mathcal{C}^{\oint}\right) & \mathsf{F}^{\mathsf{act}}_{\mu}\left(\mathsf{Sq}(\mathcal{E}^{\mathsf{n.u.}})^{\oint} \underset{\mathsf{Sq}(\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C}))^{\oint}}{\times} \mathcal{C}^{\oint}\right) \arrow[l, "\text{(B.3.7)}"', "\sim"] \arrow[r] & \mathsf{F}^{\mathsf{act}}_{\mu}(\mathsf{Sq}(\mathcal{E}^{\mathsf{n.u.}})^{\oint}) \arrow[d] \\
(\mathcal{E}^{\mathsf{n.u.}})^{\oint} \arrow[rrr, hook] & & & \mathsf{Sq}(\mathcal{E}^{\mathsf{n.u.}})^{\oint}
\end{tikzcd}$$

in $\mathsf{Cat}_{\mathsf{cocart}/\mathbf{\Delta}^{\mathsf{op}}}$, in which the right vertical functor is the composite of diagram (B.3.4) and $\alpha^{\mathsf{n.u.}}$ lies in $\mathsf{coCart}_{\mathbf{\Delta}^{\mathsf{op}}}$ and hence defines a functor $\Phi^{\mathsf{n.u.}}(\theta^*\mathcal{E}) \xrightarrow{\alpha^{\mathsf{n.u.}}} \mathcal{E}^{\mathsf{n.u.}}$ in $\iota_1 2\mathsf{Cat}_{/\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})}$.[177] Observe further that $\alpha^{\mathsf{n.u.}}$ in fact defines a natural morphism in $\iota_1 2\mathsf{Cart}_{\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})}$, since it evidently preserves cartesian 1-morphisms and cocartesian 2-morphisms. Hence, it also defines a natural morphism $\Phi(\theta^*\mathcal{E}) \xrightarrow{\alpha} \mathcal{E}$ in the full subcategory $\iota_1 2\mathsf{Cart}_{\mathsf{r.lax}(\mathcal{C})} \subseteq \iota_1 2\mathsf{Cart}_{\mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})}$, i.e. a morphism $\Phi\theta^* \xrightarrow{\alpha} \mathsf{id}_{\iota_1 2\mathsf{Cart}_{\mathsf{r.lax}(\mathcal{C})}}$, which it remains to show is an equivalence. Now, our verification that $\theta^*\Phi \simeq \mathsf{id}_{\iota_1 \mathsf{loc.2Cart}_{\mathcal{C}}}$ shows that $\theta^*\alpha$ is an equivalence. But $\iota_1 2\mathsf{Cart}_{\mathsf{r.lax}(\mathcal{C})} \xrightarrow{\theta^*} \iota_1 \mathsf{loc.2Cart}_{\mathcal{C}}$ is conservative by Observation B.2.7(4), and so α itself is an equivalence. ☐

B.4. **Cartesian yoga.** In this subsection, we establish two enhancements of un/straightening (Theorems B.4.1 and B.4.3), as well as an enhancement of lax un/straightening to an equivalence of $(\infty, 2)$-categories (Observation B.4.2). We begin by stating the main results, and then prove them in turn based on supporting lemmas.

Theorem B.4.1. *Let $\mathcal{C}, \mathcal{D} \in \iota_1 2\mathsf{Cat}$ be $(\infty, 2)$-categories. Then, there is a natural equivalence*

$$\hom_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{C}, 2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}}) \simeq \hom_{\iota_1 2\mathsf{Cat}_{\mathsf{l.lax}}}(\mathcal{D}, 2\mathsf{Cat}_{\mathsf{loc.2cart}/\mathcal{C}^{1\mathsf{op}}})$$

in $\mathcal{S}$, which is contravariantly functorial in both variables.[178] *Moreover, it restricts to equivalences*

$$\hom_{\iota_1 2\mathsf{Cat}}(\mathcal{C}, 2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}}) \simeq \hom_{\iota_1 2\mathsf{Cat}_{\mathsf{l.lax}}}(\mathcal{D}, 2\mathsf{Cat}_{\mathsf{2cart}/\mathcal{C}^{1\mathsf{op}}}) \ ,$$
$$\hom_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{C}, 2\mathsf{Cat}_{\mathsf{2cocart}/\mathcal{D}}) \simeq \hom_{\iota_1 2\mathsf{Cat}}(\mathcal{D}, 2\mathsf{Cat}_{\mathsf{loc.2cart}/\mathcal{C}^{1\mathsf{op}}}) \ , \qquad \textit{and}$$

---

[175]The counit of this adjunction is the equivalence between the composite $\mathcal{C}^{\oint} \to \mathsf{r.lax}^{\mathsf{n.u.}}(\mathcal{C})^{\oint} \to \mathcal{C}^{\oint}$ and $\mathsf{id}_{\mathcal{C}^{\oint}}$. To see that this indeed gives an adjunction, it suffices to verify that it gives an adjunction fiberwise over $\mathbf{\Delta}^{\mathsf{op}}$. Thereafter, the Segal condition reduces the verification to the fibers over the objects $[0]^\circ, [1]^\circ \in \mathbf{\Delta}^{\mathsf{op}}$, in which cases the assertion is evident.

[176]This follows from the fact that for any locally 2-cartesian fibration $\mathcal{E}' \xrightarrow{\pi'} \mathcal{C}'$ the corresponding functor

$$\mathsf{Sq}(\mathcal{E}')^{\oint} \underset{\mathsf{Sq}(\mathcal{C}')^{\oint}}{\times} \mathcal{C}'^{\oint} \longrightarrow \mathcal{C}'^{\oint}$$

is a cocartesian fibration (as in Observation B.2.3).

[177]Informally, the corresponding functor $\Phi^{\mathsf{n.u.}}(\theta^*\mathcal{E}) \to \mathcal{E}^{\mathsf{n.u.}}$ is given by composing 1- and 2-morphisms in $\mathcal{E}^{\mathsf{n.u.}}$.

[178]In particular, it restricts to an equivalence $\hom_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{C}, 2\mathsf{Cat}_{\mathsf{loc.1cocart}/\mathcal{D}}) \simeq \hom_{\iota_1 2\mathsf{Cat}_{\mathsf{l.lax}}}(\mathcal{D}, 2\mathsf{Cat}_{\mathsf{loc.1cart}/\mathcal{C}^{1\mathsf{op}}})$.

$$\mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{C}, \mathsf{loc.2coCart}_{\mathcal{D}}) \simeq \mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{l.lax}}}(\mathcal{D}, \mathsf{loc.2Cart}_{\mathcal{C}^{1\mathsf{op}}}) \ .$$

Observation B.4.2. Given a right-lax functor $\mathcal{C} \overset{F}{\rightsquigarrow} \mathcal{D}$ between $(\infty, 2)$-categories, Theorem B.4.1 gives a functor

$$2\mathsf{Cat}_{\mathsf{loc.2cart}/\mathcal{D}} \xrightarrow{F^*} 2\mathsf{Cat}_{\mathsf{loc.2cart}/\mathcal{C}} \ ,$$

which restricts to a functor

$$\mathsf{loc.2Cart}_{\mathcal{D}} \xrightarrow{F^*} \mathsf{loc.2Cart}_{\mathcal{C}} \ :$$

namely, for any $(\infty, 2)$-category $\mathcal{X} \in \iota_1 2\mathsf{Cat}$, we obtain a natural morphism

$$\mathsf{hom}_{\iota_1 2\mathsf{Cat}}(\mathcal{X}, 2\mathsf{Cat}_{\mathsf{loc.2cart}/\mathcal{D}}) \simeq \mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{D}^{1\mathsf{op}}, 2\mathsf{Cat}_{2\mathsf{cocart}/\mathcal{X}}) \longrightarrow \mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{C}^{1\mathsf{op}}, 2\mathsf{Cat}_{2\mathsf{cocart}/\mathcal{X}}) \simeq \mathsf{hom}_{\iota_1 2\mathsf{Cat}}(\mathcal{X}, 2\mathsf{Cat}_{\mathsf{loc.2cart}/\mathcal{C}})$$

in $\mathcal{S}$. In particular, taking F to be the universal right-lax functor $\mathcal{C} \rightsquigarrow \mathsf{r.lax}(\mathcal{C})$, by lax un/straightening we find that the pullback functor

$$2\mathsf{Cat}_{2\mathsf{cart}/\mathsf{r.lax}(\mathcal{C})} \longrightarrow 2\mathsf{Cat}_{\mathsf{loc.2cart}/\mathcal{C}}$$

is an equivalence of $(\infty, 2)$-categories, contravariantly functorial in $\mathcal{C} \in \iota_1 2\mathsf{Cat}$ by pullback, which restricts to an equivalence

$$2\mathsf{Cart}_{\mathsf{r.lax}(\mathcal{C})} \xrightarrow{\sim} \mathsf{loc.2Cart}_{\mathcal{C}} \ .$$ [179]

Theorem B.4.3. *Let $\mathcal{C}, \mathcal{D} \in \iota_1 2\mathsf{Cat}$ be $(\infty, 2)$-categories. Then, unstraightening gives a monomorphism*

$$\mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{l.lax}}}(\mathcal{C}, 2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}}) \hookrightarrow \iota_0 \mathsf{loc.2coCart}_{\mathcal{C}\times\mathcal{D}}$$

in $\mathcal{S}$, whose image consists of those locally 2-cocartesian fibrations $(\mathcal{E} \downarrow (\mathcal{C} \times \mathcal{D})) \in \iota_0 \mathsf{loc.2coCart}_{\mathcal{C}\times\mathcal{D}}$ that satisfy the following condition.[180]

($*$) *For any pair of 1-morphisms $c \to c'$ in $\mathcal{C}$ and $d \to d'$ in $\mathcal{D}$, the pullback of $\mathcal{E} \downarrow (\mathcal{C} \times \mathcal{D})$ along the functor $[2] \to \mathcal{C} \times \mathcal{D}$ selecting the commutative triangle*

$$\begin{tikzcd} (c,d) \arrow[r] \arrow[dr] & (c',d) \arrow[d] \\ & (c',d') \end{tikzcd}$$

is a 2-cocartesian fibration.[181]

Moreover, the further subspace

$$\mathsf{hom}_{\iota_1 2\mathsf{Cat}}(\mathcal{C}, 2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}}) \subseteq \mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{l.lax}}}(\mathcal{C}, 2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}})$$

corresponds to the additional condition that for every object $d \in \mathcal{D}$ the functor $\mathcal{E}_d \to \mathcal{C}$ is a 2-cocartesian fibration.

Definition B.4.4. Given a locally 2-co/cartesian fibration $\mathcal{E} \xrightarrow{\pi} \mathcal{C}$, we say that a functor $\mathcal{E} \to \mathcal{D}$ ***smushes*** π if it carries all π-co/cartesian 1- and 2-morphisms to equivalences.

Lemma B.4.5. *Suppose we are given a span $\mathcal{C} \xleftarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{D}$ in $2\mathsf{Cat}$ such that F is a locally 2-cartesian fibration that G smushes. Then, the following conditions are equivalent.*

(1) *The functor G is a (locally) 2-cocartesian fibration that F smushes.*

(2) *The following two conditions are satisfied.*

[179] It is obvious that on ι_0 this equivalence coincides with that of Theorem B.3.5. In fact this is also true on ι_1, but we do not use that here.

[180] The further subspace $\mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{l.lax}}}(\mathcal{C}, 2\mathsf{Cat}_{2\mathsf{cocart}/\mathcal{D}}) \subseteq \mathsf{hom}_{\iota_1 2\mathsf{Cat}_{\mathsf{l.lax}}}(\mathcal{C}, 2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}})$ corresponds to the additional condition that for every object $c \in \mathcal{C}$ the functor $\mathcal{E}_c \to \mathcal{D}$ is a 2-cocartesian fibration, by the naturality of un/straightening.

[181] Recall Remark B.1.12.

(i) For every object $c \in \mathcal{C}$, the functor $\mathcal{E}_c \to \mathcal{D}$ is a (resp. locally) 2-cocartesian fibration.

(ii) For every 1-morphism $c_1 \to c_2$ in $\mathcal{C}$, the corresponding cartesian monodromy functor $\mathcal{E}_{c_1} \leftarrow \mathcal{E}_{c_2}$ (which lies over $\mathcal{D}$ by the assumption that G smushes F) lies in $2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}} \subseteq 2\mathsf{Cat}_{/\mathcal{D}}$ *(i.e. it preserves cartesian 2-morphisms over $\mathcal{D}$).*

Proof. We will prove the version involving the two instances of the word "locally"; in particular, we will explicitly identify the locally cocartesian 1-morphisms over $\mathcal{D}$. Using this, the version without the word "locally" follows from Lemma B.2.5.

We begin by establishing the result in the case that $\mathcal{D} \in \mathsf{Cat} \subset 2\mathsf{Cat}$ is an $(\infty, 1)$-category. Since both conditions are compatible with pullback in the variable $\mathcal{D}$, it suffices to consider the case that $\mathcal{D} = [1]$.

Now, suppose first that the functor $\mathcal{E} \downarrow (\mathcal{C} \times [1])$ satisfies condition (1). Given an object $e_0 \in \mathcal{E}_{(c,0)}$, observe that the cocartesian 1-morphism $e_0 \to e_1$ in $\mathcal{E} \downarrow [1]$ lifting $0 \to 1$ canonically lifts to a cocartesian 1-morphism in $\mathcal{E}_c \downarrow [1]$. Thus, condition (1) implies condition (2) (assuming that $\mathcal{D} \in \mathsf{Cat}$).

In the other direction, suppose that the functor $\mathcal{E} \downarrow (\mathcal{C} \times [1])$ satisfies condition (2). By Observation B.2.7(3) it suffices to show that a cocartesian 1-morphism $e \to e'$ in $\mathcal{E}_c \downarrow [1]$ lifting $0 \to 1$ is also a cocartesian 1-morphism in $\mathcal{E} \downarrow [1]$. To see this, recall that $\mathcal{C} \xleftarrow{F} \mathcal{E}$ is a locally 2-cartesian fibration, and observe that the functor $\mathcal{E} \leftarrow \mathcal{E}_1$ is a morphism in $\mathsf{loc.2Cart}_{\mathcal{C}}$. Therefore, for any $e_1 \in \mathcal{E}_1$ we obtain a commutative triangle

$$\begin{tikzcd}
\mathsf{hom}_{\mathcal{E}}(e, e_1) \arrow[dr] & & \mathsf{hom}_{\mathcal{E}_1}(e', e_1) \arrow[ll] \arrow[dl] \\
& \mathsf{hom}_{\mathcal{C}}(F(e), F(e_1)) &
\end{tikzcd} \tag{B.4.1}$$

(recall that $F(e) = c$) that defines a morphism in $\mathsf{coCart}_{\mathsf{hom}_{\mathcal{C}}(F(e),F(e_1))}$. Hence, to show that the horizontal functor in diagram (B.4.1) is an equivalence, it suffices to verify that it is an equivalence on fibers over an arbitrary object $\varphi \in \mathsf{hom}_{\mathcal{C}}(F(e), F(e_1))$. Letting $e_1' \to e_1$ be a cartesian 1-morphism in $\mathcal{E}$ lifting the 1-morphism $F(e) \xrightarrow{\varphi} F(e_1)$ in $\mathcal{C}$, we see that on fibers the horizontal functor in diagram (B.4.1) is the composite equivalence

$$\mathsf{hom}^{\varphi}_{\mathcal{E}}(e, e_1) \simeq \mathsf{hom}^{\varphi}_{\mathcal{E}_{F(e)}}(e, e_1') \simeq \mathsf{hom}_{\mathcal{E}_{(F(e),1)}}(e', e_1') \simeq \mathsf{hom}_{\mathcal{E}_1}(e', e_1) \ .$$

So indeed, condition (2) implies condition (1) (assuming that $\mathcal{D} \in \mathsf{Cat}$).

We now consider the case that $\mathcal{D} \in 2\mathsf{Cat}$ is an arbitrary $(\infty, 2)$-category. Note that conditions (1) and (2) refer to both 1- and 2-morphisms, and the above special case establishes the equivalence of the conditions on 1-morphisms. Thus, the relevant functors to $\mathcal{D}$ are locally 2-cocartesian fibrations if and only if they are strict homwise cartesian fibrations.

Now, fix any objects $e, e' \in \mathcal{E}$ and let us respectively write $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$ for their images under F and G respectively. Then, we obtain a span

$$\mathsf{hom}_{\mathcal{C}}(c, c') \xleftarrow{F_{e,e'}} \mathsf{hom}_{\mathcal{E}}(e, e') \xrightarrow{G_{e,e'}} \mathsf{hom}_{\mathcal{D}}(d, d')$$

in Cat in which $F_{e,e'}$ is a cocartesian fibration that $G_{e,e'}$ smushes. By $((-)^{1\&2\mathsf{op}}$ applied to) the above special case, the functor $G_{e,e'}$ is a cartesian fibration that $F_{e,e'}$ smushes if and only if for every object $\varphi \in \mathsf{hom}_{\mathcal{C}}(c, c')$ the functor $\mathsf{hom}^{\varphi}_{\mathcal{E}}(e, e') \to \mathsf{hom}_{\mathcal{D}}(d, d')$ is a cartesian fibration. This latter condition holds if and only if for every object $e'' \in \mathcal{E}_c$ the functor $\mathsf{hom}_{\mathcal{E}_c}(e, e'') \to \mathsf{hom}_{\mathcal{D}}(G(e), G(e''))$ is a cartesian fibration: indeed, given an object $\varphi \in \mathsf{hom}_{\mathcal{C}}(c, c')$, letting $e'' \to e'$ be a locally cartesian

1-morphism lift in $\mathcal{E}$ we obtain a commutative triangle

$$\begin{tikzcd} \mathsf{hom}^{\varphi}_{\mathcal{E}}(e,e') \arrow[dr] & & \mathsf{hom}_{\mathcal{E}_c}(e,e'') \arrow[ll, "\sim"'] \arrow[dl] \\ & \mathsf{hom}_{\mathcal{D}}(d,d') & \end{tikzcd}$$

in Cat using the 2-cartesianness of $\mathcal{E}$ over $\mathcal{C}$, so that in particular one functor is a cartesian fibration if and only if the other is. Unwinding the definitions, we see that $\mathcal{E} \xrightarrow{G} \mathcal{D}$ is a strict homwise locally cartesian fibration if and only if the functors $\mathcal{E}_c \to \mathcal{D}$ are such for all objects $c \in \mathcal{C}$ and moreover condition (2)(ii) is satisfied, which proves the claim. □

Proof of Theorem B.4.1. Fix a right-lax functor $\mathcal{C} \rightsquigarrow 2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}}$. By lax (cartesian) unstraightening, this is equivalent data to a morphism

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "{(F,G)}"] \arrow[dr, "F"'] & & \mathcal{C}^{1\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathsf{pr}"] \\ & \mathcal{C}^{1\mathsf{op}} & \end{tikzcd}$$

in $\mathsf{loc.2Cart}_{\mathcal{C}^{1\mathsf{op}}}$ such that the upper horizontal functor satisfies condition(2) of Lemma B.4.5. Therefore, by Lemma B.4.5, G is a locally 2-cocartesian fibration that F smushes. By lax (cocartesian) straightening and ($(-)^{1\&2\mathsf{op}}$ applied to) Lemma B.4.5,[182] this is equivalent data to a left-lax functor $\mathcal{D} \rightsquigarrow 2\mathsf{Cat}_{2\mathsf{cart}/\mathcal{C}^{1\mathsf{op}}}$. The functoriality in $\mathcal{C}, \mathcal{D} \in \iota_1 2\mathsf{Cat}^{\mathsf{op}}$ is clear. Moreover, the first two specializations are evident from the construction. The third specialization is implemented by imposing the following condition: for any 1-morphisms $c \to c'$ in $\mathcal{C}^{1\mathsf{op}}$ and $d \to d'$ in $\mathcal{D}$ as well as a lifted commutative square

$$\begin{tikzcd} e \arrow[r, "\alpha"] \arrow[d, "\beta"'] & e' \arrow[d, "\gamma"] & (c,d) \arrow[r] \arrow[d] & (c',d) \arrow[d] \\ e'' \arrow[r, "\delta"'] & e''' \arrow[r, phantom, "\longmapsto"] & (c,d') \arrow[r] & (c',d') \end{tikzcd}$$

in $\mathcal{E}$, such that α is locally cartesian and γ is locally cocartesian, then β is locally cocartesian if and only if δ is locally cartesian.[183] □

Lemma B.4.6. *Suppose we are given a span $\mathcal{C} \xleftarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{D}$ in $2\mathsf{Cat}$. Then, the following conditions are equivalent.*

(1) The the functor $\mathcal{E} \xrightarrow{(F,G)} \mathcal{C} \times \mathcal{D}$ is a locally 2-cocartesian fibration that satisfies condition $(*)$ of Theorem B.4.3.

(2) The following three conditions are satisfied.

(i) The functor F is a locally 2-cocartesian fibration that G smushes.

(ii) For every object $c \in \mathcal{C}$, the functor $\mathcal{E}_c \to \mathcal{D}$ is a locally 2-cocartesian fibration.

(iii) For every 1-morphism $c_1 \to c_2$ in $\mathcal{C}$, the corresponding cocartesian monodromy functor $\mathcal{E}_{c_1} \to \mathcal{E}_{c_2}$ (which lies over $\mathcal{D}$ by the assumption that G smushes F) lies in $2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}}$ (i.e. it preserves cartesian 2-morphisms over $\mathcal{D}$).

Proof. Suppose that condition (1) is satisfied.

First of all, clearly condition (2)(ii) is satisfied.

[182] That is, we interchange the words "cartesian" and "cocartesian" in the statement of Lemma B.4.5.

[183] Here, we mean e.g. that α is locally cartesian with respect to the composite $\mathcal{E} \to \mathcal{C}^{1\mathsf{op}} \times \mathcal{D} \to \mathcal{C}^{1\mathsf{op}}$, or equivalently with respect to the functor $\mathcal{E}_d \to \mathcal{C}^{1\mathsf{op}}$.

We now establish condition (2)(i). For this, fix any object $e \in \mathcal{E}$, write $(c,d) := (F(e), G(e)) \in \mathcal{C} \times \mathcal{D}$ for its image, and fix a 1-morphism $c \xrightarrow{\varphi} c'$ in $\mathcal{C}$. Let $e \to e'$ be the locally cocartesian 1-morphism lift in $\mathcal{E}$ of the 1-morphism $(c,d) \xrightarrow{(\varphi, \mathsf{id}_d)} (c',d)$ in $\mathcal{C} \times \mathcal{D}$. We claim that this is also a locally cocartesian 1-morphism lift of the 1-morphism $c \xrightarrow{\varphi} c'$ in $\mathcal{C}$. To see this, fix an object $f \in \mathcal{E}_{c'}$, and write $d' := G(f) \in \mathcal{D}$ for its image. Then, we have a commutative triangle

$$
\begin{tikzcd}
\mathsf{hom}_{\mathcal{E}_{c'}}(e', f) \arrow[rr] \arrow[dr] & & \mathsf{hom}^{\varphi}_{\mathcal{E}}(e, f) \arrow[dl] \\
& \mathsf{hom}_{\mathcal{D}}(d, d') &
\end{tikzcd}
\tag{B.4.2}
$$

in Cat defining a morphism in $\mathsf{Cart}_{\mathsf{hom}_{\mathcal{D}}(d,d')}$ (using that (F,G) is a locally 2-cocartesian fibration, so that its pullback along $[1] \times \mathcal{D} \xrightarrow{\varphi \times \mathsf{id}_{\mathcal{D}}} \mathcal{C} \times \mathcal{D}$ is as well). So to verify that the upper horizontal functor in diagram (B.4.2) is an equivalence, it suffices to check that it is an equivalence on fibers over an arbitrary object $\psi \in \mathsf{hom}_{\mathcal{D}}(d,d')$. For this, let $e' \to e''$ be a locally cocartesian 1-morphism lift in $\mathcal{E}$ of the 1-morphism $(c',d) \xrightarrow{(\mathsf{id}_{c'}, \psi)} (c',d')$ in $\mathcal{C} \times \mathcal{D}$. Then, we have equivalences

$$
\mathsf{hom}^{\psi}_{\mathcal{E}_{c'}}(e', f) \simeq \mathsf{hom}_{\mathcal{E}_{(c',d')}}(e'', f) \simeq \mathsf{hom}^{(\varphi,\psi)}_{\mathcal{E}}(e, f) \ ,
$$

the latter by condition $(*)$ of Theorem B.4.3. It follows that the functor $\mathcal{E} \xrightarrow{F} \mathcal{C}$ admits locally cocartesian 1-morphism lifts. To see that it is a homwise cartesian fibration, observe that for any pair of objects $e, e' \in \mathcal{E}$, writing $(c,d), (c',d') \in \mathcal{C} \times \mathcal{D}$ for their images under (F,G), in the composite

$$
\mathsf{hom}_{\mathcal{E}}(e, e') \longrightarrow \mathsf{hom}_{\mathcal{C} \times \mathcal{D}}((c,d),(c',d')) \simeq \mathsf{hom}_{\mathcal{C}}(c,c') \times \mathsf{hom}_{\mathcal{D}}(d,d') \longrightarrow \mathsf{hom}_{\mathcal{C}}(c,c')
$$

the first functor is a cartesian fibration by assumption and hence the composite is as well. From here, the fact that it is a strict homwise cartesian fibration follows from the fact that (F,G) is. Moreover, it follows immediately from these considerations that G smushes F.

We now establish condition (2)(iii). Choose any 1-morphism $c \xrightarrow{\varphi} c'$ in $\mathcal{C}$ and objects $e, f \in \mathcal{E}_c$, and let $e \to e'$ and $f \to f'$ be locally cocartesian 1-morphism lifts in $\mathcal{E}$ of φ. Then, the locally cocartesian monodromy functor $\mathcal{E}_c \to \mathcal{E}_{e'}$ acts on hom-∞-categories via the diagram

$$
\mathsf{hom}_{\mathcal{E}_c}(e, f) \longrightarrow \mathsf{hom}^{\varphi}_{\mathcal{E}}(e, f') \xleftarrow{\sim} \mathsf{hom}_{\mathcal{E}_{c'}}(e', f')
\tag{B.4.3}
$$

in Cat, which lies over $\mathsf{hom}_{\mathcal{D}}(G(e'), G(f'))$ because G smushes F. Hence, condition (2)(iii) follows from the fact that the left functor in diagram (B.4.3) preserves cartesian morphisms because (F,G) is a strict homwise cartesian fibration.

Now, suppose that condition (2) is satisfied.

Fix a 1-morphism $(c,d) \to (c',d')$ in $\mathcal{C} \times \mathcal{D}$ and an object $e \in \mathcal{E}_{(c,d)}$. Let $e \to e'$ be a locally F-cocartesian 1-morphism lift of the 1-morphism $c \to c'$ in $\mathcal{C}$; this lies over the object $d \in \mathcal{D}$ because G smushes F (by condition (2)(i)). Then, let $e' \to e''$ be a locally cocartesian 1-morphism lift of $d \to d'$ with respect to the locally 2-cocartesian fibration $\mathcal{E}_{c'} \to \mathcal{D}$ (using condition (2)(ii)). Evidently the composite $e \to e' \to e''$ is a locally (F,G)-cocartesian 1-morphism. Thus, locally (F,G)-cocartesian 1-morphism lifts exist, and moreover condition $(*)$ of Theorem B.4.3 is satisfied.

Now, choose any pair of objects $e, e' \in \mathcal{E}$, and respectively write $(c,d), (c',d') \in \mathcal{C} \times \mathcal{D}$ for their images. Then, we have a span

$$
\mathsf{hom}_{\mathcal{C}}(c,c') \xleftarrow{F_{e,e'}} \mathsf{hom}_{\mathcal{E}}(e,e') \xrightarrow{G_{e,e'}} \mathsf{hom}_{\mathcal{D}}(d,d')
$$

in Cat in which F is a cartesian fibration that G smushes. Now, choose any 1-morphism $c \xrightarrow{\varphi} c_1$ in $\mathcal{C}$, and write $e \to e_1$ for its locally F-cocartesian 1-morphism lift in $\mathcal{E}$. Then, we have an equivalence

$$
\mathsf{hom}^{\varphi}_{\mathcal{E}}(e, e') \simeq \mathsf{hom}_{\mathcal{E}_{c_1}}(e_1, e')
$$

in Cat, which lies over $\mathsf{hom}_{\mathcal{D}}(d,d')$ because G smushes F. Thus, the functor $\mathsf{hom}^{\varphi}_{\mathcal{E}}(e,e') \to \mathsf{hom}_{\mathcal{D}}(d,d')$ is a cartesian fibration. Moreover, for every morphism $\varphi_1 \to \varphi_2$ in $\mathsf{hom}_{\mathcal{C}}(c,c')$, the corresponding cartesian monodromy functor $\mathsf{hom}^{\varphi_1}_{\mathcal{E}}(e,e') \leftarrow \mathsf{hom}^{\varphi_2}_{\mathcal{E}}(e,e')$ defines a morphism in $\mathsf{Cart}_{\mathsf{hom}_{\mathcal{D}}(d,d')}$: it lies over $\mathsf{hom}_{\mathcal{D}}(d,d')$ since G smushes F, and moreover preserves cartesian morphisms since F is a strict homwise cartesian fibration. It follows that the functor

$$\mathsf{hom}_{\mathcal{E}}(e,e') \longrightarrow \mathsf{hom}_{\mathcal{C}}(c,c') \times \mathsf{hom}_{\mathcal{D}}(d,d') \simeq \mathsf{hom}_{\mathcal{C}\times\mathcal{D}}((c,d),(c',d'))$$

is a cartesian fibration, i.e. that $\mathcal{E} \xrightarrow{(F,G)} \mathcal{C}\times\mathcal{D}$ is a homwise cartesian fibration. To see that it is a strict homwise cartesian fibration, we observe that $\mathcal{E} \xrightarrow{F} \mathcal{C}$ is such, so it suffices to show that $\mathcal{E} \xrightarrow{G} \mathcal{D}$ is such as well, which follows from a diagram chase using conditions (2)(ii) and (2)(iii). $\square$

Proof of Theorem B.4.3. Fix a left-lax functor $\mathcal{C} \rightsquigarrow 2\mathsf{Cat}_{\mathsf{loc.2cocart}/\mathcal{D}}$. By lax unstraightening, this is equivalent data to a morphism

$$\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[dr] & & \mathcal{C}\times\mathcal{D} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

in $\mathsf{loc.2coCart}_{\mathcal{C}}$ such that the upper horizontal functor satisfies condition (2) of Lemma B.4.6. The first assertion now follows from Lemma B.4.6. For the second assertion, it suffices to observe that the functor $\mathcal{E} \to \mathcal{C}$ is a 2-cocartesian fibration if and only if the functors $\mathcal{E}_d \to \mathcal{C}$ are 2-cocartesian fibrations for all $d \in \mathcal{D}$, which follows from Lemma B.2.5 and the fact that $\mathcal{E} \to \mathcal{D}$ smushes $\mathcal{E} \to \mathcal{C}$. $\square$

B.5. **Adjunctions.** In this subsection, we discuss adjunctions in $(\infty,2)$-categories (including in $2\mathsf{Cat}$). Our main results are Lemmas B.5.7 and B.5.8, which give parametrized versions of the mate correspondence; their proofs are adapted from [GR17, Chapter 12, §§3-4].

Local Notation B.5.1. Throughout this subsection, we fix $(\infty,2)$-categories $\mathcal{C},\mathcal{D} \in 2\mathsf{Cat}$.

Definition B.5.2. A 1-morphism $c \xrightarrow{L} d$ in $\mathcal{C}$ is a ***left adjoint*** if there exist a 1-morphism $c \xleftarrow{R} d$ (a ***right adjoint***) and 2-morphisms $\mathsf{id}_c \xrightarrow{\eta} RL$ and $LR \xrightarrow{\varepsilon} \mathsf{id}_d$ (a ***unit*** and ***counit***, respectively) such that the composite 2-morphisms

$$L \simeq L\,\mathsf{id}_c \xrightarrow{\mathsf{id}_L\,\eta} LRL \xrightarrow{\varepsilon\,\mathsf{id}_L} \mathsf{id}_d\,L \simeq L \qquad \text{and} \qquad R \simeq \mathsf{id}_c\,R \xrightarrow{\eta\,\mathsf{id}_R} RLR \xrightarrow{\mathsf{id}_R\,\varepsilon} R\,\mathsf{id}_d \simeq R \quad \text{(B.5.1)}$$

are homotopic to identity 2-morphisms. We write $\mathsf{Adj} \in \iota_1 2\mathsf{Cat}$ for the object corepresenting (the space of) left adjoints.[184] We generally consider $\mathsf{Adj} \in 2\mathsf{Cat}_{[1]/}$ via the epimorphism $[1] \to \mathsf{Adj}$ corepresenting the universal left adjoint. Dually, we say that a 1-morphism $c \xleftarrow{R} d$ in $\mathcal{C}$ is a ***right adjoint*** if there exist (L,η,ε) as above.

Observation B.5.3. It is immediate from Definition B.5.2 that we have a canonical equivalence $\mathsf{Adj} \simeq \mathsf{Adj}^{1\&2\mathsf{op}}$, and moreover that $\mathsf{Adj}^{1\mathsf{op}} \simeq \mathsf{Adj}^{2\mathsf{op}}$ corepresents the space of right adjoints. Furthermore, it follows e.g. from the description of $\mathsf{Adj} \in 2\mathsf{Cat}$ given in [RV16] that there is also a canonical equivalence $\mathsf{Adj} \simeq \mathsf{Adj}^{1\mathsf{op}}$; in particular, adjoints are unique when they exist. We use these facts without further comment.

Definition B.5.4. We write

$$2\mathsf{biCart}_{\mathcal{C}} := 2\mathsf{coCart}_{\mathcal{C}} \cap 2\mathsf{Cart}_{\mathcal{C}} \qquad \text{and} \qquad 2\mathsf{Cat}_{2\mathsf{bicart}/\mathcal{C}} := 2\mathsf{Cat}_{2\mathsf{cocart}/\mathcal{C}} \cap 2\mathsf{Cat}_{2\mathsf{cart}/\mathcal{C}}$$

and refer to objects of these $(\infty,2)$-categories as ***2-bicartesian fibrations*** over $\mathcal{C}$.

Lemma B.5.5. *Pullback along the functor* $[1] \to \mathsf{Adj}$ *defines an equivalence*

$$\mathsf{hom}_{\iota_1 2\mathsf{Cat}}(\mathsf{Adj}, 2\mathsf{Cat}) \xrightarrow{\sim} \iota_0 2\mathsf{biCart}_{[1]}\ .$$

[184] The existence of Adj is clear from the presentability of $\iota_1 2\mathsf{Cat}$. For an explicit description of $\mathsf{Adj} \in 2\mathsf{Cat}$, see [RV16].

Proof. By straightening, the pullback functor $\iota_0 2\mathsf{coCart}_{\mathsf{Adj}} \to \iota_0 2\mathsf{coCart}_{[1]}$ is a monomorphism. We show that it is an equivalence onto the subspace $\iota_0 2\mathsf{biCart}_{[1]} \subset \iota_0 2\mathsf{coCart}_{[1]}$.

Suppose first that we are given a functor $[1] \to 2\mathsf{Cat}$ that selects a left adjoint $\mathcal{C} \xrightarrow{L} \mathcal{D}$ in $2\mathsf{Cat}$. Consider the cocartesian unstraightening $\mathcal{E} \downarrow [1]$ of L. Using any adjunction data (R, η, ε) extending L, we show that $\mathcal{E} \downarrow [1]$ is a 2-cartesian fibration. Given any object $d \in \mathcal{D}$, the morphism $LRd \xrightarrow{\varepsilon} d$ in $\mathcal{D}$ determines a morphism $Rd \to d$ in $\mathcal{E}$, which it is easy to see is cartesian. Hence, by Observation B.2.7(3), $\mathcal{E} \downarrow [1]$ is a 2-cartesian fibration.

In the other direction, suppose that $\mathcal{E} \downarrow [1]$ is a 2-bicartesian fibration. We must show that its cocartesian unstraightening $[1] \xrightarrow{F} 2\mathsf{Cat}$ selects a left adjoint. Let us write $\mathcal{C} \xrightarrow{L} \mathcal{D}$ for the 1-morphism selected by its cocartesian straightening and $\mathcal{C} \xleftarrow{R} \mathcal{D}$ for the 1-morphism selected by its cartesian straightening. Now, an evident composite $[2] \times [2] \xrightarrow{G} [1] \xrightarrow{F} 2\mathsf{Cat}$ classifies a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "="] \arrow[d, "\parallel"'] & \mathcal{C} \arrow[r, "="] \arrow[d, "\parallel"'] & \mathcal{C} \arrow[d, "L"] \\
\mathcal{C} \arrow[r, "="] \arrow[d, "\parallel"'] & \mathcal{C} \arrow[r, "L"] \arrow[d, "L"'] & \mathcal{D} \arrow[d, "\parallel"] \\
\mathcal{C} \arrow[r, "L"'] & \mathcal{D} \arrow[r, "="'] & \mathcal{D}
\end{tikzcd}$$

in $2\mathsf{Cat}$.[185] Write $\widetilde{\mathcal{E}} := G^*\mathcal{E}$. Clearly, the canonical functor $\widetilde{\mathcal{E}} \downarrow [2] \times [2]$ is a 2-bicartesian fibration. Applying cocartesian straightening in the second coordinate, we obtain the first functor in the composite

$$[2] \longrightarrow 2\mathsf{Cat}_{\mathsf{2bicart}/[2]} \xrightarrow{\mathsf{fgt}} 2\mathsf{Cat}_{\mathsf{2cart}/[2]} \ ,$$

which by Theorem B.4.3 gives an object of $2\mathsf{Cat}_{\mathsf{loc.2cart}/([2]^{\mathsf{op}} \times [2])}$ that is classified by a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "\parallel"'] & \mathcal{C} \arrow[l, "="'] \arrow[d, "\parallel"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{C} \arrow[l, "="'] \arrow[d, "L"] \\
\mathcal{C} \arrow[d, "\parallel"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{C} \arrow[l, "="'] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow\varepsilon"] & \mathcal{D} \arrow[l, "R"'] \arrow[d, "\parallel"] \\
\mathcal{C} & \mathcal{D} \arrow[l, "R"] & \mathcal{D} \arrow[l, "="]
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
\mathcal{C} \arrow[d, "\parallel"'] & \mathcal{C} \arrow[l, "="'] \arrow[d, "\parallel"'] \arrow[dl, "="'] & \mathcal{C} \arrow[l, "="'] \arrow[d, "L"] \arrow[dl, "="'] \\
\mathcal{C} \arrow[d, "\parallel"'] & \mathcal{C} \arrow[l, "="'] \arrow[d, "L"'] \arrow[dl, "="'] & \mathcal{D} \arrow[l, "R"'] \arrow[d, "\parallel"] \arrow[dl, "LR"'] \\
\mathcal{C} & \mathcal{D} \arrow[l, "R"] & \mathcal{D} \arrow[l, "="]
\end{tikzcd}
\tag{B.5.2}$$

in $2\mathsf{Cat}$; in particular, this defines the 2-morphisms η and ε as (straightenings of) locally 2-cartesian fibrations over $[2]$ via appropriate functors $[2] \to ([2]^{\mathsf{op}} \times [2])$. Moreover, the requisite identifications of the composite 2-morphisms (B.5.1) follow from the functoriality of un/straightening with respect to functors $([1]^{\mathsf{op}} \times [1]) \to ([2]^{\mathsf{op}} \times [2])$, namely those selecting the right half and the bottom half of diagram (B.5.2) respectively. $\square$

Corollary B.5.6. *A functor $\mathcal{C} \xrightarrow{L} \mathcal{D}$ in $2\mathsf{Cat}$ is a left adjoint if and only if for every object $d \in \mathcal{D}$ the functor $\mathsf{hom}_{\mathcal{D}}(L(-), d) \in \mathsf{Fun}(\mathcal{C}^{1\mathsf{op}}, \mathsf{Cat})$ is representable, i.e. there exist an object $c \in \mathcal{C}$ and a 1-morphism $L(c) \to d$ in $\mathcal{D}$ such that the composite*

$$\mathsf{hom}_{\mathcal{C}}(c', c) \longrightarrow \mathsf{hom}_{\mathcal{D}}(L(c'), L(c)) \longrightarrow \mathsf{hom}_{\mathcal{D}}(L(c'), d)$$

is an equivalence in Cat for all $c' \in \mathcal{C}$.

Proof. This is immediate from Lemma B.5.5. $\square$

Lemma B.5.7. *The datum of a morphism*

$$\mathcal{E}_0 \longleftarrow \mathcal{E}_1 \tag{B.5.3}$$

[185] Here and in what follows, we orient our diagrams according to the same conventions as matrices (rows (top to bottom) before columns (left to right)).

in $2\mathsf{Cat}_{1\mathsf{cocart}/\mathcal{C}}$ *that on fibers over each object* $c \in \mathcal{C}$ *is a right adjoint is equivalent to the datum of a morphism*

$$\mathcal{E}_0 \longrightarrow \mathcal{E}_1 \tag{B.5.4}$$

in $2\mathsf{Cat}_{1\mathsf{cart}/\mathcal{C}^{1\mathsf{op}}}$ *that on fibers over each object* $c \in \mathcal{C}^{1\mathsf{op}}$ *is a left adjoint, with the equivalence given fiberwise by passing to adjoints.*

Proof. Let us consider the morphism (B.5.3) as a functor

$$[1]^{\mathsf{op}} \longrightarrow 2\mathsf{Cat}_{1\mathsf{cocart}/\mathcal{C}} \ .$$

By Theorem B.4.1, this is equivalent data to a functor

$$\mathcal{C} \longrightarrow 2\mathsf{Cat}_{1\mathsf{cart}/[1]} \simeq 1\mathsf{Cat}_{1\mathsf{cart}/[1]} = \mathsf{Cat}_{\mathsf{cart}/[1]}$$

that factors through the subcategory $\mathsf{Cat}_{\mathsf{bicart}/[1]} := \mathsf{Cat}_{\mathsf{cocart}/[1]} \cap \mathsf{Cat}_{\mathsf{cart}/[1]} \subset \mathsf{Cat}_{\mathsf{cart}/[1]}$. Applying Theorem B.4.1 to the resulting composite

$$\mathcal{C} \longrightarrow \mathsf{Cat}_{\mathsf{bicart}/[1]} \xrightarrow{\mathsf{fgt}} \mathsf{Cat}_{\mathsf{cocart}/[1]} \simeq 2\mathsf{Cat}_{1\mathsf{cocart}/[1]} \ ,$$

we obtain a functor

$$[1] \longrightarrow 2\mathsf{Cat}_{1\mathsf{cart}/\mathcal{C}^{1\mathsf{op}}} \ ,$$

which selects the desired morphism (B.5.4). It is now clear that this construction indeed defines an equivalence of spaces. $\square$

Lemma B.5.8. *Given a solid diagram*

$$\begin{tikzcd} {[1]^{\mathsf{op}}} \arrow[r, "F"] \arrow[d, hook] & 2\mathsf{Cat}_{1\mathsf{cart}/\mathcal{C}} \\ \mathsf{Adj}^{1\mathsf{op}} \arrow[ur, dashed] & \end{tikzcd} \ , \tag{B.5.5}$$

there exists an extension (i.e. F selects a right adjoint in $2\mathsf{Cat}_{1\mathsf{cart}/\mathcal{C}}$*) if and only if the following conditions are satisfied.*

(1) On fibers over each object $c \in \mathcal{C}$, the functor F selects a right adjoint.

(2) The functor F factors through the subcategory $1\mathsf{Cart}_{\mathcal{C}} \subseteq 2\mathsf{Cat}_{1\mathsf{cart}/\mathcal{C}}$.

Proof. Suppose first that there exists an extension (B.5.5). It is clear that condition (1) is satisfied. To verify condition (2), let us write

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, shift left, "R"] \end{tikzcd}$$

for the adjunction in $2\mathsf{Cat}_{1\mathsf{cart}/\mathcal{C}}$ (leaving the functors to $\mathcal{C}$ implicit). We must show that R preserves cartesian 1-morphisms over $\mathcal{C}$. For this, by taking pullback along an arbitrary functor $[1] \to \mathcal{C}$ it suffices to consider the case that $\mathcal{C} = [1]$. Given a cartesian 1-morphism $f_0 \to f_1$ in $\mathcal{F}$ over the morphism $0 \to 1$ in $[1]$, we must show that the 1-morphism $R(f_0) \to R(f_1)$ in $\mathcal{E}$ is also cartesian. For any object $e \in \mathcal{E}_0$ we have the commutative diagram

$$\begin{tikzcd} \mathsf{hom}_{\mathcal{E}_0}(e, R(f_0)) \arrow[r] \arrow[d, "\wr"] & \mathsf{hom}_{\mathcal{E}}(e, R(f_1)) \arrow[d, "\wr"] \\ \mathsf{hom}_{\mathcal{F}_0}(L(e), f_0) \arrow[r, "\sim"'] & \mathsf{hom}_{\mathcal{F}}(L(e), f_1) \end{tikzcd} \ ,$$

in which the vertical equivalences arise from the adjunctions $L \dashv R$ and $L_0 \dashv R_0$ and the lower morphism is an equivalence since $f_0 \to f_1$ is cartesian.

Now, suppose that conditions (1) and (2) are satisfied. We must show that there exists an extension

$$\begin{tikzcd} {[1]^{\mathsf{op}}} \arrow[r, "F"] \arrow[d, hook] & \mathsf{1Cart}_{\mathcal{C}} \arrow[d, hook] \\ \mathsf{Adj}^{\mathsf{1op}} \arrow[r, dashed] & \mathsf{2Cat}_{\mathsf{1cart}/\mathcal{C}} \end{tikzcd} . \tag{B.5.6}$$

We claim that it suffices to assume that $\mathcal{C}$ is a thin 2-category. Indeed, to show this we verify that it suffices to assume that $\mathcal{C} = c_k$ is the free $(\infty, 2)$-category on a k-morphism for $0 \leq k \leq 2$. For this, observe the monomorphism

$$\mathsf{hom}_{\iota_1 \mathsf{2Cat}}(\mathsf{Adj}^{\mathsf{1op}}, \mathsf{2Cat}_{\mathsf{1cart}/\mathcal{C}}) \hookrightarrow \mathsf{hom}_{\iota_1 \mathsf{2Cat}}([1]^{\mathsf{op}}, \mathsf{2Cat}_{\mathsf{1cart}/\mathcal{C}})$$

in $\mathcal{S}$, which by Theorem B.4.1 is equivalent to a morphism

$$\mathsf{hom}_{\iota_1 \mathsf{2Cat}}(\mathcal{C}^{\mathsf{1op}}, \mathsf{2Cat}_{\mathsf{1cocart}/\mathsf{Adj}^{\mathsf{1op}}}) \longrightarrow \mathsf{hom}_{\iota_1 \mathsf{2Cat}}(\mathcal{C}^{\mathsf{1op}}, \mathsf{2Cat}_{\mathsf{1cocart}/[1]^{\mathsf{op}}})$$

which is therefore also a monomorphism in $\mathcal{S}$, functorially in $\mathcal{C} \in \iota_1 \mathsf{2Cat}^{\mathsf{op}}$. It follows that the pullback functor

$$\mathsf{2Cat}_{\mathsf{cocart}/\mathsf{Adj}^{\mathsf{1op}}} \longrightarrow \mathsf{2Cat}_{\mathsf{cocart}/[1]^{\mathsf{op}}}$$

is a monomorphism in $\iota_1 \mathsf{2Cat}$. Hence, again applying Theorem B.4.1, there exists an extension (B.5.5) if and only if there exists an extension after pullback along each functor $c_k \to \mathcal{C}$. (And clearly conditions (1) and (2) hold if and only if they do after pullback along each functor $c_k \to \mathcal{C}$.) Thus, we can indeed assume that $\mathcal{C}$ is thin.

Now, since $\mathcal{C}$ is thin, the functor $\mathsf{2Cat}_{\mathsf{1cart}/\mathcal{C}} \xrightarrow{\mathsf{fgt}} \mathsf{2Cat}$ is 1-full. Hence, there exists an extension (B.5.5) if and only if the composite $[1]^{\mathsf{op}} \to \mathsf{2Cat}_{\mathsf{1cart}/\mathcal{C}} \xrightarrow{\mathsf{fgt}} \mathsf{2Cat}$ selects a functor admitting a left adjoint that satisfies the condition of admitting a lift to $\mathsf{2Cat}_{\mathsf{1cart}/\mathcal{C}}$.

Consider the cartesian unstraightening $\mathcal{E} \xrightarrow{(p,q)} \mathcal{C} \times [1]$ of the functor F (the upper horizontal functor in diagram (B.5.6)). By Lemma B.5.5, it is equivalent to show that $\mathcal{E} \xrightarrow{q} [1]$ is a 2-bicartesian fibration whose cocartesian 1-morphisms are carried to equivalences by the functor $\mathcal{E} \xrightarrow{p} \mathcal{C}$. By assumption, for each object $c \in \mathcal{C}$, the functor $\mathcal{E}_c \xrightarrow{q_c} [1]$ is a 2-bicartesian fibration. Given any $e_0 \in \mathcal{E}_0 := q^{-1}(0)$, let us write $s := p(e_0) \in \mathcal{C}$ and let $e_0 \to e_1$ be a q_s-cocartesian 1-morphism. We claim that this 1-morphism is in fact q-cocartesian, which will evidently give the desired result. That is, we must show that for any object $e_1' \in \mathcal{E}_1$, writing $t := p(e_1') \in \mathcal{C}$, the upper functor in the commutative triangle

$$\begin{tikzcd} \mathsf{hom}_{\mathcal{E}_1}(e_1, e_1') \arrow[rr] \arrow[dr] & & \mathsf{hom}_{\mathcal{E}}(e_0, e_1') \arrow[dl] \\ & \mathsf{hom}_{\mathcal{C}}(s,t) & \end{tikzcd} \tag{B.5.7}$$

in Cat is an equivalence. Because $\mathcal{E} \xrightarrow{p} \mathcal{C}$ is a 2-cartesian fibration, diagram (B.5.7) defines a morphism in $\mathsf{coCart}_{\mathsf{hom}_{\mathcal{C}}(s,t)}$, so it suffices to prove that its upper horizontal functor restricts to an equivalence $\mathsf{hom}^{\varphi}_{\mathcal{E}_1}(e_1, e_1') \to \mathsf{hom}^{\varphi}_{\mathcal{E}}(e_0, e_1')$ on fibers over an arbitrary object $\varphi \in \mathsf{hom}_{\mathcal{C}}(s,t)$. Now, let $e_1'' \to e_1'$ be a (p,q)-cartesian lift of the 1-morphism (id_1, φ). Then, we have a commutative diagram

$$\begin{tikzcd} \mathsf{hom}^{\varphi}_{\mathcal{E}_1}(e_1, e_1') \arrow[r] & \mathsf{hom}^{\varphi}_{\mathcal{E}}(e_0, e_1') \\ \mathsf{hom}_{\mathcal{E}_{(1,s)}}(e_1, e_1'') \arrow[u, "\wr"'] \arrow[r, "\sim"'] & \mathsf{hom}_{\mathcal{E}_s}(e_0, e_1'') \arrow[u, "\wr"'] \end{tikzcd}$$

in Cat, where the vertical morphisms are equivalences since $e_1'' \to e_1'$ is (p,q)-cartesian and the lower horizontal morphism is an equivalence since $e_0 \to e_1$ is q_s-cocartesian. $\square$

B.6. **Lax limits.** In this subsection, we define lax limits in Cat over $(\infty,2)$-categories, and we give an alternative description (Theorem B.6.2) in the case that the base is the left-laxification of an $(\infty,1)$-category in terms of its subdivision (as introduced and studied in §A.4).

Definition B.6.1. Given an $(\infty,2)$-category $\mathcal{C} \in 2\mathsf{Cat}$ and a functor $\mathcal{C} \to \mathsf{Cat}$, its ***left-*** and ***right-lax limits*** are respectively the $(\infty,1)$-categories of sections of its cocartesian unstraightening over $\mathcal{C}$ and its cartesian unstraightening over $\mathcal{C}^{1\mathsf{op}}$.[186] Evidently, these define right adjoints

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, shift left, "(-)\times\mathcal{C}"] \arrow[r, phantom, "\perp"] & 2\mathsf{Cat}_{1\mathsf{cocart}/\mathcal{C}} \arrow[l, shift left, dashed, "\lim^{\mathsf{l.lax}}_{\mathcal{C}}"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathsf{Cat} \arrow[r, shift left, "(-)\times\mathcal{C}^{1\mathsf{op}}"] \arrow[r, phantom, "\perp"] & 2\mathsf{Cat}_{1\mathsf{cart}/\mathcal{C}^{1\mathsf{op}}} \arrow[l, shift left, dashed, "\lim^{\mathsf{r.lax}}_{\mathcal{C}}"] \end{tikzcd} .$$

Theorem B.6.2. *Given an ∞-category $\mathcal{B} \in \mathsf{Cat}$, the composite*

$$\mathsf{loc.coCart}_{\mathcal{B}} \xleftarrow{\sim} 1\mathsf{coCart}_{\mathsf{l.lax}(\mathcal{B})} \simeq 1\mathsf{Cart}_{\mathsf{l.lax}(\mathcal{B})^{1\mathsf{op}}} \hookrightarrow 2\mathsf{Cat}_{1\mathsf{cart}/\mathsf{l.lax}(\mathcal{B})^{1\mathsf{op}}} \xrightarrow{\lim^{\mathsf{r.lax}}_{\mathsf{l.lax}(\mathcal{B})}} \mathsf{Cat}$$

(where the first (leftwards) equivalence is that of Observation B.4.2) is corepresented by the object $(\mathsf{sd}(\mathcal{B}) \xrightarrow{\mathsf{max}} \mathcal{B}) \in \mathsf{loc.coCart}_{\mathcal{B}}$.

Lemma B.6.3. *Suppose we are given $(\infty,2)$-categories $\mathcal{C},\mathcal{D} \in 2\mathsf{Cat}$, right-lax functors $F,G \in \hom_{\iota_1 2\mathsf{Cat}_{\mathsf{r.lax}}}(\mathcal{C},\mathcal{D})$, and a right-lax natural transformation $F \overset{\alpha}{\rightsquigarrow} G$. Suppose we are also given a 2-cartesian fibration $\mathcal{E} \downarrow \mathcal{C}$. Then, we obtain a morphism*

$$G^*\mathcal{E} \xrightarrow{\alpha^*\mathcal{E}} F^*\mathcal{E}$$

in $2\mathsf{Cat}_{/\mathcal{D}}$ as well as a lax-commutative triangle

$$\begin{tikzcd} \Gamma_{\mathcal{C}}(\mathcal{E}) \arrow[rr, "F^*"] \arrow[dr, "G^*"'] & \Downarrow & \Gamma_{\mathcal{D}}(F^*\mathcal{E}) \\ & \Gamma_{\mathcal{D}}(G^*\mathcal{E}) \arrow[ur, "\Gamma_{\mathcal{D}}(\alpha^*\mathcal{E})"'] & \end{tikzcd}$$

in $2\mathsf{Cat}$.[187] *Moreover, these data are natural with respect to pullback along right-lax functors $\mathcal{D}' \rightsquigarrow \mathcal{D}$.*

Proof. Let $\mathcal{D} \times [1] \overset{\beta}{\rightsquigarrow} \mathcal{C}$ be the right-lax functor defining α. Note that by (the $(-)^{1\&2\mathsf{op}}$ version of) Lemma B.4.6, the composite functor $\beta^*\mathcal{E} \to \mathcal{D} \times [1] \to [1]$ is a 2-cartesian fibration.

Now, we have a solid commutative square

$$\begin{tikzcd} \mathsf{Fun}_{/[1]}([1],\beta^*\mathcal{E}) \arrow[r, shift left, "\mathsf{ev}_1"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathsf{Fun}_{/[1]}(\{1\},\beta^*\mathcal{E}) \simeq G^*\mathcal{E} \arrow[l, shift left, dashed] \arrow[d] \\ \mathsf{Fun}([1],\mathcal{D}) \arrow[r, shift left, "\mathsf{ev}_1"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, dashed] \end{tikzcd} \tag{B.6.1}$$

in $2\mathsf{Cat}$. Moreover, in diagram (B.6.1) the dashed right adjoints exist by Corollary B.5.6 (for the upper right adjoint, using also that $\beta^*\mathcal{E} \downarrow [1]$ is a 2-cartesian fibration). Furthermore, it is clear that diagram (B.6.1) commutes after omitting the left adjoints (i.e. it satisfies the Beck–Chevalley condition). Hence, we obtain a diagram

$$\begin{tikzcd} \mathsf{Fun}_{/[1]}([1],\beta^*\mathcal{E}) \underset{\mathsf{Fun}([1],\mathcal{D})}{\times} \mathcal{D} \arrow[r, shift left, "\mathsf{ev}_1"] \arrow[r, phantom, "\perp"] \arrow[d, "\mathsf{ev}_0"'] & G^*\mathcal{E} \arrow[l, shift left, "\mathsf{ev}_1^R"] \\ F^*\mathcal{E} \end{tikzcd}$$

[186] These are also the $(\infty,2)$-categories of sections: all 2-morphisms therein are invertible.

[187] In fact, it is not hard to see that $G^*\mathcal{E} \xrightarrow{\alpha^*\mathcal{E}} F^*\mathcal{E}$ is a morphism in $2\mathsf{Cat}_{2\mathsf{cart}/\mathcal{D}}$.

in $2\mathsf{Cat}_{/\mathcal{D}}$, and thereafter a lax-commutative triangle

$$\begin{tikzcd} \mathsf{Fun}_{/[1]}([1], \beta^*\mathcal{E}) \underset{\mathsf{Fun}([1],\mathcal{D})}{\times} \mathcal{D} \arrow[rr, "\mathsf{ev}_0"] \arrow[dr, "\mathsf{ev}_1"'] & \arrow[d, phantom, "\Downarrow"] & F^*\mathcal{E} \\ & G^*\mathcal{E} \arrow[ur, "\mathsf{ev}_0\mathsf{ev}_1^R"'] & \end{tikzcd}$$

in $2\mathsf{Cat}_{/\mathcal{D}}$. Taking $\alpha^*\mathcal{E} := \mathsf{ev}_0\mathsf{ev}_1^R$, applying $\Gamma_{\mathcal{D}}(-)$, and precomposing with the evident functor

$$\Gamma_{\mathcal{C}}(\mathcal{E}) \longrightarrow \Gamma_{\mathcal{D}\times[1]}(\beta^*\mathcal{E}) \simeq \Gamma_{\mathcal{D}}\left(\mathsf{Fun}_{/[1]}([1], \beta^*\mathcal{E}) \underset{\mathsf{Fun}([1],\mathcal{D})}{\times} \mathcal{D}\right)$$

establishes the claim. □

Proof of Theorem B.6.2. We begin by noting the equivalences

$$\mathsf{sd}(\mathcal{B}) := \mathsf{colim}_{([n]\downarrow\mathcal{B})\in\mathbf{\Delta}_{/\mathcal{B}}}\mathsf{sd}([n]) \qquad \text{and} \qquad \mathsf{l.lax}(\mathcal{B}) \simeq \mathsf{colim}_{([n]\downarrow\mathcal{B})\in\mathbf{\Delta}_{/\mathcal{B}}}\mathsf{l.lax}([n])$$

(the latter by Observation B.1.21), which together imply that it suffices to consider the case that $\mathcal{B} = [n]$. Throughout, we refer to the description of $\mathsf{l.lax}([n])$ given by Proposition B.1.24.[188]

We now explicitly describe the image of the object $(\mathsf{sd}([n]) \xrightarrow{\mathsf{max}} [n]) \in \mathsf{loc.coCart}_{[n]}$ under the composite equivalence

$$\mathsf{loc.coCart}_{[n]} \xleftarrow{\sim} 1\mathsf{coCart}_{\mathsf{l.lax}([n])} \simeq 1\mathsf{Cart}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}$$

(where the first (leftwards) equivalence is that of Observation B.4.2). For this, let $\mathsf{coSpan}(\mathsf{sd}([n])) \in 2\mathsf{Cat}$ denote the strict 2-category of cospans in $\mathsf{sd}([n])$ (which exists since $\mathsf{sd}([n])$ has pushouts, which are given by union of subsets of $[n]$). We define the 1-full subcategory

$$\widetilde{\mathsf{sd}} := \widetilde{\mathsf{sd}([n])} \subseteq \mathsf{coSpan}(\mathsf{sd}([n]))$$

on those 1-morphisms $I \xrightarrow{I\hookrightarrow K \hookleftarrow J} J$ that are cospans among subsets of $[n]$ in which the inclusion $I \hookrightarrow K$ is isomax and the inclusion $K \hookleftarrow J$ is both isomin and inert. We note that $\widetilde{\mathsf{sd}}$ is a thin 2-category. Now, we claim that the desired image in $1\mathsf{Cart}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}$ is given by the functor

$$\widetilde{\mathsf{sd}} \xrightarrow{\mathsf{max}} \mathsf{l.lax}([n])^{1\mathsf{op}} \tag{B.6.2}$$

characterized by the fact that it carries a 1-morphism

$$\begin{tikzcd} {[i]} \arrow[r, hook, "\alpha'"] \arrow[dr, "\alpha"'] & {[k]} \arrow[d, "\gamma"] & {[j]} \arrow[l, hook', "\beta'"'] \arrow[dl, "\beta"] \\ & {[n]} & \end{tikzcd} \tag{B.6.3}$$

in $\widetilde{\mathsf{sd}}$ to the 1-morphism in $\mathsf{l.lax}([n])^{1\mathsf{op}}$ corresponding to the 1-morphism

$$\mathsf{max}(\alpha) = \alpha(i) = \gamma(\alpha'(i)) = \gamma(k) \xleftarrow{\gamma_{|[j,k]}} \gamma(j) = \gamma(\beta'(j)) = \beta(j) = \mathsf{max}(\beta)$$

in $\mathsf{l.lax}([n])$ given by the image along γ of the interval $[j,k] = [k]_{j/} = [k]_{\beta'(j)//\alpha'(i)} \subseteq [k]$. First of all, using Observation B.2.7(2) we see that the functor (B.6.2) is indeed a 1-cartesian fibration, with cartesian 1-morphisms given by those 1-morphisms (B.6.3) for which α' is an equality. Thus, in the 1-cartesian fibration (B.6.2) the fiber over an object $a \in \mathsf{l.lax}([n])^{1\mathsf{op}}$ is $\mathsf{sd}([n])_{\{a\}/\mathsf{isomax}}$ and its cartesian monodromy functors are given by concatenation. Noting that the straightening $\mathsf{l.lax}([n]) \to \mathsf{Cat}$ factors through the full subcategory $\mathsf{Poset} \subset \mathsf{Cat}$ so that it is characterized by its values on the objects and generating 1-morphisms in $\mathsf{l.lax}([n])$ (i.e. those of the form $a < b$), the claim follows.

In order to proceed, let us observe that the functor (B.6.2) admits an evident section

$$\widetilde{\mathsf{sd}} \xleftarrow{\sigma} \mathsf{l.lax}([n])^{1\mathsf{op}}$$

[188]In particular, we use without further reference that $\mathsf{l.lax}([n])^{1\mathsf{op}}$ is a thin 2-category.

whose image is the full subcategory on the inclusions of singletons into $[n]$, which we consider as a morphism in $2\mathsf{Cat}_{1\mathsf{cart}/\mathsf{l.lax}([n])^{1\mathsf{op}}}$. It now suffices to show that for any 1-cartesian fibration $(\mathcal{E} \downarrow \mathsf{l.lax}([n])^{1\mathsf{op}}) \in 2\mathsf{Cat}_{1\mathsf{cart}/\mathsf{l.lax}([n])^{1\mathsf{op}}}$, the resulting restriction functor

$$\mathsf{hom}_{1\mathsf{Cart}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}}(\widetilde{\mathsf{sd}}, \mathcal{E}) \longrightarrow \Gamma_{\mathsf{l.lax}([n])^{1\mathsf{op}}}(\mathcal{E}) \tag{B.6.4}$$

in Cat is an equivalence. We will construct an inverse.

For this, consider the right-lax functor

$$\widetilde{\mathsf{sd}} \overset{\mathsf{min}}{\rightsquigarrow} \mathsf{l.lax}([n])^{1\mathsf{op}}$$

characterized by the fact that it carries a 1-morphism (B.6.3) in $\widetilde{\mathsf{sd}}$ to the 1-morphism in $\mathsf{l.lax}([n])^{1\mathsf{op}}$ corresponding to the 1-morphism

$$\mathsf{min}(\alpha) = \alpha(0) = \gamma(\alpha'(0)) \xleftarrow{\gamma_{|[0,\alpha'(0)]}} \gamma(0) = \gamma(\beta'(0)) = \beta(0) = \mathsf{min}(\beta)$$

in $\mathsf{l.lax}([n])$ given by the image along γ of the interval $[0, \alpha'(0)] = [k]_{/\alpha'(0)} = [k]_{\beta'(0)//\alpha'(0)} \subseteq [k]$. Observe that there exists a right-lax natural transformation

$$\mathsf{max} \rightsquigarrow \mathsf{min} \tag{B.6.5}$$

from max to min characterized by the fact that it carries an object $(I \subseteq [n]) \in \widetilde{\mathsf{sd}}$ to the 1-morphism in $\mathsf{l.lax}([n])^{1\mathsf{op}}$ corresponding to the 1-morphism $\mathsf{max}(I) \xleftarrow{I} \mathsf{min}(I)$ in $\mathsf{l.lax}([n])$.[189]

We now construct a functor

$$\mathsf{hom}_{1\mathsf{Cart}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}}(\widetilde{\mathsf{sd}}, \mathcal{E}) \longleftarrow \Gamma_{\mathsf{l.lax}([n])^{1\mathsf{op}}}(\mathcal{E}) \tag{B.6.6}$$

in the opposite direction as (B.6.4), which we will later show to be its inverse. By Lemma B.6.3, we obtain a morphism $\mathsf{min}^*\mathcal{E} \to \mathsf{max}^*\mathcal{E}$ in $2\mathsf{Cat}_{\mathsf{loc.1cart}/\widetilde{\mathsf{sd}}}$ and on sections a composite functor

$$\Gamma_{\mathsf{l.lax}([n])^{1\mathsf{op}}}(\mathcal{E}) \longrightarrow \Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{min}^*\mathcal{E}) \longrightarrow \Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{max}^*\mathcal{E}) \ . \tag{B.6.7}$$

We claim that the functor (B.6.7) lands in the subcategory

$$\mathsf{hom}_{1\mathsf{Cart}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}}(\widetilde{\mathsf{sd}}, \mathcal{E}) \subseteq \mathsf{hom}_{2\mathsf{Cat}_{/\mathsf{l.lax}([n])^{1\mathsf{op}}}}(\widetilde{\mathsf{sd}}, \mathcal{E}) \simeq \Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{max}^*\mathcal{E}) \ ;$$

this will give our functor (B.6.6). To see this, let $[1] \xrightarrow{\varphi} \widetilde{\mathsf{sd}}$ select a $(\widetilde{\mathsf{sd}} \xrightarrow{\mathsf{max}} \mathsf{l.lax}([n])^{1\mathsf{op}}$-cartesian 1-morphism. By our above description thereof, the composite (strict) functor

$$[1] \xrightarrow{\varphi} \widetilde{\mathsf{sd}} \overset{\mathsf{min}}{\rightsquigarrow} \mathsf{l.lax}([n])^{1\mathsf{op}}$$

is constant. Therefore, the functor

$$\Gamma_{\mathsf{l.lax}([n])^{1\mathsf{op}}}(\mathcal{E}) \longrightarrow \Gamma_{[1]}(\varphi^*\mathsf{min}^*\mathcal{E})$$

factors through $\Gamma^{\mathsf{cart}}_{[1]}(\varphi^*\mathsf{min}^*\mathcal{E}) \subseteq \Gamma_{[1]}(\varphi^*\mathsf{min}^*\mathcal{E})$. Moreover, pullback along φ of the right-lax natural transformation (B.6.5) yields a strict natural transformation (between strict functors), so that the morphism

$$\varphi^*\mathsf{min}^*\mathcal{E} \longrightarrow \varphi^*\mathsf{max}^*\mathcal{E}$$

lies in $1\mathsf{Cart}_{[1]} \subset 2\mathsf{Cat}_{1\mathsf{cart}/[1]}$ (i.e. it preserves cartesian 1-morphisms). This proves the claimed factorization of the functor (B.6.7).

We now conclude by showing that the functors (B.6.4) and (B.6.6) are inverses.

We first show that the composite functor (B.6.4) $\circ$ (B.6.6) is the identity: this follows from the fact that (B.6.5) $\circ\, \sigma$ is the identity natural transformation from $\mathsf{id}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}$ to itself.

[189]For instance, given a 1-morphism (B.6.3), the lax-commutative square described in Remark B.1.12 is determined by the containment $\alpha([i]) \cup \gamma([k]_{/\alpha'(0)}) \subseteq \gamma([k]_{\beta'(j)/}) \cup \beta([j])$ among subsets of $[n]$.

We now show that the composite functor (B.6.6) $\circ$ (B.6.4) is the identity. Let us first observe that the right-lax natural transformation $\sigma \circ \mathsf{max} \xrightarrow{\sigma\circ(\text{B.6.5})} \sigma \circ \mathsf{min}$ factors as a composite

$$\sigma \circ \mathsf{max} \longrightarrow \mathsf{id}_{\widetilde{\mathsf{sd}}} \rightsquigarrow \sigma \circ \mathsf{min}$$

of right-lax natural transformations among right-lax endofunctors of $\widetilde{\mathsf{sd}}$, which is determined (using that $\widetilde{\mathsf{sd}}$ is thin) by the fact that its value on an object $(I \subseteq [n]) \in \widetilde{\mathsf{sd}}$ is the composite

$$\{\mathsf{max}(I)\} \xrightarrow{\{\mathsf{max}(I)\}\hookrightarrow I \overset{=}{\hookleftarrow} I} I \xrightarrow{I \overset{=}{\hookrightarrow} I \hookleftarrow \{\mathsf{min}(I)\}} \{\mathsf{min}(I)\} \ .$$

Now, since $\mathsf{max} \circ \sigma \circ \mathsf{max} \simeq \mathsf{max}$, the (strict) natural transformation $\sigma \circ \mathsf{max} \to \mathsf{id}_{\widetilde{\mathsf{sd}}}$ induces an equivalence on pullbacks of $\mathsf{max}^*\mathcal{E} \downarrow \widetilde{\mathsf{sd}}$. It follows that the composite (B.6.6) $\circ$ (B.6.4) is obtained by restriction to the full subcategory $\mathsf{hom}_{1\mathsf{Cart}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}}(\widetilde{\mathsf{sd}}, \mathcal{E}) \subseteq \Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{max}^*\mathcal{E})$ (in both the source and the target) of the functor

$$\Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{max}^*\mathcal{E}) \longrightarrow \Gamma_{\widetilde{\mathsf{sd}}}((\sigma \circ \mathsf{min})^*\mathsf{max}^*\mathcal{E}) \longrightarrow \Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{max}^*\mathcal{E}) \tag{B.6.8}$$

given by Lemma B.6.3 applied to the right-lax natural transformation $\mathsf{id}_{\widetilde{\mathsf{sd}}} \rightsquigarrow \sigma \circ \mathsf{min}$. Lemma B.6.3 also gives a natural transformation

$$\mathsf{id}_{\Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{max}^*\mathcal{E})} \longrightarrow (\text{B.6.8}) \ , \tag{B.6.9}$$

which it remains to show is an equivalence on the full subcategory $\mathsf{hom}_{1\mathsf{Cart}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}}(\widetilde{\mathsf{sd}}, \mathcal{E}) \subseteq \Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{max}^*\mathcal{E})$. For this, suppose we are given an object

$$\varphi \in \mathsf{hom}_{1\mathsf{Cart}_{\mathsf{l.lax}([n])^{1\mathsf{op}}}}(\widetilde{\mathsf{sd}}, \mathcal{E}) \subseteq \Gamma_{\widetilde{\mathsf{sd}}}(\mathsf{max}^*\mathcal{E}) \ .$$

For an arbitrary object $(I \subseteq [n]) \in \widetilde{\mathsf{sd}}$, consider the ($\widetilde{\mathsf{sd}} \xrightarrow{\mathsf{max}} \mathsf{l.lax}([n])^{1\mathsf{op}}$)-cartesian morphism $I \xrightarrow{I \overset{=}{\hookrightarrow} I \hookleftarrow \{\mathsf{min}(I)\}} \{\mathsf{min}(I)\}$. Since both φ and (B.6.8)(φ) preserve cartesian 1-morphisms, it suffices to show that the natural transformation (B.6.9) is an equivalence on objects given by singleton subsets of $[n]$, i.e. those in the image of σ. This follows from the equivalence (B.6.4) $\circ$ (B.6.6) $\simeq$ $\mathsf{id}_{\Gamma_{\mathsf{l.lax}([n])^{1\mathsf{op}}}(\mathcal{E})}$ that we have already shown. $\square$

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