On $K(n)$-equivalences of spaces

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Abstract. Working at a fixed prime p, we show that each $K(n)_*$-equivalence of spaces is a $K(m)_*$-equivalence for $1 \le m \le n$. Our proof uses homotopical localization theory and depends on the $K(n)_*$-nonacyclicily of the highly connected infinite loop spaces in the associated Ω-spectrum of $k(m)$.

1. Introduction

In [Ra1, Theorem 2.11], Ravenel showed that a $K(n)_*$-acyclic finite CW-spectrum must also be $K(m)_*$-acyclic for $1 \le m \le n$, where $K(m)_*$ is the m^{th} Morava K-theory at a prime p. Although the corresponding result fails for arbitrary connective spectra of finite type (see, e.g., Theorem 1.3), we will show that it holds for arbitrary spaces. Our main result is

Theorem 1.1. *Each $K(n)_*$-equivalence of spaces is a $K(m)_*$-equivalence for $1 \le m \le n$.*

This will follow immediately from Theorems 1.2 and 1.3 below. We call a spectrum X *connective* when $\pi_i X = 0$ for each $i < 0$, and we let $\underline{X}_k$ denote the k^{th} space in the associated Ω-spectrum of X. Following [HRW], we call X *strongly E_*-acyclic* for a homology theory E_* if $\underline{X}_q$ is E_*-acyclic for some $q \ge 0$. This implies that $\underline{X}_j$ is E_*-acyclic for all $j \ge q$. Let $k(m)$ denote the connective cover of $K(m)$.

Theorem 1.2. *For a homology theory E_* and integer $m \ge 1$, the following are equivalent:*

(i) *each E_*-equivalence of spaces is a $K(m)_*$-equivalence;*

(ii) *the spectrum $k(m)$ is not strongly E_*-acyclic.*

This will follow from Theorem 2.7 below, and can easily be sharpened to show that if $K(Z/p, q)$ is E_*-acyclic and $\underline{k(m)}_q$ is E_*-nonacyclic for some $q \ge 0$, then each E_*-equivalence of spaces is a $K(m)_*$-equivalence. Theorem 1.1 now follows from

Theorem 1.3. *For $1 \le m \le n$, the spectrum $k(m)$ is not strongly $K(n)_*$-acyclic, although it is $K(n)_*$-acyclic when $m \ne n$.*

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This is well-known to the experts, and will be proved in 2.8 using results of [Wi 1] and [HRW]. Theorem 1.2 has an analogue with $K(m)_*$ and $k(m)$ replaced by HZ/p_* and HZ/p respectively. In fact, by [Bo 2] or [Bo 5, Lemma 9.13] and [RW], we have

Theorem 1.4. *For a homology theory E_* and integer $n \geq 1$, if $K(Z/p, n)$ is not E_*-acyclic, then each E_*-equivalence of spaces is an $H_i(-; Z/p)$-equivalence for $i \leq n$. In particular, each $K(n)_*$-equivalence of spaces is an $H_i(-; Z/p)$-equivalence for $i \leq n$.*

In view of Theorems 1.1 and 1.4, we can extend the usual notion of "type" for finite complexes (see, e.g., [Ra 2]) to cover arbitrary spaces. We say that a space X has *type* n, for some $n \geq 1$, when X is $K(i)_*$-acyclic for $1 \leq i < n$ and is $K(i)_*$-nonacyclic for $i \geq n$, and we say that X has *type* ∞ when X is $K(i)_*$-acyclic for $i \geq 1$, or equivalently when X is $H_*(-; Z/p)$-acyclic. Each space now has a unique type.

We also see that the $K(n)_*$-equivalences of spaces are closely related to other sorts of equivalences. For instance, by Theorem 1.1 and [Ra 1, Theorem 2.1], we have

Theorem 1.5. *The homology theories $K(n)_* \oplus HQ_*$, $E(n)_*$, and $v_n^{-1}BP_*$ all determine the same equivalences of spaces, and likewise so do the homology theories $K(n)_*$, $E(n)Z/p_*$, and $v_n^{-1}BPZ/p_*$.*

This suggests that the $K(n)_*$-localizations of spaces should be closely related to the $E(n)_*$-localizations. For a space X and homology theory E_*, we let $X \to X_E$ denote the E_*-localization [Bo 1]. We call a group G *prenilpotent* when the lower central series $\{\Gamma_n G\}_{n \geq 1}$ is eventually constant, and we write ΓG for this constant term. Combining Theorem 1.5 with our version [Bo 2, Proposition 7.2] of Mislin's arithmetic square theorem, we now have

Theorem 1.6. *Let $X \in Ho_*$ be a connected space. If $\pi_1 X$ is prenilpotent with $\pi_1 X / \Gamma \pi_1 X$ finite, or if X is an H-space, then for $n \geq 1$ there is a natural equivalence $(X_{E(n)})_{HZ/p} \simeq X_{K(n)}$ and a natural homotopy fiber square*

$$\begin{array}{ccc} X_{E(n)} & \longrightarrow & X_{K(n)} \\ \downarrow & & \downarrow \\ X_{HQ} & \longrightarrow & (X_{K(n)})_{HQ}. \end{array}$$

Finally, we note that Steve Wilson has found a very different proof of Theorem 1.1 for spaces of finite type [Wi 2]. We thank him and Doug Ravenel for their comments on this work.

2. Proofs of the main theorems

Our main goal is to prove a generalized version of Theorem 1.2 involving "f-localizations" in the sense of [Bo 3], [Bo 4], [Bo 5], [Ca], [DF 1], and [DF 2]. Working in the homotopy categories Ho_* and Ho^s of pointed CW-complexes and of spectra, we start by recalling the required notions.

2.1. f-localizations of spaces. For a fixed map $f : A \to B$ in Ho_*, a space $Y \in Ho_*$ is called *f-local* when $f^* : \mathrm{map}(B, Y) \simeq \mathrm{map}(A, Y)$; a map $u : X \to X'$ in Ho_* is called an *f-equivalence* when $u^* : \mathrm{map}(X', Y) \simeq \mathrm{map}(X, Y)$ for each f-local space Y; and a map $X \to X'$ is called an *f-localization* of X when it is an f-equivalence to an f-local space X'. For each map f and space X, there is a natural f-localization $\alpha : X \to L_f X$. Moreover, by [Bo 5, 2.5] or [DF 2, 1.E.4], the E_*-localization of spaces for a homology theory E_* is given by an f-localization for a suitable map f.

2.2. ϕ-localizations of spectra. Similarly, for a fixed map $\phi : I \to J$ in Ho^s, a spectrum $Y \in Ho^s$ is called *ϕ-local* when $\phi^* : F^c(J, Y) \simeq F^c(I, Y)$, where $F^c(X, Y)$ is the connective cover of the function spectrum $F(X, Y)$; a map $u : X \to X'$ in Ho^s is called a *ϕ-equivalence* when $u^* : F^c(X', Y) \simeq F^c(X, Y)$ for each ϕ-local spectrum Y; and a map $X \to X'$ is called a *ϕ-localization* of X when it is a ϕ-equivalence to a ϕ-local spectrum X'. For each map ϕ and spectrum X, there is a natural ϕ-localization $\alpha = X \to L_\phi X$ by [Bo 4, Theorem 2.1]. The following result of [Bo 4, Theorem 2.10] will allow us to determine f-localizations of infinite loop spaces by using $\Sigma^\infty f$-localizations of the associated spectra.

Theorem 2.3. *For a map $f : A \to B$ in Ho_* and a spectrum X, there is a natural equivalence $L_f(\Omega^\infty X) \simeq \Omega^\infty(L_{\Sigma^\infty f} X)$.*

To study ϕ-localizations of module spectra where $\phi : I \to J$ is a fixed map in Ho^s, we need

Lemma 2.4. *Let E be a connective spectrum.*

(i) *If $h : X \to X'$ is a ϕ-equivalence in Ho^s, then so is $1 \wedge h : E \wedge X \to E \wedge X'$.*

(ii) *If Y is a ϕ-local spectrum in Ho^s, then so is $F(E, Y)$.*

Proof. This follows since there are natural equivalences

$$F^c\big(E, F^c(X, Y)\big) \simeq F^c(E \wedge X, Y) \simeq F^c\big(X, F(E, Y)\big)$$

when E is connective. □

2.5. ϕ-localizations of module spectra. Let E be a connective ring spectrum, and let M be a (left) E-module spectrum in the elementary sense of [Ad, p. 246]. Since the map $1 \wedge \alpha : E \wedge M \to E \wedge L_\phi M$ is a ϕ-equivalence by Lemma 2.4, the multiplication map $\mu : E \wedge M \to M$ extends to a unique map $\overline{\mu} : E \wedge L_\phi M \to L_\phi M$. This makes $L_\phi M$ into an E-module spectrum such that $\alpha : M \to L_\phi M$ is an E-module homomorphism. Moreover, L_ϕ acts as a functor $L_\phi : E\text{-Mod} \to E\text{-Mod}$ on the category of E-module spectra and homomorphisms. For M, $N \in E$-Mod, let $[M, N]^E \subset [M, N]$ denote the group of E-module homomorphisms $M \to N$. When N is ϕ-local, the isomorphism $\alpha^* : [L_\phi M, N] \cong [M, N]$ restricts to an isomorphism $\alpha^* : [L_\phi M, N]^E \cong [M, N]^E$. Hence, if $h : M \to M'$ is a ϕ-equivalence in E-Mod and if $N \in E$-Mod is ϕ-local, then $h^* : [M', N]^E \cong [M, N]^E$. In general, we see that the ϕ-localization theory in Ho^s restricts to a ϕ-localization theory in E-Mod.

For $m \geq 1$, recall that $k(m)$ is a ring spectrum with $\pi_* k(m) = Z/p[v_m]$ where $|v_m| = 2p^m - 2$.

Lemma 2.6. *For a map $\phi : I \to J$ of connective spectra, if $\Sigma^q HZ/p$ is ϕ-acyclic and if $\Sigma^q k(m)$ is ϕ-nonacyclic for some $q \geq 0$, then $k(m)$ is ϕ-local.*

Proof. Since $\Sigma^q k(m)$ is a module spectrum over the connective ring spectrum $k(m)$, so is $L_\phi \Sigma^q k(m)$. Since $\Sigma^q HZ/p$ is ϕ-acyclic, the (left) $k(m)$-module homomorphism $v_m : \Sigma^{j+2p^m-2} k(m) \to \Sigma^j k(m)$ is a ϕ-equivalence for $j \geq q+1$ and induces an isomorphism

$$v_m : \pi_j L_\phi \Sigma^q k(m) \cong \pi_{j+2p^m-2} L_\phi \Sigma^q k(m)$$

by 2.5. Moreover, each map $\Sigma^q HZ/p \to L_\phi \Sigma^q k(m)$ is nullhomotopic since $\Sigma^q HZ/p$ is ϕ-acyclic, and the map $\alpha : \Sigma^q k(m) \to L_\phi \Sigma^q k(m)$ is essential since $\Sigma^q k(m)$ is ϕ-nonacyclic. Hence, the groups $\pi_j L_\phi \Sigma^q k(m)$ cannot be trivial for all $j \geq q+1$, and the q-connected section of $L_\phi \Sigma^q k(m)$ is a nontrivial wedge of copies of $\Sigma^j k(m)$ for $j \geq q+1$. Thus $k(m)$ can be constructed from $L_\phi \Sigma^q k(m)$ using loopings, connective sections, and retracts. Consequently, $k(m)$ is also ϕ-local. □

We can now prove a generalized version of Theorem 1.2. For a map of spaces $f : A \to B$, a connective spectrum X will be called *strongly f-acyclic* if the infinite loop space $\underline{X}_q$ is f-acyclic for some $q \geq 0$. This implies that $\underline{X}_j$ is f-acyclic for all $j \geq q$.

Theorem 2.7. *For a map of spaces $f : A \to B$ and integer $m \geq 1$, the following are equivalent:*

(i) *each f-equivalence of spaces is a $K(m)_*$-equivalence;*

(ii) *the spectrum $k(m)$ is not strongly f-acyclic.*

Proof. First suppose that HZ/p is not strongly f-acyclic. Then $K(Z/p, q)$ is f-local for all $q \geq 1$ by [Bo 5, Lemma 9.13], and thus each f-equivalence of spaces is an $H_*(-; Z/p)$-equivalence. Hence, conditions (i) and (ii) both hold. Next suppose that $k(m)$ is strongly f-acyclic. Then, for some q, $\underline{k(m)}_q$ is f-acyclic, although it is not $K(m)^*$-acyclic since there is an essential map $\underline{k(m)}_q \to \underline{K(m)}_q$. Hence, conditions (i) and (ii) both fail. Finally, suppose that HZ/p is strongly f-acyclic and $k(m)$ is not. Then, by Theorem 2.3, $\Sigma^q HZ/p$ is $\Sigma^\infty f$-acyclic and $\Sigma^q k(m)$ is $\Sigma^\infty f$-nonacyclic for some $q \geq 0$. Hence, $k(m)$ is $\Sigma^\infty f$-local by Lemma 2.6, and $\underline{K(m)}_0 = \Omega^\infty k(m)$ is f-local. Thus conditions (i) and (ii) both hold in this final case. □

We conclude with

2.8. Proof of Theorem 1.3. Suppose that $\underline{k(m)}_q$ is $K(n)_*$-acyclic for some $q \geq 0$. Then the Postnikov map $\underline{K(m)}_q \to P^{q-1}\underline{K(m)}_q$ is a $K(n)_*$-equivalence and $K(n)_*(\underline{K(m)}_q)$ is even degree by [HRW, Corollary 1.3]. For $1 \leq m \leq n$, this contradicts Wilson's calculation of $K(n)_*(\underline{K(m)}_q)$ in [Wi 1], and we conclude that $k(m)$ is not strongly $K(n)_*$-acyclic. Finally, $k(m)$ is $K(n)_*$-acyclic for $m \neq n$ since $K(m)$ and HZ/p are $K(n)_*$-acyclic by [Ra 1, Theorem 2.1]. □

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