ALGEBRAIC K-THEORY OF REAL TOPOLOGICAL K-THEORY

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Abstract. We calculate the $A(1)$-homotopy of the topological cyclic homology of the connective real topological K-theory spectrum ko, and show that it is a finitely generated and free $\mathbb{F}_2[v_2^{32}]$-module of even rank between 390 and 444, on explicit generators in stems $-1 \le * \le 198$. This is achieved by using syntomic cohomology of ko as introduced by Hahn–Raksit–Wilson, extending work of Bhatt–Morrow–Scholze from the case of classical rings to $\mathbb{E}_\infty$ rings. In our case there are nontrivial differentials in the motivic spectral sequence from syntomic cohomology to topological cyclic homology, unlike in the case of complex K-theory at odd primes that was studied by Hahn–Raksit–Wilson.

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1. Introduction

Work of Hahn–Raksit–Wilson [HRW22] extends the notions of prismatic cohomology and syntomic cohomology to the setting of $\mathbb{E}_\infty$ rings. This produces a new tool for computing topological cyclic homology and, consequently, algebraic K-theory. In the present paper, we use this tool to compute the $A(1)$-homotopy of algebraic K-theory of the $\mathbb{E}_\infty$ ring ko known as connective real topological K-theory (cf. Notation 2.16). Throughout, we work at the prime $p = 2$.

This paper continues the program of the second and third authors of understanding the arithmetic of ring spectra through the lens of telescopically localized algebraic K-theory. In particular, the second and third authors conjectured, in a family of predictions known as the redshift conjectures [AR08], that algebraic K-theory increases chromatic complexity by one. This has now been proven in a

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qualitative form for all $\mathbb{E}_\infty$ rings in a tour de force [BSY22] building on [Yua21], [CMNN20], and [LMMT20].

To better understand the arithmetic of $\mathbb{E}_\infty$ rings, it is still desirable to prove more quantitative forms of the redshift conjectures, such as the one originally appearing as the "chromatic redshift problem" in [Rog00]. In the present paper, we solve this problem in the case of ko. Explicitly, we prove the following theorem.

Theorem A (Theorem 6.1)**.** *There is a subset $S \subset \mathbb{Z}_{\geq 0}$ of the form $S = T + 32\,\mathbb{Z}_{\geq 0}$, with $T \subset \{0, 1, \dots, 31\}$ satisfying $0, 1, 4 \in T$ and $2, 3 \notin T$, such that*

$$\begin{aligned} A(1)_* \operatorname{TC}(\mathrm{ko}) \cong \mathbb{F}_2&\{v_2^i \mid i \in S\} \\ &\oplus \mathbb{F}_2[v_2]\{\partial, \varsigma, \nu, \lambda_1', w, \lambda_2\} \\ &\oplus \mathbb{F}_2[v_2]\{\varsigma\nu, \nu^2, \partial\lambda_2, \nu w, \nu\lambda_2, \lambda_1'\lambda_2\} \\ &\oplus \mathbb{F}_2\{v_2^j \nu^2 w \mid j \geq 0, j+2 \in S\} \end{aligned}$$

is a finitely generated free $\mathbb{F}_2[v_2^{32}]$-module of rank $384 + 2\operatorname{card}(T)$.

Conjecture B (Eight-deck blackjack)**.** $S = \{i \geq 0 \mid i \equiv 0, 1 \mod 4\}$.

As a consequence, we determine the $A(1)$-homotopy of the algebraic K-theory of ko, with the same indeterminacy as Theorem A.

Theorem C (Theorem 6.4)**.** *There is an exact sequence of $\mathbb{F}_2[v_2^{32}]$-modules*

$$0 \to \Sigma^3\mathbb{F}_2 \longrightarrow A(1)_* \operatorname{K}(\mathrm{ko}) \xrightarrow{\mathrm{trc}} A(1)_* \operatorname{TC}(\mathrm{ko}) \longrightarrow \mathbb{F}_2\{\partial, \varsigma\} \to 0\,,$$

with $|\partial| = -1$ and $|\varsigma| = 1$.

Let $K(n)$ denote the 2-primary height n Morava K-theory, which has coefficients $K(n)_* = \mathbb{F}_2[v_n^{\pm 1}]$. Let L_n denote Bousfield localization at $K(0) \oplus \cdots \oplus K(n)$. Let L_n^f denote Bousfield localization at $T(0) \oplus \cdots \oplus T(n)$, where $T(m)$ is the v_m-telescope $v_m^{-1}F(m)$ of a spectrum $F(m)$ of type m, i.e., a finite 2-local spectrum such that $K(m)_*F(m) \neq 0$ and $K(m-1)_*F(m) = 0$.

Definition 1.1. We say that X satisfies the height n telescope conjecture (at the prime 2) if the canonical map $L_n^f X \to L_n X$ is an equivalence.

By recent groundbreaking work of Burklund–Hahn–Levy–Schlank [BHLS], we know that not all spectra satisfy the height n telescope conjecture. However, it is still interesting to consider the question of which spectra satisfy the height n telescope conjecture, for example see [MR99, Conjecture 7.3].

Theorem D (Theorem 7.1)**.** *The spectrum $\operatorname{TC}(\mathrm{ko})_2^\wedge$ satisfies the height 2 telescope conjecture.*

In [AR08], the second and third authors phrased the redshift conjecture in terms of telescopic complexity. We say that a 2-local spectrum X has telescopic complexity n if the map $X \to L_n^f X$ is an equivalence in all sufficiently large degrees.

Theorem E (Theorem 7.2)**.** *The spectrum $\operatorname{K}(\mathrm{ko})_{(2)}$ has telescopic complexity 2.*

Conventions. We let $\mathcal{A}$ and $\mathcal{A}^\vee$ denote the mod 2 Steenrod algebra and its dual, respectively. We write $H_*(X) := H_*(X; \mathbb{F}_2)$ and

$$\nu \colon H_*(X) \longrightarrow \mathcal{A}^\vee \otimes H_*(X)$$

for the $\mathcal{A}^\vee$-coaction. We write $\mathcal{A}(1)$ for the subalgebra $\langle \mathrm{Sq}^1, \mathrm{Sq}^2\rangle$ of $\mathcal{A}$ and note that $\mathcal{A}(1)^\vee = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$.

Let $\mathbb{Z}^{\mathrm{op}}$ be the category whose objects are integers, such that $\mathrm{Hom}_{\mathbb{Z}^{\mathrm{op}}}(n, m) = *$ if $n \geq m$ and empty otherwise. Let $\mathbb{Z}^\delta$ be the integers as a discrete category. Given a presentably symmetric monoidal stable ∞-category $\mathcal{C}$, we write

$$\mathcal{C}^{\mathrm{fil}} := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}) \text{ and } \mathcal{C}^{\mathrm{gr}} := \mathrm{Fun}(\mathbb{Z}^\delta, \mathcal{C})\,.$$

Given $I^\star \in \mathcal{C}^{\mathrm{fil}}$ and $J^* \in \mathcal{C}^{\mathrm{gr}}$, we write $I^w := I(w)$ and $J^w := J(w)$. We recall that there is a monoidal functor

$$\mathrm{gr}^* \colon \mathcal{C}^{\mathrm{fil}} \longrightarrow \mathcal{C}^{\mathrm{gr}}$$

defined on an object I by $\mathrm{gr}^w I = I^w/I^{w+1}$. As in the notation above, for consistency we use the superscript $\star$ as in $I^\star$ for a filtered object, the superscript $*$ as in J^* for a graded object, and a superscript $\bullet$ as in $K^\bullet$ for a cosimplicial object.

We use the terminology from [Isa19, Definition 4.1.2] for (hidden) extensions in spectral sequences.

In Sections 2–5, we use the following conventions: We implicitly 2-complete each of the following invariants: $\mathrm{THH} := \mathrm{THH}(-)^\wedge_2$, $\mathrm{TC}^- := \big(\mathrm{THH}(-)^{h\mathbb{T}}\big)^\wedge_2$, $\mathrm{TP} := \big(\mathrm{THH}(-)^{t\mathbb{T}}\big)^\wedge_2$, $\mathrm{TC} := \mathrm{TC}(-)^\wedge_2$, and $\mathrm{K} := \mathrm{K}(-)^\wedge_2$. We write ko, ku, KU, MU, MUP and $\mathbb{S}$ for the 2-completions of connective real topological K-theory, connective complex K-theory, periodic complex K-theory, complex cobordism, periodic complex cobordism and the sphere spectrum. We will simply write $\mathbb{Z}$ for the 2-adic integers. We write Sp_2 for the ∞-category of 2-complete spectra with symmetric monoidal product $\otimes$. Note that our smash product is implicitly 2-completed in Sections 2–5 so that $\otimes_R := (- \otimes_R -)^\wedge_2$ for any 2-complete $\mathbb{E}_\infty$ ring R and we omit R from the notation when R is the 2-complete sphere spectrum. We also write $\otimes$ for the (underived) tensor product over the 2-adic integers and expect the intended meaning to be clear from context. We write $\mathbb{T}$ for the circle regarded as the subgroup $\mathbb{T} \subset \mathbb{C}^\times$ of the units in the complex numbers. We write $\mathcal{C}^{B\mathbb{T}} := \mathrm{Fun}(B\mathbb{T}, \mathcal{C})$.

As in Section 1, we explicitly include notation for 2-completion in Sections 6 and 7. Therefore, we write $\otimes_R$ for the usual relative smash product over an $\mathbb{E}_\infty$ ring R in these sections. We also write $\otimes$ for the usual tensor product over the integers and expect the meaning to be clear from context. Note that the canonical map $\mathrm{TC}(\mathrm{ko})^\wedge_2 \to \mathrm{TC}(\mathrm{ko}^\wedge_2)^\wedge_2$ is an equivalence by [Mad94, p. 274-275] (cf. [NS18, p. 351-352]) so we do not include 2-completion in the argument of $\mathrm{TC}(-)^\wedge_2$.

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2. Hochschild homology and motivic filtrations

We first introduce filtrations on THH, TC^-, TP, and TC. The reader is encouraged to read [BHM93, BM94, BM95, HN20] for background on these invariants and [HRW22] for a thorough account of the filtrations that we use in this paper.

Definition 2.1. Given a map $f\colon A \to B$ of 2-complete $\mathbb{E}_\infty$ rings, we write $C^\bullet(B/A)$ for the associated cosimplicial Amitsur complex with $C^q(B/A) = B^{\otimes_A 1+q}$.

Recall that there is a map of $\mathbb{E}_\infty$ rings $c\colon \mathrm{ko} \to \mathrm{ku}$ called the complexification map. By restricting the $\mathbb{E}_\infty$ ring map $\mathrm{MUP} \to \mathrm{KU}$ of [HY20, Theorem 4.3], we produce an $\mathbb{E}_\infty$ ring map $\mathrm{MU} \to \mathrm{ku}$ (cf. [HL18, p. 3]). We choose the generators x_i of MU_* once and for all such that the image of x_i is zero in $\mathrm{ku}_* = \mathbb{Z}[u]$ for $i \geq 2$. This can be arranged since $x_1 \mapsto u$ and we can correct x_i by elements in the kernel of the canonical quotient map $\mathrm{ku}_* \to \mathbb{Z}$ to produce these generators.

This provides an $\mathbb{E}_\infty$ $C^q(\mathrm{MU}/\mathbb{S})$-algebra structure on $C^q(\mathrm{ku}/\mathrm{ko})$, compatibly for each $q \geq 0$. We write

$$\tau_{\geq \star}\colon \mathrm{Sp}_2 \longrightarrow \mathrm{Sp}_2^{\mathrm{fil}}$$

for the monoidal Whitehead filtration [Lur17, Proposition 1.4.3.6, Example 2.2.1.10].

Definition 2.2. We define $\mathbb{E}_\infty$-algebras in $\mathrm{Sp}_2^{\mathrm{fil}}$

$$\begin{aligned} \mathrm{fil}^\star_{\mathrm{mot}} F(\mathrm{ko}) &:= \mathrm{Tot}\left(\tau_{\geq 2\star} F(C^\bullet(\mathrm{ku}/\mathrm{ko})/C^\bullet(\mathrm{MU}/\mathbb{S}))\right), \\ \mathrm{gr}^*_{\mathrm{mot}} F(\mathrm{ko}) &:= \mathrm{gr}^*\left(\mathrm{fil}^\star_{\mathrm{mot}} F(\mathrm{ko})\right) \end{aligned}$$

for $F \in \{\mathrm{THH}, \mathrm{TC}^-, \mathrm{THH}^{tC_2}, \mathrm{TP}\}$. Here Tot denotes the totalization of a cosimplicial object. We refer to $\pi_* \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$ as the prismatic cohomology of ko.

Remark 2.3. We will show in Proposition 2.14 that $\pi_* \mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$ agrees with prismatic cohomology in the sense of [HRW22, Definition 1.2.4] building on [BMS19].

We also fix terminology for gradings.

Definition 2.4. Given $M \in \mathrm{Sp}_2^{\mathrm{gr}}$ and $x \in \pi_n M^w$ then we say that x has stem n (= topological degree n), weight w (= half internal degree w) and motivic filtration $2w-n$, and write $\|x\| = (n, 2w-n)$. We refer to each of the spectral sequences

$$E_2^{n,2w-n} = \pi_n \mathrm{gr}^w_{\mathrm{mot}} F(\mathrm{ko}) \implies \pi_n F(\mathrm{ko})$$

for $F \in \{\mathrm{THH}, \mathrm{THH}^{tC_2}, \mathrm{TC}^-, \mathrm{TP}\}$ as the motivic spectral sequence. We plot the motivic spectral sequence with stem on the horizontal axis and motivic filtration on the vertical axis. Given this convention, if $\|x\| = (n, 2w-n)$ then $\|d_r(x)\| = (n-1, 2w-n+r)$. Note that we write $E_r^{n,2w-n}$ where it is also standard to write $E_r^{2w-n,2w}$ in the literature. With these conventions, the motivic spectral sequence $E_r^{n,s}$-term is concentrated in degrees $n+s = 2w$ and, consequently, $d_r = 0$ for all even integers $r \geq 2$.

We will now show that the filtrations from Definition 2.2 agree with the motivic filtrations considered in [HRW22, Variant 4.2.2] (where stem=degree and motivic filtration=Adams weight).

Definition 2.5 ([HRW22, Definition 2.2.1])**.** A map of (discrete) 2-complete commutative rings $A \to B$ is discretely 2-completely faithfully flat if for every A-module C the spectrum $HC \otimes_{HA} HB$ is discrete and $(HC \otimes_{HA} HB) \otimes_{HA} HA/2$ is a faithfully flat $HA/2$ module concentrated in homological degree zero.

Definition 2.6 ([HRW22, Definition 2.2.13])**.** A map $A \to B$ of 2-complete $\mathbb{E}_\infty$ rings is eff if, for all even $\mathbb{E}_\infty$ A-algebras C, the graded commutative ring $\pi_*(B \otimes_A C)$ is concentrated in even degrees and the ungraded C_*-module $\pi_*(B \otimes_A C)$ is 2-completely faithfully flat over $\pi_* C = C_*$.

Warning 2.7. *We only consider discretely* 2*-completely eff maps in this paper and therefore we omit "discretely* 2*-completely" from our terminology throughout.*

Definition 2.8. A map $A \to B$ of 2-complete $\mathbb{E}_\infty$ rings is evenly free if for all even $\mathbb{E}_\infty$ A-algebras $C \neq 0$ the graded commutative ring $\pi_*(B \otimes_A C)$ is concentrated in even degrees and the ungraded C_*-module $\pi_*(B \otimes_A C)$ is free and non-zero.

Remark 2.9. An evenly free map $A \to B$ of 2-complete $\mathbb{E}_\infty$ rings is eff.

Lemma 2.10. *The complexification map* $\mathrm{ko} \to \mathrm{ku}$ *is evenly free. Moreover, there is an isomorphism of C_*-algebras*

$$\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*[b_1]/(\hat{b}_1^2)$$

where $\hat{b}_1^2 = b_1^2 + c_2 b_1 + c_4$ *for some* $c_2, c_4 \in C_*$ *where* $|c_2| = 2$ *and* $|c_4| = 4$.

Proof. Let $C \neq 0$ be an $\mathbb{E}_\infty$ ko-algebra. Then consider the Wood cofiber sequence

$$\Sigma\,\mathrm{ko} \xrightarrow{\eta} \mathrm{ko} \xrightarrow{c} \mathrm{ku} \xrightarrow{R} \Sigma^2\,\mathrm{ko}$$

where we use the (non-standard) notation R for the map satisfying $R(u \cdot -) = \Sigma^2 r$ where r is the realification map $\mathrm{ku} \to \mathrm{ko}$. The resulting cofiber sequence

$$\Sigma C \xrightarrow{\eta} C \longrightarrow \mathrm{ku} \otimes_{\mathrm{ko}} C \xrightarrow{R} \Sigma^2 C$$

induces a short exact sequence of C_*-modules

$$0 \to C_* \longrightarrow \pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \xrightarrow{R} \pi_*\Sigma^2 C \to 0 \tag{2.1}$$

since C is even and therefore $\pi_*\Sigma C \xrightarrow{\eta} C_*$ is zero. We conclude that $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C)$ is concentrated in even degrees.

A choice of class $b_1 \in \pi_2(\mathrm{ku} \otimes_{\mathrm{ko}} C)$ with $R(b_1) = \Sigma^2 1$ is equivalent to a choice of splitting of (2.1) so that $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*\{1, b_1\}$ is non-zero and free over C_*. Writing $\hat{b}_1^2 = b_1^2 + c_2 b_1 + c_4$ for the polynomial in $C_*\{1, b_1\}$ that vanishes, then there is an isomorphism

$$\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*[b_1]/(\hat{b}_1^2)$$

of C_*-algebras. □

Remark 2.11. Note that c_2 and c_4 in the statement of Lemma 2.10 need not be zero. For example, when $C = \mathrm{ku}$ with $C_* = \mathrm{ku}_* = \mathbb{Z}[u]$ and $|u| = 2$ then $\hat{b}_1^2 = b_1^2 - u b_1$ as in [DLR22, Lemma 5.1].

Proposition 2.12. *The map* $\mathrm{THH}(\mathrm{ko}) \to \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$ *induced by the complexification map* $c\colon \mathrm{ko} \to \mathrm{ku}$ *and the unit map* $\mathbb{S} \to \mathrm{MU}$ *is evenly free.*

Proof. Let $C \neq 0$ be an even $\mathbb{E}_\infty$ THH(ko)-algebra. Then $C = \mathrm{THH}(C/C)$ and

$$\begin{aligned}\mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} C &= \mathrm{THH}(\mathrm{ku}/\mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{ko})} \mathrm{THH}(C/C) \\ &\simeq \mathrm{THH}(\mathrm{ku} \otimes_{\mathrm{ko}} C/\mathrm{MU} \otimes C)\,,\end{aligned}$$

where the last equivalence holds because THH commutes with pushouts of $\mathbb{E}_\infty$ rings. Since C is an even $\mathbb{E}_\infty$ THH(ko)-algebra, we have an isomorphism of C_*-algebras $\pi_*(\mathrm{MU} \otimes C) \cong C_*[b_k : k \geq 1]$ for some choice of generators b_k for $k \geq 1$. By Lemma 2.10, we know that $\pi_*(\mathrm{ku} \otimes_{\mathrm{ko}} C) \cong C_*[b_1]/(\hat{b}_1^2)$ as a C_*-module. Note that the unit map $\pi_*(\mathrm{MU}\otimes C) \to \pi_*(\mathrm{ku}\otimes_{\mathrm{ko}} C)$ sends b_1 to b_1. To see this, consider the commutative diagram

$$\begin{tikzcd}
\pi_2(\mathrm{MU}\otimes\tau_{\geq 0}C) \arrow[r, "\cong"] \arrow[d] & \pi_2(\mathrm{ku}\otimes_{\mathrm{ko}}\tau_{\geq 0}C) \arrow[d] \\
\pi_2((\mathrm{MU}/\mathbb{S})\otimes\tau_{\geq 0}C) \arrow[r, "\cong"] & \pi_2(\Sigma^2\tau_{\geq 0}C)
\end{tikzcd}$$

where $\mathrm{MU}/\mathbb{S}$ denotes the cofiber of the unit map $\mathbb{S}\to\mathrm{MU}$. The horizontal maps in the commutative diagram above are isomorphisms because the map $\mathrm{MU}\to\mathrm{ku}$ is 4-connected and the map $\mathbb{S}\to\mathrm{ko}$ is 3-connected. We know $\pi_*(\mathrm{MU}\otimes\tau_{\geq 0}C) = \pi_*(\tau_{\geq 0}C)[b_k : k\geq 1]$ because $\tau_{\geq 0}C$ is an even $\mathbb{E}_\infty$ ring. Consequently, we can adjust our choice of generator b_1 by dividing by a unit in order to ensure that $b_1\in\pi_2(\mathrm{MU}\otimes\tau_{\geq 0}C)$ maps to $b_1\in\pi_2(\mathrm{ku}\otimes_{\mathrm{ko}}\tau_{\geq 0}C)$. We then choose $b_1\in\pi_2(\mathrm{MU}\otimes C)$ to be the image of $b_1\in\pi_2(\mathrm{MU}\otimes\tau_{\geq 0}C)$. We can also choose our algebra generators b_k for $k\geq 2$ so that they map to zero in $\pi_*(\mathrm{ku}\otimes_{\mathrm{ko}}C)$ by subtracting a linear polynomial in b_1 from our original choice of generators. From now on in this proof, we write b_k for the generators chosen as above by a slight abuse of notation.

We then apply the Künneth spectral sequence

$$\mathrm{Tor}_*^{C_*[b_k\,:\,k\geq 1]}(C_*[b_1]/(\hat{b}_1^2), C_*[b_1]/(\hat{b}_1^2)) \implies \pi_*((C\otimes_{\mathrm{ko}}\mathrm{ku})\otimes_{(C\otimes\mathrm{MU})}(C\otimes_{\mathrm{ko}}\mathrm{ku})).$$

Since $\hat{b}_1^2$ and b_k for $k\geq 2$ act trivially on $C_*[b_1]/(\hat{b}_1^2)$, we compute

$$\mathrm{Tor}_*^{C_*[b_k:k\geq 1]}(C_*[b_1]/(\hat{b}_1^2), C_*[b_1]/(\hat{b}_1^2)) \cong C_*[b_1]/(\hat{b}_1^2)\otimes\Lambda(\sigma\hat{b}_1^2, \sigma b_k : k\geq 2)$$

where the triviality of the product $(\sigma\hat{b}_1^2)^2$ follows from the fact that the product in Tor is given by the shuffle product. Since the algebra generators are all in filtration ≤ 1, the Künneth spectral sequence collapses at the E^2-term. Note that this is a homological spectral sequence associated to an increasing filtration. Since the E^∞-term is a non-zero free C_*-module after forgetting the grading and the abutment is also a C_*-module after forgetting the grading, we conclude that the abutment is a free non-zero C_*-module after forgetting the grading.

We can rule out all potential hidden multiplicative extensions because Tor_0 splits off from the abutment and all the classes in Künneth filtration one are in odd degree. Therefore, we have

$$D_* := \pi_*((C\otimes_{\mathrm{ko}}\mathrm{ku})\otimes_{(C\otimes\mathrm{MU})}(C\otimes_{\mathrm{ko}}\mathrm{ku})) \cong C_*[b_1]/(\hat{b}_1^2)\otimes\Lambda(\sigma\hat{b}_1^2, \sigma b_k : k\geq 2)$$

We then apply the Künneth spectral sequence

$$E^2_{*,*} = \mathrm{Tor}_*^{D_*}(C_*[b_1]/(\hat{b}_1^2), C_*[b_1]/(\hat{b}_1^2)) \implies \pi_*\,\mathrm{THH}(\mathrm{ku}\otimes_{\mathrm{ko}}C/\,\mathrm{MU}\otimes C).$$

The E^2-term of this spectral sequence is concentrated in even degrees so the spectral sequence collapses at the E^2-term and the abutment is also concentrated in even degrees.

Since the E^∞-term is a non-zero free C_*-module after forgetting the grading and the abutment is also a C_*-module after forgetting the grading, we conclude that the abutment is a free non-zero C_*-module after forgetting the grading. □

Corollary 2.13. *The spectrum* $C^q(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))$ *is an even* $\mathbb{E}_\infty$ ku-*algebra whose homotopy groups are free as a* ku_**-module.*

Proof. By Proposition 2.12, it suffices to show $\pi_* \operatorname{THH}(\mathrm{ku}/\mathrm{MU})$ is free as a ku_*-module and concentrated in even degrees. We compute that

$$\pi_*(\mathrm{ku} \otimes_{\mathrm{MU}} \mathrm{ku}) \cong \mathbb{Z}[u] \otimes \Lambda(\sigma x_i \mid i \geq 2)$$

using the Künneth spectral sequence, which collapses at the E_2-term. Here, we rule out all hidden multiplicative extensions because Tor_0 splits off of the abutment and all generators in Künneth filtration 1 are in odd degrees. We then apply the Künneth spectral sequence

$$\mathrm{Tor}_*^{\pi_*(\mathrm{ku}\otimes_{\mathrm{MU}}\mathrm{ku})}(\mathrm{ku}_*, \mathrm{ku}_*) \implies \mathrm{THH}_*(\mathrm{ku}/\mathrm{MU})\,.$$

We compute the input by filtering $\pi_*(\mathrm{ku} \otimes_{\mathrm{MU}} \mathrm{ku})$ such that the associated graded is the E_∞-term of the Künneth spectral sequence above. This produces a spectral sequence

$$\mathrm{Tor}_*^{\mathrm{ku}_*[b_1]/(\hat{b}_1^2)\otimes\Lambda(\sigma b_k|k\geq 2)}(\mathrm{ku}_*, \mathrm{ku}_*) \implies \mathrm{Tor}_*^{\pi_*(\mathrm{ku}\otimes_{\mathrm{MU}}\mathrm{ku})}(\mathrm{ku}_*, \mathrm{ku}_*)$$

whose input is $\mathrm{ku}_* \otimes \Gamma(\sigma^2 x_i \mid i \geq 2)$, which is a free ku_*-module concentrated in even degrees. Consequently, the Künneth spectral sequence collapses and the same argument as in the proof of Proposition 2.12 applies to show that $\mathrm{THH}_*(\mathrm{ku}/\mathrm{MU})$ is free as a ku_*-module concentrated in even degrees. □

Proposition 2.14. *The filtration from Definition 2.2 agrees with the motivic filtration considered in* [HRW22, Definition 4.2.1]*. Moreover there exist filtered maps*

$$\mathrm{can}, \varphi\colon \operatorname{fil}^\star_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) \longrightarrow \operatorname{fil}^\star_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}).$$

that converge to the canonical map and Frobenius map from [NS18]*.*

Proof. The first statement follows from [HRW22, Corollary 2.2.14] and Proposition 2.12 where $A = M = \mathrm{THH}(\mathrm{ko})$ and $B = \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$. In this case

$$M \otimes_A B^{\otimes_A 1+q} = A \otimes_A B^{\otimes_A 1+q} = B^{\otimes_A 1+q}$$

and $B^{\otimes_A 1+q}$ is even by Corollary 2.13 so that the even filtration agrees with the double-speed Whitehead filtration. In light of this, the second statement follows from [HRW22, Theorem 4.2.10]. □

Definition 2.15. In light of Proposition 2.14, we define

$$\begin{aligned}
\operatorname{fil}^\star_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) &:= \operatorname{fib}(\mathrm{can} - \varphi\colon \operatorname{fil}^\star_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) \longrightarrow \operatorname{fil}^\star_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}))\\
\operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) &:= \operatorname{gr}^*\left(\operatorname{fil}^\star_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})\right)
\end{aligned}$$

and in light of [HRW22, §5] and [BMS19], we refer to $\operatorname{gr}^w_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ as the syntomic cohomology of connective real topological K-theory and we refer to the spectral sequence

$$E_2^{n,2w-n} = \pi_n \operatorname{gr}^w_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \implies \pi_n \mathrm{TC}(\mathrm{ko})$$

as the motivic spectral sequence and follow the same conventions as Definition 2.2.

We now introduce the relevant coefficients. Let $C\eta$ denote the cofiber of the Hopf map $\eta\colon \Sigma\mathbb{S} \to \mathbb{S}$ and let $C2$ denote the cofiber of the map $2\colon \mathbb{S} \to \mathbb{S}$. By [DM81, Proposition 2.1], there exist four choices of finite spectrum $(C2 \otimes C\eta)/v_1$ each characterized by the Sq^4-action in its mod 2 cohomology.

Notation 2.16. Following [BEM17, §1], we write $A(1)[ij]$ with $i, j \in \{0, 1\}$ for the spectrum $(C2 \otimes C\eta)/v_1$ where Sq^4 on the generator in degree 0 of its mod 2 cohomology is i times the generator in degree 4, and Sq^4 on the generator in degree 2 of its mod 2 cohomology is j times the generator in degree 6. We write $A(1)$ in place of $A(1)[ij]$ when implicitly the statement holds for any choice of $i, j \in \{0, 1\}$.

The mod 2 cohomology $H^*(A(1); \mathbb{F}_2)$ is free of rank one over $\mathcal{A}(1)$ and there is an equivalence $A(1) \otimes \mathrm{ko} \simeq \mathbb{F}_2$. Hence $A(1) \otimes \mathrm{ko}$ admits a unique $\mathbb{E}_\infty$ ko-algebra structure.

Remark 2.17. If R is an $\mathbb{E}_\infty$ ko-algebra, then there are identifications

$$A(1) \otimes R = A(1) \otimes \mathrm{ko} \otimes_{\mathrm{ko}} R = H\mathbb{F}_2 \otimes_{\mathrm{ko}} R$$

and we may therefore regard $A(1) \otimes R$ as an $\mathbb{E}_\infty$ ko-algebra. For example, we identify

$$A(1) \otimes \mathrm{THH}(\mathrm{ko}) = \mathrm{THH}(\mathrm{ko}, \mathbb{F}_2)$$

and may therefore consider $A(1) \otimes \mathrm{THH}(\mathrm{ko})$ as an $\mathbb{E}_\infty$ ko-algebra.

Lemma 2.18. *Let $A := \mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2)$ and $B := A(1) \otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$. Then there is an isomorphism*

$$A_* \cong \mathbb{F}_2[\mu] \otimes \Lambda(\lambda_1', \lambda_2)$$

of graded $\mathbb{F}_2$-algebras where $|\lambda_1'| = 5$, $|\lambda_2| = 7$, and $|\mu| = 8$. There is also an isomorphism

$$B_* \cong \mathbb{F}_2[\mu] \otimes \Lambda(\bar{\xi}_1^2) \otimes P$$

of A_*-algebras where P is a polynomial algebra with generators in even degrees, $|\bar{\xi}_1^2| = 2$, and $|\mu| = 8$. The A_*-algebra structure is determined by the map $A_* \to B_*$ sending μ to μ and mapping λ_1' and λ_2 trivially.

Proof. The complexification map $c\colon \mathrm{ko} \to \mathrm{ku}$ and the unique $\mathbb{E}_\infty$ ring map $\mathrm{ku} \to H\mathbb{F}_2$ induce monomorphisms

$$H_*(\mathrm{ko}) = \mathbb{F}_2[\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_k : k \geq 3] \to H_*(\mathrm{ku}) = \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k : k \geq 3] \to \mathcal{A}^\vee$$

of $\mathcal{A}^\vee$-comodule algebras. By Milnor's construction of the ξ_i, the map $\mathrm{MU} \to \mathrm{ku} \to \mathbb{F}_2$ induces the homomorphism $H_*(\mathrm{MU}) = \mathbb{F}_2[b_i \mid i \geq 1] \to H_*(\mathbb{F}_2) = \mathcal{A}^\vee$ given by $b_{2^i-1} \mapsto \xi_i^2$ and $b_j \mapsto 0$ for $j \neq 2^i - 1$. Hence these formulas also hold in $H_*(\mathrm{ku})$. Let $\bar{b}_i = \chi b_i$ denote the conjugate classes in $H_*(\mathrm{MU})$, so that $\bar{b}_{2^i-1} \mapsto \bar{\xi}_i^2$ and $\bar{b}_j \mapsto 0$ for $j \neq 2^i - 1$. Standard Hochschild homology computations (cf. [MS93, Proposition 2]) produce

$$\mathrm{HH}_*(H_*(\mathrm{ko})) = H_*(\mathrm{ko}) \otimes \Lambda(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_k : k \geq 3) \text{ and}$$
$$\mathrm{HH}_*(H_*(\mathrm{ku})) = H_*(\mathrm{ku}) \otimes \Lambda(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_k : k \geq 3).$$

The usual argument for hidden extensions (cf. [AR05, Theorem 6.2]) implies

$$\sigma\bar{\xi}_3^{2^{k-3}} = \sigma\bar{\xi}_k$$

in the Bökstedt spectral sequence and produces identifications

$$H_*(\mathrm{THH}(\mathrm{ko})) \cong H_*(\mathrm{ko}) \otimes \Lambda(\sigma\bar{\xi}_1^4, \sigma\bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma\bar{\xi}_3] \text{ and}$$
$$H_*(\mathrm{THH}(\mathrm{ku})) \cong H_*(\mathrm{ku}) \otimes \Lambda(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2) \otimes \mathbb{F}_2[\sigma\bar{\xi}_3].$$

By Remark 2.17, we have

$$H_*(A(1) \otimes \mathrm{ku}) = H_*(H\mathbb{F}_2 \otimes_{\mathrm{ko}} \mathrm{ku})$$

and then applying a Künneth spectral sequence that collapses at the E_2-term, we have an identification

$$H_*(A(1) \otimes \mathrm{ku}) \cong \mathcal{A}^\vee \otimes \Lambda(\xi_1^2)$$

where hidden extensions are ruled out for filtration and bidegree reasons. Here ξ_1^2 denotes the $\mathcal{A}^\vee$-comodule primitive class so that $\pi_*(A(1) \otimes \mathrm{ku}) = \Lambda(\xi_1^2)$. We conclude that

$$\pi_* \operatorname{THH}(\mathrm{ko}, H\mathbb{F}_2) = \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu] \text{ and}$$
$$\pi_*(A(1) \otimes \operatorname{THH}(\mathrm{ku})) = \Lambda(\xi_1^2, \lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

where the map

$$\pi_* \operatorname{THH}(\mathrm{ku}, H\mathbb{F}_2) = \pi_*(A(1) \otimes \operatorname{THH}(\mathrm{ko})) \longrightarrow \pi_*(A(1) \otimes \operatorname{THH}(\mathrm{ku}))$$

induced by the complexification map sends λ_2 to λ_2, λ_1' to zero and μ to μ.

Next, we compute $\pi_*(\operatorname{THH}(\mathrm{ku}/\mathrm{MU}) \otimes A(1))$ using the fact that

$$A(1) \otimes \operatorname{THH}(\mathrm{ku}/\mathrm{MU}) \simeq (A(1) \otimes \operatorname{THH}(\mathrm{ku})) \otimes_{\operatorname{THH}(\mathrm{MU})} \mathrm{MU}\,.$$

We know

$$\pi_* \operatorname{THH}(\mathrm{MU}) \cong \mathrm{MU}_* \otimes \Lambda(\sigma \bar{b}_i : i \geq 1)$$

by [MS93, Remark 4.3] (cf. [Rog20, Proposition 4.5]). We expand

$$\operatorname{THH}(\mathrm{MU}) \longrightarrow \operatorname{THH}(\mathrm{ku}) \longrightarrow A(1) \otimes \operatorname{THH}(\mathrm{ku})$$

as the composite

$$\begin{aligned} \mathrm{MU} \otimes_{\mathrm{MU} \otimes \mathrm{MU}} \mathrm{MU} &\longrightarrow \mathrm{ku} \otimes_{\mathrm{ku} \otimes \mathrm{ku}} \mathrm{ku} \\ &\longrightarrow A(1) \otimes \mathrm{ku} \otimes_{A(1) \otimes \mathrm{ku} \otimes \mathrm{ku}} A(1) \otimes \mathrm{ku} \end{aligned}$$

where

$$\begin{aligned} \mathrm{MU}_*[\bar{b}_i \mid i \geq 1] = \pi_*(\mathrm{MU} \otimes \mathrm{MU}) &\longrightarrow \pi_*(\mathrm{ku} \otimes \mathrm{ku}) \\ &\longrightarrow \pi_*(A(1) \otimes \mathrm{ku} \otimes \mathrm{ku}) = \Lambda(\xi_1^2) \otimes H_*(\mathrm{ku}) \end{aligned}$$

takes $\bar{b}_{2^i-1}$ to $\bar{\xi}_i^2$ and $\bar{b}_j$ to 0 for $j \neq 2^i - 1$. Hence $\pi_* \operatorname{THH}(\mathrm{MU}) \to \pi_*(A(1) \otimes \operatorname{THH}(\mathrm{ku}))$ is given by $\sigma \bar{b}_1 \mapsto \sigma \bar{\xi}_1^2 = \lambda_1$, $\sigma \bar{b}_3 \mapsto \sigma \bar{\xi}_2^2 = \lambda_2$, and $\sigma \bar{b}_i \mapsto 0$ for $i \notin \{1, 3\}$. (This uses that $\sigma \bar{b}_{2^i-1} \mapsto \sigma \bar{\xi}_i^2 = \sigma \bar{\xi}_i \cdot \bar{\xi}_i + \bar{\xi}_i \cdot \sigma \bar{\xi}_i = 0$ for $i \geq 3$, while $\bar{\xi}_1$ and $\bar{\xi}_2$ do not exist in $H_*(\mathrm{ku})$). The $\pi_* \operatorname{THH}(\mathrm{MU})$-algebra structure on MU_* is given by mapping x_i to x_i and mapping $\sigma \bar{b}_i$ trivially for all $i \geq 1$. The Künneth spectral sequence therefore has E^2-term

$$\Lambda(\xi_1^2) \otimes \mathbb{F}_2[\mu] \otimes \Gamma(\sigma^2 \bar{b}_{2^i-1} \mid i \geq 3) \otimes \Gamma(\sigma^2 \bar{b}_j \mid j \neq 2^i - 1)\,.$$

This spectral sequence is concentrated in even degrees and therefore collapses at the E^2-term. We resolve the hidden multiplicative extensions using Steinberger's computation [BMMS86, III.2] of the Dyer–Lashof operations on $H_*(H\mathbb{F}_2) \cong \mathcal{A}^\vee$ and Kochman's computation [Koc74, Theorem 6] of the Dyer–Lashof operations on $H_*(\mathrm{BU}) \cong H_*(\mathrm{MU})$, as in the proof of [HW22, Lemma 2.4.1]. (Note that [BMMS86, III.2] is used for the $\sigma^2 \bar{b}_{2^i-1}$, while [Koc74, Theorem 6] is used for the remaining $\sigma^2 \bar{b}_j$). This produces the identification

$$\pi_*(A(1) \otimes \operatorname{THH}(\mathrm{ku}/\mathrm{MU})) = \Lambda(\xi_1^2) \otimes \mathbb{F}_2[\mu] \otimes P$$

where

$$P := \mathbb{F}_2[w_i \mid i \geq 0] \otimes \mathbb{F}_2[y_{j,i} \mid j \geq 2 \text{ even}, i \geq 0]$$

is a polynomial algebra with algebra generators in even degrees, where w_i is any choice of lift of $\gamma_{2^i}(\sigma^2\bar{b}_7)$ and $y_{j,i}$ is any choice of lift of $\gamma_{2^i}(\sigma^2\bar{b}_j)$. We conclude that $A(1)\otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$ is an even $\mathbb{E}_\infty$ ring and the $\pi_*(A(1)\otimes \mathrm{THH}(\mathrm{ko}))$-algebra structure is determined by λ_1' and λ_2 mapping trivially (for degree reasons) and μ mapping to μ by the first half of this proof. □

Definition 2.19. We define

$$\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2) = \mathrm{Tot}\left(\tau_{\geq 2\star}\left(A(1)\otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\right)\right).$$

Corollary 2.20. *The map $A(1)\otimes \mathrm{THH}(\mathrm{ko}) \to A(1)\otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$ induced by the complexification map $c\colon \mathrm{ko}\to\mathrm{ku}$ and the unit map $\mathbb{S}\to\mathrm{MU}$ is eff.*

Proof. This can be proven directly using Lemma 2.18, but instead we simply point out that it follows from Proposition 2.12 and Remark 2.17 by base change. □

Convention 2.21. To be consistent with our conventions, we write $\mathrm{fil}^\star_{\mathrm{ev}}$ for the functor denoted $\mathrm{fil}^\star_{\mathrm{ev},2}$ in [HRW22, Variant 2.1.7].

Remark 2.22. By [HRW22, Corollary 2.2.14], Corollary 2.20, an Remark 2.17, we can identify

$$\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2) \simeq \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2)$$

in the sense of [HRW22, Construction 2.1.2].

Theorem 2.23. *We can identify*

$$\pi_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2) \cong \mathbb{F}_2[\mu]\otimes\Lambda(\lambda_1',\lambda_2)$$

as a bigraded $\mathbb{F}_2$-algebra with $\|\lambda_1'\| = (5,1)$, $\|\lambda_2\| = (7,1)$, and $\|\mu\| = (8,0)$.

Proof. We closely follow [HW22] and [HRW22]. Starting with the proof of [HW22, Proposition 6.1.6], let $A := \mathrm{THH}(\mathrm{ko},\mathbb{F}_2) = A(1)\otimes\mathrm{THH}(\mathrm{ko})$ and $B := A(1)\otimes \mathrm{THH}(\mathrm{ku}/\mathrm{MU})$, so that $\pi_*A = A_*\cong\mathbb{F}_2[\mu]\otimes\Lambda(\lambda_1',\lambda_2)$ and $\pi_*B = B_*\cong\mathbb{F}_2[\mu]\otimes\Lambda(\xi_1^2)\otimes P$ by Lemma 2.18. The descent spectral sequence associated to the cosimplicial Amitsur resolution $C^\bullet(B/A) = B^{\otimes_A 1+\bullet}$ for $A\to B$ has E_1-term

$${}^qE_1(B/A) = \pi_*(B^{\otimes_A 1+q})$$

for $q\geq 0$, and converges to A_*. Since B_* is concentrated in even stems, Proposition 2.12 implies that $\Sigma := \pi_*(B\otimes_A B)$ is even and free over B_*, so that (B_*,Σ) is a flat Hopf algebroid. Let $C^*_\Sigma(B_*,B_*)$ denote the associated cobar complex. It follows by induction on q that the natural homomorphism

$$\begin{aligned} C^q_\Sigma(B_*,B_*) = \Sigma\otimes_{B_*}\cdots\otimes_{B_*}\Sigma \xrightarrow{\cong} \pi_*((B\otimes_A B)\otimes_B\cdots\otimes_B(B\otimes_A B))\\ \cong \pi_*(B\otimes_A\cdots\otimes_A B) = {}^qE_1(B/A)\end{aligned}$$

is an isomorphism for each $q\geq 0$, since the relevant Künneth spectral sequences collapse. Passing to cohomology, we obtain an isomorphism

$$\mathrm{Ext}^*_\Sigma(B_*,B_*)\cong E_2^*(B/A)\,,$$

identifying the descent spectral sequence E_2-term with the Hopf algebroid cohomology of (B_*,Σ). We claim that in each stem this E_2-term has the same finite order as A_*, so that the descent spectral sequence for $A\to B$ must collapse at $E_2 = E_\infty$.

By convergence, the descent E_2-term is an upper bound for A_*. To show that the bound is exact, we consider the multiplicative Whitehead filtrations $\tau_{\geq *}A$ and $\tau_{\geq *}B$

of A and B, respectively. For each $q \geq 0$ we equip $B^{\otimes_A q+1}$ with the relative convolution filtration

$$\mathrm{fil}^* B^{\otimes_A q+1} = (\tau_{\geq *}B)^{\otimes_{(\tau_{\geq *}A)} q+1},$$

having associated graded $\mathbb{E}_\infty$-ring

$$\mathrm{gr}^* B^{\otimes_A q+1} = H\pi_* B^{\otimes_{H\pi_* A} q+1}.$$

Here $H\pi_* A$ and $H\pi_* B$ can be interpreted as the graded $\mathbb{E}_\infty$-rings $\mathrm{gr}^* \tau_{\geq *}A$ and $\mathrm{gr}^* \tau_{\geq *}B$. We proved in Lemma 2.18 that $A_* \to B_*$ is given by $\mu \mapsto \mu$, $\lambda_1' \mapsto 0$ and $\lambda_2 \mapsto 0$, so that

$$\bar{\Sigma} := \pi_*(H\pi_* B_* \otimes_{H\pi_* A} H\pi_* B) \cong \mathbb{F}_2[\mu] \otimes \Lambda(\xi_1^2) \otimes P \otimes \Lambda(\xi_1^2) \otimes P \otimes \Gamma(\sigma\lambda_1', \sigma\lambda_2)$$

is even and free over B_*. Hence $(B_*, \bar{\Sigma})$ is a flat Hopf algebroid, and as above we have compatible isomorphisms

$$C^q_{\bar{\Sigma}}(B_*, B_*) = \bar{\Sigma} \otimes_{B_*} \cdots \otimes_{B_*} \bar{\Sigma} \xrightarrow{\cong} H\pi_* B^{\otimes_{H\pi_* A} q+1}$$

for all $q \geq 0$. Since these bigraded groups are concentrated in even stems, and differentials reduce the stem by one, the convolution filtration spectral sequence

$$\pi_*(H\pi_* B^{\otimes_{H\pi_* A} q+1}) \implies \pi_*(B^{\otimes_A q+1})$$

collapses at this term. This proves that $\pi_*(B^{\otimes_A q+1}) = {}^qE_1(B/A)$ has a descending filtration with associated graded given by $C^q_{\bar{\Sigma}}(B_*, B_*)$. These filtrations are compatible for varying $q \geq 0$, so the descent E_1-term is a filtered differential graded algebra with associated graded $E_1 = C^*_{\bar{\Sigma}}(B_*, B_*)$. Passing to cohomology, we obtain the May–Ravenel spectral sequence

$$E_2 = \mathrm{Ext}_{\bar{\Sigma}}(B_*, B_*) \implies \mathrm{Ext}_{\Sigma}(B_*, B_*)$$

converging to the descent E_2-term, cf. [Rav86, Theorem A1.3.9].

We now view the Hopf algebroid $(B_*, \bar{\Sigma})$ as the tensor product of the three Hopf algebroids

$$(\mathbb{F}_2[\mu], \mathbb{F}_2[\mu]),\ (\Lambda(\xi_1) \otimes P, \Lambda(\xi_1) \otimes P \otimes \Lambda(\xi_1) \otimes P) \text{ and } (\mathbb{F}_2, \Gamma(\sigma\lambda_1', \sigma\lambda_2)).$$

These have cohomology algebras $\mathbb{F}_2[\mu]$, $\mathbb{F}_2$ and $\Lambda(\lambda_1', \lambda_2)$, respectively, with $\mu \in \mathrm{Ext}^0$ and $\lambda_1', \lambda_2 \in \mathrm{Ext}^1$. This confirms that the May–Ravenel E_2-term

$$\mathrm{Ext}_{\bar{\Sigma}}(B_*, B_*) \cong \mathbb{F}_2[\mu] \otimes \Lambda(\lambda_1', \lambda_2)$$

has the same finite order in each stem as A_*, which implies that the May–Ravenel spectral sequence and the descent spectral sequence both collapse at their E_2-terms. Moreover, there is no room for hidden multiplicative extensions, since λ_1' and λ_2 both square to zero in A_*.

We have now established that the descent spectral sequence

$$E_1^{*,q}(B/A) = \pi_*(B^{\otimes_A q+1}) \implies A_*$$

is concentrated in even internal degrees $n + q = 2w$, having E_2-term

$$E_2(B/A) = \mathrm{Ext}_{\Sigma}(B_*, B_*) = \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$$

with (n, q)-bidegrees $|\lambda_1'| = (5, 1)$, $|\lambda_2| = (7, 1)$ and $|\mu| = (8, 0)$. Following [HRW22, Example 4.2.3] we apply [HRW22, Corollary 2.2.14(1)] to the eff map $A \to B$, to see that

$$\mathrm{fil}^*_{\mathrm{ev}} A \xrightarrow{\simeq} \mathrm{Tot}\big(\mathrm{fil}^*_{\mathrm{ev}} B^{\otimes_A \bullet+1}\big) = \mathrm{Tot}\big(\tau_{\geq 2*}(B^{\otimes_A \bullet+1})\big)$$

is an equivalence. For each integer weight w there is a spectral sequence converging to $\pi_* \operatorname{Tot}\big(\tau_{\geq 2w}(B^{\otimes_A \bullet+1})\big)$, with E_1-term given by the part of the descent spectral sequence $E_1(B/A)$ that is located in internal degrees $n+q \geq 2w$. The d_1-differential preserves this part, so the E_2-term for weight w is given by the part of $E_2(B/A)$ in the same range of internal degrees. By naturality, this spectral sequence must collapse at the E_2-term, since the full descent spectral sequence does so. It follows that

$$\pi_* \operatorname{fil}^w_{\mathrm{ev}} A \longrightarrow A_*$$

maps the source isomorphically to the subgroup of classes in internal degree $\geq 2w$, and $\pi_* \operatorname{gr}^w_{\mathrm{ev}} A$ is isomorphic to the summand in A_* consisting of classes in internal degree $= 2w$. Hence

$$\pi_* \operatorname{gr}^*_{\mathrm{ev}} A \cong \Lambda(\lambda'_1, \lambda_2) \otimes \mathbb{F}_2[\mu]$$

as bigraded algebras, with $(n,\ 2w-n)$-bidegrees $\|\lambda'_1\| = (5,1)$, $\|\lambda_2\| = (7,1)$ and $\|\mu\| = (8,0)$. $\square$

Following [HRW22, Corollary 2.2.17], the map $\mathbb{S} \to \mathrm{MU}$ is eff and MU is even so that $\operatorname{fil}^\star_{\mathrm{ev}} \mathbb{S} \simeq \operatorname{Tot}\big(\tau_{\geq 2\star} C^\bullet(\mathrm{MU}/\mathbb{S})\big)$ and $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S} = \operatorname{Tot} H\pi_{2*} C^\bullet(\mathrm{MU}/\mathbb{S})$. Note that the even filtration is symmetric monoidal, so $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$ is an $\mathbb{E}_\infty$-algebra in graded 2-complete spectra, and $\operatorname{gr}^*_{\mathrm{ev}}$ is a lax symmetric monoidal functor from 2-complete spectra to $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$-modules.

Convention 2.24. We will simply write $\otimes$ for $\otimes_{\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}}$ when it is clear that we are in the category of modules over $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$ from the context.

Construction 2.25. We construct a $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$-module $\overline{A}(1)$ as follows. Note that by [GIKR22], we can identify $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$ with $C\tau$ in the $\mathbb{C}$-motivic homotopy category $\mathrm{SH}(\mathbb{C})$ (cf. [HRW22, Remark 1.1.7]). Consequently, by [GWX21, Theorem 1.13, Remark 4.15] there is a $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$-module corresponding to the $\mathrm{MU}_* \mathrm{MU}$-comodules $\mathrm{MU}_* C\eta$ and an $\mathbb{E}_\infty$ $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$-algebra corresponding to $\mathrm{MU}/(2, v_1, \dots, v_k)$ for $k \geq 0$. We write $\overline{C}\eta$ for the former $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$-module and $\overline{V}(k)$ for the latter. We then have an equivalence of $\overline{V}(1)$-modules $\overline{A}(1) \simeq \overline{V}(1) \otimes \overline{C}\eta$, corresponding to the isomorphism $\mathrm{MU}_*(A(1)) \cong \mathrm{MU}_*/(2, v_1) \otimes_{\mathrm{MU}_*} \mathrm{MU}_*(C\eta)$ of $\mathrm{MU}_*/(2, v_1)$-modules in $\mathrm{MU}_* \mathrm{MU}$-comodules. Consequently, we know that $\overline{A}(1)$ is a $\overline{V}(1)$-module. When M is a $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$-module and $\overline{V} \in \{\overline{C}\eta, \overline{V}(k), \overline{C}\eta \otimes \overline{V}(k)\}$, we write

$$\overline{V}_* M := \pi_*(\overline{V} \otimes M).$$

Note that we are applying Convention 2.24 throughout this construction.

Lemma 2.26. *The* $\mathrm{MU}_* \mathrm{MU}$*-comodule* $\mathrm{MU}_* C\eta$ *is not* $\mathrm{MU}_* \mathrm{MU}$*-comodule algebra.*

Proof. Note that the coaction $\nu\colon \mathrm{MU}_* C\eta \to \mathrm{MU}_* \mathrm{MU} \otimes_{\mathrm{MU}_*} \mathrm{MU}_* C\eta$ satisfies

$$\nu(1) = 1 \otimes 1$$
$$\nu(b_1) = b_1 \otimes 1 + 1 \otimes b_1 \,.$$

If $b_1^2 = p + q b_1$ with $p \in MU_4 = \mathbb{Z}\{x_1^2\}$ and $q \in MU_2 = \mathbb{Z}\{x_1\}$ we would get

$$\nu(b_1^2) = b_1^2 \otimes 1 + 2b_1 \otimes b_1 + 1 \otimes (p + qb_1) = \nu(p + qb_1) = p \otimes 1 + qb_1 \otimes 1 + q \otimes b_1 \,,$$

so that $b_1^2 + \eta_R(p) = p + qb_1$ and $2b_1 + \eta_R(q) = q$. From $\eta_R(x_1) = x_1 + 2b_1$ it follows that $q = -x_1$, so that $\eta_R(p) - p = -x_1 b_1 - b_1^2$. Since $\eta_R(x_1^2) - x_1^2 = 4x_1 b_1 + 4b_1^2$, this cannot happen for p an integer multiple of x_1^2. $\square$

Remark 2.27. A consequence of Lemma 2.26 is that $\overline{A}(1)$ is not an $\mathbb{E}_1$-algebra in $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-modules. See Remark 4.7 for further consequences.

Lemma 2.28. *There is an equivalence*

$$\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \simeq \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2)$$

of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$*-modules.*

Proof. Let $v_1 \colon \Sigma^2 V(0) \otimes C\eta \to V(0) \otimes C\eta$ be one of the eight v_1-maps, with cofiber one of the four spectra $A(1)$. Since 2 and η come from $\pi_* \mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$, the structure map

$$\overline{V}(0) \otimes \overline{C}\eta \otimes \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ku} \xrightarrow{\simeq} \mathrm{gr}^*_{\mathrm{ev}}(V(0) \otimes C\eta \otimes \mathrm{ku})$$

is an equivalence. The cofiber of $\mathrm{gr}^*_{\mathrm{ev}}(v_1) \otimes 1$ acting on the left is $\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ku}$, and the cofiber of $v_1 \otimes 1$ acting on $V(0) \otimes C\eta \otimes \mathrm{ku}$ is $A(1) \otimes \mathrm{ku} \simeq \mathbb{F}_2 \vee \Sigma^2 \mathbb{F}_2$. Since $\mathrm{MU}_*(V(0) \otimes C\eta \otimes \mathrm{ku})$ is concentrated in even degrees, it follows as in [GIKR22, Proposition 3.18] that applying $\mathrm{gr}^*_{\mathrm{ev}}$ to the latter cofiber sequence again gives a cofiber sequence, so that the cofiber of $\mathrm{gr}^*_{\mathrm{ev}}(v_1 \otimes 1)$ acting on the right is indeed $\mathrm{gr}^*_{\mathrm{ev}}(A(1) \otimes \mathrm{ku})$.

Consequently, there are equivalences

$$\begin{aligned}
&\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{ev}}\, C^q(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko})) \\
&\qquad \simeq \mathrm{gr}^*_{\mathrm{ev}}\big(A(1) \otimes C^q(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\big) \\
&\qquad\qquad \simeq \mathrm{gr}^*_{\mathrm{ev}}\big(C^q(A(1) \otimes \mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/A(1) \otimes \mathrm{THH}(\mathrm{ko}))\big)
\end{aligned}$$

for each $q \geq 0$ compatible with the cosimplicial structure maps. Altogether, this implies that

$$\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \simeq \mathrm{gr}^*_{\mathrm{ev}}\left(A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\right)$$

and the result follows from Corollary 2.20. $\square$

Remark 2.29. By Remark 2.27, the $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-module $\overline{A}(1)$ is not an $\mathbb{E}_1$ $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-algebra. However, by Lemma 2.28, there is an identification of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-modules

$$\overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) = \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}, H\mathbb{F}_2)$$

where the right-hand side is an $\mathbb{E}_\infty$ $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-algebra. We therefore use this to equip the left-hand side with an $\mathbb{E}_\infty$ $\mathrm{gr}^*_{\mathrm{ev}}\,\mathbb{S}$-algebra structure. Note that the left-hand side also has a canonical action of the circle $\mathbb{T}$, but this $\mathbb{T}$-action is not an action by $\mathbb{E}_\infty$ ring maps because the right-hand side is not equipped with a compatible $\mathbb{T}$-action.

Corollary 2.30. *There are preferred isomorphisms of bigraded* $\mathbb{F}_2$*-algebras*

$$\overline{A}(1)_* \, \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \cong \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$$

and

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \, \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \cong \Lambda(\varepsilon_2, \lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]$$

Proof. The first isomorphism is a direct consequence of Theorem 2.30, Lemma 2.28, and Remark 2.29. The second isomorphism holds because we can choose a null homotopy of v_2 compatible with the map

$$\overline{V}(2)_* \, \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \, \mathrm{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,.$$

$\square$

Lemma 2.31. *There is a multiplicative, strongly convergent, trigraded η-Bockstein spectral sequence*

$$\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})[\eta] \implies \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}). \tag{2.2}$$

Proof. Following [HRW22, Corollary 2.2.17], the map $\mathbb{S} \to \mathrm{MU}$ is eff and $\mathrm{MU}_* \mathrm{ko}$ is even so that

$$\operatorname{fil}^\star_{\mathrm{ev}} \mathbb{S} \simeq \operatorname{Tot}\big(\tau_{\geq 2\star}(\mathrm{ko} \otimes C^\bullet(\mathrm{MU}/\mathbb{S}))\big)$$

and

$$\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} = \operatorname{Tot}\big(H\pi_{2*}(\mathrm{ko} \otimes \mathrm{MU}^{\otimes 1+\bullet})\big)\,.$$

We first show that the η-Bockstein spectral sequence associated to the filtered object in graded spectra

$$\ldots \xrightarrow{\eta} \Sigma^{1,1}\overline{V}(1) \otimes \operatorname{gr}^*_{\mathrm{ev}}(\mathrm{ko}) \xrightarrow{\eta} \overline{V}(1) \otimes \operatorname{gr}^*_{\mathrm{ev}}(\mathrm{ko}) \tag{2.3}$$

is multiplicative. Note that the normalized Tot-filtration of

$$\operatorname{Tot} C^\bullet(\mathrm{ku}/\mathrm{ko}) \simeq \mathrm{ko}$$

is the same as the η-adic tower

$$\cdots \longrightarrow \Sigma^2 \mathrm{ko} \xrightarrow{\eta} \Sigma \mathrm{ko} \xrightarrow{\eta} \mathrm{ko}\,.$$

Equivalently, the normalized Tot-tower ends

$$\ldots \to \lim\big(\ \mathrm{ku} \rightrightarrows \mathrm{ku} \otimes_{\mathrm{ko}} \mathrm{ku}\ \big) \longrightarrow \mathrm{ku}$$

and is equivalent to

$$\ldots \to C\eta^2 \otimes \mathrm{ko} \to C\eta \otimes \mathrm{ko}\,.$$

Here we are using the Wood cofiber sequence

$$\Sigma \mathrm{ko} \xrightarrow{\eta} \mathrm{ko} \xrightarrow{c} \mathrm{ku} \xrightarrow{R} \Sigma^2 \mathrm{ko}$$

of ko-modules. The Bousfield–Kan homotopy spectral sequence for the cosimplicial $\mathbb{E}_\infty$ $\operatorname{gr}^*_{\mathrm{ev}}$ ko-algebra

$$C^\bullet(\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku} / \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko})$$

is multiplicative, and converges conditionally and strongly to $\pi_* \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}$. Its normalized E_1-term in codegree q is

$$\Sigma^{q,q} \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku}\,,$$

which is concentrated in even internal degrees (= integral weights). In the same way, the descent = Bousfield–Kan homotopy spectral sequence for the cosimplicial $\operatorname{gr}^*_{\mathrm{ev}}$ ko-algebra

$$C^\bullet(\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku} / \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}) \otimes_{\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}} \overline{V}$$

is multiplicative, and converges conditionally and strongly to $\pi_* \overline{V}$. The evenness/integrality noted above shows that its normalized E_1-term in codegree q is

$$E_1^q = \Sigma^{q,q} \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ku} \otimes_{\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}} \overline{V}\,.$$

This is the cofiber of

$$h_1 \otimes 1 \colon \Sigma^{q+1,q+1} \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \otimes_{\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}} \overline{V} \longrightarrow \Sigma^{q,q} \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \otimes_{\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}} \overline{V}$$

as well as of

$$1 \otimes h_1 \colon \Sigma^{q+1,q+1} \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \otimes_{\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}} \overline{V} \longrightarrow \Sigma^{q,q} \operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko} \otimes_{\operatorname{gr}^*_{\mathrm{ev}} \mathrm{ko}} \overline{V}\,,$$

since h_1 lies in the homotopy of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$. Hence E_1^q is equivalent to

$$\Sigma^{q,q}\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \otimes_{\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}} \overline{A} = \overline{A}\{h_1^q\}\,,$$

with $\overline{A} = \overline{A}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$. This uses the cofiber sequence

$$\Sigma^{1,1}\overline{V} \xrightarrow{h_1} \overline{V} \to \overline{A} \longrightarrow \Sigma^{2,0}\overline{V}$$

of $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$-modules, which follows from

$$\Sigma^{1,1}\overline{V}(1) \xrightarrow{h_1} \overline{V}(1) \longrightarrow \overline{A}(1) \to \Sigma^{2,0}\overline{V}(1)\,.$$

To summarize, the descent spectral sequence along $\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \to \mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ku}$ for $\overline{V}(1) \otimes \mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$ has E_1-term

$$E_1^* = \overline{A}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})[h_1]$$

and converges to $\overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$. □

Remark 2.32. Note that $\overline{A}(1)_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} = \mathbb{F}_2$. The η-Bockstein spectral sequence

$$\overline{A}(1)_*(\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko})[\eta] \implies \overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$$

collapses to give $\overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} = \mathbb{F}_2[\eta]$ with $|\eta| = (1,1)$. The v_1-Bockstein spectral sequence

$$\overline{V}(1)_*(\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko})[v_1] \to \overline{V}(0)_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$$

collapses to give $\overline{V}(0)_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} = \mathbb{F}_2[\eta, v_1]$ with $|v_1| = (2,0)$. The v_0-Bockstein spectral sequence

$$\overline{V}(0)_*(\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko})[v_0] \to \pi_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko}$$

with $|v_0| = (0,0)$ has $E_1 = \mathbb{F}_2[v_0, \eta, v_1]$ and $d_1(v_1) = v_0\eta$. It then collapses at

$$E_2 = \frac{\mathbb{F}_2[v_0, \eta]}{(v_0\eta)} \otimes \mathbb{F}_2[v_1^2]\,,$$

and gives $\pi_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} = \mathbb{Z}[\eta, v_1^2]/(2\eta)$. The motivic spectral sequence

$$E_2 = \pi_*\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{ko} \to \pi_*\,\mathrm{ko}$$

then has $d_3(v_1^2) = \eta^3$ and collapses at

$$E_4 = \big(\mathbb{Z}\{1, 2v_1^2\} \oplus \mathbb{F}_2\{\eta, \eta^2\}\big) \otimes \mathbb{Z}[v_1^4]\,,$$

and converges to $\pi_*\,\mathrm{ko} = \mathbb{Z}[\eta, A, B]/(2\eta, \eta^3, \eta A, A^2 = 4B)$, with A detected by $2v_1^2$ and B detected by v_1^4.

Proposition 2.33. *The η-Bockstein spectral sequence*

$$\overline{A}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})[\eta] \implies \overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}) \tag{2.4}$$

has differentials

$$d_1(\lambda_2) = \eta\lambda_1'$$
$$d_3(\lambda_1'\lambda_2) = \eta^3\mu$$

and no further differentials besides those generated by the Leibniz rule. Consequently, we can identify

$$\overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko}) \cong \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 = c \cdot \eta^2\mu, \eta^3\mu)}$$

as a bigraded $\mathbb{F}_2$-algebra where $\|\eta\| = (1,1)$, $\|\mu\| = (8,0)$ and $\|\lambda_1'\| = (5,1)$ and $c \in \mathbb{F}_2$. Moreover, there is no room for η extensions.

Proof. We deduce these differentials using a small part of the known (implicitly 2-complete) computation of $\pi_* \operatorname{THH}(\mathrm{ko})$ from [AHL10, §7]. The unit $\mathrm{ko} \to \operatorname{THH}(\mathrm{ko})$ and augmentation $\epsilon\colon \operatorname{THH}(\mathrm{ko}) \to \mathrm{ko}$ exhibit ko as a retract of THH(ko) in the category of $\mathbb{E}_\infty$-rings. We write $\operatorname{THH}(\mathrm{ko})/\mathrm{ko}$ for the complementary summand in ko-modules. In degrees $* < 12$ we have $H_*(\mathrm{ko})\{\sigma\bar\xi_1^4, \sigma\bar\xi_2^2, \sigma\bar\xi_3\} \cong H_*(\operatorname{THH}(\mathrm{ko})/\mathrm{ko})$, so there is an 11-connected map $\Sigma^5\mathrm{ksp} \simeq \mathrm{ko} \otimes (S^5 \cup_\eta e^7 \cup_2 e^8) \to \operatorname{THH}(\mathrm{ko})/\mathrm{ko}$. By [AHL10, Corollary 7.3, Figure 5], the η^2-multiple in $\pi_6\mathrm{ksp} \cong \pi_{11}\Sigma^5\mathrm{ksp}$ maps to zero in THH, so $\pi_*(\operatorname{THH}(\mathrm{ko})/\mathrm{ko}) \cong (\mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}/2, 0)$ for $5 \le * \le 11$.

We consider the η-Bockstein spectral sequence

$$E_1 = \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\eta, \mu] \implies \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko})$$

with $|\lambda_1'| = (5,1)$, $|\lambda_2| = (7,1)$, $|\mu| = (8,0)$ and $|\eta| = (1,1)$, the v_1-Bockstein spectral sequence

$$E_1 = \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko})[v_1] \to \overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko})$$

with $|v_1| = (2,0)$, the v_0-Bockstein spectral sequence

$$E_1 = \overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko})[v_0] \implies \pi_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko})$$

with $|v_0| = (0,0)$, and the motivic spectral sequence

$$E_2 = \pi_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko}) \implies \pi_* \operatorname{THH}(\mathrm{ko})\,.$$

In each case the spectral sequence for ko splits off as a direct summand. Taking this into account, there is no possible source or target for a differential affecting λ_1' in any of these spectral sequences. Hence λ_1' survives in bidegree $(5,1)$ to detect the generator of $\pi_5(\operatorname{THH}(\mathrm{ko})/\mathrm{ko}) \cong \mathbb{Z}$. Since $\pi_6(\operatorname{THH}(\mathrm{ko})/\mathrm{ko}) = 0$, it follows that $\eta\lambda_1'$ in bidegree $(6,2)$ is an infinite cycle that detects zero, i.e., a boundary in one of these spectral sequences. Since $\eta\lambda_1'$ is not a v_1- or v_0-multiple, it cannot be a v_1-Bockstein or v_0-Bockstein boundary. Since the motivic E_2-term is readily seen to be zero in bidegree $(7,0)$, it can also not be a motivic boundary. Hence $d_1(\lambda_2) = \eta\lambda_1'$ in the η-Bockstein spectral sequence is the only remaining possibility.

There is no room for other η-Bockstein d_1-differentials, so the next differential to be determined is $d_3(\lambda_1'\lambda_2) \in \mathbb{F}_2\{\eta^3\mu\}$. On one hand, if $d_3(\lambda_1'\lambda_2) = \eta^3\mu$ then the η-Bockstein E_∞-term (modulo the summand for ko) will be

$$\mathbb{F}_2\{\lambda_1', \mu, \eta\mu, \eta^2\mu\}$$

in stems ≤ 12. On the other hand, if $d_3(\lambda_1'\lambda_2) = 0$ then it will be

$$\mathbb{F}_2\{\lambda_1', \mu, \eta\mu, \eta^2\mu, \eta^3\mu\}$$

in stems ≤ 11, with the 12-stem concentrated in motivic filtrations ≥ 2. In either case this determines $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko})$ in these stems.

The first nonzero v_1-Bockstein differential is $d_1(\mu) = v_1\lambda_1'$. If it were not there, then $v_1\lambda_1'$ would survive to $\overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko})$ and $\pi_* \operatorname{gr}^*_{\mathrm{mot}} \operatorname{THH}(\mathrm{ko})$ to detect a nonzero class in $\pi_7(\operatorname{THH}(\mathrm{ko})/\mathrm{ko}) = 0$, which is impossible. There is no room for other v_1-Bockstein differentials affecting stems ≤ 11, so if $d_3(\lambda_1'\lambda_2) = \eta^3\mu$ then the v_1-Bockstein E_∞-term (modulo the summand for ko) will be

$$\mathbb{F}_2\{\lambda_1', \eta\mu, \eta^2\mu, v_1\eta\mu\}$$

in stems ≤ 11, while if $d_3(\lambda_1'\lambda_2) = 0$ then it will be

$$\mathbb{F}_2\{\lambda_1', \eta\mu, \eta^2\mu, \eta^3\mu, v_1\eta\mu\}$$

in these stems. In either case the 12-stem is concentrated in motivic filtrations ≥ 2, and these expressions determine $\overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ in this range of stems.

In the v_0-Bockstein spectral sequence, there is no room for differentials on (λ_1' and) $\eta\mu$. Multiplying by η^2, it follows that $\eta^3\mu$ is an infinite cycle (but possibly zero). Since it is not a v_0-multiple, it cannot be a v_0-Bockstein boundary, and since it is in motivic filtration 3, and the motivic E_2-term is now known to be zero in bidegrees $(12,0)$ and $(12,1)$, it cannot be a motivic d_r-boundary for $r \geq 2$. Hence if $d_3(\lambda_1'\lambda_2)$ were zero, then $\eta^3\mu$ would survive to $\overline{V}(0)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ and $\pi_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ to detect a nonzero class in $\pi_{11}(\mathrm{THH(ko)}/\mathrm{ko}) = 0$, which is impossible.

This contradiction shows that $d_3(\lambda_1'\lambda_2) = \eta^3\mu$, as claimed. This leaves the η-Bockstein E_4-term

$$\frac{\Lambda(\lambda_1') \otimes \mathbb{F}_2[\eta,\mu]}{(\eta\lambda_1', \eta^3\mu)} .$$

There is no room for further differentials, so this is also the E_∞-term. The only possible multiplicative extension in the abutment $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ is the one stated, with $\lambda_1' \cdot \lambda_1' \in \mathbb{F}_2\{\eta^2\mu\}$. □

Remark 2.34. In fact, we will show in Proposition 4.11 that $c = 1$ and $\lambda_1' \cdot \lambda_1' = \eta^2\mu$. We therefore determine the complete computation of $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$, including the multiplicative structure, in Corollary 4.12.

Corollary 2.35. *We can identify*

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)} \cong \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 = c \cdot \eta^2\mu, \eta^3\mu)}$$

as a bigraded $\mathbb{F}_2$-module, with $|\varepsilon_2| = (7,-1)$ for some $c \in \mathbb{F}_2$.

Proof. This follows because v_2 acts trivially on $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ and a choice of null-homotopy of v_2 produces the class ε_2. We can conclude that $\varepsilon_2^2 = 0$ in in $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH(ko)}$ because the group is trivial in this bidegree. □

3. Detection

The (classical mod 2) Adams spectral sequence

$${}^{\mathrm{Ad}}E_2(X) = \operatorname{Ext}^{*,*}_{\mathcal{A}^\vee}(\mathbb{F}_2, H_*(X)) \implies \pi_*(X_2^\wedge)$$

is strongly convergent for bounded below spectra X with $H_*(X)$ of finite type. Its E_2-term can be calculated as the cohomology of the normalized cobar complex

$$0 \longrightarrow H_*(X) \xrightarrow{d_1^0} \bar{\mathcal{A}}^\vee \otimes H_*(X) \xrightarrow{d_1^1} \bar{\mathcal{A}}^\vee \otimes \bar{\mathcal{A}}^\vee \otimes H_*(X) \longrightarrow \dots .$$

Here $\bar{\mathcal{A}}^\vee = \operatorname{cok}(\mathbb{F}_2 \to \mathcal{A}^\vee)$, and we will use the notation $[a]m = a \otimes m \in \bar{\mathcal{A}}^\vee \otimes H_*(X)$. Recall that d_1^0 is given by the normalized $\mathcal{A}^\vee$-coaction on $H_*(X)$, while d_1^1 also involves the coproduct $\psi \colon \mathcal{A}^\vee \to \mathcal{A}^\vee \otimes \mathcal{A}^\vee$.

When $X = A(1)[ij]$ as in Notation 2.16, the Adams E_2-terms

$${}^{\mathrm{Ad}}E_2 = \operatorname{Ext}_{\mathcal{A}}(H^*(A(1)[ij]), \mathbb{F}_2) \implies \pi_* A(1)[ij]$$

are readily calculated in a finite range using Bruner's `ext` software [Bru93] (cf. [BR21]). The results in stems $* \leq 28$ are shown in Figure 3.1, with the usual (stem, Adams filtration) bigrading. In each case, the 1-cochains

$$[\xi_1^4]1\ ,\ [\xi_2^2]1 + [\xi_1^4]\xi_1^2 \text{ and } [\xi_3]1 + [\xi_2^2]\xi_1 + [\xi_1^4]\xi_2 \tag{3.1}$$

in $\bar{\mathcal{A}}^\vee \otimes \mathcal{A}(1)^\vee$ are cocycles, but not coboundaries, hence represent nonzero classes in ${}^{\mathrm{Ad}}E_2(A(1)[ij])$ in (stem, Adams filtration) bidegrees $(3,1)$, $(5,1)$ and $(6,1)$, respectively. For sparsity reasons, these survive to ${}^{\mathrm{Ad}}E_\infty(A(1)[ij])$, and detect nonzero homotopy classes in stems 3, 5 and 6, denoted

$$\nu\ ,\ w \text{ and } v_2 \in \pi_* A(1)[ij]\,.$$

for each $i,j \in \{0,1\}$. In Figure 3.1, lines of bidegree $(0,1)$, $(1,1)$ and $(3,1)$ (dashed) indicate multiplications by h_0, h_1 and h_2, respectively.

Lemma 3.1. *In the Adams spectral sequences for the $A(1)[ij]$ the differentials originating in stems $* \le 24$ are all zero.*

Proof. This mostly follows from sparsity and the module structure over the Adams spectral sequence for S, using that $d_2(h_4) = h_0 h_3^2$ maps to zero under $\mathbb{S} \to A(1)$. Only the Adams d_2-differential from bidegree $(t-s, s) = (19,2)$ requires special attention, but the Novikov E_2-term shows that $\pi_{19}A(1)$ has order $2^2 = 4$, so there is no room for such an Adams differential. □

To calculate the Novikov E_2-term

$${}^{\mathrm{Nov}}E_2 = \mathrm{Ext}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_* A(1)) \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_* A(1))$$

for these spectra, we can note that $\mathrm{BP}_* A(1) = \mathrm{BP}_*/I_2\{1, t_1\}$ and use the long exact sequence obtained by applying $\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, -)$ to the $\mathrm{BP}_*\mathrm{BP}$-comodule extension

$$0 \longrightarrow \mathrm{BP}_*/I_2 \longrightarrow \mathrm{BP}_* A(1) \longrightarrow \Sigma^2 \mathrm{BP}_*/I_2 \to 0$$

classified by

$$h_{10} = [t_1] \in \mathrm{Ext}^{1,1}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*)\,.$$

The groups

$$\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_2)$$

are calculated in a range as in [Rav86, §4.4, p. 162], starting with the isomorphism

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_\infty)$$

that doubles internal degrees, followed by the v_n-Bockstein spectral sequences

$$E_1 = \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_{n+1})\,[v_n] \implies \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_n)$$

for descending $n \ge 2$. The v_2-Bockstein spectral sequence E_∞-term for BP_*/I_2 in stems $* \le 26$ is shown in Figure 3.2, corresponding to [Rav86, Fig. 4.4.23(c)]. Lines of bidegree $(1,1)$, $(3,1)$ and $(7,1)$ (dashed) indicate multiplications by $h_{10} = [t_1]$, $h_{11} = [t_1^2]$ and $h_{12} = [t_1^4]$, respectively. (Some) hidden extensions are shown in black.

Alternatively, one can start with the internal degree-doubling isomorphism

$$\mathrm{Ext}_{\mathcal{A}}(H^*(C2), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_\infty\{1, t_1\})$$

and calculate the v_n-Bockstein spectral sequences

$$E_1 = \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_{n+1}\{1, t_1\})\,[v_n] \implies \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_n\{1, t_1\})$$

for descending $n \ge 2$. The Adams E_2-term for $C2$ in stems $* \le 16$ is shown in Figure 3.3, and the resulting v_2-Bockstein E_∞-term for $\mathrm{BP}_*/I_2\{1, t_1\} = \mathrm{BP}_* A(1)$ in stems $* \le 26$ is shown in Figure 3.4. Again, (some) hidden extensions are shown in black.

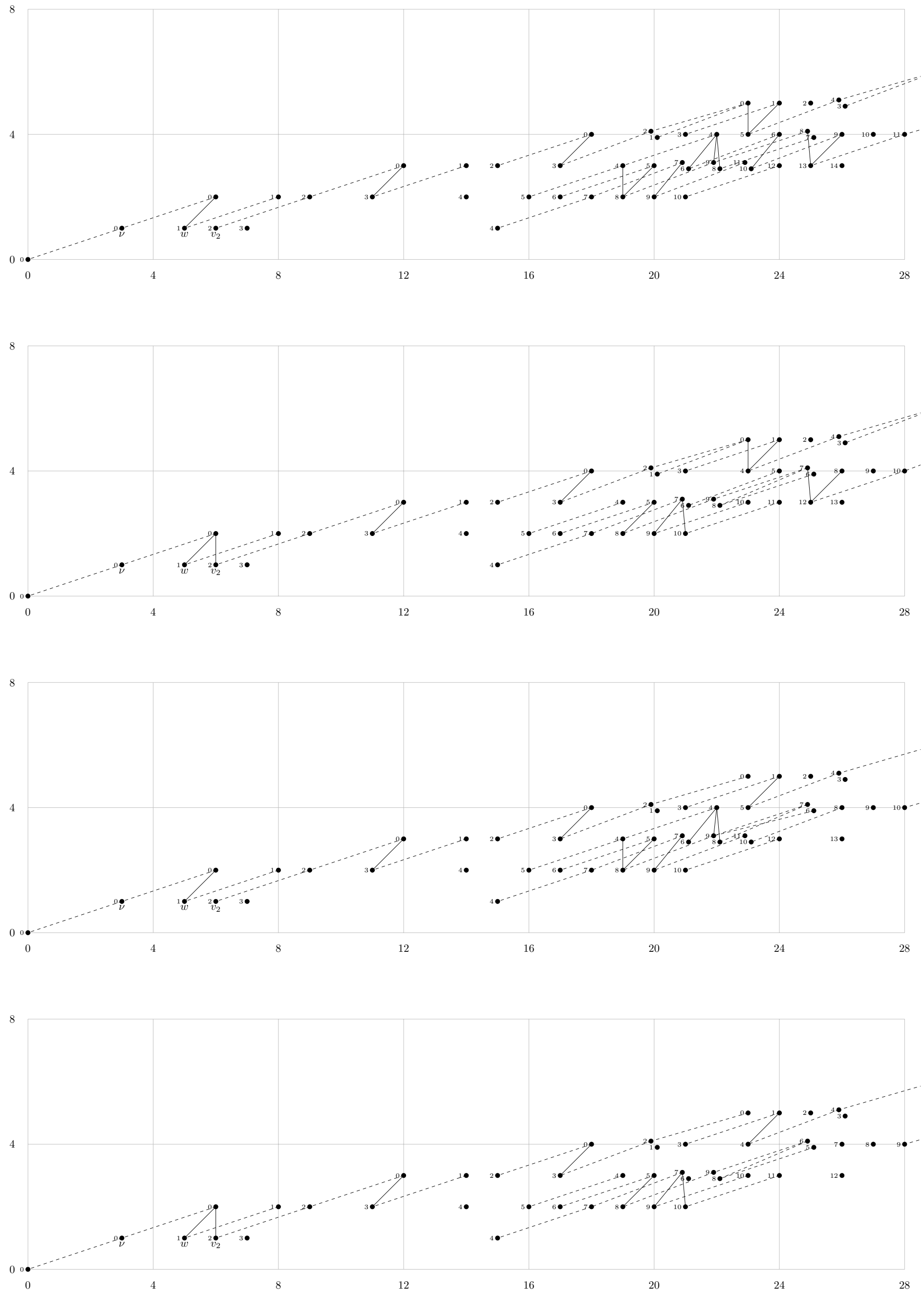

FIGURE 3.1. Adams E_2-terms for $A(1)[00]$, $A(1)[10]$, $A(1)[01]$ and $A(1)[11]$ (from top to bottom)

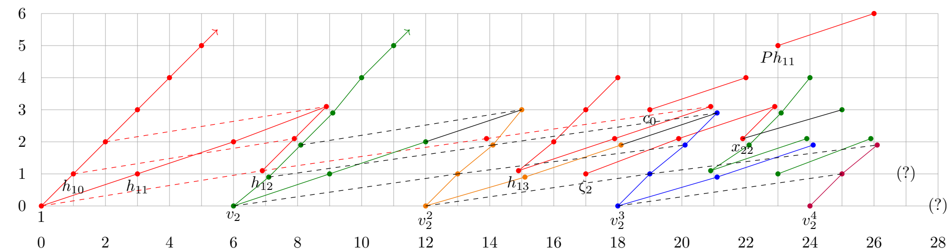

FIGURE 3.2. $E_\infty \implies \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_2)$

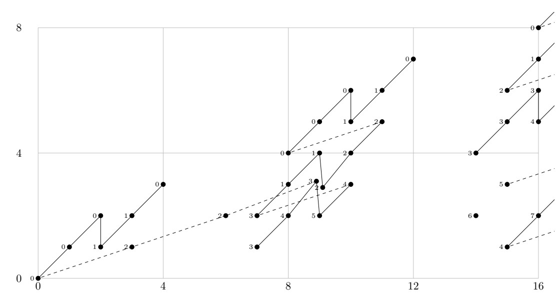

FIGURE 3.3. Adams E_2-term for $C2$

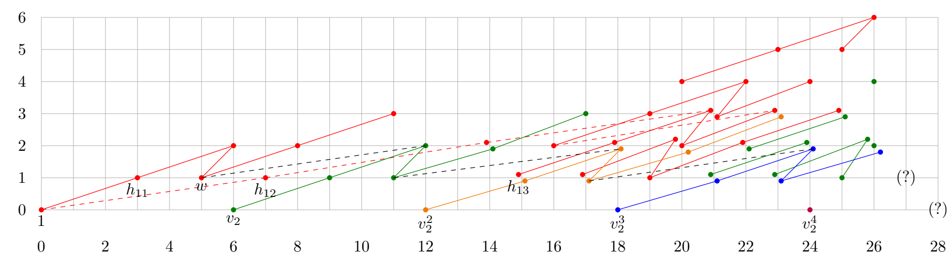

FIGURE 3.4. $E_\infty \implies \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_* A(1))$

Lemma 3.2. *In the Novikov spectral sequences for the $A(1)[ij]$ the nonzero differentials originating in stems $* \leq 22$ are*

$$d_3(v_2^2) = h_{11}^2 w \qquad \text{and} \qquad d_3(v_2^3) = v_2 h_{11}^2 w\,.$$

In the cases $A(1)[10]$ and $A(1)[11]$ there is a nonzero d_3 from bidegree $(t-s, s) = (23, 1)$.

In every case $d_3(v_2^4) = 0$ and $d_5(v_2^4) \neq 0$.

Proof. This follows by comparison of the order in each stem of the Adams E_∞-term, which equals that of the abutment $\pi_* A(1)[ij]$, with the order in each stem of the Novikov E_2-term. In particular, $\pi_{12}A(1) = \mathbb{Z}/2$ implies that v_2^2 must support a nonzero differential. Similarly, the group $\pi_{18}A(1)$ has order 2^2, so v_2^3 must support a nonzero differential. The groups $\pi_{22}A(1)[ij]$ have order $2^3 = 8$ for $i = 0$ and order $2^2 = 4$ for $i = 1$, while the groups $\pi_{23}A(1)[ij]$ have order 2^4 for $i = 0$ and 2^3 for $i = 1$. To account for this, the Novikov differential d_3 from bidegree $(t-s, s) =$

$(23, 1)$ to $(22, 4)$ must be nonzero when $i = 1$. Moreover, there must be a rank 1 Novikov differential from the 24-stem to the 23-stem. By h_{11}-linearity, it cannot originate in bidegree $(24, 2)$, hence it is either a d_3 or a d_5 starting on v_2^4.

Inspection of the Novikov E_2-term for $\mathbb{S}$ in [Rav86, Figure 4.4.45] shows that $\nu\bar{\kappa} \in \pi_{23}(\mathbb{S})$ is detected by a generator x of the $\mathbb{Z}/8$ in (stem, Novikov filtration) bidegree $(23, 5)$ of ${}^{\mathrm{Nov}}E_2(\mathbb{S})$. The unit map $\mathbb{S} \to A(1)$ takes this generator x to the generator y of the $\mathbb{Z}/2$ in the same bidegree of ${}^{\mathrm{Nov}}E_2(A(1))$, see Figure 3.4. Since $\nu\bar{\kappa}$ maps to zero in $\pi_{23}A(1)$ (by Lemma 3.1) it follows that this nonzero class y is a boundary. It cannot be a d_3-boundary, by h_{11}-linearity and the first part of this proof, so $d_5(v_2^4) = y \neq 0$ is the only possibility. In particular, we must have $d_3(v_2^4) = 0$. □

The circle group $\mathbb{T}$ acts freely on $S^1 \subset S^3 \subset \cdots \subset S^\infty = E\mathbb{T}$, and we can form the "approximate homotopy fixed point" spectrum $F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$. There is a cofiber sequence

$$\Sigma^{-2}\,\mathrm{THH}(\mathrm{ko}) \xrightarrow{i} F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \xrightarrow{p} \mathrm{THH}(\mathrm{ko}) \xrightarrow{\sigma} \Sigma^{-1}\,\mathrm{THH}(\mathrm{ko})\,, \tag{3.2}$$

where σ is induced by the $\mathbb{T}$-action on $\mathrm{THH}(\mathrm{ko})$, and a commutative diagram

$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d] & \mathrm{TC}^-(\mathrm{ko}) \arrow[r] \arrow[dr] & F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[d, "p"] \\
\mathrm{ko} \arrow[rr] & & \mathrm{THH}(\mathrm{ko})\,.
\end{tikzcd}$$

By truncating the homotopy fixed point spectral sequence

$$E^2 = A(1)_*\,\mathrm{THH}(\mathrm{ko})[t] \implies A(1)_*\,\mathrm{TC}^-(\mathrm{ko})\,,$$

where t is in stem -2, we obtain a two-column approximate homotopy fixed point spectral sequence

$$E^2 = A(1)_*\,\mathrm{THH}(\mathrm{ko})\{1, t\} \implies A(1)_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}\,, \tag{3.3}$$

which is really just the long exact sequence in $A(1)$-homotopy associated to the cofiber sequence (3.2). We have the following analogue of [AR02, Proposition 4.8].

Proposition 3.3. *The unit images in $A(1)_*\,\mathrm{TC}^-(\mathrm{ko})$ and $A(1)_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$ of the classes ν, w and $v_2 \in \pi_* A(1)$ are detected by*

$$t\lambda_1'\ ,\ t\lambda_2 \text{ and } t\mu\,,$$

respectively, in the homotopy fixed point and approximate fixed point spectral sequences.

Proof. By naturality, it suffices to prove this in the approximate fixed point case. The unit map takes the infinite cycles in (3.1), detecting ν, w and v_2 in $\pi_* A(1)$, to the 1-cocycles

$$\begin{gathered}
[\xi_1^4](1 \otimes 1) \\
[\xi_2^2](1 \otimes 1) + [\xi_1^4](\xi_1^2 \otimes 1) \\
[\xi_3](1 \otimes 1) + [\xi_2^2](\xi_1 \otimes 1) + [\xi_1^4](\xi_2 \otimes 1)
\end{gathered} \tag{3.4}$$

in $\bar{\mathcal{A}}^\vee \otimes \mathcal{A}(1)^\vee \otimes H_*(F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}})$. We claim that these are not in the image of the coboundary d_1^0 from the 0-cochains

$$\mathcal{A}(1)^\vee \otimes H_*(F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}})\,,$$

hence represent nonzero classes in ${}^{\mathrm{Ad}}E_\infty(A(1)\otimes F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}})$, detecting the (nonzero) images of ν, w and v_2 in $A(1)_*F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$.

Recall that $H_*(\mathrm{ko})=\mathbb{F}_2[\bar\xi_1^4,\bar\xi_2^2,\bar\xi_3,\dots]$, $H_*\,\mathrm{THH}(\mathrm{ko})=H_*(\mathrm{ko})\otimes\Lambda(\sigma\bar\xi_1^4,\sigma\bar\xi_2^2)\otimes\mathbb{F}_2[\sigma\bar\xi_3]$ and $\mathcal{A}(1)^\vee=\mathbb{F}_2[\xi_1,\xi_2]/(\xi_1^4,\xi_2^2)$. In the long exact sequence associated to (3.2), the map σ has kernel $\mathbb{F}_2\{1,\sigma\bar\xi_1^4,\sigma\bar\xi_2^2\}$ in degrees ≤ 7, and the image of i consists of t-multiples. In the extension

$$0\longrightarrow\mathcal{A}(1)^\vee\otimes\mathrm{im}(i)\longrightarrow\mathcal{A}(1)^\vee\otimes H_*F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}\longrightarrow\mathcal{A}(1)^\vee\otimes\ker(\sigma)\to 0$$

the coboundaries on classes in $\mathcal{A}(1)^\vee\otimes\mathrm{im}(i)$ will lie in $\bar{\mathcal{A}}^\vee\otimes\mathcal{A}(1)^\vee\otimes\mathrm{im}(i)$, hence do not contribute any terms of the form $[a](m\otimes 1)$. The classes 1, $\sigma\bar\xi_1^4$ and $\sigma\bar\xi_2^2$ are $\mathcal{A}^\vee$-comodule primitive in $\ker(\sigma)$, hence lift to classes in $H_*F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$ that are $\mathcal{A}^\vee$-comodule primitive modulo $\mathrm{im}(i)$, so also the coboundaries on (the lifts of) $\mathcal{A}(1)^\vee\otimes\mathbb{F}_2\{1,\sigma\bar\xi_1^4,\sigma\bar\xi_2^2\}$ do not contain any terms of the form in (3.4). This proves our claim.

It remains to be determined where in (3.3) the (nonzero) unit images of ν, w and v_2 are detected. Recall that $A(1)_*\,\mathrm{THH}(\mathrm{ko})=\Lambda(\lambda_1',\lambda_2)\otimes\mathbb{F}_2[\mu]$ is equal to $\mathbb{F}_2\{1,\lambda_1',\lambda_2,\mu\}$ in stems ≤ 8. The composite map

$$A(1)\longrightarrow A(1)\otimes F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}\xrightarrow{\ p\ }A(1)\otimes\mathrm{THH}(\mathrm{ko})$$

factors through $A(1)\otimes\mathrm{ko}\simeq\mathbb{F}_2$, so the images of ν, w and v_2 in $A(1)_*\,\mathrm{THH}(\mathrm{ko})$ are all zero. (This was obvious for ν and v_2.) Hence the nonzero images of ν, w and v_2 must all be detected by t-multiplies in the approximate fixed point spectral sequence, and for degree reasons the only possible detecting classes are $t\lambda_1'$, $t\lambda_2$ and $t\mu$, respectively. □

Since $\mathrm{MU}_*\,A(1)$ is even, the motivic spectral sequence

$$\pi_*\overline{A}(1)\implies\pi_*A(1)\tag{3.5}$$

can be identified with the Novikov spectral sequence

$${}^{\mathrm{Nov}}E_2=\mathrm{Ext}^{*,*}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*,\mathrm{MU}_*\,A(1))\implies\pi_*A(1)$$

by [HRW22, Corollary 2.2.17]. The spectral sequence must collapse in stems ≤ 10, for sparsity reasons, so these three classes detect ν, w and v_2, respectively, see Lemma 3.2.

Corollary 3.4. *The classes h_{11}, w and v_2 in $\pi_*\overline{A}(1)$ map by the unit to classes in*

$$\overline{A}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\mathrm{ko})\quad\textit{and}\quad\overline{A}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$$

detected by $t\lambda_1'$, $t\lambda_2$ and $t\mu$, respectively. Likewise, the images of h_{11} and w in $\pi_*\overline{V}(2)\otimes\overline{C}\eta$ are detected by $t\lambda_1'$ and $t\lambda_2$ in $(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}$.

The classes h_{11} and v_2 in $\pi_*\overline{V}(1)$ map by the unit to classes in

$$\overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}},$$

and, consequently, $\overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\mathrm{ko})$, detected by $t\lambda_1'$ and $t\mu$, respectively. Likewise, the class h_{11} in $\pi_*\overline{V}(2)$ maps by the unit to a class in

$$\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,F(S^3_+,\mathrm{THH}(\mathrm{ko}))^{\mathbb{T}},$$

and, consequently, $\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\mathrm{ko})$, detected by $t\lambda_1'$.

Proof. The motivic spectral sequence

$$\begin{aligned}\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} = \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu]\{1, t\} \\ \Longrightarrow A(1)_* F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}}\end{aligned}$$

is concentrated in filtrations $0 \le * \le 2$ and integer weights, hence collapses, and the result follows from Proposition 3.3. The claim with coefficients in $\overline{V}(2) \otimes \overline{C}\eta$ follows by passing to cofibers for multiplication by v_2.

In the case of $\overline{V}(1)$, there is no analogue of Proposition 3.3 and we must work directly with the Novikov spectral sequence. Since h_{11} and v_2 in $\overline{V}(1)$ map to h_{11} and v_2 in $\overline{A}(1)$, the claim with $\overline{V}(1)$ coefficients follows from the commuting square

$$\begin{tikzcd} \overline{V}(1) \arrow[r] \arrow[d] & \overline{A}(1) \arrow[d] \\ \overline{V}(1) \otimes F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[r] & \overline{A}(1) \otimes F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \end{tikzcd}$$

and the sparsity of the Novikov spectral sequences in this range (cf. Figure 3.2).

The claim with $\overline{V}(2)$ coefficients follows by passing to cofibers for multiplication by v_2. □

4. Prismatic cohomology

We consider the $\overline{V}(2)$-homotopy $\mathbb{T}$-Tate spectral sequence

$$\hat{E}^2 = \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,[t^{\pm 1}] \implies \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \tag{4.1}$$

with t in stem -2, and the $\overline{V}(2) \otimes \overline{C}\eta$-homotopy $\mathbb{T}$-Tate spectral sequence

$$\hat{E}^2 = (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,[t^{\pm 1}] \implies (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})\,. \tag{4.2}$$

They can be reindexed as cohomologically graded periodic t-Bockstein spectral sequences, in which case $\hat{E}^{2r}$ and d^{2r} correspond to E_r and d_r. However, we shall need to make a comparison with similar C_2-Tate spectral sequences, for which our indexing is more convenient.

Theorem 4.1 (Prismatic cohomology of ko mod $(2, v_1, v_2)$)**.** *The $\overline{V}(2)$-homotopy $\mathbb{T}$-Tate spectral sequence (4.1) is an algebra spectral sequence with E^2-term*

$$\hat{E}^2 = \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)} \otimes \mathbb{F}_2[t^{\pm 1}]$$

and differentials

$$\begin{aligned} d^2(\varepsilon_2) &= t\mu \\ d^2(t^{-1}) &= \eta \\ d^6(t^{-2}) &= t\lambda_1' \\ d^6(t^{-1}\lambda_1') &= t^2(\lambda_1')^2 = t^2\eta^2\mu\,, \end{aligned}$$

leading to

$$\hat{E}^\infty = \mathbb{F}_2\{1, t^2\lambda_1', t^4(\lambda_1')^2, \lambda_1'\} \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$

Hence there is a preferred isomorphism

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[t^{\pm 4}]\,,$$

where 1, η, η^2, λ_1' *and* $t^{\pm 4}$ *in bidegrees* $(0,0)$, $(1,1)$, $(2,2)$, $(5,1)$ *and* $(\mp 8, 0)$ *are detected by* 1, $t^2\lambda_1'$, $t^4(\lambda_1')^2 = t^4\eta^2\mu$, λ_1' *and* $t^{\pm 4}$*, respectively.*

Theorem 4.2 (Prismatic cohomology of ko mod $(2, \eta, v_1, v_2)$)**.** *The* $\overline{V}(2) \otimes \overline{C}\eta$*-homotopy* $\mathbb{T}$*-Tate spectral sequence* (4.2) *is a module spectral sequence over* (4.1)*, with* E^2*-term*

$$\hat{E}^2 = \Lambda(\varepsilon_2) \otimes \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu] \otimes \mathbb{F}_2[t^{\pm 1}]$$

and differentials

$$\begin{aligned} d^2(\varepsilon_2) &= t\mu \\ d^6(t^{-1}) &= t^2\lambda_1' \\ d^6(t^{-2}) &= t\lambda_1' \\ d^6(t^{-1}\lambda_2) &= t^2\lambda_1'\lambda_2 \\ d^6(t^{-2}\lambda_2) &= t\lambda_1'\lambda_2 \\ d^8(t^{-3}) &= t\lambda_2 \\ d^8(t^{-1}\lambda_1') &= t^3\lambda_1'\lambda_2 \end{aligned}$$

leading to

$$\hat{E}^\infty = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$

Hence there is a preferred isomorphism

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}]\,,$$

where 1, λ_1', λ_2, $\lambda_1'\lambda_2$ *and* $t^{\pm 4}$ *in bidegrees* $(0,0)$, $(5,1)$, $(7,1)$, $(12,2)$, *and* $(\mp 8, 0)$ *are detected by the classes in the* E^∞*-term with the same names.*

The proofs of these theorems will occupy the remainder of this section. We provide Figure 4.1 and Figure 4.2 for reference during the course of the proofs.

By Proposition 2.33, we can identify

$$\hat{E}^2 = \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 = c \cdot \eta^2\mu, \eta^3\mu)} \otimes \mathbb{F}_2[t^{\pm 1}]\,.$$

where we have yet to determine the coefficient $c \in \mathbb{F}_2$.

Proposition 4.3. *The spectral sequence* (4.1) *is multiplicative and it has differentials*

$$d^2(\eta) = 0\,,\ d^2(t^{-1}) = \eta\,,\ d^2(\varepsilon_1) = t\mu\,,\ d^2(\mu) = t\eta\mu\,,\ \text{ and } d^2(\lambda_1') = 0\,.$$

Consequently, we can identify

$$\hat{E}^4 = \mathbb{F}_2\{1, t\lambda_1', \lambda_1', \eta^2\mu\} \otimes \mathbb{F}_2[t^{\pm 2}]$$

with $\eta^2\mu = \eta^3\varepsilon_2$*. Moreover, the class* η *is an infinite cycle.*

Proof. The first claim follows because $\overline{V}(1) \otimes \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ is an $\mathbb{E}_\infty$ $\operatorname{gr}^*_{\mathrm{ev}} \mathbb{S}$-algebra. Using the $\mathbb{T}$-equivariant attaching maps of the standard $\mathbb{T}$-CW complex structure on $S^\infty = E\mathbb{T}$, we compute differentials

$$d^2(t^{-1}) = \eta \text{ and } d^2(\eta) = 0$$

as in [Hes96, Lemma 1.4.2].

We know $d^2(t\lambda_1') = 0$ because $t\lambda_1'$ detects ν by Corollary 3.4. Consequently, we know that $d^2(\lambda_1') = 0$ by the Leibniz rule and the fact that $\eta \cdot \lambda_1' = 0$.

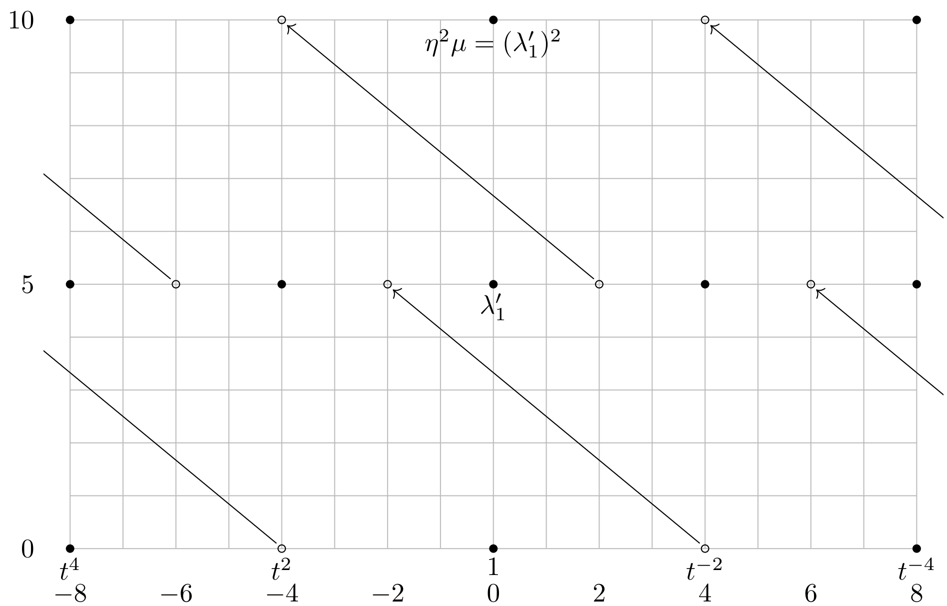

FIGURE 4.1. $\mathbb{T}$-Tate spectral sequence converging to $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$

Since ε_2 arises from a choice of null homotopy of v_2 and by Corollary 3.4 the class v_2 is detected in $\overline{V}(1)$-homotopy by $t\mu$, there is a differential $d^2(\varepsilon_2) = t\mu$. Hence $t\mu$ is a d^2-cycle, and the Leibniz rule implies that $d^2(\mu) = t\eta\mu$. The last statement follows because η is the image of η in $\pi_*\overline{V}(1)$. □

Proposition 4.4. *The classes t^4 and λ_1' are permanent cycles in the spectral sequence (4.1) for all integers k. Moreover, there is a differential*

$$d^6(t^{-2}) = t\lambda_1'$$

and there is a differential

$$d^6(t^{-1}\lambda_1') = ct^2\eta^2\mu$$

for some $c \in \mathbb{F}_2$. The spectral sequence (4.1) collapses at $\widehat{E}^7 = \widehat{E}^\infty$.

Proof. We know $t\lambda_1'$ is an infinite cycle in the spectral sequence (4.1), because it detects ν by Corollary 3.4. Let $\mathbb{C}(2)$ denote the complex 1-dimensional $\mathbb{T}$-representation where $z \in \mathbb{T}$ acts as multiplication by z^2, so that $S(\mathbb{C}(2)) \cong \mathbb{T}/C_2$. Then the $\mathbb{T}$-equivariant cofiber sequence

$$S(\mathbb{C}(2))_+ \longrightarrow S^0 \xrightarrow{e} S^{\mathbb{C}(2)}$$

induces a cofiber sequence

$$(\Sigma^{-\mathbb{C}(2)} \mathrm{THH}(\mathrm{ko}))^{h\mathbb{T}} \longrightarrow \mathrm{TC}^-(\mathrm{ko}) \xrightarrow{F} \mathrm{THH}(\mathrm{ko})^{hC_2}$$

of $\mathrm{TC}^-(\mathrm{ko})$-modules, mapping to the cofiber sequence

$$(\Sigma^{-\mathbb{C}(2)} \mathrm{THH}(\mathrm{ko}))^{t\mathbb{T}} \longrightarrow \mathrm{TP}(\mathrm{ko}) \xrightarrow{F} \mathrm{THH}(\mathrm{ko})^{tC_2}$$

of $\mathrm{TP}(\mathrm{ko})$-modules.

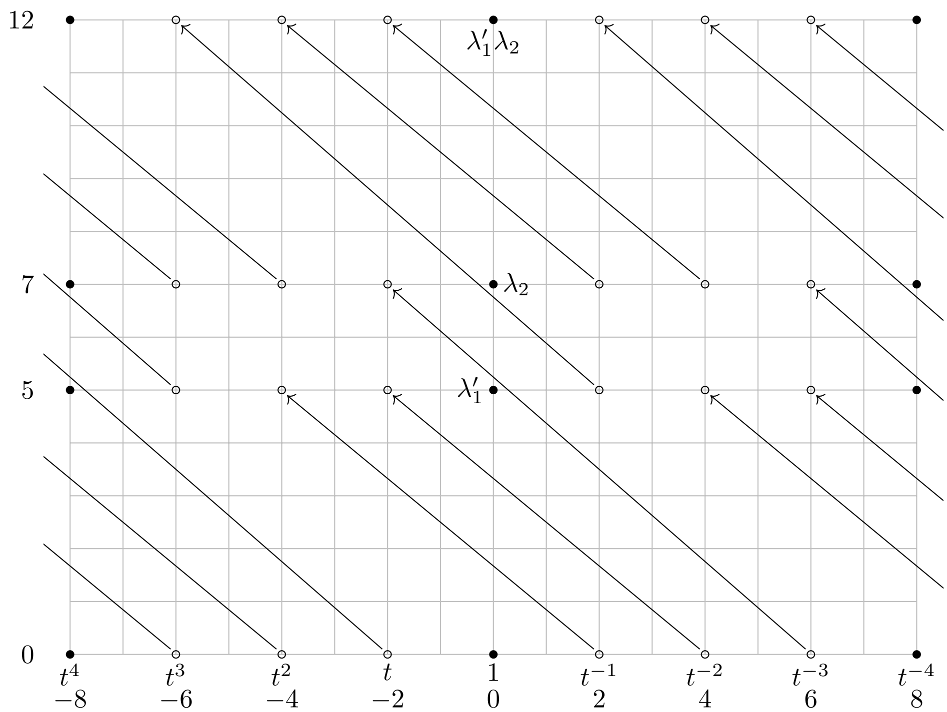

FIGURE 4.2. $\mathbb{T}$-Tate spectral sequence converging to $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$

We obtain a commutative square of spectral sequences converging to

$$\begin{array}{ccc} \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) & \xrightarrow{F} & \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{hC_2} \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) & \xrightarrow{F} & \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}\,, \end{array}$$

with E^2-terms

$$\begin{array}{ccc} \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \otimes \mathbb{F}_2[t] & \longrightarrow & \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \otimes \mathbb{F}_2[t] \otimes \Lambda(\overline{u}_1) \\ \downarrow & & \downarrow \\ \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \otimes \mathbb{F}_2[t^{\pm 1}] & \longrightarrow & \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \otimes \mathbb{F}_2[t^{\pm 1}] \otimes \Lambda(\overline{u}_1)\,. \end{array}$$

In the two right-hand cases, $\overline{u}_1$ maps by the connecting homomorphism to $\Sigma\Sigma^{-\mathbb{C}(2)}1$ and has (stem, motivic filtration) bidegree $|\overline{u}_1| = (-1, -1)$.

We know that ν maps to zero in $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$, because the target is zero in the relevant bidegree. Therefore, there must be a differential

$$d^6(t^{-2}) = t\lambda_1'\,. \tag{4.3}$$

in the $\overline{V}(1)$-homotopy C_2-Tate spectral sequence by a diagram chase in the diagram

$$\begin{tikzcd}
 & \mathbb{S} \arrow[d] & \\
\mathrm{TF}(\mathrm{ko}) \arrow[r, "F"] \arrow[d, "\Gamma"'] & \mathrm{THH}(\mathrm{ko})^{C_2} \arrow[r, "R"] \arrow[d, "\Gamma_1"'] & \mathrm{THH}(\mathrm{ko}) \arrow[d, "\hat{\Gamma}_1 = \varphi_2^{h\mathbb{T}}"] \\
\mathrm{TC}^-(\mathrm{ko}) \arrow[r, "F^h"] \arrow[d] & \mathrm{THH}(\mathrm{ko})^{hC_2} \arrow[r, "R^h = \mathrm{can}"] \arrow[d, "F^h"'] & \mathrm{THH}(\mathrm{ko})^{tC_2} \\
F(S^3_+, \mathrm{THH}(\mathrm{ko}))^{\mathbb{T}} \arrow[r, "p"] & \mathrm{THH}(\mathrm{ko}) &
\end{tikzcd}$$

as in [AR02, Theorem 5.5]. There is no earlier C_2-Tate differential of the form

$$d^r(t^{1-r}\eta^{6-r}) = t\lambda_1'$$

for $2 \leq r \leq 5$, since η is an infinite cycle. Consequently, there are differentials

$$d^6(t^{-2}) = t\lambda_1' \qquad \text{and} \qquad d^6(t^{-1}\lambda_1') = t^2(\lambda_1')^2 = c \cdot t^2\eta^2\mu$$

in the $\overline{V}(2)$-homotopy $\mathbb{T}$-Tate spectral sequence, and the classes $t^{\pm 4}$ and λ_1' are infinite cycles.

To complete the proof, we deduce $d^6(t^{-1}\lambda_1')$ from $d^6(t^{-2}) = t\lambda_1'$ and the Leibniz rule. We educe that $d^r(t^{-4}) = 0$ and $d^r(\lambda_1') = 0$ for all $r \geq 4$ by the sparsity of the E^4-term determined in Proposition 4.3.

There are no further differentials because the target groups of all d^r-differentials for $r \geq 7$ are trivial (cf. Figure 4.1). □

Remark 4.5. The coefficients denoted $c \in \mathbb{F}_2$ in Proposition 2.4, Corollary 2.35, and Proposition 4.4 are all the same. We determine that $c = 1$ in Proposition 4.11.

Even with incomplete information about $(\lambda_1')^2$ and $d^6(t^{-1}\lambda_1')$, we can extract the following computation.

Corollary 4.6. *We identify*

$$\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = \begin{cases} \mathbb{F}_2\{1\} & \text{if } n = 0\,, \\ \mathbb{F}_2\{t^2\lambda_1'\} & \text{if } n = 1\,, \\ \mathbb{F}_2\{t^4\eta^2\mu\} & \text{if } n = 2\,, \\ 0 & \text{if } n = 3, 4\,, \\ \mathbb{F}_2\{\lambda_1'\} & \text{if } n = 5\,. \end{cases}$$

Moreover, we have

$$\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = 0 \text{ if } n = 6, 7\,,$$

if $c = 1$ and

$$\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = \begin{cases} \mathbb{F}_2\{\eta^2\mu\} & \text{if } n = 6\,, \\ \mathbb{F}_2\{t^{-1}\lambda_1'\} & \text{if } n = 7\,, \end{cases}$$

if $c = 0$. These repeat 8-periodically, via multiplication by $t^{\pm 4}$.

We now move towards computing the spectral sequence (4.2). By Corollary 2.30, we can identify the E^2-term of (4.2):

$$\widehat{E}^2 = \Lambda(\varepsilon_2) \otimes \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[\mu] \otimes \mathbb{F}_2[t^{\pm 1}]\,.$$

Remark 4.7. We emphasize that the differentials in the spectral sequence (4.2) do not satisfy the Leibniz rule. This is a consequence of Remark 2.29.

Proposition 4.8. *The spectral sequence* (4.2) *is a module over the spectral sequence* (4.1)*. There are differentials*

$$\begin{aligned} d^2(\varepsilon_2) &= t\mu \\ d^6(t^{-2}) &= t\lambda_1' \\ d^8(t^{-3}) &= t\lambda_2 \end{aligned}$$

in the spectral sequence (4.2) *and multiplication by $t^{\pm 4}$ and λ_1' commutes with all differentials in this spectral sequence.*

Proof. The unit map $\overline{V}(2) \to \overline{V}(2) \otimes \overline{C}\eta$ is a map of $\overline{V}(2)$-modules, so (4.2) is a module spectral sequence over (4.1), and the map from (4.1) to (4.2) respects this module structure. This implies that multiplication by the infinite cycles $t^{\pm 4}$ and λ_1' will commute with each differential in (4.2).

For the differential $d^2(\varepsilon_2) = t\mu$ we observe that ε_2 exhibits a null-homotopy of v_2, and v_2 is detected by $t\mu$. It follows that

$$\hat{E}^4 = \Lambda(\lambda_1', \lambda_2) \otimes \mathbb{F}_2[t^{\pm 1}]\,.$$

We know ν is detected by $t\lambda_1'$ and w is detected by $t\lambda_2$, so these classes are also detected in $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{hC_2}$. However, we know that ν and w are trivial in $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ so this means that $t\lambda_1'$ and $t\lambda_2$ map trivially to $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}$. This means that they must be hit by differentials in the $\overline{V}(2)\otimes\overline{C}\eta$-homotopy C_2-Tate spectral sequence. By examination of bidegrees, the only possibility is the differentials $d^6(t^{-2}) = t\lambda_1'$ and $d^8(t^{-3}) = t\lambda_2$. Since the map of spectral sequences converging to the map

$$(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \to (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}$$

is injective in the relevant bidegrees, we also have the stated differentials in the spectral sequence (4.2). □

Proposition 4.9. *There are differentials*

$$\begin{aligned} d^6(t^{-1}) &= t^2\lambda_1' \\ d^6(t^{-1}\lambda_2) &= t^2\lambda_1'\lambda_2 \\ d^6(t^{-2}\lambda_2) &= t\lambda_1'\lambda_2 \\ d^6(\lambda_2) &= 0 \end{aligned}$$

in the spectral sequence (4.2)*.*

Proof. Since t^{-3} survives to the E^8-term in the spectral sequence (4.1) by Proposition 4.8, we know $d^6(t^{-3}) = 0$ in the spectral sequence (4.2). By Proposition 4.4 we have $d^6(t^{-2}) = t\lambda_1'$ in the spectral sequence (4.1). Using the module structure of the spectral sequence (4.2) over the spectral sequence (4.1), we have $d^6(t^{-5}) = d^6(t^{-2} \cdot t^{-3}) = t\lambda_1' \cdot t^{-3} + t^{-2} \cdot 0 = t^{-2}\lambda_1'$ and $d^6(t^{-1}) = t^2\lambda_1'$.

Since $t\lambda_2$ is a d^8-boundary by Proposition 4.8 it must be a d^6-cycle, which implies that $d^6(t^{-1}\lambda_2) = d^6(t^{-2} \cdot t\lambda_2) = t\lambda_1' \cdot t\lambda_2 + t^{-2} \cdot 0 = t^2\lambda_1'\lambda_2$.

The fact that $t\lambda_2$ is a d^8-boundary also implies that $\lambda_1' \cdot t\lambda_2 = t\lambda_1'\lambda_2$ must be a d^r-boundary for some $r \le 8$. Since $t^{-3}\lambda_1'$ is a d^6-boundary, it cannot be the source of this d^r-differential, so the only remaining possibility is that $d^6(t^{-2}\lambda_2) = t\lambda_1'\lambda_2$. Using the module structure over the spectral sequence (4.1), we also conclude that

$$d^6(\lambda_2) = d^6(t^2 \cdot t^{-2}\lambda_2) = t^5\lambda_1' \cdot t^{-2}\lambda_2 + t^2 \cdot t\lambda_1'\lambda_2 = 0\,.$$

□

Corollary 4.10. *There are isomorphisms*

$$(\overline{V}(2) \otimes \overline{C}\eta)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \begin{cases} \mathbb{F}_2\{1\} & \text{if } n = 0\,, \\ 0 & \text{if } n \in \{1,2,3\}\,, \\ \mathbb{F}_2\{t^4\lambda_1'\lambda_2\} & \text{if } n = 4\,, \\ \mathbb{F}_2\{\lambda_1'\} & \text{if } n = 5\,, \end{cases}$$

and these repeat 8-periodically, via multiplication by $t^{\pm 4}$.

Proof. This follows directly from Proposition 4.8 and Proposition 4.9. □

Proposition 4.11. *We have the following results:*

(a) *The multiplicative relation $(\lambda_1')^2 = \eta^2\mu$ holds in the abutment $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})$ of the η-Bockstein spectral sequence (2.2).*

(b) *There is a nonzero differential*

$$d^6(t^{-1}\lambda_1') = t^2\eta^2\mu$$

in the spectral sequence (4.1). Hence

$$\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = 0$$

for $n \in \{6,7\}$.

(c) *The unit images of η and η^2 are detected by $t^2\lambda_1'$ and $t^4\eta^2\mu$, respectively, in the spectral sequence (4.1).*

(d) *There is a nonzero differential*

$$d^8(t^{-1}\lambda_1') = t^3\lambda_1'\lambda_2$$

in the $\overline{V}(2) \otimes \overline{C}\eta$-homotopy $\mathbb{T}$-Tate spectral sequence (4.2). Hence

$$(\overline{V}(2) \otimes \overline{C}\eta)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = \begin{cases} 0 & \text{if } n = 6, \\ \mathbb{F}_2\{\lambda_2\} & \text{if } n = 7, \end{cases}$$

and these repeat 8-periodically, via multiplication by $t^{\pm 4}$.

Proof. The $\overline{V}(1)$-module cofiber sequence

$$\Sigma^{1,1}\overline{V}(2) \xrightarrow{\eta} \overline{V}(2) \xrightarrow{i} \overline{V}(2) \otimes \overline{C}\eta \xrightarrow{j} \Sigma^{2,0}\overline{V}(2)$$

induces a long exact sequence

$$\cdots \longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_{n+2} \operatorname{gr}^{*+1}_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \xrightarrow{j} \overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \xrightarrow{\eta}$$
$$\xrightarrow{\eta} \overline{V}(2)_{n+1} \operatorname{gr}^{*+1}_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \xrightarrow{i} (\overline{V}(2) \otimes \overline{C}\eta)_{n+1} \operatorname{gr}^{*+1}_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \longrightarrow \ldots\,.$$

By case $n = 0$ of Corollary 4.6, the cases $n \in \{0,1\}$ of Corollary 4.10, and the fact that $i(1) = 1$, we deduce from exactness that $\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) = 0$ for $n \equiv -1$

mod 8. Referring back to Proposition 4.4, this implies that $t^{-1}\lambda_1'$ in stem $7 = -1+8$ cannot survive to the E^∞-term of (4.1), so the differential $d^6(t^{-1}\lambda_1') = t^2(\lambda_1')^2 = c \cdot t^2\eta^2\mu$ must be nonzero. Hence $c = 1$, which proves that $(\lambda_1')^2 = \eta^2\mu$ and $d^6(t^{-1}\lambda_1') = t^2\eta^2\mu$. This means that the E^∞-term of (4.1), and its abutment, must be trivial in stems 6 and 7.

By the cases $n \in \{1,2,3\}$ of Corollary 4.10, and exactness, it also follows that η and η^2 generate $\overline{V}(2)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko}) \cong \mathbb{F}_2$ for $n = 1$ and 2, hence are detected by the only classes in stems 1 and 2, namely $t^2\lambda_1'$ and $t^4\eta^2\mu$, in the E^∞-term of (4.1).

By (4.11) and exactness, it follows that $(\overline{V}(2) \otimes \overline{C}\eta)_n \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TP}(\mathrm{ko})$ is 0 for $n = 6$ and $\mathbb{F}_2$ for $n = 7$. Hence $t^3\lambda_1'\lambda_2$ in stem 6 cannot survive to the E^∞-term of (4.2), and since $d^6(\lambda_2) = 0$ by Proposition 4.9 the only possible source of a differential killing it is $t^{-1}\lambda_1'$. Hence $d^8(t^{-1}\lambda_1') = t^3\lambda_1'\lambda_2$, and the lone surviving class in stem 7 of the E^∞-term of (4.2) is λ_2. Note that we have $\eta(1) = t^2\lambda_1'$ and $\eta(t^2\lambda_1') = t^4\eta^2\mu$, we have $i(1) = 1$ and $i(\lambda_1') = \lambda_1'$, and we have $j(t^4\lambda_1'\lambda_2) = t^4\eta^2\mu$ and $j(\lambda_2) = \lambda_1'$. □

Corollary 4.12. *We have a preferred isomorphism of bigraded $\mathbb{F}_2$-algebras*

$$\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \cong \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)}\,.$$

We can now prove Theorem 4.1 and Theorem 4.2.

Proof of Theorem 4.1. By Corollary 4.12, the spectral sequence (4.1) has E^2-term:

$$\hat{E}^2 = \mathbb{F}_2[t^{\pm 1}] \otimes \mathbb{F}_2[\eta, \lambda_1', \mu]/(\eta\lambda_1', (\lambda_1')^2 = \eta^2\mu) \otimes \Lambda(\varepsilon_2)\,. \tag{4.4}$$

The differentials follow from Propositions 4.3, 4.4, and 4.11 leaving

$$\hat{E}^4 = \hat{E}^6 = \mathbb{F}_2\{1, t\lambda_1', \lambda_1', (\lambda_1')^2\} \otimes \mathbb{F}_2[t^{\pm 2}]$$

and

$$\hat{E}^\infty = \mathbb{F}_2\{1, t^2\lambda_1', t^4(\lambda_1')^2, \lambda_1'\} \otimes \mathbb{F}_2[t^{\pm 4}]\,,$$

with 1, η, η^2, λ_1' and $t^{\pm 4}$ being detected by 1, $t^2\lambda_1'$, $t^4(\lambda_1')^2$, λ_1' and $t^{\pm 4}$ respectively in the E^∞-term. □

Proof of Theorem 4.2. By Corollary 2.30, the spectral sequence (4.2) has E^2-term:

$$\hat{E}^2 = \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)} \otimes \mathbb{F}_2[t^{\pm 1}]\,. \tag{4.5}$$

The differentials follow from Propositions 4.8, 4.9,and 4.11 leaving

$$\hat{E}^4 = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 1}] \text{ and}$$
$$\hat{E}^\infty = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$

□

5. Syntomic cohomology

We shall now calculate the syntomic cohomology $\pi_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ of ko (cf. Definition 2.15). We first carry out these computations in $\overline{V}(2)$- and $\overline{V}(2) \otimes \overline{C}\eta$-homotopy, and then use v_2-Bockstein spectral sequences to lift the results to $\overline{V}(1)$- and $\overline{A}(1)$-homotopy.

By restricting the $\mathbb{T}$-Tate spectral sequences (4.1) and (4.2) to the second quadrant, we obtain the $\overline{V}(2)$-homotopy $\mathbb{T}$-homotopy fixed point spectral sequence

$$\begin{aligned} E^2 &= \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,[t] \\ &= \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)} \otimes \mathbb{F}_2[t] \qquad (5.1) \\ &\Longrightarrow \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) \end{aligned}$$

and the $\overline{V}(2) \otimes \overline{C}\eta$-homotopy $\mathbb{T}$-homotopy fixed point spectral sequence

$$\begin{aligned} E^2 &= (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})\,[t] \\ &= \Lambda(\varepsilon_2) \otimes \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu] \otimes \mathbb{F}_2[t] \qquad (5.2) \\ &\Longrightarrow (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})\,. \end{aligned}$$

The former is an algebra spectral sequence, and the latter is a module spectral sequence over it. They can be reindexed as cohomologically graded t-Bockstein spectral sequences, but the current indexing is the one inherited from the homologically graded C_2- and $\mathbb{T}$-Tate spectral sequences.

Proposition 5.1. *There is an isomorphism*

$$\begin{aligned} \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko}) \cong\ & \mathbb{F}_2[t^4] \otimes \mathbb{F}_2\{1, t^2\lambda_1', \lambda_1', (\lambda_1')^2\} \\ & \oplus \mathbb{F}_2\{t\lambda_1', (t\lambda_1')^2\} \\ & \oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ & \oplus \mathbb{F}_2[\bar\mu]\{\bar\mu, \eta\bar\mu, \eta^2\bar\mu, \lambda_1'\mu\} \end{aligned}$$

with $\bar\mu = \mu + \eta\varepsilon_2$*, where* $(\lambda_1')^2 = \eta^2\mu \neq \eta^2\bar\mu$*,* $\eta \cdot \eta^2\bar\mu = \eta^4\varepsilon_2$ *and* $\bar\mu^2 = \mu^2$*.*

Proof. The map of spectral sequences induced by can: $\mathrm{TC}^-(\mathrm{ko}) \to \mathrm{TP}(\mathrm{ko})$ is given at the E^2-terms by inverting t, so the differentials in (4.1) from Theorem 4.1 lift to differentials

$$\begin{aligned} d^2(\varepsilon_2) &= t\mu \\ d^2(t) &= t^2\eta \\ d^6(t^2) &= t^5\lambda_1' \\ d^6(t^3\lambda_1') &= t^6(\lambda_1')^2 \end{aligned}$$

in (5.1). Some bookkeeping shows that

$$\begin{aligned} E^4 = \ & \mathbb{F}_2[t^2] \otimes \mathbb{F}_2\{1, t\lambda_1', \lambda_1', (\lambda_1')^2\} \\ & \oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ & \oplus \mathbb{F}_2[\bar\mu]\{\bar\mu, \eta\bar\mu, \eta^2\bar\mu, \lambda_1'\mu\} \end{aligned}$$

with $\bar\mu = \mu + \eta\varepsilon_2$ and $\eta \cdot \eta^2\bar\mu = \eta^4\varepsilon_2$, and

$$\begin{aligned} E^8 = E^\infty = \ & \mathbb{F}_2[t^4] \otimes \mathbb{F}_2\{1, t^2\lambda_1', \lambda_1', (\lambda_1')^2\} \\ & \oplus \mathbb{F}_2\{t\lambda_1', (t\lambda_1')^2\} \\ & \oplus \mathbb{F}_2[\eta]\{\eta, \eta^4\varepsilon_2\} \\ & \oplus \mathbb{F}_2[\bar\mu]\{\bar\mu, \eta\bar\mu, \eta^2\bar\mu, \lambda_1'\mu\}\,. \end{aligned}$$

□

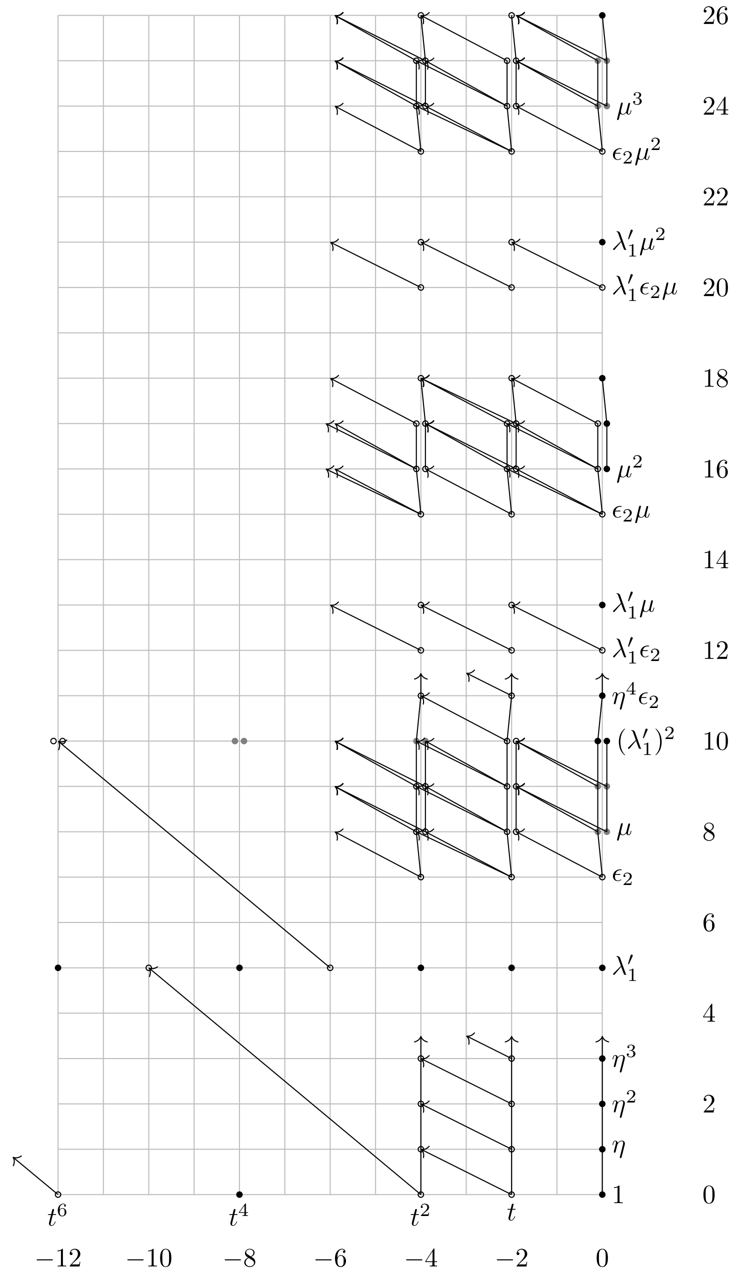

FIGURE 5.1. $\mathbb{T}$-homotopy fixed point spectral sequence converging to $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$

For a pictorial representation of the spectral sequence (5.1) see Figure 5.1.

Proposition 5.2. *There is an isomorphism*

$$(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\mathrm{ko})\cong\frac{\mathbb{F}_2[t^4,\mu]}{(t^4\mu)}\otimes\Lambda(\lambda_1')\{1,\lambda_2\}$$
$$\oplus\,\mathbb{F}_2\{t^2\lambda_1',t\lambda_1',t\lambda_2,t^3\lambda_1'\lambda_2,t^2\lambda_1'\lambda_2,t\lambda_1'\lambda_2\}\,.$$

Proof. The differentials in (4.2) from Theorem 4.2 lift over the canonical map to differentials $d^2(\varepsilon_2)=t\mu$ (repeating t-periodically) and

$$\begin{aligned}d^6(t^2)&=t^5\lambda_1'\\ d^6(t^3)&=t^6\lambda_1'\\ d^6(t^2\lambda_2)&=t^5\lambda_1'\lambda_2\\ d^6(t^3\lambda_2)&=t^6\lambda_1'\lambda_2\\ d^8(t)&=t^5\lambda_2\\ d^8(t^3\lambda_1')&=t^7\lambda_1'\lambda_2\end{aligned}$$

(repeating t^4-periodically) in (5.2). It follows that $E^4=\mathbb{F}_2[t,\mu]/(t\mu)\otimes\Lambda(\lambda_1')\{1,\lambda_2\}$ and

$$E^{10}=E^\infty=\frac{\mathbb{F}_2[t^4,\mu]}{(t^4\mu)}\otimes\Lambda(\lambda_1')\{1,\lambda_2\}\oplus\mathbb{F}_2\{t^2\lambda_1',t\lambda_1',t\lambda_2,t^3\lambda_1'\lambda_2,t^2\lambda_1'\lambda_2,t\lambda_1'\lambda_2\}\,.$$

See also Figure 5.2. □

As discussed in the proof of Proposition 4.4, there is a $\overline{V}(2)$-homotopy C_2-Tate spectral sequence

$$\begin{aligned}\hat{E}^2&=\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\otimes\Lambda(h)\otimes\mathbb{F}_2[t^{\pm1}]\\ &=\Lambda(\varepsilon_2)\otimes\frac{\mathbb{F}_2[\eta,\lambda_1',\mu]}{(\eta\lambda_1',(\lambda_1')^2+\eta^2\mu)}\otimes\Lambda(h)\otimes\mathbb{F}_2[t^{\pm1}]\\ &\Longrightarrow\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}\,.\end{aligned}\tag{5.3}$$

Similarly, we have a $\overline{V}(2)\otimes\overline{C}\eta$-homotopy C_2-Tate spectral sequence

$$\begin{aligned}\hat{E}^2&=(\overline{V}(2)\otimes\overline{C}\eta)*\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\otimes\Lambda(h)\otimes\mathbb{F}_2[t^{\pm1}]\\ &=\Lambda(\varepsilon_2)\otimes\Lambda(\lambda_1')\{1,\lambda_2\}\otimes\mathbb{F}_2[\mu]\otimes\Lambda(h)\otimes\mathbb{F}_2[t^{\pm1}]\\ &\Longrightarrow(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}\,.\end{aligned}\tag{5.4}$$

There is a map F of algebra spectral sequences from (4.1) to (5.3), and (5.4) is a module spectral sequence over (5.3).

Proposition 5.3. *There is an isomorphism*

$$\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}\cong\mathbb{F}_2\{1,\eta,\eta^2,\lambda_1'\}\otimes\Lambda(h)\otimes\mathbb{F}_2[t^{\pm4}]$$

where η, η^2 and λ_1' are detected by $t^2\lambda_1'$, $(t^2\lambda_1')^2$ and λ_1', respectively. Under this correspondence, the cyclotomic structure map

$$\varphi_2\colon\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\longrightarrow\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}$$

is given by $\varepsilon_2\mapsto ht^{-4}$, $\eta\mapsto\eta$, $\lambda_1'\mapsto\lambda_1'$ and $\mu\mapsto t^{-4}$, hence can be identified with the localization homomorphism

$$\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\longrightarrow\mu^{-1}\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$$

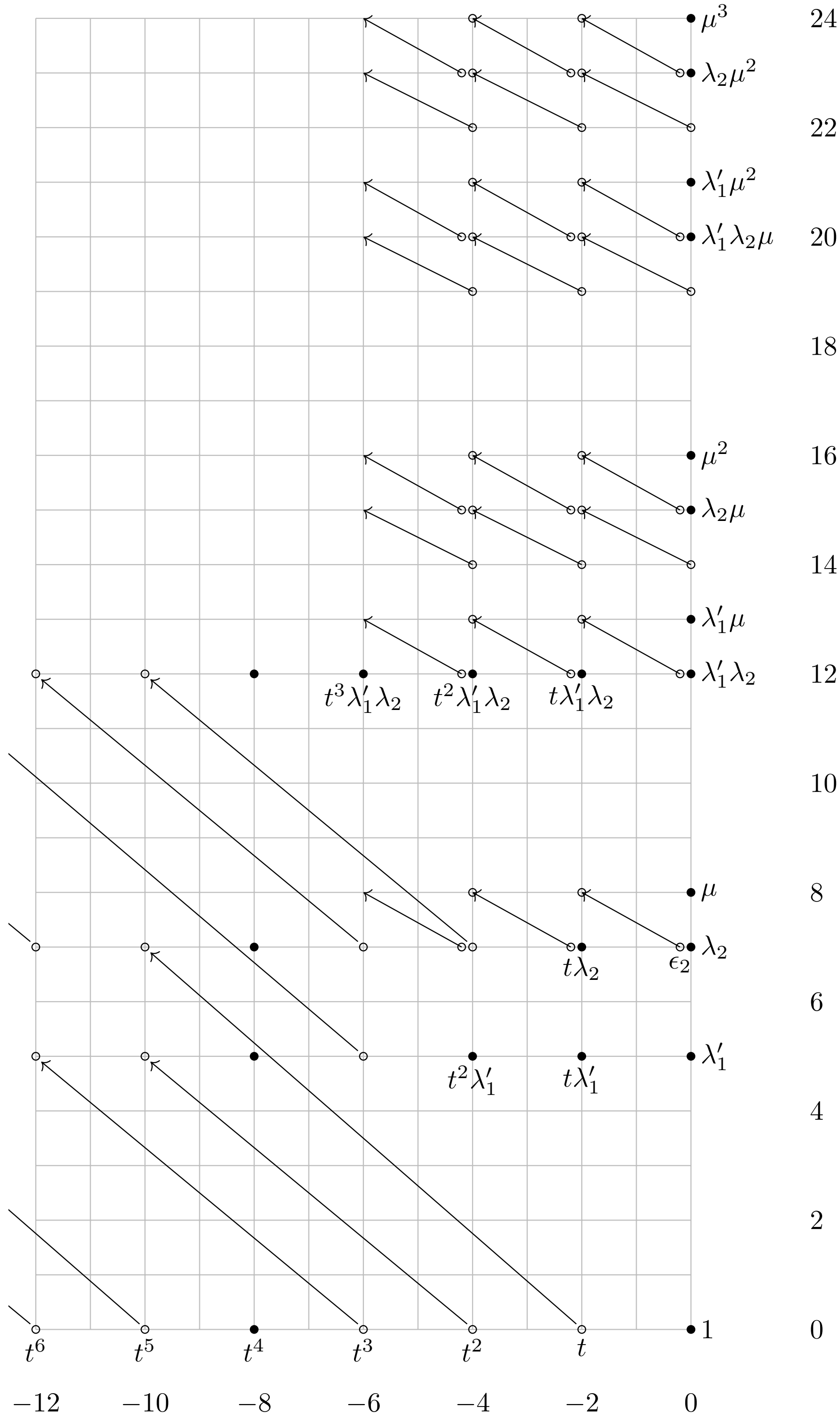

FIGURE 5.2. $\mathbb{T}$-homotopy fixed point spectral sequence converging to $(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$

that inverts μ.

Proof. Since φ_2 is a ring map, we know that v_2 acts trivially on

$$\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}\,.$$

A choice of null-homotopy compatible with this map corresponds to the class $\overline{u}_1 t^{-4}$. Consequently, $d^r(h) = 0$ for $r \leq 8$ and $d^9(ht^{-4}) = t\mu$ in the $\overline{V}(1)$-homotopy C_2-Tate spectral sequence. When combined with the differentials in (4.1) from Theorem 4.1, this shows that

$$\hat{E}^4 = \mathbb{F}_2\{1, t\lambda_1', \lambda_1', (\lambda_1')^2\} \otimes \Lambda(h) \otimes \mathbb{F}_2[t^{\pm 2}]$$

and

$$\hat{E}^\infty = \mathbb{F}_2\{1, t^2\lambda_1', (t^2\lambda_1')^2, \lambda_1'\} \otimes \Lambda(h) \otimes \mathbb{F}_2[t^{\pm 4}]\,.$$

The detection results then follow from those in Theorem 4.1.

It is clear that $\varphi_2(\eta) = \eta$. To evaluate φ_2 on ε_2 and μ we use the commutative diagram

$$\begin{CD}
\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) @>{\varphi_2}>> \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2} \\
@V{c}VV @VV{c}V \\
\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku}) @>{\varphi_2}>> \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})^{tC_2}\,.
\end{CD}$$

The complexification map

$$\begin{aligned}
\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \cong \Lambda(\varepsilon_2) \otimes \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)}& \\
\to \Lambda(\varepsilon_2) \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu] \cong \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})&
\end{aligned}$$

is given by $\varepsilon_2 \mapsto \varepsilon_2$, $\eta \mapsto 0$, $\lambda_1' \mapsto 0$ and $\mu \mapsto \mu$. To see this, note that as in the proof of Lemma 2.18 the map

$$\overline{V}(2) \otimes \overline{C}\eta_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko}) \longrightarrow \overline{V}(2) \otimes \overline{C}\eta_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})$$

is given by $\varepsilon_2 \mapsto \varepsilon_2$, $\lambda_2 \mapsto \lambda_2$, $\lambda_1' \mapsto 0$ and $\mu \mapsto \mu$. The claim then follows by the map of η-Bockstein spectral sequences, where the claim that η maps to zero follows for bidegree reasons.

It follows from [HRW22, Theorem 6.1.2] for $p = 2$ that in the ku-case the cyclotomic structure map

$$\begin{aligned}
\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku}) \cong \Lambda(\varepsilon_2) \otimes \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu]& \\
\to \Lambda(\lambda_1, \lambda_2) \otimes \Lambda(h) \otimes \mathbb{F}_2[t^{\pm 4}] \cong \overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ku})^{tC_2}&
\end{aligned}$$

satisfies $\varepsilon_2 \mapsto ht^{-4}$, $\lambda_1 \mapsto \lambda_1$, $\lambda_2 \mapsto \lambda_2$ and $\mu \mapsto t^{-4}$. Where the claim about ϵ_2 follows because ht^{-4} is the only non-trivial class in the correct bidegree to be the image of ε_2 and we know that we can choose a null homotopy of v_2 compatibly with the cyclotomic structure map so that ε_2 maps non-trivially.

Hence the cyclotomic structure map in the ko-case must likewise map ε_2 to ht^{-4} and μ to t^{-4}. Lastly, the relation $(\lambda_1')^2 = \eta^2\mu$ shows that $\varphi_2(\lambda_1')^2$ must be detected by $(t^2\lambda_1')^2 \cdot t^{-4} = (\lambda_1')^2$, which can only happen if $\varphi_2(\lambda_1')$ is (detected by) λ_1'.

The claim about localization then amounts to the isomorphism

$$\mu^{-1} \frac{\mathbb{F}_2[\eta, \lambda_1', \mu]}{(\eta\lambda_1', (\lambda_1')^2 + \eta^2\mu)} \cong \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[\mu^{\pm 1}]\,.$$

□

Proposition 5.4. *There is an isomorphism*

$$(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}\cong\Lambda(\lambda_1')\{1,\lambda_2\}\otimes\Lambda(h)\otimes\mathbb{F}_2[t^{\pm4}]$$

where 1, λ_1', λ_2 *and* $\lambda_1'\lambda_2$ *are detected by classes with the same names. Under this correspondence, the cyclotomic structure map*

$$\varphi_2\colon(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\longrightarrow(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}$$

is given by $\varepsilon_2\mapsto ht^{-4}$, $\lambda_1'\mapsto\lambda_1'$, $\lambda_2\mapsto\lambda_2$ *and* $\mu\mapsto t^{-4}$, *hence can be identified with the localization homomorphism*

$$(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})\longrightarrow\mu^{-1}(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})$$

that inverts μ.

Proof. As before, $d^r(h)=0$ for $r\le 8$. When combined with the differentials in (4.2) from Theorem 4.2, this shows that

$$\hat{E}^4=\Lambda(\lambda_1')\{1,\lambda_2\}\otimes\Lambda(h)\otimes\mathbb{F}_2[t^{\pm1}]$$

and

$$\hat{E}^\infty=\Lambda(\lambda_1')\{1,\lambda_2\}\otimes\Lambda(h)\otimes\mathbb{F}_2[t^{\pm4}]\,.$$

The detection results then follow from those in Theorem 4.2.

The evaluation of φ_2 on ε_2, λ_1' and μ follows from that in Proposition 5.3 by comparison along $\overline{V}(2)\to\overline{V}(2)\otimes\overline{C}\eta$, while that of φ_2 on λ_2 follows by comparison along $\overline{V}(2)\otimes\overline{C}\eta\to\Sigma^2\overline{V}(2)$, which maps λ_2 to $\Sigma^{2,0}\lambda_1'$. □

Remark 5.5. The computations in Theorem 4.1, Theorem 4.2, Proposition 5.3, and Proposition 5.4 are consistent with the isomorphisms

$$\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\mathrm{ko})\cong\overline{V}(1)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2},\ \text{and}$$
$$(\overline{V}(2)\otimes\overline{C}\eta)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TP}(\mathrm{ko})\cong\overline{A}(1)\otimes_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}$$

in analogy with [HRW22, Theorem 6.2.1].

Theorem 5.6 (Syntomic cohomology of ko mod $(2,v_1,v_2)$)**.** *We have an algebra isomorphism*

$$\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}(\mathrm{ko})\cong\mathbb{F}_2[\eta]\{1,\eta^4\epsilon_2\}\oplus\mathbb{F}_2\{\partial,\nu,\lambda_1',\partial\lambda_1',\nu^2,(\lambda_1')^2\}\,,$$

of $\mathbb{F}_2[\eta]$*-modules with generators in bidegrees* $\|\partial\|=(-1,1)$, $\|\eta\|=(1,1)$, $\|\nu\|=(3,1)$, $\|\lambda_1'\|=(5,1)$ *and* $\|\eta^4\epsilon_2\|=(11,3)$. *See also Figure 5.3.*

Proof. To calculate the effect in $\overline{V}(2)$-homotopy of can: $\mathrm{TC}^-(\mathrm{ko})\to\mathrm{TP}(\mathrm{ko})$, we use the map of spectral sequences from (5.1) to (4.1), described in Proposition 5.1 and Theorem 4.1, given at E^2 by inverting t. To calculate the effect of $\varphi_2^{h\mathbb{T}}\colon\mathrm{TC}^-(\mathrm{ko})\to(\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}$ we appeal to Proposition 5.3 to see that there is a $\mathbb{T}$-homotopy fixed point spectral sequence

$$\begin{aligned}\mu^{-1}E^2&=\overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{THH}(\mathrm{ko})^{tC_2}\,[t]\\&=\Lambda(\epsilon_2)\otimes\mathbb{F}_2\{1,\eta,\eta^2,\lambda_1'\}\otimes\mathbb{F}_2[\mu^{\pm1}]\otimes\mathbb{F}_2[t]\\&\Longrightarrow\ \overline{V}(2)_*\,\mathrm{gr}^*_{\mathrm{mot}}(\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}\,,\end{aligned}\tag{5.5}$$

and $\varphi_2^{h\mathbb{T}}$ is calculated by the map of spectral sequences from (5.1) to (5.5) that is given at E^2 by inverting μ. The differentials

$$d^2(\epsilon_2) = t\mu \qquad \text{and} \qquad d^2(\mu) = t\eta\mu$$

carry over from the proof of Proposition 5.1, leaving

$$\mu^{-1}E^4 = \mu^{-1}E^\infty = \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[\bar{\mu}^{\pm 1}]$$

concentrated on the vertical axis. As before, $\bar{\mu} = \mu + \eta\epsilon_2$. The $\overline{V}(2)$-homotopy isomorphism

$$\mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[t^{\pm 4}] \xrightarrow{\cong} \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \otimes \mathbb{F}_2[\bar{\mu}^{\pm 1}]$$

induced by the equivalence $G\colon \mathrm{TP}(\mathrm{ko}) \to (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}$ can then only be given by $\eta \mapsto \eta$, $\lambda_1' \mapsto \lambda_1'$ and $t^{\pm 4} \mapsto \bar{\mu}^{\mp 1}$. Note that we know that $G\colon \mathrm{TP}(\mathrm{ko}) \to (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}}$ is an equivalence a priori by [BBLNR14, Proposition 3.8] (cf. [NS18, Lemma II.4.2]).

We claim that the map can $-\varphi$ induces isomorphisms

$$\mathbb{F}_2[t^4]\{t^4\} \otimes \mathbb{F}_2\{1, t^2\lambda_1', (t^2\lambda_1')^2, \lambda_1'\} \xrightarrow{\cong} \mathbb{F}_2[\bar{\mu}^{-1}]\{\bar{\mu}^{-1}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1') \tag{5.6}$$

$$\mathbb{F}_2[\bar{\mu}]\{\bar{\mu}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \xrightarrow{\cong} \mathbb{F}_2[\bar{\mu}]\{\bar{\mu}\} \otimes \mathbb{F}_2\{1, \eta, \eta^2, \lambda_1'\} \tag{5.7}$$

$$\mathbb{F}_2\{t^2\lambda_1', (t^2\lambda_1')^2\} \xrightarrow{\cong} \mathbb{F}_2\{\eta, \eta^2\} \tag{5.8}$$

and the zero homomorphism

$$\mathbb{F}_2[\eta]\{1, \eta^4\epsilon_2\} \oplus \mathbb{F}_2\{t\lambda_1', \lambda_1', (t\lambda_1')^2, (\lambda_1')^2\} \xrightarrow{0} \mathbb{F}_2\{1, \lambda_1'\}\,.$$

The isomorphism (5.6) occurs in horizontal degrees (= filtrations) where inverting t (or t^4) is an isomorphism, and $\varphi_2^{h\mathbb{T}}$ is zero. The isomorphism (5.7) occurs in vertical degrees where inverting μ (or $\bar{\mu}$) is an isomorphism, and can is zero. The isomorphism (5.8) uses that η and η^2 in (5.1) are detected by $t^2\lambda_1'$ and $(t^2\lambda_1')^2$ in (4.1), but map to zero in (5.5). The homomorphisms can and $\varphi_2^{h\mathbb{T}}$ agree on classes coming from $\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathbb{S}$, such as 1, η and ν, hence their difference is zero on $\mathbb{F}_2[\eta]\{1\}$ and $\mathbb{F}_2\{t\lambda_1', (t\lambda_1')^2\}$. Both $\mathrm{can}(\lambda_1')$ and $\varphi_2^{h\mathbb{T}}(\lambda_1')$ are detected by λ_1', hence agree in $\overline{V}(2)$-homotopy since there are no other classes in the same total degree, which implies that can $-\varphi_2^{h\mathbb{T}}$ is zero on λ_1' and its square. Both can and $\varphi_2^{h\mathbb{T}}$ take $\eta^4\epsilon_2 = \eta^3\bar{\mu}$ to zero, so their difference is zero on $\mathbb{F}_2[\eta]\{\eta^4\epsilon_2\}$.

Hence we have an isomorphism

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2[\eta]\{1, \eta^4\epsilon_2\} \oplus \mathbb{F}_2\{\partial, \partial\lambda_1', t\lambda_1', \lambda_1', (t\lambda_1')^2, (\lambda_1')^2\}\,.$$

The classes $t\lambda_1'$ and $(t\lambda_1')^2$ detect ν and ν^2, respectively. The algebra structure is evident from the notation and sparsity, except for the fact that $\eta \cdot \lambda_1' = 0$, which follows from Theorem 5.7 below. □

Next we compute the v_2-Bockstein spectral sequence

$$\overline{V}(2)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})[v_2] \implies \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})\,. \tag{5.9}$$

Proposition 5.7. *In the spectral sequence (5.9) there is a d_1-differential*

$$d_1(\eta^4\varepsilon_2) = v_2\eta^4$$

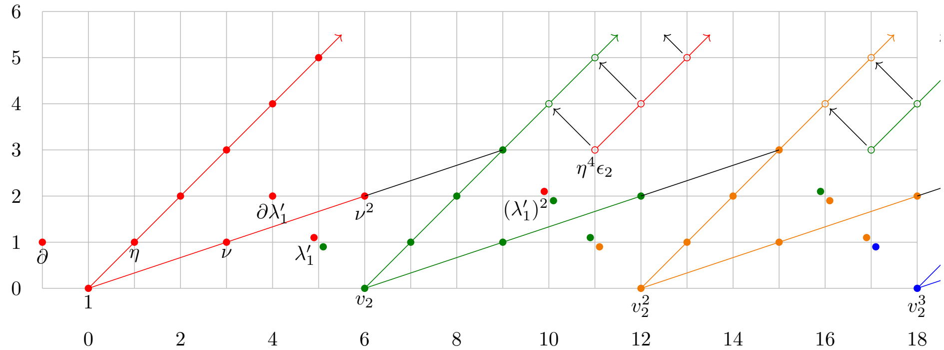

FIGURE 5.3. $E_1 \implies \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC(ko)}$

together with its various η- and v_2-power multiples. This produces an algebra isomorphism

$$\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC(ko)} \cong \frac{\Lambda(\partial) \otimes \mathbb{F}_2[\eta, \nu, \lambda_1', v_2]}{(\partial\eta, \partial\nu, \eta\nu, \eta\lambda_1', \nu\lambda_1', \nu^3 = v_2\eta^3 = \partial(\lambda_1')^2, (\lambda_1')^3 = d \cdot v_2^2\eta^3)},$$

where $d \in \mathbb{F}_2$ and we have not resolved this indeterminacy.

Proof. The unit map $\mathbb{S} \to \mathrm{TC(ko)}$ induces a map of v_2-Bockstein spectral sequences, from

$$\mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_3)\,[v_2] \implies \mathrm{Ext}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*/I_2) \tag{5.10}$$

shown in Figure 3.2 to (5.9) shown in Figure 5.3. Since $v_2^2 h_{10}^4 = 0$ in the abutment of the former, we must have that $v_2^2\eta^4$ is a boundary in the latter. Considering bidegrees and v_2-powers, this can only happen if $d_1(v_2\eta^4\epsilon_2) = v_2^2\eta^4$. Hence $d_1(v_2^i\eta^j\epsilon_2) = v_2^{i+1}\eta^j$ for all $i \geq 0$ and $j \geq 4$, as claimed. There is no room for other v_2-Bockstein differentials, so $E_2 = E_\infty$ in (5.9).

The relation $v_2h_{11}^3 = v_2^2h_{10}^3$ in the abutment of (5.10) also implies that $v_2\nu^3 = v_2^2\eta^3$ in the abutment of (5.9). Hence we have hidden ν-extensions from $v_2^i\nu^2$ to $v_2^{i+1}\eta^3$ for all $i \geq 0$. The products $\partial\eta$ and $\eta\lambda_1'$ lie in trivial groups. The well-known relation $\eta\nu = 0$ implies the vanishing of $\partial\nu$ and $\nu\lambda_1'$. We postpone the proof that $\partial(\lambda_1')^2$ is equal to $\nu^3 = v_2\eta^3$ to Remark 5.10. We have not determined whether $(\lambda_1')^3 \in \mathbb{F}_2\{v_2^2\eta^3\}$ is zero or not □

Proposition 5.8. *We have an isomorphism*

$$\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC(ko)} \cong \mathbb{F}_2[v_2] \otimes \Big(\Lambda(\partial)\{1, \lambda_1', \lambda_2, \lambda_1'\lambda_2\} \oplus \mathbb{F}_2\{t^2\lambda_1', t\lambda_1', t\lambda_2, t^3\lambda_1'\lambda_2, t^2\lambda_1'\lambda_2, t\lambda_1'\lambda_2\}\Big)$$

of finitely generated free $\mathbb{F}_2[v_2]$-modules, where $\|v_2\| = (6,0)$, $\|\partial\| = (-1,1)$, $\|\lambda_1'\| = (5,1)$, $\|\lambda_2\| = (7,1)$ and $\|t\| = (-2,0)$. See also Figure 5.4.

Proof. This proof is similar to that of Theorem 5.6, to which we refer for a more elaborate review of some of the notations. To calculate the effect of can in $\overline{V}(2)\otimes\overline{C}\eta$-homotopy we use the map of spectral sequences from (5.2) to (4.2), described in Proposition 5.2 and Theorem 4.2, given at E^2 by inverting t. To calculate the effect

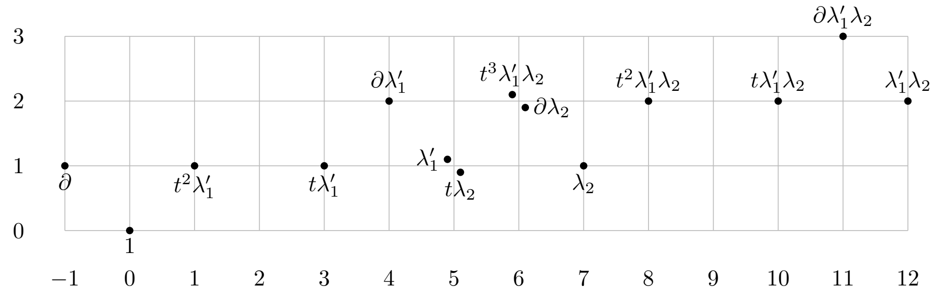

FIGURE 5.4. $\mathbb{F}_2[v_2]$-basis for $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$

of $\varphi_2^{h\mathbb{T}}$ we use Proposition 5.4 to see that there is a $\mathbb{T}$-homotopy fixed point spectral sequence

$$\begin{aligned}
\mu^{-1}E^2 &= (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{THH}(\mathrm{ko})^{tC_2}\,[t] \\
&= \Lambda(\epsilon_2) \otimes \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu^{\pm 1}] \otimes \mathbb{F}_2[t] \\
&\implies (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} (\mathrm{THH}(\mathrm{ko})^{tC_2})^{h\mathbb{T}} ,
\end{aligned} \tag{5.11}$$

and $\varphi_2^{h\mathbb{T}}$ is given by the map of spectral sequences from (5.2) to (5.11) that is given at E^2 by inverting μ. The differential $d^2(\epsilon_2) = t\mu$ carries over from the proof of Proposition 5.2, leaving

$$\mu^{-1}E^4 = \mu^{-1}E^\infty = \Lambda(\lambda_1')\{1, \lambda_2\} \otimes \mathbb{F}_2[\mu^{\pm 1}]$$

concentrated on the vertical axis. The $\overline{V}(2) \otimes \overline{C}\eta$-homotopy isomorphism

$$\Lambda(\lambda_1')\{1, \lambda_2) \otimes \mathbb{F}_2[t^{\pm 4}] \xrightarrow{\cong} \Lambda(\lambda_1')\{1, \lambda_2) \otimes \mathbb{F}_2[\mu^{\pm 1}]$$

induced by the equivalence G must thus be given by $\lambda_1' \mapsto \lambda_1'$, $\lambda_2 \mapsto \lambda_2$ and $t^{\pm 4} \mapsto \mu^{\mp 1}$.

The map can $-\varphi_2^{h\mathbb{T}}$ induces isomorphisms

$$\begin{aligned}
\mathbb{F}_2[t^4]\{t^4\} \otimes \Lambda(\lambda_1')\{1, \lambda_2\} &\xrightarrow{\cong} \mathbb{F}_2[\mu^{-1}]\{\mu^{-1}\} \otimes \Lambda(\lambda_1')\{1, \lambda_2\} \\
\mathbb{F}_2[\mu]\{\mu\} \otimes \Lambda(\lambda_1')\{1, \lambda_2\} &\xrightarrow{\cong} \mathbb{F}_2[\mu]\{\mu\} \otimes \Lambda(\lambda_1')\{1, \lambda_2\}
\end{aligned}$$

and the zero homomorphism

$$\Lambda(\lambda_1')\{1, \lambda_2\} \oplus \mathbb{F}_2\{t^2\lambda_1', t\lambda_1', t\lambda_2, t^3\lambda_1'\lambda_2, t^2\lambda_1'\lambda_2, t\lambda_1'\lambda_2\} \xrightarrow{0} \Lambda(\lambda_1')\{1, \lambda_2\} ,$$

by the same arguments as in the proof of Theorem 5.6. Hence we have an additive isomorphism

$$\begin{aligned}
(\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \cong \Lambda(\partial)\{1, \lambda_1', \lambda_2, \lambda_1'\lambda_2\}& \\
\oplus \mathbb{F}_2\{t^2\lambda_1', t\lambda_1', t\lambda_2, t^3\lambda_1'\lambda_2, t^2\lambda_1'\lambda_2, t\lambda_1'\lambda_2\}& ,
\end{aligned}$$

There is no room for differentials in the v_2-Bockstein spectral sequence

$$E_1 = (\overline{V}(2) \otimes \overline{C}\eta)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})\,[v_2] \implies \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) .$$

$\square$

Lemma 5.9. *The Hurewicz images in $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ of the classes 1, h_{11}, w, $h_{11}^2 = h_{10}w$, $h_{10}w$ and $h_{11}^2 w$ in $\pi_*\overline{A}(1)$ are detected by 1, $t\lambda_1'$, $t\lambda_2$, $t^3\lambda_1'\lambda_2 \mod \partial\lambda_2$, $t^2\lambda_1'\lambda_2$ and $\partial\lambda_1'\lambda_2$, respectively. The product $h_{10}\lambda_1'$ is detected by $t\lambda_1'\lambda_2$.*

Proof. The cofiber sequence

$$\Sigma^{1,1}\overline{V}(1) \xrightarrow{\eta} \overline{V}(1) \xrightarrow{i} \overline{A}(1) \xrightarrow{j} \Sigma^{2,0}\overline{V}(1)$$

induces a long exact sequence

$$\dots \xrightarrow{\eta} \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \xrightarrow{i} \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \\ \xrightarrow{j} \Sigma^{2,0}\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \xrightarrow{\eta} \dots$$

of $\mathbb{F}_2[v_2]$-modules, see Figures 5.3 and 5.4. Having chosen $\lambda_1' \in \overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ we choose $\lambda_1', \lambda_2 \in \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ so that $i(\lambda_1') = \lambda_1'$ and $j(\lambda_2) = \Sigma^{2,0}\lambda_1'$. By exactness, i is then given by

$$\begin{aligned} 1 &\longmapsto 1 \\ \partial &\longmapsto \partial \\ \nu &\longmapsto t\lambda_1' \\ \lambda_1' &\longmapsto \lambda_1' \\ \partial\lambda_1' &\longmapsto \partial\lambda_1' \\ \nu^2 &\longmapsto t^3\lambda_1'\lambda_2 \mod \partial\lambda_2 \\ (\lambda_1')^2 &\longmapsto t\lambda_1'\lambda_2 \mod v_2\partial\lambda_1' \,, \end{aligned}$$

while j is given by

$$\begin{aligned} t^2\lambda_1' &\longmapsto \Sigma^{2,0}\partial \\ t\lambda_2 &\longmapsto \Sigma^{2,0}\nu \\ \lambda_2 &\longmapsto \Sigma^{2,0}\lambda_1' \\ \partial\lambda_2 &\longmapsto \Sigma^{2,0}\partial\lambda_1' \\ t^2\lambda_1'\lambda_2 &\longmapsto \Sigma^{2,0}\nu^2 \\ \lambda_1'\lambda_2 &\longmapsto \Sigma^{2,0}(\lambda_1')^2 \\ \partial\lambda_1'\lambda_2 &\longmapsto \Sigma^{2,0}\nu^3 \,. \end{aligned}$$

The formulas for i imply the claims for 1, $\nu = h_{11}$ and $\nu^2 = h_{11}^2$. We know from Corollary 3.4 that w is detected by $t\lambda_2$, so the formulas for j imply the claims for $\nu w = h_{11}w$ and $\nu^2 w = h_{11}^2 w$.

The $\overline{V}(1)$-module action on $\overline{A}(1)$ shows that $\nu \cdot \lambda_2 = h_{11}\lambda_2$ is detected by $t\lambda_1' \cdot \lambda_2$, since the latter product is nonzero in $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}^-(\mathrm{ko})$. □

Remark 5.10. We can now complete the unfinished business in the proof of Theorem 5.7. Since $\nu^2 w$ is detected by $\partial\lambda_1'\lambda_2$, and j maps w to $\Sigma^{2,0}\nu$ and λ_2 to $\Sigma^{2,0}\lambda_1'$, it follows that $\Sigma^{2,0}\nu^3$ is detected by $\Sigma^{2,0}\partial(\lambda_1')^2$, so $\partial(\lambda_1')^2$ is equal to $\nu^3 = v_2\eta^3$ in $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$.

Lemma 5.11. *Let $\varsigma \in \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$ be the class in bidegree $(1,1)$ detected by $t^2\lambda_1'$. Then $\nu\varsigma$ is the class in bidegree $(4,2)$ detected by $\partial\lambda_1'$.*

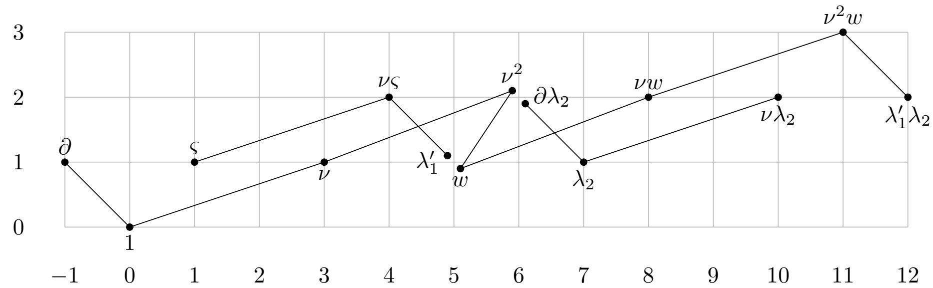

FIGURE 5.5. $\mathbb{F}_2[v_2]$-basis for $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$, with lines of slope -1, 1 and $1/3$ indicating multiplication by ∂, η and ν, respectively.

Proof. By [BHM93, Theorem 5.17], [Rog02, Cor. 1.21] there is a 2-complete equivalence $\mathrm{TC}(\mathbb{S}) \simeq \mathbb{S} \vee \Sigma\mathbb{C}P^\infty_{-1}$, and by [BM94, Proposition 10.9], [Dun97, Main Theorem] the 3-connected map $\mathbb{S} \to \mathrm{ko}$ induces a 4-connected map $\mathrm{TC}(\mathbb{S}) \to \mathrm{TC}(\mathrm{ko})$. For each $i \geq -1$ let $\Sigma\beta_i \in H_{2i+1}(\Sigma\mathbb{C}P^\infty_{-1})$ denote the generator. The Atiyah–Hirzebruch spectral sequence

$$E^2 = H_*(\Sigma\mathbb{C}P^\infty_{-1}; \pi_* A(1)) \implies A(1)_*(\Sigma\mathbb{C}P^\infty_{-1})$$

has nonzero differentials $d^4(\Sigma\beta_1) = \nu\Sigma\beta_{-1}$ and $d^6(\Sigma\beta_2) = w\Sigma\beta_{-1}$. This follows from [Mos68, Proposition 5.2, Proposition 5.4], using that $w \in \langle \nu, \eta, \iota \rangle$ in $\pi_* A(1)$, where ι is the class of $\mathbb{S} \to A(1)$. Hence

$$A(1)_* \mathrm{TC}(\mathbb{S}) \cong \mathbb{F}_2\{\Sigma\beta_{-1}, \iota, \Sigma\beta_0, \nu\iota, \nu\Sigma\beta_0\}$$

in stems $-1 \leq * \leq 4$, mapping isomorphically to

$$A(1)_* \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2\{\partial, 1, t^2\lambda_1', t\lambda_1', \partial\lambda_1'\}$$

in this range. It follows that $\Sigma\beta_0$ maps to the class ς detected by $t^2\lambda_1'$ and $\nu\Sigma\beta_0$ to the class detected by $\partial\lambda_1'$, which must therefore be equal to $\nu\varsigma$. □

Theorem 5.12 (Syntomic cohomology of ko mod $(2, \eta, v_1)$)**.** *We have an isomorphism*

$$\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \cong \mathbb{F}_2[v_2] \otimes \Big(\mathbb{F}_2\{1, \partial, \nu, w, \nu^2 = \eta w, \nu w, \lambda_1'\lambda_2, \nu^2 w = \partial\lambda_1'\lambda_2\} \oplus \mathbb{F}_2\{\varsigma, \lambda_1', \nu\varsigma = \partial\lambda_1'\} \oplus \mathbb{F}_2\{\lambda_2, \partial\lambda_2, \nu\lambda_2\} \Big)$$

of $\overline{V}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$-modules, where the (stem, motivic filtration) bidegrees and detecting classes of the $\mathbb{F}_2[v_2]$-module generators are as in Table 5.1. See also Figure 5.5.

Proof. This summarizes the results of Proposition 5.8 and Lemmas 5.9 and 5.11. The lift of $t^3\lambda_1'\lambda_2$ over $\pi\colon \mathrm{TC}(\mathrm{ko}) \to \mathrm{TC}^-(\mathrm{ko})$ is only defined modulo $\partial\lambda_2$ in the image under $\partial\colon \Sigma^{-1}\mathrm{TP}(\mathrm{ko}) \to \mathrm{TC}(\mathrm{ko})$, but the image of ν^2 specifies one such choice of lift. □

generator	bidegree	detecting class
1	$(0,0)$	1
∂	$(-1,1)$	∂
ς	$(1,1)$	$t^2\lambda_1'$
ν	$(3,1)$	$t\lambda_1'$
w	$(5,1)$	$t\lambda_2$
λ_1'	$(5,1)$	λ_1'
λ_2	$(7,1)$	λ_2
$\nu\varsigma$	$(4,2)$	$\partial\lambda_1'$
$\partial\lambda_2$	$(6,2)$	$\partial\lambda_2$
ν^2	$(6,2)$	$t^3\lambda_1'\lambda_2 \mod \partial\lambda_2$
νw	$(8,2)$	$t^2\lambda_1'\lambda_2$
$\nu\lambda_2$	$(10,2)$	$t\lambda_1'\lambda_2$
$\lambda_1'\lambda_2$	$(12,2)$	$\lambda_1'\lambda_2$
$\nu^2 w$	$(11,3)$	$\partial\lambda_1'\lambda_2$

TABLE 5.1. Bidegrees and detecting classes for $\mathbb{F}_2[v_2]$-module generators of $\overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko})$

6. Topological cyclic homology and algebraic K-theory

We now use the motivic spectral sequence

$$E_2 = \overline{A}(1)_* \operatorname{gr}^*_{\mathrm{mot}} \mathrm{TC}(\mathrm{ko}) \implies A(1)_* \mathrm{TC}(\mathrm{ko}) \tag{6.1}$$

to compute the $A(1)$-homotopy of the topological cyclic homology of ko. The E_2-term is concentrated in even total degrees and motivic filtrations $0 \le * \le 3$, so the only possibly nonzero differentials are

$$d_3(v_2^i) \in \mathbb{F}_2\{v_2^{i-2}\nu^2 w\}$$

for $i \ge 2$. We show that some, but not all, of these differentials are nonzero. This contrasts with the motivic spectral sequence converging to $V(1)_* \mathrm{TC}(\ell)$ at odd primes p, which was shown to collapse at the E_2-term by Hahn–Raksit–Wilson in [HRW22, Cor. 1.3.3]

Theorem 6.1. *In the motivic spectral sequence (6.1) there are nonzero differentials*

$$d_3(v_2^2) = \nu^2 w \qquad \textit{and} \qquad d_3(v_2^3) = v_2\nu^2 w\,,$$

while $d_3(1) = d_3(v_2) = d_3(v_2^4) = 0$. *Let*

$$S = \{i \ge 0 \mid d_3(v_2^i) = 0\} \qquad \textit{and} \qquad T = \{i \in S \mid i < 32\}\,.$$

Then $S = T + 32\,\mathbb{Z}_{\ge 0}$, $0, 1, 4 \in T$ *and* $2, 3 \notin T$. *Moreover*

$$\begin{aligned} A(1)_* \mathrm{TC}(\mathrm{ko}) \cong\ & \mathbb{F}_2\{v_2^i \mid i \in S\} \\ & \oplus \mathbb{F}_2[v_2]\{\partial, \varsigma, \nu, \lambda_1', w, \lambda_2\} \\ & \oplus \mathbb{F}_2[v_2]\{\nu\varsigma, \nu^2, \partial\lambda_2, \nu w, \nu\lambda_2, \lambda_1'\lambda_2\} \\ & \oplus \mathbb{F}_2\{v_2^j \nu^2 w \mid j \ge 0, j+2 \in S\} \end{aligned}$$

is a finitely generated free $\mathbb{F}_2[v_2^{32}]$*-module of rank* $384 + 2\,\mathrm{card}(T)$.

Proof. Bhattacharya–Egger–Mahowald [BEM17, Main Theorem] proved (for each version of $A(1)$) that there exists a v_2^{32}-self map $\Sigma^{192}A(1) \to A(1)$, which induces multiplication by v_2^{32} on (6.1) and its abutment. Hence $d_3(v_2^{32}v_2^i) = v_2^{32}d_3(v_2^i)$ for all $i \geq 0$, which shows that $S = T + 32\,\mathbb{Z}_{\geq 0}$.

The unit map $\mathbb{S} \to \mathrm{TC}(\mathrm{ko})$ induces a map from the Novikov spectral sequence for $A(1)$, as discussed in Lemma 3.2, to the motivic spectral sequence (6.1). By Lemma 5.9 this map of E_2-terms sends v_2^i to v_2^i and $v_2^i h_{11}^2 w$ to $v_2^i \partial\lambda_1'\lambda_2$ for each $i \geq 0$. Since $d_3(v_2^2) = h_{11}^2 w$, $d_3(v_2^3) = v_2 h_{11}^2 w$ and $d_3(1) = d_3(v_2) = d_3(v_2^4) = 0$ in the Novikov spectral sequence, we must have $d_3(v_2^2) = \partial\lambda_1'\lambda_2$, $d_3(v_2^3) = v_2\partial\lambda_1'\lambda_2$ and $d_3(1) = d_3(v_2) = d_3(v_2^4) = 0$ in the motivic spectral sequence, as asserted.

It follows that all classes in motivic filtrations 1 and 2 survive to E_∞. In filtrations 0 and 3, only the classes v_2^i with $i \in S$ and $v_2^{i-2}\nu^2 w$ with $i \geq 2$ and $i \in S$ survive. Setting $j = i - 2$ gives the asserted formulas. Since the E_∞-term is free over $\mathbb{F}_2[v_2^{32}]$, and is concentrated in finitely many filtrations, it follows that the abutment is also free over $\mathbb{F}_2[v_2^{32}]$, on the same number of generators. There are $\mathrm{card}(T)$, 192, 192 and $\mathrm{card}(T)$ of these in motivic filtrations 0, 1, 2 and 3, respectively, adding to $384 + 2\,\mathrm{card}(T)$. □

Remark 6.2. We expect that $d_3(v_2^i)$ in fact repeats 4-periodically, rather than with the period 32 guaranteed by [BEM17]. If this is correct, then $S = \{i \geq 0 \mid i \equiv 0, 1 \mod 4\}$ and $\mathrm{card}(T) = 16$, so that $A(1)_* \mathrm{TC}(\mathrm{ko})$ is a free $\mathbb{F}_2[v_2^{32}]$-module of rank 416 (or a free $\mathbb{F}_2[v_2^4]$-module of rank 52). This is the content of Conjecture B.

Theorem 6.1 allows us to determine the $A(1)$-homotopy of the algebraic K-theory spectra of ko and $\mathrm{ko}_2^\wedge$ with the same residual indeterminacy. We begin with the 2-complete case.

Theorem 6.3. *There is an exact sequence of $\mathbb{F}_2[v_2^{32}]$-modules*

$$0 \to \Sigma^1\mathbb{F}_2 \oplus \Sigma^3\mathbb{F}_2 \longrightarrow A(1)_* \mathrm{K}(\mathrm{ko}_2^\wedge) \xrightarrow{\mathrm{trc}} A(1)_* \mathrm{TC}(\mathrm{ko}) \longrightarrow \mathbb{F}_2\{\partial, \varsigma\} \to 0\,,$$

with $|\partial| = -1$ and $|\varsigma| = 1$.

Proof. By [HM97, Theorem D] and [Dun97, Main Theorem] (cf. [DGM13, Theorem 7.3.1.8]) applied to the 1-connected (structured) ring spectrum map $\mathrm{ko}_2^\wedge \to H\mathbb{Z}_2$ there is a homotopy cofiber sequence

$$\mathrm{K}(\mathrm{ko}_2^\wedge)_2^\wedge \xrightarrow{\mathrm{trc}} \mathrm{TC}(\mathrm{ko})_2^\wedge \xrightarrow{p} \Sigma^{-1}H\mathbb{Z}_2^\wedge\,.$$

The associated long exact sequence in $A(1)$-homotopy breaks up into four-term exact sequences, as above.

In more detail, the 3-connected map $A(1) \to H = H\mathbb{F}_2$ identifies $A(1)_*H\mathbb{Z}_2^\wedge$ with $\mathbb{F}_2\{1, \xi_1^2, \bar\xi_2, \xi_1^2\bar\xi_2\} \subset H_*H\mathbb{Z}_2^\wedge \subset \mathcal{A}^\vee$. By [BM94, Proposition 10.9], $\mathrm{K}(\mathrm{ko}_2^\wedge) \to \mathrm{K}(\mathbb{Z}_2)$ is 2-connected, where $\mathrm{K}_0(\mathbb{Z}_2^\wedge) = \mathbb{Z}$ and $\mathrm{K}_1(\mathbb{Z}_2^\wedge) = (\mathbb{Z}_2^\wedge)^\times$, so that $A(1)_0 \mathrm{K}(\mathrm{ko}_2^\wedge) \cong A(1)_0 \mathrm{K}(\mathbb{Z}_2^\wedge) = \mathbb{Z}/2$ and $A(1)_1 \mathrm{K}(\mathrm{ko}_2^\wedge) \cong A(1)_1 \mathrm{K}(\mathbb{Z}_2^\wedge) = (\mathbb{Z}_2^\wedge)^\times/(\pm((\mathbb{Z}_2^\wedge)^\times)^2) \cong \mathbb{Z}/2$, generated by any $u \in (\mathbb{Z}_2^\wedge)^\times$ congruent to 3 or 5 modulo 8. This uses that $\eta \in \pi_1(\mathbb{S})$ maps to $-1 \in (\mathbb{Z}_2^\wedge)^\times \cong \mathrm{K}_1(\mathbb{Z}_2^\wedge)$. By exactness, we know $p\colon \partial \mapsto \Sigma^{-1}1$ and $p\colon \varsigma \mapsto \Sigma^{-1}\xi_1^2$. Multiplication by ν acts trivially on $H\mathbb{Z}_2^\wedge$, so $p\colon \nu\varsigma \mapsto 0$ does not hit $\Sigma^{-1}\xi_1^2\bar\xi_2$. There is no class in degree 2 that p could map to $\Sigma^{-1}\bar\xi_2$. Hence these two classes instead appear as $\Sigma^{-2}\bar\xi_2$ and $\Sigma^{-2}\xi_1^2\bar\xi_2$ in $A(1)_* \mathrm{K}(\mathrm{ko}_2^\wedge)$, in degrees 1 and 3, respectively. □

The proof in the integral case relies on the proven Lichtenbaum–Quillen conjecture for $\mathbb{Z}[1/2]$, cf. [Voe03] and [RW00].

Theorem 6.4. *There is an exact sequence of $\mathbb{F}_2[v_2^{32}]$-modules*

$$0 \to \Sigma^3\mathbb{F}_2 \longrightarrow A(1)_* \mathrm{K}(\mathrm{ko}) \xrightarrow{\mathrm{trc}} A(1)_* \mathrm{TC}(\mathrm{ko}) \longrightarrow \mathbb{F}_2\{\partial, \varsigma\} \to 0\,,$$

with $|\partial| = -1$ *and* $|\varsigma| = 1$.

Proof. By [Rog02, Theorem 3.13] there are two homotopy cofiber sequences

$$\mathrm{K}(\mathrm{ko})_2^\wedge \xrightarrow{\mathrm{trc}} \mathrm{TC}(\mathrm{ko})_2^\wedge \xrightarrow{q} X$$
$$\Sigma^{-2}\,\mathrm{ku}_2^\wedge \xrightarrow{\delta} \Sigma^4\,\mathrm{ko}_2^\wedge \longrightarrow X$$

with equivalent third terms. Passing to $A(1)$-homotopy, the second cofiber sequence ensures that $A(1)_*X = \mathbb{F}_2\{x_{-1}, x_1, x_4\}$, where $|x_i| = i$. The long exact sequence associated to the first cofiber sequence then breaks up into four-term exact sequences, as shown.

This time, the details are as follows. The 3-connected $\mathbb{E}_\infty$ ring map $\mathbb{S} \to \mathrm{ko}$ induces a 4-connected map $\mathrm{K}(\mathbb{S}) \to \mathrm{K}(\mathrm{ko})$, where

$$\mathrm{K}(\mathbb{S}) \simeq \mathbb{S} \vee \mathrm{Wh}^{\mathrm{Diff}}(*)\,.$$

Here $\mathrm{Wh}^{\mathrm{Diff}}(*)$ is 2-connected with $\pi_3\,\mathrm{Wh}^{\mathrm{Diff}}(*) = \mathbb{Z}/2$, cf. [Rog02, Theorem 5.8]. Hence $A(1)_0\,\mathrm{K}(\mathrm{ko}) \cong A(1)_0\,\mathrm{K}(\mathbb{S}) = \mathbb{Z}/2\{1\}$, $A(1)_1\,\mathrm{K}(\mathrm{ko}) \cong A(1)_1\,\mathrm{K}(\mathbb{S}) = 0$, $A(1)_2\,\mathrm{K}(\mathrm{ko}) \cong A(1)_2\,\mathrm{K}(\mathbb{S}) = 0$ and $A(1)_3\,\mathrm{K}(\mathrm{ko}) \cong A(1)_3\,\mathrm{K}(\mathbb{S}) = \mathbb{Z}/2\{\nu\} \oplus \mathbb{Z}/2$. By exactness, we know $q\colon \partial \mapsto x_{-1}$ and $q\colon \varsigma \mapsto x_1$, while x_4 must contribute to $A(1)_3K(\mathrm{ko})$ and cannot be in the image of q. (It follows that $\nu x_1 = 0 \neq x_4$.) □

7. The Lichtenbaum–Quillen property and the telescope conjecture

Recall from Definition 1.1 that a spectrum X satisfies the height n telescope conjecture (at the prime 2) if the canonical map $L_n^f X \to L_n X$ is an equivalence.

Theorem 7.1. *The spectrum* $\mathrm{TC}(\mathrm{ko})_2^\wedge$ *satisfies the height* 2 *telescope conjecture.*

Proof. The argument in [HRW22, Theorem 6.6.4] applies directly since the $\mathbb{E}_\infty$ ring $C^q(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))$ is even for each $q \geq 0$ by Proposition 2.12, and the Bousfield–Kan/motivic spectral sequence associated to the cosimplicial spectrum $A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko})))$ has a horizontal vanishing line by Theorem 2.23 and Lemma 2.28.

We spell out the details for completeness. By the fiber sequence

$$\mathrm{TC}(\mathrm{ko})_2^\wedge \longrightarrow \mathrm{TC}^-(\mathrm{ko})_2^\wedge \xrightarrow{\mathrm{can} - \varphi_2^{h\mathbb{T}}} \mathrm{TP}(\mathrm{ko})_2^\wedge$$

from [NS18, Corollary 1.5] it suffices to prove that the telescope conjecture holds for the spectrum $\mathrm{TC}^-(\mathrm{ko})_2^\wedge$ and the spectrum $\mathrm{TP}(\mathrm{ko})_2^\wedge$. Since ko is an $\mathbb{E}_\infty$ ring, we know that $\mathrm{TP}(\mathrm{ko})_2^\wedge$ is a $\mathrm{TC}^-(\mathrm{ko})_2^\wedge$-module. Since both L_2^f and L_2 are smashing localizations, any module over a spectrum satisfying the height 2 telescope conjecture also satisfies the height 2 telescope conjecture. Therefore, it suffices to prove that $\mathrm{TC}^-(\mathrm{ko})_2^\wedge$ satisfies the height 2 telescope conjecture.

We know $L_1^f = L_1$ at $p=2$ by [Mah81] and [Mah82, Theorem 1.2]. By the map of chromatic fracture squares (cf. [ACB22]) from

$$\begin{tikzcd} L_2^f X \arrow[r] \arrow[d] & L_{T(2)} X \arrow[d] \\ L_1^f X \arrow[r] & L_1^f L_{T(2)} X \end{tikzcd}$$

to

$$\begin{tikzcd} L_2 X \arrow[r] \arrow[d] & L_{K(2)} X \arrow[d] \\ L_1 X \arrow[r] & L_1 L_{K(2)} X \end{tikzcd}$$

for $X = \mathrm{TC}^-(\mathrm{ko})_2^\wedge$, it suffices to prove that the map

$$L_{T(2)}\,\mathrm{TC}^-(\mathrm{ko})_2^\wedge \longrightarrow L_{K(2)}\,\mathrm{TC}^-(\mathrm{ko})_2^\wedge \tag{7.1}$$

is an equivalence. Since $K(2)$-local spectra are $T(2)$-local it suffices to prove that the map (7.1) is an equivalence after $T(2)$-localization. By the thick subcategory theorem, it therefore suffices to prove that the map (7.1) is an equivalence after applying the functor $v_2^{-1}A(1) \otimes -$.

We claim that there are equivalences

$$\begin{aligned} A(1) \otimes \mathrm{TC}^-(\mathrm{ko}) &\simeq A(1) \otimes \left(\mathrm{Tot}\left(C^\bullet\left(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko})\right)\right)\right)^{h\mathbb{T}} \\ &\simeq \left(\mathrm{Tot}\left(A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\right)\right)^{h\mathbb{T}}. \end{aligned}$$

The first equivalence holds because the map $\mathrm{THH}(\mathrm{ko}) \to \mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})$ is 1-connected and therefore the relevant Bousfield–Kan/descent sequence converges. The second equivalence holds because $A(1)$ is a finite spectrum.

By Proposition 5.2 the motivic spectral sequence computing $A(1)_*\,\mathrm{TC}^-(\mathrm{ko})$ is concentrated in motivic filtrations $0 \le s \le 2$ (note that t^k and v_2^j have motivic filtration 0 for all $k \in \mathbb{Z}$ and $j \ge 0$). This motivic spectral sequence agrees with the Bousfield–Kan spectral sequence associated to the cosimplicial spectrum

$$\left(A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))\right)^{h\mathbb{T}}$$

by Proposition 2.14. We know that whenever X is an even $\mathbb{E}_\infty$ algebra in $\mathrm{Sp}^{B\mathbb{T}}$, such as $X = C^q(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko}))$, then

$$v_2^{-1}A(1) \otimes X^{h\mathbb{T}} \simeq L_n^f A(1) \otimes X^{h\mathbb{T}} \simeq L_n A(1) \otimes X^{h\mathbb{T}} \tag{7.2}$$

because $X^{h\mathbb{T}}$ is an even $\mathbb{E}_\infty$ ring and therefore it is complex oriented it admits the structure of an MU-module by [CM15]. The horizontal vanishing line in the motivic/Bousfield–Kan spectral sequence implies that we can apply [CM21, Lemma 2.34] to determine that the composite map

$$\begin{aligned} & v_2^{-1}A(1) \otimes \mathrm{TC}^-(\mathrm{ko}) \\ &\quad \longrightarrow v_2^{-1}A(1) \otimes \mathrm{Tot}\left(C^\bullet(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko})\right) \\ &\quad \longrightarrow v_2^{-1}\,\mathrm{Tot}\left(A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko})\right) \\ &\qquad\qquad \longrightarrow \mathrm{Tot}\left(v_2^{-1}A(1) \otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko})\right) \end{aligned} \tag{7.3}$$

is an equivalence. The target of the composite map (7.3) is equivalent to

$$\mathrm{Tot}\,(L_2A(1)\otimes C^\bullet(\mathrm{THH}(\mathrm{ku}/\,\mathrm{MU})/\,\mathrm{THH}(\mathrm{ko})))$$

by (7.2). Since totalizations of L_2-local spectra are L_2-local, this implies that $v_2^{-1}A(1)\otimes\mathrm{TC}^-(\mathrm{ko})$ is L_2-local and, consequently, the canonical map

$$v_2^{-1}A(1)\otimes\mathrm{TC}^-(\mathrm{ko})\simeq L_2^fA(1)\otimes\mathrm{TC}^-(\mathrm{ko})\longrightarrow L_2A(1)\otimes\mathrm{TC}^-(\mathrm{ko})$$

is an equivalence. □

Theorem 7.2. *For $X\in\{\mathrm{K}(\mathrm{ko}),\mathrm{K}(\mathrm{ko}_2^\wedge),\mathrm{TC}(\mathrm{ko})\}$ and $Y\in\{X_{(2)},X_2^\wedge\}$, the canonical map $Y\to L_2^fY$ is an equivalence in all sufficiently large degrees.*

Proof. Let $X\in\{\mathrm{K}(\mathrm{ko}),\mathrm{K}(\mathrm{ko}_2^\wedge),\mathrm{TC}(\mathrm{ko})\}$. Then by Theorem 6.4, Theorem 6.3, and Theorem 6.1 respectively we know that $(A(1)/(v_2^{32}))_*X_2^\wedge$ is finite. This implies that $X_2^\wedge$ has fp-type 2 in the sense of [MR99, p. 5] by [MR99, Proposition 3.2]. By [MR99, Theorem 8.2], the spectrum $IC_2^fX_2^\wedge$ is bounded below and, consequently, $C_2^fX_2^\wedge$ is bounded above and the map $X_2^\wedge\to L_2^fX_2^\wedge$ is an equivalence in all sufficiently large degrees (cf. [HW22, Theorem 3.1.3]). By the pullback

$$\begin{tikzcd} X_{(2)} \arrow[r] \arrow[d] & X_2^\wedge \arrow[d] \\ X_{(2)}[1/2] \arrow[r] & X_2^\wedge[1/2]\,, \end{tikzcd}$$

and the fact that $X_{(2)}[1/2]$ and $X_2^\wedge[1/2]$ are L_2^f-local, we conclude that the canonical map $X_{(2)}\to L_2^fX_{(2)}$ is also an equivalence in all sufficiently large degrees. □

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