Summary

Wednesday, April 27, 2016 8:56 AM

For the proof of the Kervaire invariant theorem we need a spectrum Ω with 3 properties
1) Detection theorem
2) Periodicity theorem
3) Gap theorem

All 3 are proved by certain calculations which require certain constructions in Sp^G, the category of G-equivariant spectra.

Ten steps
1) We need to define such spectra precisely. This requires enriched category theory, Kan extensions and the Day convolution. The Mandell-May definition is

2002 $Sp^G = [\mathcal{J}_G, \mathcal{T}^G]$, the category of enriched functors $\mathcal{J}_G \to \mathcal{T}^G$, where
$\mathcal{T}^G$ = pointed G-spaces and equivariant maps (enriched over $\mathcal{T}$)
$\mathcal{T}_G$ = pointed G-spaces and continuous maps (enriched over $\mathcal{T}^G$)

$\mathcal{J}$ = category of pointed spaces

$\mathcal{J}_G$ = Mandell-May G = finite gp

objects are findim orth reps V of G

morphisms spaces are certain Thom spaces.

The categories $Sp^G = [\mathcal{J}_G, \mathcal{T}^G]$ and $Sp_G [\mathcal{J}_G, \mathcal{T}_G]$ are both closed symmetric monoidal under $\wedge$ with unit S^0, the sphere spectrum.

We could replace $\mathcal{J}_G$ by any full subcategory $\mathcal{J}'$ whose object set is closed under $\oplus$ and includes the trivial 1-dim representation. Then one can show $[\mathcal{J}', \mathcal{T}^G]$ is equivalent to $[\mathcal{J}_G, \mathcal{T}^G]$. Two interesting examples

1) $\mathcal{J}$ = category of reps with trivial G-action.

$[\mathcal{J}, Sp_G] = Sp_G^{naive}$

= category of ordinary with G-action

2) $\mathcal{J}^+$ = category of reps V with $V^G \neq 0$.

$[\mathcal{J}^+, Sp_G] =:$ category of positive G-spectra

② Define a model category structure on $\mathcal{S}p^G$ (we cannot do this on $\mathcal{S}p_G$ because the fiber / cofiber of a non-equiv map does not have a G-action).

On $\mathcal{T}^G$ one has a cofibrantly generated MC structure with generating sets

$$I = \{ G_+ \wedge_H (S_+^{n-1} \to D_+^n) : n \geq 0, \ H \subseteq G \}$$

$$J = \{ G_+ \wedge_H (I_+^{n-1} \to I_+^n) : \quad '' \quad \}$$

Hence we get a the <u>strict</u> positive complete MC structure on $[\mathcal{J}^+, \mathcal{T}^G]$ with

$$\mathcal{I} = \{ G_+ \wedge_H S^{-V} \wedge (S_+^{n-1} \to D_+^n) : V = \text{rep of } H \subset G \text{ with } V^H \neq 0, \ n \geq 0 \}$$

$$\mathcal{J} = \{ G_+ \wedge S^{-V} \wedge (I_+^{n-1} \to I_+^n) : \text{same} \}$$

A map $X \xrightarrow{f} Y$ of positive G-spectra is a weak equiv if $f_V : X_V \to Y_V$ is one for each V in $\mathcal{J}_G^+$. This is <u>not</u> the structure we want in stable homotopy theory.

We want to expand the class of weak equivalences to include the maps

$$\left\{ G_+ \underset{H}{\wedge} \left(S^{-V\oplus W} \wedge S^{W} \xrightarrow{e_{V,W}} S^{-V} \right) : \begin{array}{l} V \text{ is a rep of } G \\ \text{with } V^H \neq 0 \\ W \text{ any rep of } H \end{array} \right\}$$

This means we have to enlarge J, the generating set of trivial cofibrations by adjoining corner maps involving the factorization

$$S^{-V\oplus W} \xrightarrow{e_{V,W}} S^{-V}$$

$$\text{trivial cofibration} = \tilde{e}_{V,W} : S^{-V\oplus W} \to \tilde{S}^{V,W}, \qquad \hat{e}_{V,W} : \tilde{S}^{V,W} \to S^{-V} = \text{trivial fibration}$$

$\tilde{S}^{V,W}$ = mapping cylinder of $e_{V,W}$

Corner maps we want are

$$G_+ \underset{H}{\wedge} (S^{n-1} \to D^n) \,\square\, \hat{e}_{V,W}$$

This gives the positive complete MC structure on Sp^G.

(3) Show that the following constructions are homotopical on cofibrant objects — behave

a) indexed wedges

b) indexed smash products, e.g. norm

c) Symmetric powers
(this requires dim $V > 0$ above)

d) Geometric fixed points
(this requires dim $V^H > 0$ above)

PROOFS ARE TECHNICAL
MANY PAGES

(4) Define a MC structure on $Comm^G$, the category of commutative algebras in Sp^G, i.e. comm ring spectra

This requires Sym^n to preserve cofibrant objects. There is an adjunction for $H \leq G$

$$N_H^G : Comm^H \rightleftarrows Comm^G : i_H^G$$

norm = indexed smash product

restriction

There is a similar adjunction

$$G_+\wedge_H(-) : Sp^H \rightleftarrows Sp^G : i_H^G$$

← indexed wedge

(5) Construct $MU_{\mathbb{R}}$, the star of the show using an equivalence of categories

$$Sp_{\mathbb{R}} := [J_{\mathbb{R}}, T^{C_2}] \simeq [J_{C_2}, T^{C_2}] =: Sp^{C_2}$$

$$\text{Comm } Sp_{\mathbb{R}} \simeq \text{Comm}^{C_2}$$

show $\Phi^{C_2} MU_{\mathbb{R}} \simeq MO$

$MU_{\mathbb{R}}$ is the cofibrant (in Comm^{C_2}) of the comm ring built out of the Thom spaces $MU(n)$ equipped with C_2 action via complex conjugation

(6) Construct $\tilde{\Omega} = D^{-1} N_{C_2}^{C_8}(MU_{\mathbb{R}})$

where D is a certain element in $\pi_{19\rho_8}^{C_8} N_{C_2}^{C_8}(MU_{\mathbb{R}})$ ρ_8 = reg rep of C_8

IDNCT One needs to show that inverting D

IDNCT One needs to show that inverting D via a colimit preserves the comm ring structure

(7) Show $\tilde{\Omega}^{C_8} \simeq \tilde{\Omega}^{hC_8}$ IDNCT

(8) Define the slice filtration of Sp^G, i.e. define a collection of localizing categories $\tau_n^G \leq Sp^G$ ($\tau_n = (n-1)$-connected spectra)

Properties

i) $S^{\rho_G} \wedge \tau_n^G \cong \tau^G_{n+|G|}$

$\tau_n^G \cong S^{-\rho_G} \wedge \tau^G_{n+|G|}$

ii) $N_H^G \tau_n^H \hookrightarrow \tau^G_{n|G/H|}$

These mean we can describe the slice filtration on $\tilde{\Omega} = D^{-1} N_{C_2}^{C_8} MU_{\mathbb{R}}$ in terms of the one on $MU_{\mathbb{R}}$.

(9) Identify the slices of $MU_{\mathbb{R}}$ and its relatives, i.e. $N_{C_2}^{C_8} MU_{\mathbb{R}}$ and $\tilde{\Omega}$.

IDNCT.

Def Let X be a G spectrum with underlying $\pi_n X$ free abelian. Hence there is a map $\bigvee S^n \xrightarrow{f} X$ inducing an isomorphism on π_n^u.

A *refinement* of f is an equivariant map $W \longrightarrow X$ where W is a wedge of slice spheres of dim n as previously defined.

Recall $\hat{S}(m, K) = G_+ \wedge_K S^{m\rho_K}$ for $m \in \mathbb{Z}$ and $K \leq G$. This is a slice sphere of dim $m|K|$.

It turns out that the slices of $MU_{\mathbb{R}}$ + its relatives all have the form

$$P_n^n X \simeq \begin{cases} * & \text{for } n \text{ odd} \\ H\underline{\mathbb{Z}} \wedge W & \text{for } n \text{ even} \end{cases}$$

where W is as above.

The key step in identifying the slices is the Reduction Thm which says

$$P^0 N_{C_2}^{C_{2^j}} MU_{\mathbb{R}} = H\underline{\mathbb{Z}}.$$

Another way to say this:

Note $\pi_*^u N_2^{2^j} MU_{\mathbb{R}} = \mathbb{Z}[x_{n,i} : \begin{smallmatrix} n > 0 \\ 0 \le i < 2^{j-1} \end{smallmatrix}]$

$$x_{n,i} \in \pi_{2n}$$

For a generator γ of C_{2^j},

$$\gamma(x_{n,i}) = \begin{cases} x_{n,i+1} & \text{for } 0 \le i < 2^{j-1} - 1 \\ (-1)^n x_{n,0} & \text{for } i = 2^{j-1} - 1 \end{cases}$$

e.g. for $G = C_8$

$$x_{n,0} \to x_{n,1} \to x_{n,2} \to x_{n,3} \to (-1)^n x_{n,0}$$

For each generator we can form a twisted monoid ring

$$S^{-0}[x_{n,i}] = \bigvee_{k \ge 0} S^{2nk}$$

It is an associative algebra in Sp

We can make a G-spectrum out of $\bigwedge_{0\le i<2^{j-1}} S[x_{n,i}]$

It is not comm because e.g.

$$\begin{array}{ccc} S^2\wedge S^2 & \longrightarrow & S^4 \\ \text{swap}\downarrow & \nearrow & \\ S^2\wedge S^2 & & \end{array}$$

does <u>not</u> commute

Let $R = \bigwedge_{n>0} \bigwedge_{0\le i<2^{j-1}} S[x_{n,i}]$

Example $n=1$, $G = C_8$

$$\bigwedge_{0\le i\le 3} \bar{S}^0[x_{1,i}] = \bar{S}^0 \vee \hat{S}(1, C_2) \vee \cdots$$

$$\hat{S}(1, C_2) = C_{8+} \wedge_{C_2} S^{\rho C_2} \simeq \bigvee_4 S^2$$

In dim 4 we have a wedge of 10 copies of S^4

$\{x_{1,i}^2\}$ $\{x_{1,i}\,\gamma(x_{1,i})\}$ $\{x_{10}x_{12}, x_{11}x_{13}\}$

$$x_{10}x_{12} \longrightarrow x_{11}x_{13} \longrightarrow -x_{12}x_{10}$$

(10) Prove the 3 theorems.