Goal: Define a MC structure on

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$Comm^G$ = category of commutative algebras in Sp^G, i.e. G-equivariant comm ring spectra

Idea: apply the Kan transfer thm (3.9.11) to the adjunction

$$Sym: Sp^G \rightleftarrows Comm^G : U$$

where U is the forgetful functor and

$$Sym\, X = \coprod_{n \geq 0} Sym^n X \quad \text{and}$$

$$Sym^n X = X^{\wedge n} / \Sigma_n$$

Σ_n = symmetric gp on n letters

Difficulty: Sym^n is not homotopical in general. We need an MC structure where it is homotopical on cofibrant objects.

Technical generalization

Let T be a finite G-set

$B_T G$ = category with object set T and morphisms induced by action of G

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i.e. $\forall t \in T$ and $\gamma \in G$ we have a morphism $t \longrightarrow \gamma(t)$.

For $T = G/G$ we denote this by BG.

Consider the functor category $Sp^{B_T G}$ (i.e. T-shaped diagrams of spectra), where Sp denotes the category of orthogonal spectra. Then

① $Sp^{BG} = Sp^G_{naive} \simeq Sp^G$

Sp^G_{naive} = ordinary spectra with G-action; Sp^G = genuine G-spectra

② $Sp^{B_{G/H} G} \simeq Sp^{BH} = Sp^H_{naive} \simeq Sp^H$

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Any finite G-set has the form $\coprod G/G_t$, so

$$Sp^{\mathcal{B}_T G} \simeq \prod_t Sp^{\mathcal{B}_{G/G_t} G} \simeq \prod_t Sp^{G_t}$$

G_t = stabilizer gp for the tth orbit of T.

This product of categories has a positive complete MC structure induced from the ones on each factor.

Suppose we have a surjective map $T \longrightarrow T'$ of finite G-sets, e.g. $T' = G/G$.

This leads to functors

$$Sp^{\mathcal{B}_T G} \underset{p_*^{\wedge}}{\overset{p_*^{\vee}}{\rightrightarrows}} Sp^{\mathcal{B}_{T'} G},$$

indexed wedges and indexed smash products.

One can show that the indexed wedge is homotopical, and that the the indexed smash product is homotopical on cofibrant objects.

This means the functor (for $T \twoheadrightarrow T'$)

$$Sp^{B_T G} \xrightarrow{p_*^\wedge} Sp^{B_{T'} G} \longrightarrow ho\, Sp^{B_{T'} G}$$

has a left derived functor that is obtained by cofibrant replacement.

<u>Generalization of the n-fold symmetric power functor</u>:

Let T be a finite G-set

Σ_T = permutation gp of T (non-equiv)

G acts on Σ_T by conjugation.

Let $\Sigma \subset \Sigma_T$ that is G-invariant.

$$Sym_\Sigma^T X = X^{\wedge T}/\Sigma$$

Let $\tilde{G} = \Sigma \rtimes G$ = semi-direct product

$E_G \Sigma$ = a universal space $E\mathcal{F}$ for the family $\mathcal{F}$ of subgps of $\tilde{G}$ having trivial intersection with the normal subgp Σ.

For $G=e$, this is $E\Sigma$, a contractible free Σ-space.

Given a finite G-set T, we have and action of $\tilde{G} = \Sigma \rtimes G$ for a G-invariant subgp $\Sigma \subset \Sigma_T$. Let $X \longrightarrow Y$ be a map between cofibrant objects in $Sp^{B_T G}$. Then we get maps

$$Sym_\Sigma^T X \longrightarrow Sym_\Sigma^T Y$$

and

$$E_G\Sigma_+ \underset{\Sigma}{\wedge} X^{\wedge T} \longrightarrow E_G\Sigma_+ \underset{\Sigma}{\wedge} Y^T$$

General observation

Suppose Σ acts on a pointed space W. Then one has an orbit space W/Σ and $=: W_\Sigma$ a homotopy orbit space

$$E\Sigma_+ \underset{\Sigma}{\wedge} W =: W_{h\Sigma}$$

This is a pointed version of the Borel construction

$$W \longrightarrow E\Sigma \underset{\Sigma}{\times} W \longrightarrow B\Sigma$$

W = pointed Σ-space

There is a map
$$\begin{tikzcd} W \arrow[r] \arrow[d, equal, "h\Sigma"] & W \arrow[d, equal, "\Sigma"] \\ E\Sigma_+ \wedge_{\Sigma} W & W/\Sigma \end{tikzcd}$$

It is nice when it is a homotopy equivalence. This happens when the action of Σ is free away from the basepoint. This is good.

Suppose K is a pointed CW-cx, $\Sigma = \Sigma_n$ acting on $W = K^{\wedge n}$.

The action is never free for $n > 1$ and $K \neq *$. This is bad.

How to fix this in the world of spectra.

Consider $S^{-1} \wedge K$, so $(S^{-1} \wedge K)^{\wedge n} = S^{-n} \wedge K^{\wedge n}$ has a Σ_n-action

$$(S^{-n} \wedge K^{\wedge n})_V = J(n, V) \wedge K^{\wedge n}$$

$J(n, V)$ has a free action of $O(n)$ for large enough V, and hence a free action of its subgp Σ_n

Hence $S^{-n} \wedge K^{\wedge n}$ has a free Σ_n-action

Recall the positive (complete) model structure on Sp has generating cofibrations

$$I = \{ S^{-i} \wedge (S^{n-1}_+ \to D^n_+) : i > 0,\ n \geq 0 \}$$

so $S^{-1} \wedge K$ is cofibrant for any pointed CW-complex K, but $S^{-0} \wedge K = \Sigma^\infty K$ is not cofibrant.

$\rightsquigarrow$ If X is a cofibrant spectrum then $X^{\wedge n}$ has a free Σ_n-action.

This is the reason for Jeff Smith's positivity condition.

By 8.5.17, cofibrant spectra X have nice cofibrant approximations to their symmetric powers. This means that we can use the Kan transfer theorem to get a MC structure on $Comm^G$, the category of G-equiv commutative algebras, e.g. $MU_{\mathbb{R}}$

Why do we need $V^H \neq 0$

in $J = \{ G_+ \wedge_H S^{-V} \wedge (S^{n-1}_+ \to D^n_+) \}$?

H = any subgp

V = any rep of H with $V^H \neq 0$

We want Φ^H (cofibrant object) to be cofibrant.

$$\Phi^H (G_+ \wedge_H S^{-V}) = S^{-V^H}$$