More about commutative ring spectra

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Recall lax monoidal functors of 2.6.8

<u>Prop 2.11.9</u> Let $(\mathcal{C}, \oplus, 0)$ and $(\mathcal{D}, \otimes, 1)$ be SMC's and let $[\mathcal{C}, \mathcal{D}]$ be the category of enriched functors $\mathcal{C} \to \mathcal{D}$. Assume both $\mathcal{C}$ and $\mathcal{D}$ are enriched over $\mathcal{D}$. Then the category of commutative monoids (e.g. comm. ring spectra) iso to the category of lax symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$.

The spectrum MO

Given an orth rep V we have spaces

$O(V)$ = its orthogonal gp

$EO(V)$ = contractible free $O(V)$-space

$BO(V) = EO(V)_+ \wedge_{O(V)} S^0$ = orbit space of $EO(V)$

$\quad$ = classifying space of $O(V)$

$MO(V) = EO(V)_+ \wedge_{O(V)} S^V$

$O(V,W)$ = space of orthogonal embeddings $V \to W$ = Stiefel manifold

$O(V,W) \times BO(V) \longrightarrow BO(W)$ structure map.

The spectrum MO is the functor $V \longmapsto MO(V)$

It is lax symm monoidal

The spectrum MO is the functor $V \longmapsto MO(V)$
It is lax symm monoidal, i.e.
there is a natural map
$$MO(V) \wedge MO(W) \longrightarrow MO(V \oplus W)$$
with suitable properties.
This means MO is a commutative ring spectrum.

How to construct other Thom spectra

Suppose we have a collection of maps
$$X_V \xrightarrow{f_V} BO(V) \quad \text{with}$$

(6.6.4)
$$\begin{tikzcd} O(V,W) \times BO(V) \arrow[r] & BO(W) \\ O(V,W) \times X_V \arrow[r] \arrow[u, "O(V,W) \times f_V"] & X_W \arrow[u, "f_W"'] \end{tikzcd}$$

commuting for all V, W. Then we get a spectrum T where T_V is the Thom space associated with f_V.

A fancier formulation:

\mathscr{I}_G

Def The Striefel category $\mathscr{I}_G$ has reps V as objects with morphism spaces
$$\mathscr{I}_G(V,W) = O(V,W)$$
$$(\mathscr{J}_G(V,W) = \text{Thom}(O(V,W), W-V))$$
A Striefel space is a functor

A Stiefel space is a function

$$\mathcal{J}_G \longrightarrow \mathcal{T}_G$$

Example BO is the Striefel space defined by $BO_V = BO(V)$ as defined above.

In (6.6.4), X is a Striefel space over BO, i.e. X is equipped with a map $X \xrightarrow{b} BO$ of Striefel spaces.

A Striefel space X over BO leads to a Thom spectrum T as described above, e.g. $BO \rightsquigarrow MO$.

How to construct $MU_{\mathbb{R}}$.

Def The complex Mandell-May category

$\mathcal{J}_{\mathbb{C}}$ has fin dim Hermitian vector spaces as objects with

$$\mathcal{J}_{\mathbb{C}}(A, B) = \mathrm{Thom}(U(A,B); B-A)$$

where $U(A,B)$ is the space of unitary embeddings $A \to B$, i.e. a complex Striefel manifold. $\mathcal{J}_{\mathbb{C}}$ is enriched over $\mathcal{T}$, i.e. pointed spaces.

The real Mandell-May category $\mathcal{J}_{\mathbb{R}}$ has orth vector spaces (fin dim) as

The <u>real Mandell-May category</u> $\mathcal{J}_{\mathbb{R}}$ has fin dim orthogonal vector spaces as objects with $\mathcal{J}_{\mathbb{R}}(V, W) := \mathcal{J}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}}) \circlearrowleft C_2$

where $V_{\mathbb{C}}$ denotes $V \otimes_{\mathbb{R}} \mathbb{C} = V \otimes \rho_{C_2}$

<u>Complex</u> and <u>real spectra</u> are functors $\mathcal{J}_{\mathbb{C}} \longrightarrow \mathcal{T}$ and $\mathcal{J}_{\mathbb{R}} \longrightarrow \mathcal{T}_{C_2}$

Example $MU_{\mathbb{R}}$ is the real spectrum defined by $\mathbb{R}^n \longmapsto MU(n) \supset MO(n)$.

We want a genuine C_2-spectrum $MU_{\mathbb{R}}$, i.e. functor

$$\begin{tikzcd} \mathcal{J}_{\mathbb{R}} \arrow[rr, "MU"] \arrow[dr, "k"'] & & \mathcal{T}_{C_2} \\ & \mathcal{J}_{C_2} \arrow[ur, dashed, "\text{left Kan extension}"'] & \end{tikzcd}$$

where k is given by $\mathbb{R}^n \longmapsto \mathbb{R}^n \otimes \rho_{C_2}$

$MU_{\mathbb{R}}$ is a commutative C_2-ring spectrum

This is due to Schwede.

More about commutative ring spectra

Remark about model category structures on Sp^G:

1) The positive complete model structure has as its set of generating cofibrations

$\{G_+ \wedge_H S^{-V} \wedge (S^{n-1}_+ \longrightarrow D^n_+) : n \geq 0, \; H \subseteq G,$ $V = \ldots$ of H with $\ldots$

$\dim V^H > 0$

2) The stable complete model structure has as its set of generating cofibrations

$$\left\{ G_+ \wedge_H S^{-V} \wedge (S^{n-1}_+ \to D^n_+) : \begin{array}{l} n \geq 0,\ H \subseteq G \\ V = \text{rep of } H \text{ with} \end{array} \right\}$$

3) The semi positive complete model structure

$$\left\{ \text{as above with } \dim V > 0 \right\}$$

The condition $\dim V > 0$ is needed to get a MC structure on the category of comm ring spectra, for reasons to be explained later.

The condition $\dim V^H > 0$ is needed to get good behavior of geometric fixed points.

<u>Back to comm ring spectra</u>

Let $Comm^G = Comm\ Sp^G$ denote the category of comm G-ring spectra. Want to use the Kan transfer theorem to make it a CGMC.

$$Sym : Sp^G \rightleftarrows Comm^G : U = \text{forgetful functor}$$

where $Sym^n X = X^{\wedge n}/\Sigma_n$ = nth symmetric smash power of X and

$$Sym\, X := \bigvee_{n\geq 0} Sym^n X$$

where $Sym^0 X = S^{-0}$ and $Sym^1 X = X$

$Sym\, X$ is the free comm ring spectrum generated by X.

Disturbing example. Consider $S^{-1}\wedge S^1 \xrightarrow{e_1} S^{-0}$

It is a stable equivalence, but for $n>1$

$$Sym^n(S^{-1}\wedge S^1) \longrightarrow Sym^n(S^{-0}) = S^{-0}$$

is not an equivalence because

$$Sym^n(S^{-1}\wedge S^1) \simeq \Sigma^\infty B\Sigma_n$$

This is the reason we cannot let S^{-0} be cofibrant.