Yan on the detection thm (continued)

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$\mathrm{Ext}(-)$ is short for $\mathrm{Ext}_{MU_*MU}(MU_* -)$

and Ext_a " $\mathrm{Ext}_a(\mathbb{Z}/2, \mathbb{Z}/2)$

We have a diagram

$$\begin{tikzcd}
\mathrm{Ext}^{2,2^{i+1}}(MU_*) \arrow[rrr] \arrow[ddd] & & & H^2(C_8; \pi_{2^{i+1}} \Omega_{\mathbb{O}}) \arrow[ddd] \\
& & & \\
& \mathrm{Ext}^{1,2^{i+1}}(MU_*/2) \arrow[r] \arrow[uul] \arrow[dl] & H^1(C_8; \pi_{2^{i+1}} \Omega_{\mathbb{O}}/2) \arrow[uur] \arrow[d] & \\
\mathrm{Ext}_a^{2,2^{i+1}} \arrow[rr, "\cong"] & & H^2(C_8; A_{2^{i+1}}/(\pi)) \arrow[r] \arrow[d, phantom, "\cong"] & H^2(C_8; A_{2^{i+1}}) \arrow[d, phantom, "\cong"] \\
& & \mathbb{Z}/2 & A/(8)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "\pi"] & A_* \arrow[r] & A_*/(\pi) \arrow[r] & 0 \\
0 \arrow[r] & \pi_* \Omega_0 \arrow[r, "2"] \arrow[u, "2/\pi"] & \pi_* \Omega_0 \arrow[r] \arrow[u] & \pi_* \Omega_0/2 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

<u>Prop</u> The maps $\mathrm{Ext}^{1,2^{i+1}}(MU_*/2) \longrightarrow H^1(C_8; A_{2^{i+1}}/(\pi)) = \mathbb{Z}/2$

and " $\longrightarrow \mathrm{Ext}_a^{2,2^{i+1}} = \mathbb{Z}/2$

are surjective and have the same kernel.

The second statement was proved in MRW. The first one was new in HHR.

Commutative ring spectra

Prior to EKMM, a "commutative" ring spectrum R had structure maps $S^0 \xrightarrow{i} R$ (unit) $R \wedge R \xrightarrow{m} R$ (multiplication) such that

(1)
$$\begin{tikzcd}
S^0 \wedge R \arrow[d, "\eta_L"'] \arrow[dr, "1"] & \\
R \wedge R \arrow[r, "m"] & R \\
R \wedge S^0 \arrow[u, "\eta_R"] \arrow[ur, "1"'] &
\end{tikzcd}$$
unit

(2)
$$\begin{tikzcd}
R \wedge R \wedge R \arrow[r, "R \wedge m"] \arrow[d, "m \wedge R"'] & R \wedge R \arrow[d, "m"] \\
R \wedge R \arrow[r, "m"] & R
\end{tikzcd}$$
associativity

(3)
$$\begin{tikzcd}
R \wedge R \arrow[d, "\tau"'] \arrow[dr, "m"] & \\
R \wedge R \arrow[r, "m"'] & R
\end{tikzcd}$$
commutativity

These need to commute up to homotopy or up to higher homotopy described in terms of operads.

Today we can require these diagrams to commute pointwise. Interesting examples include MO and MU.

(Symmetric) monoidal categories are defined in 2.6.1. One example is $(\mathcal{Sp}^G, \wedge, S^0)$

Recall that $\mathcal{Sp}^G$ is the category of enriched functors $\mathcal{J}_G \longrightarrow \mathcal{T}_G$ where

$\mathcal{T}_G$ = category of pointed G-spaces

$\mathcal{J}_G$ = Mandell-May category whose objects are reps V with $\mathcal{J}_G(V, W)$ is a certain Thom space and an object in $\mathcal{T}_G$

$\mathcal{J}_G$ and $\mathcal{T}_G$ are both enriched / $\mathcal{T}_G$

$(\mathcal{J}_G, \oplus, 0)$ is a SMC and $(\mathcal{T}_G, \wedge, S^0)$ is a closed SMC

Lax (symmetric) monoidal functors are defined in 2.6.8.

<u>Def</u> A <u>monoid object</u> R in a SMC $(\mathcal{Sp}, \wedge, S^0)$ is one equipped with morphisms $S^0 \xrightarrow{i} R$ and $R \wedge R \xrightarrow{m} R$ with the expected properties. It is <u>commutative</u> if

$$\begin{tikzcd} R \wedge R \arrow[rr, "\tau_{R,R}"] \arrow[dr, "m"'] & & R \wedge R \arrow[dl, "m"] \\ & R & \end{tikzcd}$$

commutes.

<u>Example</u> A (comm) monoid in $(\mathcal{Ab}, \otimes, \mathbb{Z})$ is a (commutative) ring.

This leads to categories

2.6.24 $\mathcal{A}ssoc\, \mathcal{S}p$ = category of monoids in $\mathcal{S}p$

and $\mathcal{C}omm\, \mathcal{S}p$ = category of comm. monoids in $\mathcal{S}p$

2.6.25 The forgetful functors

$\mathcal{A}ssoc\, \mathcal{S}p \longrightarrow \mathcal{S}p$ and

$\mathcal{C}omm\, \mathcal{S}p \longrightarrow \mathcal{S}p$

have left adjoints

$$X \longmapsto T(X) := \bigvee_{n \geq 0} X^{\wedge n}$$

and $$X \longmapsto Sym(X) := \bigvee_{n \geq 0} (X^{\wedge n})_{\Sigma_n}$$

where the Σ_n action on $X^{\wedge n}$ is derived from the symmetric monoidal structure on $\mathcal{S}p$ and $(X^{\wedge n})_{\Sigma_n}$ is the colimit of the corresponding diagram, the orbit space.

Assume the $\mathcal{S}p$ is cocomplete.