CM, UY, SD, QZ

Yan on Detection Theorem

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Everything is 2-local

Thm 1 (Topological Detection thm) If $\theta_j \in \pi_{2^{j+1}-2} S^0$ exists for $j > 2$, then its Hurewicz image in $\pi_* \Omega$ is $\neq 0$.

Notation: $MU^{((C_8))} = \bigwedge_{C_8/C_2} MU_{\mathbb{R}}$, $\Omega_{\mathbb{O}} = D^{-1} MU^{((C_8))}$

$D \in \pi_{19\rho_8} MU^{((C_8))}$, $\Omega = \Omega_{\mathbb{O}}^{C_8} \simeq \Omega_{\mathbb{O}}^{hC_8}$.

There is a HFPSS converging to $\pi_* \Omega^{hC_8}$

and the ANSS " $\pi_* S^0$

Thm 2 (Algebraic Detection Thm). If $x \in \mathrm{Ext}^{2, 2^{j+1}}_{MU_* MU}(MU_*, MU_*)$ maps $h_j^2 \in \mathrm{Ext}_{\mathcal{A}}^{2, 2^{j+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$ then its image in $H^2(C_8; \pi_{2^{j+1}} \Omega_{\mathbb{O}})$ is nontrivial.

The map will be explained later and will show ADT $\Longrightarrow$ DT.

$$x \in \mathrm{Ext}^{2, 2^{j+1}}_{MU_* MU}(MU_*, MU_*) \xrightarrow{f} H^2(C_8; \pi_{2^{j+1}} \Omega_{\mathbb{O}})$$

If x is a perm cycle in the ANSS, so is $f(x)$ in the HFPSS.

We know x is not a boundary because there are no elts in Ext^1 or Ext^0 that could kill it.

We know $\mathrm{Ext}^{0, 2^{j+1}-1}(\mathbb{Z}/2, \mathbb{Z}/2) = 0$

$$\mathrm{Ext}^{0,2i+1}_{MU_* MU_*}(MU_*, MU_*) = 0$$

For $f(x)$, $d_2 : H^0(C_8; \pi_{2^{j+1}-1} \Omega_{\mathbb{O}}) \longrightarrow H^2(C_8; \pi_{2^{j+1}} \Omega_{\mathbb{O}})$ is trivial because its source is $\cdot$ $\pi_* \Omega_{\mathbb{O}}$ is even dimensional.

<u>What is f?</u>

$$\begin{array}{ccc} \mathrm{Ext}^{*,*}_{MU_*MU_*}(MU_*, MU_*) & \longrightarrow & H^*(C_8; \pi_* \Omega_{\mathbb{O}}) \\ \downarrow & & \\ \mathrm{Ext}^{*,*}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2) & & \end{array}$$

We define a ring $A = \mathbb{Z}_2[\zeta]/(\zeta^4+1)$, $\zeta = 8$th root of unity.

$A_* = A[u]$ with $|u| = 2$

A_* admits a C_8-action, where a generator $\gamma \in C_8$ acts trivially on $a \in A$ and $\gamma(u) = \zeta u$.

<u>Prop 3</u> A is a discrete valuation ring with maximal ideal (π) with $\pi = 1 - \zeta$.

<u>Prop 4</u> The image of $D \in \pi_{19\rho_8} MU^{((C_8))}$ under the composition

$$\pi_{19\rho_8} MU^{((C_8))} \longrightarrow \pi_{152} N_2^8 MU_{\mathbb{R}} \longrightarrow A_{152}$$

forgetful functor

is a unit.

We have a ring maps

$$\begin{array}{ccc} \pi_* \Omega_{\mathbb{O}} & \longrightarrow & A_* \\ \uparrow & \nearrow & \\ \pi_* MU^{((C_8))} & & \end{array}$$

Prop 5 $MU_{\mathbb{R}}$ admits the universal C_2-eqv formal group law, i.e. there is a FGL / $\pi_* MU_{\mathbb{R}}$ where conjugation action is related to the formal inverse.

Prop 6 $\pi_* MU^{((C_8))}$ admits the universal C_8-eqv FGL that extends the C_2-action. It has an automorphism whose fourth power is the formal inverse.

Thm (Lubin-Tate) If R is a (complete local) DVR with maximal ideal (π) with $R/(\pi) = \mathbb{F}_2$ and $\exists\, f(x) \in R[[x]]$ with

(i) $f(x) \equiv \pi x \mod (x^2)$

(ii) $f(x) \equiv x^2 \mod (\pi)$

Then $\exists$ FGL F_f and a ring hom $R \to \mathrm{End}(F_f)$ with

$$\pi \longmapsto [\pi](x) := f(x).$$

In our case $R = A = \mathbb{Z}_2[\zeta]/(\zeta^4+1)$ and $f(x) = \pi x + x^2$

Lubin-Tate gives us a hom

$$\begin{aligned} A &\longrightarrow \mathrm{End}(F_f) \\ 2 &\longmapsto [2](x) \\ \pi &\longmapsto f(x) \\ \zeta^4 &\longmapsto [-1]\, x \end{aligned}$$

Thm 7 There is a bijection between

$$\{\text{homogeneous FGLs over } A_*\} \longleftrightarrow \{\text{FGLs } / A\}$$

Def Let $\mathcal{M}_{FG}$ be the category whose objects are pairs (R, F) where F is a FGL $/R$ and a morphism $(R_1, F_1) \to (R_2, F_2)$ is a pair (f, ψ) with $f: R_1 \to R_2$ and an iso $\psi: F_2 \to f_* F_1$ of FGLs over R_2. (It is a stack.) Let $\mathcal{M}^h_{FG}$ be a homogeneous analog for $R_1[u]$ and $R_2[u]$ with $|u| = 2$

The map ANSS $\to$ HFPSS is given by the following

If $G = C_8$ acts on an FGL fixing R, then

$$\begin{tikzcd}
(A_*, F_f^u) \arrow[r] \arrow[d, squiggly] & (\mathcal{M}^u_{FG}) \arrow[d, squiggly] \\
(R_*, C(C_8, R_*)) & (MU_*, MU_*MU_*) \arrow[l]
\end{tikzcd}$$

where $A_* = A[u]$.

ring of R_*-valued functions on C_8

map of Hopf algebroids

The map above induces f : ANSS $\to$ HFPSS.

Big diagram

$$\begin{tikzcd}
\text{ANSS} & x \in \mathrm{Ext}^{2,2^{j+1}}_{MU_*MU_*}(MU_*, MU_*) \arrow[rrr] \arrow[dd, "\text{Thom reduction}"] & & & H^2(C_8; \pi_{2^{j+1}}\Omega_0) \arrow[dd] \\
& & \mathrm{Ext}^{1,2^{j+1}}_{MU_*MU}(MU_*, MU_*/2) \arrow[ul, "\delta"'] \arrow[dl] \arrow[r] & H^1(C_8; \pi_{2^{j+1}}\Omega_0/2) \arrow[ur] \arrow[d] & \\
\text{ASS} & h_1^2 \in \mathrm{Ext}^{2,2^{j+1}}_{\mathcal{A}}(\mathbb{Z}/2, \mathbb{Z}/2) \arrow[rr, equal] & & H^1(C_8; A_{2^{j+1}}/(\pi)) \arrow[r] & H^2(C_8, A_{2^{j+1}})
\end{tikzcd}$$

$\mathbb{Z}/2, \mathbb{Z}/2$ $\overset{\shortparallel}{\mathbb{Z}/2}$

δ is conn hom for $0 \to MU_* \xrightarrow{2} MU_* \to MU_*/2 \to 0$

To be continued.