More about the slice filtration

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Recall the isotropy separation sequence

$$E\mathcal{P}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{P}$$ cofiber sequence of G-spaces

$$E\mathcal{P}_+ \wedge X \longrightarrow X \longrightarrow \tilde{E}\mathcal{P} \wedge X$$ for a G-spectrum X

Lemma 7.3.16 For a map $X \xrightarrow{f} Y$ of G-spectra

i) $\Phi^H X \simeq *$ for all $H \subseteq G \iff X \simeq *$

ii) $\Phi^H(f)$ is an equiv for all $H \iff f$ is an equiv

iii) $E\mathcal{P}_+ \wedge X \simeq * \iff \Phi^H X \simeq *$ for all $H \subset G$.

iv) $\tilde{E}\mathcal{P} \wedge X \simeq * \iff \Phi^G X \simeq *$

Lemma 7.3.17 If a pointed G-space T is obtained from T_0 by attaching induced cells (i.e. $G_+ \underset{H}{\wedge} D^m_+$ for $H \subset G$), then the map $T_0 \longrightarrow T$ induces an iso

$$[T, \tilde{E}\mathcal{P} \wedge X]^G_* \xrightarrow{\cong} [T_0, \tilde{E}\mathcal{P} \wedge X]^G_*$$ for any X.

e.g. $T_0 = T^G$

The slice filtration

Recall $\hat{S}(m,K) = G_+ \wedge_K \begin{cases} \Sigma^\infty S^{m\rho_G} & \text{for } m \geq 0 \\ S^{-m\rho_G} & \text{for } m \leq 0 \end{cases}$

$m \in \mathbb{Z}$
$K \subset G$

= slice sphere

Let $Sp^G_{\geq n}$ be the subcategory of Sp^G generated by $\{\hat{S}(m,K) : m|K| \geq n\}$

$X \in Sp^G_{\geq n} \Leftrightarrow S^{\rho_G} \wedge X \in Sp^G_{\geq n+|G|}$

In the HHR paper this was defined to be the category generated by

$$\{\Sigma^{-\varepsilon} \hat{S}(m,K) : m|K| - \varepsilon \geq n,\ \varepsilon = 0,1\}$$

<u>Def</u> Let $\mathcal{T}^G_n \subset Sp^G$ be the subcategory generated by all G-spectra X with

$$\pi_k(\Phi^H X) = 0 \quad \text{for } k < n/|H| \text{ for all } H \subseteq G.$$

GEOMETRIC CONNECTIVITY

Will show $\mathcal{T}^G_n = Sp^G_{\geq n}$

<u>Proof</u> i) $\mathcal{T}^G_n$ is a localizing subcat.

ii) $\Sigma^\infty S^{\rho_G}$ is in $\mathcal{T}^G_{|G|}$

iii) For $X \in \mathcal{T}^G_m$ and $Y \in \mathcal{T}^G_n$, then

$X \wedge Y \in \mathcal{T}^G_{m+n}$

iv) Smashing with S^{ρ_G} induces an equiv between τ_n^G and $\tau_{n+|G|}^G$.

Prop $\hat{S}(m,K) \in \mathcal{T}^G_{m|K|}$

Pf Suffices to prove this for $m \geq 0$

$$\Phi^H \hat{S}(m,K) = \Phi^H\left(\Sigma^\infty G_+ \wedge_K S^{m\rho_K}\right)$$

$$= \Sigma^\infty \left(G_+ \wedge_K S^{m\rho_K}\right)^H$$

$$= \begin{cases} * & \text{for } H \not\leq K \\ \Sigma^\infty G_+ \wedge_K S^{m|K/H|} & \text{for } H \leq K \end{cases}$$

so $\hat{S}(m,K) \in \mathcal{T}^G_{m|K|}$. QED

This implies $Sp^G_{\geq n} \subset \mathcal{T}^G_n$.

Will prove the converse by induction on $|G|$. For $G = e$, both categories are those of $(n-1)$-connected ~~spectra~~.

Lemma 9.3.5 (Using the inductive hyp)

If $\mathcal{T}^H_n = Sp^H_{\geq n}$ for all $H < G$, $Y \in \mathcal{T}^G_n$, $\Phi^G Y \simeq *$, then $Y \in Sp^G_{\geq n}$

Lemma 9.3.6 (The inductive step)

If $Y \in \mathcal{T}^G_n$ and $\Phi^H Y \simeq *$ for all $H < G$ then $Y \in Sp^G_{\geq n}$.

Thus if X is in $\mathcal{T}_n^G$, it is in $Sp^G_{\geq n}$.

Proof assuming the 2 lemmas:

Consider the cofiber sequence

$$EP_+ \wedge X \longrightarrow X \longrightarrow \widetilde{EP} \wedge X$$

If $X \in \mathcal{T}_n^G$, so are $EP_+ \wedge X$ and $\widetilde{EP} \wedge X$

Then $EP_+ \wedge X \in Sp^G_{\geq n}$ by first lemma

$\widetilde{EP} \wedge X$ " second "

so $X \in Sp^G_{\geq n}$ QED

Proof of lemma 9.3.5 If $\Phi^G Y \simeq *$, so

$\widetilde{EP} \wedge Y \simeq *$ $EP_+ \wedge Y \simeq Y$, so Y is built entirely of induced G-cells, i.e. cells induced up from $\mathcal{T}_n^H = Sp^H_{\geq n}$ for $H \subset G$, so they are all in $Sp^G_{\geq n}$. QED

Proof of lemma 9.3.5. We have $\Phi^H Y \simeq *$ for all $H \subset G$, so $Y \simeq \widetilde{EP} \wedge Y$. Since $Y \in \mathcal{T}_n^G$, $\pi_k \Phi^G Y = 0$ for $k < n/|G|$ and hence for $k < \lceil n/|G| \rceil :=$ smallest integer $\geq n/|G|$.

$\lceil n/|G| \rceil = m$

Hence $Y \in \mathcal{T}^G_{m|G|}$. We also know

$$[G_+ \wedge_H S^\ell, \tilde{E}\mathcal{P} \wedge Y]^G = 0$$ for any $H \subset G$ and any $\ell \in \mathbb{Z}$

It follows that the Postnikov filtration of Y (after rescaling by |G|) coincides with both the slice filtration and the geometric connectivity filtration.

$Y \in Sp^G_{\geq m|G|} \subset Sp^G_{\geq n}$

since $m|G| \geq n$. QED

Prop 9.3.8 Let V be a rep of G of degree d

i) $\Sigma^\infty S^V \in \mathcal{T}^G_d \iff \dim V^H \geq \lfloor d/|H| \rfloor$ for all $H \leq G$.

ii) $S^{-V} \in \mathcal{T}^G_{-d} \iff \dim V^H \leq \lfloor d/|H| \rfloor$ for all $H \leq G$.

<u>Cor 9.3.11</u> Let V be as above and $n \in \mathbb{Z}$ with $\lfloor \frac{n}{|H|} \rfloor + \dim V^H = \lceil \frac{n+d}{|H|} \rceil$ for all $H \leq G$

Then $S^V \wedge (-) : \mathcal{T}^G_n \longrightarrow \mathcal{T}^G_{n+d}$ is an equivalence of categories.

<u>Exercise</u> Let $V = \rho_G$, then the hypotheses of 9.3.11 hold for all n.