Zhu on homotopy limits + colimits

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Example In Top_* consider the diagrams

① $A \xleftarrow{f} C \xrightarrow{g} B$. The colimit is the pushout $A \vee B / f(c) \sim g(c)$. e.g. $* \leftarrow S^2 \rightarrow *$, colim $= *$.

② For $X \xrightarrow{f} Z \xleftarrow{g} Y$, the limit is the pullback $\{(x,y) \in X \times Y : f(x) = g(y)\}$

For $* \xrightarrow{0} [0,1] \xleftarrow{0} *$, lim $= *$

$* \xrightarrow{0} [0,1] \xleftarrow{1} *$, lim $= \emptyset$ } in Top

These show lim and colim are not homotopical.

For ① we define hocolim to be the colim of

$A \xleftarrow{f} C \xrightarrow{0} C \times [0,1] \xleftarrow{1} C \xrightarrow{g} B$, i.e.

$A \vee C \times I \vee B / \{(c,0) \sim f(c), (c,1) \sim g(c)\}$

For $A = B = *$, hocolim $= \Sigma C$.

For ② we define holim to be lim of

$Z \longrightarrow Z \longleftarrow Z^{[0,1]} \longrightarrow Z \longleftarrow Y$, namely

$\{(x, m: [0,1] \to Z, y) \mid f(x) = m(0), g(y) = m(1)\}$

For $X = Y = *$, $f = g$, we have holim $= \Omega Y$.

To do this for a general diagram we need simplicial sets.

Let $\mathcal{C}$ be a simplicial model category, i.e. one enriched / sSet. For any object X in $\mathcal{C}$ and simp set A, $X \otimes A$ is an object in $\mathcal{C}$ such that

TENSOR

$\mathcal{C}(X \otimes A, Y) \simeq sSet(A, \mathcal{C}(X, Y))$. Recall that for simp sets A and B, $sSet(A, B)[n] \overset{?}{=} Set(A[n], B[n])$

POWER

We also define X^A to be an object in $\mathcal{C}$ such that

$$\mathcal{C}(Y, X^A) = sSet(A, \mathcal{C}(X, Y))$$

For a small category I, $\mathcal{C}^I$ is the diagram category

For $X \in \mathcal{C}^I$ and $A \in sSet^{I^{op}}$,

$$X \otimes_I A := \int^{i \in I} X(i) \otimes A(i)$$

$$= \operatorname{colim}\left(\coprod_{i \xrightarrow{\alpha} j} X(i) \otimes A(j) \overset{\alpha_*}{\underset{\alpha^*}{\rightrightarrows}} \coprod_i X(i) \otimes A(i) \right)$$

If $A \in sSet^I$,

$$\hom^I(A, X) = \int_I X(i)^{A(i)}$$

$$= \lim\left(\prod_i X(i)^{A(i)} \rightrightarrows \prod_{j \xrightarrow{\alpha} i} X(i)^{A(j)} \right)$$

Then $\operatorname{colim}_I X = X \otimes_I *_I$ where $*_I$ is the constant $*$-valued simp set

and $\lim_I X = \hom^I(*_I, X)$

We want to replace $*_I$ by a contractible space N encoding I, so

$$X \otimes_I N \simeq X \otimes_I (*_I)$$

and $\hom^I(*_I, X) \simeq \hom^I(N, X)$

<u>Def</u> The <u>over category</u> I/α is the category whose objects are morphisms $\beta \xrightarrow{\alpha} \alpha$ in I and whose morphisms are diagrams

$$\begin{tikzcd} \beta \arrow[rr, "\delta"] \arrow[dr, "\sigma"'] & & \beta' \arrow[dl, "\sigma'"] \\ & \alpha & \end{tikzcd}$$

The <u>under category</u> α/I is dually defined. Objects are morphisms $\alpha \xrightarrow{\sigma} \beta$ and morphisms are diagrams

$$\begin{tikzcd} & \alpha \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ \beta \arrow[rr, "\delta"] & & \beta' \end{tikzcd}$$

<u>Def</u> For a small category I, the <u>nerve of I</u>, NI is the simplicial set with

$$(NI)_n = \{ \sigma : \alpha_0 \xrightarrow{\sigma_0} \alpha_1 \to \cdots \xrightarrow{\sigma_{n-1}} \alpha_n \}$$

where degeneracy + face maps are

$$d_i \sigma = \begin{cases} \alpha_1 \xrightarrow{\sigma_1} \cdots \xrightarrow{\sigma_{n-1}} \alpha_n & : i = 0 \\ \alpha_0 \to \cdots \alpha_{i-1} \longrightarrow \alpha_{i+1} \to \cdots \alpha_n & : 0 < i < n \\ \alpha_0 \to \cdots \to \alpha_{n-1} & : i = n \end{cases}$$

$$s_i \sigma = \alpha_0 \to \cdots \alpha_i \xrightarrow{1} \alpha_i \to \cdots \alpha_n$$

Thus we have $N(I/i)$ and $N(i/I)^{op}$, so $\exists$ functors

$N(I/-) : I \longrightarrow sSet$

$N(-/I)^{op} : I^{op} \longrightarrow sSet$.

<u>Def</u> $\mathrm{hocolim}_I X = X \otimes_I N(-/I)^{op}$ simp set

$= \mathrm{colim}\left(\coprod_{i \xrightarrow{\alpha} j} X(i) \otimes N(j/I)^{op} \rightrightarrows \coprod_i X(i) \otimes N(i/I)^{op} \right)$

$\mathrm{holim}_I X = \hom_I (N(I/-), X)$

Example $A \xleftarrow{f} C \xrightarrow{g} B$ Let I be $1 \leftarrow 3 \rightarrow 2$

Earlier we said

$\mathrm{hocolim} = \mathrm{colim}\left(A \xleftarrow{f} C \xrightarrow{0} C \times [0,1] \xleftarrow{1} C \xrightarrow{g} B \right)$

Note $1/I$ has one object, $1 \xrightarrow{id} 1$ so

$2/I$ " " , $2 \rightarrow 2$

$3/I$ has three objects $\left\{ \begin{array}{l} 3 \xrightarrow{a} 1 \\ 3 \xrightarrow{b} 2 \\ 3 \xrightarrow{c} 3 \end{array} \right\}$

so $|N(1/I)^{op}| = * = |N(2/I)^{op}|$

and $|N(3/I)^{op}| = N(3/I)^{op}_0 \times \Delta^0 \amalg N(3/I)^{op}_1 \times \Delta^1 / \sim$

$\simeq [0,1]$

Thus for $X = \{A \xleftarrow{f} C \xrightarrow{g} B\}$,

$$\operatorname{hocolim}_I X \simeq \operatorname{colim}\left(\coprod_{i \xrightarrow{\alpha} j} X(i) \otimes N(j/I)^{op} \rightrightarrows \coprod X(i) \otimes N(i/I)^{op} \right)$$

$$\begin{tikzcd}
 & C = X(3)\otimes N(1) \arrow[dl] \arrow[d] & \coprod & X(3)\otimes N(2) = C \arrow[dll] \arrow[d] \\
X(1)\otimes N(1) = A & X(3)\otimes N(3) = C \times I & & X(2)\otimes N(2) = B
\end{tikzcd}$$

This is what we said before.

Now we define holim and hocolim for spectra.

For X in $(Sp^G)^I$, for any rep V of G

$$(\operatorname{hocolim}_I X)_V := \operatorname{hocolim}_I (X_V) \quad \text{where } X_V \in (T^G)^I.$$

$\operatorname{holim}_I X$ is similarly defined.

<u>Rem</u> holim and hocolim may not be homotopy invariant under stable equivalence.

<u>Lemma</u> (Hirschhorn) Let $F_{obj} X$ be objectwise fibrant replacement of X in C^I for a model cat C and small cat I. Then the functor

$X \mapsto \hom^I(N(I/-), F_{obj} X)$ is homotopy invariant. (cofibrant; FIBRANT)

Dually, $X \mapsto C_{obj} X \otimes N(-/I)^{op}$ is htpy invariant.

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where $C_{obj} X$ is objectwise cofibrant replacement.

The canonical homotopy presentation

Let $\cdots V_n \hookrightarrow V_{n+1} \hookrightarrow V_{n+2} \cdots$

be an exhaustive sequence of reps of G and let $W_n = V_{n+1} - V_n$ = orth complement

$$\begin{array}{ccc} S^{-V_{n+1}} \wedge J_G(V_n, V_{n+1}) \wedge X_{V_n} & \longrightarrow & S^{-V_{n+1}} \wedge X_{V_{n+1}} \\ \downarrow & & \\ S^{-V_n} \wedge X_{V_n} & & \end{array}$$

Let X_n be CW-approx of X_{V_n} so we have

$$\begin{array}{ccc} S^{-V_{n+1}} \wedge J_G(V_n, V_{n+1}) \wedge X_n & \longrightarrow & S^{-V_{n+1}} \wedge X_{n+1} = A_{n+1} \\ \downarrow & & \\ S^{-V_n} \wedge X_n =: A_n & & \end{array}$$

We also have $S^{W_n} \longrightarrow J_G(V_n, V_{n+1})$

and let $B_n = S^{-V_{n+1}} \wedge S^{W_n} \wedge X_n$

so we have $\begin{array}{ccc} B_n & \longrightarrow & A_{n+1} \\ \downarrow & & \\ A_n & & \end{array}$ and we get

diagram of cofibrant objects

$$A_0 \xleftarrow{\simeq} B_0 \longrightarrow A_1 \xleftarrow{\simeq} B_1 \longrightarrow A_2 \xleftarrow{\simeq} B_2 \longrightarrow A_3 \cdots$$

Let $C_n := \operatorname{hocolim}(A_0 \leftarrow \cdots \longrightarrow A_n)$

$\simeq A_n$

Then

$\mathrm{hocolim}(C_0 \to C_1 \to C_2 \to \cdots\cdots)$

$= \mathrm{hocolim}(A_0 \xleftarrow{\simeq} B_0 \to A_1 \cdots\cdots)$

and $\pi_* \, \mathrm{hocolim}(C_n) = \mathrm{colim}\, \pi_* C_n = \mathrm{colim}\, \pi_* A_n$

$X \simeq \mathrm{hocolim}\, C_n$.

Is this hocolim cofibrant?