Geometric fixed points

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Let $\mathcal{F}$ be a family of subgps of G that is closed under conjugation + inclusion.

Examples

1) all subgps not containing any conjugate of a given subgp K
2) All abelian subgps
3) all p-subgroups
4) $\mathcal{P}$ = family of all proper subgps
5) all subgps
6) $\{e\}$

For each $\mathcal{F}$ there is a G-space $E\mathcal{F}$ with

$$(E\mathcal{F})^H \simeq \begin{cases} * & \text{for } H \in \mathcal{F} \\ \emptyset & \text{for } H \notin \mathcal{F} \end{cases}$$

It can be constructed by taking infinite joins of G/H for all $H \in \mathcal{F}$.

e.g. $\mathcal{F} = \{e\}$, then $E\mathcal{F} = \underset{\infty}{*}\, G = EG$

EG is a contractible free G-space.

$(EG)^H = \emptyset$ for $H \neq e$, and $EG^{\{e\}} \simeq *$.

We will use this for $\mathcal{F} = \mathcal{P}$. Consider the

ISOTROPY SEPARATION SEQUENCE

$$E\mathcal{P}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{P} = \text{cone on } E\mathcal{P}$$

Fixed point spectra

Let X be a G-spectrum. The ordinary spectrum X^G, the <u>fixed point spectrum</u>, defined $(X^G)_n = (X_n)^G$

WARNING. This is not homotopical.

<u>Example</u> Let $G = C_2$, $X = S^{-\sigma} \wedge S^{\sigma}$, $Y = S^{-0}$

There is map $X \to Y$ which is a weak equivalence of G-spectra

$(Y^G)_n = (S^n)^G = S^n$, so $Y^G = Y = S^{-0}$

$X_n = J(\sigma, n) = J(\sigma, \mathbb{R}^n)$. G acts freely away from the basepoint on this space

$J(\sigma, \mathbb{R}^0) = *$, $J(\sigma, \mathbb{R}^1) = O(1)_+$ where σ acts on $O(1)$ via the iso $C_2 \cong O(1)$.

Hence $(S^{-\sigma} \wedge S^{\sigma})^G = *$

Recall our definition of $\pi_*^G(X)$. Choosing and exhaustive sequence of reps

$V_0 \subset V_1 \subset V_2 \subset V_3 \cdots$

(every V is contained in some V_n)

e.g $V_n = n\rho_G$

Then

$$\pi_R^G X = \operatorname{colim}_n \pi_R\left(\Omega^{V_n} X_{V_n}\right)^G$$

This is not the same as $\pi_R(X^G)$ for general X, but it is for fibrant X because for such X,

$$\Omega^{V_n} X_{V_n} \simeq \Omega^{V_{n+1}} X_{V_{n+1}} \quad \text{for } n \gg 0$$

$(-)^G$ is homotopical on fibrant spectra

Def The geometric fixed point spectrum $\Phi^G X$ is

$$\left(\left(E\hat{P} \wedge X\right)_f\right)^G$$

We can define $\Phi^H X = \Phi^H(i_H^G X)$ for $H \subset G$. $\Phi^e X = (i_e^G X)_f$

Thm Properties of $\underline{\Phi}^G$

1) The functor $\underline{\Phi}^G$ is homotopical
2) It commutes with filtered homotopy colimits (to be defined later)
3) For a pointed G-space A,

$$\underline{\Phi}^G(S^{-V} \wedge A) \simeq S^{-V^G} \wedge A^G$$

e.g. $\underline{\Phi}^G(\Sigma^\infty A) \simeq \Sigma^\infty A^G$

4) There is a natural chain of weak equivalences relating $\underline{\Phi}^G(X \wedge Y)$ with $\underline{\Phi}^G(X) \wedge \underline{\Phi}^G(Y)$.

Proof of 1) The functor $\underline{\Phi}^G$ is the composite

$$X \rightsquigarrow \tilde{E}\mathcal{P} \wedge X \rightsquigarrow (\tilde{E}\mathcal{P} \wedge X)_f \rightsquigarrow ((\tilde{E}\mathcal{P} \wedge X)_f)^G$$

Each of these is homotopical.

2) Straightfoward

4) Every spectrum X has a canonical homotopy presentation

$$X = \operatorname{hocolim} S^{-V} \wedge X_V$$

$$Y = \operatorname{hocolim} S^{-W} \wedge Y_W$$

assuming 3), $\Phi^G(S^{-V} \wedge X_V) \simeq S^{-V^G} \wedge (X_V)^G$

and $\Phi^G(S^{-W} \wedge Y_W) \simeq S^{-W^G} \wedge (Y_W)^G$

This leads to the result. QED

Thm Let $f: X \longrightarrow Y$ be an equivariant map of G-spectra

i) $\tilde{E}\mathcal{P} \wedge f$ is an equivalence $\Longleftrightarrow \Phi^G(f)$ is one

ii) $E\mathcal{P}_+ \wedge f$ " $\Longleftrightarrow \Phi^H(f)$ is one for each proper $H \subset G$.

iii) $X \simeq * \Longleftrightarrow \Phi^H X \simeq *$ for all $H \leq G$

iv) f is an equiv $\Longleftrightarrow \Phi^H(f)$ is one for all $H \leq G$.

Lemma i) $E\mathcal{P}_+ \wedge E\mathcal{P}_+ \simeq E\mathcal{P}_+$

ii) $\tilde{E}\mathcal{P} \wedge \tilde{E}\mathcal{P} \simeq \tilde{E}\mathcal{P}$

iii) $E\mathcal{P}_+ \wedge \tilde{E}\mathcal{P} \simeq *$

Proof Recall the 2 space are characterized (up to equiv equivalence) by

$$(E\mathcal{P}_+)^H \simeq \begin{cases} * & H = G \\ S^0 & H \subset G \end{cases} \quad \text{and} \quad \tilde{E}\mathcal{P}^H \simeq \begin{cases} S^0 & H = G \\ * & H \subset G \end{cases}$$

The equivalence of i) is $E\mathcal{P}_+ \wedge (E\mathcal{P}_+ \longrightarrow S^0)$

ii) $\tilde{E}\mathcal{P} \wedge (\tilde{E}\mathcal{P} \longleftarrow S^0)$

iii) $E\mathcal{P}_+ \wedge \tilde{E}\mathcal{P} \longrightarrow *$ QED

Proof of Theorem We have

$$(E\mathcal{P}_+ \wedge X)^H_f \simeq \begin{cases} \Phi^H X & \text{for } H \subset G \\ * & \text{for } H = G \end{cases}$$

$$(\tilde{E}\mathcal{P} \wedge X)^H_f \simeq \begin{cases} * & \text{for } H \subset G \\ \Phi^G X & \text{for } H = G \end{cases}$$

Hence if $\Phi^H X \simeq *$ for all $H \le G$, then $E\mathcal{P}_+ \wedge X$ and $E\tilde{\mathcal{P}} \wedge X$ are both G-contractible, so X is G-contractible.

For iv), apply iii) to the cofiber C_f. Similar arguments work for i) and ii).

QED

<u>Next</u>: We define the normal N_H^G as an indexed smash product. Given an H-spectrum X, consider the smash product $\bigwedge_{G/H} X$, i.e.

$$\bigwedge_{\alpha \in G/H} X_\alpha$$

where G permutes the factors, each is invariant under H.

We see that that this functor is homotopical on cofibrant X.

Then For a cofibrant H-spectrum X

Wed QZ on hocolim