# On the gap theorem

Thursday, March 31, 2016 8:56 AM

For any cyclic 2-gp $G = C_{2^n}$ we can form the norm $X = N_2^{2^n} MU_{\mathbb{R}} = MU^{((2^n))}$, invert a suitable $D \in \pi^G_{\star \rho_G} X$ and get a periodic spectrum. For $n \geq 3$, it will detect $\theta_j$ (?), but not for $n < 3$. The periodicity grows rapidly with $n$. For $n = 1, 2, 3, 4$ it is $8, 32, 256$ and $2^{13}$.

The gap theorem holds in all cases and is proved in the same way. It uses the slice SS.

Recall for a finite gp $G$, for $H \leq G$ and an integer $m$,

$$\hat{S}(m, H) = \begin{cases} G_+ \wedge_H \Sigma^\infty S^{m\rho_H} & \text{for } m > 0 \\ G_+ \wedge_H S^{m\rho_H} & \text{for } m \leq 0 \end{cases}$$

space

YONEDA SPECTRUM

where $\rho_H$ is the regular representation of $H$.

These are <u>slice cells</u>.

Let $Sp^G_{\geq n}$ be the localizing subcategory of $Sp^G$ generated by $\{\hat{S}(m, H) : m|H| \geq n\}$

Denote the corresponding localization functor by $P^{n-1}$, the analog of the $(n-1)$th Postnikov section. Let $P_n X$ be the fiber of $X \to P^{n-1}X$, the analog of the $(n-1)$-connected cover.

Since $Sp^{G}_{\geq n+1} \subset Sp^{G}_{\geq n}$, we have maps $P^n X \longrightarrow P^{n-1} X$ and denote the fiber by $P^n_n X$, the <u>nth slice</u> of $X$.

Thm (Slice theorem) For $X = N_2^{2^l} MU_{\mathbb{R}}$ or $X = D^{-1}$(same), then

$$P^n_n X = \begin{cases} * & \text{for } n \text{ odd} \\ W_n \wedge H\mathbb{Z} & \text{for } n \text{ even} \end{cases}$$

where $W_n$ is a wedge of isotropic $n$-dimensional slice cells, i.e. cells $\hat{S}(m, H)$ with $m|H| = n$ for $H \neq e$.

This is not easy to prove. It is related to the Reduction theorem, which says

$$P^0_0 N_2^{2^l} MU_{\mathbb{R}} = H\underline{\mathbb{Z}} = \text{Integer EM spectrum}$$

The gap theorem follows

Lemma $\pi_{-2}^G \hat{S}(m,H) \wedge HZ = 0$ for all $m$ and all $H \neq e$.

For the main theorem, we have $\overline{X} = D^{-1} MU^{((8))}$

$$\Omega = \left(D^{-1} N_2^8 MU_{\mathbb{R}}\right)^{C_8}$$

$\pi_* \Omega = \pi_*^G \left(D^{-1} N_2^8 MU_{\mathbb{R}}\right)$, which can be computed with the slice SS with

$$E_2^{**} = \pi_*^G P_n^n X$$

where each $P_n^n X$ is $HZ \wedge W$ as above.

The lemma implies $\pi_{-2}^G P_n^n X = 0$ for all $n$.

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CLAIM For $m > 0$, $S^{m\rho_H}$ is a $H$-CW complex with cells in dimensions ranging from $m$ to $m|H|$.

Note $\rho_H = 1 + \overline{\rho}_H$ where $\overline{\rho}_H$ is the reduced regular rep. $\rho_H$ is $\mathbb{R}^{|H|}$ with basis $\{[\gamma] : \gamma \in H\}$. The subspace generated by $\sum_{\gamma \in H} [\gamma]$ is fixed by $H$, so it generated an invariant 1-dimensional summand

$S^{\bar\rho_H}$ is an $H$-CW cx with cells in dimensions
$0$ thru $|H|-1$.
$S^{\rho_H} = \Sigma S^{\bar\rho_H}$ has cells in dimensions
$1$ thru $|H|$.
$S^{m\rho_H} = (S^{\rho_H})^{\wedge m}$ has cells in dimensions
$m$ thru $m|H|$.
A similar statement holds for the spectrum $\Sigma^\infty S^{m\rho_H}$. The Yoneda spectrum $S^{-m\rho_H}$ has "cells" in dimensions

$-m|H|$ thru $-m$.
We are interested $\pi_*^G(\hat{S}(m,H) \wedge H\underline{\mathbb{Z}})$, which is related to $H_*(\hat{S}(m,H))$ as follows.
Given a $G$-CW cx $X$ we get a cellular chain complex $C_*(X)$ of $\mathbb{Z}[G]$-modules. For any $\mathbb{Z}[G]$-module $M$ we define a Mackey functor $\underline{M}$ by $\underline{M}(G/H) = M^H$. Hence from $C_*(X)$, a chain complex of $\mathbb{Z}[G]$-modules, we get a chain complex $\underline{C}_*(X)$ of Mackey functors. Its homology is a graded Mackey functor $\underline{H}_* X$

We want to find $\underline{H}_*(G_+ \underset{H}{\wedge} S^{m\rho_H})(G/G)$

$$\|$$

$$\pi_*^G(G_+ \wedge S^{m\rho_H} \wedge H\underline{\mathbb{Z}})$$

The cellular chain complex for $S^{-m\rho_H}$ is dual to that for $S^{m\rho_H}$ for $m>0$. Each is nontrivial only in a certain range of dimensions.

The only cases where we have any cells in dimension $-2$ is $m=-1$ and $m=-2$.

For simplicity let $G=H=C_2$. Need to consider $S^{-\rho_2}$ and $S^{-2\rho_2}$. For these we first look at $S^{\rho_2}$ and $S^{2\rho_2}$. The cellular chain complexes are

$$\begin{array}{lcccc}
 & 1 & 2 & 3 & 4 \\
S^{\rho_2} & \mathbb{Z} & \xleftarrow{\nabla} \mathbb{Z}G & & \\
 & 0 & \longleftarrow\!\shortmid \gamma-1 & & \\
S^{2\rho_2} & & \mathbb{Z} & \xleftarrow{\nabla} \mathbb{Z}G & \xleftarrow{1-\gamma} \mathbb{Z}G \\
 & & & 0 & \longleftarrow\!\shortmid \gamma+1
\end{array}$$

$\mathbb{Z}G = \mathbb{Z}[C_2]$

$\gamma$ = gen of $C_2$

$\nabla(\gamma) = 1$

$\nabla(1) = 1$

$\mathbb{Z}$ has trivial $G$-action

For $S^{-\rho_2}$ and $S^{-2\rho_2}$ the chain complexes are

$$\begin{array}{lcccc}
 & -1 & -2 & 3 & 4 \\
S & & & &
\end{array}$$

For $S^{-\rho_2}$ and $S^{-2\rho_2}$ we have

$$
\begin{array}{lcccc}
 & -1 & -2 & -3 & -4 \\
S^{-\rho_2} & \mathbb{Z} & \xrightarrow{\Delta} \mathbb{Z}G & & \\
 & 1 & \longmapsto 1+\gamma & & \\
S^{-2\rho_2} & & \mathbb{Z} & \xrightarrow{\Delta} \mathbb{Z}G & \xrightarrow{1-\gamma} \mathbb{Z}G \\
 & & 1 & \longmapsto 1+\gamma & \longmapsto 0 \\
 & & & 1 & \longmapsto 1-\gamma
\end{array}
$$

trivial $G$-action

$(\mathbb{Z}G)^G$ = subgp gen'd by $1+\gamma$

complexesare

applying $(-)^{C_2}$ to these gives

$$
\begin{array}{lcccc}
 & -1 & -2 & -3 & -4 \\
S^{-\rho_2} & \mathbb{Z} & \xrightarrow{1} \mathbb{Z} & & \\
S^{-2\rho_2} & & \mathbb{Z} & \xrightarrow{1} \mathbb{Z} & \xrightarrow{0} \mathbb{Z}
\end{array}
$$

In both cases $H_{-2} = 0$, so

$$\pi_{-2}^{G}(H\mathbb{Z} \wedge S^{-\rho_2}) = 0 \quad \text{and}$$

$$\pi_{-2}^{G}(H\mathbb{Z} \wedge S^{-2\rho_2}) = 0$$

We get the same answer for larger $G$ and $H$.

This proves the gap theorem.

Next meeting Monday 4/11/16.