Zeng on Periodicity Theorem II

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We want to construct a spectrum Ω with

① It is 256-periodic (Periodicity Theorem)

② $\pi_{-2}\Omega = 0$ (Gap theorem)

③ Ω detects θ_j. (Detection theorem)

$\Omega := (D^{-1} MU^{((8))})^{hC_8}$ for a certain $D \in \pi_{19\rho_8}^{C_8} MU^{((8))}$

① and ③ are about htpy fixed pt, while ② is about actual fixed point spectrum. We need a theorem equating them.

<u>Lemma</u> Let X be a G-spectrum and suppose a G-map $\Delta : \Sigma^n X \to X$ induces an iso of π_*^u. Then Δ^{hG} is a weak equivalence.

Proof is not in paper.

<u>Strategy for finding the right D</u>

Find a D divisible by $N_2^8(\bar{r}_1)$ and a $u_{2k\rho_8}$ being a permanent cycle in slice SS for $D^{-1}MU^{((8))}$

Then $\Delta := u_{2k\rho_8}(N_2^8 \bar{r}_1^2)^k$ is permanent cycle of integer degree which becomes invertible after forgetting G-action.

Notation $\bar{\partial}_k^{G} = N_2^{G}(\bar{r}_{2^k-1}) \in \pi^{G}_{(2^k-1)\rho_G} MU^{((G))}$. Recall the adjunction

$$N_H^G : \mathrm{Comm}^H \rightleftarrows \mathrm{Comm}^G : i_H^G$$

$$\begin{array}{ccc} MU^{((G))} & \xrightarrow{1} & MU^{((G))} \\ \| & & \| \\ N_H^G MU^{((H))} & \longrightarrow & N_H^G MU^{((H))} \\ & \Downarrow \text{adjoint} & \\ MU^{((H))} & \longrightarrow & i_H^G MU^{((G))} \end{array}$$

$$\bar{r}_1 : S^{i\rho_2} \longrightarrow i_2^G MU^{((G))}$$

$$\begin{array}{ccc} N_2^G S^{i\rho_G} & \longrightarrow & N_2(\ ) \\ & & \downarrow \\ & & MU^{((G))} \end{array}$$

$f_i := a_{\bar{\rho}_G}^{i} N_2^G \bar{r}_i$, e.g. $f_{2^k-1} = a_{\bar{\rho}_G}^{2^k-1}\bar{\partial}_k$

so $f_{2^{k+1}-1}\bar{\partial}_k = f_{2^k-1} a_{\bar{\rho}_G}^{2^k}\bar{\partial}_{k+1}$

Cor In the slice SS of $MU^{((G))}$, $\bar{\partial}_k u_{2\sigma}^{2^k}$ is perm. cycle.

Proof $\bar{\partial}_k$ is perm cycle and $u_{2\sigma}^{2^k}$ supports a slice diff, so

$$d_{1+(2^{k+1}-1)g}(\bar{\partial}_k u_{2\sigma}^{2^k}) = \bar{\partial}_k d_{1+(2^{k+1}-1)g}(u_{2\sigma}^{2^k})$$

$$= \bar{\partial}_k a_\sigma^{2^{k+1}} f_{2^{k+1}-1} = a_\sigma^{2^{k+1}} f_{2^k-1} a_{\bar{\rho}_G}^{2^k}\bar{\partial}_{k+1}$$

$$= a_\sigma^{2^k} a_{\bar{\rho}_G}^{2^k}\bar{\partial}_{k+1} d_{1+(2^k-1)g}(u_{2\sigma}^{2^{k-1}})$$

$$= 0 \text{ in } E_{1+(2^{k+1}-1)g}$$

Need to show there is no higher differential. This is a technical vanishing line argument.

QED

We have a map $MU^{((H))} \longrightarrow i_H^G MU^{((G))}$ which we can norm up.

<u>More notation</u> Write $\bar{r}_1^H$ $\bar{\partial}_k^H$ for the image in $\pi_\star^H MU^{((G))}$

$g = |G|$
$h = |H|$

<u>Thm</u> Let $D \in \pi_{\ell\rho_G}^G MU^{((G))}$. If

① For nontrivial $H \subseteq G$, $N_H^G i_H^G D$ divides a power of D

② $\bar{\partial}_{g/h}^H \mid i_H^G H$ for all $e \neq H \subseteq G$

then $u_{2\rho_G}^{2^{g/2}}$ is a perm cycle in slice SS for $D^{-1} MU^{((G))}$

Proof: By the Con, $\bar{\partial}_{g/H}^H u_{2\sigma_H}^{2^{g/h}}$ is perm cycle.

Since $\bar{\partial}_{g/H}^H$ divides $i_H^* D$, $u_{2\sigma_H}^{g/h}$ is perm cycle in $D^{-1} MU^{((G))}$

$$u_{2\rho_G} = \prod_{e \neq H \subseteq G} N_H^G \left(u_{2\sigma_H}^{h/2}\right) \text{ and } 2^{g/h} \leq hk/2 \text{ for some } k \text{ and all } H \neq e$$

For $h = 2$, $k = 2^{g/2}$ is minimal value.

$$u_{2\rho_G}^k = \prod_{e \neq H \subset G} N_H^G \left(u_{2\sigma_H}^{hk/2}\right)$$

Hence $\Delta = (u \quad (\partial_1^G)^2)^{2^{g/2}}$ is a perm cycle

Hence $\Delta = \left(M_{2\rho_G} (\partial_1^G)^2 \right)^{2^{g/2}}$ is a perm cycle

For $G = C_8$, degree $2 \cdot 8 = 16$ and $2^{g/2} = 16$, so degree is 256.

When D satisfies ① of Thm, then $D^{-1}MU^{((G))}$ is a ring spectrum by separate result of HHR (Mult. closure)

How to find the right D

Assume $D = N_2^{G} x$ for some x.

Then $N_H^G i_H D = N_H^G i_H^* N_{C_2}^G x = N_H^G (i_H^* N_H^G N_2^H x)$

$= N_H^G (N_H^{C_2} x)^{g/h} = N_2^g (x^{g/h}) = D^{g/h}$

$\bar{\partial}_{g/h}^H := N_2^h (\bar{r}^H_{2^{g/h}-1})$

$D = N_H^G (\bar{\partial}^H_{g/h})$

$D = N_2^8(\partial_4^{C_2}) N_4^8(\partial_2^{C_4}) \partial_1^{C_8}$

For general G the periodicity dimension is $2 \cdot g 2^{g/2}$.

This value appears to be minimal.

We know $\Phi^G N_2^8 \bar{r}_1 = h_1 = 0 \in \pi_1 MO$, so $\Phi^G D = 0$

and $\Phi^G D^{-1} MU^{((8))}$