McTague continued

Thursday, March 3, 2016 8:49 AM

Correction: A left deformation is always homotopical

Pf: $f : X \xrightarrow{\sim} Y$

$$\begin{tikzcd} Q(X) \arrow[r, "Q(f)"] \arrow[d, "q", "\sim"'] & Q(Y) \arrow[d, "q", "\sim"'] \\ X \arrow[r, "f", "\sim"'] & Y \end{tikzcd}$$

Now $q\,Q(f) = f q$ is a weak equiv so $Q(f)$ is by 2-of-3.

Prop Given

$$(*) \quad \begin{tikzcd} X_1 \arrow[d, "\sim"'] & A_1 \arrow[l, "b"'] \arrow[r, "b"] \arrow[d, "\sim"] & Y_1 \arrow[d, "\sim"] \\ X_2 & A_2 \arrow[l] \arrow[r, "b"] & Y_2 \end{tikzcd},$$

the map $X_1 \cup_{A_1} Y_1 \longrightarrow X_2 \cup_{A_2} Y_2$.

The rows are "spans"

Proof ① We did it for the case $A_1 = A_2 = A$ last time.

② General case. Expand the diagram above to

$$\begin{tikzcd} & X_1 \cup_{A_1} Y_1 & \\ X_1 \arrow[ur, "b"] \arrow[d, "\sim"'] \arrow[dd, bend right=60, "\simeq"'] & A_1 \arrow[l, "b"'] \arrow[r, "b"] \arrow[u, "b"] \arrow[d, "\sim"] & Y_1 \arrow[ul, "b"'] \arrow[d, "\sim"] \arrow[dd, bend left=60, "\simeq"] \\ X_1 \cup_{A_1} A_2 \arrow[d] & A_2 \arrow[l, "b"'] \arrow[r, "b"] \arrow[d, equal] & A_2 \cup_{A_1} X_2 \arrow[d] \\ X_2 & A_2 \arrow[l] \arrow[r, "b"] & Y_2 \end{tikzcd}$$

flatness is preserved by cobase change

Will the top 2 rows have equivalent pushouts

$$(X_1 \cup_{A_1} Y_1) \simeq A_1 \cup_{A_1} (X_1 \cup_{A_1} Y_1) \xrightarrow{\simeq} A_2 \cup_{A_1} (X_1 \cup_{A_1} Y_1) = \text{colimit of second}.$$

Then ① implies the result. QED row

If our category has factorization

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "b"'] & & Y \\ & Y' \arrow[ur, "\simeq"'] & \end{tikzcd}$$

then we can weaken the hypothesis, by requiring only one map in top row of (*) to be flat e.g.

$$\begin{tikzcd} & & Y_1' \arrow[dr, "\simeq"] \arrow[ddr, bend left, "\simeq"] & \\ X_1 \arrow[d, "\sim"'] & A_1 \arrow[l, "b"'] \arrow[r] \arrow[d, "\sim"'] \arrow[ur, "b"] & Y_1 \arrow[d, "\sim"] & \\ X_2 & A_2 \arrow[l] \arrow[r, "b"] & Y_2 & \end{tikzcd}$$

and conclude that $X_1 \cup_{A_1} Y_1' \xrightarrow{\sim} X_2 \cup_{A_2} Y_2$

$$\begin{tikzcd} X_1 \arrow[d, equal] & A_1 \arrow[l, "b"'] \arrow[r, "b"] \arrow[d, equal] & Y_1' \arrow[d, "\sim"] \arrow[dd, bend left, "\sim"] \\ X_1 \arrow[d, "\sim"'] & A_1 \arrow[l, "b"'] \arrow[r] \arrow[d, "\sim"'] & Y_1 \arrow[d, "\sim"'] \\ X_2 & A_2 \arrow[l] \arrow[r] & Y_2 \end{tikzcd}$$

Claim first 2 rows have equivalent pushouts

Sp^G with stable weak equivs have the factorization property above.

Rem Suppose $(M, \wedge, 1)$ is homotopical SMC in which every object Z admits a weak equiv from a flat object $\tilde{Z}$. Then if $X \xrightarrow{f} Y$ is weak equiv, so is $f \wedge Z$ for any Z.

CLAIM

of flat objects

Proof By def $\tilde{Z} \otimes (-)$ preserves limits and weak equiv

$$\begin{tikzcd} X \wedge \tilde{Z} \arrow[r, "\sim", "?"'] \arrow[d, "\sim"'] & X \wedge Z \arrow[d] \\ Y \wedge \tilde{Z} \arrow[r, "\sim", "?"'] & Y \wedge Z \end{tikzcd}$$

Notation $i_{n+} : S^{n-1}_+ \longrightarrow D^n_+$, $j_{n+} : I^n_+ \longrightarrow I^{n+1}_+$ $n \geq 0$
$(\partial D_0 = \emptyset)$

Then $\mathcal{T}$ (category of pointed spaces) has a cofibrantly generated model category structure with generating sets

$\mathcal{I} = \{ i_{n+} : n \geq 0 \}$ and $\mathcal{J} = \{ j_{n+} : n \geq 0 \}$
= generating cofibrations = generating trivial cofibrations

$\mathcal{T}^G$ = category of pointed G-spaces and equivariant maps

It has a CGMC structure with

$$\mathcal{I} = \{ G_+ \underset{H}{\wedge} i_{n+} : n \geq 0, H \subseteq G \}$$

$$\mathcal{J} = \{ G_+ \underset{H}{\wedge} j_{n+} : n \geq 0, H \subseteq G \}$$

Weak equivalences are maps $X \xrightarrow{f} Y$ such that $f^H : X^H \longrightarrow Y^H$ is a weak equivalence $\forall H \subseteq G$.

Let Sp^G = category of G-spectra and equivariant maps.

A map $f: X \to Y$ is a <u>strict equivalence</u> if $f_V: X_V \to Y_V$ is a weak equiv in $\mathcal{T}^G$ for each rep. V. TOO RIGID

We get a CGMC structure with

$$I = \{ G_+ \underset{H}{\wedge} S^{-V} \wedge i_{n+} : \begin{array}{l} n \geq 0, \; H \leq G, \\ V = \text{rep of } H \end{array} \}$$

Recall the Yoneda spectrum S^{-V} is defined by $(S^{-V})_W = \mathcal{J}_H(V, W)$ = certain Thom space

$$J = \{ G_+ \underset{H}{\wedge} S^{-V} \wedge j_{n+1} : \text{ same } \}$$

To define stable homotopy equivalences, choose a sequence of reps of G

(*) $V_1 \hookrightarrow V_2 \hookrightarrow V_3 \hookrightarrow \cdots$

which is <u>exhaustive</u>, i.e. every finite dimensional rep V is contained in some V_n, e.g. $V_n = n\rho_G$. For a spectrum X, let

$$RX = \operatorname{colim} \left(\Omega^{V_1} S^{V_1} \wedge X \to \Omega^{V_2} S^{V_2} \wedge X \to \cdots \right)$$

It is independent of the choice of *

FIBRANT REPLACEMENT

<u>Def</u> A map $f: X \to Y$ is a <u>stable equivalence</u> if $Rf: RX \to RY$ is a strict equivalence.

Remarks: 1) RX is an "Ω-spectrum", i.e. one where $Y_V \simeq \Omega^W Y_{V \oplus W}$ for all V, W

2) Define $\pi_V X = \operatorname{colim} \pi_{V+V_n} X_{V_n}$
stable homotopy gps

We will define a CGMC structure with this notion of weak equiv and $\perp$ as before. We need more generating trivial cofibrations than before.

Let W be a rep of G and define

$$e_W : S^{-W} \wedge S^W \longrightarrow S^{-0}$$

G-space

Yoneda spectra

$$(S^{-W} \wedge S^W)_V = (S^{-W})_V \wedge S^W$$
$$= \mathcal{J}_G(W, V) \wedge \mathcal{J}_G(0, W)$$

$\downarrow$ composition

$$(S^{-0})_v = d_G(0, v)$$

__Thm__ e_W is a stable equivalence and so is $e^W \wedge S^{-V} : S^{-W \oplus V} \longrightarrow S^{-V}$

We can use the mapping cylinder construction (equivalently the small object argument based on $\mathcal{I}$-cofibs) to factor

$$\begin{tikzcd}
S^{-V\oplus W} \wedge S^{W} \arrow[rr, "S^{-V}\wedge e_W"] \arrow[dr, "\tilde{e}_{V,W}"', "\text{cofibration}"] & & S^{-V} \\
& \tilde{S}^{V,W} \arrow[ur, "\hat{e}_{V,W}"', "\text{true fibration}"] &
\end{tikzcd}$$

$\tilde{S}^{V,W}$ $=$ mapping cylinder of $S^{-V}\wedge e_W$

$\hat{e}_{V,W}$ is a stable equiv since $S^{-V}\wedge e_W$ is one and $\tilde{e}_{V,W}$ is a strict equivalence.

Recall __corner maps__: Given a diagram (in any category with pushouts)

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

There is a map $X \cup_A B \longrightarrow Y$, the __corner map__.

In a SMC let $X \xrightarrow{f} Y$ and $g : A \longrightarrow B$.

$$\begin{tikzcd}
A \wedge X \arrow[rr, "A \wedge f"] \arrow[d, "g \wedge X"'] \arrow[rr, phantom] & & A \wedge Y \arrow[dl] \arrow[d, "g \wedge Y"] \\
 & \text{pushout} \arrow[dr] & B \wedge Y \\
B \wedge X \arrow[ur] \arrow[rru, "B \wedge f"'] & &
\end{tikzcd}$$

The corner map is denoted by $f \square g$.

We then define

$$J_{stable} = J \cup \left\{ G_+ \wedge_H \left(i_{n+} \square \hat{e}_{V,W} \right) : \begin{array}{l} n \geq 0 \\ H \leq G \\ V, W \text{ reps} \\ \text{of } H \end{array} \right\}$$

This defines the <u>complete model category structure</u> on Sp^G

For technical reasons to be explained later we require (in both J and J_{stable}) the <u>positivity condition</u>

$V^H \neq 0$

Weird consequence of positivity

S^{-0} is <u>not</u> cofibrant.

Its cofibrant replacement is $S^{-1} \wedge S^1$!

NEXT MEETING MARCH 21