UY, CM, YQ, IB, YZ, SD, QZ, MZ

M. Zeng on the periodicity theorem

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Background

We need $MU_{\mathbb{R}}$

Thm There is a C_2-~~spectrum~~ ring $MU_{\mathbb{R}}$ s.t.

1) MU is underlying spectrum
2) $\Phi^{C_2} MU_{\mathbb{R}} \simeq MO$ = unoriented cobordism spectrum
3) $MU_{\mathbb{R}}$ is cofibrant

The proof is not easy. Need to know that $Comm^G$, the category of comm G-ring spectrum, has a MC structure.

We need the norm.

Def Let G be a finite cyclic 2-gp with $e \neq H \subset G$, and let X be an H-spectrum. Then the norm

$N_H^G X := \bigwedge_{G/H} X$ where, for a generator γ_G of G,

$$\gamma(x_1, \dots x_{|G/H|}) = (\gamma_H(x_{|G/H|}), x_1, \dots x_{|G/H|-1})$$

Prop $N_H^G : Comm^H \rightleftarrows Comm^G : i_H^*$ is a Quillen pair

The monoidal geometric fixed point we need:

$\Phi_M^G : Sp^G \longrightarrow Sp$ such that

① Φ_M^G preserves (trivial) cofibrations

② $\Phi_M^G \Sigma^\infty A = \Sigma^\infty A^G$ for pointed G-space A

(3) There is a map

$\Phi^G_M(X) \wedge \Phi^G_M(Y) \longrightarrow \Phi^G_M(X \wedge Y)$ that is an iso

if X or Y is cellular (cofibrant in stable complete MC)

(4) Φ^G preserves colimits

(5) $\Phi^G_M(S^{-V} \wedge A) = S^{-V^G} \wedge A^G$ for A as above

We can define $\Phi^G = L\Phi^G_M$ (left derived functor)

(not done in KI paper)

Proving all this is technical and is done in App. B of KI paper

Thm If X is a cofibrant H-ring spectrum, then

$\Phi^G N_H^G X \simeq \Phi^H X$

e.g. $\Phi^G N_{C_2}^G MU_{\mathbb{R}} \simeq \Phi^{C_2} MU_{\mathbb{R}} \simeq MO$

$\boxed{\begin{array}{l}\Phi^G(N_2^G \bar{r}_1): S^i \to MO \\ \text{is } h_i \text{, so} \\ \Phi(N_2^G \bar{r}_{2^k-1}) = 0\end{array}}$

Recall $\pi_* MO = \mathbb{Z}/2[h_i : i \neq 2^k - 1]$ for $h_i \in \pi_i$

$g = |G|$

Thm $\pi_*^u N_2^g MU_{\mathbb{R}} = \mathbb{Z}[G \cdot r_1, G \cdot r_2, \ldots]$ where r_i is the map underlying $S^{i\rho_2} \xrightarrow{\bar{r}_1} N_2^g MU_{\mathbb{R}}$

and $G \cdot r_i := \{r_i, \gamma_G r_i, \ldots\}$ with $\gamma_G^{g/2}(r_i) = (-1)^i r_i$.

Thm For a G-spectrum X, $\operatorname{colim} P^n X = *$ and

$X \simeq \lim P^n X$,

This implies that the slice SS converges to $\pi_\bigstar X$.

Thm (Slice theorem) $P_n^n N_2^g MU_{\mathbb{R}}$ is the n-dimensional summand of $N_2^g S^0[\bar{r}_1, \bar{r}_2 \ldots]$ smashed with $H\underline{\mathbb{Z}}$.

Given a map $S^V \xrightarrow{x} E$ = ring spectrum, $S^0[x] = \bigvee_{n=0}^{\infty} S^{nV}$ with $S^0[x] \longrightarrow E$. These can be smashed together for various x.

We need to know $\underline{\pi}_\star H\underline{\mathbb{Z}}$. It contains the following

$G = C_2$, $S^{n\sigma}$ has cellular chain cx

$$\mathbb{Z} \leftarrow \mathbb{Z}G \leftarrow \mathbb{Z}G \cdots\cdots \leftarrow \mathbb{Z}G$$

$$\mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow \mathbb{Z} \cdots$$

leading to $a_{n\sigma} \in \underline{H}_0 S^{n\sigma}(G/G)$

We also have $a_V : S^0 \to S^V$ for each actual rep V.

We use same notation for its Hurewicz image in $\underline{H}_0 S^V(G/G) = \pi^G_{-V} H\underline{\mathbb{Z}}$

Given an orientable rep V, there is a class $u_V \in \pi^G_{|V|-V} H\underline{\mathbb{Z}}$ representing an orientation.

The slice thm says 0-slice of $N_2^g MU_{\mathbb{R}}$ is $H\underline{\mathbb{Z}}$.

Let $f_i = a_{\bar\rho}^i N_2^g(\overline{r_i}) \in \pi^G_i P^{ig}_{ig} N_2^g MU_{\mathbb{R}}$ has filtration $i(g-1)$.

$a_\sigma \in \pi^G_{-\sigma} P^0_0 N_2^g MU_{\mathbb{R}}$ has filt 1, where σ = sign rep.

$u_{2\sigma} \in \pi^G_{2-2\sigma} P^0_0 N_2^g MU_{\mathbb{R}}$ has filtration 0.

Then f_i and a_σ are permanent cycles, but $u_{2\sigma}$ is not

Thm In the slice SS for $N_2^g MU_{\mathbb{R}}$,

$d_i\left(u_{2\sigma}^{2^{k-1}}\right) = 0$ for $i < 1 + (2^k-1)g$ and

$d_{1+(2^k-1)g}(\text{same}) = a_\sigma^{2^k} f_{2^k-1}$.

$2^k + (2^k-1)(g-1)$