Y. Zou on the positive complete model category structure on Sp^G.

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Goal:

$$I = \{ G_+ \wedge_H S^{-V} \wedge i_{n+} : n \geq 0,\ H \subseteq G,\ V^H \neq 0 \}$$

$i_n : S^{n-1} \to D^n$

$j_n : I^n \to I^{n+1}$

= generating set of cofibrations

$$J = \{ G_+ \wedge_H S^{-V} \wedge j_{n+} : \text{same} \} \cup \{ G_+ \wedge_H i_{n+} \,\square\, \tilde{e}_{V,W} \}$$

= generating set of trivial cofibrations

$\square$ denotes pushout corner map

W = any rep of H

$$\begin{tikzcd}
S^{-V\oplus W} \wedge S^W \arrow[rr] \arrow[dr, "I\text{-cofib } \tilde{e}_{V,W}"'] & & S^{-V} \\
& \tilde{S}^{V,W} \arrow[ur, "\hat{e}_{V,W}\ (\text{triv fib})"'] &
\end{tikzcd}$$

$\tilde{S}^{V,W}$ = mapping cylinder

Corner map fibs

(1)

$$\begin{tikzcd}
S^{n-1}_+ \wedge S^{-V\oplus W} \wedge S^W \arrow[r, "S^{n-1}_+ \wedge \tilde{e}_{V,W}", "\simeq"'] \arrow[d, "i_{n+} \wedge -"'] & S^{n-1}_+ \wedge \tilde{S}^{V,W} \arrow[d, "\beta"] \arrow[ddr, bend left] & \\
D_{n+} \wedge S^{-V\oplus W} \wedge S^W \arrow[r, "\alpha", "\simeq"'] \arrow[drr, "\simeq"'] & \text{pushout} \arrow[dr, "\square", "\simeq"'] & \\
& & D_{n+} \wedge \tilde{S}^{V,W}
\end{tikzcd}$$

__Main Theorem__ I and J above define a cofibrantly generated model category structure on Sp^G.

Proof Will use Kan recognition theorem, which requires 4 conditions

1) Smallness of I and J, technical but easy

(2) Each elt of J is stable equiv

(3) Every map with RLP for I is weak equiv and has RLP for J

(4) A weak equiv and I-cofib is also a J-cofib

or

A weak equiv with RLP for J has it for I.

We need to show these 4 conditions are met

1) For each H-rep V with $V^H \neq 0$, $S^{-V} \wedge i^n_{n+}$ and corner map of ① are all small. $\hat{e}_{V,W}$ is in the saturated class generated by I, so it is small relative to I. Pushout is colimit which preserves smallness.

2) Need to show J-cofibs are I-cofibs and weak equivs. For the noncorner maps in J, this is easy. We need to show $\hat{e}_{V,W}$ is an I-cofib. This is true objectwise. Same goes for $i_{n+} \square \hat{e}_{V,W}$. The map $\hat{e}_{V,W}$ is a weak equiv since $S^{-V} \wedge e_W$ is one. α in ① is weak equiv so corner map is.

3) Let $f: X \to Y$ have RLP for I. Need to show it has it for J and is a weak equiv. RLP for $I \Rightarrow$ strict equiv.

$$\begin{tikzcd} G_+ \wedge_H S^{-V} \wedge S^{n-1}_+ \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ G_+ \wedge_H S^{-V} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \iff \begin{tikzcd} S^{-V} \wedge S^{n-1}_+ \arrow[r] \arrow[d] & i_H^G X \arrow[d, "i_H^G f"] \\ S^{-V} \wedge D^n_+ \arrow[r] \arrow[ur, dashed] & i_H^G Y \end{tikzcd} \iff \begin{tikzcd} S^{n-1}_+ \arrow[r] \arrow[d] & (i_H^G X)_V \arrow[d, "(i_H^G f)_V"] \\ D^n_+ \arrow[r] \arrow[ur, dashed] & (i_H^G Y)_V \end{tikzcd}$$

so $(i_H^G f)_V$ is a trivial fibration, so f is weak equiv.

Showing RLP for J is harder.

Prop $f: X \to Y$ has RLP for J iff $\forall H \subseteq G$ and rep V of H with $V^H \neq 0$,

$$(i_H^G X)_V \longrightarrow (i_H^G Y)_V \quad \text{is a fibration}$$

and for each H-rep W the diagram

$$\begin{tikzcd}
(i_H^G X)_V \arrow[r, "(i_G^H f)_V"] \arrow[d] & (i_H^G Y)_V \arrow[d] \\
\Omega^W (i_H^G X)_{V \oplus W} \arrow[r, "\Omega^W (i_H^G f)_{V \oplus W}"] & \Omega^W (i_H^G Y)_{V \oplus W}
\end{tikzcd}$$

is homotopy Cartesian i.e. upper left space is pull back.

(Proof omitted)

The map $(i_G^H f)_V$ is a trivial fibr hence a fibration

Both horizontal maps above are weak equivs.

This concludes (3).

For (4) we need

Prop If $f: X \to Y$ is a weak equiv with RLP for J then it has RLP for I.

~~Proof~~ The RLP for $J \Rightarrow (i_H^G f)_V$ is fibration and we have a htpy Cartesian diagram

$$\begin{tikzcd}
(i_H^G X)_V \arrow[r, "\simeq"] \arrow[d] & (i_H^G Y)_V \arrow[d] \\
\operatorname{hocolim}_W \Omega^W (i_H^G X)_{V \oplus W} \arrow[r, "\simeq"] & \operatorname{hocolim}_W \Omega^W (i_H^G Y)_{V \oplus W}
\end{tikzcd}$$

Hence f is trivial fibration QED.