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<u>Def</u> A <u>localizing subcategory</u> $\mathcal{C}$ of a model category $\mathcal{M}$ is a full subcat with the following properties

1) If $X \in \mathcal{C}$ and $X \simeq Y$ then $Y \in \mathcal{C}$
2) If X and Y are in $\mathcal{C}$ and $X \xrightarrow{f} Y \to Z$ is any map, then its cofiber $Z = C_f$ is in $\mathcal{C}$
3) If $X \longrightarrow Y \longrightarrow Z$ is a cofiber seq with $X, Z \in \mathcal{C}$, then $Y \in \mathcal{C}$.
4) An arbitrary coproduct (wedge) of objects in $\mathcal{C}$ is in $\mathcal{C}$.

e.g. the category C_n of n-connected spaces in $\mathcal{T}$.

C_n is the smallest such subcategory containing S^{n+1}.

We get a localization associated with map $S^{n+1} \longrightarrow *$ in which fibrant approximation is $X \longrightarrow P^n X$, the nth Postnikov section. The fiber of this map is $P_{n+1} X$, the n-connected cover of X.

One can show that any such category

is closed under retracts.

Suppose C is the smallest such subcategory containing each object in a set $\{X_\alpha\}$. Then it is also the smallest one containing $X = \bigvee_\alpha X_\alpha$. Use the map $X \xrightarrow{b} *$ as above to construct fibrant replacement.

Back to the Postnikov example.

Let C_n = category of n-connected spectra. Clearly $C_{n+1} \subset C_n$ so we have

$$\cdots \to P^{n+2}X \to P^{n+1}X \longrightarrow P^n X \longrightarrow P^{n-1}X \to \cdots$$

Denote the fiber of $P^n X \to P^{n-1}X$ by $P_n^n X$. It is $K(\pi_n X, n)$.

We can also do this in the category Sp of spectra. Want to do it for Sp^G for a finite gp G.

<u>Def</u> The <u>slice cell</u> $\hat{S}(m, H)$, for $m \in \mathbb{Z}$ and $H \leq G$, is

$$\begin{cases} G_+ \wedge_H \Sigma^\infty S^{m\rho_H} & m \geq 0 \\ G_+ \wedge_H S^{m\rho_H} & m < 0 \end{cases}$$

where ρ_H is the regular rep of H

$\dim \mathcal{S}(m, H) = m\,|H|$

Def $Sp^G_{>n}$ is the localizing subcat generated by all $\hat{S}(m,H)$ with $m|H|>n$. This is the analog of n-connected spectra. Can also write $Sp^G_{\geq n+1}$

Def Let $\mathcal{T}^G_n$ be the category of G-spectra X with $\pi_k^H X = 0$ for $k < \left\lfloor \frac{n}{|H|} \right\rfloor$ for all H- where $\pi_k^H X$ is the H-equivariant htpy gp of X and $\lfloor x \rfloor$ is the floor of $x \in \mathbb{Q}$, largest integer $\leq x$.

Can show $\mathcal{T}^G_n = Sp^G_{\geq n}$

We can define maps $P^n X \longrightarrow P^{n-1} X$ as before with fiber $P^n_n X$, the <u>nth slice</u> of X. The diagram

$$\cdots \to P^n X \longrightarrow P^{n-1} X \longrightarrow \cdots$$

is the <u>slice tower</u> of X. It leads leads to the <u>slice spectral sequence</u>

The input is information about the slices $P^n_n X$ and the output is

slices "$P_n^n X$" and the output is
info about X

Classically this SS is useless because the input is $\{\pi_\star P_n^n X = \pi_n X\}$
Equivariantly $P_n^n X$ is more interesting

Example $X = MU_{\mathbb{R}}$, a C_2-spectrum

It is the complex cobordism MU equipped with a C_2-action induced by complex conjugation.

$\pi_\star MU = \mathbb{Z}[x_1, x_2, \ldots]$ where $x_i \in \pi_{2i}$

a generator $\gamma \in C_2$ acts on this by

$$\gamma(x_i) = (-1)^i x_i$$

It turns out that that

$$P_n^n X = \begin{cases} * & \text{for } n \text{ odd or } n<0 \\ H\mathbb{Z} \wedge (\text{wedge of copies of } S^{n\rho_2/2} & \text{for } n \text{ even} \end{cases}$$

(This is a nontrivial theorem)

What is $\pi_\star^H H\mathbb{Z} \wedge S^{m\rho_2}$? ρ_2 = reg rep of C_2

How to compute it. Consider $S^{m\rho_2}$ as $m \geq 0$ a G-CW complex. This leads to a reduced cellular chain complex of $\mathbb{Z}[G]$-modules. In this case it has the form

$$C_i = \begin{cases} 2 & \text{for } i = m \\ 2\ln & \text{for } m < i \le 2m \\ 0 & \text{else} \end{cases}$$

Example $m = 2$ $\qquad \mathbb{Z}G = \mathbb{Z}[G] = \mathbb{Z}[\gamma]/(\gamma^2 - 1)$

$$2 \qquad 3 \qquad 4$$

$$\mathbb{Z} \xleftarrow{d_3} \mathbb{Z}G \xleftarrow{d_4} \mathbb{Z}G$$

The $\mathbb{Z}G$-linear boundary operator is determined by the fact that $H_* C = \bar{H}_* S^{2m}$

$$\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}G \xleftarrow{1-\gamma} \mathbb{Z}G$$

$$1 \longleftarrow\!\shortmid \gamma$$

$m = 3$ $\qquad 3 \qquad 4 \qquad 5 \qquad 6$

$$\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}G \xleftarrow{1-\gamma} \mathbb{Z}G \xleftarrow{1+\gamma} \mathbb{Z}G$$

$\mathbb{Z} \in C_n$ has trivial G-action.

To find $\pi_*^G(H\mathbb{Z} \wedge S^{m\rho_2}) = H_*(C_*^G)$

For $m = 3$ we get

$$\begin{array}{lccccccc}
C^G & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} \\
& \downarrow 1 & & \downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta \\
C & \mathbb{Z} & \xleftarrow[\nabla]{} & \mathbb{Z}G & \xleftarrow[1-\gamma]{} & \mathbb{Z}G & \xleftarrow[1+\gamma]{} & \mathbb{Z}G
\end{array}$$

$$0 \longleftarrow\!\shortmid 1-\gamma$$

$\Delta(1) = 1 + \gamma \in \mathbb{Z}$

Passing to homology we get

$$\mathbb{Z}/2 \qquad 0 \qquad \mathbb{Z}/2 \qquad 0$$

$0 \quad 0 \quad 0 \quad \rightarrow \quad z_-$

z with signification of c_z

$m = 4$

$$\begin{tikzcd}[column sep=large]
 & 4 & 5 & 6 & 7 & 8 \\
C^{h} & \mathbb{Z} \arrow[d, "1"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, "\Delta"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, "\Delta"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, "\Delta"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, "\Delta"'] \\
C & \mathbb{Z} & \mathbb{Z}G \arrow[l, "\nabla"'] & \mathbb{Z}G \arrow[l, "1-\gamma"'] & \mathbb{Z}G \arrow[l, "1+\gamma"'] & \mathbb{Z}G \arrow[l, "1-\gamma"']
\end{tikzcd}$$

$0 \longleftarrow 1+\gamma$

In H_* we have

$$\begin{tikzcd}
\mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z} \arrow[d, "1"'] \\
0 & 0 & 0 & 0 & \mathbb{Z}
\end{tikzcd}$$

$\mathbb{Z}$ with trivial C_2-action

In this way we get the E_2-term for the slice SS for MU

There are differentials in the SS ANOTHER TIME.