Mingkong to speak on periodicity thm ??? to verifying PC model category structure on Sp^G.

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Bousfield localization 1975

Idea. Let M be a topological model cat i.e. one enriched over $\mathcal{T}$.

Lemma 3.13.1 Let $X \xrightarrow{f} X'$ be a map of cofibrant objects. Then f is a weak equivalence $\Leftrightarrow$ for each fibrant object Y, the map $f^*: M(X', Y) \to M(X, Y)$ is equivalence.

Idea of Bousfield locazn.

Let M be a top cofib gen model category with generating sets I and J.

Will construct a new model cat M' with the same underlying category as M but with more weak equivalences. Cofibrations in M' are the same as cofibs in M; more of them are trivial.

Fibrations are defined in terms of lifting properties. There are fewer in M' than in M. Fibrant replacement is more drastic than before.

The hard part in proving that $\mathcal{M}'$ is a model category is verifying MC5, the factorization axiom. Recall any morphism $X \xrightarrow{f} Y$ can be factored in 2 ways:

i) $X \xrightarrow{\text{cofib}} X' \xrightarrow[\text{fib}]{\text{triv}} Y$ (same as in $\mathcal{M}$)

ii) $X \xrightarrow[\text{cofib}]{\text{triv}} X'' \xrightarrow{\text{fib}} Y$ (HARD)

$\mathcal{M}'$ is cofibrantly generated with the same set I as before, but a bigger generating set J' of trivial cofibrations. There is no good general description of J'.

Example ① (original example of Bousfield)

$\mathcal{M} = \mathcal{T} =$ pointed top spaces.

h_* = some generalized homology theory e.g. K-theory, $H_*(-;\mathbb{Z}/p)$, $K(n)_*$, BP_*, etc.

We say $f: X \longrightarrow X'$ is a weak equivalence if it induces an isomorphism in $h_*(-)$.

A space Y is h_*-local if any h_*-equivalence f induces an equivalence

$$\tilde{J}(X', Y) \xrightarrow{f^*} \tilde{J}(X, Y)$$

so Y must be fibrant.

An h_*-<u>localization</u> of Y' is a map $Y' \xrightarrow{e} Y$ such that Y is h_*-local and e is an h_*-equivalence

Thm (Bousfield) Localizations exist in all cases, i.e. for all homology theories h_* and all spaces Y.

He proved it using fibrant replacement. This can also be done in the category of spectra. This led to chromatic homotopy theory.

<u>Example 2</u> Let $\mathcal{S}p$ be the category of orthogonal spectra, i.e functors $\mathcal{J} \to \mathcal{T}$. Given a CGMC $\mathcal{M}$ (such as $\mathcal{T}$) and a small category J (such as the Mandell-May category $\mathcal{J}$) one can define a CGMC structure on the functor category $\mathcal{M}^J$ = category of functors $J \to \mathcal{M}$. in which a map $X \longrightarrow Y$ is a weak equiv/fibration if $X_j \longrightarrow Y_j$ is one for each j.

This is the <u>strict</u> MC structure on $\mathcal{M}^J$.

e.g. J = Mandell-May's $\mathcal{J}$ \mathscr{J}

$\mathcal{M} = \mathcal{T}$ = cat of pointed spaces

$\mathcal{I} = \{ S^{-k} \wedge (S^{n-1}_+ \to D^n_+) : k, n \geq 0 \}$

\mathcal{J} $= \{ S^{-k} \wedge (I^n_+ \to I^{n+1}_+) : k, n \geq 0 \}$

All spectra are fibrant.
CW-spectra are cofibrant.

Def A map $f: X \longrightarrow Y$ of spectra is a <u>stable equivalence</u> if

$$\operatorname{colim}_k \Omega^k X_{n+k} \longrightarrow \operatorname{colim}_k \Omega^k Y_{n+k}$$

is a weak equivalence for all $n \geq 0$.

This notion is weaker than strict equivalence. AKB localization works here. The fibrant objects are the "Ω-spectra", i.e. spectra X with $X_n \xrightarrow{\simeq} \Omega^k X_{n+k}$ an equivalence for all n and k. Fibrant replacement is $X_n \longmapsto \operatorname{colim}_k \Omega^k X_{n+k}$

Example (3)

Fix a map $A \xrightarrow{f} B$ in $\mathcal{M}$.

An ~~objects~~ object W is <u>f-local</u> if

$$\mathcal{M}(B,W) \xrightarrow{f^*} \mathcal{M}(A,W) \text{ is a weak equiv}$$

A map $g: X \to Y$ is an <u>f-local equivalence</u>

if $g^*: \mathcal{M}(Y,W) \to \mathcal{M}(X,W)$ is a weak equivalence for all f-local W.

The case $\mathcal{M} = \mathcal{T}$ was studied by Dror Farjoun in 1996. We get a new model cat $\mathcal{M}'$ in which the fibrant objects are the f-local objects.

<u>subexample</u> $\mathcal{M} = \mathcal{T}$ $\qquad S^{n+1} \underset{\leftarrow}{\xrightarrow{f}} *$

A space W is f-local if $\Omega^{n+1} W \simeq *$.

i.e. $\pi_i W = 0$ for $i > n$.

For a space X, the fibrant replacement $L_f X$ is $P^n X$ = nth Postnikov section of X

= space obtained from X by attaching cells to kill all π_i for $i > n$.

The fiber of $X \to P^n X$ is the fiber is $P_{n+1} X$, the n-connected cover of X.

The full subcat $\mathcal{C}_n$ of n-connected spaces has the following properties

1) If $X \simeq Y$ and $Y \in \mathcal{C}_n$ then $X \in \mathcal{C}_n$

2) Given a cofiber sequence $W \to X \to Y$ with $W \in \mathcal{C}_n$ then $X \in \mathcal{C}_n \Leftrightarrow Y \in \mathcal{C}_n$

3) Any wedge of objects in C_n is in C_n

4) C_n is closed under retracts

5) C_n " filtered colimits