CM MZ YQ YZ SD

McTague on homotopic categories II

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Recall a homotopical category $\mathcal{M}$ is equipped with a subcat (of weak equivs) $\mathcal{W}$ satisfying the 2-of-6 condition.

We can localize w.r. to $\mathcal{W}$ and get a functor
$\mathcal{M} \xrightarrow{\gamma} \mathrm{Ho}(\mathcal{M})$ (one sending weak equivs to isos)

Every homotopy functor from $\mathcal{M}$ factors uniquely from $\mathcal{M}$. A homotopical functor $\mathcal{M} \to \mathcal{N}$ induces $\mathrm{Ho}(\mathcal{M}) \longrightarrow \mathrm{Ho}(\mathcal{N})$.

We want to consider the Yoneda functor
$h^X := \mathcal{M}(X, -)$ from $\mathcal{M}$ to Set

<u>Prop</u> The natural trans^n (induced by γ) $\mathcal{M}(X,-) \Longrightarrow \mathrm{Ho}\,\mathcal{M}(\gamma X, \gamma(-))$ is the universal transformation h^X to a homotopy functor. That is given a nat trans $h^X \Longrightarrow F$ (homotopy functor $\mathcal{M} \to$ Set)

(1)
$$\begin{tikzcd} h^X \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\gamma"'] & F \\ \mathrm{Ho}\,\mathcal{M}(\gamma X, \gamma(-)) \arrow[ur, Rightarrow, dashed, "\exists!"'] & \end{tikzcd}$$

Proof Every homotopy functor F from $\mathcal{M}$ factors uniquely as $\hat{F}\gamma$. This implies for homotopy functors $F, G : \mathcal{M} \to$ Set ,

$$\{\check{F} \Rightarrow G\} \longleftrightarrow \{F \Rightarrow G\}$$

By Yoneda lemma $\{h^X \Rightarrow F\} \longleftrightarrow \{F(X)\}$

$$\eta \longmapsto \eta_X(1_X)$$

and $\{Ho\,\mathcal{M}(\gamma(X), \gamma(-)) \Rightarrow F\}$

$$\underset{\gamma}{\longleftrightarrow} \{Ho\,\mathcal{M}(\gamma X, -) \Rightarrow \tilde{F}\}$$

$$\underset{\text{Yoneda}}{\longleftrightarrow} \tilde{F}(\gamma X) = F(X).$$

Claim the map $F(X) \to F(X)$ obtained by the 2 ways of going around ① is the identity.

QED

Cor If $\mathcal{M}$ is a homotopical category and $X \in \mathcal{M}$ has the property that $\mathcal{M}(X,-)$ is a homotopy functor then $\mathcal{M}(X,-) \to Ho\,\mathcal{M}(X,-)$ is a bijection.

We can discuss derived functors here as we did in model categories. We need a method to guarantee their existence.

Def Given functor $F: \mathcal{M} \to \mathcal{C}$ from a homotopical $\mathcal{M}$ we define LF and RF as right and left Kan extensions of

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{C} \\ Ho(\mathcal{M}) \arrow[ur, dashed] & \end{tikzcd}$$

We need an analog of fibrant + cofibrant replacement, i.e. a good family of objects for a given functor.

Def A left deformation on a homotopical cat $\mathcal{M}$ is a functor $Q: \mathcal{M} \to \mathcal{M}$ with nat trans $q: Q \Rightarrow 1$ inducing a weak equiv on each object.

Def A left deformation retract $\mathcal{M}_Q \subseteq \mathcal{M}$ is a full the? subcat whose objects are in image of Q. TYPO in ESHT? (Analog of cofibrant objects.) It is is a left F-deformation retract if the restriction of F to $\mathcal{M}_Q$ is homotopical. [Compare existence result for $\mathbb{L}F$: $F\begin{pmatrix}\text{tw cofibs of} \\ \text{cofibrant objects}\end{pmatrix}$ = weak equivalences

Def A left deformation of a functor $F: \mathcal{M} \to \mathcal{N}$ between homotopical categories is a left deformation on $\mathcal{M}$ such that F is homotopical on a (the?) subcat $\mathcal{M}_Q$ of cofibrant objects.

Def If F admits a left deformation retract then it is left deformable.

Ex Suppose F is a functor on $\mathcal{T}$ which is homotopical on spaces with nondegen base points, then adding a whisker is a left points, then adding a whisker is a left deformation for F.

Note Q is always homotopical. Given $X \xrightarrow[\sim]{f} Y$

$$\begin{tikzcd} Q(X) \arrow[r, "Q(f)"] \arrow[d, "q"', "\simeq"] & Q(Y) \arrow[d, "q"', "\simeq"] \\ X \arrow[r, "f", "\sim"'] & Y \end{tikzcd} \quad (?)$$

Ex When $\mathcal{M}$ is a model cat, cofibrant replacement Q is a left deformation for any left Quillen functor

Thm (Existence of left derived functor) If $F: \mathcal{M} \to \mathcal{N}$ is a functor between homotopical categories has a left deformation Q, then F has a left derived functor $\mathbb{L}F = FQ$

Proof
$$\begin{tikzcd} \mathcal{M} \arrow[r, "Q"] \arrow[drr, bend right] & \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathcal{N} \\ & Ho(\mathcal{M}) \arrow[ur, dashed] & \end{tikzcd}$$

FQ is homotopy functor so it factors uniquely thru γ. QED

Flat functors and flat maps

Def A *flat functor* $\mathcal{M} \xrightarrow{F} \mathcal{N}$ between homotopical complete categories is one which is homotopical and limit preserving.

Def A *flat map* $f: A \to X$ in such a cat is one with the property that $\forall$ map $A \to B$ and weak equiv $B \to B'$, the map $X \cup_A B \to X \cup_A B'$ is a weak equiv is a weak equiv.

Def A <u>flat object</u> X in a symm monoidal complete homotopical category $\mathcal{M}$ is one for which $X \otimes -$ is a flat functor.

<u>Future example</u> In a left proper (to be defined later) model cat, all cofibrations are flat.

<u>Properties of flat maps</u>

i) all finite coproducts of flat maps are flat.

ii) Compositions of flat maps are flat

iii) Any cobase change of a flat map is flat.

$$\begin{tikzcd} A \arrow[r, "\text{flat}"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d] \\ X \arrow[r, "\text{flat}"'] & X \cup B \end{tikzcd}$$

iv) If weak equivs are closed under retraction, so are flat maps.

flat symbol

Prop Given

$$\begin{tikzcd} X_1 \arrow[d, "\simeq"'] & A_1 \arrow[l, "\flat"'] \arrow[r, "\flat"] \arrow[d, "\simeq"'] & Y_1 \arrow[d, "\simeq"'] \\ X_2 & A_2 \arrow[l] \arrow[r, "\flat"] & Y_2 \end{tikzcd}$$

Each row is called a "tent"?

then induce map of pushouts is a weak equiv

Recall first example of 2/29

Pf First assume $A_1 = A_2 = A$. Then

$X_1 \cup_A Y_1 \longrightarrow X_1 \cup_A Y_2$ is weak equiv since $A \xrightarrow{b} X_1$

The map $X_1 \longrightarrow X_1 \cup_A Y_2$ is flat by cobase change property

$$\begin{tikzcd}
Y_2 \arrow[r] & X_1 \cup_A Y_2 \\
A \arrow[u, "\text{flat}"] \arrow[r] \arrow[ur, phantom, "\Longrightarrow"] & X_1 \arrow[u, "\text{flat}"']
\end{tikzcd}$$

This implies $X_1 \cup_A Y_2 \cong X_1 \cup_{X_1} (X_1 \cup_A Y_2)$

$$\downarrow \simeq$$

$$X_2 \cup_{X_1} (X_1 \cup_A Y_2) \cong X_2 \cup_A Y_2$$

Hence $X_1 \cup_A Y_1 \longrightarrow X_1 \cup_A Y_2 \longrightarrow X_2 \cup_A Y_2$ is weak equiv.

Now stretch the original diagram to

$$\begin{tikzcd}
X_1 \arrow[d, "\simeq"'] & A_1 \arrow[l, "b"'] \arrow[r, "b"] \arrow[d, "\simeq"] \arrow[dl, phantom, "\Downarrow"] \arrow[dr, phantom, "\Downarrow"] & Y_1 \arrow[d, "\simeq"] \\
X_1 \cup_{A_1} A_2 \arrow[d] & A_2 \arrow[l, "b"'] \arrow[r, "b"] \arrow[d, equal] & A_2 \cup_{A_1} Y_1 \arrow[d] \\
X_2 & A_2 \arrow[l] \arrow[r, "b"] & Y_2
\end{tikzcd}$$

- flatness of $A_1 \to X_1, Y_1$ implies $\simeq$
- Cobase change implies $A_1 \to X_1 \cup_{A_1} Y_1$ is flat