The definition of orthogonal G-spectra

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The original definition (1959). A spectrum E is a collection of spaces E_n for $n \geq 0$ with maps $\varepsilon_n : \Sigma E_n \to E_{n+1}$. A map $E \xrightarrow{f} F$ is a collection of maps $f_n : E_n \to F_n$ with

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d, "\varepsilon_n"'] & \Sigma F_n \arrow[d, "\varepsilon_n"] \\ E_{n+1} \arrow[r, "f_{n+1}"] & F_{n+1} \end{tikzcd}$$

all maps are continuous

The structure $\varepsilon_n : \Sigma E_n \to E_{n+1}$ is adjoint to a map $E_n \xrightarrow{\bar{\varepsilon}_n} \Omega E_{n+1}$. We can iterate this

$$E_n \xrightarrow{\bar{\varepsilon}_n} \Omega E_{n+1} \xrightarrow{\Omega \bar{\varepsilon}_{n+1}} \Omega^2 E_{n+2} \to \cdots .$$

We define $\pi_k E = \pi_k \operatorname{colim} \Omega^n E_n$

$$E_0 \to \Omega E_1 \to \Omega^2 E_2 \to \cdots$$

Problem: Smash products are awkward.

In 1969 Boardman-Vogt described the homotopy category spectra $Ho(\mathcal{S})$. It is still used today.

EKMM Elmendorf, Kriz, Mandell, May 1997 gave a definition of spectra as a closed symmetric monoidal category

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The modern definition is due to Mandell-May 2002. It uses enriched category theory. The smash product is defined as a left Kan extension.

A spectrum is a functor $\mathcal{J}_G \xrightarrow{E} \mathcal{T}_G$ where $\mathcal{T}_G$ is the category of pointed G-spaces, which is enriched over itself.

$\mathcal{J}_G$ is SMC enriched over $\mathcal{T}_G$.

The objects of $\mathcal{J}_G$ are finite dimensional real orthogonal representations of G.

The morphism object $\mathcal{J}_G(V,W)$ as follows:

Let $O(V,W)$ be the space of orthogonal embeddings $V \hookrightarrow W$ (not required to be equivariant). It could be empty.

Such an embedding $V \xrightarrow{t} W$ defines an orthogonal complement $W - t(V)$. This defines a vector bundle $/\,O(V,W)$ of dimension $\dim W - \dim V$. $\mathcal{J}_G(V,W)$ is its Thom space. It is a pointed G-space.

Composition is a map $\qquad U \hookrightarrow V \hookrightarrow W$

$$\mathcal{J}_G(V,W) \wedge \mathcal{J}_G(U,V) \longrightarrow \mathcal{J}_G(U,W)$$

for objects U, V and W.

$\mathcal{J}_G$ is a category enriched over

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$(\mathcal{T}_G, \wedge, S^0)$

$\mathcal{J}_G$ is also a symmetric monoidal category under $\oplus$

We have $\mathcal{J}_G(V', W') \wedge \mathcal{J}_G(V'', W'') \longrightarrow \mathcal{J}_G(V' \oplus V'', W' \oplus W'')$

$V' \hookrightarrow W', \ V'' \hookrightarrow W'' \rightsquigarrow V' \oplus V'' \hookrightarrow W' \oplus W''$

Note $\mathcal{T}_G$ is a <u>closed</u> symmetric monoidal category, i.e. it has an internal hom functor that coincides with the categorical hom functor

$\mathcal{J}_G$ is <u>not</u> a closed symmetric monoidal category.

<u>Main Definition</u> An orthogonal G-spectrum E is a functor

$$\begin{aligned} \mathcal{J}_G &\xrightarrow{E} \mathcal{T}_G \\ V &\longmapsto E_V \end{aligned}$$

There are equivariant structure maps $\mathcal{J}_G(V, W) \wedge E_V \to E_W$ for each V, W.

<u>Examples</u>

1) For $\dim W < \dim V$, $\mathcal{J}_G(V, W) = *$.

disjoint base pt

2) For $\dim W = \dim V$, $\mathcal{J}_G(V, W) = O(V, W)_+$

2) For $\dim W = \dim V$, $J_G(V,W) = O(V,W)_+$

3) $J_G(0,W) = S^W :=$ one point compactification of W

For $\dim W > \dim V$, $\mathcal{J}_G(V,W)$ is $(\dim W - \dim V - 1)$-connected, and a choice of embedding $t: V \to W$ leads to a map $S^{W-t(V)} \longrightarrow \mathcal{J}_G(V,W)$

e.g. $V = \mathbb{R}^n$, $W = \mathbb{R}^{n+1}$ and we have chosen $t: \mathbb{R}^n \to \mathbb{R}^{n+1}$. Then we have $S^1 \to \mathcal{J}(\mathbb{R}^n, \mathbb{R}^{n+1})$

$$\begin{array}{l} \Sigma E_n \longrightarrow \mathcal{J}(\mathbb{R}^n, \mathbb{R}^{n+1}) \wedge E_n \longrightarrow E_{n+1} \\ \quad \shortparallel \\ S^1 \wedge E_n \end{array}$$

$\mathcal{J}(\mathbb{R}, \mathbb{R}^{n+1})$ = Thom space of the tangent bundle of S^n

$\mathcal{J}(\mathbb{R}^n, \mathbb{R}^{n+1})$ = Thom space of normal bundle of $S^n \hookrightarrow \mathbb{R}^{n+1}$

<u>Example</u> The Yoneda spectrum S^{-V} is defined by $(S^{-V})_W = \mathcal{J}_G(V,W)$

e.g. $(S^{-0})_W = \mathcal{J}_G(0,W) = S^W$

S^{-0} is the sphere spectrum.

$\mathcal{S}_G$ = category of G-spectra as defined + continuous maps

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Description of $\mathcal{S}_G(E,F)$ (categorical hom). It is a pointed G-space since $\mathcal{S}_G$ is enriched over $\mathcal{T}_G$. It is also the set of of natural transformations $F \Rightarrow G$. It is the end

$$\int_{V\in \mathcal{J}_G} \mathcal{T}_G(E_V, F_V) \hookrightarrow \prod_V \mathcal{T}_G(E_V, F_V)$$

The smash product $E\wedge F$ is

$$\begin{tikzcd} \mathcal{J}_G\times\mathcal{J}_G \arrow[r, "E\times F"] \arrow[dr, "\oplus"'] & \mathcal{T}_G\times\mathcal{T}_G \arrow[r, "\wedge"] & \mathcal{T}_G \\ & \mathcal{J}_G \arrow[ur, dashed, "\mathrm{Lan}_{\oplus}(\wedge\cdot E\times F)"'] & \end{tikzcd}$$

$$E\wedge F = \int^{(V',V'')\in \mathcal{J}_G\times\mathcal{J}_G} S^{-V'\oplus V''}\wedge E_{V'}\wedge E_{V''}$$

We can define the smash product of a spectrum E with a space X by $(E\wedge X)_V = E_V\wedge X$.

<u>Example</u> $S^{-0}\wedge E \cong E$

$S^{-V}\wedge S^{-W} \cong S^{-V\oplus W}$

$$E \wedge F = \int^{\mathfrak{d}_G \times \mathfrak{d}_G} S^{-V' \oplus V''} \wedge E_{V'} \wedge F_{V''}$$

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$$(E \wedge F)_W = \int^{\mathfrak{d}_G \times \mathfrak{d}_G} \left(S^{-V' \oplus V''}\right)_W \wedge E_{V'} \wedge F_{V''}$$

$$= \int^{\mathfrak{d}_G \times \mathfrak{d}_G} \mathfrak{d}_G(V' \oplus V'', W) \wedge E_{V'} \wedge F_{V''}$$

Qiaofeng to lecture on 3.1–3.2

Mingcong " 3.3–3.4

Carl " 3.5–3.8