YU, QZ, QS, SD, CM, YQ

Vogt on enriched categories and weighted limits and colimits

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Some categories have natural structures in their morphism sets.

<u>Def</u> Let $V=(V_0, \otimes, I)$ be a symmetric monoidal category. A <u>V-category</u> C consists of a collection of objects, for each pair $X, Y \in ob\,C$ a morphism object $C(X,Y) \in ob\,V_0$, for each object X an identity morphism $I \xrightarrow{Id_X} C(X,X)$ in V_0, and for each tuple $X, Y, Z \in ob\,C$ a composition law $C(Y,Z) \otimes C(X,Y) \xrightarrow{\circ} C(X,Z)$.

These data are required to satisfy associativity property

$$\begin{tikzcd}
C(Z,W)\otimes C(Y,Z)\otimes C(X,Y) \arrow[r, "C(Z,W)\otimes \circ"] \arrow[d, "\circ \otimes C(X,Y)"'] & C(Z,W)\otimes C(X,Y) \arrow[d, "\circ"] \\
C(Y,W)\otimes C(X,Y) \arrow[r, "\circ"] & C(X,W)
\end{tikzcd}$$

and unit property

$$\begin{tikzcd}
C(X,Y)\otimes I \arrow[r, "C(X,Y)\otimes Id_X"] \arrow[dr, "\cong"'] & C(X,Y)\otimes C(X,X) \arrow[d, "\circ"] \\
& C(X,Y)
\end{tikzcd}$$

and similarly for $I \otimes C(X,Y)$.

There is an <u>underlying ordinary category</u> C_0 with same objects as C with $C_0(X,Y) = V_0(I, C(X,Y))$

Example

① $(\mathcal{S}et, \times, *)$ is enriched over itself

② For a gp G, the category Top^G of G-spaces and equivariant maps. It is enriched over $(Top, \times, *)$ [Riehl denotes the space of such maps by $\underline{Top}^G(X,Y)$ and the underlying set by $Top^G(X,Y)$.] Top^G is symmetric monoidal under Cartesian product with diagonal action.

For G-spaces X, Y, $\underline{Top}(X,Y)$ is a G-space under conjugation. For $f: X \to Y$ and $g \in G$, $(gf)(x) := gf(g^{-1}(x))$. We denote the resulting category by $Top_G(X,Y)$, which is enriched over Top^G and over Top_G.

As a $\underline{Top}^G$-enriched category, $\underline{Top}_G$ is underlain by Top^G.

$$Top^G(*, \underline{Top}_G(X,Y)) = Top^G(X,Y) = Top(*, \underline{Top}^G(X,Y))$$

Def Let $\mathcal{C}$ be a category enriched over a SMC $\mathcal{V}$. Then $\mathcal{C}$ is <u>tensored (or copowered)</u> over $\mathcal{V}$ if $\forall$ object K in $\mathcal{V}$ and X in $\mathcal{C}$ there is an object $K \cdot X$ in $\mathcal{C}$ with a natural isomorphism in $\mathcal{V}$

$$\mathcal{C}(K \cdot X, Y) \cong \mathcal{V}(K, \mathcal{C}(X,Y)) \quad \text{for each object } Y \text{ in } \mathcal{C}$$

In other words tensoring with X as a functor $\mathcal{V} \to \mathcal{C}$ is left adjoint of $\mathcal{C}(X,-) : \mathcal{C} \to \mathcal{V}$

Dually $\mathcal{C}$ is <u>cotensored (or **powered**) over $\mathcal{V}$</u> if there is an object Y^K with natural isomorphisms

$$\mathcal{C}(X, Y^K) \cong \mathcal{V}(K, \mathcal{C}(X,Y))$$

<u>Limits and colimits in enriched category</u>

Let $F : J \to \mathcal{C}$ for small category J

We define a J-set (functor $J \to \mathrm{Set}$) $\mathcal{C}(c,F)$ by $j \mapsto \mathcal{C}(c, F_j)$ and a J^{op} set $\mathcal{C}(F,c)$ by $j \mapsto \mathcal{C}(F_j, c)$

We have a $*$-valued J-set and J^{op} set

Then the limit and colimit of F (if they exist) are characterized by

$$(1)\quad \begin{cases} \mathcal{C}(c, \lim F) \cong \mathrm{Set}^J(*, \mathcal{C}(c,F)) & \text{and} \\ \mathcal{C}(\operatorname{colim} F, c) \cong \mathrm{Set}^{J^{op}}(*, \mathcal{C}(F,c)) \end{cases}$$

(This is a restatement of adjunctions $\operatorname{colim} \dashv \Delta$ and $\Delta \dashv \lim$.)

We can generalized (1) by replacing $*$ by a J-set (J^{op}-set) W called the <u>weight</u>

and define weight limit / colimit, $\lim^W$, colim^W as in 1.

Assume $\mathcal{C}$ is complete. Then

(2) $\mathbf{Set}^J(W, \mathcal{C}(c,F)) \cong \int_J \mathbf{Set}(W_j, \mathcal{C}(c,F_j))$

Prop 2.4.7 Suppose we have two functors $F, G : \mathcal{C} \to \mathcal{E}$ where $\mathcal{C}$ is small. Let $H : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ by $H(c,c') = \mathcal{E}(F(c), G(c'))$

Then $\int_{\mathcal{C}} H(c,c') = \int_{\mathcal{C}} \mathcal{E}(F(c), G(c'))$

$= \mathrm{Nat}(F,G) = [\mathcal{C}, \mathcal{E}](F,G)$

For $\mathcal{E} = \mathbf{Set}$ this gives (2) and

$$\mathbf{Set}^J(W, \mathcal{C}(c,F)) \cong \int_J \mathbf{Set}(W_j, \mathcal{C}(c,F_j))$$
$$\cong \int_J \mathcal{C}(c, F_j^{W_j})$$ cotensor
$$\cong \mathcal{C}(c, \int_J F_j^{W_j})$$

so $\lim^W F = \int_J F_j^{W_j}$

In particular, $\lim^W F$ is just the set of natural transformations from W to F.

<u>Example</u> ① By Yoneda lemma, the limit of F weighted by the representable functor $\mathcal{C}(c,-)$ is Fc

② For $\mathcal{C}$ complete and J small, for any functor $F: J \to \mathcal{C}$ and $K: J \to \mathcal{D}$, $\mathrm{Ran}_K F$ is defined by

$$\mathrm{Ran}_K F(d) = \int_J F_j^{\mathcal{D}(d,K_j)} = \lim_J^{\mathcal{D}(d,K-)} F$$

③ Dually

$$\mathrm{Lan}_K F(d) = \int^J \mathcal{D}(K_j, d)\cdot F_j = \mathrm{colim}^{\mathcal{D}(K-,d)} F$$

<u>Def</u> A <u>V-functor</u> $F: \mathcal{D} \to \mathcal{C}$ of V-categories consists of a function $F: ob\,\mathcal{D} \to ob\,\mathcal{C}$ and for each pair of objects X, Y a V-morphism $F: \mathcal{D}(X,Y) \to \mathcal{C}(FX, FY)$ compatible with unit and composition

$$\begin{tikzcd}
\mathcal{D}(Y,Z)\otimes\mathcal{D}(X,Y) \arrow[r, "\circ"] \arrow[d, "F_{Y,Z}\otimes F_{X,Y}"'] & \mathcal{D}(X,Z) \arrow[d, "F_{X,Z}"] \\
\mathcal{C}(FY,FZ)\otimes\mathcal{C}(FX,FY) \arrow[r, "\circ"] & \mathcal{C}(FX,FZ)
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "id_X"] \arrow[d, "Id"'] & \mathcal{D}(X,X) \arrow[d, "F_{X,X}"] \\
{} & {}
\end{tikzcd}$$

$$\begin{tikzcd} \ldots(X,X) \arrow[d, "F_{X,X}"] \\ \mathcal{C}(FX,FX) \end{tikzcd}$$

Id_{FX}

A $\mathcal{V}$-natural transformation $T : F \Rightarrow G$ assigns to each object X a morphism $T_X : 1 \to \mathcal{C}(FX, FY)$ making the diagram

$$\begin{tikzcd}
\mathcal{D}(X,Y) \arrow[r, "T_Y \otimes F"] \arrow[d, "G \otimes T_X"'] & \mathcal{C}(FX, GY) \otimes \mathcal{C}(FX, FY) \arrow[d, "\circ"] \\
\mathcal{C}(GX, GY) \otimes \mathcal{C}(FX, GY) \arrow[r, "\circ"] & \mathcal{C}(FX, GY)
\end{tikzcd}$$

commute for all X, Y.